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THE RADON TRANSFORM OF CERTAIN OSCILLATORY DISTRIBUTIONS

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Abstract. We study the Radon transform in the class of oscillatory distributions $\mathcal{K}'(\mathbb{R}^n)$. We show that the support theorems for rapidly decreasing functions also hold in this space of oscillatory distributions.

1. Introduction

The study of support results for the Radon transform has attracted the attention of several authors, starting with the pioneering work of Helgason [11] and Ludwig [13]. Helgason established that if f is a continuous function with suitable fast decay at infinity, whose Radon transform vanishes at all hyperplanes that do not intersect a given compact convex set K, then actually $\operatorname{supp} f \subset K$.

Many extensions and generalizations have been given through the following years. Simple examples (just take $n = 2$, $f(z) = z^{-m}$, $m \in \mathbb{N}$, $m \geq 2$, and K any compact set that contains the origin) show that the fast decay at infinity is indispensable to obtain support theorems,[1] but one can relax the smoothness of f quite a bit, asking, for instance, that it be a distribution [2]. More information on these matters can be found in [18].

In [19] Strichartz gave a nice method to prove that if f has *compact* support and if its Radon transform vanishes at all hyperplanes that do not intersect a given compact convex set K, then $\operatorname{supp} f \subset K$. In this note we show that a small extra argument in the method of [19] allows us to obtain support theorems in much larger classes of functions, not only for rapidly decreasing continuous functions, but actually in case f belongs to the class of oscillatory kernels $\mathcal{K}'(\mathbb{R}^n)$ explained in Section 2.

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[1]There are rather extreme negative examples: the existence of harmonic functions in $\mathbb{R}^n$ whose Radon transform vanishes *everywhere* has been shown for $n = 2$ in [21] and for general n in [1].

The plan of this note is as follows. In Section 2 we recall the definition of the space of distributions $\mathcal{K}'(\mathbb{R}^n)$, and several of its properties. In Section 3 we show how one can define the Radon transform in this space and give transformation formulas for a reparametrization of the Radon transform. The main idea of our procedure is explained in Section 4, while the support theorem in $\mathcal{K}'(\mathbb{R}^n)$ is given in Section 5.

2. Preliminaries

We shall work in the space[2] $\mathcal{K}'(\mathbb{R}^n)$, dual of the space of test functions $\mathcal{K}(\mathbb{R}^n)$, which is the inductive limit

$$\mathcal{K}(\mathbb{R}^n) = \operatorname*{ind\,lim}_{q\to\infty} \mathcal{K}_q(\mathbb{R}^n), \tag{2.1}$$

where $\phi \in \mathcal{K}_q(\mathbb{R}^n)$ if and only for each multiindex $\boldsymbol{\alpha} \in \mathbb{N}^n$,

$$\mathbf{D}^{\boldsymbol{\alpha}}\phi(\mathbf{x}) = O\big(|\mathbf{x}|^{q-|\boldsymbol{\alpha}|}\big) \quad \text{as } |\mathbf{x}| \to \infty, \tag{2.2}$$

where $\mathbf{D} = (\partial_{x_1}, \dots, \partial_{x_n})$ and $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_n$. The topology of $\mathcal{K}_q(\mathbb{R}^n)$ is given by the family of seminorms

$$\|\phi\|_{q,\boldsymbol{\alpha}} = \max_{\mathbf{x}\in\mathbb{R}^n}(1+|\mathbf{x}|^2)^{-(q-|\boldsymbol{\alpha}|)/2}|\mathbf{D}^{\boldsymbol{\alpha}}\phi(\mathbf{x})|, \quad \boldsymbol{\alpha} \in \mathbb{N}^n, \tag{2.3}$$

so that $\mathcal{K}_q(\mathbb{R}^n)$ is a Fréchet space. The *topological* vector space $\mathcal{K}(\mathbb{R}^n)$ has the inductive limit topology, so that it is an *LF* space [20, Chap. 13]. We shall usually employ the strong or the weak topologies of the dual space $\mathcal{K}'(\mathbb{R}^n)$, topologies for which the usual operations on distributions will be continuous.

