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ON A CAPUTO FRACTIONAL DIFFERENTIAL INCLUSION WITH INTEGRAL BOUNDARY CONDITION FOR CONVEX-COMPACT AND NONCONVEX-COMPACT VALUED MULTIFUNCTIONS

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Abstract. In this paper, we investigate a Caputo fractional differential inclusion with integral boundary condition under different conditions. First, we investigate it for L^1-Caratheodory convex-compact valued multifunction. Then, we investigate it for nonconvex-compact valued multifunction via some conditions. Also we give two examples to illustrate our results.

1. Introduction

As we know, it has been published many papers about the existence of solution for different fractional differential equations (see for example, [8–16] and the references there in). Also, it has been appeared many works on fractional differential inclusions (see for example, [1–7, 18–23, 26, 29, 31–33, 36] and the references there in). One can find more details about necessary notions in [25, 30, 34]. Let $\eta, \nu, \beta \in (0,1)$ and $\alpha \in (1,2]$ be such that $\Gamma(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0$ and $\alpha - \beta > 1$. In this paper, we investigate the existence of solutions for the Caputo fractional differential inclusion

$$^cD^\alpha x(t) \in F(t, x(t), {}^c D^\beta x(t), x'(t)), \tag{1.1}$$

for almost all $t \in [0,1]$, via the integral boundary value conditions

$$x(0) + x'(0) + {}^cD^\beta x(0) = \int_0^\eta x(s)ds$$

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and

$$x(1) + x'(1) + {}^cD^\beta x(1) = \int_0^\nu x(s)ds,$$

where $F : [0,1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a compact valued multifunction and ${}^cD^\alpha$ is the Caputo differential operator of order $\alpha \in (1,2]$, that is, ${}^cD^\alpha x(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{x''(s)}{(t-s)^{\alpha-1}} ds$. Let (X,d) be a metric space. It is well known that the Pompeiu-Hausdorff metric (see [17]) $H_d : 2^X \times 2^X \to [0,\infty)$ is defined by

$$H_d(A,B) = \max\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\},$$

where $d(A,b) = \inf_{a\in A} d(a,b)$. Then $(CB(X), H_d)$ is a metric space and $(C(X), H_d)$ is a generalized metric space, where $CB(X)$ is the set of closed and bounded subsets of X and $C(X)$ is the set of closed subsets of X ([26]). Denote the set of compact and convex subsets of X by $P_{cp,cv}(X)$. Let $T : X \to 2^X$ be a multifunction. An element $x \in X$ is called an fixed point of T whenever $x \in Tx$ [24]. A multifunction $T : X \to C(X)$ is called a contraction whenever there exists $\gamma \in (0,1)$ such that $H_d(N(x), N(y)) \leq \gamma d(x,y)$ for all $x, y \in X$. In 1970, Covitz and Nadler proved that each closed valued contractive multifunction on a complete metric space has a fixed point [21]. A multifunction $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t))$ is measurable for all $y \in \mathbb{R}$, where $J = [0,1]$ [22]. We say that $F : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is a Caratheodory multifunction whenever $t \mapsto F(t,x,y,z)$ is measurable for all $x,y,z \in \mathbb{R}$ and $(x,y,z) \mapsto F(t,x,y,z)$ is upper semi-continuous for almost all $t \in J$ [7,22,26]. Also, a Caratheodory multifunction $F : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ is called L^1-Caratheodory whenever for each $\rho > 0$ there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\| F(t,x,y,z) \| = \sup\{|v| : v \in F(t,x,y,z)\} \leq \phi_\rho(t),$$

for all $|x|, |y|, |z| \leq \rho$ and for almost all $t \in J$ [7, 22, 26]. By using main idea of [5, 6, 31, 36], we define the set of selections of F by

$$S_{F,x} := \{v \in AC[0,1](J,\mathbb{R}) : \ v(t) \in F(t, x(t), {}^cD^\beta x(t), x'(t)) \text{ for almost all } t \in J\},$$

for all $x \in C(J,\mathbb{R})$. Let E be a nonempty closed subset of a Banach space X and $G : E \to 2^X$ a multifunction with nonempty closed values. We say that the multifunction G is lower semi-continuous whenever the set $\{y \in E : G(y) \cap B \neq \emptyset\}$ is open for all open set B in X [24]. It has been proved that each completely continuous multifunction is lower semi-continuous [24]. Denote by $AC[0,1]$ the space of all the absolutely continuous functions defined on $[0,1]$. Let $AC^2[0,1] = \{w \in C^1[0,1] : w' \in L[0,1]\}$. Recall that $I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x(s)}{(t-s)^{1-\alpha}} ds$ is said to be the Riemann-Liouville fractional integral of order α (for more details see [34, 37]). We use the followings in our main results.

Lemma 1.1 ([28]). *Let X be a Banach space, $F : J \times X \to P_{cp,cv}(X)$ an L^1-Caratheodory multifunction and Θ a linear continuous mapping from $L^1(J,X)$ to*

$C(J,X)$. *Then the operator* $\Theta o S_F : C(J,X) \to P_{cp,cv}(C(J),X)$ *defined by* $(\Theta o S_F)(x) = \Theta(S_{F,x})$ *is a closed graph operator in* $C(J,X) \times C(J,X)$.

