COMPACTNESS OF MATRIX OPERATORS ON SOME SEQUENCE SPACES DERIVED BY FIBONACCI NUMBERS

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ABSTRACT. In this paper, we apply the Hausdorff measure of noncompactness to obtain the necessary and sufficient conditions for certain matrix operators on the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ to be compact, where $1 \leq p < \infty$.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{R}$ be the set of all real numbers. We shall write $\lim_k$, $\sup_k$, $\inf_k$ and $\sum_k$ instead of $\lim_{k \to \infty}$, $\sup_{k \in \mathbb{N}}$, $\inf_{k \in \mathbb{N}}$ and $\sum_{k=0}^{\infty}$, respectively. Let ω be the vector space of all real sequences $x = (x_k)_{k \in \mathbb{N}}$. By the term *sequence space*, we shall mean any linear subspace of ω. Let φ, ℓ_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences, respectively. We write $\ell_p = \{x \in \omega : \sum_k |x_k|^p < \infty\}$ for $1 \leq p < \infty$. Also, we shall use the conventions that $e = (1, 1, \ldots)$ and $e^{(n)}$ is the sequence whose only non-zero term is 1 in the nth place for each $n \in \mathbb{N}$. For any sequence $x = (x_k)$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section. Morever, we write bs and cs for the sets of series with bounded and convergent partial sums, respectively.

The *Fibonacci numbers* are the sequence of numbers $\{f_n\}_{n=0}^{\infty}$ defined by the linear recurrence equations

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2}; \quad n \geq 2.$$

Fibonacci numbers have many interesting properties and applications in arts, sciences and architecture. For example, the ratio sequences of Fibonacci numbers converge to the golden ratio which is important in sciences and arts. Also, some basic properties of Fibonacci numbers can be found in [19].

A $B-space$ is a complete normed space. A topological sequence space in which all coordinate functionals π_k, $\pi_k(x) = x_k$, are continuous is called a $K-space$. A $BK-space$ is defined as a $K-$ space which is also a $B-$ space, that is, a $BK-$ space is a Banach space with continuous coordinates. A $BK-$ space $X \supset \varphi$ is said to have AK if every sequence $x = (x_k) \in X$ has a unuqiue representation $x = \sum_k x_k e^{(k)}$. For example, the space ℓ_p $(1 \leq p < \infty)$ is $BK-$ space with $\|x\|_p = (\sum_k |x_k|^p)^{1/p}$ and c_0, c and ℓ_∞ are $BK-$ spaces with $\|x\|_\infty = \sup_k |x_k|$. Further, the $BK-$ spaces c_0 and ℓ_p have AK, where $1 \leq p < \infty$ ([22, 4]).

A sequence (b_n) in a normed space X is called a *Schauder basis* for X if for every $x \in X$ there is a unique sequence (α_n) of scalars such that $x = \sum_n \alpha_n b_n$, i.e., $\lim_m \|x - \sum_{n=0}^m \alpha_n b_n\| = 0$.

The β-dual of a sequence space X is defined by

$$X^\beta = \left\{ a = (a_k) \in \omega : ax = (a_k x_k) \in cs \text{ for all } x = (x_k) \in X \right\}.$$

Let $A = (a_{nk})_{n,k=0}^\infty$ be an infinite matrix of real numbers a_{nk}, where $n, k \in \mathbb{N}$. We write A_n for the sequence in the nth row of A, that is $A_n = (a_{nk})_{k=0}^\infty$ for every $n \in \mathbb{N}$. In addition, if $x = (x_k)_{k=0}^\infty \in \omega$ then we define the *A-transform of x* as the sequence $Ax = \{A_n(x)\}_{n=0}^\infty$, where

$$(1.1) \qquad\qquad A_n(x) = \sum_{k=0}^\infty a_{nk} x_k; \quad n \in \mathbb{N}$$

provided the series on the right side converges for each $n \in \mathbb{N}$.

For arbitrary subsets X and Y of ω, we write (X, Y) for the class of all infinite matrices that map X into Y. Thus, $A \in (X, Y)$ if and only if $A_n \in X^\beta$ for all $n \in \mathbb{N}$ and $Ax \in Y$ for all $x \in X$.

The matrix domain X_A of an infinite matrix A in a sequence space X is defined by

$$(1.2) \qquad\qquad X_A = \{x = (x_k) \in \omega : Ax \in X\}$$

which is a sequence space ([4]).

