Kragujevac Journal of Mathematics
Volume 41(1) (2017), Pages 93–103.

RECTIFYING SUBMANIFOLDS OF RIEMANNIAN MANIFOLDS AND TORQUED VECTOR FIELDS

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Abstract. Recently, the author defined and classified rectifying submanifolds in Euclidean spaces in [12]; extending his earlier work on rectifying curves in Euclidean 3-space done in [6]. In this article, first the author introduces the notion of rectifying submanifolds in an arbitrary Riemannian manifold. Then he defines torqued vector fields on Riemannian manifolds and classifies Riemannian manifolds which admit a torqued vector field. Finally, he characterizes and studies rectifying submanifolds in a Riemannian manifold equipped with a torqued vector field. Some related results and applications are also presented.

1. Introduction

Let $\mathbb{E}^3$ denote Euclidean 3-space with its inner product $\langle\ ,\ \rangle$. Consider a unit-speed space curve $x : I \to \mathbb{E}^3$, where $I = (a, b)$ is a real interval. Let $\mathbf{x}$ denote the position vector field of the curve and let $\mathbf{x}'$ be denoted by $\mathbf{t}$.

It is possible, in general, that $\mathbf{t}'(s) = 0$ for some s; however, we assume that this never happens. Then we can introduce a unique vector field $\mathbf{n}$ and positive function κ so that $\mathbf{t}' = \kappa\mathbf{n}$. We call $\mathbf{t}'$ the *curvature vector field*, $\mathbf{n}$ the *principal normal vector field*, and κ the *curvature* of the curve. Since $\mathbf{t}$ is of constant length, $\mathbf{n}$ is orthogonal to $\mathbf{t}$. The *binormal vector field* is defined by $\mathbf{b} = \mathbf{t} \times \mathbf{n}$, which is a unit vector field orthogonal to both $\mathbf{t}$ and $\mathbf{n}$. One defines the *torsion* τ by the equation $\mathbf{b}' = -\tau\mathbf{n}$.

Key words and phrases. Rectifying submanifolds, torqued vector field, rectifying curves, twisted product, concircular vector field, concurrent vector field.

2010 *Mathematics Subject Classification.* Primary: 53C40. Secondary: 53C42, 53B20.

Received: August 4, 2016.

Accepted: August 16, 2016.

The famous Frenet-Serret equations are given by

$$\begin{cases} \mathbf{t}' = \kappa \mathbf{n}, \\ \mathbf{n}' = -\kappa \mathbf{t} + \tau \mathbf{b}, \\ \mathbf{b}' = -\tau \mathbf{n}. \end{cases} \tag{1.1}$$

At each point of the curve, the planes spanned by $\{\mathbf{t}, \mathbf{n}\}$, $\{\mathbf{t}, \mathbf{b}\}$, and $\{\mathbf{n}, \mathbf{b}\}$ are known as the *osculating plane*, the *rectifying plane*, and the *normal plane*, respectively.

From elementary differential geometry it is well known that a curve in $\mathbb{E}^3$ lies in a plane if its position vector lies in its osculating plane at each point, and lies on a sphere if its position vector lies in its normal plane at each point.

In view of these basic facts, the author asked the following simple geometric question in [6]:

QUESTION: *When does the position vector of a space curve $\mathbf{x} : I \to \mathbb{E}^3$ always lie in its rectifying plane?*

The author called such a curve a *rectifying curve* in [6]. The author derived many fundamental properties of rectifying curves. In particular, he classifies all rectifying curves. It is known that rectifying curves are related with centrodes, constant-ratio curves and convolution manifolds (cf. [3–5, 7, 8]). For a recent survey on rectifying curves, see [13].

In [12], the author extended the notion of rectifying curves to the notion of rectifying submanifolds in a Euclidean space. Rectifying Euclidean submanifolds are characterized and classified in [12].

