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A GENERALIZATION OF HERMITE-HADAMARD'S INEQUALITY

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Abstract. In literature the Hermite-Hadamard inequality was eligible for many reasons, one of the most surprising and interesting that the Hermite-Hadamard inequality combine the midpoint and trapezoid formulae in an inequality. In this work, a Hermite-Hadamard like inequality that combines the composite trapezoid and composite midpoint formulae is proved. So that, the classical Hermite-Hadamard inequality becomes a special case of the presented result. Some Ostrowski's type inequalities for convex functions are proved as well.

1. Introduction

Let $f : [a, b] \to \mathbb{R}$, be a twice differentiable mapping such that $f''(x)$ exists on (a, b) and $\|f''\|_\infty = \sup_{x\in(a,b)} |f''(x)| < \infty$. Then the midpoint inequality is known as:

$$\left| \int_a^b f(x)\,dx - (b-a) f\left(\frac{a+b}{2}\right) \right| \le \frac{(b-a)^3}{24} \|f''\|_\infty ,$$

and, the trapezoid inequality

$$\left| \int_a^b f(x)\,dx - (b-a) \frac{f(a)+f(b)}{2} \right| \le \frac{(b-a)^3}{12} \|f''\|_\infty ,$$

also hold. Therefore, the integral $\int_a^b f(x)\,dx$ can be approximated in terms of the midpoint and the trapezoidal rules, respectively such as:

$$\int_a^b f(x)\,dx \cong (b-a) f\left(\frac{a+b}{2}\right),$$

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and

$$\int_a^b f(x)\,dx \cong (b-a)\frac{f(a)+f(b)}{2},$$

which are combined in a useful and famous relationship, known as the Hermite-Hadamard's inequality. That is,

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.1}$$

which hold for all convex functions f defined on a real interval $[a,b]$.

The real beginning was (almost) in the last twenty five years, where, in 1992 Dragomir [9] published his article about (1.1). The main result in [9] was

Theorem 1.1. *Let $f:[a,b]$ is convex function one can define the following mapping on $[0,1]$ such as:*

$$H(t) = \frac{1}{(b-a)}\int_a^b f\left(tx+(1-t)\frac{a+b}{2}\right)dx,$$

then:

(a) *H is convex and monotonic non-decreasing on $[0,1]$;*
(b) *one has the bounds for H*

$$\sup_{t\in[0,1]} H(t) = \frac{1}{(b-a)}\int_a^b f(x)\,dx = H(1),$$

and

$$\inf_{t\in[0,1]} H(t) = f\left(\frac{a+b}{2}\right) = H(0).$$

Few years after 1992, many authors have took (a real) attention to the Hermite-Hadamard inequality and sequence of several works under various assumptions for the function involved such as bounded variation, convex, differentiable functions whose n-derivative(s) belong to $L_p[a,b]$; $(1\le p\le\infty)$, Lipschitz, monotonic, etc., have been published. For a comprehensive list of results and excellent bibliography we recommend the interested to refer to [3, 4, 13].

In 1997, Yang and Hong [15], continued on Dragomir result (Theorem 1.1) and they proved the following theorem.

Theorem 1.2. *Suppose that $f:[a,b]\to\mathbb{R}$, is convex and the mapping $F:[0,1]\to\mathbb{R}$ is defined by*

$$F(t) = \frac{1}{b-a}\int_a^b \left[f\left(\frac{1+t}{2}a+\frac{1-t}{2}u\right)+f\left(\frac{1+t}{2}b+\frac{1-t}{2}u\right)\right]du,$$

then:

(a) *the mapping F is convex and monotonic nondecreasing on* $[0,1]$*;*
(b) *we have the bounds*

$$\inf_{t\in[0,1]} F(t,s) = \frac{1}{(b-a)}\int_a^b f(x)\,dx = F(0),$$

$$\sup_{t\in[0,1]} F(t) = \frac{f(a)+f(b)}{2} = F(1).$$

For other closely related results see [1, 5–8, 10–12].