Let Δ denote the matrix $\Delta = (\Delta_{nk})$ defined by

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & (n-1 \leq k \leq n), \\ 0, & (0 \leq k < n-1) \text{ or } (k > n), \end{cases}$$

or

$$\Delta_{nk} = \begin{cases} (-1)^{n-k}, & (n \leq k \leq n+1), \\ 0, & (0 \leq k < n) \text{ or } (k > n+1). \end{cases}$$

In the literature, the matrix domain λ_Δ is called the *difference sequence space* whenever λ is a normed or paranormed sequence space. The idea of difference sequence

space was introduced by Kızmaz [17]. In 1981, Kızmaz [17] defined the sequence spaces

$$X(\Delta) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in X\}$$

for $X = \ell_\infty$, c and c_0. The difference space bv_p, consisting of all sequnces (x_k) such that $(x_k - x_{k-1})$ is in the sequence space ℓ_p, was studied in the case $0 < p < 1$ by Altay and Başar [2] and in the case $1 \leq p \leq \infty$ by Başar and Altay [5] and Çolak et al. [12]. The paranormed difference sequence space

$$\Delta\lambda(p) = \{x = (x_k) \in \omega : (x_k - x_{k+1}) \in \lambda(p)\}$$

was examined by Ahmad and Mursaleen [1] and Malkowsky [23], where $\lambda(p)$ is any of the paranormed spaces $\ell_\infty(p)$, $c(p)$ and $c_0(p)$ defined by Simons [37] and Maddox [20].

Recently, Başar et al. [3] have defined the sequence spaces $bv(u, p)$ and $bv_\infty(u, p)$ by

$$bv(u, p) = \{x = (x_k) \in \omega : \sum_k |u_k(x_k - x_{k-1})|^{p_k} < \infty\}$$

and

$$bv_\infty(u, p) = \{x = (x_k) \in \omega : \sup_{k \in \mathbb{N}} |u_k(x_k - x_{k-1})|^{p_k} < \infty\},$$

where $u = (u_k)$ is an arbitrary fixed sequence and $0 < p_k \leq H < \infty$ for all $k \in \mathbb{N}$. Also in [11, 13, 14, 18, 24, 25, 27, 28, 36, 38], authors studied some difference sequence spaces.

Let S_X denote the unit sphere in a normed linear space X. If $X \supset \varphi$ is a BK space and $a = (a_k) \in \omega$, then we write

(1.3)
$$\|a\|_X^* = \sup_{x \in S_X} \left| \sum_k a_k x_k \right|$$

provided the expression on the right side is defined and finite which is the case whenever $a \in X^\beta$.

The following results are very important in our study.

Lemma 1.1. [22, Theorem 1.29] *Let $1 < p < \infty$ and $q = p/(p-1)$. Then, we have $\ell_\infty^\beta = \ell_1$, $\ell_1^\beta = \ell_\infty$ and $\ell_p^\beta = \ell_q$. Furthermore, let X denote any of the spaces ℓ_∞, ℓ_1 or ℓ_p. Then, we have $\|a\|_X^* = \|a\|_{X^\beta}$ for all $a \in X^\beta$, where $\|.\|_{X^\beta}$ is the natural norm on the dual space X^β.*

Lemma 1.2. [22, Theorem 1.23 (a)] *Let X and Y be BK-spaces. Then we have $(X, Y) \subset B(X, Y)$, that is, every matrix $A \in (X, Y)$ defines a linear operator $L_A \in B(X, Y)$ by $L_A(x) = Ax$ for all $x \in X$, where $B(X, Y)$ denotes the set all bounded (continuous) linear operators $L : X \to Y$.*

Lemma 1.3. [22, Lemma 2.2] *Let $X \supset \phi$ be BK-space and Y be any of the spaces c_0, c or ℓ_∞. If $A \in (X, Y)$, then*

$$\|L_A\| = \|A\|_{(X,\ell_\infty)} = \sup_n \|A_n\|_X^* < \infty.$$

By M_X, we denote the collection of all bounded subsets of a metric space (X, d). If $Q \in M_X$, then the *Hausdorff measure of noncompactness* of the set Q, denoted by $\chi(Q)$, is defined by

$$\chi(Q) = \inf\left\{\varepsilon > 0 : Q \subset \overset{n}{\underset{i=1}{\cup}} B(x_i, r_i), \; x_i \in X, r_i < \varepsilon, (i = 1, 2, \ldots, n), n \in \mathbb{N} - \{0\}\right\}.$$

The function $\chi : M_X \to [0, \infty)$ is called the *Hausdorff measure of noncompactness*.

