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THE OPTIMAL HYPERBALL PACKINGS RELATED TO THE SMALLEST COMPACT ARITHMETIC 5-ORBIFOLDS

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Abstract. The smallest three hyperbolic compact arithmetic 5-orbifolds can be derived from two compact Coxeter polytopes that are combinatorially simplicial prisms (i.e. complete orthoschemes of degree $d = 1$) in five dimensional hyperbolic space $\mathbf{H}^5$ (see [2] and [4]). The corresponding hyperbolic tilings are generated by reflections through their delimiting hyperplanes. These involve our studies for the relating densest hyperball (hypersphere) packings with congruent hyperballs.

The analogous problem was discussed in [13] and [14] in hyperbolic spaces $\mathbf{H}^n$ ($n = 3, 4$). In this paper we extend this procedure to determine the optimal hyperball packings of the above 5-dimensional prism tilings. We compute their metric data and the densities of their optimal hyperball packings, moreover, we formulate a conjecture for the candidates for the densest hyperball packings in 5-dimensional hyperbolic space $\mathbf{H}^5$.

1. Introduction

Let $\mathrm{Isom}(\mathbf{H}^5)$ denote the group of isometries in 5-dimensional hyperbolic space and let $\mathrm{Isom}^+(\mathbf{H}^5)$ denote its orientation preserving subgroup of index two. In [2] the lattice of smallest covolume was determined among cocompact arithmetic lattices of $\mathrm{Isom}^+(\mathbf{H}^5)$.

In [4] the second and third values in the volume spectrum of compact orientable arithmetic hyperbolic 5-orbifolds are determined and it has been proved the following

Theorem 1.1 (Emery-Kellerhals). *The lattices Γ'_0, Γ'_1, Γ'_2 (ordered by increasing covolume) are the three cocompact arithmetic lattices in* $\mathrm{Isom}^+(\mathbf{H}^5)$ *of minimal covolume. They are unique, in the sense that any cocompact arithmetic lattice in* $\mathrm{Isom}^+(\mathbf{H}^5)$

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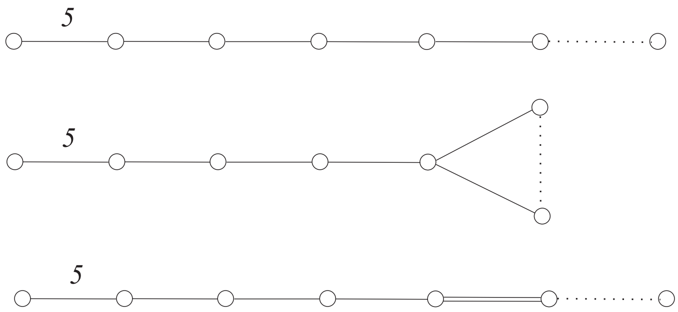

FIGURE 1.

of covolume smaller than or equal to Γ'_2 *is conjugate in* $\mathrm{Isom}(\mathbf{H}^5)$ *to one of the* Γ'_i *(*$i = 0, 1, 2$*).*

The above lattices Γ'_0, Γ'_1, Γ'_2 can be derived by compact Coxeter polytopes $\mathcal{S}_i$ ($i = 0, 1, 2$) in $\mathbf{H}^5$, characterized by the extended Coxeter symbols $[5, 3, 3, 3, 3]$, $[5, 3, 3, 3, 3^{1,1}]$ $[5, 3, 3, 3, 4]$ and Figure 1.

The Coxeter group generated by the reflections through the hyperplanes delimiting $\mathcal{S}_i$ ($i = 0, 1, 2$) will be denoted by Δ_i. The following version has also been formulated in [4]:

Theorem 1.2 (Emery-Kellerhals)**.** *For* $i = 0, 1, 2$ *let* Δ_i^+ *be the lattice* $\Delta_i \cap \mathrm{Isom}^+(\mathbf{H}^5)$*, which is of index two in* Δ_i*. Then* Δ_i^+ *is conjugate to* Γ'_i *in* $\mathrm{Isom}(\mathbf{H}^5)$*. In particular,* Δ_0 *realizes the smallest covolume among the cocompact arithmetic lattices in* $\mathrm{Isom}(\mathbf{H}^5)$*.*

Polytopes $\mathcal{S}_0$ and $\mathcal{S}_2$ are complete Coxeter orthoschemes of degree $d = 1$ (see next section) that were classified by Im Hof in [5] and [6]. Moreover, polytope $\mathcal{S}_1$ can be derived by the reflection of $\mathcal{S}_0$ in its 4-facet of symbol $[5, 3, 3, 3]$.

