

## INTUITIONISTIC FUZZY GRAPH STRUCTURES

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**ABSTRACT.** In this paper, we introduce the concept of an intuitionistic fuzzy graph structure (IFGS). We discuss certain notions, including intuitionistic fuzzy  $B_i$ -cycles, intuitionistic fuzzy  $B_i$ -trees and  $\phi$ -complement of an intuitionistic fuzzy graph structure with several examples. We also present  $\phi$ -complement of an intuitionistic fuzzy graph structure along with self-complementary and strong self-complementary intuitionistic fuzzy graph structures.

### 1. INTRODUCTION

Fuzzy set was introduced by Zadeh in 1965. A fuzzy set gives the degree of membership of an object in a given set. Kaufmann's initial definition of a fuzzy graph [10] was based on Zadeh's fuzzy relations [22]. The fuzzy relations between fuzzy sets were considered by Rosenfeld and he developed the structure of fuzzy graphs, obtaining analogs of several graph theoretical concepts. Later on, Bhattacharya [7] gave some remarks on fuzzy graphs and some operations on fuzzy graphs were introduced by Mordeson and Peng [14]. In 1983, Atanassov [5] extended the idea of a fuzzy set and introduced the concept of an intuitionistic fuzzy set. He added a new component, degree of non-membership, in the definition of a fuzzy set with the condition that sum of two degrees must be less or equal to one. Atanassov [6] also introduced the concept of intuitionistic fuzzy graphs and intuitionistic fuzzy relations. Shannon and Atanassov investigated some properties of intuitionistic fuzzy relations and intuitionistic fuzzy graphs in [20]. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [16]. Karunambigai et al. used intuitionistic fuzzy graphs to find shortest paths in networks [11]. Akram et al. [1–4] introduced many

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*Key words and phrases.* Intuitionistic fuzzy graph structure (IFGS), strong IFGS, Intuitionistic fuzzy  $B_i$ -cycles, Intuitionistic fuzzy  $B_i$ -trees,  $\phi$ -complement of an IFGS.

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new concepts, including strong intuitionistic fuzzy graphs, intuitionistic fuzzy trees, intuitionistic fuzzy hypergraphs, and intuitionistic fuzzy digraphs in decision support systems.

Fuzzy graph theory is finding an increasing number of applications in modeling real time systems where the level of information inherent in the system varies with different levels of precision. Fuzzy models are becoming useful because of their aim in reducing the differences between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. Intuitionistic fuzzy set has got an advantage over fuzzy set because of its additional component which explains the deficiency of knowledge in assigning the degree of membership to an object because there is a fair chance of the existence of a non-zero hesitation part at each moment of evaluation of anything. The advantages of intuitionistic fuzzy sets and graphs are that they give more accuracy into the problems, reduce the cost of implementation and improve efficiency. Intuitionistic fuzzy sets are very useful in providing a flexible model to describe uncertainty and vagueness involved in decision making, so intuitionistic fuzzy graphs are playing a substantial role in chemistry, economics, computer sciences, engineering, medicine and decision making problems, now a days. Graph structures or generalized graph structures introduced by Sampathkumar in 2006 [19], are a generalization of graphs which is quite useful in studying signed graphs and graphs in which every edge is labeled or colored because they help to study various relations and corresponding edges simultaneously. Dinesh and Ramakrishnan [9] introduced fuzzy graph structures. In this paper, we have worked on intuitionistic fuzzy graph structures, some of their fundamental concepts and properties due to the improved influence of intuitionistic fuzzy sets and particular use of graph structures. In this paper, we introduce the concept of an intuitionistic fuzzy graph structure (IFGS). We discuss certain notions, including intuitionistic fuzzy  $B_i$ -cycles, intuitionistic fuzzy  $B_i$ -trees and  $\phi$ -complement of an intuitionistic fuzzy graph structure with several examples. We also present  $\phi$ -complement of an intuitionistic fuzzy graph structure along with self-complementary and strong self-complementary intuitionistic fuzzy graph structures.

## 2. PRELIMINARIES

We first review some definitions from [19] that are necessary for this paper.

A *graph structure*  $G^* = (U, E_1, E_2, \dots, E_k)$ , consists of a non-empty set  $U$  together with relations  $E_1, E_2, \dots, E_k$  on  $U$ , which are mutually disjoint such that each  $E_i$  is irreflexive and symmetric. If  $(u, v) \in E_i$  for some  $i, 1 \leq i \leq k$ , we call it an  $E_i$ -edge and write it as " $uv$ ". A graph structure  $G^* = (U, E_1, E_2, \dots, E_k)$  is *complete*, if

- (i) each edge  $E_i, 1 \leq i \leq k$  appears at least once in  $G^*$ ;
- (ii) between each pair of vertices  $uv$  in  $U$ ,  $uv$  is an  $E_i$ -edge for some  $i, 1 \leq i \leq k$ .

A graph structure  $G^* = (U, E_1, E_2, \dots, E_k)$  is *connected*, if the underlying graph is connected. In a graph structure,  $E_i$ -path between two vertices  $u$  and  $v$ , is the path

which consists of only  $E_i$ -edges for some  $i$ , and similarly,  $E_i$ -cycle is the cycle, which consists of only  $E_i$ -edges for some  $i$ . A graph structure is a *tree*, if it is connected and contains no cycle or equivalently the underlying graph of  $G^*$  is a tree.  $G^*$  is an  $E_i$ -tree, if the subgraph structure induced by  $E_i$ -edges is a tree.

Similarly,  $G^*$  is a  $E_1E_2 \dots E_k$ -tree, if  $G^*$  is a  $E_j$ -tree for each  $j$ ,  $1 \leq j \leq k$ .

A graph structure is an  $E_i$ -forest, if the subgraph structure induced by  $E_i$ -edges is a forest, i.e., if it has no  $E_i$ -cycles. Let  $S \subseteq U$ , then the subgraph structure  $\langle S \rangle$  induced by  $S$ , has vertex set  $S$ , where two vertices  $u$  and  $v$  in  $\langle S \rangle$  are joined by an  $E_i$ -edge if, and only if, they are joined by an  $E_i$ -edge in  $G^*$  for  $1 \leq i \leq k$ . For some  $i$ ,  $1 \leq i \leq k$ , the  $E_i$ -subgraph induced by  $S$ , is denoted by  $E_i\text{-}\langle S \rangle$  and it has only  $E_i$ -edges joining the vertices in  $S$ . If  $T$  is a subset of edge set in  $G^*$ , then subgraph structure  $\langle T \rangle$  induced by  $T$  has the vertex set, the end vertices in  $T$ , and whose edges are those in  $T$ . Let  $G^* = (U_1, E_1, E_2, \dots, E_m)$  and  $H^* = (U_2, E'_1, E'_2, \dots, E'_n)$  be graph structures then  $G^*$  and  $H^*$  are *isomorphic*, if  $m = n$  and there exists a bijection  $f : U_1 \rightarrow U_2$  and a permutation  $\phi : \{E_1, E_2, \dots, E_m\} \rightarrow \{E'_1, E'_2, \dots, E'_n\}$ , say  $E_i \rightarrow E'_j$ ,  $1 \leq i, j \leq n$ , such that for all  $u, v \in U_1$ ,  $uv \in E_i$  implies  $f(u)f(v) \in E'_j$ .