The basic properties of the Hausdorff measure of noncompactness can be found in [22].

The following result gives an estimate for the Hausdorff measure of noncompactness in the BK space ℓ_p for $1 \leq p < \infty$.

Lemma 1.4. [35, Theorem 2.8] *Let $1 \leq p < \infty$ and $Q \in M_{\ell_p}$. If $P_m : \ell_p \to \ell_p \; (m \in \mathbb{N})$ is the operator defined by $P_m(x) = (x_0, x_1, \ldots, x_m, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_p$, then we have*

$$\chi(Q) = \lim_{m \to \infty} \left(\sup_{x \in Q} \|(I - P_m)(x)\|_{\ell_p}\right),$$

where I is the identity operator on ℓ_p.

Let X and Y be Banach spaces. Then, a linear operator $L : X \to Y$ is said to be *compact* if the domain of L is all of X and $L(Q)$ is a totally bounded subset of Y for every $Q \in M_X$. Equivalently, we say that L is compact if its domain is all of X and for every bounded sequence (x_n) in X, the sequence $(L(x_n))$ has a convergent subsequence in Y.

The idea of compact operators between Banach spaces is closely related to the Hausdorff measure of noncompactness, and it can be given as follows.

Let X and Y be Banach spaces and $L \in B(X, Y)$. Then, the Hausdorff measure of noncompactness of L, is denoted by $\|L\|_\chi$, can be given by

(1.4) $$\|L\|_\chi = \chi(L(S_X))$$

and we have

(1.5) $$L \text{ is compact if and only if } \|L\|_\chi = 0.$$

The Hausdorff measure of noncompactness has various applications in the theory of sequence spaces, one of them is to obtain necessary and sufficient conditions for matrix operators between BK spaces to be compact. Recently, several authors have studied compact operators on the sequence spaces and given very important results related to the Hausdorff measure of noncompactness of a linear operator. For example [6-10,16,21,26,29-34].

In this paper, we derive some identities for the Hausdorff measure of noncompactness on the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ defined by Kara [15].

We also apply the Hausdorff measure of noncompactness to obtain the necessary and sufficient conditions for such operators to be compact.

2. The Fibonacci Difference Sequence Spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$

Throughout, let $1 \le p \le \infty$ and q denote the conjugate of p, that is, $q = p/(p-1)$ for $1 < p < \infty$, $q = \infty$ for $p = 1$ or $q = 1$ for $p = \infty$.

Recently, Kara[15] has defined the Fibonacci difference sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ by

$$\ell_p(\widehat{F}) = \left\{ x = (x_n) \in \omega : \sum_n \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right|^p < \infty \right\}; \quad 1 \le p < \infty$$

and

$$\ell_\infty(\widehat{F}) = \left\{ x = (x_n) \in \omega : \sup_{n \in \mathbb{N}} \left| \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1} \right| < \infty \right\}.$$

With the notation of (1.2), the sequence spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ may be redefined by

$$(2.1) \qquad \ell_p(\widehat{F}) = (\ell_p)_{\widehat{F}} \ (1 \le p < \infty) \ \text{ and } \ \ell_\infty(\widehat{F}) = (\ell_\infty)_{\widehat{F}},$$

where the matrix $\widehat{F} = (\widehat{f}_{nk})$ is defined by

$$(2.2) \qquad \widehat{f}_{nk} = \begin{cases} -\frac{f_{n+1}}{f_n}, & k = n-1, \\ \frac{f_n}{f_{n+1}}, & k = n, \\ 0, & 0 \le k < n-1 \text{ or } k > n, \end{cases} \qquad n, k \in \mathbb{N}.$$

Further, it is clear that the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$ are BK spaces with the norms given by

$$(2.3) \qquad \|x\|_{\ell_p(\widehat{F})} = \left(\sum_n |y_n|^p \right)^{1/p}; \quad 1 \le p < \infty \ \text{ and } \ \|x\|_{\ell_\infty(\widehat{F})} = \sup_n |y_n|,$$

where the sequence $y = (y_n) = (\widehat{F}_n(x))$ is the $\widehat{F}$-transform of a sequence $x = (x_n)$, i.e.,

$$(2.4) \qquad y_n = \widehat{F}_n(x) = \begin{cases} \frac{f_0}{f_1} x_0 = x_0, & n = 0, \\ \frac{f_n}{f_{n+1}} x_n - \frac{f_{n+1}}{f_n} x_{n-1}, & n \ge 1, \end{cases} \qquad n \in \mathbb{N}.$$

