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NEW HADAMARD'S INEQUALITY FOR (s_1, s_2)-PREINVEX FUNCTIONS ON CO-ORDINATES

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ABSTRACT. In this paper the authors introduce a new classes of preinvexity called (s_1, s_2)-preinvex functions on co-ordinates in the first and the second sense and establish some new Hermite-Hadamard type inequalities for those new concepts.

1. INTRODUCTION

One of the most well-known inequalities in mathematics for convex functions is so called Hermite-Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a real continuous convex function on the finite interval $[a, b]$. If the function f is concave, then (1.1) holds in the reverse direction (see [19]).

The Hermite-Hadamard inequality play an important role in nonlinear analysis and optimization. The above double inequality has attracted many researchers, various generalizations, refinements, extensions and variants of (1.1) have appeared in the literature, we can mention the works [1–6, 8, 11–15, 21–24] and the references cited therein.

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. Hanson in [10], introduced a new class of generalized convex functions, called invex functions, many authors studied some properties and the applications in mathematical programming and optimizations about invexity and preinvexity we refer readers to [7, 9, 16–18, 20, 25–27].

Key words and phrases. Hermite-Hadamard inequality, co-ordinates preinvex, (s_1, s_2)-preinvex functions on co-ordinates.

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Sarikaya et al. [24] proved the following Hadamard's type inequalities for co-ordinated convex functions

Theorem 1.1. [24, Theorem 2] *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ is a convex function on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int\limits_a^b\int\limits_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|}{4}\right),$$

where

$$A = \frac{1}{2(b-a)}\int_a^b [f(x,c)+f(x,c+\eta_2(d,c))]\,dx$$
$$+\frac{1}{2(d-c)}\int_c^d [f(a,y)+f(a+\eta_1(b,a),y)]\,dy.$$

Theorem 1.2. [24, Theorem 3] *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$, $q > 1$, is a convex function on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{4(p+1)^{\frac{2}{q}}}\left(\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},$$

where A is as defined in Theorem 1.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.3. [24, Theorem 4] *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ be a partial differentiable mapping on $\Delta = [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$, $c < d$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$, $q \geq 1$, is a convex function on the co-ordinates on Δ, then one has the inequalities*

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}+\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dxdy - A\right|$$
$$\leq \frac{(b-a)(d-c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q}{4}\right)^{\frac{1}{q}},$$

where A is as defined in Theorem 1.1.

Latif et al. [14] proved the following Hadamard's type inequalities for co-ordinated preinvex functions

Theorem 1.4. [14, Theorem 8] *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial\lambda\partial t} \in L\left([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]\right)$$

with $\eta_1(b,a) \neq 0$, $\eta_2(d,c) \neq 0$, where $a, b \in K_1$ and $c, d \in K_2$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ is preinvex on the co-ordinates on $K_1 \times K_2$, then the following inequality holds

$$\left|\frac{f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))}{4}\right.$$
$$\left. + \frac{1}{\eta_1(b,a)\,\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$
$$\leq \frac{\eta_1(b,a)\,\eta_2(d,c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right| + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|}{4}\right),$$

where

$$A = \frac{1}{2\eta_1(b,a)} \int_a^{a+\eta_1(b,a)} \left[f(x,c) + f(x, c+\eta_2(d,c))\right] dx$$
$$+ \frac{1}{2\eta_2(d,c)} \int_c^{c+\eta_2(d,c)} \left[f(a,y) + f(a+\eta_1(b,a), y)\right] dy.$$

Theorem 1.5. [14, Theorem 9] *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial\lambda\partial t} \in L\left([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]\right)$$

with $\eta_1(b,a) \neq 0$, $\eta_2(d,c) \neq 0$, where $a, b \in K_1$ and $c, d \in K_2$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ is preinvex on the co-ordinates on $K_1 \times K_2, q \in (1, \infty)$, then the following inequality holds

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{q}}}\left(\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|}{4}\right)^{\frac{1}{q}},$$

where A is as defined in Theorem 1.4 *and* $\frac{1}{p}+\frac{1}{q}=1$.

Theorem 1.6. [14, Theorem 10] *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial\lambda\partial t} \in L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$$

with $\eta_1(b,a) \neq 0$, $\eta_2(d,c) \neq 0$, where $a, b \in K_1$ and $c, d \in K_2$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ is preinvex on the co-ordinates on $K_1 \times K_2, q \in [1,\infty)$, then the following inequality holds

$$\left|\frac{f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))}{4}\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{16}\left(\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|}{4}\right)^{\frac{1}{q}},$$

where A is as defined in Theorem 1.4.

Motivated by the above results, we first introduce a new classes of convexity called (s_1, s_2)-preinvex functions on co-ordinates in the first and the second sense, then we establish some new Hadamard's type inequalities.

2. Preliminaries

Let us recall some known results concerning our work. Let $\Delta := [a,b] \times [c,d]$ the bidimensional interval in $\mathbb{R}^2$ with $a < b$ and $c < d$ and K_1, K_2 be a nonempty closed subsets in $\mathbb{R}^n$. Let $f : K_1 \times K_2 \to \mathbb{R}$, $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$ be continuous functions.

Definition 2.1. [8] A function $f : \Delta \to \mathbb{R}$ is said to be convex on Δ, if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda f(x,y) + (1-\lambda)f(z,w),$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$.

Definition 2.2. [8] A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leq & \lambda t f(x,y) + \lambda(1-t)f(x,w) + (1-\lambda)tf(z,y) \\ & + (1-\lambda)(1-t)f(z,w), \end{aligned}$$

holds, for all $(x,y),(z,w),(x,w),(z,y) \in \Delta$ and $\lambda, t \in [0,1]$.