Two graph structures  $G^* = (U, E_1, E_2, \dots, E_k)$  and  $H^* = (U, E'_1, E'_2, \dots, E'_k)$ , on the same vertex set  $U$ , are *identical*, if there exists a bijection  $f : U \rightarrow U$ , such that for all  $u$  and  $v$  in  $U$  and an  $E_i$ -edge  $uv$  in  $G^*$ ,  $f(u)f(v)$  is an  $E'_i$ -edge in  $H^*$ , where  $1 \leq i \leq k$  and  $E_i \simeq E'_i$  for all  $i$ . Let  $\phi$  be a permutation on  $\{E_1, E_2, \dots, E_k\}$  then the  $\phi$ -cyclic complement of  $G^*$  denoted by  $(G^*)^{\phi c}$  is obtained by replacing  $E_i$  with  $\phi(E_i)$  for  $1 \leq i \leq k$ . Let  $G^* = (U, E_1, E_2, \dots, E_k)$  be a graph structure and  $\phi$  be a permutation on  $\{E_1, E_2, \dots, E_k\}$ , then

- $G^*$  is  $\phi$ -self complementary, if  $G^*$  is isomorphic to  $(G^*)^{\phi c}$ , the  $\phi$ -cyclic complement of  $G^*$  and  $G^*$  is self-complementary, if  $\phi \neq$  identity permutation;
- $G^*$  is strong  $\phi$ -self complementary, if  $G^*$  is identical to  $(G^*)^{\phi c}$ , the  $\phi$ -complement of  $G^*$  and  $G^*$  is strong self-complementary, if  $\phi \neq$  identity permutation.

**Definition 2.1** ([6]). An *intuitionistic fuzzy set* (IFS) on an universe  $X$  is an object of the form

$$A = \{ \langle x, \mu_A(x), \nu_A(x) \rangle \mid x \in X \},$$

where  $\mu_A(x) (\in [0, 1])$  is called degree of membership of  $x \in A$ ,  $\nu_A(x) (\in [0, 1])$  is called degree of nonmembership of  $x \in A$ , and  $\mu_A$  and  $\nu_A$  satisfy the following condition: for all  $x \in X$ ,  $\mu_A(x) + \nu_A(x) \leq 1$ .

**Definition 2.2** ([6]). An intuitionistic fuzzy relation  $R = (\mu_R(x, y), \nu_R(x, y))$  in an universe  $X \times Y (R(X \rightarrow Y))$  is an intuitionistic fuzzy set of the form

$$R = \{ \langle (x, y), \mu_R(x, y), \nu_R(x, y) \rangle \mid (x, y) \in X \times Y \},$$

where  $\mu_R : X \times Y \rightarrow [0, 1]$  and  $\nu_R : X \times Y \rightarrow [0, 1]$ . The intuitionistic fuzzy relation  $R$  satisfies  $\mu_R(x, y) + \nu_R(x, y) \leq 1$  for all  $x, y \in X$ .

**Definition 2.3** ([9]). Let  $G^* = (U, E_1, E_2, \dots, E_k)$  be a graph structure and  $\nu, \rho_1, \rho_2, \dots, \rho_k$  be the fuzzy subsets of  $U, E_1, E_2, \dots, E_k$ , respectively such that

$$0 \leq \rho_i(xy) \leq \nu(x) \wedge \nu(y), \quad \text{for all } x, y \in U \text{ and } i = 1, 2, \dots, k.$$

Then  $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$  is a fuzzy graph structure of  $G^*$ .

**Definition 2.4** ([9]). Let  $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$  be a fuzzy graph structure of a graph structure  $G^* = (U, E_1, E_2, \dots, E_k)$ . Then  $F = (\nu, \tau_1, \tau_2, \dots, \tau_k)$  is a partial fuzzy spanning subgraph structure of  $G$  if,  $\tau_i \subseteq \rho_i$  for  $i = 1, 2, \dots, k$ .

**Definition 2.5** ([9]). Let  $G^*$  be a graph structure and  $G$  be a fuzzy graph structure of  $G^*$ . If  $xy \in \text{supp}(\rho_i)$ , then  $xy$  is said to be a  $\rho_i$ -edge of  $G$ .

**Definition 2.6** ([9]). The strength of a  $\rho_i$ -path  $x_0x_1 \dots x_n$  of a fuzzy graph structure  $G$  is  $\bigwedge_{j=1}^n \rho_i(x_{j-1}x_j)$  for  $i = 1, 2, \dots, k$ .

**Definition 2.7** ([9]). In any fuzzy graph structure  $G$ ,

$$\begin{aligned} \rho_i^2(xy) &= \rho_i \circ \rho_i(xy) = \bigvee_z \{ \rho_i(xz) \wedge \rho_i(zx) \}, \\ \rho_i^j(xy) &= (\rho_i^{j-1} \circ \rho_i)(xy) = \bigvee_z \{ \rho_i^{j-1}(xz) \wedge \rho_i(zx) \}, \end{aligned}$$

$j = 2, 3, \dots, m$ , for any  $m \geq 2$ . Also  $\rho_i^\infty(xy) = \bigvee \{ \rho_i^j(xy), j = 1, 2, \dots \}$ .

**Definition 2.8** ([9]).  $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$  is a  $\rho_i$ -cycle iff  $(\text{supp}(\nu), \text{supp}(\rho_1), \text{supp}(\rho_2), \dots, \text{supp}(\rho_k))$  is a  $E_i$ -cycle.

**Definition 2.9** ([9]).  $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$  is a fuzzy  $\rho_i$ -cycle iff  $(\text{supp}(\nu), \text{supp}(\rho_1), \text{supp}(\rho_2), \dots, \text{supp}(\rho_k))$  is a  $E_i$ -cycle and there exists no unique  $xy$  in  $\text{supp}(\rho_i)$  such that

$$\rho_i(xy) = \bigwedge \{ \rho_i(uv) \mid uv \in \text{supp}(\rho_i) \}.$$

**Definition 2.10** ([9]).  $G = (\nu, \rho_1, \rho_2, \dots, \rho_k)$  is a fuzzy  $\rho_i$ -tree if it has a partial fuzzy spanning subgraph structure,  $\check{F}_i = (\nu, \tau_1, \tau_2, \dots, \tau_k)$  which is a  $\tau_i$ -tree where for all  $\rho_i$ -edges not in  $\check{F}_i$ ,  $\rho_i(xy) < \tau_i^\infty(xy)$ .