Moreover, it is obvious by (2.2) that $\widehat{F}$ is a triangle. Thus, it has a unique inverse $\widehat{F}^{-1} = (\widehat{f}_{nk}^{-1})$ which is also a triangle and the elements of $\widehat{F}^{-1}$ are given by

$$(2.5) \qquad \widehat{f}_{nk}^{-1} = \begin{cases} \frac{f_{n+1}^2}{f_k f_{k+1}}, & 0 \le k \le n, \\ 0, & k > n, \end{cases}$$

for all $n, k \in \mathbb{N}$. Therefore, we have by (2.4) that

$$(2.6) \qquad x_n = \sum_{k=0}^{n} \frac{f_{n+1}^2}{f_k f_{k+1}} y_k; \quad n \in \mathbb{N}.$$

In [15], the β-duals of the sequence spaces $\ell_p(\widehat{F})$ ($1 \le p < \infty$) and $\ell_\infty(\widehat{F})$ have been determined and some related matrix classes characterized. Now, by taking into account that the inverse of $\widehat{F}$ is given by (2.5), we have the following lemma which is immediate by [15, Theorem 4.6].

Lemma 2.1. *Let $1 \le p \le \infty$. If $a = (a_k) \in \{\ell_p(\widehat{F})\}^\beta$, then $\bar{a} = (\tilde{a}_k) \in \ell_q$ and we have*

$$(2.7) \qquad \sum_k a_k x_k = \sum_k \tilde{a}_k y_k$$

for all $x = (x_k) \in \ell_p(\widehat{F})$ with $y = \widehat{F}x$, where

$$(2.8) \qquad \tilde{a}_k = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_j; \quad k \in \mathbb{N}.$$

Now, we prove the following results which will be needed in the sequel.

Lemma 2.2. *Let $1 < p < \infty$, $q = p/(p-1)$ and $\bar{a} = (\tilde{a}_k)$ be the sequence defined by (2.8). Then, we have*

(a) *If $a = (a_k) \in \{\ell_\infty(\widehat{F})\}^\beta$, then $\|a\|_{\ell_\infty(\widehat{F})}^* = \sum_k |\tilde{a}_k| < \infty$.*
(b) *If $a = (a_k) \in \{\ell_1(\widehat{F})\}^\beta$, then $\|a\|_{\ell_1(\widehat{F})}^* = \sup_k |\tilde{a}_k| < \infty$.*
(c) *If $a = (a_k) \in \{\ell_p(\widehat{F})\}^\beta$, then $\|a\|_{\ell_p(\widehat{F})}^* = \left(\sum_k |\tilde{a}_k|^q\right)^{1/q} < \infty$.*

Proof. (a) Let $a = (a_k) \in \{\ell_\infty(\widehat{F})\}^\beta$. Then, it follows by Lemma 2.1 that $\bar{a} = (\tilde{a}_k) \in \ell_1$ and the equality (2.7) holds for all sequences $x = (x_k) \in \ell_\infty(\widehat{F})$ and $y = (y_k) \in \ell_\infty$ which are connected by the relation (2.4). Further, we have by (2.3) that $x \in S_{\ell_\infty(\widehat{F})}$ if and only if $y \in S_{\ell_\infty}$. Therefore, we derive from (1.3) and (2.7) that

$$\|a\|_{\ell_\infty(\widehat{F})}^* = \sup_{x \in S_{\ell_\infty(\widehat{F})}} \left| \sum_k a_k x_k \right| = \sup_{y \in S_{\ell_\infty}} \left| \sum_k \tilde{a}_k y_k \right| = \|\bar{a}\|_{\ell_\infty}^*.$$

Hence, by using Lemma 1.1, we have that

$$\|a\|_{\ell_\infty(\widehat{F})}^* = \|\bar{a}\|_{\ell_\infty}^* = \|\bar{a}\|_{\ell_1}.$$

This completes the proof of part (a).

