

ON THE CHARACTERIZATION OF A CLASS OF FOUR DIMENSIONAL MATRICES AND STEINHAUS TYPE THEOREMS

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ABSTRACT. In the present paper, we give the definitions of four dimensional conull and coregular matrices, and characterize the classes $(\mathcal{M}_u : \mathcal{C}_\vartheta)$, where $\vartheta \in \{p, p0, f\}$ and \mathcal{C}_f denotes the space of all almost convergent double sequences. Additionally, we give two Steinhaus type theorems related to the coercive/almost coercive four dimensional sequence to sequence matrix transformations.

1. INTRODUCTION

Referring Eizen and Laush [1] and Başar [2], we characterize the classes $(\mathcal{M}_u : \mathcal{C}_\vartheta)$, where $\vartheta \in \{p, p0, f\}$. We quote some lemmas and also, we define the concept of conullity and coregularity of a four dimensional matrix. Additionally, as a new development of Steinhaus type theorems we prove that there is no matrix belonging to both of the classes $(\mathcal{M}_u : \mathcal{C}_\vartheta)$ and $(\mathcal{C}_\vartheta : \mathcal{C}_\vartheta; P)$ with $\vartheta \in \{bp, f\}$ of four dimensional matrices.

2. DEFINITIONS, NOTATIONS AND BACKGROUND

We denote the set of all complex valued double sequences by Ω which is a vector space with coordinate-wise addition and scalar multiplication. Any vector subspace of Ω is called as a *double sequence space*. A double sequence $x = (x_{mn})$ of complex numbers is said to be *bounded* if $\|x\|_\infty = \sup_{m,n \in \mathbb{N}} |x_{mn}| < \infty$, where $\mathbb{N} = \{0, 1, 2, \dots\}$. It is known that the space \mathcal{M}_u of all bounded double sequences is a Banach space with the norm $\|\cdot\|_\infty$. If there is an $l \in \mathbb{C}$ such that for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N} \ni |x_{mn} - l| < \varepsilon$ for all $m, n > n_0$, then we call that the double

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sequence $x = (x_{mn}) \in \Omega$ is *convergent* in the *Pringsheim's sense* to the limit l and write $p - \lim_{m,n \rightarrow \infty} x_{mn} = l$, [3]; where \mathbb{C} denotes the complex field. By \mathcal{C}_p , we denote the space of all convergent double sequences in the Pringsheim's sense. It is well-known that there are such sequences in the space \mathcal{C}_p but not in the space \mathcal{M}_u . Indeed, if we define the sequence $x = (x_{kl})$ by

$$x_{kl} := \begin{cases} k, & k \in \mathbb{N}, l = 0, \\ l, & l \in \mathbb{N}, k = 0, \\ 0, & k, l \in \mathbb{N} \setminus \{0\}, \end{cases}$$

for all $k, l \in \mathbb{N}$, then it is trivial that $x \in \mathcal{C}_p \setminus \mathcal{M}_u$, since $p - \lim_{k,l \rightarrow \infty} x_{kl} = 0$ but $\|x\|_\infty = \infty$. So, we can consider the space \mathcal{C}_{bp} of the double sequences which are both convergent in the Pringsheim's sense and bounded, i.e., $\mathcal{C}_{bp} = \mathcal{C}_p \cap \mathcal{M}_u$. A sequence in the space \mathcal{C}_p is said to be *regularly convergent* if it is a single convergent sequence with respect to each index and denote the space of all such sequences by \mathcal{C}_r . By \mathcal{C}_{p0} , we denote the space of all double sequences converging to 0 in the *Pringsheim's sense*. Also by \mathcal{C}_{bp0} and \mathcal{C}_{r0} , we denote the spaces of all double null sequences contained in the sequence spaces \mathcal{C}_{bp} and \mathcal{C}_r , respectively. Móricz [4] proved that \mathcal{C}_{bp} , \mathcal{C}_{bp0} , \mathcal{C}_r and \mathcal{C}_{r0} are Banach spaces with the norm $\|\cdot\|_\infty$.

