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EXPLICIT SOLUTION TO MODULAR OPERATOR EQUATION $TXS^* - SX^*T^* = A$

M. MOHAMMADZADEH KARIZAKI[1], M. HASSANI[2], AND S. S. DRAGOMIR[3]

Abstract. In this paper, by using some block operator matrix techniques, we find explicit solution of the operator equation $TXS^* - SX^*T^* = A$ in the general setting of the adjointable operators between Hilbert C^*-modules. Furthermore, we solve the operator equation $TXS^* - SX^*T^* = A$, when $\operatorname{ran}(\mathrm{T}) + \operatorname{ran}(\mathrm{S})$ is closed.

1. Introduction and Preliminaries

The equation $TXS^* - SX^*T^* = A$ was studied by Yuan [8] for finite matrices and Xu et al. [7] generalized the results to Hilbert C^*-modules, under the condition that ran(S) is contained in ran(T). When T equals an identity matrix or identity operator, this equation reduces to $XS^* - SX^* = A$, which was studied by Braden [1] for finite matrices, and Djordjevic [3] for the Hilbert space operators. In this paper, by using block operator matrix techniques and properties of the Moore-Penrose inverse, we provide a new approach to the study of the equation $TXS^* - SX^*T^* = A$ for adjointable Hilbert module operators than those with closed ranges. Furthermore, we solve the operator equation $TXS^* - SX^*T^* = A$, when $\operatorname{ran}(\mathrm{T}) + \operatorname{ran}(\mathrm{S})$ is closed.

Throughout this paper, $\mathcal{A}$ is a C^*-algebra. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules, and $\mathcal{L}(\mathcal{X},\mathcal{Y})$ be the set of the adjointable operators from $\mathcal{X}$ to $\mathcal{Y}$. For any $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$, the range and the null space of T are denoted by ran(T) and $\ker(T)$ respectively. In case $\mathcal{X} = \mathcal{Y}$, $\mathcal{L}(\mathcal{X},\mathcal{X})$ which we abbreviate to $\mathcal{L}(\mathcal{X})$, is a C^*-algebra. The identity operator on $\mathcal{X}$ is denoted by $1_{\mathcal{X}}$ or 1 if there is no ambiguity.

Theorem 1.1. [5, Theorem 3.2] *Suppose that $T \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ has closed range. Then*

- $\ker(T)$ *is orthogonally complemented in $\mathcal{X}$, with complement* $\operatorname{ran}(\mathrm{T}^*)$*;*

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- ran(T) *is orthogonally complemented in* $\mathcal{Y}$*, with complement* $\ker(T^*)$*;*
- *The map* $T^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ *has closed range.*

Xu and Sheng [6] showed that a bounded adjointable operator between two Hilbert $\mathcal{A}$-modules admits a bounded Moore-Penrose inverse if and only if it has closed range. The Moore-Penrose inverse of T, denoted by $T^\dagger$, is the unique operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ satisfying the following conditions:

$$TT^\dagger T = T, \quad T^\dagger TT^\dagger = T^\dagger, \quad (TT^\dagger)^* = TT^\dagger, \quad (T^\dagger T)^* = T^\dagger T.$$

It is well-known that $T^\dagger$ exists if and only if ran(T) is closed, and in this case $(T^\dagger)^* = (T^*)^\dagger$. Let $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a closed range, then $TT^\dagger$ is the orthogonal projection from $\mathcal{Y}$ onto ran(T) and $T^\dagger T$ is the orthogonal projection from $\mathcal{X}$ onto ran(T*). Projection, in the sense that they are self adjoint idempotent operators.

A matrix form of a bounded adjointable operator $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ can be induced by some natural decompositions of Hilbert C^*-modules. Indeed, if $\mathcal{M}$ and $\mathcal{N}$ are closed orthogonally complemented submodules of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and $\mathcal{X} = \mathcal{M} \oplus \mathcal{M}^\perp, \quad \mathcal{Y} = \mathcal{N} \oplus \mathcal{N}^\perp$, then T can be written as the following 2×2 matrix

$$T = \begin{bmatrix} T_1 & T_2 \\ T_3 & T_4 \end{bmatrix},$$

where, $T_1 \in \mathcal{L}(\mathcal{M}, \mathcal{N})$, $T_2 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N})$, $T_3 \in \mathcal{L}(\mathcal{M}, \mathcal{N}^\perp)$ and $T_4 \in \mathcal{L}(\mathcal{M}^\perp, \mathcal{N}^\perp)$. Note that $P_\mathcal{M}$ denotes the projection corresponding to $\mathcal{M}$.

