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ON A CONVEXITY PROPERTY

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Abstract. In this article we proved an interesting property of the class of continuous convex functions. This leads to the form of pre-Hermite-Hadamard inequality which in turn admits a generalization of the famous Hermite-Hadamard inequality. Some further discussion is also given.

1. Introduction

Most general class of convex functions is defined by the inequality

$$\frac{\phi(x)+\phi(y)}{2} \geq \phi\left(\frac{x+y}{2}\right).$$

A function which satisfies this inequality in a certain closed interval I is called *convex* in that interval. Geometrically it means that the midpoint of any chord of the curve $y=\phi(x)$ lies above or on the curve.

Denote now by Q the family of *weights* i.e., non-negative real numbers summing to 1. If ϕ is continuous, then the inequality

$$p\phi(x)+q\phi(y) \geq \phi(px+qy) \tag{1.1}$$

holds for any $p,q \in Q$. Moreover, the equality sign takes place only if $x=y$ or ϕ is linear (cf. [1]).

The same is valid for so-called *Jensen functional*, defined as

$$\mathcal{J}_\phi(\mathbf{p},\mathbf{x}) := \sum p_i\phi(x_i) - \phi\left(\sum p_i x_i\right),$$

where $\mathbf{p}=\{p_i\}_1^n \in Q$, $\mathbf{x}=\{x_i\}_1^n \in I$, $n \geq 2$.

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Geometrically, the inequality (1.1) asserts that each chord of the curve $y = \phi(x)$ lies above or on the curve.

2. Results and Proofs

Main contribution of this paper is the following

Proposition 2.1. *Let $f(\cdot)$ be a continuous convex function defined on a closed interval $[a, b] := I$. Denote*

$$F(s,t) := f(s) + f(t) - 2f\left(\frac{s+t}{2}\right).$$

Prove that

$$\max_{s,t\in I} F(s,t) = F(a,b). \tag{2.1}$$

Proof. It suffices to prove that the inequality

$$F(s,t) \le F(a,b)$$

holds for $a < s < t < b$.

In the sequel we need the assertion stated in Lemma 2.1 (which is of independent interest).

Lemma 2.1. *Let $f(\cdot)$ be a continuous convex function on some interval $I \subseteq \mathbb{R}$. If $x_1, x_2, x_3 \in I$ and $x_1 < x_2 < x_3$, then*
(i) $\frac{f(x_2)-f(x_1)}{2} \le f\left(\frac{x_2+x_3}{2}\right) - f\left(\frac{x_1+x_3}{2}\right)$;
(ii) $\frac{f(x_3)-f(x_2)}{2} \ge f\left(\frac{x_1+x_3}{2}\right) - f\left(\frac{x_1+x_2}{2}\right)$.

We shall prove the first part of the lemma; proof of the second part goes along the same lines.

Since $x_1 < x_2 < (x_2 + x_3)/2 < x_3$, there exist $p, q;\ 0 \le p,\ q \le 1,\ p + q = 1$ such that $x_2 = px_1 + q(\frac{x_2+x_3}{2})$. Hence,

$$\begin{aligned}
\frac{f(x_1)-f(x_2)}{2} + f\left(\frac{x_2+x_3}{2}\right) &= \frac{1}{2}\left[f(x_1) - f\left(px_1 + q\frac{x_2+x_3}{2}\right)\right] + f\left(\frac{x_2+x_3}{2}\right)\\
&\ge \frac{1}{2}\left[f(x_1) - \left(pf(x_1) + qf\left(\frac{x_2+x_3}{2}\right)\right)\right] + f\left(\frac{x_2+x_3}{2}\right)\\
&= \frac{q}{2}f(x_1) + \frac{2-q}{2}f\left(\frac{x_2+x_3}{2}\right)\\
&\ge f\left(\frac{q}{2}x_1 + \frac{2-q}{2}\left(\frac{x_2+x_3}{2}\right)\right)\\
&= f\left(\frac{q}{2}x_1 + \left(\frac{x_2+x_3}{2}\right) - \frac{1}{2}(x_2 - px_1)\right)\\
&= f\left(\frac{x_1+x_3}{2}\right).
\end{aligned}$$

For the proof of second part we can take $x_2 = p\left(\frac{x_1+x_2}{2}\right) + qx_3$ and proceed as above.

