KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 39(2) (2015), PAGES 133–139.

AN UPPER BOUND ON THE DOUBLE DOMINATION NUMBER OF TREES

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ABSTRACT. In a graph G, a vertex dominates itself and its neighbors. A set S of vertices in a graph G is a *double dominating set* if S dominates every vertex of G at least twice. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set in G. The *annihilation number* $a(G)$ is the largest integer k such that the sum of the first k terms of the non-decreasing degree sequence of G is at most the number of edges in G. In this paper, we show that for any tree T of order $n \geq 2$, different from P_4, $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$.

1. INTRODUCTION

In this paper, G is a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The order $|V|$ of G is denoted by $n = n(G)$. For every vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum degree* of a graph G is denoted by $\delta = \delta(G)$. A *leaf* of a tree T is a vertex of degree 1, a *support vertex* is a vertex adjacent to a leaf and a *strong support vertex* is a vertex adjacent to at least two leaves. A strong support vertex is said to be *end-stem* if all its neighbors except one of them are leaves. We write P_n for a path of order n. For a subset $S \subseteq V(G)$, we let $\sum(S, G) = \sum_{v \in S} \deg_G(v)$. For notation and graph theory terminology, we in general follow [10].

The concept of domination in graphs, with its many variations, is now well studied in graph theory and the literature on this subject has been surveyed and detailed in the two books by Haynes, Hedetniemi, and Slater [10, 11]. A *dominating set* of a

Key words and phrases. domination number, double dominating set, double domination number, annihilation number, tree.
2010 *Mathematics Subject Classification.* 05C69.
Received: 23 April, 2015.
Accepted: 18 July, 2015.

graph G is a set S of vertices of G such that every vertex in $V(G) - S$ is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

Harary and Haynes [9] defined a generalization of domination as follows: a subset S of V is a *k-tuple dominating set* of G if for every vertex $v \in V$, $|N[v] \cap S| \geq k$. The *k-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of a k-tuple dominating set of G. Obviously, $\gamma(G) = \gamma_{\times 1}(G) \leq \gamma_{\times k}(G)$. For a graph to have a k-tuple dominating set, its minimum degree must be at least $k - 1$. Hence for trees, we have $k \leq 2$. A 2-tuple dominating set is called a *double dominating set* (DDS). A $\gamma_{\times 2}(G)$-set is a DDS of cardinality $\gamma_{\times 2}(G)$. The redundancy involved in k-tuple domination makes it useful in many applications.

Let $d_1, d_2, \ldots, d_n$ be the degree sequence of a graph G arranged in non-decreasing order, and so $d_1 \leq d_2 \leq \cdots \leq d_n$. The *annihilation number* of G, denoted $a(G)$, is the largest integer k such that the sum of the first k terms of the degree sequence is at most half the sum of the degrees in the sequence. Equivalently, the annihilation number is the largest integer k satisfying the condition that $\sum_{i=1}^{k} d_i \leq \sum_{i=k+1}^{n} d_i$. It is immediate from the definition that if G has m edges and annihilation number k, then $\sum_{i=1}^{k} d_i \leq m$. The annihilation number was introduced by Pepper in [15] and has been studied by several authors [2, 6, 7, 8, 13, 14, 16]. As an immediate consequence of the definition of the annihilation number, Larson and Pepper [14] observed that for any graph G of order n, $a(G) \geq \lfloor \frac{n}{2} \rfloor$.

In [15] and [16], Pepper proved that the annihilation number is an upper bound on the independence number of a graph and in [14] the case for equality of the upper bound was characterized by Larson and Pepper.

The relation between annihilation number and some graph parameters have been studied by several authors. For instance, DeLaViña et al. presented an upper bound on 2-domination number in terms of annihilation number for some classes of graphs [6], Dehgardi et al. investigated the relation between some domination parameters and the annihilation number of trees [3, 4, 5], Desormeaux et al. proved that for any tree T, $a(T) + 1$ is an upper bound on the total domination number [8].

If G is a connected graph, different from C_5, of order n with minimum degree at least two, then it is known [12] that $\gamma_{\times 2}(G) \leq \frac{3n}{4}$. Hence, if $G \neq C_5$ is a connected graph of order n with minimum degree at least 2, then $\gamma_{\times 2}(G) \leq \frac{3a(T)}{2}$ because $a(G) \geq \lfloor \frac{n}{2} \rfloor$. On the other hand, we have $\gamma_{\times 2}(C_5) = 4 = \frac{3a(C_5)+2}{2}$.