In fact $T_1 = P_\mathcal{N} T P_\mathcal{M}$, $T_2 = P_\mathcal{N} T(1 - P_\mathcal{M})$, $T_3 = (1 - P_\mathcal{N}) T P_\mathcal{M}$ and $T_4 = (1 - P_\mathcal{N}) T (1 - P_\mathcal{M})$.

Lemma 1.1 (see [4, Corollary 1.2.]). *Suppose that* $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ *has closed range. Then* T *has the following matrix decomposition with respect to the orthogonal decompositions of closed submodules* $\mathcal{X} = \mathrm{ran}(T^*) \oplus \ker(T)$ *and* $\mathcal{Y} = \mathrm{ran}(T) \oplus \ker(T^*)$*:*

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix},$$

where T_1 *is invertible. Moreover*

$$T^\dagger = \begin{bmatrix} T_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran}(T) \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran}(T^*) \\ \ker(T) \end{bmatrix}.$$

Remark 1.1. Let $\mathcal{X}$ and $\mathcal{Y}$ be two Hilbert $\mathcal{A}$-modules, we use the notation $\mathcal{X} \oplus \mathcal{Y}$ to denote the direct sum of $\mathcal{X}$ and $\mathcal{Y}$, which is also a Hilbert $\mathcal{A}$-module whose $\mathcal{A}$-valued inner product is given by

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle = \langle x_1, x_2 \rangle + \langle y_1, y_2 \rangle,$$

for $x_i \in \mathcal{X}$ and $y_i \in \mathcal{Y}$, i = 1, 2. To simplify the notation, we use $x \oplus y$ to denote $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{X} \oplus \mathcal{Y}$.

Lemma 1.2 (see [2, Lemma 3.]). *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $S, T \in \mathcal{L}(\mathcal{X})$. Then* $\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{X})$ *has closed range if and only if* $\mathrm{ran(T)} + \mathrm{ran(S)}$ *is closed, and*

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} T^*(TT^* + SS^*)^{\dagger} & 0 \\ S^*(TT^* + SS^*)^{\dagger} & 0 \end{bmatrix}.$$

Lemma 1.3 (see [2, Corollary 4.]). *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module and $S, T \in \mathcal{L}(\mathcal{X})$. Then* $\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{X} \oplus \mathcal{X})$ *has closed range if and only if* $\mathrm{ran(T^*)} + \mathrm{ran(S^*)}$ *is closed, and*

$$\begin{bmatrix} T & 0 \\ S & 0 \end{bmatrix}^{\dagger} = \begin{bmatrix} (T^*T + S^*S)^{\dagger}T^* & (T^*T + S^*S)^{\dagger}S^* \\ 0 & 0 \end{bmatrix}.$$

2. Solutions to $TXS^* - SX^*T^* = A$

In this section, we will study the operator equation $TXS^* - SX^*T^* = A$ in the general context of the Hilbert C^*-modules. First, in the following theorem we solve to the operator equation $TXS^* - SX^*T^* = A$, in the case when S and T are invertible operators.

Theorem 2.1. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:*
(a) *There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* - SX^*T^* = A$;*
(b) *$A = -A^*$.*
If (a) or (b) is satisfied, then any solution to

$$TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}), \tag{2.1}$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfy $Z^* = Z$.

Proof. (a) $\Rightarrow$ (b): Obvious.
$(b) \Rightarrow (a)$: Note that, if $A = -A^*$ then $X = \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}Z(S^*)^{-1}$ is a solution to (2.1). The following sentences state this claim

$$\begin{aligned} &T\left(\frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}Z(S^*)^{-1}\right)S^* - S\left(\frac{1}{2}S^{-1}A^*(T^*)^{-1} + S^{-1}Z^*(T^*)^{-1}\right)T^* \\ =&\frac{1}{2}(TT^{-1}A(S^*)^{-1}S^* - SS^{-1}A^*(T^*)^{-1}T^*) + TT^{-1}Z(S^*)^{-1}S^* - SS^{-1}Z^*(T^*)^{-1}T^* \\ =&A + Z - Z^* \\ =&A. \end{aligned}$$