In this article, first the author introduces the notion of rectifying submanifolds in an arbitrary Riemannian manifold. Then he defines torqued vector fields on Riemannian manifolds and classifies Riemannian manifolds which admit a torqued vector field. Finally, he characterizes and studies rectifying submanifolds in a Riemannian manifold equipped with a torqued vector field. Some related results and applications are also presented.

2. Preliminaries

Let $x : M \to \tilde{M}$ be an isometric immersion of a Riemannian manifold M into another Riemannian manifold $\tilde{M}$. For each point $p \in M$, we denote by T_pM and $T_p^\perp M$ the tangent and the normal spaces at p.

There is a natural orthogonal decomposition:

$$T_p\tilde{M} = T_pM \oplus T_p^\perp M. \tag{2.1}$$

Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections of M and $\tilde{M}$, respectively. The formulas of Gauss and Weingarten are given respectively by (cf. [1, 9])

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y),$$
$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi,$$

for vector fields X, Y tangent to M and ξ normal to M, where h is the second fundamental form, D the normal connection, and A the shape operator of M.

For a given point $p \in M$, the *first normal space*, of M in $\mathbb{E}^m$, denoted by $\operatorname{Im} h_p$, is the subspace defined by

$$\operatorname{Im} h_p = \operatorname{Span}\{h(X,Y) : X, Y \in T_pM\}.$$

For each normal vector ξ at p, the shape operator A_ξ is a self-adjoint endomorphism of T_pM. The second fundamental form h and the shape operator A are related by

$$\langle A_\xi X, Y\rangle = \langle h(X,Y), \xi\rangle,$$

where $\langle\ ,\ \rangle$ is the inner product on M as well as on the ambient space $\tilde{M}$.

The covariant derivative $\bar{\nabla} h$ of h with respect to the connection on $TM \oplus T^\perp M$ is defined by

$$(\bar{\nabla}_X h)(Y,Z) = D_X(h(Y,Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

For a given point $p \in M$, we put

$$\operatorname{Im}(\bar{\nabla} h_p) = \{(\bar{\nabla}_X h)(Y,Z) : X, Y, Z \in T_pM\}.$$

The subspace $\operatorname{Im} \bar{\nabla} h_p$ is called the *second normal space at* p.

The *equation of Codazzi* is

$$(\tilde{R}(X,Y)Z)^\perp = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z),$$

where $(\tilde{R}(X,Y)Z)^\perp$ denotes the normal component of $\tilde{R}(X,Y)Z$.

It follows from the definition of a rectifying curve $x : I \to \mathbb{E}^3$ that the position vector field $\mathbf{x}$ of x satisfies

$$\mathbf{x}(s) = \lambda(s)\mathbf{t}(s) + \mu(s)\mathbf{b}(s),$$

for some functions λ and μ.

For a curve $x : I \to \mathbb{E}^3$ with $\kappa(s_0) \neq 0$ at $s_0 \in I$, the first normal space at s_0 is the line spanned by the principal normal vector $\mathbf{n}(s_0)$. Hence, the rectifying plane at s_0 is nothing but the plane orthogonal to the first normal space at s_0. Therefore, for a submanifold M of $\mathbb{E}^m$ and a point $p \in M$, we call the subspace of $T_p\mathbb{E}^m$, orthogonal complement to the first normal space $\operatorname{Im} \sigma_p$, the *rectifying space of* M at p.

Now, we introduce the following definitions.

Definition 2.1. Let V is a non-vanishing vector field on a Riemannian manifold $\tilde{M}$ and let M be a submanifold of $\tilde{M}$ such that the normal component V^N of V is nowhere zero on M. Then M is called a *rectifying submanifold* (with respect to V) if and only if

$$\langle V(p), \operatorname{Im} h_p\rangle = 0$$

holds at every $p \in M$.

Definition 2.2. A submanifold M of a Riemannian manifold $\tilde{M}$ is called *twisted* if

$$\operatorname{Im} \bar{\nabla} h_p \not\subseteq \operatorname{Im} h_p$$

holds for $p \in M$.