Now, applying the part (i) with $x_1 = a$, $x_2 = s$, $x_3 = b$ and the part (ii) with $x_1 = s$, $x_2 = t$, $x_3 = b$, we get

$$\frac{f(s)-f(a)}{2} \le f\left(\frac{s+b}{2}\right) - f\left(\frac{a+b}{2}\right); \tag{2.2}$$

$$\frac{f(b)-f(t)}{2} \ge f\left(\frac{s+b}{2}\right) - f\left(\frac{s+t}{2}\right), \tag{2.3}$$

respectively.

Subtracting (2.2) from (2.3), the desired inequality follows. □

Remark 2.1. A challenging task is to find a geometric proof of the property (2.1).

We shall quote now a couple of important consequences. The first one is used in a number of articles although we never saw a proof of it.

Corollary 2.1. *Let f be defined as above. If $x, y \in [a, b]$ and $x + y = a + b$, then*

$$f(x) + f(y) \le f(a) + f(b).$$

Proof. Obvious, as a simple application of Proposition 2.1. □

Corollary 2.2. *Under the conditions of Proposition* 2.1*, the double inequality*

$$2f\left(\frac{a+b}{2}\right) \le f(pa+qb) + f(pb+qa) \le f(a) + f(b) \tag{2.4}$$

holds for arbitrary weights $p, q \in Q$.

Proof. Applying Proposition 2.1 with $s = pa + qb$, $t = pb + qa$; $s, t \in I$ we get the right-hand side of (2.4). The left-hand side inequality is obvious since, by definition,

$$\frac{f(pa+qb)+f(pb+qa)}{2} \ge f\left[\frac{(pa+qb)+(pb+qa)}{2}\right] = f\left(\frac{a+b}{2}\right).$$

□

Remark 2.2. The relation (2.4) represents a kind of pre-Hermite-Hadamard inequalities. Indeed, integrating both sides of (2.4) over $p \in [0, 1]$, we obtain the form of Hermite-Hadamard inequality (cf. [2]),

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(t)dt \le \frac{f(a)+f(b)}{2}.$$

Moreover, the inequality (2.4) admits a generalization of the Hermite-Hadamard inequality.

Proposition 2.2. *Let g be an arbitrary non-negative and integrable function on I. Then, with f defined as above, we get*

$$2f\left(\frac{a+b}{2}\right)\int_a^b g(t)dt \le \int_a^b (g(t)+g(a+b-t))f(t)dt \le (f(a)+f(b))\int_a^b g(t)dt. \tag{2.5}$$

Proof. Multiplying both sides of (2.4) with $g(pa+qb)$ and integrating over $p \in [0,1]$, we obtain

$$2f\left(\frac{a+b}{2}\right)\frac{\int_a^b g(t)dt}{b-a} \le \frac{\int_a^b (f(t)+f(a+b-t))g(t)dt}{b-a} \le (f(a)+f(b))\frac{\int_a^b f(t)dt}{b-a},$$

and, because

$$\int_a^b (f(t)+f(a+b-t))g(t)dt = \int_a^b (g(t)+g(a+b-t))f(t)dt,$$

the inequality (2.5) follows. □

We shall give in the sequel some illustrations of this proposition.