### 3. INTUITIONISTIC FUZZY GRAPH STRUCTURES

**Definition 3.1.** Let  $\{E_i : i = 1, 2, \dots, n\}$  be a set of irreflexive, symmetric and mutually disjoint relations on a non-empty set  $U$ . An *intuitionistic fuzzy graph structure (IFGS)* with underlying vertex set  $U$  is denoted by  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$ , where

- (i)  $A$  is an intuitionistic fuzzy set of  $U$  with  $\mu_A : U \rightarrow [0, 1]$  and  $\nu_A : U \rightarrow [0, 1]$ , namely the degree of membership and the degree of nonmembership of  $x \in U$ , respectively, such that

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1, \quad \text{for all } x \in U.$$

- (ii) Each  $B_i$  is an intuitionistic fuzzy set of  $E_i$  such that the functions  $\mu_{B_i} : E_i \rightarrow [0, 1]$  and  $\nu_{B_i} : E_i \rightarrow [0, 1]$  are defined by

$$\mu_{B_i}(xy) \leq \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) \leq \nu_A(x) \vee \nu_A(y)$$

and

$$0 \leq \mu_{B_i}(xy) + \nu_{B_i}(xy) \leq 1, \quad \text{for all } xy \in U \times U, \quad i = 1, 2, \dots, n.$$

Equivalently, an IFGS of a graph structure may be defined in the following way.

Let  $G^* = (U, E_1, E_2, \dots, E_n)$  be a graph structure and let  $A, B_1, B_2, \dots, B_{n-1}$  and  $B_n$  be intuitionistic fuzzy subsets of  $U, E_1, E_2, \dots, E_{n-1}$  and  $E_n$ , respectively. Then  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  is called an IFGS of  $G^*$ , if

$$\mu_{B_i}(xy) \leq \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) \leq \nu_A(x) \vee \nu_A(y),$$

for all  $xy \in E_i, i = 1, 2, \dots, n$ , and

$$\mu_{B_i}(xy) + \nu_{B_i}(xy) \leq 1, \quad \text{for all } xy \in U \times U.$$

*Example 3.1.* Let  $G^* = (U, E_1, E_2)$  be a graph structure such that  $U = \{a_1, a_2, a_3, a_4\}$ ,  $E_1 = \{a_1a_2, a_2a_3\}$  and  $E_2 = \{a_3a_4, a_1a_4\}$ . Let  $A, B_1$  and  $B_2$  be intuitionistic fuzzy subsets of  $U, E_1$  and  $E_2$ , respectively, such that

$$\begin{aligned} A &= \{(a_1, 0.5, 0.2), (a_2, 0.7, 0.3), (a_3, 0.4, 0.3), (a_4, 0.7, 0.3)\}, \\ B_1 &= \{(a_1a_2, 0.5, 0.3), (a_2a_3, 0.4, 0.3)\}, \\ \text{and } B_2 &= \{(a_3a_4, 0.4, 0.3), (a_1a_4, 0.1, 0.2)\}. \end{aligned}$$

Then  $\check{G}_s = (A, B_1, B_2)$  is an IFGS of  $G^*$  as shown in Fig. 1.

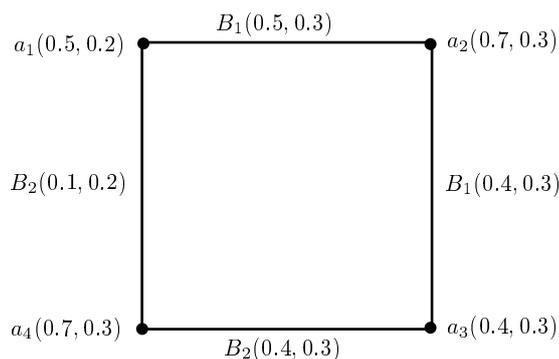


FIGURE 1. IFGS  $\check{G}_s = (A, B_1, B_2)$

**Definition 3.2.** An IFGS  $\check{H}_s = (C, D_1, D_2, \dots, D_n)$  is said to be an intuitionistic fuzzy subgraph structure of an IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ , if  $C \subseteq A$  and  $D_i \subseteq C_i$  for all  $i$ , that is

$$\mu_C(x) \leq \mu_A(x), \quad \nu_C(x) \geq \nu_A(x), \quad \text{for all } x \in U,$$

and for  $i = 1, 2, \dots, n$

$$\mu_{D_i}(xy) \leq \mu_{B_i}(xy), \quad \nu_{D_i}(xy) \geq \nu_{B_i}(xy), \quad \text{for all } xy \in U \times U.$$

$\check{H}_s$  is called an intuitionistic fuzzy spanning subgraph structure of an IFGS  $\check{G}_s$ , if  $C = A$ .

$\check{H}_s$  is called an intuitionistic fuzzy partial spanning subgraph structure of an IFGS  $\check{G}_s$ , if it excludes some edges of  $\check{G}_s$ .