In this paper we continue the study of double domination in trees and we prove that for any tree T of order at least two $\gamma_{\times 2}(T) \leq \frac{3a(T)+2}{2}$ and the equality holds if and only if $T = P_4$.

The value of $\gamma_{\times 2}(P_n)$ for a path P_n is established in [1].

Proposition 1.1. *$\gamma_{\times 2}(P_n) = 2\left\lceil \frac{n}{3} \right\rceil + 1$ if $n \equiv 0 \pmod 3$ and $\gamma_{\times 2}(P_n) = 2\left\lceil \frac{n}{3} \right\rceil$ otherwise.*

The annihilation number is easy to compute for paths and we have the following observation.

Observation 1.1. *For* $n \geq 2$, $a(P_n) = \lceil \frac{n}{2} \rceil$.

The next result is an immediate consequence of Proposition 1.1 and Observation 1.1.

Proposition 1.2. *For* $n \geq 3$, $\gamma_{\times 2}(P_n) \leq \frac{3a(T)+2}{2}$ *with equality if and only if* $n = 4$.

The next result is immediate from definitions.

Observation 1.2. *Every leaf and every support vertex of a graph* G *is in every* $\gamma_{\times 2}(G)$-*set.*

A *subdivision* of an edge uv is obtained by replacing the edge uv with a path uwv, where w is a new vertex. The *subdivision graph* $S(G)$ is the graph obtained from G by subdividing each edge of G. The subdivision star $S(K_{1,t})$ for $t \geq 2$, is called a *healthy spider* $S_{t,t}$. A *wounded spider* $S_{t,q}$ is the graph formed by subdividing q of the edges of a star $K_{1,t}$ for $t \geq 2$ where $q \leq t-1$. Note that stars are wounded spiders. A *spider* is a healthy or wounded spider.

Proposition 1.3. *If* T *is a spider different from* P_4, *then* $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$ *with equality if and only if* $T \cong S_{3,1}$ or $S_{4,3}$.

Proof. If $T = S_{t,t}$ is a healthy spider for some $t \geq 2$, then obviously $\gamma_{\times 2}(T) = 2t$ and $a(T) = t + \lfloor \frac{t}{2} \rfloor$, and we have $\gamma_{\times 2}(T) \leq \frac{3a(T)}{2}$.

Now let $T = S_{t,q}$ be a wounded spider obtained from $K_{1,t}$ $(t \geq 2)$ by subdividing $0 \leq q \leq t-1$ edges. Then $\gamma_{\times 2}(T) = n(T)$ by Observation 1.2. If $q = 0$, then T is a star and clearly $a(T) = t$ implying that $\gamma_{\times 2}(T) \leq \frac{3a(T)}{2}$. Assume $q > 0$. Since $T \neq P_4$, we have $q \neq 1$ or $t \neq 2$. It is easy to see that $a(T) = t + \lfloor \frac{q}{2} \rfloor$ and so $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$ with equality if and only if $T \cong S_{3,1}$ or $S_{4,3}$. This completes the proof. □

For $p, q \geq 1$, a double star $DS_{p,q}$ is a tree with exactly two vertices that are not leaves, with one adjacent to p leaves and the other to q leaves.

Proposition 1.4. *For the double star* $DS_{p,q}$, *different from* P_4,

$$\gamma_{\times 2}(DS_{p,q}) \leq \frac{3a(T)+1}{2}$$

with equality if and only if $T = DS_{1,2}$.

Proof. We may assume without loss of generality that $p \leq q$. Since $T \neq P_4$, $q \geq 2$. By Observation 1.2, $\gamma_{\times 2}(DS_{p,q}) = p + q + 2$. On the other hand, $a(DS_{p,q}) = p + q$. This implies that $\gamma_{\times 2}(DS_{p,q}) \leq \frac{3a(T)+1}{2}$ with equality if and only if $p = 1$ and $q = 2$, that is, $T = DS_{1,2}$. □

2. An Upper Bound

In this section we establish an upper bound on the double domination number of trees in terms of their annihilation number.