In terms of composite numerical integration, we recall *the composite midpoint rule* [2, p. 202]

$$\int_a^b f(x)\,dx = 2h\sum_{j=0}^{n/2} f(x_{2j}) + \frac{(b-a)}{6}h^2 f''(\mu),$$

for some $\mu \in (a,b)$, where $f \in C^2[a,b]$, n is even, $h = \frac{b-a}{n+2}$ and $x_j = a+(j+1)h$, for each $j = -1, 0, \dots, n+1$; and, *the composite trapezoid rule* [2, p. 203]

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] - \frac{(b-a)}{12}h^2 f''(\mu),$$

for some $\mu \in (a,b)$, where $f \in C^2[a,b]$, $h = \frac{b-a}{n}$ and $x_j = a + jh$, for each $j = 0, 1, \dots, n$.

The main purpose of this work, is to combine the composite trapezoid and composite midpoint formulae in an inequality that is similar to the classical Hermite-Hadamard inequality (1.1) for convex functions defined on a real interval $[a,b]$. In this way, we establish a conventional generalization of (1.1) which is in turn most useful and has a very constructional form.

2. A Generalization of Hermite-Hadamard's Inequality

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, then the double inequality*

$$h\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) \le \int_a^b f(t)\,dt \le \frac{h}{2}\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right], \tag{2.1}$$

holds, where $x_k = a + k\frac{b-a}{n}$, $k = 0, 1, 2, \dots, n$; with $h = \frac{b-a}{n}$, $n \in \mathbb{N}$. The constant '1' in the left-hand side and '$\frac{1}{2}$' in the right-hand side are the best possible for all $n \in \mathbb{N}$. If f is concave then the inequality is reversed.

Nota bene: after revision of this paper, the anonymous referee informed us that the inequality (2.1) was proved in more general case in [14] (see also [13, p. 22–23]). We appreciate this remark from the reviewer.

Proof. Since f is convex on $[a,b]$, then f so is on each subinterval $[x_{j-1},x_j]$, $j=1,\ldots,n$, then for all $t\in[0,1]$, we have

$$f\left(tx_{j-1}+(1-t)\,x_j\right)\le tf\left(x_{j-1}\right)+(1-t)\,f\left(x_j\right). \tag{2.2}$$

Integrating (2.2) with respect to t on $[0,1]$ we get

$$\int_0^1 f\left(tx_{j-1}+(1-t)\,x_j\right)dt\le\frac{f\left(x_{j-1}\right)+f\left(x_j\right)}{2}. \tag{2.3}$$

Substituting $u=tx_{j-1}+(1-t)\,x_j$, in the left hand side of (2.3) , we get

$$\int_{x_{j-1}}^{x_j} f(u)\,du\le\frac{x_j-x_{j-1}}{2}\left(f\left(x_{j-1}\right)+f\left(x_j\right)\right).$$

Taking the sum over j from 1 to n, we get

$$\begin{aligned}
&\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} f(u)\,du\\
&=\int_a^b f(u)\,du\\
&\le\sum_{j=1}^{n}\frac{x_j-x_{j-1}}{2}\left(f\left(x_{j-1}\right)+f\left(x_j\right)\right)\\
&\le\frac{1}{2}\max_j\left\{x_j-x_{j-1}\right\}\cdot\sum_{j=1}^{n}\left(f\left(x_{j-1}\right)+f\left(x_j\right)\right)\\
&=\frac{h}{2}\left[f\left(x_0\right)+f\left(x_1\right)+\sum_{j=2}^{n-1}\left\{f\left(x_{j-1}\right)+f\left(x_j\right)\right\}+f\left(x_{n-1}\right)+f\left(x_n\right)\right]\\
&=\frac{h}{2}\left[f(a)+2\sum_{j=1}^{n-1}f\left(x_j\right)+f(b)\right].
\end{aligned} \tag{2.4}$$

On the other hand, again since f is convex on I_x, then for $t\in[0,1]$, we have

$$\begin{aligned}
f\left(\frac{x_{j-1}+x_j}{2}\right)&=f\left(\frac{tx_j+(1-t)\,x_{j-1}}{2}+\frac{(1-t)\,x_j+tx_{j-1}}{2}\right)\\
&\le\frac{1}{2}\left[f\left(tx_j+(1-t)\,x_{j-1}\right)+f\left((1-t)\,x_j+tx_{j-1}\right)\right].
\end{aligned} \tag{2.5}$$