The concept of almost convergence for single sequences was introduced by Lorentz [6] and for double sequences by Móricz and Rhoades [5]. A double sequence $x = (x_{mn})$ of complex numbers is said to be *almost convergent* to a generalized limit L if

$$p - \lim_{q,r \rightarrow \infty} \left| \frac{1}{(q+1)(r+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+r} x_{kl} - L \right| = 0 \text{ uniformly in } m, n.$$

In this case, L is called the f_2 -limit of x . We denote the space of all almost convergent double sequences by \mathcal{C}_f .

If we define the sequence $x = (x_{kl})$ with respect to the natural numbers q and r by

$$x_{kl} := \begin{cases} kl, & 0 \leq k \leq q, 0 \leq l \leq r, \\ 0, & \text{otherwise,} \end{cases}$$

for all $k, l \in \mathbb{N}$, one can easily see that $x \in \mathcal{C}_p$. But, since the limit

$$p - \lim_{q,r \rightarrow \infty} \frac{1}{(q+1)(r+1)} \sum_{k=m}^{m+q} \sum_{l=n}^{n+r} x_{kl} = p - \lim_{q,r \rightarrow \infty} \left(m + \frac{q}{2} \right) \left(n + \frac{r}{2} \right)$$

does not exist for each m and n , $x \notin \mathcal{C}_f$. So, we note that a convergent double sequence in the Pringsheim's sense need not be almost convergent. However every bounded convergent double sequence is almost convergent and every almost convergent double sequence is bounded. Four dimensional matrices mapping every almost convergent double sequence into a bp -convergent double sequence with the same limit are said to be strongly regular and considered by Móricz and Rhoades [5].

Let λ be a space of double sequences converging with respect to some linear convergence rule $\vartheta - \lim : \lambda \rightarrow \mathbb{C}$. The sum of a double series $\sum_{k,l} x_{kl}$ with respect to

this rule is defined by $\vartheta - \sum_{k,l} x_{kl} = \vartheta - \lim_{m,n \rightarrow \infty} \sum_{k,l=0}^{m,n} x_{kl}$. For short, throughout the text the summations without limits run from 0 to ∞ , for example $\sum_{k,l} x_{kl}$ means that $\sum_{k,l=0}^{\infty} x_{kl}$.

The $\beta(\vartheta)$ -dual $\lambda^{\beta(\vartheta)}$ with respect to the ϑ -convergence of a double sequence space λ is defined by

$$\lambda^{\beta(\vartheta)} := \left\{ (a_{kl}) \in \Omega : \vartheta - \sum_{k,l} a_{kl} x_{kl} \text{ exists for all } (x_{kl}) \in \lambda \right\}.$$

Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, a double sequence $x = (x_{kl})$ is said to be ϑ -summable with respect to A if and only if

$$(2.1) \quad (Ax)_{mn} = \vartheta - \sum_{k,l} a_{mnkl} x_{kl}$$

exists for each $m, n \in \mathbb{N}$. We define the ϑ -summability domain $\lambda_A^{(\vartheta)}$ of A in a double sequence space λ by

$$\lambda_A^{(\vartheta)} = \left\{ x = (x_{kl}) \in \Omega : Ax = \left(\vartheta - \sum_{k,l} a_{mnkl} x_{kl} \right)_{m,n \in \mathbb{N}} \text{ exists and is in } \lambda \right\}.$$

Here and after, unless stated otherwise we assume that ϑ denotes any of the symbols p, bp or r . Let λ and μ be two spaces of double sequences, and A be a four dimensional matrix. Then, we say with the notation (2.1) that A maps the space λ into the space μ if $\lambda \subset \mu_A^{(\vartheta)}$ and we denote the set of all four dimensional matrices, transforming the space λ into the space μ , by $(\lambda : \mu)$. Thus, $A = (a_{mnkl}) \in (\lambda : \mu)$ if and only if the double series on the right side of (2.1) converges in the sense of ϑ for each $m, n \in \mathbb{N}$, i.e, $A_{mn} \in \lambda^{\beta(\vartheta)}$ for all $m, n \in \mathbb{N}$ and every $x \in \lambda$, and we have $Ax \in \mu$ for all $x \in \lambda$; where $A_{mn} = (a_{mnkl})_{k,l \in \mathbb{N}}$ for all $m, n \in \mathbb{N}$. Also by $(\lambda : \mu; P)$, we denote the subclass of $(\lambda : \mu)$ of four dimensional matrices which preserve the limit or sum. We say that A is \mathcal{C}_ϑ -conservative if $\mathcal{C}_\vartheta \subset (\mathcal{C}_\vartheta)_A$, and is \mathcal{C}_ϑ -regular if it is \mathcal{C}_ϑ -conservative and $\vartheta - \lim_A x = \vartheta - \lim_{m,n \rightarrow \infty} (Ax)_{mn} = \vartheta - \lim_{m,n \rightarrow \infty} x_{mn}$, where $x = (x_{mn}) \in \mathcal{C}_\vartheta$.