Definition 2.3. [1] A nonnegative function $f : \Delta \subset [0,\infty)^2 \to \mathbb{R}$ is said to be s-convex in the first sense on Δ, if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x,y) + (1-\lambda^s)f(z,w),$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$, for some fixed $s \in (0,1]$.

Definition 2.4. [3] A nonnegative function $f : \Delta \subset [0,\infty)^2 \to \mathbb{R}$ is said to be s-convex in the second sense on Δ, if the following inequality

$$f(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda^s f(x,y) + (1-\lambda)^s f(z,w),$$

holds, for all $(x,y),(z,w) \in \Delta$ and $\lambda \in [0,1]$, for some fixed $s \in (0,1]$.

Definition 2.5. [1] A nonnegative function $f : \Delta \subset [0,\infty)^2 \to \mathbb{R}$ is said to be s-convex in the first sense on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leq & \lambda^s t^s f(x,y) + \lambda^s(1-t^s)f(x,w) + (1-\lambda^s)t^s f(z,y) \\ & + (1-\lambda^s)(1-t^s)f(z,w), \end{aligned}$$

holds, for all $(x,y),(z,w),(x,w),(z,y) \in \Delta$ and $\lambda, t \in [0,1]$, for some fixed $s \in (0,1]$.

Definition 2.6. [3] A nonnegative function $f : \Delta \subset [0,\infty)^2 \to \mathbb{R}$ is said to be s-convex in the second sense on the co-ordinates on Δ, if the following inequality

$$\begin{aligned} f(\lambda x + (1-\lambda)z, ty + (1-t)w) \leq & \lambda^s t^s f(x,y) + \lambda^s(1-t)^s f(x,w) + (1-\lambda)^s t^s f(z,y) \\ & + (1-\lambda)^s(1-t)^s f(z,w), \end{aligned}$$

holds, for all $(x,y),(z,w),(x,w),(z,y) \in \Delta$ and $\lambda, t \in [0,1]$, for some fixed $s \in (0,1]$.

Definition 2.7. [15] Let K_1, K_2 be nonempty subsets of $\mathbb{R}^n$, $(u,v) \in K_1 \times K_2$. We say $K_1 \times K_2$ is invex at (u,v) with respect to η_1 and η_2, if for each $(x,y) \in K_1 \times K_2$ and $t, s \in [0,1]$, we have

$$(u + t\eta_1(x,u), v + s\eta_2(y,v)) \in K_1 \times K_2,$$

$K_1 \times K_2$ is said to be an invex set with respect to $\eta_1$and η_2 if $K_1 \times K_2$ is invex at each $(u,v) \in K_1 \times K_2$.

Definition 2.8. [14] Let $K_1 \times K_2$ be invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex if for every $(x, y), (u, v) \in K_1 \times K_2$ and $t \in [0, 1]$, we have

$$f\left(u + t\eta_1\left(x, u\right), v + t\eta_2\left(y, v\right)\right) \leq (1-t)f(u, v) + \ tf(x, y).$$

Definition 2.9. [14] Let $K_1 \times K_2$ be invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be preinvex on the co-ordinates if for every $(x, y), (x, v), (u, y), (u, v) \in K_1 \times K_2$ and $\lambda, t \in [0, 1]$, we have

$$\begin{aligned} f\left(u + \lambda\eta_1\left(x, u\right), v + t\eta_2\left(y, v\right)\right) \leq &(1-\lambda)(1-t)f(u, v) + (1-\lambda)tf(u, y) \\ &+ (1-t)\lambda f(x, v) + \lambda t f(x, y). \end{aligned}$$

Lemma 2.1. [14, Lemma 2] *Let $K_1 \times K_2$ be an open invex subset of $\mathbb{R}^2$ with respect to the mappings $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be a twice partial differentiable mapping such that*

$$\frac{\partial^2 f}{\partial\lambda\partial t} \in L_1\left(\left[a, a + \eta_1(b, a)\right] \times \left[c, c + \eta_2(d, c)\right]\right)$$

with $\eta_1(b, a) \neq 0$, $\eta_2(d, c) \neq 0$, where $a, b \in K_1$ and $c, d \in K_2$. Then the following equality holds

$$\begin{aligned} &\frac{1}{4}\left[f\left(a, c\right) + f\left(a, c + \eta_2\left(d, c\right)\right) + f\left(a + \eta_1\left(b, a\right), c\right) + f\left(a + \eta_1\left(b, a\right), c + \eta_2(d, c)\right)\right] \\ &\quad + \frac{1}{\eta_1\left(b, a\right)\eta_2\left(d, c\right)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f\left(x, y\right) dxdy - A \\ &= \frac{\eta_1\left(b, a\right)\eta_2\left(d, c\right)}{4} \int_0^1\int_0^1 \left(1 - 2\lambda\right)\left(1 - 2t\right) \frac{\partial^2 f}{\partial t\partial\lambda}\left(a + \lambda\eta_1\left(b, a\right), c + t\eta_2(d, c)\right) d\lambda dt, \end{aligned}$$

where

$$\begin{aligned} A = &\frac{1}{2\eta_1\left(b, a\right)} \int_a^{a+\eta_1(b,a)} \left[f\left(x, c\right) + f\left(x, c + \eta_2(d, c)\right)\right] dx \\ &+ \frac{1}{2\eta_2\left(d, c\right)} \int_c^{c+\eta_2(d,c)} \left[f\left(a, y\right) + f\left(a + \eta_1\left(b, a\right), y\right)\right] dy. \end{aligned}$$

3. Main Results

We now introduce the classes of (s_1, s_2)-preinvexity on co-ordinates in the first and second sense.