Observe that any polynomial p in n variables belongs to $\mathcal{K}(\mathbb{R}^n)$, so that the evaluation $\langle f, p\rangle$ is well defined if $f \in \mathcal{K}'(\mathbb{R}^n)$. In particular, if $f \in \mathcal{K}'(\mathbb{R}^n)$ then it has well defined moments,

$$\mu_{\mathbf{k}} = \langle f(\mathbf{x}), \mathbf{x}^{\mathbf{k}}\rangle, \quad \mathbf{k} \in \mathbb{N}^n. \tag{2.4}$$

The space $\mathcal{K}'(\mathbb{R}^n)$ is very important in the theory of asymptotic expansion of generalized functions [8]. Indeed [7,8], a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $\mathcal{K}'(\mathbb{R}^n)$ if and only if it satisfies the *Moment Asymptotic Expansion*,

$$f(\lambda\mathbf{x}) \sim \sum_{j=0}^{\infty}\sum_{|\mathbf{k}|=j}(-1)^j\frac{\mu_{\mathbf{k}}\,\mathbf{D}^{\mathbf{k}}\delta(\mathbf{x})}{\mathbf{k}!\,\lambda^{n+j}} \quad \text{as } \lambda \to \infty, \tag{2.5}$$

where the constants $\mu_{\mathbf{k}}$ are given by (2.4). Therefore, $f \in \mathcal{K}'(\mathbb{R}^n)$ if and only if for each test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\phi$, the smooth function $F_\phi(\lambda) = \langle f(\lambda\mathbf{x}), \phi(\mathbf{x})\rangle$, $\lambda \in (0,\infty)$, satisfies $F_\phi(\lambda) = o(\lambda^{-k})$ as $\lambda \to \infty$ for all $k > 0$; actually this characterization holds if we ask that $F_\phi(\lambda) = o(\lambda^{-k})$ for all $k > 0$ as $\lambda \to \infty$ but just for all $\phi \in \mathcal{D}(\mathbb{R}^n)$ with $0 \notin \operatorname{supp}\phi$.

Moreover, the elements of $\mathcal{K}'(\mathbb{R}^n)$ admit the following alternative characterization [4,8]. A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ belongs to $\mathcal{K}'(\mathbb{R}^n)$ if and only if f is of rapid decay

[2]This is the space of so-called GLS symbols, introduced in [10].

at infinity in the *Cesàro* sense: $f(\mathbf{x}) = o(|\mathbf{x}|^{-k})$ as $|\mathbf{x}| \to \infty$ in the Cesàro sense for all $k > 0$. Please see [8, Chap. 6] for details on the Cesàro behaviour of distributions.

Hence, we can say that the elements of $\mathcal{K}'(\mathbb{R}^n)$ are of rapid decay in the parametric sense, as well as in the Cesàro sense, but in the ordinary sense they are usually oscillatory functions or distributions.

It should be mentioned that, unfortunately, the term "rapidly decreasing at infinity in the distributional sense" is not a standard one. Indeed, Horváth [12] calls a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ "rapidly decreasing" if $f * \phi \in \mathcal{S}(\mathbb{R}^n)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$. Similarly, the support theorems of Boman and Lindskog [2] apply to distributions of "rapid decay" in the sense that $f * \phi \in \mathcal{S}(\mathbb{R}^n)$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$. It can be shown that rapid decay in the sense of Horváth and rapid decay in the sense of Boman and Lindskog are equivalent,[3] and indeed, such condition is equivalent to $f \in \mathcal{O}'_C(\mathbb{R}^n)$, where $\mathcal{O}'_C(\mathbb{R}^n) = \mathcal{F}(\mathcal{O}_M(\mathbb{R}^n))$ is the space of Fourier transforms of multipliers of $\mathcal{S}(\mathbb{R}^n)$ [12]. Since $\mathcal{O}'_C(\mathbb{R}^n) \subset \mathcal{K}'(\mathbb{R}^n)$, distributions that are of rapid decay in the Horváth–Boman–Lindskog sense are of rapid decay in the sense of [8], but there are distributions of $\mathcal{K}'(\mathbb{R}^n)$ that are not of rapid decay in the H-B-L sense, such as, among many, the oscillatory distributions $f(\mathbf{x}) = \sin|\mathbf{x}|$ or $g(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^n}(-1)^{|\mathbf{k}|}\delta(\mathbf{x}-\mathbf{k})$.