Integrating inequality (2.5) with respect to t on $[0,1]$ we get

$$\begin{aligned}(2.6)\qquad & f\left(\frac{x_{j-1}+x_j}{2}\right)\\ &\leq \frac{1}{2}\int_0^1 \left[f\left(tx_j+(1-t)\,x_{j-1}\right)+f\left((1-t)\,x_j+tx_{j-1}\right)\right]dt\\ &= \frac{1}{2}\int_0^1 f\left(tx_j+(1-t)\,x_{j-1}\right)dt+\frac{1}{2}\int_0^1 f\left((1-t)\,x_j+tx_{j-1}\right)dt.\end{aligned}$$

By putting $1-t=s$ in the second integral on the right-hand side of (2.6), we have

$$\begin{aligned}(2.7)\qquad & f\left(\frac{x_{j-1}+x_j}{2}\right)\\ &\leq \frac{1}{2}\int_0^1 f\left(tx_j+(1-t)\,x_{j-1}\right)dt+\frac{1}{2}\int_0^1 f\left((1-t)\,x_j+tx_{j-1}\right)dt\\ &= \int_0^1 f\left(tx_j+(1-t)\,x_{j-1}\right)dt.\end{aligned}$$

Substituting $u=tx_j+(1-t)\,x_{j-1}$, in the left hand side of (2.7), and then taking the sum over j from 1 to n, we get

$$h\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right)\leq \int_a^b f(t)\,dt. \tag{2.8}$$

From (2.4) and (2.8), we get the desired inequality (2.1).

To prove the sharpness let (2.1) hold with another constants $C_1, C_2>0$, which gives

$$\begin{aligned}(2.9)\qquad C_1\cdot h\sum_{k=1}^{n} f\left(\frac{x_{k-1}+x_k}{2}\right) &\leq \int_a^b f(t)\,dt\\ &\leq C_2\cdot h\left[f(a)+2\sum_{k=1}^{n-1} f(x_k)+f(b)\right].\end{aligned}$$

Let $f:[a,b]\to\mathbb{R}$ be the identity map $f(x)=x$, then the right-hand side of (2.9) reduces to

$$\begin{aligned}\frac{b^2-a^2}{2} &\le C_2\cdot h\left[a+2\sum_{k=1}^{n-1}x_k+b\right]\\ &= C_2\cdot h\left[a+2\sum_{k=1}^{n-1}\left(a+k\frac{b-a}{n}\right)+b\right]\\ &= C_2\cdot h\left[a+2\sum_{k=1}^{n-1}a+2\frac{b-a}{n}\sum_{k=1}^{n-1}k+b\right]\\ &= C_2\cdot\frac{b-a}{n}\left[a+2\left(n-1\right)a+2\frac{b-a}{n}\cdot\frac{n\left(n-1\right)}{2}+b\right]\\ &= C_2\cdot\left(b-a\right)\left(a+b\right).\end{aligned}$$

It follows that $\frac{1}{2}\le C_2$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.1).

For the left-hand side, we have

$$\begin{aligned}\frac{b^2-a^2}{2} &\ge C_1\cdot h\sum_{k=1}^{n}\frac{x_{k-1}+x_k}{2}\\ &= C_1\cdot h\sum_{k=1}^{n}\left\{a+\left(2k-1\right)\frac{b-a}{2n}\right\}.\\ &= C_1\cdot\frac{b-a}{n}\cdot\left[na+\frac{b-a}{2n}\left(2\cdot\frac{n\left(n+1\right)}{2}-n\right)\right]\\ &= C_1\cdot\frac{b^2-a^2}{2},\end{aligned}$$

which means that $1\ge C_1$, and thus 1 is the best possible constant in the left-hand side of (2.1). Thus the proof of Theorem 2.1 is completely finished. □

Remark 2.1. In Theorem 2.1, if we take $n=1$, then we refer to the original Hermite-Hadamard inequality (1.1).

In viewing of (2.1), next we give direct sharp refinements of Hermite-Hadamard's type inequalities for convex functions defined on a real interval $[a,b]$, according to the number of division 'n' (in our case $n=1,2,3,4$) in Theorem 2.1.