On the other hand, let X be any solution to (2.1). Then $X = T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1}$. We have

$$\begin{aligned} X &= T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1} \\ &= \frac{1}{2}T^{-1}A(S^*)^{-1} + \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}SX^*T^*(S^*)^{-1} \\ &= \frac{1}{2}T^{-1}A(S^*)^{-1} + T^{-1}(\frac{1}{2}A + SX^*T^*)(S^*)^{-1}. \end{aligned}$$

Taking $Z = \frac{1}{2}A + SX^*T^*$, we get $Z^* = Z$. □

Corollary 2.1. *Suppose that $\mathcal{Y}, \mathcal{Z}$ are Hilbert $\mathcal{A}$-modules and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is invertible operator, $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:*
(a) *There exists a solution $X \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ to the operator equation $TX - X^*T^* = A$;*
(b) *$A = -A^*$.*
If (a) or (b) is satisfied, then any solution to

$$TX - X^*T^* = A, \quad X \in \mathcal{L}(\mathcal{Y}, \mathcal{Z}),$$

has the form

$$X = \frac{1}{2}T^{-1}A + T^{-1}Z,$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfy $Z^* = Z$.

In the following theorems we obtain explicit solutions to the operator equation

$$TXS^* - SX^*T^* = A, \tag{2.2}$$

via matrix form and complemented submodules.

Theorem 2.2. *Suppose $S \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ is an invertible operator and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ has closed range and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:*
(a) *There exists a solution $X \in \mathcal{L}(\mathcal{Z})$ to (2.2);*
(b) *$A = -A^*$ and $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$.*
If (a) or (b) is satisfied, then any solution to (2.2) has the form

$$X = \frac{1}{2}T^\dagger ATT^\dagger(S^*)^{-1} + T^\dagger ZTT^\dagger(S^*)^{-1} + T^\dagger A(1 - TT^\dagger)(S^*)^{-1} + (1 - T^\dagger T)Y(S^*)^{-1}, \tag{2.3}$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^*(Z - Z^*)T = 0$, and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary.

Proof. (a) ⇒ (b): Obviously, $A = -A^*$. Also,

$$\begin{aligned} (1 - TT^\dagger)A(1 - TT^\dagger) &= (1 - TT^\dagger)(TXS^* - SX^*T^*)(1 - TT^\dagger) \\ &= (T - TT^\dagger T)XS^*(1 - TT^\dagger) - (1 - TT^\dagger)SX^*(T^* - T^*TT^\dagger) \\ &= 0. \end{aligned}$$

(b) ⇒ *(a)*: Note that the condition $(1 - TT^\dagger)A(1 - TT^\dagger) = 0$ is equivalent to $A = ATT^\dagger + TT^\dagger A - TT^\dagger ATT^\dagger$. On the other hand, since $T^*(Z - Z^*)T = 0$, then $(Z - Z^*)T \in \ker(T^*) = \ker(T^\dagger)$. Therefore $T^\dagger(Z - Z^*)T = 0$.

Hence we have

$$\begin{aligned}&\frac{1}{2}TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger + TT^\dagger A(1-TT^\dagger) + T(1-T^\dagger T)Y - \frac{1}{2}TT^\dagger A^*(T^\dagger)^*T^*\\ &- TT^\dagger Z^*(T^\dagger)^*T^* - (1-TT^\dagger)A^*(T^\dagger)^*T^* - Y^*(1-T^\dagger T)T^*\\ =&ATT^\dagger + TT^\dagger A - TT^\dagger ATT^\dagger + TT^\dagger ZTT^\dagger - TT^\dagger Z^*(T^\dagger)^*T^*\\ =&A.\end{aligned}$$

That is, any operator X of the form (2.3) is a solution to (2.2).