Lemma 1.2 ([24]). *Let* E *be a Banach space,* C *a closed convex subset of* E, U *an open subset of* C *and* $0 \in U$. *Suppose that* $F : \overline{U} \to P_{cp,cv}(C)$ *is a upper semi-continuous compact map, where* $P_{cp,cv}(C)$ *denotes the family of nonempty, compact convex subsets of* C. *Then either* F *has a fixed point in* $\overline{U}$ *or there exist* $u \in \partial U$ *and* $\lambda \in (0,1)$ *such that* $u \in \lambda F(u)$.

2. Main Results

Now, we are ready to provide our results about the existence of solutions of the inclusion problem (1.1). Define $x_v(t) = I^\alpha v(t) - c_{0v} - c_{1v}t$, where

$$\begin{aligned} c_{0v} = &- \frac{1}{\Gamma(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m)dmds \\ &+ \frac{(2-\eta^2)(\nu-1)}{2\gamma\Gamma(\alpha)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m)dmds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds \\ &+ \frac{(2-\eta^2)(1-\eta)}{2\gamma\Gamma(\alpha)} \int_0^\nu \int_0^s (s-m)^{\alpha-1} v(m)dmds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} v(s)ds + \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)ds \end{aligned}$$

and

$$\begin{aligned} c_{1v} = &- \frac{(1-\nu)t}{\gamma\Gamma(\alpha)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m)dmds - \frac{(1-\eta)t}{\gamma\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds \\ &- \frac{(\eta-1)t}{\gamma\Gamma(\alpha)} \int_0^\nu \int_0^s (s-m)^{\alpha-1} v(m)dmds - \frac{(1-\eta)t}{\gamma\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} v(s)ds \\ &- \frac{(1-\eta)t}{\gamma\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)ds. \end{aligned}$$

It is easy to check that $x_v \in AC^2[0,1]$ is well-define and x_v', ${}^cDx_v$, $\int_0^\eta x_v(s)ds$ exist whenever $v \in AC[0,1]$ (see [27]).

Lemma 2.1. *Let* $v \in AC[0,1]$, $\beta, \eta, \nu \in (0,1)$ *and* $\alpha \in (1,2]$ *with* $\alpha - \beta > 1$ *and*

$$\Gamma(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0.$$

Then $x_v(t)$ *is the unique solution for the problem* ${}^cD^\alpha x(t) = v(t)$ *with the integral boundary value conditions* $x(0) + x'(0) + {}^cD^\beta x(0) = \int_0^\eta x(s)ds$ *and* $x(1) + x'(1) + {}^cD^\beta x(1) = \int_0^\nu x(s)ds$.

Proof. It is known that the general solution of the equation ${}^cD^\alpha x(t) = v(t)$ is

$$x(t) = I^\alpha v(t) - a_0 - a_1 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds - a_0 - a_1 t,$$

where a_0, a_1 are arbitrary constants and $t \in J$ [25, 34]. Thus,

$${}^cD^\beta x(t) = I^{\alpha-\beta} v(t) - \frac{t^{1-\beta} a_1}{\Gamma(2-\beta)} = \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} v(s) ds - \frac{t^{1-\beta} a_1}{\Gamma(2-\beta)}$$

and $x'(t) = I^{\alpha-1} v(t) - a_1 = \frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} v(s) ds - a_1$. Hence by using an easy calculation, we get $x(0) + {}^cD^\beta x(0) + x'(0) = -a_0 - a_1$ and

$$\begin{aligned} x(1) +^c D^\beta x(1) + x'(1) = &\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ &+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} v(s) ds \\ &\times \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s) ds - \frac{\Gamma(2) a_1}{\Gamma(2-\beta)} - 2a_1 - a_0. \end{aligned}$$

By using the boundary conditions, we obtain

$$a_0(\eta - 1) - a_1 \left(\frac{\eta^2}{2} - 1 \right) = \frac{1}{\Gamma(\alpha)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m) dm ds$$

and

$$\begin{aligned} &a_0(\nu - 1) + a_1 \left(\frac{\nu^2}{2} - 2 - \frac{\Gamma(2)}{\Gamma(2-\beta)} \right) \\ = &- \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ &- \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} v(s) ds - \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s) ds \\ &+ \frac{1}{\Gamma(\alpha)} \int_0^\nu \int_0^s (s-m)^{\alpha-1} v(m) dm ds. \end{aligned}$$

Thus,

$$\begin{aligned} a_0 = c_{0v} = &- \frac{1}{\Gamma(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m) dm ds \\ &+ \frac{(2-\eta^2)(\nu-1)}{2\gamma\Gamma(\alpha)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m) dm ds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s) ds \\ &+ \frac{(2-\eta^2)(1-\eta)}{2\gamma\Gamma(\alpha)} \int_0^\nu \int_0^s (s-m)^{\alpha-1} v(m) dm ds \end{aligned}$$

$$+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}v(s)ds$$
$$+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)ds$$

and

$$\begin{aligned}a_1=a_{1v}= & -\frac{(1-\nu)t}{\gamma\Gamma(\alpha)}\int_0^\eta\int_0^s(s-m)^{\alpha-1}v(m)dmds-\frac{(1-\eta)t}{\gamma\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}v(s)ds\\ & -\frac{(\eta-1)t}{\gamma\Gamma(\alpha)}\int_0^\nu\int_0^s(s-m)^{\alpha-1}v(m)dmds\\ & -\frac{(1-\eta)t}{\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}v(s)ds-\frac{(1-\eta)t}{\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)ds.\end{aligned}$$