Typical elements of $\mathcal{K}'(\mathbb{R})$ are the following *oscillatory* functions: $\sin x$, $\cos x$, $J_\nu(x)$, and $\sum_{n=-\infty}^{\infty}(-1)^n\delta(x-n)$. In several variables we may mention the following elements of $\mathcal{K}'(\mathbb{R}^n)$: those of the form $f_1(x_1)\cdots f_n(x_n)$, where the f_j belong to $\mathcal{K}'(\mathbb{R})$,[4] or those of the form $f(|\mathbf{x}|)$ where $f \in \mathcal{K}'(\mathbb{R})$.[5]

3. The Radon Transform in $\mathcal{K}'(\mathbb{R}^n)$

The standard Radon transform of a function f defined in $\mathbb{R}^n$ is the function $R(f)(H)$, whose arguments are hyperplanes H of $\mathbb{R}^n$, defined as

$$R(f)(H) = \int_H f(\mathbf{u})\,\mathrm{d}\mathbf{u}, \tag{3.1}$$

where $\mathrm{d}\mathbf{u}$ is the surface measure on H. The hyperplanes of $\mathbb{R}^n$ can be parametrized in terms of the parameters $\boldsymbol{\omega} \in \mathbb{S}^{n-1}$, the unit sphere of $\mathbb{R}^n$, and $t \in \mathbb{R}$ as $H = H_{(\boldsymbol{\omega},t)}$, the hyperplane given by the equation $\mathbf{x}\cdot\boldsymbol{\omega} = t$. Observe that $H_{(\boldsymbol{\omega},t)} = H_{(-\boldsymbol{\omega},-t)}$. Thus we may treat the Radon transform as an *even* function $Rf(\boldsymbol{\omega}, t)$ defined on the cylinder $\mathbb{S}^{n-1}\times\mathbb{R}$.

Clearly the integral definition (3.1) can be applied only for functions of suitable decay at infinity, but as with any integral transform, it can be extended to more general functions, and distributions, by employing duality. Indeed, if $\mathcal{A}(\mathbb{R}^n)$ is a

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[3] This is not obvious, since in general [14] the S-asymptotics in $\mathcal{D}$ may not hold in $\mathcal{S}$.

[4] Observe, however, that in general if the ϕ_j belong to $\mathcal{K}(\mathbb{R})$ then $\phi_1(x_1)\cdots\phi_n(x_n)$ does not belong to $\mathcal{K}(\mathbb{R})$.

[5] Interestingly, no restriction of f at the origin is required, but the distribution of one variable f that represents the radial distribution $f(|\mathbf{x}|)$ is not unique, since $g(|\mathbf{x}|) = 0$ if $g(x) = \sum_{j=0}^{n-1} a_j\,\delta^{(j)}(x)$. See [5].

space of smooth functions, that is, $\mathcal{A}$ can be $\mathcal{D}$, $\mathcal{S}$, $\mathcal{E}$ or $\mathcal{K}$, among others, then the corresponding space $\mathcal{A}(\mathbb{S}^{n-1}\times\mathbb{R})$ is formed by those smooth functions φ defined in the cylinder that satisfy that $\phi_{\pm}(\mathbf{x})\rho(\mathbf{x})=\varphi\big(\mathbf{x}/|\mathbf{x}|,\pm|\mathbf{x}|\big)\rho(\mathbf{x})$ belong to $\mathcal{A}(\mathbb{R}^n)$ for every smooth function ρ defined in $\mathbb{R}^n$ with $\rho(\mathbf{x})=0$, $|\mathbf{x}|<a$; $\rho(\mathbf{x})=1$, $|\mathbf{x}|>b$, for some $a,b>0$. The spaces of test functions $\mathcal{A}(\mathbb{S}^{n-1}\times\mathbb{R})$ can be given a canonical topology, and this allows us to consider the spaces of distributions $\mathcal{A}'(\mathbb{S}^{n-1}\times\mathbb{R})$ on the cylinder. If $\phi\in\mathcal{S}(\mathbb{R}^n)$ and $\varphi\in\mathcal{S}(\mathbb{S}^{n-1}\times\mathbb{R})$, then