Now, suppose that

$$X = X_0(S^*)^{-1}, \quad X_0 \in \mathcal{L}(\mathcal{Y},\mathcal{Z}), \tag{2.4}$$

is a solution to (2.2). We let (2.4) in (2.2). Hence (2.2) get into the following equation

$$TX_0 - X_0^*T^* = A, \quad X_0 \in \mathcal{L}(\mathcal{Y},\mathcal{Z}). \tag{2.5}$$

Since T has closed range, we have $\mathcal{Z} = \mathrm{ran(T^*)} \oplus \mathrm{ker(T)}$ and $\mathcal{Y} = \mathrm{ran(T)} \oplus \mathrm{ker(T^*)}$. Now, T has the matrix form

$$T = \begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where T_1 is invertible. On the other hand, $A = -A^*$ and $(1-TT^\dagger)A(1-TT^\dagger) = 0$ imply that A has the form

$$A = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix},$$

where $A_1 = -A_1^*$. Let X_0 have the form

$$X_0 = \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix}.$$

Now by using matrix form for operators T, X_0 and A, we have

$$\begin{bmatrix} T_1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} - \begin{bmatrix} X_1^* & X_3^* \\ X_2^* & X_4^* \end{bmatrix}\begin{bmatrix} T_1^* & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix},$$

or equivalently

$$\begin{bmatrix} T_1X_1 - X_1^*T_1^* & T_1X_2 \\ -X_2^*T_1^* & 0 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ -A_2^* & 0 \end{bmatrix}.$$

Therefore

$$T_1X_1 - X_1^*T_1^* = A_1,$$

and

$$T_1X_2 = A_2. \tag{2.6}$$

Now, we obtain X_1 and X_2. By Lemma 1.1, T_1 is invertible. Hence, Corollary 2.1 implies that

$$X_1 = \frac{1}{2}T_1^{-1}A_1 + T_1^{-1}Z_1,$$

where $Z_1 \in \mathcal{L}(\mathrm{ran(T)})$ satisfy $Z_1^* = Z_1$. Now, multiplying T_1^{-1} from the left to (2.6), we get

$$X_2 = T_1^{-1}A_2. \tag{2.7}$$

Hence $\begin{bmatrix} \frac{1}{2}T_1^{-1}A_1 + T_1^{-1}Z_1 & T_1^{-1}A_2 \\ X_3 & X_4 \end{bmatrix}$ is a solution to (2.5), such that X_3, X_4 are arbitrary operators. Let

$$Y = \begin{bmatrix} Y_1 & Y_2 \\ X_3 & X_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T^*)} \\ \ker(T) \end{bmatrix},$$

and

$$Z = \begin{bmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{bmatrix} : \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix} \to \begin{bmatrix} \mathrm{ran(T)} \\ \ker(T^*) \end{bmatrix}.$$

Then

$$\frac{1}{2}T^\dagger ATT^\dagger = \begin{bmatrix} \frac{1}{2}T_1^{-1}A_1 & 0 \\ 0 & 0 \end{bmatrix}, \quad T^\dagger ZTT^\dagger = \begin{bmatrix} T_1^{-1}Z_1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$T^\dagger A(1 - TT^\dagger) = \begin{bmatrix} 0 & T^{-1}A_2 \\ 0 & 0 \end{bmatrix}, \quad (1 - T^\dagger T)Y = \begin{bmatrix} 0 & 0 \\ X_3 & X_4 \end{bmatrix}.$$

Consequently, $X_0 = \frac{1}{2}T^\dagger ATT^\dagger + T^\dagger ZTT^\dagger + T^\dagger A(1 - TT^\dagger) + (1 - T^\dagger T)Y$, where $Z \in \mathcal{L}(\mathcal{Y})$ satisfies $T^*(Z - Z^*)T = 0$, and $Y \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ is arbitrary. ☐

In the following theorem we obtain explicit solution to the operator equation $TXS^* - SX^*T^* = A$ when $(1 - P_{\mathrm{ran(S)}})T$ and S have closed ranges.

Theorem 2.3. *Suppose that $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ are Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ and $A \in \mathcal{L}(\mathcal{Y})$ and such that $(1 - SS^\dagger)T$ and S have closed ranges. If the equation*

$$TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z}), \tag{2.8}$$

is solvable, then

$$\begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}^\dagger \tag{2.9}$$

Proof. Taking $H = \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} : \mathcal{Z} \oplus \mathcal{X} \to \mathcal{Y} \oplus \mathcal{Y}$. The operator H has closed range, since let $\{z_n \oplus x_n\}$ be sequence chosen in $\mathcal{Z} \oplus \mathcal{X}$, $\{z_n\}, \{x_n\}$ be sequences chosen in $\mathcal{Z}$ and $\mathcal{X}$, respectively such that $T(z_n) + S(x_n) \to y$ for some $y \in \mathcal{Y}$. Then

$$(1 - SS^\dagger)T(z_n) = (1 - SS^\dagger)(T(z_n) + S(x_n)) \to (1 - SS^\dagger)(y).$$

