

SOME REVERSES OF THE CAUCHY-SCHWARZ AND TRIANGLE INEQUALITIES IN 2-INNER PRODUCT SPACES

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ABSTRACT. In this paper, we give some reverses of the Cauchy-Schwarz inequality and triangle inequality in 2-inner product spaces. Applications for determinantal integral inequalities are also provided.

1. INTRODUCTION

The Cauchy-Schwarz inequality plays an important role in the theory of inner product spaces (see, for instance, [22, 23]), which is one of the classical inequalities. It is well known that, in a semi-inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$, the Cauchy-Schwarz inequality has the form

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle,$$

for all $x, y \in \mathcal{X}$.

In recent years, many authors have studied some related topics such as the reverse of the Cauchy-Schwarz inequality, the triangle and Bessel inequality as well as Grüss inequality (see [7, 10, 11, 18]). The probably first reverse of the Cauchy-Schwarz inequality for positive real numbers was obtained by Pólya and Szegő in 1925 (see [20, p. 57 and 213–214] and [21, p. 71–72 and 253–255]). Since then, there exist a lot of generalizations of the reverse of the Cauchy-Schwarz inequality. For example, in 2007, Dragomir [6, Chapter 2] contributed much to the reverses of the Cauchy-Schwarz inequality and also similar results for integrals, isotonic functionals as well as generalizations in the setting of inner product spaces are well-studied and understood

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(see the book [5]). Some other interesting inequalities for the reverse of the Cauchy-Schwarz inequality can be found in [8, 9, 12].

In this paper, we continue and complement this research by proving some new reverses of the Cauchy-Schwarz inequality in framework of 2-inner product spaces. Furthermore, as applications, some reverse results for the generalized triangle inequality, i.e., upper bounds for the quantity

$$(0 \leq) \sum_{i=1}^n \|x_i, z\| - \left\| \sum_{i=1}^n x_i, z \right\|$$

under various assumptions for the vectors $z, x_i \in \mathcal{X}$, $i \in \{1, \dots, n\}$, are established and also, some applications for the generalized triangle inequality are given.

2. PRELIMINARIES

The concept of 2-normed spaces was introduced by Gähler [16] in 1963. After that, in 1973 and 1977, Diminnie, Gähler and White introduced the concept of 2-inner product spaces [3, 4]. For more details on 2-inner product spaces, see [2, 15, 17, 19]. A systematic presentation of the recent results related to the theory of 2-inner product spaces as well as an extensive list of the related references can be found in the book [1]. Here we give the basic definitions and the elementary properties of 2-inner product spaces.

Let \mathcal{X} be a linear space of dimension greater than 1 over the field $\mathbb{K} = \mathbb{R}$ of real numbers or the field $\mathbb{K} = \mathbb{C}$ of complex numbers. Suppose that $\langle \cdot, \cdot | \cdot \rangle$ is a \mathbb{K} -valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ satisfying the following conditions:

- (2I-1) $\langle x, x | z \rangle \geq 0$, and $\langle x, x | z \rangle = 0$ if and only if x and z are linearly dependent;
- (2I-2) $\langle x, x | z \rangle = \langle z, z | x \rangle$;
- (2I-3) $\langle y, x | z \rangle = \langle x, y | z \rangle$;
- (2I-4) $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$, for any $\alpha \in \mathbb{K}$;
- (2I-5) $\langle x + x', y | z \rangle = \langle x, y | z \rangle + \langle x', y | z \rangle$.

$\langle \cdot, \cdot | \cdot \rangle$ is called a *2-inner product* on \mathcal{X} and $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ is called a *2-inner product space* (or *2-pre-Hilbert space*). Some basic properties of 2-inner products $\langle \cdot, \cdot | \cdot \rangle$ can be immediately obtained as follows:

- (P-1) If $\mathbb{K} = \mathbb{R}$, then (2I-3) reduces to $\langle y, x | z \rangle = \langle x, y | z \rangle$;
- (P-2) $\langle 0, y | z \rangle = \langle x, 0 | z \rangle = \langle x, y | 0 \rangle = 0$;
- (P-3) $\langle x, y | \alpha z \rangle = |\alpha|^2 \langle x, y | z \rangle$, for all $x, y, z \in \mathcal{X}$ and $\alpha \in \mathbb{K}$.