Hence,

$$\begin{aligned}x(t)= & x_vt\\ = & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}v(s)ds+\frac{1}{\Gamma(\alpha)(1-\eta)}\int_0^\eta\int_0^s(s-m)^{\alpha-1}v(m)dmds\\ & +\frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma(\alpha)}\int_0^\eta\int_0^s(s-m)^{\alpha-1}v(m)dmds\\ & +\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}v(s)ds\\ & +\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma(\alpha)}\int_0^\nu\int_0^s(s-m)^{\alpha-1}v(m)dmds\\ & +\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}v(s)ds\\ & +\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)ds\\ & +\frac{(1-\nu)t}{\gamma\Gamma(\alpha)}\int_0^\eta\int_0^s(s-m)^{\alpha-1}v(m)dmds\\ & +\frac{(1-\eta)t}{\gamma\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}v(s)ds+\frac{(\eta-1)t}{\gamma\Gamma(\alpha)}\int_0^\nu\int_0^s(s-m)^{\alpha-1}v(m)dmds\\ & +\frac{(1-\eta)t}{\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}v(s)ds+\frac{(1-\eta)t}{\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)ds\\ = & I^\alpha v(t)-c_{0v}-c_{1v}t.\end{aligned}$$

Conversely, it is clear that $x_v'(t)=I^{\alpha-1}v(t)+c_{1v}$ and $x_v''(t)=(I^{\alpha-1}v(t))'={}^RD^{2-\alpha}v(t)$ for almost all $t\in J$. Since $2-\alpha\in(0,1]$, we get

$$^cD^\alpha x_v(t)=I^{2-\alpha}x_v''(t)=I^{2-\alpha}({}^RD^{2-\alpha}v(t))=v(t).$$

Similar to the last part, we obtain

$$x_v(0) + x_v'(0) +^c D^\beta x_v(0) = -c_{0v} - c_{1v} = \int_0^\eta x(s)ds$$

and

$$\begin{aligned} x_v(1) + x_v'(1) +^c D^\beta x_v(1) = &\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds \\ &+ \frac{1}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} v(s)ds \\ &\times \frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-s)^{\alpha-2} v(s)ds - \frac{\Gamma(2)a_1}{\Gamma(2-\beta)} - 2c_{1v} - c_{0v} \\ = &\int_0^\nu x(s)ds. \end{aligned}$$

This completes the proof. □

An element $x \in AC^2([0,1],\mathbb{R})$ is called a solution of the problem (1.1) whenever it satisfies the integral boundary conditions and there exists a function $v \in S_{F,x}$ such that

$$x(t) = I^\alpha v(t) - c_{0v} - c_{1v}t,$$

for all $t \in J$. Let $\mathcal{X} = \{x : x, x',^c D^\beta x \in C(J,\mathbb{R}) \text{ for all } \beta \in (0,1)\}$ endowed with the norm $\|x\| = \sup_{t\in J} |x(t)| + \sup_{t\in J} |^c D^\beta x(t)| + \sup_{t\in J} |x'(t)|$. Then, $(\mathcal{X}, \|.\|)$ is a Banach space [35]. For investigation of the problem (1.1) we provide two different methods.

Theorem 2.1. *Suppose that $F : J \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to P_{cp,cv}(\mathbb{R})$ is a L^1-Caratheodory multifunction and there exist a bounded continuous non-decreasing map $\psi : [0,\infty) \to (0,\infty)$ and a continuous function $p : J \to (0,\infty)$ such that*

$$\|F(t, x(t),^c D^\beta x(t), x'(t))\| = \sup\{|v| : \ v \in F(t, x(t),^c D^\beta x(t), x'(t))\} \le p(t)\psi(\|x\|),$$

for all $t \in J$ and $x \in \mathcal{X}$. Then the inclusion problem (1.1) *has at least one solution.*

Proof. Define the operator $N : \mathcal{X} \to 2^{\mathcal{X}}$ by

$$N(x) = \left\{ h \in \mathcal{X} : \text{ there exists } v \in S_{F,x} \text{ such that } h(t) = I^\alpha v(t) - c_{0v} - c_{1v}t, \ t \in J \right\}.$$