It follows from Definition 2.2 that a curve in $\mathbb{E}^3$ is twisted if and only if the curve has nonzero torsion at every point.

We recall the following definition.

Definition 2.3. The twisted product $B \times_\lambda F$ of two pseudo Riemannian manifolds (B, g_B) and (F, g_F) is the product manifold $B \times F$ equipped with the pseudo Riemannian metric

$$g = g_B + \lambda^2 g_F, \tag{2.2}$$

where λ is a positive function on $B \times F$, which is called the *twisting function.*

In particular, if the function λ in (2.2) depending only B, the it is called a *warped product* and the function λ is called the *warping function.*

Definition 2.4. A twisted product $B \times_\lambda F$ of two pseudo Riemannian manifolds (B, g_B) and (F, g_F) is called *proper* if the twisting function λ cannot be expressed as the product of a function f depending only on B and another function k depending only on F.

3. Torqued Vector Fields

According to K. Yano, a vector field v on a pseudo Riemannian manifold M is called *torse-forming* if it satisfies

$$\nabla_X v = \phi X + \gamma(X) v,$$

for some function ϕ and a 1-form γ and any vector $X \in TM$, where ∇ is the Levi-Civita connection of M (cf. [19,20]). The 1-form γ is called the *generating form* and the function ϕ is called the *conformal scalar* of v (see [16]).

For simplicity, we make the following definition.

Definition 3.1. A *nowhere zero vector field $\mathcal{T}$ on a (pseudo) Riemannian manifold satisfying the following two conditions*

$$\nabla_X \mathcal{T} = \varphi X + \alpha(X)\mathcal{T} \quad \textit{and} \quad \alpha(\mathcal{T}) = 0, \tag{3.1}$$

is called a *torqued vector field.* The function φ and the 1-form α are called the *torqued function* and the *torqued form* of the torqued vector field $\mathcal{T}$, respectively.

A torqued vector field $\mathcal{T}$ is called *proper* if it is not a concircular vector field, i.e., its torqued form is non-trivial.

The main result of this section is the following.

Theorem 3.1. *A Riemannian n-manifold M admits a torqued vector field if and only if it is locally a twisted product $I \times_\lambda F$, where I is an open interval, F is a Riemannian $(n-1)$-manifold and λ is the twisting function.*

Proof. Assume that $\mathcal{T}$ is a torqued vector field on a Riemannian n-manifold M. Let $\rho = |\mathcal{T}|$. Then we have

$$\mathcal{T} = \rho e_1, \tag{3.2}$$

where e_1 is a unit vector field on M. It follows from (3.1) and (3.2) that

$$\rho\varphi e_1 = \nabla_{\mathcal{T}}\mathcal{T} = (\mathcal{T}\rho)e_1 + \rho^2\nabla_{e_1}e_1.$$

Since $\nabla_{e_1}e_1$ is perpendicular to e_1, we find

$$\nabla_{e_1}e_1 = 0 \quad \text{and} \quad \mathcal{T}(\ln\rho) = \varphi. \tag{3.3}$$

The first equation in (3.3) shows that the integrable curves of e_1 are geodesics in M. Thus, if we put $\mathcal{D} = \text{Span}\,\{e_1\}$, then $\mathcal{D}$ is a totally geodesic foliation, i.e., $\mathcal{D}$ is an integrable distribution whose leaves are totally geodesic in M.

We may extend the unit vector field e_1 to a local orthonormal frame $e_1, \dots, e_n$ on M. If we put

$$\nabla_{e_j}e_i = \sum_{k=1}^{n}\omega_i^k(e_j)e_k, \quad i = 1, \dots, n, \tag{3.4}$$

then we have $\omega_i^k = -\omega_k^i$.