For a vertex v in a rooted tree T, let $C(v)$ denote the set of children of v, $D(v)$ denote the set of descendants of v and $D[v] = D(v) \cup \{v\}$, and the depth of v, depth(v), is the largest distance from v to a vertex in $D(v)$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v. In the sequel, we denote by $T - T_v$, the tree obtained from a rooted tree T by deleting all vertices of $D[v]$.

In the proof of next theorem, we will always consider trees T' formed from T by removing a set of vertices. For such a tree T' of order n', let $d'_1, d'_2, \dots, d'_{n'}$ be the non-decreasing degree sequence of T', and let S' be a set of vertices corresponding to the first $a(T')$ terms in the degree sequence of T'. In fact, if $u_1, u_2, \dots, u_{n'}$ are the vertices of T' such that $\deg(u_i) = d'_i$ for each i, then $S' = \{u_1, u_2, \dots, u_{a(T')}\}$. We denote the size of T' by m'.

Theorem 2.1. *For any nontrivial tree T, different from P_4, $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$.*

Proof. The proof is by induction on n. The statement holds for all trees of order $n \leq 4$. For the induction hypothesis, let $n \geq 5$ and suppose that for every nontrivial tree T of order less than n the result is true. Let T be a tree of order n. By Propositions 1.2, 1.3 and 1.4, we may assume diam$(T) \geq 4$ and that T is not a path. We proceed further with a series of claims that we may assume satisfied by the tree.

Claim 1. T has no end-steam.

Proof. Let T have an end-stem u and let $N(u) = \{v, u_1, u_2, \dots, u_k\}$ where v is not a leaf. Assume $T' = T - \{u, u_1, \dots, u_k\}$. If $T' = P_4$, then clearly $\gamma_{\times 2}(T) = 4 + k$ and $a(T) = k + 2$ implying that $\gamma_{\times 2}(T) \leq \frac{3a(T)}{2}$. Let $T' \neq P_4$. Then obviously $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + k + 1$. If $v \notin S'$, then $\sum(S', T) = \sum(S', T')$ and if $v \in S'$, then $\sum(S', T) = \sum(S', T') + 1$. Thus, $\sum(S', T) - 1 \leq \sum(S', T') \leq m' = m - k - 1$, and hence $\sum(S', T) \leq m - k$. Let $S = S' \cup \{u_1, \dots, u_k\}$. Then $\sum(S, T) = \sum(S', T) + k \leq m$ implying that $a(T) \geq a(T') + k$. By inductive hypothesis, we have

$$\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + k + 1 \leq \frac{3a(T')+1}{2} + k + 1 \leq \frac{3(a(T)-k)+1}{2} + k + 1 \leq \frac{3a(T)+1}{2},$$

as desired. □

Let $v_1 v_2 \dots v_D$ be a diametral path in T. By Claim 1, $\deg(v_2) = \deg(v_{D-1}) = 2$ and all neighbors of v_3 with exception of v_4 (resp. all neighbors of v_{D-2} with exception of v_{D-3}), are leaves or support vertices of degree 2. If diam$(T) = 4$, then T is a spider and the result follows by Proposition 1.3. Hence, we assume diam$(T) \geq 5$. Root T at v_D.

Claim 2. $\deg_T(v_3) \leq 3$.

Proof. Assume that $\deg_T(v_3) \geq 4$. Let first v_3 be adjacent to a support vertex $w_2 \notin \{v_2, v_4\}$. By Claim 1, we have $\deg_T(w_2) = 2$. Suppose w_1 is the leaf adjacent to w_2 and let $T' = T - \{v_1, v_2, w_1, w_2\}$. If $T' = P_4$, then clearly $\gamma_{\times 2}(T) \leq 8$ and $a(T) = 5$ implying that $\gamma_{\times 2}(T) \leq 8 \leq \frac{3a(T)+1}{2}$. Let $T' \neq P_4$. Then every double dominating set of T' can be extended to a double dominating set of T by adding v_1, w_1, v_2, w_2 and hence $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 4$. Assume that $S = S' \cup \{v_1, v_2, w_1\}$ when $v_3 \notin S'$ and $S = (S' - \{v_3\}) \cup \{v_1, v_2, w_1, w_2\}$ if $v_3 \in S'$. Clearly $\sum(S, T) \leq m$ that implies $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. It follows from the induction hypothesis that

$$\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 4 \leq \frac{3a(T')+1}{2} + 4 \leq \frac{3(a(T)-3)+1}{2} + 4 < \frac{3a(T)+1}{2}.$$