We show that the operator N has a fixed point. First, we show that N maps bounded sets of $\mathcal{X}$ into bounded sets. Suppose that $r > 0$ and $B_r = \{x \in \mathcal{X} : \|x\| \le r\}$. Let $x \in B_r$ and $h \in N(x)$. Choose $v \in S_{F,x}$ such that $h(t) = I^\alpha v(t) - c_{0v} - c_{1v}t$ for almost all $t \in J$. Thus,

$$\begin{aligned} |h(t)| \le &\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)|ds + \frac{1}{\Gamma(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} |v(m)|dmds \\ &+ \left| \frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma(\alpha)} \right| \int_0^\eta \int_0^s (s-m)^{\alpha-1} |v(m)|dmds \end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}|v(s)|ds\\
&+\left|\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma(\alpha)}\right|\int_0^\nu\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta)}\right|\int_0^1(1-s)^{\alpha-\beta-1}|v(s)|ds\\
&+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}|v(s)|ds\\
&+\left|\frac{(1-\nu)t}{\gamma\Gamma(\alpha)}\right|\int_0^\eta\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(1-\eta)t}{\gamma\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}|v(s)|ds+\left|\frac{(\eta-1)t}{\gamma\Gamma(\alpha)}\right|\int_0^\nu\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(1-\eta)t}{\gamma\Gamma(\alpha-\beta)}\right|\int_0^1(1-s)^{\alpha-\beta-1}|v(s)|ds+\left|\frac{(1-\eta)t}{\gamma\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}|v(s)|ds\\
&\le\Lambda_1\|p\|_\infty\psi(\|x\|),
\end{aligned}
$$

$$
\begin{aligned}
|{}^cD^\beta h(t)|\le&\frac{1}{\Gamma(\alpha-\beta)}\int_0^t(t-s)^{\alpha-\beta-1}|v(s)|ds\\
&+\left|\frac{(1-\nu)t^{1-\beta}}{\gamma\Gamma(\alpha)\Gamma(2-\beta)}\right|\int_0^\eta\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma(\alpha)\Gamma(2-\beta)}\right|\int_0^1(1-s)^{\alpha-1}|v(s)|ds\\
&+\left|\frac{(\eta-1)t^{1-\beta}}{\gamma\Gamma(\alpha)\Gamma(2-\beta)}\right|\int_0^\nu\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma(\alpha-\beta)\Gamma(2-\beta)}\right|\int_0^1(1-s)^{\alpha-\beta-1}|v(s)|ds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma(\alpha-1)\Gamma(2-\beta)}\right|\int_0^1(1-s)^{\alpha-2}|v(s)|ds\\
\le&\Lambda_2\|p\|_\infty\psi(\|x\|)
\end{aligned}
$$

and

$$
\begin{aligned}
|h'(t)|\le&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|v(s)|ds+\left|\frac{(1-\nu)}{\gamma\Gamma(\alpha)}\right|\int_0^\eta\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}|v(s)|ds+\left|\frac{(\eta-1)}{\gamma\Gamma(\alpha)}\right|\int_0^\nu\int_0^s(s-m)^{\alpha-1}|v(m)|dmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-\beta)}\right|\int_0^1(1-s)^{\alpha-\beta-1}|v(s)|ds+\left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}|v(s)|ds
\end{aligned}
$$

$$\leq \Lambda_3 \|p\|_\infty \psi(\|x\|)$$

for all $t \in J$, where $\|p\|_\infty = \sup_{t\in J} |p(t)|$,

$$\begin{aligned}
\Lambda_1 = \Bigg[& \frac{1}{\Gamma(\alpha+1)} + \frac{\eta^{\alpha+1}}{\Gamma(\alpha+2)(1-\eta)} + \left|\frac{(\eta^2-2)(\nu-1)\eta^{\alpha+1}}{2\gamma\Gamma(\alpha+2)}\right| + \left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha+1)}\right| \\
& + \left|\frac{(\eta^2-2)(1-\eta)\nu^{\alpha+1}}{2\gamma\Gamma(\alpha+2)}\right| + \left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta+1)}\right| + \left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha)}\right| \\
& + \left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma(\alpha+2)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha+1)}\right| + \left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma(\alpha+2)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-\beta+1)}\right| \\
& + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha)}\right|\Bigg], \\
\Lambda_2 = \Bigg[& \frac{1}{\Gamma(\alpha-\beta+1)} + \left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma(\alpha+2)\Gamma(2-\beta)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha+1)\Gamma(2-\beta)}\right| \\
& + \left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma(\alpha+2)\Gamma(2-\beta)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-\beta+1)\Gamma(2-\beta)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha)\Gamma(2-\beta)}\right|\Bigg], \\
\Lambda_3 = \Bigg[& \frac{1}{\Gamma(\alpha)} + \left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma(\alpha+2)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha+1)}\right| + \left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma(\alpha+2)}\right| + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-\beta+1)}\right| \\
& + \left|\frac{(1-\eta)}{\gamma\Gamma(\alpha)}\right|\Bigg].
\end{aligned}$$

Hence,

$$\|h\| = \max_{t\in J} |h(t)| + \max_{t\in J} |{}^cD^\beta h(t)| + \max_{t\in J} |h'(t)| \leq (\Lambda_1 + \Lambda_2 + \Lambda_3)\|p\|_\infty \psi(\|x\|).$$