Definition 3.1. Let $K_1 \times K_2 \subset [0, \infty)^n \times [0, \infty)^n$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A nonnegative function $f : K_1 \times K_2 \to \mathbb{R}$

is said to be s-preinvex in the first sense for some fixed $s \in (0,1]$, if for every $(x,y),(u,v) \in K_1 \times K_2$ and $t \in [0,1]$, we have

$$f\left(u + t\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \le (1-t^s)f(u,v) + t^s f(x,y).$$

Definition 3.2. Let $K_1 \times K_2 \subset [0,\infty)^n \times [0,\infty)^n$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A nonnegative function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be s-preinvex in the second sense for some fixed $s \in (0,1]$, if for every $(x,y),(u,v) \in K_1 \times K_2$ and $t \in [0,1]$, we have

$$f\left(u + t\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \le (1-t)^s f(u,v) + t^s f(x,y).$$

Definition 3.3. Let $K_1 \times K_2 \subset [0,\infty)^n \times [0,\infty)^n$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A nonnegative function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be s-preinvex in the first sense on co-ordinates for some fixed $s \in (0,1]$, if for every $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$ and $\lambda, t \in [0,1]$, we have

$$\begin{aligned} f\left(u + \lambda\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \le &(1-\lambda^s)(1-t^s)f(u,v) + (1-\lambda^s)t^s f(u,y) \\ &+ (1-t^s)\lambda^s f(x,v) + \lambda^s t^s f(x,y). \end{aligned}$$

Definition 3.4. Let $K_1 \times K_2 \subset [0,\infty)^n \times [0,\infty)^n$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A nonnegative function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be s-preinvex in the second sense on co-ordinates for some fixed $s \in (0,1]$, if for every $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$ and $\lambda, t \in [0,1]$, we have

$$\begin{aligned} f\left(u + \lambda\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \le &(1-\lambda)^s(1-t)^s f(u,v) + (1-\lambda)^s t^s f(u,y) \\ &+ (1-t)^s\lambda^s f(x,v) + \lambda^s t^s f(x,y). \end{aligned}$$

Definition 3.5. Let $K_1 \times K_2 \subset [0,\infty)^n \times [0,\infty)^n$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A nonnegative function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be (s_1,s_2)-preinvex in the first sense on co-ordinates for some fixed $s_1, s_2 \in (0,1]$, if for every $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$ and $\lambda, t \in [0,1]$, we have

$$\begin{aligned} f\left(u + \lambda\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \le &(1-\lambda^{s_1})(1-t^{s_2})f(u,v) + (1-\lambda^{s_1})t^{s_2} f(u,y) \\ &+ (1-t^{s_2})\lambda^{s_1} f(x,v) + \lambda^{s_1} t^{s_2} f(x,y). \end{aligned}$$

Definition 3.6. Let $K_1 \times K_2 \subset [0,\infty)^n \times [0,\infty)^n$ be an invex set with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}^n$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}^n$. A nonnegative function $f : K_1 \times K_2 \to \mathbb{R}$ is said to be (s_1,s_2)-preinvex in the second sense on co-ordinates for some fixed $s_1, s_2 \in (0,1]$, if for every $(x,y),(x,v),(u,y),(u,v) \in K_1 \times K_2$ and $\lambda, t \in [0,1]$, we have

$$\begin{aligned} f\left(u + \lambda\eta_1\left(x,u\right), v + t\eta_2\left(y,v\right)\right) \le &(1-\lambda)^{s_1}(1-t)^{s_2} f(u,v) + (1-\lambda)^{s_1} t^{s_2} f(u,y) \\ &+ (1-t)^{s_2}\lambda^{s_1} f(x,v) + \lambda^{s_1} t^{s_2} f(x,y). \end{aligned}$$

Statement of the results.

Theorem 3.1. *Let $K_1 \times K_2$ be an open invex subset of $[0,\infty) \times [0,\infty)$ with respect to $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ and $\eta_2 : K_2 \times K_2 \to \mathbb{R}$, $a, b \in K_1, c, d \in K_2$ with $\eta_1(b,a) > 0$ and $\eta_2(d,c) > 0$. Suppose $f : K_1 \times K_2 \to \mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial\lambda\partial t} \in L\left([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)]\right)$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ is (s_1, s_2)-preinvex in the second sense on co-ordinates on $K_1 \times K_2$, for some fixed $s_1, s_2 \in (0,1]$, then the following inequality holds*

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left[f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))\right] \\
&\qquad + \frac{1}{\eta_1(b,a)\,\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg| \\
&\qquad \le \frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\left[s_1 + \left(\frac{1}{2}\right)^{s_1}\right]\left[s_2 + \left(\frac{1}{2}\right)^{s_2}\right] \\
&\qquad \times \left[\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right| + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right| + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right| + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\right],
\end{aligned} \tag{3.1}$$

where A is defined as in Lemma 2.1.