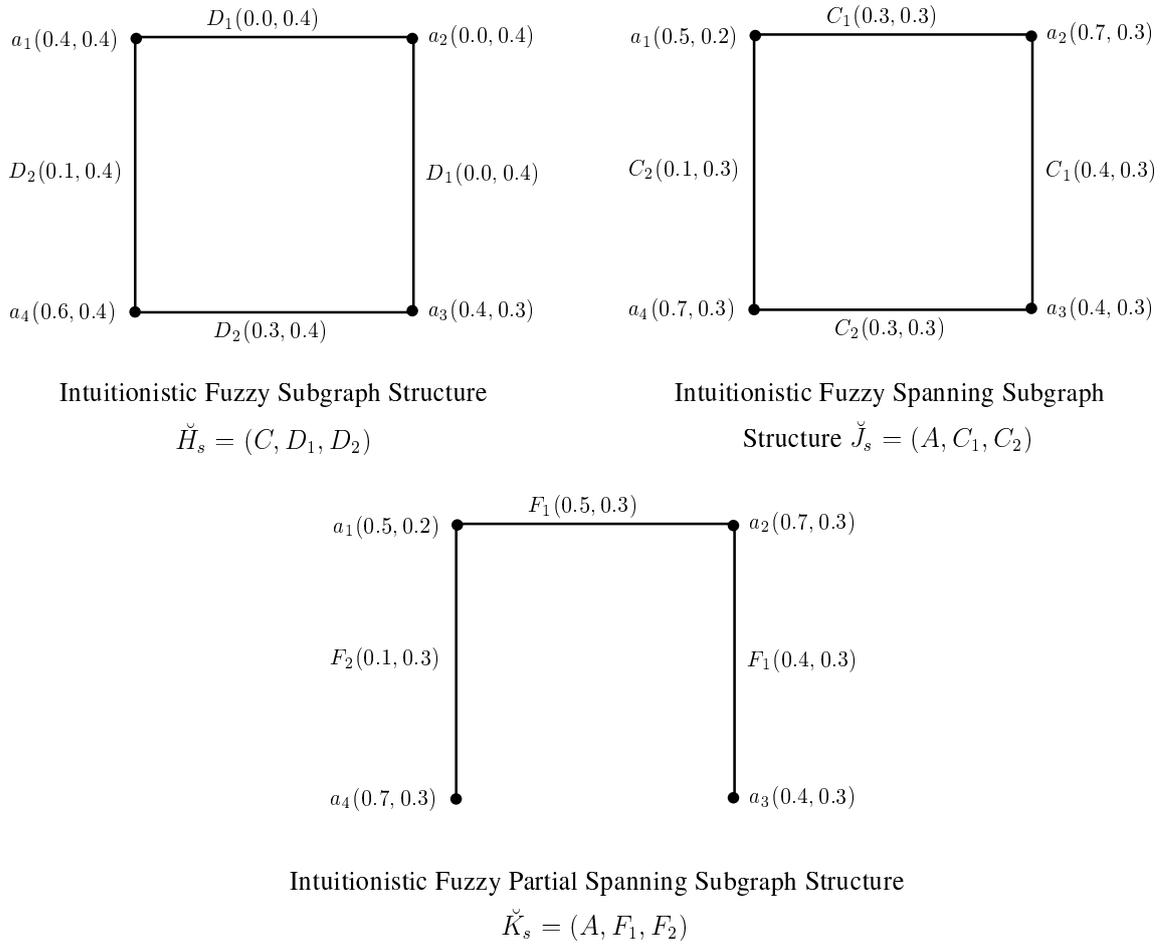


FIGURE 2. Intuitionistic Fuzzy Subgraph Structures

*Example 3.2.* Consider an IFGS  $\check{G}_s = (A, B_1, B_2)$ , as shown in Fig. 1. Let

$$C = \{(a_1, 0.4, 0.4), (a_2, 0.0, 0.4), (a_3, 0.4, 0.3), (a_4, 0.6, 0.4)\},$$

$$D_1 = \{(a_1a_2, 0, 0.4), (a_2a_3, 0, 0.4)\},$$

$$D_2 = \{(a_3a_4, 0.3, 0.4), (a_1a_4, 0.1, 0.4)\},$$

$$\begin{aligned} C_1 &= \{(a_1a_2, 0.3, 0.3), (a_2a_3, 0.4, 0.3)\}, \\ C_2 &= \{(a_3a_4, 0.3, 0.3), (a_1a_4, 0.1, 0.3)\}, \\ F_1 &= \{(a_1a_2, 0.5, 0.3), (a_2a_3, 0.4, 0.3)\}, \\ \text{and } F_2 &= \{(a_1a_4, 0.1, 0.3)\}. \end{aligned}$$

By routine calculations, it is easy to see that  $\check{H}_s = (C, D_1, D_2)$ ,  $\check{J}_s = (A, C_1, C_2)$  and  $\check{K}_s = (A, F_1, F_2)$  are respectively the intuitionistic fuzzy subgraph structure, intuitionistic fuzzy spanning subgraph structure and intuitionistic fuzzy partial spanning subgraph structure of  $\check{G}_s$ . Their respective drawings are shown in Fig. 2.

**Definition 3.3.** Let  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  be an IFGS with underlying vertex set  $U$ . Then there is a  $B_i$ -edge between two vertices  $x$  and  $y$  of  $U$ , if one of the following is true:

- (i)  $\mu_{B_i}(xy) > 0$  and  $\nu_{B_i}(xy) > 0$ ,
- (ii)  $\mu_{B_i}(xy) > 0$  and  $\nu_{B_i}(xy) = 0$ ,
- (iii)  $\mu_{B_i}(xy) = 0$  and  $\nu_{B_i}(xy) > 0$ ,

for some  $i$ .

**Definition 3.4.** For an intuitionistic fuzzy graph structure  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with vertex set  $U$ , support of  $B_i$  is given by:

$$\text{supp}(B_i) = \{xy \in U \times U : \mu_{B_i}(xy) \neq 0 \text{ or } \nu_{B_i}(xy) \neq 0\}, \quad i = 1, 2, \dots, n.$$

**Definition 3.5.**  $B_i$ -path of an IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ , is a sequence of distinct vertices  $v_1, v_2, \dots, v_m \in U$  (except the choice  $v_m = v_1$ ), such that  $v_{j-1}v_j$  is a  $B_i$ -edge for all  $j = 2, 3, \dots, m$ .

**Definition 3.6.** In an IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ , two vertices  $x$  and  $y$  of  $U$  are said to be  $B_i$ -connected, if they are joined by a  $B_i$ -Path, for some  $i \in \{1, 2, 3, \dots, n\}$ .

**Definition 3.7.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ , is said to be  $B_i$ -strong, if for all  $B_i$ -edges  $xy$

$$\mu_{B_i}(xy) = \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i}(xy) = \nu_A(x) \vee \nu_A(y),$$

for some  $i \in \{1, 2, 3, \dots, n\}$ .

*Example 3.3.* Consider the IFGS  $\check{G}_s = (A, B_1, B_2)$ , as shown in Fig. 1. Then

- (i)  $a_1a_2, a_2a_3$  are  $B_1$ -edges and  $a_3a_4, a_1a_4$  are  $B_2$ -edges;
- (ii)  $a_1a_2a_3$  and  $a_3a_4a_1$  are  $B_1$ - and  $B_2$ -paths, respectively;
- (iii)  $a_1$  and  $a_3$  are  $B_1$ -connected vertices of  $U$ ;
- (iv)  $\check{G}_s$  is  $B_1$ -strong, since  $\text{supp}(B_1) = \{a_1a_2, a_2a_3\}$  and

$$\begin{aligned} \mu_{B_1}(a_1a_2) &= 0.5 = (\mu_A(a_1) \wedge \mu_A(a_2)), \\ \nu_{B_1}(a_1a_2) &= 0.3 = (\nu_A(a_1) \vee \nu_A(a_2)), \end{aligned}$$

$$\mu_{B_1}(a_2a_3) = 0.4 = (\mu_A(a_2) \wedge \mu_A(a_3)),$$

$$\text{and } \nu_{B_1}(a_2a_3) = 0.3 = (\nu_A(a_2) \vee \nu_A(a_3)).$$

**Definition 3.8.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  is said to be strong, if it is  $B_i$ -strong for all  $i \in \{1, 2, 3, \dots, n\}$ .

