EXPLICIT SOLUTION TO MODULAR OPERATOR EQUATION

$TXS^* - SX^*T^* = A$

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Abstract. In this paper, by using some block operator matrix techniques, we find explicit solution of the operator equation $TXS^* - SX^*T^* = A$ in the general setting of the adjointable operators between Hilbert $C^*$-modules. Furthermore, we solve the operator equation $TXS^* - SX^*T^* = A$, when $\text{ran}(T) + \text{ran}(S)$ is closed.

1. Introduction and Preliminaries

The equation $TXS^* - SX^*T^* = A$ was studied by Yuan [8] for finite matrices and Xu et al. [7] generalized the results to Hilbert $C^*$-modules, under the condition that $\text{ran}(S)$ is contained in $\text{ran}(T)$. When $T$ equals an identity matrix or identity operator, this equation reduces to $XS^* - SX^* = A$, which was studied by Braden [1] for finite matrices, and Djordjevic [3] for the Hilbert space operators. In this paper, by using block operator matrix techniques and properties of the Moore-Penrose inverse, we provide a new approach to the study of the equation $TXS^* - SX^*T^* = A$ for adjointable Hilbert module operators than those with closed ranges. Furthermore, we solve the operator equation $TXS^* - SX^*T^* = A$, when $\text{ran}(T) + \text{ran}(S)$ is closed.

Throughout this paper, $A$ is a $C^*$-algebra. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $A$-modules, and $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ be the set of the adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, the range and the null space of $T$ are denoted by $\text{ran}(T)$ and $\ker(T)$ respectively. In case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X}, \mathcal{X})$ which we abbreviate to $\mathcal{L}(\mathcal{X})$, is a $C^*$-algebra. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or $1$ if there is no ambiguity.

Theorem 1.1. [5, Theorem 3.2] Suppose that $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ has closed range. Then

- $\ker(T)$ is orthogonally complemented in $\mathcal{X}$, with complement $\text{ran}(T^*)$;
• \( \text{ran}(T) \) is orthogonally complemented in \( \mathcal{Y} \), with complement \( \ker(T^*) \);
• The map \( T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X}) \) has closed range.

Xu and Sheng [6] showed that a bounded adjointable operator between two Hilbert \( \mathcal{A} \)-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse of \( T \), denoted by \( T^\dagger \), is the unique operator \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) satisfying the following conditions:

\[
TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.
\]

It is well-known that \( T^\dagger \) exists if and only if \( \text{ran}(T) \) is closed, and in this case \( (T^\dagger)^* = (T^*)^\dagger \). Let \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) be a closed range, then \( TT^\dagger \) is the orthogonal projection from \( \mathcal{Y} \) onto \( \text{ran}(T) \) and \( T^\dagger T \) is the orthogonal projection from \( \mathcal{X} \) onto \( \text{ran}(T^*) \). Projection, in the sense that they are self adjoint idempotent operators.

A matrix form of a bounded adjointable operator \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) can be induced by some natural decompositions of Hilbert \( C^* \)-modules. Indeed, if \( \mathcal{M} \) and \( \mathcal{N} \) are closed orthogonally complemented submodules of \( \mathcal{X} \) and \( \mathcal{Y} \), respectively, and \( \mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp \), then \( T \) can be written as the following \( 2 \times 2 \) matrix

\[
T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},
\]

where, \( T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N}), \quad T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}), \quad T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp) \) and \( T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp) \). Note that \( P_\mathcal{M} \) denotes the projection corresponding to \( \mathcal{M} \).

In fact \( T_1 = P_\mathcal{N}TP_\mathcal{M}, T_2 = P_\mathcal{N}T(1-P_\mathcal{M}), T_3 = (1-P_\mathcal{N})TP_\mathcal{M} \) and \( T_4 = (1-P_\mathcal{N})T(1-P_\mathcal{M}) \).