Now, we show that N maps bounded sets into equi-continuous subsets of $\mathcal{X}$. Let $x \in B_r$ and $t_1, t_2 \in J$ with $t_1 < t_2$. Then, we have

$$\begin{aligned}
|h(t_2) - h(t_1)| = \Bigg| & \frac{1}{\Gamma(\alpha)} \int_0^{t_2} (t_2 - s)^{\alpha-1} v(s)ds - \frac{1}{\Gamma(\alpha)} \int_0^{t_1} (t_1 - s)^{\alpha-1} v(s)ds \\
& + \frac{(1-\nu)t_2}{\gamma\Gamma(\alpha)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m)dmds \\
& - \frac{(1-\nu)t_1}{\gamma\Gamma(\alpha)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} v(m)dmds \\
& + \frac{(1-\eta)t_2}{\gamma\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds - \frac{(1-\eta)t_1}{\gamma\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} v(s)ds \\
& \times \frac{(\eta-1)t_2}{\gamma\Gamma(\alpha)} \int_0^\nu \int_0^s (s-m)^{\alpha-1} v(m)dmds \\
& - \frac{(\eta-1)t_1}{\gamma\Gamma(\alpha)} \int_0^\nu \int_0^s (s-m)^{\alpha-1} v(m)dmds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(1-\eta)t_2}{\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}v(s)ds\\
&-\frac{(1-\eta)t_1}{\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}v(s)ds\\
&+\frac{(1-\eta)t_2}{\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)ds-\frac{(1-\eta)t_1}{\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}v(s)ds\Bigg|\\
\le{}&\|p\|_\infty\psi(\|x\|)\left[\left|\frac{t_2^\alpha-t_1^\alpha}{\Gamma(\alpha+1)}\right|+\left|\frac{(1-\nu)\eta^{\alpha+1}(t_2-t_1)}{\gamma\Gamma(\alpha+2)}\right|+\left|\frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma(\alpha+1)}\right|\right.\\
&\left.+\left|\frac{(\eta-1)\nu^{\alpha+1}(t_2-t_1)}{\gamma\Gamma(\alpha+2)}\right|+\left|\frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma(\alpha-\beta+1)}\right|+\left|\frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma(\alpha)}\right|\right],
\end{aligned}
$$

$$
\begin{aligned}
|^cD^\beta h(t_2)-{}^cD^\beta h(t_1)|\le\|p\|_\infty\psi(\|x\|)&\left[\left|\frac{t_2^{\alpha-\beta}-t_1^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\right|+\left|\frac{(t_2^{1-\beta}-t_1^{1-\beta})(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma(\alpha+2)\Gamma(2-\beta)}\right|\right.\\
&+\left|\frac{(t_2^{1-\beta}-t_1^{1-\beta})(1-\eta)}{\gamma\Gamma(\alpha+1)\Gamma(2-\beta)}\right|+\left|\frac{(t_2^{1-\beta}-t_1^{1-\beta})(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma(\alpha+2)\Gamma(2-\beta)}\right|\\
&\left.+\left|\frac{(t_2^{1-\beta}-t_1^{1-\beta})(1-\eta)}{\gamma\Gamma(\alpha-\beta+1)\Gamma(2-\beta)}\right|+\left|\frac{(t_2^{1-\beta}-t_1^{1-\beta})(1-\eta)}{\gamma\Gamma(\alpha)\Gamma(2-\beta)}\right|\right]
\end{aligned}
$$

and $|h'(t_2)-h'(t_1)|\le\|p\|_\infty\psi(\|x\|)\frac{(t_2^{\alpha-1}-t_1^{\alpha-1})}{\Gamma(\alpha)}$ for all $h\in N(x)$. Hence, $\lim_{t_2\to t_1}|h(t_2)-h(t_1)|=\lim_{t_2\to t_1}|^cD^\beta h(t_2)-{}^cD^\beta h(t_1)|=\lim_{t_2\to t_1}|h'(t_2)-h'(t_1)|=0$ and so by using the Arzela-Ascoli theorem, N is completely continuous. Now, we show that N has a closed graph. Let $x_n\to x_0$, $h_n\in N(x_n)$ for all n and $h_n\to h_0$. We prove that $h_0\in N(x_0)$. For each n, choose $v_n\in S_{F,x_n}$ such that $h_n(t)=I^\alpha v_n(t)-c_{0v_n}-c_{1v_n}t$ for all $t\in J$. Consider the continuous linear operator $\theta:L^1(J,\mathbb{R})\to\mathfrak{X}$ defined by $\theta(v)(t)=I^\alpha v(t)-c_{0v}-c_{1v}t$. By using Lemma 1.1, θoS_F is a closed graph operator. Since $x_n\to x_0$ and $h_n\in\theta(S_{F,x_n})$ for all n, there exists $v_0\in S_{F,x_0}$ such that $h_0(t)=I^\alpha v_0(t)-c_{0v}-c_{1v_0}t$. Thus, N has a closed graph. In this section, we show that $N(x)$ is convex for all $x\in\mathfrak{X}$. Let $h_1,h_2\in N(x)$ and $w\in[0,1]$. Choose $v_1,v_2\in S_{F,x}$ such that $h_i(t)=I^\alpha v_i(t)-c_{0v_i}-c_{1v_i}t$ for almost all $t\in J$ and $i=1,2$. Then,