$$\int_0^\infty\int_{\mathbb{S}^{n-1}} R(\phi)(\boldsymbol{\omega},t)\,\varphi(\boldsymbol{\omega},t)\,\mathrm{d}\sigma(\boldsymbol{\omega})\,\mathrm{d}t=\int_{\mathbb{R}^n}\phi(\mathbf{x})\,R^*(\varphi)(\mathbf{x})\,\mathrm{d}\mathbf{x}, \tag{3.2}$$

where the transpose $R^*(\varphi)$ is defined as

$$R^*(\varphi)(\mathbf{x})=\int_{\mathbb{S}^{n-1}}\varphi(\boldsymbol{\omega},\boldsymbol{\omega}\cdot\mathbf{x})\,\mathrm{d}\sigma(\boldsymbol{\omega}). \tag{3.3}$$

Thus we may define the Radon transform of certain classes of distributions by employing (3.2), that is,

$$\langle R(f),\varphi\rangle_{\mathbb{S}^{n-1}\times\mathbb{R}}=\frac{1}{2}\langle f,R^*(\varphi)\rangle_{\mathbb{R}^n}\,, \tag{3.4}$$

defines $R\colon\mathcal{A}'(\mathbb{R}^n)\to\mathcal{B}'(\mathbb{S}^{n-1}\times\mathbb{R})$ whenever R^* is an operator from $\mathcal{B}(\mathbb{S}^{n-1}\times\mathbb{R})$ to $\mathcal{A}(\mathbb{R}^n)$. Indeed, this construction gives the Radon transform of distributions of compact support [13], $R\colon\mathcal{E}'(\mathbb{R}^n)\to\mathcal{E}'(\mathbb{S}^{n-1}\times\mathbb{R})$; on the other hand, it is not possible to define the Radon transform as an operator from $\mathcal{S}'(\mathbb{R}^n)$ to itself [15,16]. In the case of the space $\mathcal{K}'(\mathbb{R}^n)$, we have the following result.

Lemma 3.1. *The dual Radon transform R^* is a continuous map from $\mathcal{K}(\mathbb{S}^{n-1}\times\mathbb{R})$ to $\mathcal{K}(\mathbb{R}^n)$. The Radon transform is an operator from $\mathcal{K}'(\mathbb{R}^n)$ to $\mathcal{K}'(\mathbb{S}^{n-1}\times\mathbb{R})$, continuous if both dual spaces carry the weak or both the strong topologies.*

Proof. Indeed, a smooth function φ defined in the cylinder $\mathbb{S}^{n-1}\times\mathbb{R}$ belongs to $\mathcal{K}_q(\mathbb{S}^{n-1}\times\mathbb{R})$ if and only if for each differential operator $\mathbf{L}$ with smooth coefficients in $\mathbb{S}^{n-1}$ we have

$$\frac{\partial^m}{\partial t^m}\mathbf{L}\varphi(\boldsymbol{\omega},t)=O(t^{q-m})\quad\text{as } t\to\infty, \tag{3.5}$$

uniformly with respect to $\boldsymbol{\omega}$. The formula (3.3) hence immediately gives that R^* sends $\mathcal{K}_q(\mathbb{S}^{n-1}\times\mathbb{R})$ continuously to $\mathcal{K}_q(\mathbb{R}^n)$, and consequently $\mathcal{K}(\mathbb{S}^{n-1}\times\mathbb{R})$ to $\mathcal{K}(\mathbb{R}^n)$. The second part then follows by standard functional analysis arguments [12,20]. □