**Lemma 1.1** (see [4, Corollary 1.2]). Suppose that \( T \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \) has closed range. Then \( T \) has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules \( \mathcal{X} = \text{ran}(T^*) \oplus \ker(T) \) and \( \mathcal{Y} = \text{ran}(T) \oplus \ker(T^*) \):

\[
T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix},
\]

where \( T_1 \) is invertible. Moreover

\[
T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \ker(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \ker(T) \end{bmatrix}.
\]

**Remark 1.1.** Let \( \mathcal{X} \) and \( \mathcal{Y} \) be two Hilbert \( \mathcal{A} \)-modules, we use the notation \( \mathcal{X} \oplus \mathcal{Y} \) to denote the direct sum of \( \mathcal{X} \) and \( \mathcal{Y} \), which is also a Hilbert \( \mathcal{A} \)-module whose \( \mathcal{A} \)-valued inner product is given by

\[
\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle,
\]

for \( x_1 \in \mathcal{X} \) and \( y_i \in \mathcal{Y}, \ i = 1, 2 \). To simplify the notation, we use \( x \oplus y \) to denote \( \begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X} \oplus \mathcal{Y} \).
Lemma 1.2 (see [2, Lemma 3.]). Suppose that $X$ is a Hilbert $A$-module and $S,T \in \mathcal{L}(X)$. Then $egin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(X \oplus X)$ has closed range if and only if $\text{ran}(T) + \text{ran}(S)$ is closed, and
\[
\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger = \begin{bmatrix} T^*(TT^* + SS^*)^\dagger & 0 \\ S^*(TT^* + SS^*)^\dagger & 0 \end{bmatrix}.
\]

Lemma 1.3 (see [2, Corollary 4.]). Suppose that $X$ is a Hilbert $A$-module and $S,T \in \mathcal{L}(X)$. Then $egin{bmatrix} T & 0 \\ S & 0 \end{bmatrix} \in \mathcal{L}(X \oplus X)$ has closed range if and only if $\text{ran}(T^*) + \text{ran}(S^*)$ is closed, and
\[
\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}^\dagger = \begin{bmatrix} (T^*T + S^*S)^\dagger T^* & (T^*T + S^*S)^\dagger S^* \\ 0 & 0 \end{bmatrix}.
\]

2. Solutions to $TXS^* - SX^*T^* = A$

In this section, we will study the operator equation $TXS^* - SX^*T^* = A$ in the general context of the Hilbert $C^*$-modules. First, in the following theorem we solve to the operator equation $TXS^* - SX^*T^* = A$, in the case when $S$ and $T$ are invertible operators.

Theorem 2.1. Let $X, Y, Z$ be Hilbert $A$-modules, $S \in \mathcal{L}(X, Y)$ and $T \in \mathcal{L}(Z, Y)$ be invertible operators and $A \in \mathcal{L}(Y)$. Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(X, Z)$ to the operator equation $TXS^* - SX^*T^* = A$;
(b) $A = -A^*$.

If (a) or (b) is satisfied, then any solution to
\[
(2.1) \quad TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(X, Z),
\]
has the form
\[
X = \frac{1}{2} T^{-1} A(S^*)^{-1} + T^{-1} Z(S^*)^{-1},
\]
where $Z \in \mathcal{L}(Y)$ satisfy $Z^* = Z$.

Proof. (a) $\Rightarrow$ (b): Obvious.

(b) $\Rightarrow$ (a): Note that, if $A = -A^*$ then $X = \frac{1}{2} T^{-1} A(S^*)^{-1} + T^{-1} Z(S^*)^{-1}$ is a solution to (2.1). The following sentences state this claim
\[
T\left(\frac{1}{2} T^{-1} A(S^*)^{-1} + T^{-1} Z(S^*)^{-1}\right)S^* - S\left(\frac{1}{2} S^{-1} A^*(T^*)^{-1} + S^{-1} Z^*(T^*)^{-1}\right)T^*
= \frac{1}{2} (TT^{-1} A(S^*)^{-1} S^* - SS^{-1} A^*(T^*)^{-1} T^*) + TT^{-1} Z(S^*)^{-1} S^* - SS^{-1} Z^*(T^*)^{-1} T^*
= A + Z - Z^*
= A.
\]
On the other hand, let \( X \) be any solution to (2.1). Then \( X = T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1} \). We have
\[
X = T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1} \\
= \frac{1}{2}T^{-1}A(S^*)^{-1} + \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1} \\
= \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}\left(\frac{1}{2}A + SX^*T^*\right)(S^*)^{-1}.
\]
Taking \( Z = \frac{1}{2}A + SX^*T^* \), we get \( Z^* = Z \).