Now let all neighbors of v_3 with exception of v_2, v_4, are leaves. Suppose w is a leaf adjacent to v_3 and let $T' = T - \{v_1, v_2, w\}$. Since daim$(T) \geq 5$, we have $T' \neq P_4$. Clearly, every double dominating set of T' can be extended to a double domination set of T by adding the vertices v_1, v_2, w and hence $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 3$. Assume that $S = S' \cup \{v_1, w\}$ when $v_3 \notin S'$ and $S = (S' - \{v_3\}) \cup \{v_1, v_2, w\}$ if $v_3 \in S'$. Then $\sum(S, T) \leq m$ that implies $a(T) \geq |S| = |S'| + 2 = a(T') + 2$. As above, we have $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$. □

Claim 3. $\deg_T(v_3) = 2$.

Proof. Let $\deg(v_3) = 3$. Suppose first that v_3 is adjacent to a support vertex $w_2 \notin \{v_2, v_4\}$. Let w_1 be the leaf adjacent to w_2 and $T' = T - T_{v_3}$. If $T' = P_4$, then clearly $\gamma_{\times 2}(T) = 8$ and $a(T) = 5$ implying that $\gamma_{\times 2}(T) = \frac{3a(T)+1}{2}$. Let $T' \neq P_4$. Then every double dominating set of T' can be extended to a double dominating set of T by adding v_1, w_1, v_2, w_2 and hence $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 4$. Assume that $S = S' \cup \{v_1, v_2, w_1\}$. Then $\sum(S, T) \leq m$ and so $a(T) \geq |S| = |S'| + 3 = a(T') + 3$. Now the result follows by the induction hypothesis.

Now let v_3 be adjacent to a leaf w and let $T' = T - \{v_1, v_2, w\}$. If $T' = P_4$, then clearly $\gamma_{\times 2}(T) = 6$ and $a(T) = 5$ implying that $\gamma_{\times 2}(T) = 6 < \frac{3a(T)+1}{2}$. Let $T' \neq P_4$. Using an argument similar to that described in Claim 2, we obtain $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$ as desired. □

Claim 4. There is no path $v_4 z_3 z_2 z_1$ in T such that $z_3 \notin \{v_3, v_5\}$.

Proof. Assume there is a path $v_4 z_3 z_2 z_1$ in T where $z_3 \notin \{v_3, v_5\}$. By Claims 1–3, we may assume $\deg_T(z_2) = \deg_T(z_3) = 2$. Assume $T' = T - \{v_1, v_2, v_3, z_1, z_2\}$. If $T' = P_4$, then clearly $\gamma_{\times 2}(T) = 7$ and $a(T) = 5$, implying that $\gamma_{\times 2}(T) < \frac{3a(T)+1}{2}$. Let $T' \neq P_4$. By Observation 1.2, v_4 is in every $\gamma_{\times 2}(T)$-set because v_4 is a support vertex. Hence, every $\gamma_{\times 2}(T')$-set can be extended to a double dominating set of T by adding the vertices v_1, v_2, z_1, z_2 which implies that $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 4$. Assume $S = S' \cup \{v_1, v_2, z_1\}$ when $v_4 \notin S'$ and $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3, z_1\}$ when $v_4 \in S'$. Then $\sum(S, T) \leq m$ and hence $a(T) \geq a(T') + 3$. Now the result follows by inductive hypothesis. □

Claim 5. v_4 is not a support vertex.

Proof. Assume v_4 is a support vertex. Let $T' = T - \{v_1, v_2, v_3\}$. If $T' = P_4$, then it is easy to verify that $\gamma_{\times 2}(T) \leq \frac{3a(T)+1}{2}$. Let $T' \neq P_4$. By Observation 1.2, v_4 is in every $\gamma_{\times 2}(T')$-set and so every $\gamma_{\times 2}(T')$-set can be extended to a double dominating set of T by adding the vertices v_1, v_2 implying that $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 2$. Assume $S = S' \cup \{v_1, v_2\}$ when $v_4 \notin S'$ and $S = (S' - \{v_4\}) \cup \{v_1, v_2, v_3\}$ when $v_4 \in S'$. Clearly $\sum(S, T) \leq m$ and hence $a(T) \geq a(T') + 2$. Applying the induction hypothesis we obtain $\gamma_{\times 2}(T) < \frac{3a(T)+1}{2}$. □

Claim 6. v_4 is not adjacent to a support vertex other than v_5.