Since $\mathrm{ran((1 - SS^\dagger)T)}$ is assumed to be closed, $(1 - SS^\dagger)(y) = (1 - SS^\dagger)T(z_1)$ for some $z_1 \in \mathcal{Z}$. It follows that $y - T(z_1) \in \ker(1 - SS^\dagger) = \mathrm{ran(S)}$, hence $y = T(z_1) + S(x)$ for some $x \in \mathcal{X}$. Therefore H has closed range. Let $Y = \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} : \mathcal{Z} \oplus \mathcal{X} \to \mathcal{Z} \oplus \mathcal{X}$

and $H^* = \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Z} \oplus \mathcal{X}$ and $B = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} : \mathcal{Y} \oplus \mathcal{Y} \to \mathcal{Y} \oplus \mathcal{Y}$, therefore (2.8) get into

$$HYH^* = B. \tag{2.10}$$

Since H has closed range and (2.8) has solution, then (2.10) has solution, therefore with multiplying $HH^\dagger$ on the left and multiply $H^*(H^*)^\dagger$ on the right, we have

$$Y = H^\dagger B (H^*)^\dagger.$$

□

The proof of the following remark is the same as in the matrix case.

Remark 2.1. Let $T \in \mathcal{L}(\mathcal{Y}, \mathcal{Z})$ and $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ have closed ranges, and $A \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$. Then the equation

$$TXS = A, \quad X \in \mathcal{L}(\mathcal{Y}), \tag{2.11}$$

has a solution if and only if

$$TT^\dagger A S^\dagger S = A.$$

In which case, any solution of (2.11) has the form

$$X = T^\dagger A S^\dagger.$$

Now, we solve to the operator equation $TXS^* - SX^*T^* = A$ in the case when $\mathrm{ran(T)} + \mathrm{ran(S)}$ is closed.

Theorem 2.4. *Suppose that $\mathcal{X}$ is a Hilbert $\mathcal{A}$-module, $S, T, A \in \mathcal{L}(\mathcal{X})$ such that* $\mathrm{ran(T)} + \mathrm{ran(S)}$ *is closed. Then the following statements are equivalent:*
(a) *There exists a solution $X \in \mathcal{L}(\mathcal{X})$ to the operator equation $TXS^* - SX^*T^* = A$;*
(b) *$A = -A^*$ and $P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran(TT^*+SS^*)}} = A$.*
If (a) or (b) is satisfied, then any solution to

$$TXS^* - SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}), \tag{2.12}$$

has the form

$$X = T^*(TT^* + SS^*)^\dagger A (TT^* + SS^*)^\dagger S. \tag{2.13}$$

Proof. (a) $\Rightarrow$ (b) Suppose that (2.12) has a solution $X \in \mathcal{L}(\mathcal{X})$. Then obviously $A = -A^*$. On the other hand, (2.12) get into

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}. \tag{2.14}$$

Since (2.12) is solvable, then (2.14) is solvable. Since $\mathrm{ran(T)} + \mathrm{ran(S)}$ is closed, then Lemma 1.2 implies that $\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger$ exists. Hence, by Remark 2.1 we have

$$\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.$$

By applying Lemma 1.2, Corollary 1.3 are shown that

$$\begin{aligned}
&\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^*(TT^*+SS^*)^\dagger & 0 \\ S^*(TT^*+SS^*)^\dagger & 0 \end{bmatrix}\begin{bmatrix} A(TT^*+SS^*)^\dagger T & A(TT^*+SS^*)^\dagger S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} (TT^*+SS^*)(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger(TT^*+SS^*) & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran((TT^*+SS^*)^*)}} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran(TT^*+SS^*)}} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

That is, $P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran(TT^*+SS^*)}} = A$.
$(b) \Rightarrow (a)$: If $P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran(TT^*+SS^*)}} = A$, then we have

$$\begin{aligned}
\begin{bmatrix} P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran((TT^*+SS^*)^*)}} & 0 \\ 0 & 0 \end{bmatrix} &= \begin{bmatrix} P_{\mathrm{ran(TT^*+SS^*)}} A P_{\mathrm{ran(TT^*+SS^*)}} & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},
\end{aligned}$$

or equivalently

$$\begin{aligned}
&\begin{bmatrix} (TT^*+SS^*)(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger(TT^*+SS^*) & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}^\dagger \begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned}$$

Therefore, Remark 2.1 implies that (2.14) is solvable and hence (2.12) is solvable.