Notice that if $f\in\mathcal{K}'(\mathbb{R}^n)$ and $\phi\in\mathcal{D}(\mathbb{R}^{n-1})$, then the distribution of one variable, x, given by $g(x)=\langle f(\mathbf{y},x),\phi(\mathbf{y})\rangle_{\mathbf{y}}$ belongs to $\mathcal{D}'(\mathbb{R})$:

$$\langle g(x),\varphi(x)\rangle_x=\big\langle f(\mathbf{y},x),\phi(\mathbf{y})\varphi(x)\big\rangle_{(\mathbf{y},x)}\,,\quad\varphi\in\mathcal{D}(\mathbb{R}), \tag{3.6}$$

but in general it does not belong to $\mathcal{K}'(\mathbb{R})$, and its meaning if $\phi\in\mathcal{K}(\mathbb{R}^{n-1})$ is not clear at all. However, if $\psi\in\mathcal{D}(\mathbb{S}^{n-1})$ and $f\in\mathcal{K}'(\mathbb{R}^n)$ then the evaluation

$h(t) = \langle f(t\boldsymbol{\omega}), \psi(\boldsymbol{\omega})\rangle_{\boldsymbol{\omega}}$ gives[6] an element of $\mathcal{K}'(0,\infty)$:

$$\langle h(t), \rho(t)\rangle_t = \left\langle f(\mathbf{x}), \psi(\mathbf{x}/|\mathbf{x}|)\,\rho(|\mathbf{x}|)\,|\mathbf{x}|^{1-n}\right\rangle_{\mathbf{x}}, \quad \rho \in \mathcal{K}(0,\infty), \tag{3.7}$$

where $\rho \in \mathcal{K}(0,\infty)$ if it has a smooth extension $\widetilde{\rho} \in \mathcal{K}(\mathbb{R})$ with $\widetilde{\rho}^{(j)}(0) = 0$ for all $j \in \mathbb{N}$. Similar considerations apply in the cylinder, so that employing Lemma 3.1 we obtain the following useful result.

Lemma 3.2. *If $f \in \mathcal{K}'(\mathbb{R}^n)$ and $\psi \in \mathcal{D}(\mathbb{S}^{n-1})$ then the evaluation $\langle R(f)(t,\boldsymbol{\omega}), \psi(\boldsymbol{\omega})\rangle_{\boldsymbol{\omega}}$ belongs to $\mathcal{K}'(0,\infty)$.*

It will prove convenient to employ other parameters instead of $(t,\boldsymbol{\omega})$ in the Radon transform. Denote the elements of $\mathbb{R}^n$ as $(\mathbf{y}, x_n)$ where $\mathbf{y} \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We would like a parametrization around the hyperplane $x_n = c$, that is to say, around the standard parameter $(t,\boldsymbol{\omega}) = (c, (\mathbf{0}, 1))$. The hyperplanes near $x_n = c$ can be described as $x_n = \mathbf{a}\cdot\mathbf{y} + b$, for b close to c and for $\mathbf{a} \in \mathbb{R}^{n-1}$ in a neighborhood of the origin of $\mathbb{R}^{n-1}$. What this means is that we may take $(\mathbf{a}, b) \in \mathbb{R}^{n-1}\times\mathbb{R}$ as the parameters to describe the Radon transform of a given function. Let us denote as $G_f(\mathbf{a}, b)$ the following reparametrization of the Radon transform of f, namely

$$G_f(\mathbf{a}, b) = c_{\mathbf{a}}\, R(f)(c_{\mathbf{a}}(-\mathbf{a}, 1), c_{\mathbf{a}} b), \tag{3.8}$$