**Corollary 2.1.** Suppose that \( y, Z \) are Hilbert \( A \)-modules and \( T \in \mathcal{L}(Z, Y) \) is invertible operator, \( A \in \mathcal{L}(Y) \). Then the following statements are equivalent:
(a) There exists a solution \( X \in \mathcal{L}(Y, Z) \) to the operator equation \( TX - X^*T^* = A \);
(b) \( A = -A^* \).
If (a) or (b) is satisfied, then any solution to
\[
TX - X^*T^* = A, \quad X \in \mathcal{L}(Y, Z),
\]
has the form
\[
X = \frac{1}{2}T^{-1}A + T^{-1}Z,
\]
where \( Z \in \mathcal{L}(Y) \) satisfy \( Z^* = Z \).

In the following theorems we obtain explicit solutions to the operator equation
(2.2)
\[
TXS^* - SX^*T^* = A,
\]
via matrix form and complemented submodules.

**Theorem 2.2.** Suppose \( S \in \mathcal{L}(Z, Y) \) is an invertible operator and \( T \in \mathcal{L}(Z, Y) \) has closed range and \( A \in \mathcal{L}(Y) \). Then the following statements are equivalent:
(a) There exists a solution \( X \in \mathcal{L}(Z) \) to (2.2);
(b) \( A = -A^* \) and \( (1 - TT^\dagger)A(1 - TT^\dagger) = 0 \).
If (a) or (b) is satisfied, then any solution to (2.2) has the form
(2.3)
\[
X = \frac{1}{2}T^\dagger ATT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger A(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1},
\]
where \( Z \in \mathcal{L}(Y) \) satisfies \( T^*(Z - Z^*)T = 0 \), and \( Y \in \mathcal{L}(Y, Z) \) is arbitrary.

**Proof.** (a) \( \Rightarrow \) (b): Obviously, \( A = -A^* \). Also,
\[
(1 - TT^\dagger)A(1 - TT^\dagger) = (1 - TT^\dagger)(TXS^* - SX^*T^*)(1 - TT^\dagger)
= (T - TT^\dagger)XS^*(1 - TT^\dagger) - (1 - TT^\dagger)SX^*(T^* - T^*TT^\dagger)
= 0.
\]
(b) \( \Rightarrow \) (a): Note that the condition \( (1 - TT^\dagger)A(1 - TT^\dagger) = 0 \) is equivalent to \( A = ATT^\dagger + TT^\dagger A - TT^\dagger ATT^\dagger \). On the other hand, since \( T^*(Z - Z^*)T = 0 \), then \( (Z - Z^*)T \in \ker(T^*) = \ker(T^\dagger) \). Therefore \( T^\dagger(Z - Z^*)T = 0 \).
Hence we have
\[
\frac{1}{2}TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger + TT^\dagger A(1 - TT^\dagger) + T(1 - T^\dagger T)Y - \frac{1}{2}TT^\dagger A^*(T^\dagger)^*T^* \\
- TT^\dagger Z^*(T^\dagger)^*T^* - (1 - TT^\dagger)A^*(T^\dagger)^*T^* - Y^*(1 - T^\dagger T)T^* \\
= ATT^\dagger + TT^\dagger A - TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger - TT^\dagger Z^*(T^\dagger)^*T^* \\
= A.
\]

That is, any operator \( X \) of the form (2.3) is a solution to (2.2).