Now, we may give the following two lemmas characterizing the classes $(\mathcal{C}_{bp} : \mathcal{C}_{bp}; P)$ and $(\mathcal{M}_u : \mathcal{C}_{bp})$ of four dimensional matrices.

Lemma 2.1. [7, Theorem 2.2 (b)] *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{C}_{bp} : \mathcal{C}_{bp}; P)$ if and only if the following conditions hold:*

$$(2.2) \quad \begin{aligned} & \sup_{m,n \in \mathbb{N}} \sum_{k,l} |a_{mnkl}| < \infty, \\ & bp - \lim_{m,n \rightarrow \infty} a_{mnkl} = 0 \text{ for all } k, l \in \mathbb{N}, \\ & bp - \lim_{m,n \rightarrow \infty} \sum_{k,l} a_{mnkl} = 1, \\ & \text{there exists } l_0 \in \mathbb{N} \text{ such that } bp - \lim_{m,n \rightarrow \infty} \sum_k |a_{mnkl_0}| = 0, \\ & \text{there exists } k_0 \in \mathbb{N} \text{ such that } bp - \lim_{m,n \rightarrow \infty} \sum_l |a_{mnk_0l}| = 0. \end{aligned}$$

Lemma 2.2. [8, Lemma 3.2] *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{M}_u : \mathcal{C}_{bp})$ if and only if the condition (2.2) holds and*

$$(2.3) \quad \text{there exists } a_{kl} \in \mathbb{C} \text{ such that } bp - \lim_{m,n \rightarrow \infty} a_{mnkl} = a_{kl} \text{ for all } k, l \in \mathbb{N},$$

$$(2.4) \quad \begin{aligned} & bp - \lim_{m,n \rightarrow \infty} \sum_{k,l} |a_{mnkl} - a_{kl}| = 0, \\ & bp - \lim_{m,n \rightarrow \infty} \sum_{l=0}^n a_{mnkl} \text{ exists for each } k \in \mathbb{N}, \\ & bp - \lim_{m,n \rightarrow \infty} \sum_{k=0}^m a_{mnkl} \text{ exists for each } l \in \mathbb{N}, \\ & \sum_{k,l} |a_{mnkl}| \text{ converges.} \end{aligned}$$

Definition 2.1. [9,10] A four dimensional matrix A is said to be *RH-regular* if it maps every bounded p -convergent double sequence into a p -convergent double sequence with the same p -limit.

Lemma 2.3. [11, Theorem 3.2] *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{M}_u : \mathcal{M}_u)$ if and only if the condition (2.2) holds.*

Mursaleen [12] introduced the concept of almost strong regularity for double sequences. A four dimensional matrix $A = (a_{mnkl})$ is said to be *almost strongly regular* if for every $x = (x_{kl}) \in \mathcal{C}_f$, $Ax \in \mathcal{C}_f$ with $f_2 - \lim Ax = f_2 - \lim x$, i.e., it transforms every almost convergent double sequence into an almost convergent double sequence with the same limit. In the following lemma Mursaleen characterized the almost strongly regular matrices.

Lemma 2.4. [12, Theorem 2.2] *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{C}_f : \mathcal{C}_f; P)$ if and only if condition (2.2) holds and*

$$(2.5) \quad bp - \lim_{q,r \rightarrow \infty} \alpha(k, l, q, r, m, n) = 0 \text{ uniformly in } m, n \in \mathbb{N} \text{ for each } k, l \in \mathbb{N},$$

$$(2.6) \quad bp - \lim_{q,r \rightarrow \infty} \sum_{k,l} \alpha(k, l, q, r, m, n) = 1 \text{ uniformly in } m, n \in \mathbb{N},$$