Corollary 2.1. *In Theorem* 2.1, *we have*

(a) *if* $n=1$, *then*

$$\left(b-a\right)f\left(\frac{a+b}{2}\right)\le\int_a^b f\left(t\right)dt\le\left(b-a\right)\frac{f\left(a\right)+f\left(b\right)}{2};$$

(b) *if* $n = 2$, *then*

$$\begin{aligned}&\frac{(b-a)}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right]\\&\leq\int_a^b f(t)\,dt\\&\leq\frac{(b-a)}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right];\end{aligned}$$

(c) *if* $n = 3$, *then*

$$\begin{aligned}&\frac{(b-a)}{3}\left[f\left(\frac{5a+b}{6}\right)+f\left(\frac{a+b}{2}\right)+f\left(\frac{a+5b}{6}\right)\right]\\&\leq\int_a^b f(t)\,dt\\&\leq\frac{(b-a)}{6}\left[f(a)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)+f(b)\right];\end{aligned}$$

(d) *if* $n = 4$, *then*

$$\begin{aligned}&\frac{(b-a)}{4}\left[f\left(\frac{7a+b}{8}\right)+f\left(\frac{5a+3b}{8}\right)+f\left(\frac{3a+5b}{8}\right)+f\left(\frac{a+7b}{8}\right)\right]\\&\leq\int_a^b f(t)\,dt\\&\leq\frac{(b-a)}{12}\left[f(a)+2f\left(\frac{3a+b}{4}\right)+2f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)+f(b)\right].\end{aligned}$$

Now, let $f : [a.b] \to \mathbb{R}$ be a convex function on $[a, b]$. Define the mappings $H_j, F_j : [0, 1] \to \mathbb{R}$, given by

$$H_j(t)=\frac{1}{h}\int_{x_{j-1}}^{x_j} f\left(tu+(1-t)\frac{x_{j-1}+x_j}{2}\right)du, \quad u\in[x_{j-1},x_j], \tag{2.10}$$

and

$$F_j(t)=\frac{1}{h}\int_{x_{j-1}}^{x_j}\left[f\left(\frac{1+t}{2}x_{j-1}+\frac{1-t}{2}u\right)+f\left(\frac{1+t}{2}x_j+\frac{1-t}{2}u\right)\right]du, \tag{2.11}$$

where $u \in [x_{j-1}, x_j]$.

Applying Theorems 1.1 and 1.2, for $f : [x_{j-1}, x_j] \to \mathbb{R}$, $j = 1, 2, \ldots, n$. Then the following statements hold:

(a) $H_j(t)$ and $F_j(t)$ are convex for all $t \in [0, 1]$ and $u \in [x_{j-1}, x_j]$;
(b) $H_j(t)$ and $F_j(t)$ are monotonic nondecreasing for all $t \in [0, 1]$ and $u \in [x_{j-1}, x_j]$.
(c) we have the following bounds for $H_j(t)$

$$\frac{1}{h}\int_{x_{j-1}}^{x_j} f(u)\,du = H_j(1), \tag{2.12}$$

and

$$f\left(\frac{x_{k-1}+x_k}{2}\right)=H_j\left(0\right). \tag{2.13}$$

and the following bounds for $F_j(t)$

$$\frac{f\left(x_{j-1}\right)+f\left(x_j\right)}{2}=F_j\left(1\right), \tag{2.14}$$

and

$$\frac{1}{h}\int_{x_{j-1}}^{x_j} f\left(u\right)du=F_j\left(0\right). \tag{2.15}$$

Hence, we may establish two related mappings for the inequality (2.1).

Proposition 2.1. *Let f be as in Theorem 2.1, define the mappings $H,F:[0,1]\to\mathbb{R}$, given by*

$$H\left(t\right)=\sum_{j=1}^{n}H_j\left(t\right) \quad \text{and} \quad F\left(t\right)=\sum_{j=1}^{n}F_j\left(t\right),$$

where $H_j(t)$ and $F_j(t)$ are defined in (2.10) and (2.11), respectively; then the following statements hold:

(a) *$H(t)$ and $F(t)$ are convex for all $t\in[0,1]$ and $u\in[a,b]$;*
(b) *$H(t)$ and $F(t)$ are monotonic nondecreasing for all $t\in[0,1]$ and $u\in[a,b]$;*
(c) *We have the following bounds for $H(t)$*

$$\sup_{t\in[0,1]}H\left(t\right)=\frac{1}{h}\int_a^b f\left(u\right)du=H\left(1\right),$$

and

$$\inf_{t\in[0,1]}H\left(t\right)=\sum_{k=1}^{n}f\left(\frac{x_{k-1}+x_k}{2}\right)=H\left(0\right),$$

and the following bounds for $F(t)$

$$\sup_{t\in[0,1]}F\left(t\right)=\frac{1}{2}\left[f\left(a\right)+2\sum_{k=1}^{n-1}f\left(x_k\right)+f\left(b\right)\right]=F\left(1\right),$$

and

$$\inf_{t\in[0,1]}F\left(t\right)=\frac{1}{h}\int_a^b f\left(u\right)du=F\left(0\right).$$