**Definition 3.9.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ , is called *complete* or  $B_1B_2 \dots B_n$ -*complete* if

- (i)  $\check{G}_s$  is a strong IFGS;
- (ii)  $\text{supp}(B_i) \neq \emptyset$  for all  $i = 1, 2, 3, \dots, n$ ;
- (iii) For each pair of vertices  $x, y \in U$ ,  $xy$  is a  $B_i$ -edge for some  $i$ .

*Example 3.4.* Let  $\check{G}_s = (A, B_1, B_2)$  shown in Fig. 3, be IFGS of the graph structure  $G^* = (U, E_1, E_2)$  where  $U = \{a_1, a_2, a_3, a_4\}$ ,  $E_1 = \{a_1a_3, a_3a_4, a_1a_4\}$  and  $E_2 = \{a_1a_2, a_2a_3, a_2a_4\}$ . Then  $\check{G}_s$  is a strong IFGS since it is both  $B_1$ -strong and  $B_2$ -strong.

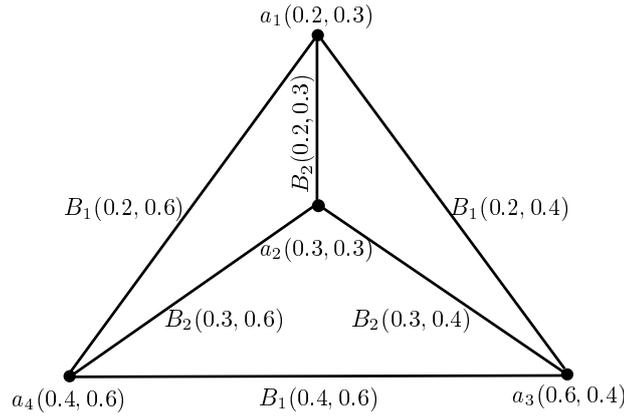


FIGURE 3. IFGS  $\check{G}_s = (A, B_1, B_2)$

Moreover  $\text{supp}(B_1) \neq \emptyset$ ,  $\text{supp}(B_2) \neq \emptyset$ , every pair of vertices belonging to  $U$ , is either a  $B_1$ -edge or a  $B_2$ -edge, so  $\check{G}_s$  is a complete or  $B_1B_2$ -complete IFGS as well.

**Definition 3.10.** In an IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ ,  $\mu_{B_i}$ - and  $\nu_{B_i}$ -strengths of a  $B_i$ -path " $\underline{P}_{B_i} = v_1v_2 \dots v_m$ ", are denoted by  $\delta.\underline{P}_{B_i}$  and  $\Delta.\underline{P}_{B_i}$ , respectively, such that

$$\delta.\underline{P}_{B_i} = \bigwedge_{j=2}^m [\mu_{B_i}(v_{j-1}v_j)] \quad \text{and} \quad \Delta.\underline{P}_{B_i} = \bigvee_{j=2}^m [\nu_{B_i}(v_{j-1}v_j)].$$

Then we write, strength of the path  $\underline{P}_{B_i} = (\delta.\underline{P}_{B_i}, \Delta.\underline{P}_{B_i})$ .

*Example 3.5.* In  $\check{G}_s = (A, B_1, B_2)$  shown in Fig. 3,  $\underline{P}_1 = a_1a_3a_4a_1$  is a  $B_1$ -path and  $\underline{P}_2 = a_3a_2a_4$  is a  $B_2$ -path and

$$\delta.\underline{P}_1 = \mu_{B_1}(a_1a_3) \wedge \mu_{B_1}(a_3a_4) \wedge \mu_{B_1}(a_4a_1) = 0.2 \wedge 0.4 \wedge 0.2 = 0.2,$$

$$\Delta.P_1 = \nu_{B_1}(a_1a_3) \vee \nu_{B_1}(a_3a_4) \vee \nu_{B_1}(a_4a_1) = 0.4 \vee 0.6 \vee 0.6 = 0.6,$$

$$\delta.P_2 = \mu_{B_2}(a_3a_2) \wedge \mu_{B_2}(a_2a_4) = 0.3 \wedge 0.3 = 0.3,$$

$$\Delta.P_2 = \nu_{B_2}(a_3a_2) \vee \nu_{B_2}(a_2a_4) = 0.4 \vee 0.6 = 0.6.$$

Thus strength of  $B_1$ -path  $P_1 = (\delta.P_1, \Delta.P_1) = (0.2, 0.6)$ , strength of  $B_2$ -path  $P_2 = (\delta.P_2, \Delta.P_2) = (0.3, 0.6)$ .

**Definition 3.11.** In an IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  with underlying vertex set  $U$ :

- (i)  $\mu_{B_i}$ -strength of connectedness between  $x$  and  $y$ , is defined by  $\mu_{B_i}^\infty(xy) = \bigvee_{j \geq 1} \{\mu_{B_i}^j(xy)\}$ , where  $\mu_{B_i}^j(xy) = (\mu_{B_i}^{j-1} \circ \mu_{B_i})(xy)$  for  $j \geq 2$  and  $\mu_{B_i}^1(xy) = (\mu_{B_i} \circ \mu_{B_i})(xy) = \bigvee_z \{\mu_{B_i}(xz) \wedge \mu_{B_i}(zy)\}$ ;
- (ii)  $\nu_{B_i}$ -strength of connectedness between  $x$  and  $y$ , is defined by  $\nu_{B_i}^\infty(xy) = \bigvee_{j \geq 1} \{\nu_{B_i}^j(xy)\}$ , where  $\nu_{B_i}^j(xy) = (\nu_{B_i}^{j-1} \circ \nu_{B_i})(xy)$  for  $j \geq 2$  and  $\nu_{B_i}^1(xy) = (\nu_{B_i} \circ \nu_{B_i})(xy) = \bigwedge_z \{\nu_{B_i}(xz) \vee \nu_{B_i}(zy)\}$ .