Proof. From Lemma 2.1 and modulus we have

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left[f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))\right] \\
&\qquad + \frac{1}{\eta_1(b,a)\,\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg| \\
&\qquad \le \frac{\eta_1(b,a)\,\eta_2(d,c)}{4} \\
&\qquad \times \int_0^1\int_0^1 |1-2\lambda|\,|1-2t| \left|\frac{\partial^2 f}{\partial\lambda\partial t}(a+\lambda\eta_1(b,a), c+t\eta_2(d,c))\right| d\lambda dt,
\end{aligned} \tag{3.2}$$

using (s_1, s_2)-preinvexity in the second sense, we get

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left[f(a,c) + f(a, c+\eta_2(d,c)) + f(a+\eta_1(b,a), c) + f(a+\eta_1(b,a), c+\eta_2(d,c))\right] \\
&\qquad + \frac{1}{\eta_1(b,a)\,\eta_2(d,c)} \int_a^{a+\eta_1(b,a)} \int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg|
\end{aligned}$$

$$\leq \frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\left[(1-\lambda)^{s_1}(1-t)^{s_2}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|\right.$$
$$+(1-\lambda)^{s_1}t^{s_2}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+(1-t)^{s_2}\lambda^{s_1}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|$$
$$\left.+\lambda^{s_1}t^{s_2}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|\right]d\lambda dt$$

$$(3.3)\qquad = \frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-\lambda)^{s_1}(1-t)^{s_2}d\lambda dt$$
$$+\frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-\lambda)^{s_1}t^{s_2}d\lambda dt$$
$$+\frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-t)^{s_2}\lambda^{s_1}d\lambda dt$$
$$+\frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,\lambda^{s_1}t^{s_2}d\lambda dt.$$

Taking into account that

$$\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-\lambda)^{s_1}(1-t)^{s_2}d\lambda dt = \int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-\lambda)^{s_1}t^{s_2}d\lambda dt$$
$$= \int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-t)^{s_2}\lambda^{s_1}d\lambda dt$$
$$(3.4)\qquad = \int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,\lambda^{s_1}t^{s_2}d\lambda dt$$
$$= \frac{\left[s_1+\left(\frac{1}{2}\right)^{s_1}\right]\left[s_2+\left(\frac{1}{2}\right)^{s_2}\right]}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}.$$

With (3.3) and (3.4) we get the desired inequality in (3.1). The proof is completed. □

Remark 3.1. Theorem 3.1, will be reduced to Theorem 8 in [14] if we take $s_1 = s_2 = 1$, moreover if $\eta_1(b,a) = b-a$ and $\eta_2(d,c) = d-c$, then we obtain Theorem 2 in [24].

Corollary 3.1. *Under the same assumptions of Theorem* 3.1, *if* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ *is s-preinvex in the second sense on co-ordinates on* $K_1 \times K_2$, *for some fixed* $s \in (0,1]$ *then the* *following inequality holds*

$$\left|\frac{1}{4}\left[f\left(a,c\right)+f\left(a,c+\eta_2\left(d,c\right)\right)+f\left(a+\eta_1\left(b,a\right),c\right)+f\left(a+\eta_1\left(b,a\right),c+\eta_2(d,c)\right)\right]\right.$$

$$\left.+\frac{1}{\eta_1\left(b,a\right)\eta_2\left(d,c\right)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)}f\left(x,y\right)dxdy-A\right|$$

$$\leq\frac{\eta_1\left(b,a\right)\eta_2\left(d,c\right)}{4}\left[s+\left(\frac{1}{2}\right)^s\right]^2$$

$$\times\left[\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}\left(a,c\right)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|}{\left[\left(s+1\right)\left(s+2\right)\right]^2}\right],$$

where A is defined as in Lemma 2.1.

Remark 3.2. Corollary 3.1, will be reduced to Theorem 8 in [14] if we take $s=1$, moreover if $\eta_1\left(b,a\right)=b-a$ and $\eta_2(d,c)=d-c$ then we obtain Theorem 2 in [24].

Theorem 3.2. *Let $K_1\times K_2$ be an open invex subset of $[0,\infty)\times[0,\infty)$ with respect to $\eta_1:K_1\times K_1\to\mathbb{R}$ and $\eta_2:K_2\times K_2\to\mathbb{R}$, $a,b\in K_1, c,d\in K_2$ with $\eta_1(b,a)>0$ and $\eta_2(d,c)>0$. Suppose $f:K_1\times K_2\to\mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial\lambda\partial t}\in L\left([a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)]\right)$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ is (s_1,s_2)-preinvex in the first sense on co-ordinates on $K_1\times K_2$, for some fixed $s_1,s_2\in(0,1]$, then the following inequality holds*

$$\left|\frac{1}{4}\left[f\big(a,c\big)+f\left(a,c+\eta_2\left(d,c\right)\right)+f\left(a+\eta_1\left(b,a\right),c\right)+f\left(a+\eta_1\left(b,a\right),c+\eta_2(d,c)\right)\right]\right.$$

$$\left.+\frac{1}{\eta_1\left(b,a\right)\eta_2\left(d,c\right)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)}f\left(x,y\right)dxdy-A\right|$$

$$\leq\frac{\eta_1\left(b,a\right)\eta_2\left(d,c\right)}{4}\left[\left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{\left(s_1+1\right)\left(s_1+2\right)}\right]\right. \tag{3.5}$$

$$\times\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{\left(s_2+1\right)\left(s_2+2\right)}\right]\left|\frac{\partial^2 f}{\partial\lambda\partial t}\left(a,c\right)\right|$$

$$+\left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{\left(s_1+1\right)\left(s_1+2\right)}\right]\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{\left(s_2+1\right)\left(s_2+2\right)}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|$$

$$+\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{\left(s_1+1\right)\left(s_1+2\right)}\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{\left(s_2+1\right)\left(s_2+2\right)}\right]\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|$$

$$+\frac{\left[s_1+\left(\frac{1}{2}\right)^{s_1}\right]\left[s_2+\left(\frac{1}{2}\right)^{s_2}\right]}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|\Bigg],$$

where A is defined as in Lemma 2.1.