Since parts (b) and (c) can also be proved by analogy with part (a), we leave the detailed proof to the reader. $\qquad \square$

Throughout this paper, if $A = (a_{nk})$ is an infinite matrix, we define the associated matrix defined $\tilde{A} = (\tilde{a}_{nk})$ by

$$(2.9) \qquad \tilde{a}_{nk} = \sum_{j=k}^{\infty} \frac{f_{j+1}^2}{f_k f_{k+1}} a_{nj}; \quad n, k \in \mathbb{N}$$

provided the series on the right side converges for all $n, k \in \mathbb{N}$ which is the case whenever $A_n \in \{\ell_p(\widehat{F})\}^{\beta}$ for all $n \in \mathbb{N}$, where $1 \leq p \leq \infty$. Then, we have:

Lemma 2.3. *Let $1 \leq p \leq \infty$, X be a sequence space and $A = (a_{nk})$ be an infinite matrix. If $A \in (\ell_p(\widehat{F}), X)$, then $\tilde{A} \in (\ell_p, X)$ such that $Ax = \tilde{A}y$ for all $x \in \ell_p(\widehat{F})$ with $y = \widehat{F}x$, where $\tilde{A} = (\tilde{a}_{nk})$ is the associated matrix defined by (2.9).*

Proof. Suppose that $A \in (\ell_p(\widehat{F}), X)$ and let $x \in \ell_p(\widehat{F})$. Then $A_n \in \{\ell_p(\widehat{F})\}^{\beta}$ for all $n \in \mathbb{N}$. Thus, it follows by Lemma 2.1 that $\tilde{A}_n \in \ell_q$ for all $n \in \mathbb{N}$ and the equality $Ax = \tilde{A}y$ holds which yields that $\tilde{A}y \in X$, where $y = \widehat{F}x$. Since every $y \in \ell_p$ is the assocaited sequence of some $x \in \ell_p(\widehat{F})$, we obtain that $\tilde{A} \in (\ell_p, X)$. This concludes the proof. □

Lemma 2.4. *Let $1 \leq p < \infty$. If $A \in (\ell_1(\widehat{F}), \ell_p)$, then*

$$\|L_A\| = \|A\|_{(\ell_1(\widehat{F}), \ell_p)} = \sup_k \left(\sum_n |\tilde{a}_{nk}|^p \right)^{1/p} < \infty.$$

Proof. The proof is elementary and left to the reader. □

3. COMPACT OPERATORS ON THE SPACES $\ell_p(\widehat{F})$ AND $\ell_\infty(\widehat{F})$

In this section, we give some classes of compact operators on the spaces $\ell_p(\widehat{F})$ and $\ell_\infty(\widehat{F})$, where $1 \leq p < \infty$.

The following lemma gives necessary and sufficient conditions for a matrix transformation from a BK space X to c_0, c and ℓ_∞ to be compact (the only sufficient condition for ℓ_∞).

Lemma 3.1. *[29, Theorem 3.7] Let $X \supset \varphi$ be a BK space. Then, we have*

(a) *If $A \in (X, \ell_\infty)$, then $0 \leq \|L_A\|_\chi \leq \limsup\limits_n \|A_n\|_X^*$.*

(b) *If $A \in (X, c_0)$, then $\|L_A\|_\chi = \limsup\limits_n \|A_n\|_X^*$.*

(c) *If X has AK or $X = \ell_\infty$ and $A \in (X, c)$, then*

$$\frac{1}{2} \cdot \limsup_n \|A_n - \alpha\|_X^* \leq \|L_A\|_\chi \leq \limsup_n \|A_n - \alpha\|_X^*,$$

where $\alpha = (\alpha_k)$ with $\alpha_k = \lim_n a_{nk}$ for all $k \in \mathbb{N}$.

Now, let $A = (a_{nk})$ be an infinite matrix and $\tilde{A} = (\tilde{a}_{nk})$ the associated matrix defined by (2.9). Then, by combining Lemmas 2.2, 2.3 and 3.1, we have the following result.

Theorem 3.1. *Let $1 < p < \infty$ and $q = p/(p-1)$. Then we have:*

(a) *if $A \in (\ell_p(\widehat{F}), \ell_\infty)$, then*

$$(3.1) \qquad 0 \leq \|L_A\|_\chi \leq \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q}$$

and

$$(3.2) \qquad L_A \text{ is compact if } \lim_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q} = 0;$$

(b) *if $A \in (\ell_p(\widehat{F}), c_0)$, then*

$$(3.3) \qquad \|L_A\|_\chi = \limsup_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q}$$

and

$$(3.4) \qquad L_A \text{ is compact if and only if } \lim_n \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q} = 0;$$

(c) *if $A \in (\ell_p(\widehat{F}), c)$, then*

$$(3.5) \qquad \frac{1}{2} \cdot \limsup_n \left(\sum_k |\tilde{a}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q} \leq \|L_A\|_\chi \leq \limsup_n \left(\sum_k |\tilde{a}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q}$$

and

$$(3.6) \qquad L_A \text{ is compact if and only if } \lim_n \left(\sum_k |\tilde{a}_{nk} - \tilde{\alpha}_k|^q \right)^{1/q} = 0,$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ with $\tilde{\alpha}_k = \lim_n \tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

Proof. It is obvious that (3.2), (3.4) and (3.6) are respectively obtained from (3.1), (3.3) and (3.5) by using (1.5). Thus, we have to prove (3.1), (3.3) and (3.5).