$$
\begin{aligned}
&wh_1+(1-w)h_2](t)\\
={}&\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}[wv_1(s)+(1-w)v_2(s)]ds\\
&+\frac{1}{\Gamma(\alpha)(1-\eta)}\int_0^\eta\int_0^s(s-m)^{\alpha-1}[wv_1(m)+(1-w)v_2(m)]dmds\\
&+\frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma(\alpha)}\int_0^\eta\int_0^s(s-m)^{\alpha-1}[wv_1(m)+(1-w)v_2(m)]dmds
\end{aligned}
$$

$$+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}[wv_1(s)+(1-w)v_2(s)]ds$$
$$+\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma(\alpha)}\int_0^{\nu}\int_0^s(s-m)^{\alpha-1}[wv_1(m)+(1-w)v_2(m)]dmds$$
$$+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}[wv_1(s)+(1-w)v_2(s)]ds$$
$$+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}[wv_1(s)+(1-w)v_2(s)]ds$$
$$+\frac{(1-\nu)t}{\gamma\Gamma(\alpha)}\int_0^{\eta}\int_0^s(s-m)^{\alpha-1}[wv_1(m)+(1-w)v_2(m)]dmds$$
$$+\frac{(1-\eta)t}{\gamma\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}[wv_1(s)+(1-w)v_2(s)]ds$$
$$+\frac{(\eta-1)t}{\gamma\Gamma(\alpha)}\int_0^{\nu}\int_0^s(s-m)^{\alpha-1}[wv_1(m)+(1-w)v_2(m)]dmds$$
$$+\frac{(1-\eta)t}{\gamma\Gamma(\alpha-\beta)}\int_0^1(1-s)^{\alpha-\beta-1}[wv_1(s)+(1-w)v_2(s)]ds$$
$$+\frac{(1-\eta)t}{\gamma\Gamma(\alpha-1)}\int_0^1(1-s)^{\alpha-2}[wv_1(s)+(1-w)v_2(s)]ds,$$

for all $t \in J$. Since F has convex values, $S_{F,x}$ is convex and so $wh_1+(1-w)h_2 \in N(x)$. If there exists $\lambda \in (0,1)$ such that $x \in \lambda N(x)$, then there exists $v \in S_{F,x}$ such that $x(t) = I^{\alpha}v(t) - c_{0v} - c_{1v}t$ for almost all $t \in J$. Choose $L > 0$ such that $\frac{L}{(\Lambda_1+\Lambda_2+\Lambda_3)\|p\|_\infty \psi(\|x\|)} > 1$ for all $x \in \mathfrak{X}$. Thus, $\|x\| < L$. Now, put $U = \{x \in \mathfrak{X} : \|x\| < L+1\}$. Note that, there are no $x \in \partial U$ and $\lambda \in (0,1)$ such that $x \in \lambda N(x)$ and the operator $N : \overline{U} \to P_{cp,cv}(\overline{U})$ is upper semi-continuous because it is completely continuous. Now by using Lemma 1.2, N has a fixed point in $\overline{U}$ which is a solution of the inclusion problem (1.1). This completes the proof. □

Here, we provide an example for the result.

Example 2.1. Consider the fractional differential inclusion

$$ {}^cD^{\frac{5}{2}}x(t) \in F(t, x(t), {}^cD^{\frac{1}{2}}x(t), x'(t)), \tag{2.1}$$

with the boundary value conditions

$$x(0)+x'(0)+{}^cD^{\frac{1}{2}}x(0) = \int_0^{\frac{1}{2}} x(s)ds \text{ and } x(1)+x'(1)+{}^cD^{\frac{1}{2}}x(1) = \int_0^{\frac{1}{3}} x(s)ds.$$

Put, $\alpha = \frac{5}{2}$, $\beta = \frac{1}{2}$, $\eta = \frac{1}{2}$, $\nu = \frac{1}{3}$ and consider the multifunction $F : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ defined by

$$F(t,x_1,x_2,x_3) = \left[\cos t + \frac{|x_1|}{1+|x_1|} + \sin x_2, 5 + t^2 + \frac{t}{1+e^{|x_3|}}\right].$$

Note that, $\|F(t,x_1,x_2,x_3)\| = \sup\{|y| : y \in F(t,x_1,x_2,x_3)\} \le 7$. If $p(t)=1$, and $\psi(t)=7$, then one can check that the assumptions of Theorem 2.1 hold and so the problem (2.1) has at least one solution.

Here, we provide another result about the existence of solutions for the problem (1.1) by changing the assumption of convex values for the multifunction.