Using the above properties, we can prove the Cauchy-Schwarz inequality

$$|\langle x, y | z \rangle|^2 \leq \langle x, x | z \rangle \langle y, y | z \rangle.$$

In any given 2-inner product space $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$, we can define a function $\| \cdot, \cdot \|$ on $\mathcal{X} \times \mathcal{X}$ by

$$(2.1) \quad \|x, z\| = \sqrt{\langle x, x | z \rangle},$$

for all $x, z \in \mathcal{X}$. It is easy to see that this function satisfies the following conditions:

- (2N-1) $\|x, z\| \geq 0$ and $\|x, z\| = 0$ if and only if x and z are linearly dependent;
- (2N-2) $\|x, z\| = \|z, x\|$;
- (2N-3) $\|\alpha x, z\| = |\alpha| \|x, z\|$ for any scalar $\alpha \in \mathbb{K}$;
- (2N-4) $\|x + x', z\| \leq \|x, z\| + \|x', z\|$.

Any function $\|\cdot, \cdot\|$ defined on $\mathcal{X} \times \mathcal{X}$ and satisfying the above conditions is called a *2-norm* on \mathcal{X} and $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a *linear 2-normed space*. Whenever a 2-inner product space $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ is given, we consider it as a linear 2-norm space $(\mathcal{X}, \|\cdot, \cdot\|)$ with the 2-norm defined by (2.1).

3. SOME REVERSES OF THE CAUCHY-SCHWARZ INEQUALITY

First, we have the following.

Theorem 3.1. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} and $x, y, z \in \mathcal{X}$, $r_1, r_2 > 0$ are such that*

$$(3.1) \quad r_1 \leq \|x, z\| - \|y, z\| \leq \|x - y, z\| \leq r_2,$$

then

$$(3.2) \quad \begin{aligned} \|x, z\| \|y, z\| - |\langle x, y | z \rangle| &\leq \|x, z\| \|y, z\| - |\operatorname{Re} \langle x, y | z \rangle| \\ &\leq \|x, z\| \|y, z\| - \operatorname{Re} \langle x, y | z \rangle \\ &\leq \frac{1}{2} (r_2^2 - r_1^2). \end{aligned}$$

Proof. Taking the square in the third inequality in (3.1), we have

$$\|x, z\|^2 - 2 \operatorname{Re} \langle x, y | z \rangle + \|y, z\|^2 \leq r_2^2,$$

which is equivalent to

$$(3.3) \quad 2 (\|x, z\| \|y, z\| - \operatorname{Re} \langle x, y | z \rangle) + (\|x, z\| - \|y, z\|)^2 \leq r_2^2.$$

Using the first inequality in (3.1), we have

$$(3.4) \quad r_1^2 \leq (\|x, z\| - \|y, z\|)^2.$$

Therefore, from (3.3) and (3.4), we have (3.2). This completes the proof. \square

Corollary 3.1. *With all the assumptions of Theorem 3.1, the following holds:*

$$(3.5) \quad \|x, z\| + \|y, z\| - \|x + y, z\| \leq \sqrt{r_2^2 - r_1^2}.$$

Proof. It follows from (3.2) that

$$\begin{aligned} (\|x, z\| + \|y, z\|)^2 - \|x + y, z\|^2 &= 2 (\|x, z\| \|y, z\| - \operatorname{Re} \langle x, y | z \rangle) \\ &\leq r_2^2 - r_1^2 \end{aligned}$$

gives

$$(3.6) \quad (\|x, z\| + \|y, z\|)^2 \leq \|x + y, z\|^2 + r_2^2 - r_1^2.$$

Taking the square root in (3.6) and taking into account that

$$\sqrt{\alpha + \beta} \leq \sqrt{\alpha} + \sqrt{\beta}$$

for all $\alpha, \beta \geq 0$, we have the desired inequality (3.5). This completes the proof. \square

Theorem 3.2. *Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . For any $x, y, z \in \mathcal{X}$,*

$$(3.7) \quad \left\| \frac{x}{\|x, z\|} - \frac{y}{\|y, z\|}, z \right\| \leq r$$

and

$$\|x, z\| \|y, z\| - \operatorname{Re} \langle x, y|z \rangle \leq \frac{1}{2} r^2 \|x, z\| \|y, z\|$$

are equivalent.