where

$$c_{\mathbf{a}} = \frac{1}{\sqrt{1 + |\mathbf{a}|^2}}\,. \tag{3.9}$$

If $f \in \mathcal{S}(\mathbb{R}^n)$, then

$$G_f(\mathbf{a}, b) = \langle f(\mathbf{y}, \mathbf{a}\cdot\mathbf{y} + b), 1\rangle_{\mathbf{y}}\,, \tag{3.10}$$

for $\mathbf{a} \in \mathbb{R}^{n-1}$ and $b \in \mathbb{R}$. If $f \in \mathcal{K}'(\mathbb{R}^n)$ then the Radon transform $R(f)$ is a distribution, and thus it need not have values at every point; hence for a fixed $(\mathbf{a}, b)$ the *pointwise* evaluation (3.10) may or may not make sense, but it is well defined *distributionally* with respect to $(\mathbf{a}, b)$. Let us also observe the following consequence of Lemma 3.2.

Lemma 3.3. *Let $f \in \mathcal{K}'(\mathbb{R}^n)$ and $\xi \in \mathcal{D}(\mathbb{R}^{n-1})$, then the evaluation $\langle G_f(\mathbf{a}, b), \xi(\mathbf{a})\rangle_{\mathbf{a}}$ belongs to $\mathcal{K}'(0,\infty)$.*

The use of the parameter $(\mathbf{a}, b)$ allow us to write some transformation formulas for the Radon transform in a clear fashion.

Lemma 3.4. *Let $f \in \mathcal{K}'(\mathbb{R}^n)$. Let $g_j(\mathbf{x}) = x_j f(\mathbf{x})$. If $j < n$, then*

$$\frac{\partial G_{g_j}}{\partial b} = \frac{\partial G_f}{\partial a_j}\,, \tag{3.11}$$

[6]It is possible to define this distribution at 0, following the ideas of [5,6], but we shall not need such an extension presently.

while

$$\frac{\partial G_{g_n}}{\partial b} = \sum_{j=1}^{n-1} a_j \frac{\partial G_f}{\partial a_j} + G_f + b\,\frac{\partial G_f}{\partial b}\,. \tag{3.12}$$

Proof. Suppose first that $f \in \mathcal{S}(\mathbb{R}^n)$. If $j < n$, then $g_j(\mathbf{y}, x_n) = y_j f(\mathbf{y}, x_n)$; thus,

$$\frac{\partial}{\partial b}\Big(\langle y_j f(\mathbf{y}, \mathbf{a}\cdot\mathbf{y} + b), 1\rangle_{\mathbf{y}}\Big) = \left\langle y_j \frac{\partial f}{\partial x_n}(\mathbf{y}, \mathbf{a}\cdot\mathbf{y} + b), 1\right\rangle_{\mathbf{y}} = \frac{\partial G_f(\mathbf{a}, b)}{\partial a_j}\,, \tag{3.13}$$

and since $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{K}'(\mathbb{R}^n)$, this formula is still valid in $\mathcal{K}'(\mathbb{R}^n)$. The formula for $j = n$ follows by a simple computation since $G_{g_n}(\mathbf{a}, b) = \sum_{j=1}^{n-1} a_j\, G_{g_j}(\mathbf{a}, b) + b\, G_f(\mathbf{a}, b)$. □

4. Several Lemmas

Let us start with a rather simple result.

Lemma 4.1. *Let $\mathcal{C}$ be a subset of $\mathcal{D}'(\mathbb{R}^n)$ that satisfies:*

(a) *if $f \in \mathcal{C}$ then $\int_{|\boldsymbol{\theta}|=r} f(\boldsymbol{\theta})\,\mathrm{d}\sigma(\boldsymbol{\theta}) = 0$, for $r > a$, where $\mathrm{d}\sigma$ is the surface measure on the sphere;*

(b) *if $f \in \mathcal{C}$ and p is a polynomial in n variables, then $pf \in \mathcal{C}$.*

Then

$$f(\mathbf{x}) = 0 \quad \textit{for} \quad |\mathbf{x}| > a, \quad \textit{whenever} \quad f \in \mathcal{C}. \tag{4.1}$$