Corollary 2.3. *For any f that is convex and continuous on $I := [a,b], 0 < a < b$ and $\alpha \in \mathbb{R}/\{0\}$, we have*

$$2f\left(\frac{a+b}{2}\right) \le \frac{\alpha}{b^\alpha - a^\alpha}\int_a^b \left[t^{\alpha-1} + (a+b-t)^{\alpha-1}\right] f(t)dt \le f(a)+f(b).$$

Also, for $\alpha \to 0$, we get

Corollary 2.4.

$$2f\left(\frac{a+b}{2}\right)\frac{\log(b/a)}{a+b} \le \int_a^b \frac{f(t)}{t(a+b-t)}dt \le [f(a)+f(b)]\frac{\log(b/a)}{a+b}.$$

Similarly,

Corollary 2.5.

$$2f\left(\frac{\pi}{2}\right) \le \int_0^\pi f(t)\sin t dt \le f(0)+f(\pi);$$

$$2f\left(\frac{\pi}{4}\right) \le \int_0^{\pi/2} [\sin t + \cos t] f(t)dt \le f(0) + f\left(\frac{\pi}{2}\right).$$

Estimations of the convolution of symmetric kernel on a symmetric interval are also of interest.

Corollary 2.6. *Let f and g be defined as above on a symmetric interval $[-a,a], a > 0$. Then we have that*

$$2f(0)\int_{-a}^a g(t)dt \le \int_{-a}^a [g(-t)+g(t)]f(t)dt \le [f(-a)+f(a)]\int_{-a}^a g(t)dt.$$

Remark 2.3. There remains the question of possible extensions of the relation (2.1). In this sense one can try to prove, along the lines of the proof of (2.1), that

$$\max_{p,q\in Q; x,y\in[a,b]} F^*(p,q;x,y) = F^*(p,q;a,b),$$

where

$$F^*(p,q;x,y) := pf(x) + qf(y) - f(px+qy).$$

Anyway the result will be wrong, as simple examples show (apart from the case $f(x) = x^2$).

On the other hand, it was proved in [3] that for $p_i \in Q$ and $x_i \in [a, b]$ there exist $p, q \in Q$ such that

$$\mathcal{J}_f(\mathbf{p}, \mathbf{x}) = \sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le pf(a) + qf(b) - f(pa + qb), \tag{2.6}$$

for any continuous function f which is convex on $[a, b]$.

Therefore, an important conclusion follows.

Corollary 2.7. *For arbitrary $p_i \in Q$ and $x_i \in [a, b]$, we have that*

$$\sum p_i f(x_i) - f\left(\sum p_i x_i\right) \le \max_p [pf(a) + qf(b) - f(pa + qb)] := T_f(a, b),$$

where $T_f(a, b)$ is an optimal upper global bound, depending only on a and b (cf. [3]*).*

An answer to the above remark is given by the next

Proposition 2.3. *If f is continuous and convex on $[a, b]$, then*

$$\max_{p,q \in Q; x,y \in [a,b]} F^*(p, q; x, y) \le F(a, b).$$

Proof. We shall prove just that

$$F^*(p, q; x, y) \le F(x, y),$$

for all $p, q \in Q$ and $x, y \in [a, b]$.

Indeed,

$$\begin{aligned} F(x, y) - F^*(p, q; x, y) &= qf(x) + pf(y) + f(px + qy) - 2f\left(\frac{x+y}{2}\right) \\ &\ge f(qx + py) + f(px + qy) - 2f\left(\frac{x+y}{2}\right) \\ &\ge 2f\left(\frac{(qx + py) + (px + qy)}{2}\right) - 2f\left(\frac{x+y}{2}\right) \\ &= 0. \end{aligned}$$

The rest of the proof is an application of Proposition 2.1. □

Putting there $x = a$, $y = b$ and combining with (2.6), we obtain another global bound for Jensen functional.

Corollary 2.8. *We have that*

$$\mathcal{J}_f(\mathbf{p}, \mathbf{x}) \le f(a) + f(b) - 2f\left(\frac{a+b}{2}\right) := T'_f(a, b).$$

The bound $T'_f(a, b)$ is not so precise as $T_f(a, b)$ but is much easier to calculate.

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