In hyperbolic space $\mathbf{H}^n$ ($n \geq 3$) a regular prism is the convex hull of two congruent $(n-1)$-dimensional regular polytopes in ultraparallel hyperplanes, (i.e. $(n-1)$-planes), related by "translation" along the line joining their centres that is the common perpendicular. Each vertex of such a tiling is either proper point or every vertex lies on the absolute quadric of $\mathbf{H}^n$, in this case the prism tiling is called fully asymptotic. Thus the prism is a polytope having at each vertex one $(n-1)$-dimensional regular polytope and some $(n-1)$-dimensional prisms, meeting at this vertex.

From definitions of regular prism tilings and complete orthoschemes of degree $d = 1$ (see next section) it follows that a prism tiling exists in n-dimensional hyperbolic space $\mathbf{H}^n$, $n \geq 3$ if and only if there exists an appropriate complete Coxeter orthoscheme of degree $d = 1$. The formulas for the hyperbolic covolumes of the considered 5-dimensional Coxeter tilings are determined in [4] (see also formula (4.3)) therefore, it is possible to compute the volume of the regular prism and the density of the corresponding hyperball packing.

In [13] and [14] we have studied the regular prism tilings and their optimal hyperball packings in $\mathbf{H}^n$ ($n = 3, 4$).

n	Coxeter symbol	The known maximal density
3	$[7,3,3]$	≈ 0.82251367
4	$[3,5,3,3]$	≈ 0.57680322
5	$[5,3,3,3,3]$	≈ 0.50514481

Table 1.

In this paper we extend the method developed in earlier papers to 5-dimensional hyperbolic space and construct for each interesting Coxeter tiling described above a regular prism tiling in $\mathbf{H}^5$, study the corresponding optimal hyperball packings. Moreover, we determine their metric data and their densities (see Tables 1 and 2). In hyperbolic plane $\mathbf{H}^2$ the universal upper bound of the hypercycle packing density is $\frac{3}{\pi}$ determined by I. Vermes in [17] and recently, (to the author's best knowledge) the candidates for the densest hyperball (hypersphere) packings in $3,4$ and 5-dimensional hyperbolic spaces are derived by the regular prism tilings in [13,14] and in the present paper.

2. The Projective Model of Complete Orthoschemes

We use for $\mathbf{H}^n$ and for its polytopes the projective model in the Lorentz space $\mathbf{E}^{1,n}$, i.e. it denotes the real vector space $\mathbf{V}^{n+1}$ equipped with the bilinear form (scalar product) of signature $(1,n)$

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x^0y^0 + x^1y^1 + \dots + x^ny^n.$$

Here and later on non-zero vectors

$$\mathbf{x} = (x^0, x^1, \dots, x^n) \in \mathbf{V}^{n+1} \quad \text{and} \quad \mathbf{y} = (y^0, y^1, \dots, y^n) \in \mathbf{V}^{n+1},$$

are determined up to real factors, for representing points of $\mathcal{P}^n(\mathbf{R})$. Then $\mathbf{H}^n$ can be interpreted as the interior of the quadric

$$Q = \{[\mathbf{x}] \in \mathcal{P}^n | \langle \mathbf{x}, \mathbf{x} \rangle = 0\} =: \partial \mathbf{H}^n$$

in the real projective space $\mathcal{P}^n(\mathbf{V}^{n+1}, \boldsymbol{V}_{n+1})$, i.e. $X[\mathbf{x}]$ is proper point of $\mathbf{H}^n$ iff $\langle \mathbf{x}, \mathbf{x} \rangle < 0$.

A point of the boundary $\partial \mathbf{H}^n$ is also called point at infinity of $\mathbf{H}^n$, points lying outside $\partial \mathbf{H}^n$ are said to be outer points relative to Q. Let $P([\mathbf{p}]) \in \mathcal{P}^n$, a point $[\mathbf{y}] \in \mathcal{P}^n$ is said to be conjugate to $[\mathbf{p}]$ relative to Q if $\langle \mathbf{p}, \mathbf{y} \rangle = 0$ holds. The set of all points which are conjugate to $P([\mathbf{p}])$ form a projective (polar) hyperplane

$$\mathrm{pol}(P) := \{[\mathbf{y}] \in \mathcal{P}^n | \langle \mathbf{p}, \mathbf{y} \rangle = 0\}.$$

Thus the quadric Q induces a bijection (linear polarity $\mathbf{V}^{n+1} \to \boldsymbol{V}_{n+1}$) from the points of $\mathcal{P}^n$ onto its hyperplanes.