Let us put $\mathcal{D}^\perp = \text{Span}\,\{e_2, \dots, e_n\}$. Then we derive from (3.1) and (3.2) that

$$\varphi e_j + \alpha(e_j)\rho e_1 = \nabla_{e_j}\mathcal{T} = (e_j\rho)e_1 + \rho\nabla_{e_j}e_1. \tag{3.5}$$

for $j = 2, \dots, n$. We find from (3.4) and (3.5) that

$$\omega_1^k(e_j) = \frac{\varphi}{\rho}\delta_{jk}, \quad j, k = 2, \dots, n, \tag{3.6}$$

$$\alpha(e_j) = e_j(\ln\rho), \quad j = 2, \dots, n. \tag{3.7}$$

Also, (3.1) and (3.2) give

$$\alpha(e_1) = 0. \tag{3.8}$$

Equation (3.6) implies that $\mathcal{D}^\perp$ is an integrable distribution whose leaves are totally umbilical hypersurfaces of M. Therefore, it follows from a result of R. Ponge and H. Reckziegel [17] that M is locally a twisted product $I \times_\lambda F$, where I is an open interval, F is a Riemannian $(n-1)$-manifold and λ is a positive function on $I \times F$, so that the metric tensor g of M takes the form

$$g = ds^2 + \lambda^2 g_F, \tag{3.9}$$

with $e_1 = \partial/\partial s$.

From (3.2) and (3.3), we know that the torqued function of $\mathcal{T}$ satisfies $\varphi = \partial\rho/\partial s$. Moreover, it follows from (3.7) and (3.8) that the torqued form α of $\mathcal{T}$ is the dual 1-form of $d\pi_F(\nabla(\ln f))$, where $\pi_F : I \times_\lambda F \to F$ is the natural projection and $\nabla(\ln f)$ is the gradient of $\ln f$.

Conversely, suppose that M is the twisted product $I \times_\lambda F$ of an open interval and a Riemannian $(n-1)$-manifold F so that the metric of M is given by (3.9). Then we have (cf. [9, 15, 17])

$$\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} = 0, \quad \nabla_V \frac{\partial}{\partial s} = \left(\frac{\partial \ln \lambda}{\partial s} \right) V, \tag{3.10}$$

for V tangent to F.

Let us put

$$v = \lambda \frac{\partial}{\partial s}. \tag{3.11}$$

Then it follows from (3.10) and (3.11) that

$$\nabla_{\frac{\partial}{\partial s}} v = \left(\frac{\partial \lambda}{\partial s} \right) \frac{\partial}{\partial s} \tag{3.12}$$

$$\nabla_V v = (V\lambda) \frac{\partial}{\partial s} + \left(\frac{\partial \lambda}{\partial s} \right) V, \quad V \perp \frac{\partial}{\partial s}. \tag{3.13}$$

Now, let us define a scalar function φ on M by

$$\varphi = \frac{\partial \lambda}{\partial s} \tag{3.14}$$

and define a 1-form α on M by

$$\begin{cases} \alpha(\frac{\partial}{\partial s}) = 0, \\ \alpha(V) = V(\ln \lambda), \quad \text{if } V \perp \frac{\partial}{\partial s}. \end{cases} \tag{3.15}$$

Then we obtain from (3.11)–(3.15) that

$$\nabla_X v = \varphi X + \alpha(X) v, \quad \text{for all } X \in TM,$$
$$\alpha(v) = 0,$$

Consequently, the twisted product $I \times_\lambda F$ admits a torqued vector field given by $v = \lambda \frac{\partial}{\partial s}$. □

By applying the same method as given in the proof of Theorem 3.1, we also have the following result.

Theorem 3.2. *A Lorentzian n-manifold M admits a time-like torqued vector field if and only if it is locally a twisted product $I \times_\lambda F$, where I is an open interval, F is a Riemannian $(n-1)$-manifold so that the metric of M takes the form*

$$g = -ds^2 + \lambda^2 g_F,$$

where λ is a positive function on $I \times F$.