Now, suppose that
\[
X = X_0(S^*)^{-1}, \quad X_0 \in \mathcal{L}(Y, Z),
\]
is a solution to (2.2). We let (2.4) in (2.2). Hence (2.2) get into the following equation
\[
TX_0 - X_0 T^* = A, \quad X_0 \in \mathcal{L}(Y, Z).
\]

Since \( T \) has closed range, we have \( Z = \text{ran}(T^*) \oplus \text{ker}(T) \) and \( Y = \text{ran}(T) \oplus \text{ker}(T^*) \).

Now, \( T \) has the matrix form
\[
T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T^*) \\ \text{ker}(T) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \text{ker}(T^*) \end{bmatrix},
\]
where \( T_1 \) is invertible. On the other hand, \( A = -A^* \) and \((1 - TT^\dagger)A(1 - TT^\dagger) = 0\) imply that \( A \) has the form
\[
A = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \text{ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \text{ker}(T^*) \end{bmatrix},
\]
where \( A_1 = -A_1^* \). Let \( X_0 \) have the form
\[
X_0 = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \text{ker}(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T^*) \\ \text{ker}(T) \end{bmatrix}.
\]

Now by using matrix form for operators \( T, X_0 \) and \( A \), we have
\[
\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} = \begin{bmatrix} X_1^* & X_2^* \\ X_3^* & X_4^* \end{bmatrix} \begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -A_2 & A_1 \\ 0 & 0 \end{bmatrix},
\]
or equivalently
\[
\begin{bmatrix} T_1 X_1 - X_1^* T_1^* \\ -X_2^* T_1^* \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix}.
\]

Therefore
\[
T_1 X_1 - X_1^* T_1^* = A_1,
\]
and
\[
T_1 X_2 = A_2.
\]

Now, we obtain \( X_1 \) and \( X_2 \). By Lemma 1.1, \( T_1 \) is invertible. Hence, Corollary 2.1 implies that
\[
X_1 = \frac{1}{2}T_1^{-1}A_1 + T_1^{-1}Z_1,
\]
where \( Z_1 \in \mathcal{L}(\text{ran}(T)) \) satisfy \( Z_1^* = Z_1 \). Now, multiplying \( T_1^{-1} \) from the left to (2.6), we get
\[
(2.7) \quad X_2 = T_1^{-1}A_2.
\]
Hence
\[
\left[ \frac{1}{2}T_1^{-1}A_1 + T_1^{-1}Z_1 \quad T_1^{-1}A_2 \right]_{X_3 \quad X_4}
\]
is a solution to (2.5), such that \( X_3, X_4 \) are arbitrary operators. Let
\[
Y = \begin{bmatrix} Y_1 & Y_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T^*) \\ \ker(T) \end{bmatrix},
\]
and
\[
Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} : \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix} \rightarrow \begin{bmatrix} \text{ran}(T) \\ \ker(T^*) \end{bmatrix}.
\]
Then
\[
\frac{1}{2}T^\dagger TT^\dagger = \begin{bmatrix} \frac{1}{2}T_1^{-1}A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T^\dagger ZTT^\dagger = \begin{bmatrix} T_1^{-1}Z_1 & 0 \\ 0 & 0 \end{bmatrix}
\]
and
\[
T^\dagger A(1 - TT^\dagger) = \begin{bmatrix} 0 & T^{-1}A_2 \\ 0 & 0 \end{bmatrix}, \quad (1 - T^\dagger T)Y = \begin{bmatrix} 0 & 0 \\ X_3 & X_4 \end{bmatrix}.
\]
Consequently, \( X_3 = \frac{1}{2}T^\dagger ATT^\dagger + T^\dagger ZTT^\dagger + T^\dagger A(1 - TT^\dagger) + (1 - T^\dagger T)Y \), where \( Z \in \mathcal{L}(y) \) satisfies \( T^*(Z - Z^*)T = 0 \), and \( Y \in \mathcal{L}(y, z) \) is arbitrary. \( \square \)

In the following theorem we obtain explicit solution to the operator equation \( TXS^* - SX^*T^* = A \) when \( (1 - P_{\text{ran}(S)})T \) and \( S \) have closed ranges.