A point $X[\mathbf{x}]$ and a hyperplane $\alpha[\boldsymbol{a}]$ are called incident if $\mathbf{x}\boldsymbol{a} = 0$ i.e. the value of linear form $\boldsymbol{a}$ on vector $\mathbf{x}$ is equal to zero ($\mathbf{x} \in \mathbf{V}^{n+1} \setminus \{\mathbf{0}\}$, $\boldsymbol{a} \in \boldsymbol{V}_{n+1} \setminus \{\mathbf{0}\}$). Straight lines of $\mathcal{P}^n$ are characterized by 2-subspaces of $\mathbf{V}^{n+1}$ or dually in $\boldsymbol{V}_{n+1}$, i.e. by 2 points or dually by $n-1$ hyperplanes, respectively [12].

Let $P \subset \mathbf{H}^n$denote a convex polytope bounded by finitely many hyperplanes H^i, which are characterized by unit normal vectors $\boldsymbol{b}^i \in \boldsymbol{V}_{n+1}$ directed inwards to P:

$$H^i := \{\mathbf{x} \in \mathbf{H}^n | \langle \mathbf{x}, \boldsymbol{b}^i \rangle = 0\}, \qquad \text{with } \langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1.$$

We always assume that P is of finite volume and described as follows.

The Gram matrix $G(P) := (\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle)$, $i, j \in \{0, 1, 2 \dots n\}$ of the normal vectors $\boldsymbol{b}^i$ associated to P is an indecomposable symmetric matrix of signature $(1, n)$ with entries $\langle \boldsymbol{b}^i, \boldsymbol{b}^i \rangle = 1$ and $\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle \leq 0$ for $i \neq j$, having the following geometrical meaning
(2.1)

$$\langle \boldsymbol{b}^i, \boldsymbol{b}^j \rangle = \begin{cases} 0, & \text{if } H^i \perp H^j, \\ -\cos \alpha^{ij}, & \text{if } H^i, H^j \text{ intersect on } P \text{ at angle } \alpha^{ij}, \\ -1, & \text{if } H^i, H^j \text{ are parallel}, \\ -\cosh l^{ij}, & \text{if } H^i, H^j \text{ admit a common perpendicular of length } l^{ij}. \end{cases}$$

Definition 2.1. A simplex $\mathcal{S}$ in $\mathbf{H}^n$ is an orthoscheme iff the $n+1$ vertices of $\mathcal{S}$ can be labelled by $A_0, A_1, \dots, A_n$ in such a way that

$$\operatorname{span}(A_0, \dots, A_i) \perp \operatorname{span}(A_i, \dots, A_n), \qquad \text{for } 0 < i < n-1.$$

Here we indicated the subspaces spanned by the corresponding vertices.

An orthoscheme of degree d in $\mathbf{H}^n$ is bounded by $n+d+1$ hyperplanes $H^0, H^1, \dots, H^{n+d}$ such that $H^i \perp H^j$ for $j \neq i-1,\ i,\ i+1$, where, for $d=2$, indices are taken modulo $n+3$. For a usual orthoscheme we denote by H^i $(0 \leq i \leq n)$ the $(n+1)$ hyperfaces (or facets), each opposite to vertex A_i, respectively. An orthoscheme $\mathcal{S}$ has n dihedral angles which are not right angles. Let α^{ij} denote the dihedral angle of $\mathcal{S}$ between the facets H^i and H^j. Then we have

$$\alpha^{ij} = \frac{\pi}{2}, \quad \text{if } 0 \leq i < j-1 \leq n.$$

The n remaining dihedral angles $\alpha^{i,i+1}$, $(0 \leq i \leq n-1)$ are called the essential angles of $\mathcal{S}$. Geometrically, complete orthoschemes of degree d can be described as follows.

(a) For $d=0$, they coincide with the class of classical orthoschemes introduced by Schläfli (see Definitions 2.1). The initial and final vertices, A_0 and A_n of the orthogonal edge-path A_iA_{i+1}, $i = 0, \dots, n-1$, are called principal vertices of the orthoscheme.

(b) A complete orthoscheme of degree $d=1$ can be interpreted as an orthoscheme with one outer principal vertex, say A_n, which is truncated by its polar plane $\operatorname{pol}(A_n) = \boldsymbol{a}_n$ (see Fig. 2–3). In this case the orthoscheme is called simply truncated with outer vertex A_n.