Proof. From Lemma 2.1 and modulus we have

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$
$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\Bigg|$$
$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1|1-2\lambda||1-2t|\left|\frac{\partial^2 f}{\partial t\partial\lambda}(a+\lambda\eta_1(b,a),c+t\eta_2(d,c))\right|d\lambda dt,$$

using (s_1,s_2)-preinvexity in the first sense, we get

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$
$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\Bigg|$$

$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\int_0^1\int_0^1|1-2\lambda||1-2t|\Bigg[(1-\lambda^{s_1})(1-t^{s_2})\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right| \tag{3.6}$$
$$+(1-\lambda^{s_1})t^{s_2}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+(1-t^{s_2})\lambda^{s_1}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|$$
$$+\lambda^{s_1}t^{s_2}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|\Bigg]d\lambda dt$$
$$=\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|\int_0^1\int_0^1|1-2\lambda||1-2t|(1-\lambda^{s_1})(1-t^{s_2})d\lambda dt$$
$$+\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|\int_0^1\int_0^1|1-2\lambda||1-2t|(1-\lambda^{s_1})t^{s_2}d\lambda dt$$
$$+\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|\int_0^1\int_0^1|1-2\lambda||1-2t|(1-t^{s_2})\lambda^{s_1}d\lambda dt$$

$$+\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,\lambda^{s_1}t^{s_2}d\lambda dt.$$

A simple computation gives

$$\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-\lambda^{s_1})(1-t^{s_2})d\lambda dt = \left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\right] \times\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\right]$$

and

$$\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,(1-\lambda^{s_1})t^{s_2}d\lambda dt = \left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\right]\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)},$$

$$\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|(1-t^{s_2})\lambda^{s_1}d\lambda dt$$

$$= \frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\right], \tag{3.7}$$

$$\int_0^1\int_0^1 |1-2\lambda|\,|1-2t|\,\lambda^{s_1}t^{s_2}d\lambda dt = \frac{\left[s_1+\left(\frac{1}{2}\right)^{s_1}\right]\left[s_2+\left(\frac{1}{2}\right)^{s_2}\right]}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}.$$

Substituting (3.7) into (3.6) we obtain the required inequality in (3.5). The proof is completed. □

Remark 3.3. Theorem 3.2, will be reduced to Theorem 8 in [14] if we take $s_1 = s_2 = 1$, moreover if $\eta_1(b,a) = b-a$ and $\eta_2(d,c) = d-c$ then we obtain Theorem 2 in [24].

Corollary 3.2. *Under the same assumptions of Theorem* 3.2*, if* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|$ *is s-preinvex in the first sense on co-ordinates on* $K_1\times K_2$*, for some fixed* $s\in(0,1]$*, then the following inequality holds*

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$

$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg|$$

$$\le \frac{\eta_1(b,a)\eta_2(d,c)}{4}$$

$$\times\left[\left[\frac{1}{2}-\frac{s+\left(\frac{1}{2}\right)^{s}}{(s+1)(s+2)}\right]^2\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|+\left[\frac{1}{2}-\frac{s+\left(\frac{1}{2}\right)^{s}}{(s+1)(s+2)}\right]\frac{s+\left(\frac{1}{2}\right)^{s}}{(s+1)(s+2)}\right.$$

$$\times\left[\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|\right]+\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^2\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|\Bigg],$$

where A is defined as in Lemma 2.1.

Remark 3.4. Corollary 3.2, will be reduced to Theorem 8 in [14] if we take $s=1$, moreover if $\eta_1(b,a)=b-a$ and $\eta_2(d,c)=d-c$ then we obtain Theorem 2 in [24].

Theorem 3.3. *Let $K_1\times K_2$ be an open invex subset of $[0,\infty)\times[0,\infty)$ with respect to $\eta_1:K_1\times K_1\to\mathbb{R}$ and $\eta_2:K_2\times K_2\to\mathbb{R}$, $a,b\in K_1,c,d\in K_2$ with $\eta_1(b,a)>0$ and $\eta_2(d,c)>0$. Suppose $f:K_1\times K_2\to\mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial\lambda\partial t}\in L\left([a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)]\right)$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ is (s_1,s_2)-preinvex in the second sense on co-ordinates on $K_1\times K_2$, for some fixed $s_1,s_2\in(0,1]$, $q\in(1,\infty)$, then the following inequality holds*

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$

$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\Bigg|$$

$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}}\tag{3.8}$$

$$\times\left[\frac{\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q}{(1+s_1)(1+s_2)}\right]^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder's inequality, we have

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$

$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\Bigg|$$

$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1|1-2t|^p|1-2\lambda|^p\,dtd\lambda\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q dtd\lambda\right)^{\frac{1}{q}}.$$

Using (s_1, s_2)-preinvexity in the second sense, we get

$$\left|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$

$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\right|$$

$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1 |1-2t|^p|1-2\lambda|^p\,dtd\lambda\right)^{\frac{1}{p}} \tag{3.9}$$

$$\times\left(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\int_0^1\int_0^1(1-t)^{s_1}(1-\lambda)^{s_2}\,dtd\lambda\right.$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q\int_0^1\int_0^1(1-t)^{s_1}\lambda^{s_2}dtd\lambda$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\int_0^1\int_0^1 t^{s_1}(1-\lambda)^{s_2}\,dtd\lambda$$

$$\left.+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\int_0^1\int_0^1 t^{s_1}\lambda^{s_2}dtd\lambda\right)^{\frac{1}{q}}.$$