Proof. It is obvious by taking the square in (3.7) and performing the required calculations. \square

Remark 3.1. Since

$$\begin{aligned} \left\| \|y, z\| x - \|x, z\| y, z \right\| &= \left\| \|y, z\| (x - y) + (\|y, z\| - \|x, z\|) y, z \right\| \\ &\leq \|y, z\| \|x - y, z\| + \left| \|y, z\| - \|x, z\| \right| \|y, z\| \\ &\leq 2 \|y, z\| \|x - y, z\|, \end{aligned}$$

the sufficient condition for (3.7) to hold is

$$\|x - y, z\| \leq \frac{r}{2} \|x, z\|.$$

Theorem 3.3. *Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . Then, for any $x, y, z \in \mathcal{X}$ and $p \geq 1$,*

$$(3.8) \quad \begin{aligned} 0 &\leq \|x, z\| \|y, z\| - |\langle x, y|z \rangle| \\ &\leq \|x, z\| \|y, z\| - |\operatorname{Re} \langle x, y|z \rangle| \\ &\leq \frac{1}{2} \begin{cases} (\|x, z\| + \|y, z\|)^{2p} - \|x + y, z\|^{2p} \\ (\|x - y, z\|^{2p} - \left| \|x, z\| - \|y, z\| \right|^{2p}) \end{cases}^{\frac{1}{p}}. \end{aligned}$$

Proof. Firstly, observe that

$$2(\|x, z\| \|y, z\| - \operatorname{Re} \langle x, y|z \rangle) = (\|x, z\| + \|y, z\|)^2 - \|x + y, z\|^2.$$

Denoting

$$D := \|x, z\| \|y, z\| - \operatorname{Re} \langle x, y|z \rangle,$$

we have

$$(3.9) \quad 2D + \|x + y, z\|^2 = (\|x, z\| + \|y, z\|)^2.$$

Taking in (3.9) the power $p \geq 1$ and using the elementary inequality

$$(a + b)^p \geq a^p + b^p, \quad a, b \geq 0,$$

we have

$$(\|x, z\| + \|y, z\|)^{2p} = (2D + \|x + y, z\|^2)^p \geq 2^p D^p + \|x + y, z\|^{2p}.$$

This implies that

$$(3.10) \quad D^p \leq \frac{1}{2^p} ((\|x, z\| + \|y, z\|)^{2p} - \|x + y, z\|^{2p}),$$

which is clearly equivalent to the first branch of the third inequality in (3.8).

With the above notation, we also have

$$(3.11) \quad 2D + (\|x, z\| - \|y, z\|)^2 = \|x - y, z\|^2.$$

Taking the power $p \geq 1$ in (3.11) and using the inequality (3.10), we have

$$\|x - y, z\|^{2p} \geq 2^p D^p + \left| \|x, z\| - \|y, z\| \right|^{2p}$$

and so we have the last part of (3.8). This completes the proof. \square

We state the following result that provides an invariant property for the constant in the Cauchy-Schwarz inequality.

Theorem 3.4. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . For any $x, y, z \in \mathcal{X}$ and $\lambda \in \mathbb{K}$,*

$$\|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2 = \|x - \lambda y, z\|^2 \|y, z\|^2 - |\langle x - \lambda y, y | z \rangle|^2.$$

Proof. By properties of 2-inner product, it follows that, for any $x, y, z \in \mathcal{X}$ and $\lambda \in \mathbb{K}$,

$$\begin{aligned} & \|x - \lambda y, z\|^2 \|y, z\|^2 - |\langle x - \lambda y, y | z \rangle|^2 \\ &= (\|x, z\|^2 - 2 \operatorname{Re}(\bar{\lambda} \langle x, y | z \rangle) + |\lambda|^2 \|y, z\|^2) \|y, z\|^2 \\ &\quad - |\langle x, y | z \rangle - \lambda \|y, z\|^2|^2 \\ &= \|x, z\|^2 \|y, z\|^2 - 2 \|y, z\|^2 \operatorname{Re}(\bar{\lambda} \langle x, y | z \rangle) + |\lambda|^2 \|y, z\|^4 \\ &\quad - |\langle y, x | z \rangle|^2 + 2 \|y, z\|^2 \operatorname{Re}(\bar{\lambda} \langle x, y | z \rangle) - |\lambda|^2 \|y, z\|^4 \\ &= \|y, z\|^2 \|x, z\|^2 - |\langle y, x | z \rangle|^2. \end{aligned}$$