(c) A complete orthoscheme of degree $d=2$ can be interpreted as an orthoscheme with two outer principal vertices, A_0, A_n, which is truncated by its polar hyperplanes $\operatorname{pol}(A_0) = \boldsymbol{a}_0$ and $\operatorname{pol}(A_n) = \boldsymbol{a}_n$. In this case the orthoscheme

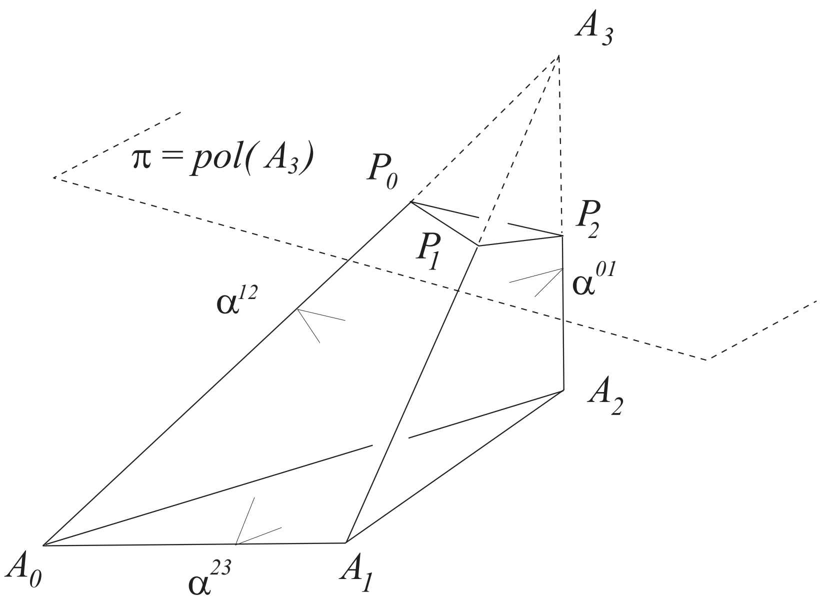

FIGURE 2. A 3-dimensional complete orthoscheme of degree $d = 1$.

is called doubly truncated. (In this case we distinguish two different types of orthoschemes but I will not enter into the details (see [7,8]).)

For complete Coxeter orthoschemes $\mathcal{S} \subset \mathbf{H}^n$ we adopt the usual conventions: If two nodes are related by the weight $\cos\frac{\pi}{p}$ then they are joined by a $(p-2)$-fold line for $p = 3, 4$ and by a single line marked p for $p \geq 5$. In the hyperbolic case if two bounding hyperplanes of S are parallel, then the corresponding nodes are joined by a line marked ∞. If they are ultraparallel, with weight $\cosh(l^{ij})$ in (2.1), then their nodes are joined by a dotted line.

The so called Coxeter-Schläfli matrix of the orthoscheme S with parameters p, q, r, s, and t provides all essential data in the sense of formulas (2.1), (2.2):

$$(c^{ij}) := \begin{pmatrix} 1 & -\cos\frac{\pi}{p} & 0 & 0 & 0 & 0 \\ -\cos\frac{\pi}{p} & 1 & -\cos\frac{\pi}{q} & 0 & 0 & 0 \\ 0 & -\cos\frac{\pi}{q} & 1 & -\cos\frac{\pi}{r} & 0 & 0 \\ 0 & 0 & -\cos\frac{\pi}{r} & 1 & -\cos\frac{\pi}{s} & 0 \\ 0 & 0 & 0 & -\cos\frac{\pi}{s} & 1 & -\cos\frac{\pi}{t} \\ 0 & 0 & 0 & 0 & -\cos\frac{\pi}{t} & 1 \end{pmatrix}. \tag{2.2}$$

3. REGULAR PRISM TILINGS AND THEIR OPTIMAL HYPERBALL PACKINGS IN $\mathbf{H}^5$

3.1. **The Structure of the 5-dimensional Regular Prism Tilings.** In hyperbolic space $\mathbf{H}^n$ $(n \geq 3)$ a regular prism is the convex hull of two congruent $(n-1)$-dimensional regular polytopes in ultraparallel hyperplanes, (i.e. $(n-1)$-planes), related by "translation" along the line joining their centres that is the common perpendicular.