Theorem 2.2. *Let $m \in C(J,\mathbb{R}^+)$ be such that $\|m\|_\infty(\Lambda_1+\Lambda_2+\Lambda_3) < 1$. Suppose that $F : J\times\mathbb{R}\times\mathbb{R}\times\mathbb{R} \to P_{cp}(\mathbb{R})$ is an integrable bounded multifunction such that the map $t \vdash F(t,x,y,z)$ is measurable and $H_d(F(t,x_1,x_2,x_3), F(t,y_1,y_2,y_3)) \le m(t)(|x_1-y_1|+|x_2-y_2|+|x_3-y_3|)$ for almost all $t \in J$ and $x,y,z,x_1,x_2,x_3,y_1,y_2,y_3 \in \mathbb{R}$. Then the problem $(*)$ has a solution.*

Proof. Note that, the multivalued map $t \vdash F(t,x(t),^c D^B x(t), x'(t))$ is measurable and closed valued for all $x \in \mathfrak{X}$. Hence, it has a measurable selection and so the set $S_{F,x}$ is nonempty. Now, consider the operator $N : \mathfrak{X} \to 2^{\mathfrak{X}}$ defined by

$$N(x) = \left\{ h \in \mathfrak{X} : \text{ there exists } v \in S_{F,x} \text{ such that } h(t) = I^\alpha v(t) - c_{0v} - c_{1v}t,\ t \in J \right\},$$

for all $t \in J$. First, we show that $N(x)$ is a closed subset of $\mathfrak{X}$ for all $x \in \mathfrak{X}$. Let $x \in \mathfrak{X}$ and $\{u_n\}_{n\ge1}$ be a sequence in $N(x)$ with $u_n \to u$. For each n, choose $v_n \in S_{F,x}$ such that $u_n(t) = I^\alpha v_n(t) - c_{0v_n} - c_{1v_n}t$ for almost all $t \in J$. Since F has compact values, $\{v_n\}_{n\ge1}$ has a subsequence which converges to some $v \in L^1(J,\mathbb{R})$. Denote the subsequence again by $\{v_n\}_{n\ge1}$. It is easy to check that $v \in S_{F,x}$ and $u_n(t) \to u(t) = I^\alpha v(t) - c_{0v} - c_{1v}t$ for all $t \in J$. This implies that $u \in N(x)$. Thus, the multifunction N has closed values. Now, we show that N is a contractive multifunction with constant $l := \|m\|_\infty(\Lambda_1+\Lambda_2+\Lambda_3) < 1$. Let $x, y \in \mathfrak{X}$ and $h_1 \in N(y)$. Choose $v_1 \in S_{F,y}$ such that $h_1(t) = I^\alpha v_1(t) - c_{0v_1} - c_{1v_1}t$ for almost all $t \in J$. Since

$$\begin{aligned} &H_d(F(t,x(t),{}^cD^\beta x(t),x'(t)), F(t,y(t),{}^cD^\beta y(t),y'(t)) \\ &\le m(t)(|x(t)-y(t)| + |{}^cD^\beta x(t) - {}^cD^\beta y(t)| + |x'(t)-y'(t)|), \end{aligned}$$

for almost all $t \in J$, there exists $w \in F(t,x(t),^c D^\beta x(t), x'(t))$ such that

$$|v_1(t) - w| \le m(t)(|x(t)-y(t)| + |{}^cD^\beta x(t) -^c D^\beta y(t)| + |x'(t)-y'(t)|),$$

for almost all $t \in J$. Define the multifunction $U : J \to 2^{\mathbb{R}}$ by

$$\begin{aligned} U(t) = \{ &w \in \mathbb{R} : |v_1(t) - w| \le m(t)(|x(t)-y(t)| + |{}^cD^\beta x(t) -^c D^\beta y(t)| \\ &+ |x'(t)-y'(t)|) \ \text{ for almost all } t \in J\}. \end{aligned}$$

It is easy to check that the multifunction $U(.) \bigcap F(.,x(.),^c D^\beta x(.), x'(.))$ is measurable. Thus, we can choose $v_2 \in S_{F,x}$ such that

$$|v_1(t) - v_2(t)| \le m(t)(|x(t)-y(t)| + |{}^cD^\beta x(t) -^c D^\beta y(t)| + |x'(t)-y'(t)|),$$

for almost all $t \in J$. Now, consider $h_2 \in N(x)$ which is defined by $h_2(t) = I^\alpha v(t) - c_{0v_2} - c_{1v_2} t$. Hence, we get

$$\begin{aligned}
|h_1(t) - h_2(t)| \leq & \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\
& + \frac{1}{\Gamma(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds \\
& + \left| \frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma(\alpha)} \right| \int_0^\eta \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds \\
& + \left| \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha)} \right| \int_0^1 (1-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma(\alpha)} \right| \int_0^\nu \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds \\
& + \left| \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-\beta)} \right| \int_0^1 (1-s)^{\alpha-\beta-1} |v(s)| ds \\
& + \left| \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma(\alpha-1)} \right| \int_0^1 (1-s)^{\alpha-2} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{(1-\nu)t}{\gamma\Gamma(\alpha)} \right| \int_0^\eta \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds \\
& + \left| \frac{(1-\eta)t}{\gamma\Gamma(\alpha)} \right| \int_0^1 (1-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{(\eta-1)t}{\gamma\Gamma(\alpha)} \right| \int_0^\nu \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds \\
& + \left| \frac{(1-\eta)t}{\gamma\Gamma(\alpha-\beta)} \right| \int_0^1 (1-s)^{\alpha-\beta-1} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{(1-\eta)t}{\gamma\Gamma(\alpha-1)} \right| \int_0^1 (1-s)^{\alpha-2} |v_1(s) - v_2(s)| ds \\
\leq & \Lambda_1 \|m\|_\infty \|x - y\|,
\end{aligned}$$