Proof. Taking the sum over j from 1 to n, in (2.12)–(2.15) we get the required results, and we shall omit the details. □

Remark 2.2. The inequality (2.1) may written in a convenient way as follows:

$$\sum_{k=1}^{n}f\left(\frac{x_{k-1}+x_k}{2}\right)-\sum_{k=1}^{n-1}f\left(x_k\right)\le\frac{1}{h}\int_a^b f\left(t\right)dt-\sum_{k=1}^{n-1}f\left(x_k\right)\le\frac{f\left(a\right)+f\left(b\right)}{2},$$

which is of Ostrowski's type.

Some sharps Ostrowski's type inequalities for convex functions defined on a real interval $[a,b]$, are proposed in the next theorems.

Theorem 2.2. *Let $f:[a,b]\to\mathbb{R}_+$ be a convex function on $[a,b]$, then the inequality*

$$\int_a^b f(x)\,dx-(b-a)\,f(y)\le\frac{h}{2}\left[f(a)+2\sum_{k=1}^{n-1}f(x_k)+f(b)\right], \tag{2.16}$$

holds for all $y\in[a,b]$. where, $x_k=a+k\frac{b-a}{n}$, $k=0,1,2,\ldots,n$; with $h=\frac{b-a}{n}$, $n\in\mathbb{N}$. The constant $\frac{1}{2}$ in the right-hand side is the best possible, in the sense that it cannot be replaced by a smaller one for all $n\in\mathbb{N}$. If f is concave then the inequality is reversed.

Proof. Fix $y\in[x_{j-1},x_j]$, $j=1,\ldots,n$. Since f is convex on $[a,b]$, then f so is on each subinterval $[x_{j-1},x_j]$, in particular on $[x_{j-1},y]$, then for all $t\in[0,1]$, we have

$$f(tx_{j-1}+(1-t)\,y)\le tf(x_{j-1})+(1-t)\,f(y),\quad j=1,\ldots,n. \tag{2.17}$$

Integrating (2.17) with respect to t on $[0,1]$ we get

$$\int_0^1 f(tx_{j-1}+(1-t)\,y)\,dt\le\frac{f(x_{j-1})+f(y)}{2}. \tag{2.18}$$

Substituting $u=tx_{j-1}+(1-t)\,y$, in the left hand side of (2.18), we get

$$\int_{x_{j-1}}^y f(u)\,du\le\frac{y-x_{j-1}}{2}\left(f(x_{j-1})+f(y)\right). \tag{2.19}$$

Now, we do similarly for the interval $[y,x_j]$, we therefore have

$$f(ty+(1-t)\,x_j)\le tf(y)+(1-t)\,f(x_j),\quad j=1,\ldots,n. \tag{2.20}$$

Integrating (2.20) with respect to t on $[0,1]$ we get

$$\int_0^1 f(ty+(1-t)\,x_j)\,dt\le\frac{f(y)+f(x_j)}{2}. \tag{2.21}$$

Substituting $u=ty+(1-t)\,x_j$, in the left hand side of (2.21), we get

$$\int_y^{x_j} f(u)\,du\le\frac{x_j-y}{2}\left(f(y)+f(x_j)\right). \tag{2.22}$$

Adding the inequalities (2.19) and (2.22), we get

$$\begin{aligned}
(2.23)\qquad & \int_{x_{j-1}}^{y} f(u)\,du + \int_{y}^{x_j} f(u)\,du \\
&= \int_{x_{j-1}}^{x_j} f(u)\,du \\
&\le \frac{y - x_{j-1}}{2}\left(f(x_{j-1}) + f(y)\right) + \frac{x_j - y}{2}\left(f(y) + f(x_j)\right) \\
&\le \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_{j-1}) + (x_j - x_{j-1}) f(y) \\
&\le \frac{x_j - x_{j-1}}{2}\left(f(x_{j-1}) + f(x_{j-1})\right) + hf(y).
\end{aligned}$$