It is convenient to observe, before we give the proof, that since f is a *distribution*, then the spherical mean $F(r) = \int_{|\boldsymbol{\theta}|=r} f(\boldsymbol{\theta})\,\mathrm{d}\sigma(\boldsymbol{\theta})$ is likewise a *distribution* of one variable, given in the interval $(0, \infty)$ as

$$\langle F(r), \phi(r)\rangle_r = \big\langle f(\mathbf{x}), |\mathbf{x}|^{n-1}\phi(|\mathbf{x}|)\big\rangle_{\mathbf{x}}\,, \tag{4.2}$$

for $\phi \in \mathcal{D}((0, \infty))$. One can consider F in intervals that contain the origin [5, 6], but we shall not need to do so presently.

Proof. Let $\phi \in \mathcal{D}((a, \infty))$ be a test function with compact support contained in (a, ∞). If $f \in \mathcal{C}$ then the distribution $f_\phi \in \mathcal{D}'(\mathbb{R}^n)$ given as $f_\phi(\mathbf{x}) = \phi(|\mathbf{x}|)f(\mathbf{x})$ has *compact support*. Our assumptions imply that $\langle f_\phi, p\rangle = 0$ for all polynomials of n variables p, and consequently[7] $f_\phi = 0$; since ϕ is arbitrary, we obtain that $f = 0$ in the open set $|\mathbf{x}| > a$. □

We shall now prove that if $\mathcal{C}$ is the class of distributions of $\mathcal{K}'(\mathbb{R}^n)$ whose Radon transform vanishes on hyperplanes contained in the exterior region $|\mathbf{x}| > a$, then it satisfies conditions (a) and (b) of Lemma 4.1, thus obtaining a support theorem for Radon transforms in the class of oscillatory distributions $\mathcal{K}'(\mathbb{R}^n)$.

[7] There are non-zero distributions of $\mathcal{D}'(\mathbb{R}^n)$, whose support is not compact, of course, that vanish when evaluated at any polynomial: just take the Fourier transform of an element of $\mathcal{D}(\mathbb{R}^n)$ whose support does not contain the origin.

Lemma 4.2. *Let $f \in \mathcal{K}'(\mathbb{R}^n)$ such that the Radon transform of f vanishes on all hyperplanes contained in the open set $|\mathbf{x}| > a$. Then $\int_{|\boldsymbol{\theta}|=r} f(\boldsymbol{\theta})\,\mathrm{d}\sigma(\boldsymbol{\theta}) = 0$, for $r > a$.*

Proof. Indeed, if F and G are the spherical means of f and of its Radon transform, then [17, 18] $G = C_n\, I_{-,2}^{(n-1)/2}(F)$, where C_n is a constant and where $I_{-,2}^{\alpha}(F)$ is the distributional extension of the Erdélyi–Köber fractional integral:

$$I_{-,2}^{\alpha}\{F(s);t\} = \frac{2}{\Gamma(\alpha)} \int_t^{\infty} \frac{F(s)\, s\, \mathrm{d}s}{(s^2 - t^2)^{1-\alpha}}\,, \tag{4.3}$$

so that, in this case, $I_{-,2}^{(n-1)/2}(F) = 0$ for $r > a$. This yields $F(r) = 0$ for $r > a$, since $F \in \mathcal{K}'(\mathbb{R})$.[8] □

We shall now prove condition (b) of Lemma 4.1 for the class of distributions of $\mathcal{K}'(\mathbb{R}^n)$ whose Radon transforms vanish on hyperplanes contained in the exterior region $|\mathbf{x}| > a$. We employ a stronger version of the argument[9] given by Strichartz [19] and also used by [2].