A simple computation gives

$$\int_0^1\int_0^1 |1-2t|^p|1-2\lambda|^p\,dtd\lambda=\frac{1}{(p+1)^2},$$

$$\int_0^1\int_0^1 (1-t)^{\alpha_1}(1-\lambda)^{\alpha_2}\,dtd\lambda=\frac{1}{(1+s_1)(1+s_2)},$$

$$\int_0^1\int_0^1 (1-t)^{s_1}\lambda^{s_2}dtd\lambda=\frac{1}{(1+s_1)(1+s_2)},$$

$$\int_0^1\int_0^1 t^{s_1}(1-\lambda)^{s_2}\,dtd\lambda=\frac{1}{(1+s_1)(1+s_2)},$$

$$\int_0^1\int_0^1 t^{s_1}\lambda^{s_2}dtd\lambda = \frac{1}{(1+s_1)(1+s_2)}. \tag{3.10}$$

Substituting (3.10) into (3.9), we obtain the desired inequality in (3.8). The proof is completed. □

Remark 3.5. Theorem 3.3, will be reduced to Theorem 9 in [14] if we take $s_1 = s_2 = 1$, moreover if $\eta_1(b,a) = b - a$ and $\eta_2(d,c) = d - c$ then we obtain Theorem 3 in [24].

Corollary 3.3. *Under the same assumptions of Theorem* 3.3*, if* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ *is s-preinvex in the second sense on co-ordinates on* $K_1 \times K_2$*, for some fixed* $s \in (0,1], q \in (1,\infty)$*, then the following inequality holds*

$$\left|\frac{1}{4}\left[f(a,c) + f(a,c+\eta_2(d,c)) + f(a+\eta_1(b,a),c) + f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}(1+s)^{\frac{2}{q}}}$$
$$\times\left[\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\right]^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1 and $\frac{1}{p} + \frac{1}{q} = 1$.

Remark 3.6. Corollary 3.3, will be reduced to Theorem 9 in [14] if we take $s = 1$, moreover if $\eta_1(b,a) = b - a$ and $\eta_2(d,c) = d - c$ then we obtain Theorem 3 in [24].

Theorem 3.4. *Let* $K_1 \times K_2$ *be an open invex subset of* $[0,\infty)\times[0,\infty)$ *with respect to* $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ *and* $\eta_2 : K_2 \times K_2 \to \mathbb{R}$*,* $a, b \in K_1, c, d \in K_2$ *with* $\eta_1(b,a) > 0$ *and* $\eta_2(d,c) > 0$*. Suppose* $f : K_1 \times K_2 \to \mathbb{R}$ *be partially differentiable mapping such that* $\frac{\partial^2 f}{\partial\lambda\partial t} \in L([a, a+\eta_1(b,a)] \times [c, c+\eta_2(d,c)])$*. If* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ *is* (s_1,s_2)*-preinvex in the first sense on co-ordinates on* $K_1 \times K_2$*, for some fixed* $s_1, s_2 \in (0,1]$*,* $q \in (1,\infty)$*, then the following inequality holds*

$$\left|\frac{1}{4}\left[f(a,c) + f(a,c+\eta_2(d,c)) + f(a+\eta_1(b,a),c) + f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$
$$\leq \frac{\eta_1(b,a)\eta_2(d,c)}{4(p+1)^{\frac{2}{p}}} \tag{3.11}$$

$$\times\left[\frac{s_1 s_2\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q+s_1\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q+s_2\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q}{(1+s_1)(1+s_2)}\right]^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Lemma 2.1 and Hölder's inequality, we have

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$
$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\Bigg|$$
$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1|1-2t|^p|1-2\lambda|^p\,dtd\lambda\right)^{\frac{1}{p}}$$
$$\times\left(\int_0^1\int_0^1\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q dtd\lambda\right)^{\frac{1}{q}}.$$

Using (s_1,s_2)-preinvexity in the first sense, we get

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$
$$+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\Bigg|$$
$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1|1-2t|^p|1-2\lambda|^p\,dtd\lambda\right)^{\frac{1}{p}} \tag{3.12}$$
$$\times\Bigg(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\int_0^1\int_0^1(1-t^{s_1})(1-\lambda^{s_2})\,dtd\lambda$$
$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q\int_0^1\int_0^1(1-t^{s_1})\lambda^{s_2}dtd\lambda$$
$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\int_0^1\int_0^1 t^{s_1}(1-\lambda^{s_2})\,dtd\lambda$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q \int_0^1\int_0^1 t^{s_1}\lambda^{s_2}dtd\lambda\Bigg)^{\frac{1}{q}}.$$

A simple computation gives

$$\int_0^1\int_0^1 |1-2t|^p\,|1-2\lambda|^p\,dtd\lambda = \frac{1}{(p+1)^2},$$

$$\int_0^1\int_0^1 (1-t^{s_1})(1-\lambda^{s_2})\,dtd\lambda = \frac{s_1 s_2}{(1+s_1)(1+s_2)},$$

$$\int_0^1\int_0^1 (1-t^{s_1})\lambda^{s_2}dtd\lambda = \frac{s_1}{(1+s_1)(1+s_2)},$$

$$\int_0^1\int_0^1 t^{s_1}(1-\lambda^{s_2})\,dtd\lambda = \frac{s_2}{(1+s_1)(1+s_2)},$$

$$\int_0^1\int_0^1 t^{s_1}\lambda^{s_2}dtd\lambda = \frac{1}{(1+s_1)(1+s_2)}. \tag{3.13}$$

Substituting (3.13) into (3.12), we obtain the desired inequality in (3.11). The proof is completed. □

Remark 3.7. Theorem 3.4, will be reduced to Theorem 9 in [14] if we take $s_1 = s_2 = 1$, moreover if $\eta_1(b,a) = b-a$ and $\eta_2(d,c) = d-c$ then we obtain Theorem 3 in [24].