$$bp - \lim_{q,r \rightarrow \infty} \sum_k |\alpha(k, l, q, r, m, n)| = 0 \text{ uniformly in } m, n \in \mathbb{N} \text{ for each } l \in \mathbb{N},$$

$$bp - \lim_{q,r \rightarrow \infty} \sum_l |\alpha(k, l, q, r, m, n)| = 0 \text{ uniformly in } m, n \in \mathbb{N} \text{ for each } k \in \mathbb{N},$$

$$bp - \lim_{q,r \rightarrow \infty} \sum_k \sum_l |\Delta_{10} \alpha(k, l, q, r, m, n)| = 0 \text{ uniformly in } m, n \in \mathbb{N},$$

$$bp - \lim_{q,r \rightarrow \infty} \sum_l \sum_k |\Delta_{01} \alpha(k, l, q, r, m, n)| = 0 \text{ uniformly in } m, n \in \mathbb{N},$$

where

$$\alpha(k, l, q, r, m, n) = \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r a_{m+i, n+j, kl},$$

$$\Delta_{10} \alpha(k, l, q, r, m, n) = \alpha(k, l, q, r, m, n) - \alpha(k+1, l, q, r, m, n),$$

$$\Delta_{01} \alpha(k, l, q, r, m, n) = \alpha(k, l, q, r, m, n) - \alpha(k, l+1, q, r, m, n).$$

Definition 2.2. If A is a \mathcal{C}_ϑ -conservative matrix, then we define the characteristic $\chi(A)$ of A by

$$\chi(A) = \vartheta - \lim_{m,n \rightarrow \infty} \sum_{k,l} a_{mnkl} - \sum_{k,l} \vartheta - \lim_{m,n \rightarrow \infty} a_{mnkl}.$$

Definition 2.3. A \mathcal{C}_ϑ -conservative matrix A is called conull if $\chi(A) = 0$ and is called coregular if $\chi(A) \neq 0$.

Remark 2.1. One can easily conclude by Lemma 2.1 and Lemma 2.4 that $\chi(A) = 1$ whenever A is a \mathcal{C}_{bp} -regular sequence to sequence four dimensional matrix or is an almost strongly regular four dimensional matrix. This leads to the fact that the class $(\mathcal{C}_\vartheta : \mathcal{C}_\vartheta; P)$ of four dimensional matrices forms a subclass of the class of co-regular four dimensional matrices, where $\vartheta \in \{bp, f\}$.

Following Zeltser [13], we define the double sequences $\mathbf{e}^{kl} = (e_{mn}^{kl})$, \mathbf{e}^l , \mathbf{e}_k , and \mathbf{e} by

$$(2.7) \quad e_{mn}^{kl} := \begin{cases} 1, & (k, l) = (m, n), \\ 0, & \text{otherwise,} \end{cases}$$

$$\mathbf{e}^l := \sum_k \mathbf{e}^{kl}, \quad \mathbf{e}_k := \sum_l \mathbf{e}^{kl}, \quad \text{and } \mathbf{e} := \sum_{k,l} \mathbf{e}^{kl}$$

for all $k, l, m, n \in \mathbb{N}$.

The original Steinhaus theorem was about the sequence to sequence matrix transformations which can be stated as “*There is no matrix belonging to the classes of both Schur matrices and Toeplitz matrices*”. Later, the corresponding results were given by considering almost convergence instead of ordinary convergence, (cf. [2] and [18]). Moreover, Steinhaus type theorems for series to sequence and series to series matrix transformations were given by Başar and Solak [19].

In this study, our main goal is to give the corresponding Steinhaus type theorems for four dimensional matrix transformations. In order to achieve this, we characterize the classes $(\mathcal{M}_u : \mathcal{C}_\vartheta)$ with $\vartheta \in \{p, p0, f\}$. However, since the characterization of the classes $(\mathcal{BS} : \mathcal{C}_f)$, $(\mathcal{BS} : \mathcal{CS}_\vartheta)$, $(\mathcal{CS}_\vartheta : \mathcal{C}_\vartheta; P)$ and $(\mathcal{CS}_\vartheta : \mathcal{CS}_\vartheta; P)$ with $\vartheta \in \{p, p0, bp, r, f\}$ are still open, we are not able to present Steinhaus type theorems for series to sequence and series to series four dimensional matrix transformations.