Now, by applying Remark 2.1 and Lemma 1.2, Corollary 1.3 imply that

$$\begin{aligned}
&\begin{bmatrix} 0 & X \\ -X^* & 0 \end{bmatrix} \\
&= \begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}^\dagger \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T^* & 0 \\ S^* & 0 \end{bmatrix}^\dagger \\
&= \begin{bmatrix} T^*(TT^*+SS^*)^\dagger & 0 \\ S^*(TT^*+SS^*)^\dagger & 0 \end{bmatrix}\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} (TT^*+SS^*)^\dagger T & (TT^*+SS^*)^\dagger S \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} T^*(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger T & T^*(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger S \\ S^*(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger T & S^*(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger S \end{bmatrix}.
\end{aligned}$$

Therefore

$$T^*(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger T = S^*(TT^*+SS^*)^\dagger A(TT^*+SS^*)^\dagger S = 0.$$

Consequently, X has the form (2.13). □

Using exactly similar arguments, we obtain the following analogue of Theorem 2.1, in which to (2.1) is replaced by

$$TXS^* + SX^*T^* = A. \tag{2.15}$$

All results of this section can be rewritten for to (2.15), considering the following theorem.

Theorem 2.5. *Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be Hilbert $\mathcal{A}$-modules, $S \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ and $T \in \mathcal{L}(\mathcal{Z}, \mathcal{Y})$ be invertible operators and $A \in \mathcal{L}(\mathcal{Y})$. Then the following statements are equivalent:*
(a) *There exists a solution $X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ to the operator equation $TXS^* + SX^*T^* = A$.*
(b) *$A = A^*$.*
If (a) or (b) is satisfied, then any solution to

$$TXS^* + SX^*T^* = A, \quad X \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$$

has the form

$$X = \frac{1}{2}T^{-1}A(S^*)^{-1} - T^{-1}Z(S^*)^{-1},$$

where $Z \in \mathcal{L}(\mathcal{Y})$ satisfy $Z^* = -Z$.

References

[1] H. Braden, *The equations $A^TX \pm X^TA = B$*, SIAM J. Matrix Anal. Appl. **20** (1998), 295–302.
[2] C.Y. Deng, H. K. Du, *Representations of the Moore-Penrose inverse for a class of 2 by 2 block operator valued partial matrices*, Linear Multilinear Algebra, **58** (2010), 15–26.
[3] D. S. Djordjevic, *Explicit solution of the operator equation $A^*X + X^*A = B$*, J. Comput. Appl. Math. **200**(2007), 701–704.
[4] M. Mohammadzadeh Karizaki, M. Hassani, M. Amyari and M. Khosravi, *Operator matrix of Moore-Penrose inverse operators on Hilbert C^*-modules*, Colloq. Math. **140** (2015), 171–182
[5] E. C. Lance, *Hilbert C^*-Modules*, LMS Lecture Note Series 210, Cambridge Univ. Press, 1995.
[6] Q. Xu, L. Sheng, *Positive semi-definite matrices of adjointable operators on Hilbert C^*-modules*, Linear Algebra Appl. **428** (2008), 992–1000.
[7] Q. Xu, L. Sheng, Y. Gu, *The solutions to some operator equations*, Linear Algebra Appl. **429** (2008), 1997–2024.
[8] Y. Yuan, *Solvability for a class of matrix equation and its applications*, J. Nanjing Univ. (Math. Biquart.) **18** (2001), 221–227.

[1]University of Torbat-e-Heydarieh, Iran
E-mail address: mohammadzadehkarizaki@gmail.com

[2]Department of Mathematics, Mashhad Branch,
Islamic Azad University, Mashhad, Iran
E-mail address: mhassanimath@gmail.com

[3]Mathematics, College of Engineering & Science Victoria University,
PO Box 14428, Melbourne City, MC 8001, Australia
E-mail address: Sever.Dragomir@vu.edu.au