*Example 3.6.* Let  $\check{G}_s = (A, B_1, B_2)$ , as shown in Fig. 4, be IFGS of graph structure  $G^* = (U, E_1, E_2)$ , such that  $U = \{a_1, a_2, a_3\}$ ,  $E_1 = \{a_1a_2, a_1a_3\}$  and  $E_2 = \{a_2a_3\}$ . Since  $\mu_{B_1}(a_1a_2) = 0.3, \mu_{B_1}(a_1a_3) = 0.3, \mu_{B_1}(a_2a_3) = 0$ , therefore

$$\mu_{B_1}^2(a_1a_2) = (\mu_{B_1} \circ \mu_{B_1})(a_1a_2) = \mu_{B_1}(a_1a_3) \wedge \mu_{B_1}(a_3a_2) = 0.3 \wedge 0.0 = 0,$$

$$\mu_{B_1}^2(a_2a_3) = (\mu_{B_1} \circ \mu_{B_1})(a_2a_3) = \mu_{B_1}(a_2a_1) \wedge \mu_{B_1}(a_1a_3) = 0.3 \wedge 0.3 = 0.3,$$

$$\mu_{B_1}^2(a_1a_3) = (\mu_{B_1} \circ \mu_{B_1})(a_1a_3) = \mu_{B_1}(a_1a_2) \wedge \mu_{B_1}(a_2a_3) = 0.3 \wedge 0.0 = 0,$$

$$\mu_{B_1}^3(a_1a_2) = (\mu_{B_1}^2 \circ \mu_{B_1})(a_1a_2) = \mu_{B_1}^2(a_1a_3) \wedge \mu_{B_1}(a_3a_2) = 0.0 \wedge 0.0 = 0,$$

$$\mu_{B_1}^3(a_2a_3) = (\mu_{B_1}^2 \circ \mu_{B_1})(a_2a_3) = \mu_{B_1}^2(a_2a_1) \wedge \mu_{B_1}(a_1a_3) = 0.0 \wedge 0.3 = 0,$$

$$\mu_{B_1}^3(a_1a_3) = (\mu_{B_1}^2 \circ \mu_{B_1})(a_1a_3) = \mu_{B_1}^2(a_1a_2) \wedge \mu_{B_1}(a_2a_3) = 0.0 \wedge 0.0 = 0.$$

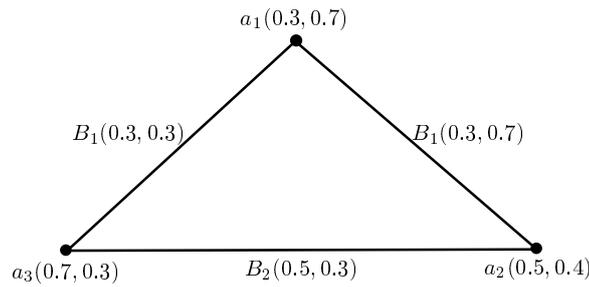


FIGURE 4. IFGS  $\check{G}_s = (A, B_1, B_2)$

Thus, we have

$$\mu_{B_1}^\infty(a_1a_2) = \bigvee \{0.3, 0.0, 0.0\} = 0.3,$$

$$\mu_{B_1}^\infty(a_2a_3) = \bigvee \{0.0, 0.3, 0.0\} = 0.3,$$

$$\mu_{B_1}^\infty(a_1a_3) = \bigvee \{0.3, 0.0, 0.0\} = 0.3.$$

Since  $\nu_{B_1}(a_1a_2) = 0.7$ ,  $\nu_{B_1}(a_1a_3) = 0.3$ ,  $\nu_{B_1}(a_2a_3) = 0$ , therefore

$$\begin{aligned}\nu_{B_1}^2(a_1a_2) &= (\nu_{B_1} \circ \nu_{B_1})(a_1a_2) = \nu_{B_1}(a_1a_3) \vee \nu_{B_1}(a_3a_2) = 0.3 \vee 0.0 = 0.3, \\ \nu_{B_1}^2(a_2a_3) &= (\nu_{B_1} \circ \nu_{B_1})(a_2a_3) = \nu_{B_1}(a_2a_1) \vee \nu_{B_1}(a_1a_3) = 0.7 \vee 0.3 = 0.7, \\ \nu_{B_1}^2(a_1a_3) &= (\nu_{B_1} \circ \nu_{B_1})(a_1a_3) = \nu_{B_1}(a_1a_2) \vee \nu_{B_1}(a_2a_3) = 0.7 \vee 0.0 = 0.7, \\ \nu_{B_1}^3(a_1a_2) &= (\nu_{B_1}^2 \circ \nu_{B_1})(a_1a_2) = \nu_{B_1}^2(a_1a_3) \vee \nu_{B_1}(a_3a_2) = 0.7 \vee 0.0 = 0.7, \\ \nu_{B_1}^3(a_2a_3) &= (\nu_{B_1}^2 \circ \nu_{B_1})(a_2a_3) = \nu_{B_1}^2(a_2a_1) \vee \nu_{B_1}(a_1a_3) = 0.3 \vee 0.3 = 0.3, \\ \nu_{B_1}^3(a_1a_3) &= (\nu_{B_1}^2 \circ \nu_{B_1})(a_1a_3) = \nu_{B_1}^2(a_1a_2) \vee \nu_{B_1}(a_2a_3) = 0.3 \vee 0.0 = 0.3,\end{aligned}$$

and

$$\begin{aligned}\nu_{B_1}^4(a_1a_2) &= (\nu_{B_1}^3 \circ \nu_{B_1})(a_1a_2) = \nu_{B_1}^3(a_1a_3) \vee \nu_{B_1}(a_3a_2) = 0.3 \vee 0.0 = 0.3, \\ \nu_{B_1}^4(a_2a_3) &= (\nu_{B_1}^3 \circ \nu_{B_1})(a_2a_3) = \nu_{B_1}^3(a_2a_1) \vee \nu_{B_1}(a_1a_3) = 0.7 \vee 0.3 = 0.7, \\ \nu_{B_1}^4(a_1a_3) &= (\nu_{B_1}^3 \circ \nu_{B_1})(a_1a_3) = \nu_{B_1}^3(a_1a_2) \vee \nu_{B_1}(a_2a_3) = 0.7 \vee 0.0 = 0.7.\end{aligned}$$

Thus, we have

$$\begin{aligned}\nu_{B_1}^\infty(a_1a_2) &= \vee \{0.7, 0.3, 0.7, 0.3\} = 0.7, \\ \nu_{B_1}^\infty(a_2a_3) &= \vee \{0.0, 0.7, 0.3, 0.7\} = 0.7, \\ \nu_{B_1}^\infty(a_1a_3) &= \vee \{0.3, 0.7, 0.3, 0.7\} = 0.7.\end{aligned}$$

By similar calculations, it can be easily checked that

$$\begin{aligned}\mu_{B_2}^\infty(a_1a_2) &= 0, \quad \mu_{B_2}^\infty(a_2a_3) = 0.5, \quad \mu_{B_2}^\infty(a_1a_3) = 0, \\ \nu_{B_2}^\infty(a_1a_2) &= 0.3, \quad \nu_{B_2}^\infty(a_2a_3) = 0.3, \quad \nu_{B_2}^\infty(a_1a_3) = 0.3.\end{aligned}$$

**Definition 3.12.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  of a graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$  is a  $B_i$ -cycle, if  $G^*$  is an  $E_i$ -cycle.