3. MAIN RESULTS

In this section, we characterize the classes $(\mathcal{M}_u : \mathcal{C}_\vartheta)$ and we give two Steinhaus type theorems, where $\vartheta \in \{p, p0, f\}$.

Following Maddox [14], we formulate the Steinhaus type theorems for four dimensional matrix transformations, as follows:

Definition 3.1. Let \mathcal{X} , \mathcal{Y} and \mathcal{Z} be given double sequence spaces such that $\mathcal{Z} \supset \mathcal{X}$ and suppose that there is some notion of limit or sum in the spaces \mathcal{X} and \mathcal{Y} . Then, we say the result of the form $(\mathcal{X} : \mathcal{Y}; P) \cap (\mathcal{Z} : \mathcal{Y}) = \emptyset$ as a Steinhaus type theorem, where \emptyset denotes the empty set.

Using convergence in the Pringsheim’s sense, Patterson [15] gave the multidimensional analogue of Steinhaus’s theorem whose original proof can be found in Cooke [16, pp. 75–76], as follows:

Theorem 3.1. [15, Theorem 3.1] *If A is an RH-regular matrix, then there exists a bounded double sequence x consisting only of 0’s and 1’s which is not A -summable.*

Now, we can give our theorem about the multidimensional analogue of Steinhaus’s theorem.

Theorem 3.2. *The classes $(\mathcal{C}_{bp} : \mathcal{C}_{bp}; P)$ and $(\mathcal{M}_u : \mathcal{C}_{bp})$ are disjoint.*

Proof. Suppose that the classes $(\mathcal{C}_{bp} : \mathcal{C}_{bp}; P)$ and $(\mathcal{M}_u : \mathcal{C}_{bp})$ are not disjoint. Then, there exists at least one four dimensional matrix A such that $A \in (\mathcal{C}_{bp} : \mathcal{C}_{bp}; P) \cap (\mathcal{M}_u : \mathcal{C}_{bp})$. Since $A \in (\mathcal{C}_{bp} : \mathcal{C}_{bp}; P)$, $\chi(A) = 1$ by Lemma 2.1. Since $A \in (\mathcal{M}_u : \mathcal{C}_{bp})$, combining the relations (2.3) and (2.4) we have $\chi(A) = 0$, a contradiction. That is to say that the classes $(\mathcal{C}_{bp} : \mathcal{C}_{bp}; P)$ and $(\mathcal{M}_u : \mathcal{C}_{bp})$ of four dimensional matrices are disjoint, as asserted. \square

Theorem 3.3. *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{M}_u : \mathcal{C}_{p0})$ if and only if*

$$(3.1) \quad p - \lim_{m,n \rightarrow \infty} a_{mnkl} = 0 \text{ for each } k, l \in \mathbb{N},$$

$$(3.2) \quad \text{for every } l \in \mathbb{N}, \text{ there exists } K \in \mathbb{N} \text{ such that } a_{mnkl} = 0 \text{ for all } m, n, k > K,$$

$$(3.3) \quad \text{for every } k \in \mathbb{N}, \text{ there exists } L \in \mathbb{N} \text{ such that } a_{mnkl} = 0 \text{ for all } m, n, l > L.$$

Proof. Suppose that the conditions (3.1)–(3.3) hold, and let $x = (x_{kl}) \in \mathcal{M}_u$. Let $n_0 = \max\{L, K\} \in \mathbb{N}$. Hence, for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} |(Ax)_{mn}| &= \left| \sum_{k,l} a_{mnkl} x_{kl} \right| \leq \|x\|_\infty \sum_{k,l} |a_{mnkl}| \\ &= \|x\|_\infty \left[\sum_{k,l=0}^{n_0} |a_{mnkl}| + \sum_{k=0}^{n_0} \sum_{l=n_0+1}^{\infty} |a_{mnkl}| + \sum_{k=n_0+1}^{\infty} \sum_{l=0}^{n_0} |a_{mnkl}| + \sum_{k,l=n_0}^{\infty} |a_{mnkl}| \right] \\ &< \varepsilon \|x\|_\infty (n_0 + 1)^2, \end{aligned}$$

for all $m, n > n_0$, that is, $Ax \in \mathcal{C}_{p0}$.