Taking the sum over j from 1 to n, we get

$$\begin{aligned}
& \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(u)\,du \\
&= \int_a^b f(u)\,du \\
&= \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{2}\left\{f(x_{j-1}) + f(x_j)\right\} + \sum_{j=1}^{n} hf(y) \\
&\le \frac{1}{2}\max_j \left\{x_j - x_{j-1}\right\} \cdot \sum_{j=1}^{n}\left(f(x_{j-1}) + f(x_j)\right) + (b-a) f(y) \\
&= \frac{h}{2}\left[f(x_0) + f(x_1) + \sum_{j=2}^{n-1}\left\{f(x_{j-1}) + f(x_j)\right\} + f(x_{n-1}) + f(x_n)\right] + (b-a) f(y) \\
&= \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + (b-a) f(y),
\end{aligned}$$

which gives that

$$\int_a^b f(u)\,du - (b-a) f(y) \le \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right],$$

for all $y \in [x_{j-1}, x_j] \subseteq [a,b]$ for all $j = 1, 2, \dots, n$, which gives the desired result (2.16).

To prove the sharpness let (2.16) hold with another constants $C > 0$, which gives

$$(2.24)\qquad \int_a^b f(x)\,dx - (b-a) f(y) \le C \cdot h\left[f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right].$$

Let $f : [0,1] \to \mathbb{R}$ be the identity map $f(x) = x$, then the right-hand side of (2.24) reduces to

$$\begin{aligned}\frac{1}{2} - y &\leq C \cdot \frac{1}{n}\left[2\sum_{k=1}^{n-1} x_k + 1\right] \\ &= C \cdot \frac{1}{n}\left[2\frac{1}{n} \cdot \frac{n(n-1)}{2} + 1\right] \\ &= C.\end{aligned}$$

Choose $y = 0$, it follows that $\frac{1}{2} \leq C$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.16). □

Theorem 2.3. *Under the assumptions of Theorem* 2.2*, we have*

$$\begin{aligned}&\int_a^b f(x)\,dx - (b-a)f(y) \\ &\leq \left[\frac{h}{2} + \max_{1\leq j\leq n}\left|y - \frac{x_{j-1}+x_j}{2}\right|\right] \cdot \left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right],\end{aligned} \tag{2.25}$$

for all $y \in [a,b]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible for all $n \in \mathbb{N}$. If f is concave then the inequality is reversed.

In particular, if $n = 1$ then

$$\int_a^b f(x)\,dx - (b-a)f(y) \leq \left[\frac{b-a}{2} + \left|y - \frac{a+b}{2}\right|\right] \cdot [f(a) + f(b)],$$

for all $y \in [a,b]$.

Proof. Repeating the steps of the proof of Theorem 2.2, therefore by (2.23)

$$\begin{aligned}\int_{x_{j-1}}^{x_j} f(u)\,du &\leq \frac{y - x_{j-1}}{2} \cdot f(x_{j-1}) + \frac{x_j - y}{2} \cdot f(x_{j-1}) + hf(y) \\ &\leq \max\left\{\frac{y - x_{j-1}}{2}, \frac{x_j - y}{2}\right\} \cdot (f(x_{j-1}) + f(x_{j-1})) + hf(y) \\ &\leq \left[\frac{x_j - x_{j-1}}{2} + \left|y - \frac{x_{j-1}+x_j}{2}\right|\right] \cdot (f(x_{j-1}) + f(x_{j-1})) \\ &\quad + hf(y)\end{aligned} \tag{2.26}$$

Taking the sum over j from 1 to n, we get

$$\begin{aligned}
&\int_a^b f(u)\,du \\
&= \sum_{j=1}^{n}\left[\frac{x_j - x_{j-1}}{2} + \left|y - \frac{x_{j-1}+x_j}{2}\right|\right]\cdot\{f(x_{j-1}) + f(x_j)\} + \sum_{j=1}^{n} hf(y) \\
&\le \max_{1\le j\le n}\left[\frac{x_j - x_{j-1}}{2} + \left|y - \frac{x_{j-1}+x_j}{2}\right|\right]\cdot\sum_{j=1}^{n}(f(x_{j-1}) + f(x_j)) + (b-a)f(y) \\
&\le \left[\frac{h}{2} + \max_{1\le j\le n}\left|y - \frac{x_{j-1}+x_j}{2}\right|\right] \\
&\quad\times\left[f(x_0) + f(x_1) + \sum_{j=2}^{n-1}\{f(x_{j-1}) + f(x_j)\} + f(x_{n-1}) + f(x_n)\right] + (b-a)f(y) \\
&= \left[\frac{h}{2} + \max_{1\le j\le n}\left|y - \frac{x_{j-1}+x_j}{2}\right|\right]\cdot\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] + (b-a)f(y),
\end{aligned}$$