**Theorem 2.3.** Suppose that \( \mathcal{X}, \mathcal{Y}, \mathcal{Z} \) are Hilbert \( \mathcal{A} \)-modules, \( S \in \mathcal{L}(\mathcal{X}, \mathcal{Y}) \), \( T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y}) \) and \( A \in \mathcal{L}(\mathcal{Y}) \) and such that \( (1 - SS^\dagger)T \) and \( S \) have closed ranges. If the equation
\[
(2.8) \quad TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}),
\]
is solvable, then
\[
(2.9) \quad \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ 0 & S^* \end{bmatrix}^\dagger
\]

**Proof.** Taking \( H = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} : \mathcal{Z} \oplus \mathcal{X} \rightarrow \mathcal{Y} \oplus \mathcal{Y} \). The operator \( H \) has closed range, since let \( \{z_n \oplus x_n\} \) be sequence chosen in \( \mathcal{Z} \oplus \mathcal{X} \), \( \{z_n\} \), \( \{x_n\} \) be sequences chosen in \( \mathcal{Z} \) and \( \mathcal{X} \), respectively such that \( T(z_n) + S(x_n) \rightarrow y \) for some \( y \in \mathcal{Y} \). Then
\[
(1 - SS^\dagger)T(z_n) = (1 - SS^\dagger)(T(z_n) + S(x_n)) \rightarrow (1 - SS^\dagger)(y).
\]
Since \( \text{ran}((1 - SS^\dagger)T) \) is assumed to be closed, \( (1 - SS^\dagger)(y) = (1 - SS^\dagger)T(z_1) \) for some \( z_1 \in \mathcal{Z} \). It follows that \( y - T(z_1) \in \ker(1 - SS^\dagger) = \text{ran}(S) \), hence \( y = T(z_1) + S(x) \) for some \( x \in \mathcal{X} \). Therefore \( H \) has closed range. Let \( Y = \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} : \mathcal{Z} \oplus \mathcal{X} \rightarrow \mathcal{Z} \oplus \mathcal{X} \).
and $H^* = \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Z} \oplus \mathcal{X}$ and $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{Y}$, therefore (2.8) get into
\begin{equation}
HYH^* = B.
\end{equation}

Since $H$ has closed range and (2.8) has solution, then (2.10) has solution, therefore with multiplying $HH^\dagger$ on the left and multiply $H^\dagger(H^\star)^\dagger$ on the right, we have
\begin{equation}
Y = H^\dagger B(H^\star)^\dagger.
\end{equation}

The proof of the following remark is the same as in the matrix case.

**Remark 2.1.** Let $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. Then the equation
\begin{equation}
TXS = A, \quad X \in \mathcal{L}(\mathcal{Y}),
\end{equation}
has a solution if and only if
\begin{equation}
TT^\dagger AS^\dagger S = A.
\end{equation}

In which case, any solution of (2.11) has the form
\begin{equation}
X = T^\dagger AS^\dagger.
\end{equation}

Now, we solve to the operator equation $TXS^* - SX^*T^* = A$ in the case when $\text{ran}(T) + \text{ran}(S)$ is closed.

**Theorem 2.4.** Suppose that $\mathcal{X}$ is a Hilbert $A$-module, $S, T, A \in \mathcal{L}(\mathcal{X})$ such that $\text{ran}(T) + \text{ran}(S)$ is closed. Then the following statements are equivalent:
(a) There exists a solution $X \in \mathcal{L}(\mathcal{X})$ to the operator equation $TXS^* - SX^*T^* = A$;
(b) $A = -A^\star$ and $P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} = A$.