Let $A \in (\ell_p(\widehat{F}), \ell_\infty)$ or $A \in (\ell_p(\widehat{F}), c_0)$. Since $A_n \in \{\ell_p(\widehat{F})\}^\beta$ for all $n \in \mathbb{N}$, we have from Part (c) of Lemma 2.2 that

$$(3.7) \qquad \|A_n\|^*_{\ell_p(\widehat{F})} = \left\| \tilde{A}_n \right\|^*_{\ell_p} = \left(\sum_k |\tilde{a}_{nk}|^q \right)^{1/q}$$

for all $n \in \mathbb{N}$. Hence, by using the equality (3.7), we get (3.1) and (3.3) from parts (a) and (b) of Lemma 3.1, respectively.

To prove (3.5), we have $A \in (\ell_p(\widehat{F}), c)$ and hence $\tilde{A} \in (\ell_p, c)$ by Lemma 2.3. Therefore, it follows by part (c) of Lemma 3.1 with Lemma 1.1 that

$$(3.8) \qquad \frac{1}{2} \cdot \limsup_n \left\| \tilde{A}_n - \tilde{\alpha} \right\|_{\ell_q} \leq \|L_{\tilde{A}}\|_\chi \leq \limsup_n \left\| \tilde{A}_n - \tilde{\alpha} \right\|_{\ell_q},$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ with $\tilde{\alpha}_k = \lim_n \tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

Now, let us write $S = S_{\ell_p(\widehat{F})}$ and $\bar{S} = S_{\ell_p}$, for short. Then, we obtain by (1.4) and Lemma 1.2 that

$$(3.9) \qquad \|L_A\|_\chi = \chi(L_A(S)) = \chi(AS)$$

and

$$(3.10) \qquad \|L_{\tilde{A}}\|_\chi = \chi(L_{\tilde{A}}(\bar{S})) = \chi(\tilde{A}\bar{S}).$$

Further, we have by (2.3) that $x \in S$ if and only if $y \in \bar{S}$ and since $Ax = \tilde{A}y$ by Lemma 2.3, we deduce that $AS = \tilde{A}\bar{S}$. This leads us with (3.9) and (3.10) to the consequence that $\|L_A\|_\chi = \|L_{\tilde{A}}\|_\chi$. Hence, we get (3.5) from (3.8). This completes the proof. $\qquad \square$

The conclusions of Theorem 3.1 still hold for $\ell_1(\widehat{F})$ or $\ell_\infty(\widehat{F})$ instead of $\ell_p(\widehat{F})$ with $q = 1$, and on replacing the summations over k by the supremums over k in the case $\ell_1(\widehat{F})$. Then, we have the following results:

Theorem 3.2. *Let* $\tilde{A} = (\tilde{a}_{nk})$ *be the associated matrix defined by* (2.9). *Then we have:*

(a) *if* $A \in (\ell_\infty(\widehat{F}), \ell_\infty)$, *then*

$$0 \le \|L_A\|_\chi \le \limsup_n \sum_k |\tilde{a}_{nk}|$$

and

$$L_A \text{ is compact if } \lim_n \sum_k |\tilde{a}_{nk}| = 0;$$

(b) *if* $A \in (\ell_\infty(\widehat{F}), c_0)$, *then*

$$\|L_A\|_\chi = \limsup_n \sum_k |\tilde{a}_{nk}|$$

and

$$L_A \text{ is compact if and only if } \lim_n \sum_k |\tilde{a}_{nk}| = 0;$$

(c) *if* $A \in (\ell_\infty(\widehat{F}), c)$, *then*

$$\frac{1}{2} \cdot \limsup_n \sum_k |\tilde{a}_{nk} - \tilde{a}_k| \le \|L_A\|_\chi \le \limsup_n \sum_k |\tilde{a}_{nk} - \tilde{a}_k|$$

and

$$L_A \text{ is compact if and only if } \lim_n \sum_k |\tilde{a}_{nk} - \tilde{a}_k| = 0,$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ *with* $\tilde{\alpha}_k = \lim_n \tilde{a}_{nk}$ *for all* $k \in \mathbb{N}$.