**Definition 3.13.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  of a graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$  is an *intuitionistic fuzzy  $B_i$ -cycle* for some  $i$ , if following conditions hold:

- (i)  $\check{G}_s$  is a  $B_i$ -cycle;
- (ii) There is no unique  $B_i$ -edge  $uv$  in  $\check{G}_s$ , such that  $\mu_{B_i}(uv) = \min\{\mu_{B_i}(xy) : xy \in E_i = \text{supp}(B_i)\}$  or  $\nu_{B_i}(uv) = \max\{\nu_{B_i}(xy) : xy \in E_i = \text{supp}(B_i)\}$ .

*Example 3.7.* IFGS  $\check{G}_s = (A, B_1, B_2)$  shown in Fig. 3, is a  $B_1$ -cycle as well as intuitionistic fuzzy  $B_1$ -cycle, since  $(\text{supp}(A), \text{supp}(B_1), \text{supp}(B_2))$  is an  $E_1$ -cycle and there are two  $B_1$ -edges with minimum degree of membership and two  $B_1$ -edges with maximum degree of nonmembership of all  $B_1$ -edges.

**Definition 3.14.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  of a graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$  is a  $B_i$ -tree, if  $(\text{supp}(A), \text{supp}(B_1), \text{supp}(B_2), \dots, \text{supp}(B_n))$  is an  $E_i$ -tree. In other words,  $\check{G}_s$  is a  $B_i$ -tree if the subgraph of  $\check{G}_s$ , induced by  $\text{supp}(B_i)$ , forms a tree.

**Definition 3.15.** An IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  of a graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$  is an *intuitionistic fuzzy  $B_i$ -tree (intuitionistic fuzzy  $B_i$ -forest)*, if  $\check{G}_s$  has an intuitionistic fuzzy partial spanning subgraph structure  $\check{H}_s = (A, C_1, C_2, \dots, C_n)$ , such that  $\check{H}_s$  is a  $C_i$ -tree ( $C_i$ -forest) and  $\mu_{B_i}(xy) < \mu_{C_i}^\infty(xy)$  and  $\nu_{B_i}(xy) < \nu_{C_i}^\infty(xy)$  for all  $B_i$ -edges not in  $\check{H}_s$ .

*Example 3.8.* The IFGS, shown in Fig. 3, is a  $B_2$ -tree but not an intuitionistic fuzzy  $B_2$ -tree. While IFGS  $\check{G}_s = (A, B_1, B_2)$ , shown in Fig. 5, is not a  $B_1$ -tree but an intuitionistic fuzzy  $B_1$ -tree, since it has an intuitionistic fuzzy partial spanning subgraph structure  $(A, B'_1, B'_2)$  as a  $B_1$ -tree, which is obtained by deleting  $B_1$ -edge  $a_1a_4$  from  $\check{G}_s$ , with  $\mu_{B_1}(a_1a_4) = 0.3 < 0.4 = \mu_{B'_1}^\infty(a_1a_4)$  and  $\nu_{B_1}(a_1a_4) = 0.5 < 0.6 = \nu_{B'_1}^\infty(a_1a_4)$ .

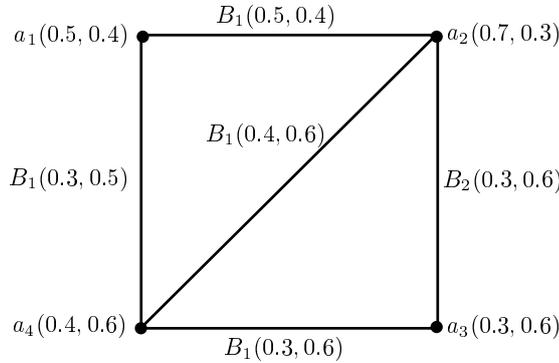


FIGURE 5. IFGS  $\check{G}_s = (A, B_1, B_2)$

**Definition 3.16.** An IFGS  $\check{G}_{s1} = (A_1, B_{11}, B_{12}, \dots, B_{1n})$  of GS  $G_1^* = (U_1, E_{11}, E_{12}, \dots, E_{1n})$  is *isomorphic* to an IFGS  $\check{G}_{s2} = (A_2, B_{21}, B_{22}, \dots, B_{2n})$  of  $G_2^* = (U_2, E_{21}, E_{22}, \dots, E_{2n})$ , if there exist a bijection  $f : U_1 \rightarrow U_2$  and a permutation  $\phi$  on the set  $\{1, 2, \dots, n\}$ , such that:

$$\mu_{A_1}(u_1) = \mu_{A_2}(f(u_1)), \quad \nu_{A_1}(u_1) = \nu_{A_2}(f(u_1)), \quad \text{for all } u_1 \in U_1$$

and for  $\phi(i) = j$

$$\mu_{B_{1i}}(u_1u_2) = \mu_{B_{2j}}(f(u_1)f(u_2)), \quad \nu_{B_{1i}}(u_1u_2) = \nu_{B_{2j}}(f(u_1)f(u_2)),$$

for all  $u_1u_2 \in E_{1i}$ ,  $i = 1, 2, \dots, n$ .

**Definition 3.17.** An IFGS  $\check{G}_{s1} = (A_1, B_{11}, B_{12}, \dots, B_{1n})$  of GS  $G_1^* = (U, E_{11}, E_{12}, \dots, E_{1n})$  is *identical* to an IFGS  $\check{G}_{s2} = (A_2, B_{21}, B_{22}, \dots, B_{2n})$  of  $G_2^* = (U, E_{21}, E_{22}, \dots, E_{2n})$ , if there exist a bijection  $f : U \rightarrow U$ , such that:

$$\mu_{A_1}(u) = \mu_{A_2}(f(u)), \quad \nu_{A_1}(u) = \nu_{A_2}(f(u)), \quad \text{for all } u \in U$$

and

$$\mu_{B_{1i}}(u_1u_2) = \mu_{B_{2i}}(f(u_1)f(u_2)), \quad \nu_{B_{1i}}(u_1u_2) = \nu_{B_{2i}}(f(u_1)f(u_2)),$$

for all  $u_1u_2 \in E_{1i}$ ,  $i = 1, 2, \dots, n$ .