Remark 3.1. Theorem 3.2 can be regarded as an extension of Theorem 1 of [10].

Remark 3.2. It is well-known in elementary differential geometry that, for a given non-closed simple curve γ in a surface, the surface admits a geodesic coordinate patch along γ. Because every geodesic coordinate patch on a surface is a twisted product metric on the surface, twisted products do exist extensively. Consequently, there exist ample examples of Riemannian manifolds which admit a torqued vector field according to Theorem 3.1.

On the other hand, it was known in [15] that if a twisted product $B \times_f F$ of (B, g_B) and (F, g_F) with $\dim F > 1$ is an Einstein manifold, then it is non-proper. In other word, $B \times_f F$ can be expressed as a warped product $B \times_\Phi F$ of (B, g_B) and $(F, \widetilde{g}_F)$ with a warping function Φ, where $\widetilde{g}_F$ is a conformally metric tensor to g_F. Consequently, not every Riemannian manifold can be expressed locally as a proper twisted product.

Remark 3.3. On a twisted product manifold $I \times_\lambda F$, let us consider a coordinate system $\{s, u_2, \ldots, u_n\}$ such that the metric is given by

$$g = \pm ds^2 + \lambda^2 g_F,$$

with $g_F = \sum_{j,k=2}^n g_{jk}^F du_j du^k$. We may regard such a coordinate system as a geodesic coordinate patch on the twisted product $I \times_\lambda F$.

A vector field on a Riemannian manifold M is called a *gradient vector field* if it is the gradient ∇f of some function f on M.

For torqued vector fields, we also have the following.

Proposition 3.1. *If a torqued vector field on a Riemannian manifold M is a gradient vector field, then it is a concircular vector field.*

Proof. Let $\mathcal{T}$ be a torqued vector field on a Riemannian manifold (M, g) with torqued function φ and torqued form α. Then it satisfied the two conditions in (3.1). Let us assume that $\mathcal{T}$ is the gradient vector field ∇f for a function f on M. Then the Hessian H^f of f satisfies

$$\begin{aligned} H^f(X, Y) &= XYf - (\nabla_X Y)f = Xg(Y, \nabla f) - g(\nabla_X Y, \nabla f) \\ &= g(Y, \nabla_X(\nabla f)) = g(Y, \nabla_X \mathcal{T}) = \varphi g(X, Y) + \alpha(X) g(Y, \mathcal{T}). \end{aligned} \tag{3.16}$$

Since the Hessian $H^f(X, Y)$ of f is a symmetric in X and Y, it follows from (3.16) that

$$\alpha(X) g(Y, \mathcal{T}) = \alpha(Y) g(X, \mathcal{T}), \tag{3.17}$$

for vector fields X and Y tangent to M.

If we choose the vector fields X and Y in such way that $X = \mathcal{T}$ and $Y \perp \mathcal{T}$, then we find from (3.17) that

$$0 = \alpha(Y) g(\mathcal{T}, \mathcal{T}).$$

Since the torqued vector field $\mathcal{T}$ is nowhere zero according to Definition 3.1, we obtain from (3.17) that $\alpha(Y) = 0$ for any Y perpendicular to $\mathcal{T}$. Therefore, after combining this with the second condition in (3.1), we get $\alpha = 0$. Consequently, the torqued vector field $\mathcal{T}$ is a concircular vector field. $\square$

4. Characterization of Rectifying Submanifolds with Respect to a Torqued Field

Let $\tilde{M}$ be a Riemannian m-manifold equipped with a torqued vector field $\mathfrak{T}$. We have the following very simple characterization of rectifying submanifolds of $\tilde{M}$ with respect to $\mathfrak{T}$.