Theorem 3.3. *Let* $\tilde{A} = (\tilde{a}_{nk})$ *be the associated matrix defined by* (2.9). *Then we have*

(a) *If $A \in (\ell_1(\widehat{F}), \ell_\infty)$, then*

$$0 \leq \|L_A\|_\chi \leq \limsup_n \left(\sup_k |\tilde{a}_{nk}| \right)$$

and

$$L_A \text{ is compact if } \lim_n \left(\sup_k |\tilde{a}_{nk}| \right) = 0.$$

(b) *If $A \in (\ell_1(\widehat{F}), c_0)$, then*

$$\|L_A\|_\chi = \limsup_n \left(\sup_k |\tilde{a}_{nk}| \right)$$

and

$$L_A \text{ is compact if and only if } \lim_n \left(\sup_k |\tilde{a}_{nk}| \right) = 0.$$

(c) *If $A \in (\ell_1(\widehat{F}), c)$, then*

$$\frac{1}{2} \cdot \limsup_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right) \leq \|L_A\|_\chi \leq \limsup_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right)$$

and

$$L_A \text{ is compact if and only if } \lim_n \left(\sup_k |\tilde{a}_{nk} - \tilde{a}_k| \right) = 0,$$

where $\tilde{\alpha} = (\tilde{\alpha}_k)$ with $\tilde{\alpha}_k = \lim_n \tilde{a}_{nk}$ for all $k \in \mathbb{N}$.

Morever, as an immediate consequence of Theorem 3.2, we have the following corollary.

Corollary 3.1. *If either $A \in (\ell_\infty(\widehat{F}), c_0)$ or $A \in (\ell_\infty(\widehat{F}), c)$, then the operator L_A is compact.*

Proof. Let $A \in (\ell_\infty(\widehat{F}), c_0)$. Then, we have by Lemma 2.3 that $\tilde{A} \in (\ell_\infty, c_0)$ which implies that $\lim_n \left(\sum_k |\tilde{a}_{nk}| \right) = 0$, [39]. This leads us with Theorem 3.2(b) to the consequence that L_A is compact. Similarly, if $A \in (\ell_\infty(\widehat{F}), c)$ then $\tilde{A} \in (\ell_\infty, c)$ and hence $\lim_n \left(\sum_k |\tilde{a}_{nk} - \tilde{a}_k| \right) = 0$, where $\tilde{\alpha} = (\tilde{\alpha}_k)$ with $\tilde{\alpha}_k = \lim_n \tilde{a}_{nk}$ for all $k \in \mathbb{N}$. Therefore, we deduce from Theorem 3.2(c) that L_A is compact. $\square$

Throughout, let $\mathcal{F}_m$ $(m \in \mathbb{N})$ be the subcollection of $\mathcal{F}$ consisting of all nonempty and finite subsets of $\mathbb{N}$ with elements that are greater than m, that is

$$\mathcal{F}_m = \{ n \in N \in \mathcal{F} : n > m \text{ for all } n \in \mathbb{N} \}; \quad m \in \mathbb{N}.$$

The next lemma [29, Theorem 3.11] gives necessary and sufficient conditions for a matrix transformation from a BK space to ℓ_1 to be compact.

Lemma 3.2. *Let $X \supset \varphi$ be a BK space. If $A \in (X, \ell_1)$, then*

$$\lim_m \left(\sup_{N \in \mathcal{F}_m} \left\| \sum_{n \in N} A_n \right\|_X^* \right) \leq \|L_A\|_\chi \leq 4 \cdot \lim_m \left(\sup_{N \in \mathcal{F}_m} \left\| \sum_{n \in N} A_n \right\|_X^* \right).$$

Now, we prove the following result.

Theorem 3.4. *Let $1 < p < \infty$ and $q = p/(p-1)$. If $A \in (\ell_p(\widehat{F}), \ell_1)$, then*

(3.11) $$\lim_m \|A\|_{(\ell_p(\widehat{F}), \ell_1)}^{(m)} \leq \|L_A\|_\chi \leq 4 \cdot \lim_m \|A\|_{(\ell_p(\widehat{F}), \ell_1)}^{(m)}$$

and

(3.12) $$L_A \text{ is compact if and only if } \lim_m \|A\|_{(\ell_p(\widehat{F}), \ell_1)}^{(m)} = 0,$$

where

$$\|A\|_{(\ell_p(\widehat{F}), \ell_1)}^{(m)} = \sup_{N \in \mathcal{F}_m} \left(\sum_k \left| \sum_{n \in N} \tilde{a}_{nk} \right|^q \right)^{1/q} ; \quad m \in \mathbb{N}.$$