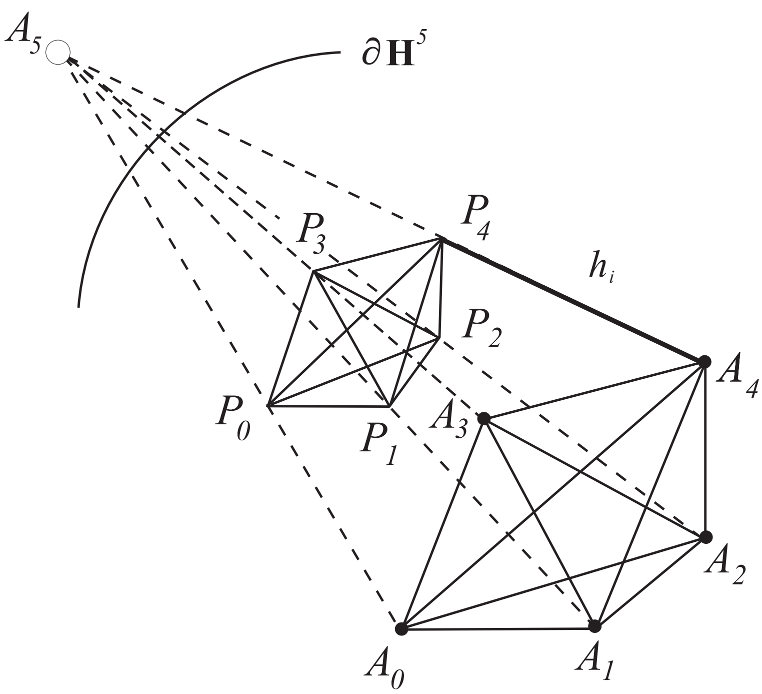

FIGURE 3. A 5-dimensional complete orthoscheme of degree $d = 1$.

In case $n = 5$ the two regular 4-faces of a regular prism are called cover-polytopes, and its other 4-dimensional facets are side-prisms.

In this section we consider the 5-dimensional regular prism tilings $\mathcal{T}_i$ $(i = 0, 2)$ derived by Coxeter tilings $[5, 3, 3, 3, 3]$ and $[5, 3, 3, 3, 4]$. (Tiling $[5, 3, 3, 3, 3^{1,1}]$ is congruent to that derived by $[5, 3, 3, 3, 3]$, explained at Fig. 1.)

Figure 3 shows a part of our 5-prism where A_4 is the centre of a 4-cover, A_3 is the centre of a 3-face of the cover. A_2 is the midpoint of its 2-face, A_1 is a midpoint of an edge of this face, and A_0 is a vertex (end) of that edge. Let B_0, B_1, B_2, B_3, and B_4 be the corresponding points of the other cover-polytope of the regular 5-prism. The midpoints of the edges which do not lie in the cover-polytopes lie in a hyperplane denoted by π. The foot points P_i $(i \in \{0, 1, 2, 3, 4\})$ of the perpendiculars dropped from the points A_i on the plane π form in both cases the same *characteristic (or fundamental) simplex* $\mathcal{S}_\pi$ with Coxeter-Schläfli symbol $[5, 3, 3, 3]$ (see Fig. 3).

Remark 3.1. In $\mathbf{H}^3$ (see [13]) the corresponding prisms are called regular p-gonal prisms $(p \geq 3)$ where the cover-faces are regular p-gons, and the side-faces are rectangles. Fig. 4 shows a part of such a prism where A_2 is the centre of a regular p-gonal face, A_1 is a midpoint of a side of this face, and A_0 is one vertex (end) of that side. Let B_0, B_1, and B_2 be the corresponding points of the other p-gonal face of the prism.

Analogously to the 3-dimensional case, it can be seen that $\mathcal{S}_i = A_0A_1A_2A_3A_4P_0P_1P_2P_3P_4$ $(i = 0, 2)$ is an complete orthoscheme with degree