This completes the proof. \square

Corollary 3.2. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . For any $x, y, z \in \mathcal{X}$ and $\lambda \in \mathbb{K}$,*

$$(3.12) \quad \|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2 \leq \|x - \lambda y, z\|^2 \|y, z\|^2.$$

The equalities holds in (3.12) if and only if $\langle x, y | z \rangle = \lambda \|y, z\|^2$.

For two parameters, we can get the following.

Theorem 3.5. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . For any $x, y, z \in \mathcal{X}$ and $\lambda, \mu \in \mathbb{K}$,*

$$(3.13) \quad \begin{aligned} & (\|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2) |\mu - \lambda|^2 \\ &= \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2 - |\langle x - \lambda y, x - \mu y | z \rangle|^2. \end{aligned}$$

Proof. Denote $w := x - \lambda y$. Using some properties of a 2-inner product, we have

$$(3.14) \quad \begin{aligned} |\langle x - \lambda y, x - \mu y | z \rangle|^2 &= |\langle w, x - \mu y | z \rangle|^2 \\ &= |\langle w, x - \lambda y + (\lambda - \mu) y | z \rangle|^2 \\ &= |\langle w, w + (\lambda - \mu) y | z \rangle|^2 \\ &= \left| \|w, z\|^2 + \overline{(\lambda - \mu)} \langle w, y | z \rangle \right|^2 \\ &= \|w, z\|^4 + 2\|w, z\|^2 \operatorname{Re}(\lambda - \mu) \overline{\langle w, y | z \rangle} \\ &\quad + |\lambda - \mu|^2 |\langle w, y | z \rangle|^2 \\ &= \|w, z\|^4 + 2\|w, z\|^2 \operatorname{Re}\left((\lambda - \mu) \overline{\langle w, y | z \rangle}\right) \\ &\quad + |\lambda - \mu|^2 \|w, z\|^2 \|y, z\|^2 \\ &\quad - |\lambda - \mu|^2 (\|w, z\|^2 \|y, z\|^2 - |\langle w, y | z \rangle|^2). \end{aligned}$$

Observe also that

$$(3.15) \quad \begin{aligned} & \|w, z\|^4 + 2\|w, z\|^2 \operatorname{Re}\left((\lambda - \mu) \overline{\langle w, y | z \rangle}\right) + |\lambda - \mu|^2 \|w, z\|^2 \|y, z\|^2 \\ &= \|w, z\|^2 \left(\|w, z\|^2 + 2 \operatorname{Re}\left(\overline{(\lambda - \mu) \langle w, y | z \rangle}\right) + |\lambda - \mu|^2 \|y, z\|^2 \right) \\ &= \|w, z\|^2 \|w + (\lambda - \mu) y, z\|^2 \\ &= \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2. \end{aligned}$$

Therefore, from (3.14) and (3.15), we have the desired result (3.13). This completes the proof. \square

Corollary 3.3. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . For any $x, y, z \in \mathcal{X}$ and $\lambda, \mu \in \mathbb{K}$,*

$$\|x, z\|^2 \|y, z\|^2 - |\langle x, y | z \rangle|^2 \leq \frac{1}{|\mu - \lambda|^2} \|x - \lambda y, z\|^2 \|x - \mu y, z\|^2.$$

As an application of Theorem 3.5, we have the following.