which gives that

$$\begin{aligned}
&\int_a^b f(u)\,du - (b-a)f(y) \\
&\le \left[\frac{h}{2} + \max_{1\le j\le n}\left|y - \frac{x_{j-1}+x_j}{2}\right|\right]\cdot\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right] m
\end{aligned}$$

for all $y \in [x_{j-1}, x_j] \subseteq [a,b]$ for all $j = 1, 2, \ldots, n$, which gives the desired result (2.25). The proof of sharpness goes likewise the proof of the sharpness of Theorem 2.2 and we shall omit the details. □

Corollary 2.2. *Let $\alpha_i \ge 0$, for all $i = 0, 1, 2, \ldots, n$, be positive real numbers such that $\sum_{i=0}^{n} \alpha_i = 1$, then under the assumptions of Theorem 2.3, we have*

$$\begin{aligned}
&\int_a^b f(x)\,dx - (b-a) f\left(\frac{1}{n+1}\sum_{i=0}^{n}\alpha_i x_i\right) \\
&\le \left[\frac{h}{2} + \max_{1\le j\le n}\left|\frac{1}{n+1}\sum_{i=0}^{n}\alpha_i x_i - \frac{x_{j-1}+x_j}{2}\right|\right]\cdot\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right],
\end{aligned}$$

for all $y \in [a,b]$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If f is concave then the inequality is reversed.

Theorem 2.4. *Under the assumptions of Theorem 2.3, we have*

$$\frac{1}{b-a}\int_a^b f(u)\,du - \frac{1}{n}\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) - \frac{1}{n}\sum_{j=1}^{n-1} f(x_j) \le \frac{f(a) + +f(b)}{2n}, \tag{2.27}$$

for all $j = 1, 2, \ldots, n$. The constant $\frac{1}{2}$ in the right-hand side is the best possible. If f is concave then the inequality is reversed.

Proof. Repeating the steps of the proof of Theorem 2.3, (2.26) if we choose $y = \frac{x_{j-1}+x_j}{2}$, then we get

$$\int_{x_{j-1}}^{x_j} f(u)\,du \le \frac{x_j - x_{j-1}}{2} \cdot (f(x_{j-1}) + f(x_{j-1})) + hf\left(\frac{x_{j-1}+x_j}{2}\right).$$

Taking the sum over j from 1 to n, we get

$$\begin{aligned}\int_a^b f(u)\,du &\le \sum_{j=1}^{n} \frac{x_j - x_{j-1}}{2} \cdot \{f(x_{j-1}) + f(x_j)\} + \sum_{j=1}^{n} hf\left(\frac{x_{j-1}+x_j}{2}\right) \\ &\le \frac{h}{2}\sum_{j=1}^{n} \{f(x_{j-1}) + f(x_j)\} + h\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right),\end{aligned}$$

which gives that

$$\int_a^b f(u)\,du - h\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \le \frac{h}{2}\left[f(a) + 2\sum_{j=1}^{n-1} f(x_j) + f(b)\right],$$

for all $j = 1, 2, \ldots, n$, which gives the desired result (2.27). The proof of sharpness goes likewise the proof of the sharpness of Theorem 2.2 and we shall omit the details. □

Theorem 2.5. *Let $I \subset \mathbb{R}$ be an open interval and $a, b \in I$, $a < b$. Let $f : I \to \mathbb{R}_+$ be an increasing convex function on $[a, b]$, then the inequality*

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2} f(y) \ge \frac{h}{2}\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \ge 0, \tag{2.28}$$

is valid for all $y \in [a, b] \subset I$. The constant $\frac{1}{2}$ in the right-hand side is the best possible, in the sense that it cannot be replaced by a greater one. If f is concave then the inequality is reversed. In particular, if $n = 1$ then

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2} f(y) \ge \frac{b-a}{2} f\left(\frac{a+b}{2}\right) \ge 0,$$

Proof. Let $y \in [x_{j-1}, x_j]$ be an arbitrary point such that $x_{j-1} < y-p \le y \le y+p < x_j$ for all $j = 1, 2, \ldots, n$ with $p > 0$.