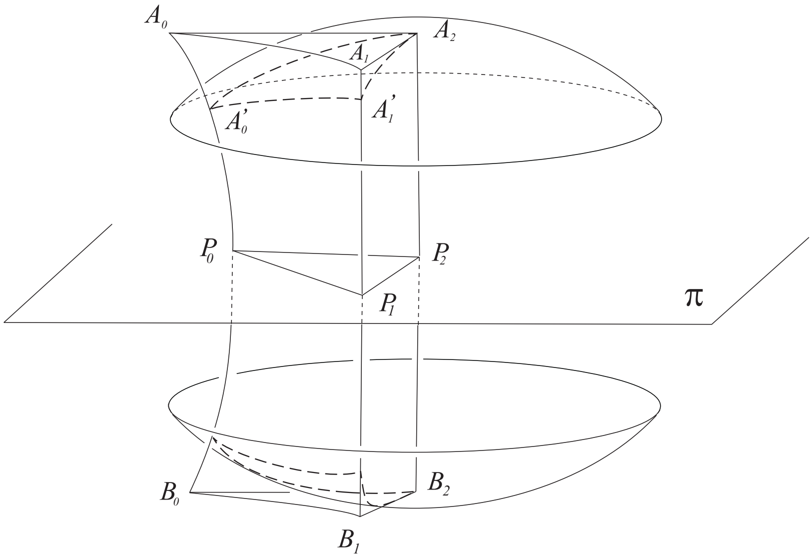

FIGURE 4. A part of a 3-dimensional regular prism.

$d = 1$ where A_5 is an outer vertex of $\mathbf{H}^5$ and points P_0, P_1, P_2, P_3, and P_4 lie in its polar hyperplane π (see Fig. 3). The corresponding regular prism $\mathcal{P}_i$ can be derived by reflections in facets of $\mathcal{S}_i$ containing point A_3.

We consider the images of $\mathcal{P}_i$ under reflections in its side facets (side prisms). The union of these 5-dimensional regular prisms (having the common π hyperplane) forms an infinite polyhedron denoted by $\mathcal{F}_i$.

From the definition of the prism tiling and the complete orthoscheme of degree $d = 1$ follows that a regular prism tiling exists in $\mathbf{H}^n$, $(n \geq 3)$ iff there exists a complete Coxeter orthoscheme of degree $d = 1$ with two ultraparallel facets.

The complete Coxeter orthoschemes were classified by Im Hof in [5] and [6] by generalizing the method of Coxeter and Böhm appropriately. He showed that they exist only for dimensions ≤ 9.

On the other hand, if a 5-dimensional regular prism tiling $[p, q, r, s, t]$ exists, then it has to satisfy the following two requirements:

(a) The orthogonal projection of the cover-polytope in hyperplane π is a regular Coxeter honeycomb with proper vertices and centres. Using the classical notation, these honeycombs are given by their Coxeter-Schläfli symbols $[p, q, r, s]$.
(b) The vertex figures about a vertex of such a prism tiling have to form a 5-dimensional regular polyhedron.

3.2. **Volume of an Orthoscheme.** A plane orthoscheme is a right-angled triangle, whose area formula can be expressed by the well known angle defect formula. For three-dimensional spherical orthoschemes, the volume differentials, depending on the differentials of the three dihedral angles that are not fixed, were found around 1850 by L. Schläfli. Already in 1836, N. I. Lobachevsky found an analogous volume formula for three-dimensional hyperbolic orthoscheme $\mathcal{S}$ [1].

The integration method for orthoschemes of dimension three was generalized by Böhm in 1962 [1] to spaces of constant nonvanishing curvature of arbitrary dimension.

R. Kellerhals in [7] derived a volume formula in 3 dimensional hyperbolic space for *complete orthoschemes of degree* d, $(d = 0, 1, 2)$ and she explicitely determined in [8] the volumes of all complete hyperbolic orthoschemes in even dimension $(n \geq 4)$. Moreover, in [10] she developed a procedure to determine the volumes of 5-dimensional hyperbolic orthoschemes.

The volumes of the above complete orthoschemes $\mathcal{S}_i$ $(i = 0, 2,$ Fig. 1) of degree $d = 1$ can be computed by the volume differential of L. Schläfli with the following formula (see [4]):

$$\mathrm{Vol}_5(\mathcal{S}_i) = \frac{1}{4}\int_{\alpha_i}^{\frac{2\pi}{5}} \mathrm{Vol}_3([5, 3, \beta(t)])dt + \frac{\zeta(3)}{3200} \tag{3.1}$$

with a compact tetrahedron $[5, 3, \beta(t)]$ whose angle parameter $0 < \beta(t) < \frac{\pi}{2}$ is given by

$$\beta(t) = \arctan\sqrt{2 - \cot^2 t}.$$

Remark 3.2. In the above formula ζ is Riemann's zeta function: $\zeta(n) := \sum_{r=1}^{\infty} \frac{1}{r^n}$.