Proposition 3.1. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . Then, for all $x, y, z, e \in \mathcal{X}$ with $\|e, z\| = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{K}$ with $\lambda \neq \mu$ and $\gamma \neq \eta$,*

$$(3.16) \quad |\langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle|$$

$$\begin{aligned}
&\leq \frac{1}{|\lambda - \mu| |\gamma - \eta|} (\|x - \lambda e, z\| \|x - \mu e, z\| \|y - \gamma e, z\| \|y - \eta e, z\| \\
&\quad - |\langle x - \lambda e, x - \mu e | z \rangle| |\langle y - \gamma e, y - \eta e | z \rangle|) \\
&\leq \frac{1}{|\lambda - \mu| |\gamma - \eta|} (\|x - \lambda e, z\| \|x - \mu e, z\| \|y - \gamma e, z\| \|y - \eta e, z\|).
\end{aligned}$$

Proof. Applying the Cauchy-Schwarz inequality for the vectors $x - \langle x, e | z \rangle e$ and $y - \langle y, e | z \rangle e$ and taking into account that

$$\langle x - \langle x, e | z \rangle e, y - \langle y, e | z \rangle e | z \rangle = \langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle,$$

and

$$\|x - \langle x, e | z \rangle e, z\|^2 = \|x, z\|^2 - |\langle x, e | z \rangle|^2,$$

and

$$\|y - \langle y, e | z \rangle e, z\|^2 = \|y, z\|^2 - |\langle y, e | z \rangle|^2,$$

we have

$$\begin{aligned}
(3.17) \quad &|\langle x, y | z \rangle - \langle x, e | z \rangle \langle e, y | z \rangle| \\
&\leq (\|x, z\|^2 - |\langle x, e | z \rangle|^2)^{\frac{1}{2}} (\|y, z\|^2 - |\langle y, e | z \rangle|^2)^{\frac{1}{2}},
\end{aligned}$$

for any $x, y, z, e \in \mathcal{X}$ with $\|e, z\| = 1$.

From (3.13), it follows that

$$\begin{aligned}
(3.18) \quad &(\|x, z\|^2 - |\langle x, e | z \rangle|^2)^{\frac{1}{2}} \\
&= \frac{1}{|\mu - \lambda|} (\|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 - |\langle x - \lambda e, x - \mu e | z \rangle|^2)^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
(3.19) \quad &(\|y, z\|^2 - |\langle y, e | z \rangle|^2)^{\frac{1}{2}} \\
&= \frac{1}{|\gamma - \eta|} (\|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 - |\langle y - \gamma e, y - \eta e | z \rangle|^2)^{\frac{1}{2}},
\end{aligned}$$

for any $x, y, z, e \in \mathcal{X}$ with $\|e, z\| = 1$ and $\lambda, \mu, \gamma, \eta \in \mathbb{K}$ with $\lambda \neq \mu$ and $\gamma \neq \eta$.

Now, if we multiply (3.18) with (3.19), then we have

$$\begin{aligned}
(3.20) \quad &(\|x, z\|^2 - |\langle x, e | z \rangle|^2)^{\frac{1}{2}} (\|y, z\|^2 - |\langle y, e | z \rangle|^2)^{\frac{1}{2}} \\
&\leq \frac{1}{|\mu - \lambda| |\gamma - \eta|} \\
&\quad \times (\|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 |\langle x - \lambda e, x - \mu e | z \rangle|^2)^{\frac{1}{2}} \\
&\quad \times (\|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 |\langle y - \gamma e, y - \eta e | z \rangle|^2)^{\frac{1}{2}}.
\end{aligned}$$

Further, if we use the elementary inequality

$$(a^2 - b^2)^{\frac{1}{2}} (c^2 - d^2)^{\frac{1}{2}} \leq ac - bd$$

for all $a \geq b \geq 0$ and $c \geq d \geq 0$, then we also have

$$(3.21) \quad \begin{aligned} & (\|x - \lambda e, z\|^2 \|x - \mu e, z\|^2 |\langle x - \lambda e, x - \mu e | z \rangle|^2)^{\frac{1}{2}} \\ & \quad \times (\|y - \gamma e, z\|^2 \|y - \eta e, z\|^2 |\langle y - \gamma e, y - \eta e | z \rangle|^2)^{\frac{1}{2}} \\ & \leq \|x - \lambda e, z\| \|x - \mu e, z\| \|y - \gamma e, z\| \|y - \eta e, z\| \\ & \quad - |\langle x - \lambda e, x - \mu e | z \rangle| |\langle y - \gamma e, y - \eta e | z \rangle|. \end{aligned}$$

Finally, using (3.17), (3.20) and (3.21), we have the second inequality (3.16). This completes the proof. \square

4. SOME REVERSES OF THE TRIANGLE INEQUALITY

In this section, we give some reverses of the triangle inequality.