Theorem 4.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ endowed with a torqued vector field $\mathfrak{T}$. If the tangential component $\mathfrak{T}^T$ of $\mathfrak{T}$ is nonzero on M, then M is a rectifying submanifold (with respect to $\mathfrak{T}$) if and only if $\mathfrak{T}^T$ is torse-forming vector field on M whose conformal scalar is the restriction of the torqued function and whose generating form is the restriction of the torqued form of $\mathfrak{T}$ on M.*

Proof. Let M be a submanifold of a Riemannian manifold $\tilde{M}$ endowed with a torqued vector field $\mathfrak{T}$. Consider the orthogonal decomposition

$$\mathfrak{T} = \mathfrak{T}^T + \mathfrak{T}^N, \quad \text{[with respect to (2.1)]} \tag{4.1}$$

of $\mathfrak{T}$ restricted to M, where $\mathfrak{T}^T$ and $\mathfrak{T}^N$ are the tangential and normal components of $\mathfrak{T}$, respectively.

From (3.1), (4.1) and the formulas of Gauss and Weingarten, we find

$$\varphi X + \alpha(X)\mathfrak{T} = \tilde{\nabla}_X \mathfrak{T} = \nabla_X \mathfrak{T}^T + h(X, \mathfrak{T}^T) - A_{\mathfrak{T}^N} X + D_X \mathfrak{T}^N \tag{4.2}$$

for any $X \in TM$. After comparing the tangential components in (4.2), we obtain

$$A_{\mathfrak{T}^N} X = \nabla_X \mathfrak{T}^T - \varphi X - \alpha(X)\mathfrak{T}^T. \tag{4.3}$$

Similarly, by comparing the normal components in (4.2) we find

$$D_X \mathfrak{T}^N = \alpha(X)\mathfrak{T}^N - h(X, \mathfrak{T}^T).$$

Now, assume that M is a rectifying submanifold with respect to $\mathfrak{T}$. Then by Definition 2.1 we know that $\mathfrak{T}^N$ is nonzero on M. Moreover, it follows from the Definition 2.1 that

$$\langle \mathfrak{T}, h(X,Y) \rangle = 0, \quad \text{for all } X, Y \in TM, \tag{4.4}$$

which gives $A_{\mathfrak{T}^N} = 0$. Hence (4.3) yields

$$\nabla_X \mathfrak{T}^T = \varphi X + \alpha(X)\mathfrak{T}^T.$$

Therefore $\mathfrak{T}^T$ is a torse-forming vector field on M whose conformal scalar is the restriction of the torqued function of $\mathfrak{T}$ on M and whose generating form is the restriction of the torqued form of $\mathfrak{T}$ on M.

Conversely, let us assume that $\mathfrak{T}^T$ is a torse-forming vector field on M whose conformal scalar is the restriction of the torqued function of $\mathfrak{T}$ on M and whose generating form is the restriction of the torqued form of $\mathfrak{T}$ on M. Then we have

$$\nabla_X \mathfrak{T}^T = \varphi X + \alpha(X)\mathfrak{T}^T, \tag{4.5}$$

for any vector X tangent to M. Therefore, after combining (4.5) with (4.3) we obtain $A_{\mathcal{T}^N} = 0$. Hence we obtain (4.4). Consequently, we conclude that M is a rectifying submanifold of $\tilde{M}$ with respect to $\mathcal{T}$. ☐

5. Some Applications

Recall that a nonzero vector field Z on a Riemannian manifold $\tilde{M}$ is called a *concircular vector field* if it satisfies

$$\tilde{\nabla}_X Z = \phi X, \quad \text{for all } X \in T\tilde{M}, \tag{5.1}$$

where ϕ is a function and $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{M}$. The function ϕ is called the *concircular function* of Z (see [10] for a brief history on concircular vector fields). Obviously, a torqued vector field with trivial torqued form, i.e., with $\alpha = 0$, is a *concircular vector field.*

A concircular vector field is called a *concurrent vector field* if the function ϕ in (5.1) is equal to one (see, e.g., [21]).

The following result follows easily from Theorem 4.1.