Conversely, suppose that $A = (a_{mnkl}) \in (\mathcal{M}_u : \mathcal{C}_{p0})$. Then, Ax exists and belongs to \mathcal{C}_{p0} for all $x \in \mathcal{M}_u$. If we take $x = \mathbf{e}^{kl}$ for each $k, l \in \mathbb{N}$, $x = \mathbf{e}^l$, $x = \mathbf{e}_k$ and $x = \mathbf{e}$ defined by (2.7), respectively, then we have

$$0 = p - \lim_{m,n \rightarrow \infty} (A\mathbf{e}^{kl})_{mn} = p - \lim_{m,n \rightarrow \infty} a_{mnkl} \text{ for each } k, l \in \mathbb{N},$$

$$0 = p - \lim_{m,n \rightarrow \infty} (A\mathbf{e}_k)_{mn} = p - \lim_{m,n \rightarrow \infty} \sum_l a_{mnkl} \text{ for each } k \in \mathbb{N},$$

$$(3.4) \quad 0 = p - \lim_{m,n \rightarrow \infty} (A\mathbf{e}^l)_{mn} = p - \lim_{m,n \rightarrow \infty} \sum_k a_{mnkl} \text{ for each } l \in \mathbb{N},$$

$$(3.5) \quad 0 = p - \lim_{m,n \rightarrow \infty} (A\mathbf{e})_{mn} = p - \lim_{m,n \rightarrow \infty} \sum_{k,l} a_{mnkl}.$$

In order to the relation (3.4) to be valid, there must exist a $K \in \mathbb{N}$ such that $a_{mnkl} = 0$ for all $m, n, k > K$, where K may depend on l . Otherwise there exist non-decreasing unbounded three sequences of positive integers (m_j) , (n_j) and (k_j) for $j \in \mathbb{N}$ such that $a_{m_j n_j k_j l} \neq 0$ for each $l \in \mathbb{N}$. If we take $x = \mathbf{e}^l$ and

$$|a_{m_j n_j k_j l}| > \left| \sum_{i=0}^{j-1} a_{m_j n_j k_i l} \right| + K,$$

then we have

$$\left| \sum_{i=0}^j a_{m_j n_j k_i l} \right| \geq |a_{m_j n_j k_j l}| - \left| \sum_{i=0}^{j-1} a_{m_j n_j k_i l} \right| > K,$$

that is, $\left| \sum_{i=0}^j a_{m_j n_j k_i l} \right| \rightarrow \infty$, as $j \rightarrow \infty$. Using the relation

$$(Ae^l)_{m_j n_j} = \sum_k a_{m_j n_j k l} = \lim_{r \rightarrow \infty} \sum_{k=0}^r a_{m_j n_j k l},$$

we can say that $(Ax)_{mn} \not\rightarrow 0$, as $m, n \rightarrow \infty$, a contradiction. That is to say that the condition (3.2) is necessary. Similarly, we obtain the necessity of the condition (3.3). Also, one can see by taking $x = \mathbf{e}$ that the relation (3.5) is the consequence of (3.2) and (3.3). This step completes the proof. \square

Theorem 3.4. *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{M}_u : \mathcal{C}_p)$ if and only if the conditions (3.2)–(3.3) hold, and*

$$(3.6) \quad \text{there exists } \alpha_{kl} \in \mathbb{C} \text{ such that } p - \lim_{m, n \rightarrow \infty} a_{mnkl} = \alpha_{kl}.$$

Proof. Suppose that $A = (a_{mnkl}) \in (\mathcal{M}_u : \mathcal{C}_p)$. Then, Ax exists and belongs to \mathcal{C}_p for any $x \in \mathcal{M}_u$. We write

$$(3.7) \quad \begin{aligned} (Ax)_{m+r, n+s} - (Ax)_{mn} &= \sum_{k, l} a_{m+r, n+s, kl} x_{kl} - \sum_{k, l} a_{mnkl} x_{kl} \\ &= \sum_{k, l} (a_{m+r, n+s, kl} - a_{mnkl}) x_{kl} \\ &= (Bx)_{mn}, \end{aligned}$$