Theorem 4.1. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . Let $z, x_i, \in \mathcal{X}$, $i \in \{1, \dots, n\}$, and $r_{ij} > 0$ for each $1 \leq i \leq j \leq n$ be such that*

$$(4.1) \quad 0 \leq \|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \leq r_{ij}.$$

Then the following quadratic reverse of the triangle inequality holds:

$$(4.2) \quad \left(\sum_{i=1}^n \|x_i, z\| \right)^2 \leq \left\| \sum_{i=1}^n x_i, z \right\|^2 + 2 \sum_{1 \leq i < j \leq n} r_{ij}.$$

The case of the equality holds in (4.2) if and only if it holds in (4.1) for each i, j with $1 \leq i \leq j \leq n$.

Proof. Observe that

$$(4.3) \quad \begin{aligned} \left(\sum_{i=1}^n \|x_i, z\| \right)^2 - \left\| \sum_{i=1}^n x_i, z \right\|^2 &= \sum_{i,j=1}^n \|x_i, z\| \|x_j, z\| - \left\langle \sum_{i=1}^n x_i, \sum_{j=1}^n x_j | z \right\rangle \\ &= \sum_{i,j=1}^n \|x_i, z\| \|x_j, z\| - \sum_{i,j=1}^n \operatorname{Re} \langle x_i, x_j | z \rangle \\ &= \sum_{i,j=1}^n (\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle) \\ &= \sum_{1 \leq i \leq j \leq n} (\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle) \\ & \quad + \sum_{1 \leq j \leq i \leq n} (\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle) \\ &= 2 \sum_{1 \leq i \leq j \leq n} (\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle). \end{aligned}$$

Using the condition (4.1), we have

$$\sum_{1 \leq i \leq j \leq n} (\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle) \leq \sum_{1 \leq i \leq j \leq n} r_{ij}$$

and, by (4.3), we have the desired inequality (4.2).

The case of the equality is obvious by the identity (4.3) and we omit the details. This completes the proof. \square

Remark 4.1. From (4.2), one may deduce the coarser inequality that might be useful in some applications:

$$0 \leq \sum_{i=1}^n \|x_i, z\| - \left\| \sum_{i=1}^n x_i, z \right\| \leq \sqrt{2} \left(\sum_{1 \leq i \leq j \leq n} r_{ij} \right)^{\frac{1}{2}}.$$

Theorem 4.2. *Let $(\mathcal{X}, \langle \cdot, \cdot | \cdot \rangle)$ be a 2-inner product space over the real or complex number field \mathbb{K} . Let $z, x_i \in \mathcal{X}$, $i \in \{1, \dots, n\}$, and $r > 0$ be such that*

$$(4.4) \quad \|x_i - x_j, z\| \leq r,$$

for each $1 \leq i \leq j \leq n$. Then

$$(4.5) \quad \left(\sum_{i=1}^n \|x_i, z\| \right)^2 \leq \left\| \sum_{i=1}^n x_i, z \right\|^2 + \frac{n(n-1)}{2} r^2.$$

The case of the equality holds in (4.5) if and only if

$$\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle = \frac{1}{2} r^2,$$

for each $1 \leq i \leq j \leq n$.

Proof. The inequality (4.4) is obviously equivalent to

$$\|x_i, z\|^2 + \|x_j, z\|^2 \leq 2 \operatorname{Re} \langle x_i, x_j | z \rangle + r^2,$$

for each $1 \leq i \leq j \leq n$. Since

$$2 \|x_i, z\| \|x_j, z\| \leq \|x_i, z\|^2 + \|x_j, z\|^2,$$

for each $1 \leq i \leq j \leq n$, we have

$$\|x_i, z\| \|x_j, z\| - \operatorname{Re} \langle x_i, x_j | z \rangle \leq \frac{1}{2} r^2,$$

for each $1 \leq i \leq j \leq n$.