Theorem 5.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ endowed with a concircular vector field $Z \neq 0$ with $Z^T \neq 0$ on M. Then M is a rectifying submanifold with respect to Z if and only if the tangential component Z^T of Z is a concircular vector field with the concircular function given by the restriction of the concircular function of Z on M.*

Proof. Let M be a submanifold of a Riemannian manifold $\tilde{M}$ endowed with a concircular vector field $Z \neq 0$ such that $Z^T \neq 0$ on M. Clearly, the concircular vector field Z is a torqued vector field with a trivial torqued form.

Suppose that M is a rectifying submanifold of $\tilde{M}$ with respect to Z. Then it follows from Theorem 4.1 that the tangential component Z^T of Z on M is a torse-forming vector field whose conformal scalar is the restriction of the torqued function of Z on M and whose generating form is the trivial 1-form, since Z has trivial torqued form.

Consequently, Z^T is a concircular vector field whose concircular function is the restriction of the concircular function of Z on M.

The converse is easy to verify. ☐

The following result is an immediate consequence of Theorem 5.1.

Corollary 5.1. *Let M be a submanifold of a Riemannian manifold $\tilde{M}$ endowed with a concurrent vector field $Z \neq 0$ such that $Z^T \neq 0$ on M. Then M is a rectifying submanifold with respect to Z if and only if the tangential component Z^T of Z is a concurrent vector field on M.*

6. Basic Properties of Rectifying Submanifolds with Respect to a Concircular Vector Field

Finally, we provide two basic properties of rectifying submanifolds of $\tilde{M}$ equipped with a concircular vector field Z.

Proposition 6.1. *Let $\tilde{M}$ be a Riemannian m-manifold endowed with a concircular vector field Z. If M is a rectifying submanifold of $\tilde{M}$ with respect to Z, then we have:*

(1) *Z^N is of constant length $\neq 0$;*
(2) *the concircular function φ of Z^T is given by $\varphi = Z^T(\ln \rho)$, where $\rho = |Z^T|$.*

Proof. Let $\tilde{M}$ be a Riemannian m-manifold endowed with a concircular vector field Z. Assume that M is a rectifying submanifold of $\tilde{M}$ with respect to Z. Then, by definition, we have $Z^N \neq 0$.

Now, by applying (5.1), equation (4.2) reduces to

$$\varphi X = \nabla_X Z^T + h\big(X, Z^T\big) - A_{Z^N} X + D_X Z^N. \tag{6.1}$$

By comparing the normal components in (6.1), we find

$$D_X Z^N = -h(X, Z^T). \tag{6.2}$$

for any $X \in TM$.

It follows from (4.4) and (6.2) that $\big\langle Z, D_X Z^N \big\rangle = 0$. Hence we obtain

$$X\big\langle Z^N, Z^N \big\rangle = 0,$$

which implies that Z^N is of constant length $\neq 0$. This proves statement (1).

Since $A_{Z^N} = 0$ on M, we find from (6.1) that

$$\nabla_X Z^T = \varphi X \tag{6.3}$$

for X tangent to M. If we put $\rho = |Z^T|$, then we have

$$Z^T = \rho e_1, \tag{6.4}$$

where e_1 is a unit vector field tangent to M.

From (6.3) with $X = e_1$ and (6.4) we find

$$\varphi e_1 = \nabla_{e_1} Z^T = (e_1 \rho) e_1 + \rho \nabla_{e_1} e_1,$$

which implies that

$$\varphi = e_1 \rho, \quad \nabla_{e_1} e_1 = 0. \tag{6.5}$$

Also, it follows from (6.3) with $X = V \perp e_1$ and (6.4) that

$$V\rho = 0, \quad \text{for all } V \perp e_1. \tag{6.6}$$

Now, it follows from (6.5) and (6.6) that there exists a function s on M such that $\rho = \rho(s)$ and $e_1 = \partial/\partial s$. Hence we obtain $\varphi = \rho'(s)$. Consequently, by (6.4) we obtain statement (2). ∎

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