Proof. Let v_4 be adjacent to a support vertex, say w and let $T' = T - \{v_1, v_2\}$. Suppose D is a $\gamma_{\times 2}(T')$-set. Then $v_3, v_4, w \in D$ and obviously $(D - \{v_3\}) \cup \{v_1, v_2\}$ is a DDS of T and hence $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 1$. Assume $S = S' \cup \{v_1\}$. Then $\sum(S, T) \leq m$ that implies $a(T) \geq a(T') + 1$. By the induction hypothesis we have $\gamma_{\times 2}(T) < \frac{3a(T)+1}{2}$. □

By Claims 4, 5, and 6 we may assume $\deg_T(v_4) = 2$. Similarly, by rooting T at v_1, we may assume that $\deg(v_{D-1}) = \deg(v_{D-2}) = \deg(v_{D-3}) = 2$.

We now return to the proof of theorem. If $\mathrm{diam}(T) = 5, 6$ or 7 then T is a path of order 6, 7 and 8, respectively, and the result is immediate by Proposition 1.2. Let $\mathrm{diam}(T) \geq 8$ and $T' = T - \{v_1, v_2, v_3, v_D, v_{D-1}, v_{D-2}\}$. If $T' = P_4$, then obviously T is a path of order 10 and the result follows by Proposition 1.2. Assume $T' \neq P_4$. Since $\deg_{T'}(v_4) = \deg_{T'}(v_{D-3}) = 1$, every $\gamma_{\times 2}(T')$-set contains v_4, v_{D-3} and can be extended to a dominating set of T by adding v_1, v_2, v_D, v_{D-1} which implies that $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 4$. Suppose $S = S' \cup \{v_1, v_2, v_D\}$. Then $\sum(S, T) \leq m$ and so $a(T) \geq a(T') + 3$. Applying the induction hypothesis, we obtain $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 4 \leq \frac{3a(T')+1}{2} + 4 \leq \frac{3(a(T)-3)+1}{2} + 4 < \frac{3a(T)+1}{2}$. This completes the proof. □

The coronal of two graphs G_1 and G_2, is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the ith vertex of G_1 is adjacent to every vertex in the ith copy of G_2.

Assume that $P_n = v_1 v_2 \ldots v_n$ is a path on n vertices and let $G_n = P_n \circ K_1$. Suppose u_i is the leaf adjacent to v_i for each i.

Theorem 2.2. *For $k \geq 3$, there exists a tree T with* $\mathrm{daim}(T) = k$ *and* $\gamma_{\times 2}(T) = \frac{3a(T)+1}{2}$.

Proof. Let $k \geq 3$ be an integer. If $k \equiv 2 \pmod 3$, then let $T = G_{k-1}$. It is easy to see that $\gamma_{\times 2}(T) = 2k - 2$ and $a(T) = \frac{4(k-2)}{3} + 1$, and so $\gamma_{\times 2}(T) = \frac{3a(T)+1}{2}$. If $k \equiv 0 \pmod 3$, then let T be the tree obtained from G_{k-1} by adding a pendant edge at v_1. It is not hard to see that $\gamma_{\times 2}(T) = 2k - 1$ and $a(T) = \frac{4k}{3} - 1$, and hence $\gamma_{\times 2}(T) = \frac{3a(T)+1}{2}$. Finally, if $k \equiv 1 \pmod 3$, then let T be the tree obtained from G_{k-1} by adding a pendant edge at v_1 and a pendant edge at v_{k-1}. It is easy to see that $\gamma_{\times 2}(T) = 2k$ and $a(T) = \frac{4(k-1)}{3} + 1$, and hence $\gamma_{\times 2}(T) = \frac{3a(T)+1}{2}$. This completes the proof. □

We conclude this paper with an open problem.

Conjecture 1. For any connected graph G of order $n \geq 2$ with $\delta(G) = 1$, $\gamma_{\times 2}(G) \leq \frac{3a(G)+2}{2}$.

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