It is well known that f is convex on I iff

$$f(y) \le \frac{1}{2p}\int_{y-p}^{y+p} f(t)\,dt,$$

for every subinterval $[y-p, y+p] \subset [a, b] \subset I$ for some $p > 0$. But since f increases on $[a, b]$, we also have

$$f(y) \le \frac{1}{2p}\int_{y-p}^{y+p} f(t)\,dt \le \frac{1}{2p}\int_{x_{j-1}}^{x_j} f(t)\,dt.$$

Choosing $p \geq \frac{h}{2}$, (this choice is available since it is true for every subinterval in I), therefore from the last inequality we get

$$f(y) \leq \frac{1}{h}\int_{y-p}^{y+p} f(t)\,dt \leq \frac{1}{h}\int_{x_{j-1}}^{x_j} f(t)\,dt.$$

Again by convexity we have

$$hf\left(\frac{x_{j-1}+x_j}{2}\right) \leq \int_{x_{j-1}}^{x_j} f(t)\,dt.$$

Adding the last two inequalities, we get

$$hf(y) + hf\left(\frac{x_{j-1}+x_j}{2}\right) \leq 2\int_{x_{j-1}}^{x_j} f(t)\,dt,$$

or we write

$$hf\left(\frac{x_{j-1}+x_j}{2}\right) \leq 2\int_{x_{j-1}}^{x_j} f(t)\,dt - hf(y).$$

Taking the sum over j from 1 to n, we get

$$h\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \leq 2\sum_{j=1}^{n}\int_{x_{j-1}}^{x_j} f(t)\,dt - \sum_{j=1}^{n} hf(y)$$

hence,

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2}f(y) \geq \frac{h}{2}\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \geq 0,$$

holds by positivity of f and this proves our assertion.

To prove the sharpness let (2.28) holds with another constant $C > 0$, which gives

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2}f(y) \geq C \cdot h\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \geq 0. \tag{2.29}$$

Let $f : [0,1] \to \mathbb{R}_+$ be the identity map $f(x) = x$, then the right-hand side of (2.29) reduces to

$$\begin{aligned}
\frac{1}{2} - \frac{1}{2}y &\geq C \cdot \frac{1}{n}\sum_{j=1}^{n}\frac{2j-1}{2n} \\
&= C \cdot \frac{1}{n}\left[\sum_{j=1}^{n}\frac{j}{n} - \sum_{j=1}^{n}\frac{1}{2n}\right] \\
&= C \cdot \frac{1}{n}\left[\frac{1}{n}\cdot\frac{n(n+1)}{2} - \frac{1}{2n}\cdot n\right] \\
&= \frac{1}{2}C.
\end{aligned}$$

Choose $y = \frac{1}{2}$, it follows that $\frac{1}{4} \geq \frac{1}{2}C$ which means that $\frac{1}{2} \geq C$, i.e., $\frac{1}{2}$ is the best possible constant in the right-hand side of (2.28). □

Corollary 2.3. *Let $\alpha_i \geq 0$, for all $i = 0, 1, 2, \ldots, n$, be positive real numbers such that $\sum_{i=0}^{n} \alpha_i = 1$, then under the assumptions of Theorem 2.5, we have*

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2} f\left(\frac{1}{n+1}\sum_{i=0}^{n} \alpha_i x_i\right) \geq \frac{h}{2}\sum_{j=1}^{n} f\left(\frac{x_{j-1}+x_j}{2}\right) \geq 0.$$

The constant $\frac{1}{2}$ in the right-hand side is the best possible. If f is concave then the inequality is reversed. In particular case if $n = 1$, then

$$\int_a^b f(t)\,dt - \frac{(b-a)}{2} f\left(\frac{\alpha a + (1-\alpha) b}{2}\right) \geq \frac{b-a}{2} f\left(\frac{a+b}{2}\right) \geq 0,$$

for all $\alpha \in [0, 1]$.

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