If (a) or (b) is satisfied, then any solution to
\begin{equation}
TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}),
\end{equation}
has the form
\begin{equation}
X = T^\dagger(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger S.
\end{equation}

**Proof.** (a) $\Rightarrow$ (b) Suppose that (2.12) has a solution $X \in \mathcal{L}(\mathcal{X})$. Then obviously $A = -A^\star$. On the other hand, (2.12) get into
\begin{equation}
\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.
\end{equation}

Since (2.12) is solvable, then (2.14) is solvable. Since $\text{ran}(T) + \text{ran}(S)$ is closed, then Lemma 1.2 implies that
$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger$ exists. Hence, by Remark 2.1 we have
$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$.\]
By applying Lemma 1.2, Corollary 1.3 are shown that

\[
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}^\dagger
\begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T^*(TT^* + SS^*)^\dagger & 0 \\
S^*(TT^* + SS^*)^\dagger & 0
\end{bmatrix}
\begin{bmatrix}
A(TT^* + SS^*)^\dagger T & A(TT^* + SS^*)^\dagger S \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
(TT^* + SS^*)(TT^* + SS^*)^\dagger & A(TT^* + SS^*)(TT^* + SS^*)^\dagger \\
0 & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} & 0 \\
0 & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} & 0 \\
0 & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\]
\]
\]
\]

That is, \(P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} = A\).

(b) \(\Rightarrow\) (a): If \(P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} = A\), then we have

\[
\begin{bmatrix}
P_{\text{ran}(TT^* + SS^*)}AP_{\text{ran}(TT^* + SS^*)} & 0 \\
0 & 0
\end{bmatrix}
\]
\]
\]
\]

or equivalently

\[
= \begin{bmatrix}
(TT^* + SS^*)(TT^* + SS^*)^\dagger & A(TT^* + SS^*)(TT^* + SS^*)^\dagger \\
0 & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}^\dagger
\begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\]
\]
\]
\]

Therefore, Remark 2.1 implies that (2.14) is solvable and hence (2.12) is solvable.

Now, by applying Remark 2.1 and Lemma 1.2, Corollary 1.3 imply that

\[
\begin{bmatrix}
0 & X \\
-X^* & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
T & S \\
0 & 0
\end{bmatrix}^\dagger
\begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
T^* & 0 \\
S^* & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
T^*(TT^* + SS^*)^\dagger & 0 \\
S^*(TT^* + SS^*)^\dagger & 0
\end{bmatrix}
\begin{bmatrix}
A & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
(TT^* + SS^*)^\dagger T & A(TT^* + SS^*)^\dagger S \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
0 & 0
\end{bmatrix}
\]
\]
\]
\]

\[
= \begin{bmatrix}
T^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger & T^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger S \\
S^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger & S^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger S
\end{bmatrix}
\]
\]
\]
\]

.
Therefore

\[ T^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)T = S^*(TT^* + SS^*)^\dagger A(TT^* + SS^*)^\dagger S = 0. \]

Consequently, \( X \) has the form (2.13).

Using exactly similar arguments, we obtain the following analogue of Theorem 2.1, in which to (2.1) is replaced by

\[
(2.15) \quad TXS^* + SX^*T^* = A.
\]

All results of this section can be rewritten for to (2.15), considering the following theorem.

**Theorem 2.5.** Let \( X, Y, Z \) be Hilbert \( A \)-modules, \( S \in \mathcal{L}(X, Y) \) and \( T \in \mathcal{L}(Z, Y) \) be invertible operators and \( A \in \mathcal{L}(Y) \). Then the following statements are equivalent:

(a) There exists a solution \( X \in \mathcal{L}(X, Z) \) to the operator equation \( TXS^* + SX^*T^* = A \).

(b) \( A = A^* \).

If (a) or (b) is satisfied, then any solution to

\[ TXS^* + SX^*T^* = A, \quad X \in \mathcal{L}(X, Z) \]

has the form

\[ X = \frac{1}{2} T^{-1} A(S^*)^{-1} - T^{-1} Z(S^*)^{-1}, \]

where \( Z \in \mathcal{L}(Y) \) satisfy \( Z^* = -Z \).

**References**


Explicit Solution to Modular Operator Equation $T X S^* - S X^* T = A$

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