Corollary 3.4. *Under the same assumptions of Theorem* 3.4, *if* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ *is s-preinvex in the first sense on co-ordinates on* $K_1\times K_2$, *for some fixed* $s\in(0,1]$, $q\in(1,\infty)$, *then the following inequality holds*

$$\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$

$$+\frac{1}{\eta_1(b,a)\,\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\Bigg|$$

$$\le \frac{\eta_1(b,a)\,\eta_2(d,c)}{4\,(p+1)^{\frac{2}{p}}\,(1+s)^{\frac{2}{q}}}$$

$$\times\left[s^2\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q + s\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q + s\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q + \left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\right]^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1 and $\frac{1}{p}+\frac{1}{q}=1$.

Remark 3.8. Corollary 3.4, will be reduced to Theorem 9 in [14] if we take $s=1$, moreover if $\eta_1(b,a)=b-a$ and $\eta_2(d,c)=d-c$ then we obtain Theorem 3 in [24].

Theorem 3.5. *Let $K_1\times K_2$ be an open invex subset of $[0,\infty)\times[0,\infty)$ with respect to $\eta_1: K_1\times K_1\to\mathbb{R}$ and $\eta_2: K_2\times K_2\to\mathbb{R}$, $a,b\in K_1, c,d\in K_2$ with $\eta_1(b,a)>0$ and $\eta_2(d,c)>0$. Suppose $f: K_1\times K_2\to\mathbb{R}$ be partially differentiable mapping such that $\frac{\partial^2 f}{\partial\lambda\partial t}\in L([a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)])$. If $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ is (s_1,s_2)-preinvex in the second sense on co-ordinates on $K_1\times K_2$, for some fixed $s_1,s_2\in(0,1]$, $q\in[1,\infty)$, then the following inequality holds*

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\\
&\qquad+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\Bigg|\\
&\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4^{2-\frac{1}{q}}}\Bigg(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\right]\\
&\qquad\times\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\right]\\
&\qquad+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d)\right|^q\left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\right]\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\\
&\qquad+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\right]\\
&\qquad+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\frac{\left[s_1+\left(\frac{1}{2}\right)^{s_1}\right]\left[s_2+\left(\frac{1}{2}\right)^{s_2}\right]}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\Bigg)^{\frac{1}{q}},
\end{aligned}\tag{3.14}$$

where A is defined as in Lemma 2.1.

Proof. In the case where $q=1$ the proof is similar to that of Theorem 3.1, now we will treat the case where $q>1$.

From Lemma 2.1 and the power-mean inequality, we have

$$\begin{aligned}
&\Bigg|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\\
&\qquad+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\Bigg|
\end{aligned}$$

$$\leq \frac{\eta_1(b,a)\,\eta_2(d,c)}{4}$$
$$\times\left(\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\left|\frac{\partial^2 f}{\partial\lambda\partial t}\left(a+t\eta_1(b,a),c+\lambda\eta_2(d,c)\right)\right|^q dtd\lambda\right)^{\frac{1}{q}}$$
$$\times\left(\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,dtd\lambda\right)^{1-\frac{1}{q}}.$$

Using (s_1,s_2)-preinvexity in the second sense, we get

$$\left|\frac{1}{4}\Big[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\Big]\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\,\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\right|$$

$$\leq \frac{\eta_1(b,a)\,\eta_2(d,c)}{4}\left(\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,dtd\lambda\right)^{1-\frac{1}{q}} \tag{3.15}$$
$$\times\left(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,t^{\alpha_1}\lambda^{\alpha_2}dtd\lambda\right.$$
$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,(1-t)^{s_1}(1-\lambda)^{s_2}\,dtd\lambda$$
$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,(1-t)^{s_1}\lambda^{\alpha_2}dtd\lambda$$
$$\left.+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,t^{\alpha_1}(1-\lambda)^{s_2}\,dtd\lambda\right)^{\frac{1}{q}}.$$

Substituting (3.4) and

$$\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,dtd\lambda=\frac{1}{4},$$

into (3.15), we get the required inequality in (3.14). The proof is completed. □

Remark 3.9. Theorem 3.5, will be reduced to Theorem 10 in [14] if we take $s_1=s_2=1$, moreover if $\eta_1(b,a)=b-a$ and $\eta_2(d,c)=d-c$ then we obtain Theorem 4 in [24].

Corollary 3.5. *Under the same assumptions of Theorem 3.5, if* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ *is s-preinvex in the second sense on co-ordinates on* $K_1 \times K_2$*, for some fixed* $s \in (0,1]$*,* $q \in [1,\infty)$*, then the following equality holds*

$$\left|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$

$$\left.+\frac{1}{\eta_1(b,a)\,\eta_2(d,c)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$

$$\leq \frac{\eta_1(b,a)\,\eta_2(d,c)}{4^{2-\frac{1}{q}}}\left(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\left[\frac{1}{2}-\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^2\right.$$

$$+\left[\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\right]\left[\frac{s+\left(\frac{1}{2}\right)^s}{2(s+1)(s+2)}-\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^2\right]$$

$$\left.+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^2\right)^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1.

Remark 3.10. Corollary 3.5, will be reduced to Theorem 10 in [14] if we take $s = 1$, moreover if $\eta_1(b,a) = b-a$ and $\eta_2(d,c) = d-c$ then we obtain Theorem 4 in [24].