$$\begin{aligned}
|{}^cD^\beta h_1(t) -{}^c D^\beta h_2(t)| \leq & \frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{(1-\nu)t^{1-\beta}}{\gamma\Gamma(\alpha)\Gamma(2-\beta)} \right| \int_0^\eta \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds \\
& + \left| \frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma(\alpha)\Gamma(2-\beta)} \right| \int_0^1 (1-s)^{\alpha-1} |v_1(s) - v_2(s)| ds \\
& + \left| \frac{(\eta-1)t^{1-\beta}}{\gamma\Gamma(\alpha)\Gamma(2-\beta)} \right| \int_0^\nu \int_0^s (s-m)^{\alpha-1} |v_1(m) - v_2(m)| dmds
\end{aligned}$$

$$\begin{aligned}
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma(\alpha-\beta)\Gamma(2-\beta)}\right|\int_0^1(1-s)^{\alpha-\beta-1}|v_1(s)-v_2(s)|ds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma(\alpha-1)\Gamma(2-\beta)}\right|\int_0^1(1-s)^{\alpha-2}|v_1(s)-v_2(s)|ds\\
&\leq\Lambda_2\|m\|_\infty\|x-y\|
\end{aligned}$$

and

$$\begin{aligned}
|h_1'(t)-h_2'(t)|\leq&\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}|v_1(s)-v_2(s)|ds\\
&+\left|\frac{(1-\nu)}{\gamma\Gamma(\alpha)}\right|\int_0^\eta\int_0^s(s-m)^{\alpha-1}|v_1(m)-v_2(m)|dmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma(\alpha)}\right|\int_0^1(1-s)^{\alpha-1}|v_1(s)-v_2(s)|ds\\
&+\left|\frac{(\eta-1)}{\gamma\Gamma(\alpha)}\right|\int_0^\nu\int_0^s(s-m)^{\alpha-1}|v_1(m)-v_2(m)|dmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-\beta)}\right|\int_0^1(1-s)^{\alpha-\beta-1}|v_1(s)-v_2(s)|ds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma(\alpha-1)}\right|\int_0^1(1-s)^{\alpha-2}|v_1(s)-v_2(s)|ds\\
&\leq\Lambda_3\|m\|_\infty\|x-y\|
\end{aligned}$$

and so $\|h_1-h_2\|\leq(\Lambda_1+\Lambda_2+\Lambda_3)\|m\|_\infty\|x-y\|=l\|x-y\|$. This implies that the multifunction N is a contraction with closed values. Thus by using the result of Covitz and Nadler, N has a fixed point which is a solution for the inclusion problem (1.1). □

Next example illustrates last result.

Example 2.2. Consider the inclusion problem

$$ {}^cD^{\frac{7}{3}}x(t)\in F(t,x(t),{}^cD^{\frac{1}{3}}x(t),x'(t)), \tag{2.2}$$

with the boundary value conditions

$$x(0)+x'(0)+{}^cD^{\frac{1}{3}}x(0)=\int_0^{\frac{1}{2}}x(s)ds \text{ and } x(1)+x'(1)+{}^cD^{\frac{1}{3}}x(1)=\int_0^{\frac{1}{3}}x(s)ds.$$

Put $\alpha=\frac{7}{3}$, $\beta=\frac{1}{3}$, $\eta=\frac{1}{2}$, $\nu=\frac{1}{3}$ and consider the multifunction $F:J\times\mathbb{R}^3\to 2^{\mathbb{R}}$ defined by

$$F(t,x_1,x_2,x_3)=\left[0,\frac{t\sin x_1}{15(5+3t^2)}+\frac{t|x_2|}{100(1+|x_2|)}+\frac{|x_3|}{100(1+|x_3|)}\right].$$

It is easy to see that

$$H_d\left(F(t,x_1,x_2,x_3),F(t,y_1,y_2,y_3)\right)\le\left(\frac{t}{15(5+30t^2)}+\frac{t}{100}+\frac{1}{100}\right)\sum_{i=1}^{3}|x_i-y_i|,$$

for all $t\in J$ and $x_1,x_2,x_3,y_1,y_2,y_3\in\mathbb{R}$. If $m(t)=\frac{t}{15(5+3t^2)}+\frac{t}{100}+\frac{1}{100}$ for all $t\in J$, then $H\left(F(t,x_1,x_2,x_3),F(t,y_1,y_2,y_3)\right)\le m(t)\sum_{i=1}^{3}|x_i-y_i|$. On the other hand, we have $L=\|m\|_\infty(\Lambda_1+\Lambda_2+\Lambda_3)\le 0.0333\times(3.654+1.263+0.963)\simeq 0.19580<1$. Thus, the assumptions of Theorem 2.2 hold and so the inclusion problem (2.2) has at least one solution.

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