Then, the volume of the 3-dimensional orthoscheme face $[5, 3, \beta(t)]$ as given by Lobachevsky's formula:

$$\begin{aligned}\mathrm{Vol}_3([5, 3, \beta(t)]) = \frac{1}{4}\Big\{&\mathcal{L}_2\left(\frac{\pi}{5} + \theta(t)\right) - \mathcal{L}_2\left(\frac{\pi}{5} - \theta(t)\right) - \mathcal{L}_2\left(\frac{\pi}{6} + \theta(t)\right)\\ &+ \mathcal{L}_2\left(\frac{\pi}{6} - \theta(t)\right) + \mathcal{L}_2\left(\beta(t) + \theta(t)\right) - \mathcal{L}_2\left(\beta(t) - \theta(t)\right)\\ &+2\mathcal{L}_2\left(\frac{\pi}{2} - \theta(t)\right)\Big\}\end{aligned} \tag{3.2}$$

where

$$\mathcal{L}(\omega) = -\int_0^{\omega} \log|2\sin t|dt, \qquad \omega \in \mathbf{R},$$

is Lobachevsky's function and

$$\theta(t) = \arctan\frac{\sqrt{1 - 4\sin^2\frac{\pi}{5}\sin^2\beta(t)}}{2\cos\frac{\pi}{5}\cos\beta(t)}.$$

3.3. **The Optimal Hyperball Packing.** The equidistance surface (or hypersphere) is a quadratic surface at a constant distance from a plane in both halfspaces. The infinite body of the hypersphere is called hyperball.

The 5-dimensional half hypersphere with distance h to a hyperplane π is denoted by $\mathcal{H}^h$. The volume of a bounded hyperball piece $\mathcal{H}^h(\mathcal{A})$ delimited by the 4-polytop $\mathcal{A} \subset \pi$, $\mathcal{H}^h$ and some to π orthogonal hyperplanes derived by the facets of $\mathcal{A}$ can be determined by the formula (3.3) that follows by the generalization of the classical method of J. Bolyai:

$$\operatorname{Vol}(\mathcal{H}^h(\mathcal{A})) = \frac{1}{16}\operatorname{Vol}(\mathcal{A})k\left(\frac{1}{2}\sinh\frac{4h}{k} + 4\sinh\frac{2h}{k}\right) + \frac{3h\operatorname{Vol}(\mathcal{A})}{8}, \tag{3.3}$$

where the volume of the hyperbolic 4-polytope $\mathcal{A}$ lying in the plane π is $\operatorname{Vol}(\mathcal{A})$. The constant $k = \sqrt{\frac{-1}{K}}$ is the natural length unit in $\mathbf{H}^n$. K denotes here the constant negative sectional curvature.

We are looking for the optimal half hyperball $\mathcal{H}^{h_i}_{opt}$ inscribed in $\mathcal{F}_i$ with maximal height.

The optimal half hypersphere $\mathcal{H}^{h_i}_{opt}$ touches the cover-faces of the regular 5-prisms contained in $\mathcal{F}_i$. Therefore, the optimal distance from the 4-midplane π will be $h_i = P_4A_4$ (Fig. 3).

We consider one from the former 5-dimensional regular prism tilings $\mathcal{T}_i$ $(i = 0, 2)$ and the infinite polyhedron $\mathcal{F}_i$ derived from that (the union of 5-dimensional regular prisms having the common hyperplane π). $\mathcal{F}_i$ and its images under reflections in its "facets" fill space $\mathbf{H}^5$ without overlap thus we obtain by the above images of $\mathcal{H}^{h_i}_{opt}$ a locally optimal hyperball packing to the tiling $\mathcal{T}_i$ $(i = 0, 2)$.

$P_4(\mathbf{p}_4)$ and $A_4(\mathbf{a}_4)$ are proper points of $\mathbf{H}^5$ and P_4 lies on the polar hyperplane $\boldsymbol{a}_5 = \operatorname{pol}(A_5)$ of the outer point A_5 thus

$$\begin{aligned}
\mathbf{p}_4 = c\cdot\mathbf{a}_5 + \mathbf{a}_4 \in \boldsymbol{a}^5 &\Leftrightarrow c\cdot\mathbf{a}_5\boldsymbol{a}^5 + \mathbf{a}_4\boldsymbol{a}^5 = 0\\
&\Leftrightarrow c = -\frac{\mathbf{a}_4\boldsymbol{a}^5}{\mathbf{a}_5\boldsymbol{a}^5}\\
&\Leftrightarrow \mathbf{p}_4 \sim -\frac{\mathbf{a}_4\boldsymbol{a}^5}{\mathbf{a}_5\boldsymbol{a}^5}\mathbf{a}_5 + \mathbf{a}_4 \sim \mathbf{a}_4(\mathbf{a}_5\boldsymbol{a}^5) - \mathbf{a}_5(\mathbf{a}_4\boldsymbol{a}^5) = \mathbf{a}_4h_{55} - \mathbf{a}_5h_{45},
\end{aligned}$$