Theorem 3.6. *Let* $K_1 \times K_2$ *be an open invex subset of* $[0,\infty)\times[0,\infty)$ *with respect to* $\eta_1 : K_1 \times K_1 \to \mathbb{R}$ *and* $\eta_2 : K_2 \times K_2 \to \mathbb{R}$*,* $a, b \in K_1, c, d \in K_2$ *with* $\eta_1(b,a) > 0$ *and* $\eta_2(d,c) > 0$*. Suppose* $f : K_1 \times K_2 \to \mathbb{R}$ *be partially differentiable mapping such that* $\frac{\partial^2 f}{\partial\lambda\partial t} \in L([a,a+\eta_1(b,a)]\times[c,c+\eta_2(d,c)])$*. If* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ *is* (s_1,s_2)*-preinvex in the first sense on co-ordinates on* $K_1 \times K_2$*, for some fixed* $s_1, s_2 \in (0,1]$*,* $q \in [1,\infty)$*, then the following inequality holds*

$$\left|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$

$$\left.+\frac{1}{\eta_1(b,a)\,\eta_2(d,c)}\int\limits_a^{a+\eta_1(b,a)}\int\limits_c^{c+\eta_2(d,c)} f(x,y)\,dxdy - A\right|$$

$$\leq \frac{\eta_1(b,a)\,\eta_2(d,c)}{4^{2-\frac{1}{q}}}\left(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\right]\right. \tag{3.16}$$

$$\times\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\right]$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q\left[\frac{1}{2}-\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\right]\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\frac{s_1+\left(\frac{1}{2}\right)^{s_1}}{(s_1+1)(s_1+2)}\left[\frac{1}{2}-\frac{s_2+\left(\frac{1}{2}\right)^{s_2}}{(s_2+1)(s_2+2)}\right]$$

$$\left.+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\frac{\left[s_1+\left(\frac{1}{2}\right)^{s_1}\right]\left[s_2+\left(\frac{1}{2}\right)^{s_2}\right]}{(s_1+1)(s_1+2)(s_2+1)(s_2+2)}\right)^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1.

Proof. In the case where $q=1$ the proof is similar to that of Theorem 3.2, now we will treat the case where $q>1$.

From Lemma 2.1 and the power-mean inequality, we have

$$\left|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$

$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\right|$$

$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left(\int_0^1\int_0^1|1-2t||1-2\lambda|\,dtd\lambda\right)^{1-\frac{1}{q}}$$

$$\times\left(\int_0^1\int_0^1|1-2t||1-2\lambda|\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a+t\eta_1(b,a),c+\lambda\eta_2(d,c))\right|^q dtd\lambda\right)^{\frac{1}{q}}.$$

Using (s_1,s_2)-preinvexity in the first sense, we get

$$\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]$$

$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)}f(x,y)\,dxdy-A\right|$$

$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4}\left[\left(\int_0^1\int_0^1|1-2t||1-2\lambda|\,dtd\lambda\right)\right. \tag{3.17}$$

$$\times\left(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\int_0^1\int_0^1|1-2t||1-2\lambda|\,t^{\alpha_1}\lambda^{\alpha_2}dtd\lambda\right.$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\int_0^1\int_0^1|1-2t||1-2\lambda|\,t^{\alpha_1}(1-\lambda^{s_2})\,dtd\lambda$$

$$+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d)\right|^q \int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,(1-t^{s_1})\lambda^{\alpha_2}dtd\lambda$$
$$\left.\left.+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q \int_0^1\int_0^1 |1-2t||1-2\lambda|(1-t^{s_1})\left(1-\lambda^{s_2}\right)dtd\lambda\right)\right]^{1-\frac{1}{q}}.$$

Substituting (3.7) and

$$\int_0^1\int_0^1 |1-2t|\,|1-2\lambda|\,dtd\lambda=\frac{1}{4},$$

into (3.17), we get the required inequality in (3.16). The proof is completed. □

Remark 3.11. Theorem 3.6, will be reduced to Theorem 10 in [14], if we take $s_1 = s_2 = 1$, moreover if $\eta_1(b,a) = b-a$ and $\eta_2(d,c) = d-c$ then we obtain Theorem 4 in [24].

Corollary 3.6. *Under the same assumptions of Theorem* 3.6*, if* $\left|\frac{\partial^2 f}{\partial\lambda\partial t}\right|^q$ *is s-preinvex in the first sense on co-ordinates on* $K_1\times K_2$*, for some fixed* $s\in(0,1]$*,* $q\in[1,\infty)$*, then the following inequality holds*

$$\left|\frac{1}{4}\left[f(a,c)+f(a,c+\eta_2(d,c))+f(a+\eta_1(b,a),c)+f(a+\eta_1(b,a),c+\eta_2(d,c))\right]\right.$$
$$\left.+\frac{1}{\eta_1(b,a)\eta_2(d,c)}\int_a^{a+\eta_1(b,a)}\int_c^{c+\eta_2(d,c)} f(x,y)\,dxdy-A\right|$$
$$\leq\frac{\eta_1(b,a)\eta_2(d,c)}{4^{2-\frac{1}{q}}}$$
$$\times\left(\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,c)\right|^q\left[\frac{1}{2}-\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^2+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,d)\right|^q\left[\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]^2\right.$$
$$\left.+\left[\left|\frac{\partial^2 f}{\partial\lambda\partial t}(a,d))\right|^q+\left|\frac{\partial^2 f}{\partial\lambda\partial t}(b,c)\right|^q\right]\left[\frac{1}{2}-\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right]\frac{s+\left(\frac{1}{2}\right)^s}{(s+1)(s+2)}\right)^{\frac{1}{q}},$$

where A is defined as in Lemma 2.1.

Remark 3.12. Corollary 3.6, will be reduced to Theorem 10 in [14] if we take $s=1$, moreover if $\eta_1(b,a)=b-a$ and $\eta_2(d,c)=d-c$ then we obtain Theorem 4 in [24].

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