Applying Theorem 4.1 for $r_{ij} := \frac{1}{2} r^2$ and taking into account that

$$\sum_{1 \leq i \leq j \leq n} r_{ij} = \frac{n(n-1)}{4} r^2,$$

we deduce the desired inequality (4.5). The case of the equality is also obvious by the Theorem 4.1 and we omit the details. This completes the proof. \square

5. APPLICATIONS FOR DETERMINANTAL INTEGRAL INEQUALITIES

Let (Ω, Σ, μ) be a measure space consisting of a set Ω , a σ -algebra Σ of subsets of Ω and a countably additive and positive measure μ on Σ with values in $\mathbb{R} \cup \{\infty\}$. Denote by $L_\rho^2(\Omega)$ the Hilbert space of all real-valued functions f defined on Ω that are 2- ρ -integrable on Ω , i.e.

$$\int_{\Omega} \rho(s) |f(s)|^2 d\mu(s) < \infty,$$

where $\rho : \Omega \rightarrow [0, \infty)$ is a measurable function on Ω .

We can introduce the following 2-inner product on $L_\rho^2(\Omega)$ by the formula

$$\langle f, g|h \rangle_\rho := \frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} \begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t),$$

where, by

$$\begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix},$$

we denote the determinant of the matrix

$$\begin{vmatrix} g(s) & g(t) \\ h(s) & h(t) \end{vmatrix},$$

generating the 2-norm on $L_\rho^2(\Omega)$ expressed by

$$\|f, h\|_\rho := \left(\frac{1}{2} \int_{\Omega} \int_{\Omega} \rho(s) \rho(t) \begin{vmatrix} f(s) & f(t) \\ h(s) & h(t) \end{vmatrix} d\mu(s) d\mu(t) \right)^{\frac{1}{2}}.$$

A simple calculation with integrals reveals that

$$(5.1) \quad \langle f, g|h \rangle_\rho = \begin{vmatrix} \int_{\Omega} \rho f g d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho g h d\mu & \int_{\Omega} \rho h^2 d\mu \end{vmatrix}$$

and

$$(5.2) \quad \|f, h\|_\rho = \left| \begin{vmatrix} \int_{\Omega} \rho f^2 d\mu & \int_{\Omega} \rho f h d\mu \\ \int_{\Omega} \rho f h d\mu & \int_{\Omega} \rho h^2 d\mu \end{vmatrix} \right|^{\frac{1}{2}},$$

where, for simplicity, instead of $\int_{\Omega} \rho(s) f(s) g(s) d\mu(s)$, we have written $\int_{\Omega} \rho f g d\mu$ (see, e.g. [13] or [14]).

Using the representations (5.1), (5.2) and the Theorem 4.1, one may state interesting determinantal integral inequality, as follows.

Proposition 5.1. *Let $f_1, \dots, f_n, g, h \in L^2_\rho(\Omega)$, where $\rho : \Omega \rightarrow [0, \infty)$ is a measurable function on Ω and $r_{ij} > 0$ such that*

$$\left| \begin{array}{cc} \int_{\Omega} \rho(f_i - f_j)^2 d\mu & \int_{\Omega} \rho(f_i - f_j) h d\mu \\ \int_{\Omega} \rho(f_i - f_j) h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right|^{\frac{1}{2}} \leq r,$$

for each $1 \leq i \leq j \leq n$, then

$$\left(\sum_{i=1}^n \left| \begin{array}{cc} \int_{\Omega} \rho f_i^2 d\mu & \int_{\Omega} \rho f_i h d\mu \\ \int_{\Omega} \rho f_i h d\mu & \int_{\Omega} \rho h^2 d\mu \end{array} \right|^{\frac{1}{2}} \right)^2 \leq \left| \begin{array}{cc} \sum_{i=1}^n \int_{\Omega} \rho f_i^2 d\mu & \sum_{i=1}^n \int_{\Omega} \rho f_i h d\mu \\ \sum_{i=1}^n \int_{\Omega} \rho f_i h d\mu & \sum_{i=1}^n \int_{\Omega} \rho h^2 d\mu \end{array} \right| + 2 \sum_{1 \leq i < j \leq n} r_{ij}.$$

Proof. The proof follows by Theorem 4.1, applied for the 2-inner product $\langle \cdot, \cdot \rangle_\rho$ and we omit the details. \square

Similar determinantal integral inequalities may be stated if one uses the other results for 2-inner products obtained above, but we do not present them here.

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