where h_{ij} is the inverse of the Coxeter-Schläfli matrix c^{ij} (see (2.2)) of the orthoscheme $\mathcal{S}_i$. The hyperbolic distance h_i can be calculated by the following formula [11]:

$$\cosh P_4A_4 = \cosh h_i = \frac{-\langle\mathbf{p}_4,\mathbf{a}_4\rangle}{\sqrt{\langle\mathbf{p}_4,\mathbf{p}_4\rangle\langle\mathbf{a}_4,\mathbf{a}_4\rangle}} = \frac{h_{45}^2 - h_{44}h_{55}}{\sqrt{h_{44}\langle\mathbf{p}_4,\mathbf{p}_4\rangle}} = \sqrt{\frac{h_{44}\ h_{55} - h_{45}^2}{h_{44}\ h_{55}}}. \tag{3.4}$$

The volume of $\mathcal{S}_i$ is denoted by $\operatorname{Vol}_5(\mathcal{S}_i)$ (see Subsection 3.2).

For the density of the packing it is sufficient to relate the volume of the optimal hyperball piece to that of its containing polytope $\mathcal{S}_i$ (see Fig. 3) because the tiling can be constructed of such polytopes. This polytope and its images in $\mathcal{F}_i$ divide $\mathcal{H}^{h_i}$ into congruent pieces whose volume is denoted by $\operatorname{Vol}(\mathcal{H}^{h_i}_{opt})$. We illustrate in the 3-dimensional case such a hyperball piece $A_2A'_0A'_1P_0P_1P_2$ in Fig. 4.

The density of the optimal hyperball packing related to the prism tiling $\mathcal{T}_i$ $(i = 0, 2)$ is defined by the following formula:

	Σ_{53333}	Σ_{53334}
$\mathrm{Vol}(S_i)$	≈ 0.00076730	≈ 0.00198469
h_i	≈ 0.38359861	≈ 0.53063753
$\mathrm{Vol}(\mathcal{H}^{h_i}_{opt})$	≈ 0.00038760	≈ 0.00059001
δ^{opt}	≈ 0.50514481	≈ 0.29727979

Table 2.

Definition 3.1.

$$\delta^{opt}(\mathcal{T}_i) := \frac{\mathrm{Vol}(\mathcal{H}^{h_i}_{opt})}{\mathrm{Vol}_5(\mathcal{S}_i)}.$$

$\delta^{opt}(\mathcal{T}_i)$ can be determined by the formulas (3.1), (3.2), (3.3) and (3.4), using that the volume $\mathrm{Vol}(\mathcal{A}) = \mathrm{Vol}(\mathcal{S}_\pi)$ (see (3.3)) of the 4-dimensional polytope $\mathcal{S}_\pi$ with Coxeter symbol $[5, 3, 3, 3]$ in both cases is

$$\mathrm{Vol}(\mathcal{S}_\pi) = \frac{\pi^2}{10800} \approx 0.00091385.$$

Finally we get the results in Table 2.

Remark 3.3. The optimal density of the Coxeter tiling $\mathcal{T}_1$ with Coxeter symbol $[5, 3, 3, 3, 3^{1,1}]$ is equal to $\delta^{opt}(\mathcal{T}_0) \approx 0.50514481$.

The next conjecture can be formulated for the optimal hyperball packings.

Conjecture 3.1. The above optimal hyperball packings to Coxeter tilings $[5, 3, 3, 3, 3]$ and $[5, 3, 3, 3, 3^{1,1}]$ provide the densest hyperball packings in $\mathbf{H}^5$.

Remark 3.4. Regular hyperbolic honeycombs exist only up to 5 dimensions [3]. Therefore regular prism tilings can exist up to 6 dimensions. From the definitions of the prism tilings and the complete orthoschemes of degree $d = 1$ it follows that prism tilings exist in $\mathbf{H}^n$, $n \geq 3$ if and only if there exist complete Coxeter orthoschemes of degree $d = 1$ with two ultraparallel facets. From paper [6] it follows that in $\mathbf{H}^5$ there is a further type, and in $\mathbf{H}^6$ there is no such Coxeter orthoschem.

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