Proof. It is obvious that (3.11) is obtained by combining Lemmas 2.2(c), 2.3 and 3.2. Also, by using (1.5), we get (3.12) from (3.11). □

Theorem 3.5. *Let $1 \leq p < \infty$. If $A \in (\ell_1(\widehat{F}), \ell_p)$, then*

(3.13) $$\|L_A\|_\chi = \lim_m \left(\sup_k \left(\sum_{n=m}^\infty |\tilde{a}_{nk}|^p \right)^{1/p} \right)$$

and

(3.14) $$L_A \text{ is compact if and only if } \lim_m \left(\sup_k \left(\sum_{n=m}^\infty |\tilde{a}_{nk}|^p \right)^{1/p} \right) = 0,$$

Proof. Let us remark that the limit in (3.13) exists by Lemma 2.4.

Now, we write $S = S_{\ell_1(\widehat{F})}$. Then, we have by Lemma 1.2 that $L_A(S) = AS \in M_{\ell_p}$. Thus, it follows from (1.4) and Lemma 1.4 that

(3.15) $$\|L_A\|_\chi = \chi(AS) = \lim_m \left(\sup_{x \in S} \|(I - P_m)(Ax)\|_{\ell_p} \right),$$

where $P_m : \ell_p \to \ell_p$ ($m \in \mathbb{N}$) is the operator defined by $P_m(x) = (x_0, x_1, \ldots, x_m, 0, 0, \ldots)$ for all $x = (x_k) \in \ell_p$ and I is the identity operator on ℓ_p.

On the other hand, let $x \in \ell_1(\widehat{F})$ be given. Then $y \in \ell_1$ and since $A \in (\ell_1(\widehat{F}), \ell_p)$, we obtain from Lemma 2.3 that $\tilde{A} \in (\ell_1, \ell_p)$ and $Ax = \tilde{A}y$. Thus, we have for every

$m \in \mathbb{N}$ that

$$\|(I - P_m)(Ax)\|_{\ell_p} = \left\|(I - P_m)(\tilde{A}y)\right\|_{\ell_p}$$

$$= \left(\sum_{n=m+1}^{\infty} \left| \tilde{A}_n(y) \right|^p \right)^{1/p}$$

$$= \left(\sum_{n=m+1}^{\infty} \left| \sum_k \tilde{a}_{nk} y_k \right|^p \right)^{1/p}$$

$$\leq \sum_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk} y_k|^p \right)^{1/p}$$

$$\leq \|y\|_{\ell_1} \left(\sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right)$$

$$= \|x\|_{\ell_1(\widehat{F})} \left(\sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right).$$

This yields that

$$\sup_{x \in S} \|(I - P_m)(Ax)\|_{\ell_p} \leq \sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} ; \quad m \in \mathbb{N}.$$

Therefore, we deduce from (3.15) that

$$(3.16) \qquad \|L_A\|_\chi \leq \lim_m \left(\sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right).$$

To prove the converse inequality, let $c^{(k)} \in \ell_1(\widehat{F})$ be such that $\widehat{F}c^{(k)} = e^{(k)}$ ($k \in \mathbb{N}$), that is, $e^{(k)}$ is the $\widehat{F}$-transform of $c^{(k)}$ for each $k \in \mathbb{N}$. Then, we have by Lemma 2.3 that $Ac^{(k)} = \tilde{A}e^{(k)}$ for every $k \in \mathbb{N}$.

Now, let $U = \{c^{(k)} : k \in \mathbb{N}\}$. Then $U \subset S$ and hence $AU \subset AS$ which implies that $\chi(AU) \leq \chi(AS) = \|L_A\|_\chi$.

Further, it follows by applying Lemma 1.4 that

$$\chi(AU) = \lim_m \left(\sup_k \left(\left(\sum_{n=m+1}^{\infty} \left| A_n(c^{(k)}) \right|^p \right)^{1/p} \right) \right)$$

$$= \lim_m \left(\sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right).$$

Thus, we obtain that

$$(3.17) \qquad \lim_m \left(\sup_k \left(\sum_{n=m+1}^{\infty} |\tilde{a}_{nk}|^p \right)^{1/p} \right) \leq \|L_A\|_\chi.$$

Hence, we get (3.13) by combining (3.16) and (3.17). This completes the proof, since (3.14) is immediate by (1.5) and (3.13). $\qquad\qquad\qquad\qquad\Box$

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