for all $m, n \in \mathbb{N}$, where the matrix $B = (b_{mnkl}^{rs})$ is defined by $b_{mnkl}^{rs} := a_{m+r, n+s, kl} - a_{mnkl}$ for all $m, n, r, s, k, l \in \mathbb{N}$. By letting p -limit in the relation (3.7) as $m, n \rightarrow \infty$, we obtain that $p - \lim_{m, n \rightarrow \infty} (Bx)_{mn} = 0$. Hence, $B \in (\mathcal{M}_u : \mathcal{C}_{p0})$. From the condition (3.1), $p - \lim_{m, n \rightarrow \infty} a_{mnkl}$ exists for each $k, l \in \mathbb{N}$, say α_{kl} which is in \mathbb{C} . So, the condition (3.6) is necessary. Following the similar way used in the proof of Theorem 3.3, one can establish the necessity of the conditions (3.2) and (3.3).

Conversely, suppose that the conditions (3.2), (3.3) and (3.6) hold, and let $x = (x_{kl}) \in \mathcal{M}_u$. Let $n_0 = \max\{L, K\} \in \mathbb{N}$. Hence, for every $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon) \in \mathbb{N}$ such that

$$\begin{aligned} \left| \sum_{k, l} a_{mnkl} x_{kl} - \sum_{k, l=0}^{n_0} \alpha_{kl} x_{kl} \right| &= \left| \sum_{k, l=0}^{n_0} a_{mnkl} x_{kl} - \sum_{k, l=0}^{n_0} \alpha_{kl} x_{kl} \right| \\ &= \left| \sum_{k, l=0}^{n_0} (a_{mnkl} - \alpha_{kl}) x_{kl} \right| \\ &\leq \|x\|_\infty \sum_{k, l=0}^{n_0} |a_{mnkl} - \alpha_{kl}| \\ &< \varepsilon \|x\|_\infty (n_0 + 1)^2, \end{aligned}$$

for all $m, n > n_0$, that is, $p - \lim_{m, n \rightarrow \infty} (Ax)_{mn} = \sum_{k, l=0}^{n_0} \alpha_{kl} x_{kl}$, as desired. \square

Theorem 3.5. *Let $A = (a_{mnkl})$ be any four dimensional matrix. Then, $A \in (\mathcal{M}_u : \mathcal{C}_f)$ if and only if the condition (2.2) holds and there exists $a_{kl} \in \mathbb{C}$ such that*

$$(3.8) \quad f_2 - \lim_{m,n \rightarrow \infty} a_{mnkl} = a_{kl} \text{ for all } k, l \in \mathbb{N},$$

$$(3.9) \quad \text{for every } m, n, l \in \mathbb{N},$$

$$(3.10) \quad \text{there exists } K \in \mathbb{N} \ni \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r a_{m+i, n+j, kl} = 0 \text{ for all } q, r, k > K,$$

$$\text{for every } m, n, k \in \mathbb{N},$$

$$\text{there exists } L \in \mathbb{N} \ni \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r a_{m+i, n+j, kl} = 0 \text{ for all } q, r, l > L.$$

Proof. Suppose that $A = (a_{mnkl}) \in (\mathcal{M}_u : \mathcal{C}_f)$. Then, Ax exists and belongs to \mathcal{C}_f for all $x \in \mathcal{M}_u$. Also, it is known by [17] that the inclusion $\mathcal{C}_f \subset \mathcal{M}_u$ holds. So, the inclusion $(\mathcal{M}_u : \mathcal{C}_f) \subset (\mathcal{M}_u : \mathcal{M}_u)$ holds which gives the necessity of the condition (2.2) by Lemma 2.3. Since Ax belongs to \mathcal{C}_f for all $x \in \mathcal{M}_u$, if we take $x = \mathbf{e}^{kl}$ for each $k, l \in \mathbb{N}$, then we obtain the necessity of the conditions (3.8). If we consider the sequences $x = \mathbf{e}^l$, $x = \mathbf{e}_k$ and $x = \mathbf{e}$, then we derive the conditions

$$(3.11) \quad f_2 - \lim_{m,n \rightarrow \infty} \sum_l a_{mnkl} \text{ exists for each } k \in \mathbb{N},$$

$$(3.12) \quad f_2 - \lim_{m,n \rightarrow \infty} \sum_k a_{mnkl} \text{ exists for each } l \in \mathbb{N},$$

$$(3.13) \quad f_2 - \lim_{m,n \rightarrow \infty} \sum_{k,l} a_{mnkl} \text{ exists.}$$