Lemma 4.3. *Let $f \in \mathcal{K}'(\mathbb{R}^n)$. Let K be a compact set in $\mathbb{R}^n$ and suppose that the Radon transform of f vanishes on all hyperplanes that do not meet K. Then if p is a polynomial in n variables, the Radon transform of fp also vanishes on all hyperplanes that do not meet K.*

Proof. Denote the elements of $\mathbb{R}^n$ as $(\mathbf{y}, x_n)$ where $\mathbf{y} \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. We may suppose that the compact set K is contained in the half-space $x_n < 0$, and we shall prove that the Radon transform of fp vanishes in a neighborhood of the hyperplane $x_n = 0$, namely, that if G_{fp} is the reparametrization (3.8), then $G_{fp}(\mathbf{a}, b) = 0$ in a neighborhood of $(\mathbf{a}, b) = (\mathbf{0}, 0)$. We shall actually show that the Radon transform of fp vanishes at the hyperplane $x_n = \mathbf{a} \cdot \mathbf{y} + b$ in a set of the form $U = \{\,(\mathbf{a}, b) \in \mathbb{R}^{n-1} \times \mathbb{R} : |\mathbf{a}| < \varepsilon,\ b > -\varepsilon\,\}$ for some $\varepsilon > 0$.

Notice now that our hypotheses imply that there exists $\varepsilon > 0$ such that $G_f(\mathbf{a}, b) = 0$ if $|\mathbf{a}| < \varepsilon$ and $b > -\varepsilon$. Use of Lemma 3.4 gives that if $g_j(\mathbf{x}) = x_j f(\mathbf{x})$, then $\partial G_{g_j}/\partial b = 0$ in U for any j, $1 \le j \le n$. Let $\xi \in \mathcal{D}(\mathbb{R}^{n-1})$ with $\operatorname{supp}\xi \subset \{\,\mathbf{a} \in \mathbb{R}^{n-1} : |\mathbf{a}| < \varepsilon\,\}$. Then $(d/db)\langle G_{g_j}, \xi\rangle_{\mathbf{a}} = 0$ for $b > -\varepsilon$, so that $\langle G_{g_j}, \xi\rangle_{\mathbf{a}}$ is constant for $b > -\varepsilon$; but Lemma 3.3 shows that $\langle G_{g_j}, \xi\rangle_{\mathbf{a}}$ belongs to $\mathcal{K}'(\mathbb{R})$ for $b > 0$, and hence it follows that $\langle G_{g_j}, \xi\rangle_{\mathbf{a}} = 0$ for $b > -\varepsilon$. Since ξ is an arbitrary test function with $\operatorname{supp}\xi \subset \{\,\mathbf{a} \in \mathbb{R}^{n-1} : |\mathbf{a}| < \varepsilon\,\}$, we conclude that $G_{g_j}(\mathbf{a}, b) = 0$ in U. Applying an inductive argument we therefore obtain that $G_{fp}(\mathbf{a}, b) = 0$ in U for all polynomials p. □

[8] This is easily proved by taking Mellin transforms, as explained in [9], and employing the fact [3], [8, Sect. 6.11] that the Mellin transform of an element of $\mathcal{K}'(\mathbb{R})$ is analytic in a right half plane.

[9] The basic argument of [19] gives support theorems for the Radon transform of functions of *compact support* only.

5. The Support Theorem

The lemmas of the previous section immediately yield the following support theorem for oscillatory distributions.

Theorem 5.1. *Let $f \in \mathcal{K}'(\mathbb{R}^n)$. Let K be a compact convex set in $\mathbb{R}^n$ and suppose that the Radon transform of f vanishes on all hyperplanes that do not meet K. Then f vanishes in $\mathbb{R}^n \setminus K$.* □

Naturally, since any continuous rapidly decreasing function f defined in $\mathbb{R}^n$ belongs to $\mathcal{K}'(\mathbb{R}^n)$, we obtain the standard support theorem for rapidly decreasing functions[10] [11, 13]. However, Theorem 5.1 applies to the more general situation of functions, and distributions, that "decrease rapidly at infinity" in the parametric or in the Cesàro sense [4, 7, 8], and, in particular, when there is rapid decrease at infinity in the H-B-L sense [2, 12].

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[10] The proof of Lemma 4.3 for continuous rapidly decreasing functions is rather easy, and thus our method gives a very simple alternative proof of the support theorem of Helgason.

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