Define the double sequence (B_{mn}) of four dimensional matrices for all $m, n, q, r, k, l \in \mathbb{N}$, as follows:

$$(3.14) \quad b_{qrkl}^{mn} := \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r a_{m+i, n+j, kl}.$$

Therefore, we obtain for $x = (x_{kl}) \in \mathcal{M}_u$ that

$$(3.15) \quad \begin{aligned} (B_{mn}x)_{qr} &= \sum_{k,l} b_{qrkl}^{mn} x_{kl} \\ &= \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r \sum_{k,l} a_{m+i, n+j, kl} x_{kl} \\ &= \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r (Ax)_{m+i, n+j}. \end{aligned}$$

Then, since Ax belongs to \mathcal{C}_f for all $x \in \mathcal{M}_u$, $f_2 - \lim Ax$ exists, that is,

$$(3.16) \quad p - \lim_{q,r \rightarrow \infty} \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r (Ax)_{m+i, n+j}$$

exists, uniformly in m and n . From the relations (3.15) and (3.16), we conclude that $B_{mn}x$ exists for all $x \in \mathcal{M}_u$, that is, $B_{mn} \in (\mathcal{M}_u : \mathcal{C}_p)$. So, by Theorem 3.4 we have

$$(3.17) \quad \text{there exists } a_{kl} \in \mathbb{C} \text{ such that } p - \lim_{q,r \rightarrow \infty} b_{qrkl}^{mn} = a_{kl} \text{ for all } m, n, k, l \in \mathbb{N},$$

$$(3.18) \quad \text{for every } m, n, l \in \mathbb{N}, \text{ there exists } K \in \mathbb{N} \text{ such that } b_{qrkl}^{mn} = 0 \text{ for all } q, r, k > K,$$

$$(3.19) \quad \text{for every } m, n, k \in \mathbb{N}, \text{ there exists } L \in \mathbb{N} \text{ such that } b_{qrkl}^{mn} = 0 \text{ for all } q, r, l > L.$$

The relation (3.17) is equivalent to the condition (3.8). Also, the conditions (3.8) and (3.18) include the condition (3.11). Similarly, the conditions (3.8) and (3.19) include the condition (3.12). Finally, the conditions (3.8), (3.18) and (3.19) include the condition (3.13).

The sufficiency of the theorem can be proved by the similar way used in the proof of Theorem 3.3. To avoid the repetition of the similar statements, we omit the detail. \square

Theorem 3.6. *The classes $(\mathcal{M}_u : \mathcal{C}_f)$ and $(\mathcal{C}_f : \mathcal{C}_f; P)$ are disjoint.*

Proof. Suppose that $(\mathcal{M}_u : \mathcal{C}_f) \cap (\mathcal{C}_f : \mathcal{C}_f; P) \neq \emptyset$. Then, there exists at least one four dimensional matrix A such that $A \in (\mathcal{M}_u : \mathcal{C}_f) \cap (\mathcal{C}_f : \mathcal{C}_f; P)$. Since $A = (a_{mnkl}) \in (\mathcal{C}_f : \mathcal{C}_f; P)$, the condition (2.5) holds, i.e.

$$bp - \lim_{q,r \rightarrow \infty} \frac{1}{(q+1)(r+1)} \sum_{i=0}^q \sum_{j=0}^r a_{m+i, n+j, kl} = 0 \text{ uniformly in } m, n \in \mathbb{N}.$$

Hence, $bp - \lim_{q,r \rightarrow \infty} b_{qrkl}^{mn} = 0$, where b_{qrkl}^{mn} is given by (3.14). Let $n_0 = \max\{K, L\} \in \mathbb{N}$. Since $A \in (\mathcal{M}_u : \mathcal{C}_f)$, the conditions (3.9) and (3.10) hold. So, for every $m, n \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$ such that $b_{qrkl}^{mn} = 0$ for all $q, r, k, l > n_0$. Hence,

$$bp - \lim_{q,r \rightarrow \infty} \sum_{k,l} b_{qrkl}^{mn} = bp - \lim_{q,r \rightarrow \infty} \sum_{k,l=0}^{n_0} b_{qrkl}^{mn} = 0$$

uniformly in $m, n \in \mathbb{N}$ which contradicts the condition (2.6). \square

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