*Example 3.9.*  $\check{G}_{s1}$  and  $\check{G}_{s2}$ , as shown in Fig. 6 and Fig. 7, are IFGSs of graph structures  $G^*_1 = (U_1, E_1, E_2, E_3, E_4)$  and  $G^*_2 = (U_2, E'_1, E'_2, E'_3, E'_4)$ , respectively, where

$$\begin{aligned} U_1 &= \{a_1, a_2, a_3, a_4, a_5\}, & E_1 &= \{a_1a_2, a_2a_5\}, \\ E_2 &= \{a_2a_3, a_2a_4\}, & E_3 &= \{a_1a_3, a_4a_5\}, \\ E_4 &= \{a_1a_5, a_3a_4\}, & U_2 &= \{b_1, b_2, b_3, b_4, b_5\}, \\ E'_1 &= \{b_2b_4, b_3b_4\}, & E'_2 &= \{b_1b_4, b_4b_5\}, \\ E'_3 &= \{b_1b_2, b_3b_5\}, & E'_4 &= \{b_1b_5, b_2b_3\}. \end{aligned}$$

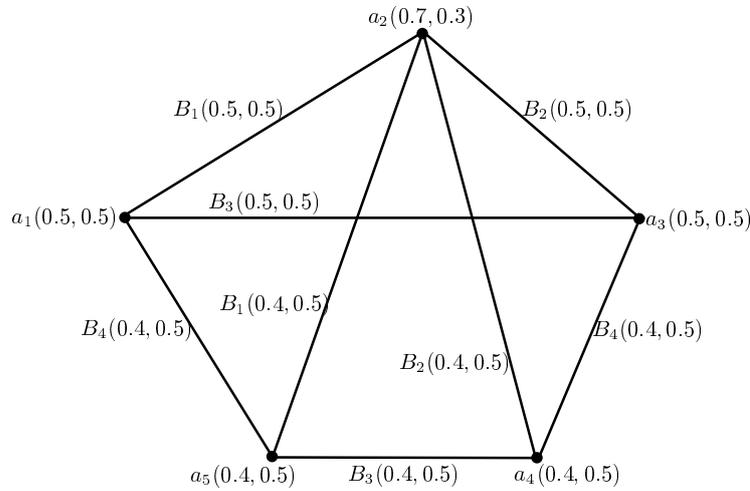


FIGURE 6. IFGS  $\check{G}_{s1} = (A_1, B_1, B_2, B_3, B_4)$

Then  $\check{G}_{s1}$  is isomorphic to  $\check{G}_{s2}$  under the mapping  $f : U_1 \rightarrow U_2$ , given by

$$f(a_1) = b_5, \quad f(a_2) = b_4, \quad f(a_3) = b_3, \quad f(a_4) = b_2, \quad f(a_5) = b_1,$$

and a permutation  $\phi$  given by

$$\phi(1) = 2, \quad \phi(2) = 1, \quad \phi(3) = 3, \quad \phi(4) = 4,$$

such that

$$\mu_{A_1}(a_i) = \mu_{A_2}(f(a_i)), \quad \nu_{A_1}(a_i) = \nu_{A_2}(f(a_i))$$

for all  $a_i \in U_1$ , and

$$\mu_{B_k}(a_i a_j) = \mu_{B_{\phi(k)}}(f(a_i) f(a_j)), \quad \nu_{B_k}(a_i a_j) = \nu_{B_{\phi(k)}}(f(a_i) f(a_j)),$$

for all  $a_i a_j \in E_k, k = 1, 2, 3, 4$ .

Also,  $\check{G}_{s1}$  is identical with  $\check{G}_{s2}$  under the mapping  $f : U_1 \rightarrow U_2$ , given by

$$f(a_1) = b_3, \quad f(a_2) = b_4, \quad f(a_3) = b_5, \quad f(a_4) = b_1, \quad f(a_5) = b_2,$$

such that

$$\mu_{A_1}(a_i) = \mu_{A_2}(f(a_i)), \quad \nu_{A_1}(a_i) = \nu_{A_2}(f(a_i)),$$

for all  $a_i \in U_1$ , and

$$\mu_{B_k}(a_i a_j) = \mu_{B'_k}(f(a_i)f(a_j)), \quad \nu_{B_k}(a_i a_j) = \nu_{B'_k}(f(a_i)f(a_j)),$$

for all  $a_i a_j \in E_k, k = 1, 2, 3, 4$ .

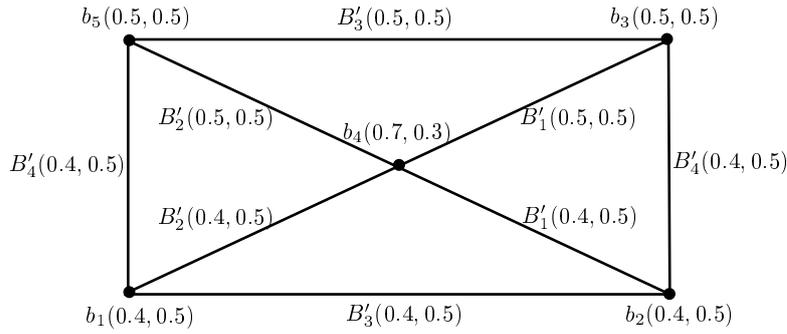


FIGURE 7. IFGS  $\check{G}_{s_2} = (A_2, B'_1, B'_2, B'_3, B'_4)$

*Remark 3.1.* Identical IFGSs are always isomorphic but the converse is not necessarily true. As IFGS shown in Fig. 3 is isomorphic to IFGS shown in Fig. 8 but they are not identical.

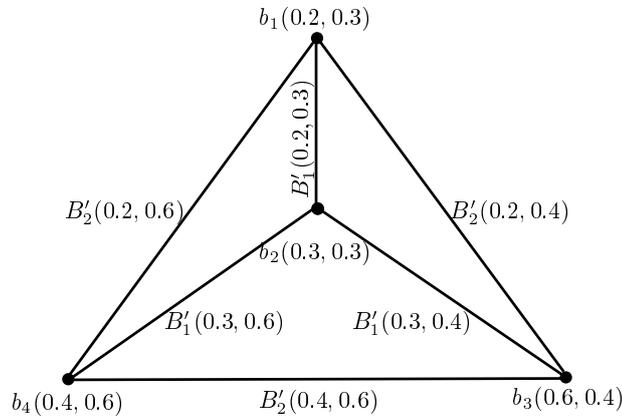


FIGURE 8. IFGS  $\check{G}_{s_1} = (A_1, B'_1, B'_2)$

**Definition 3.18.** Let  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  be an intuitionistic fuzzy graph structure of a graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$ . Let  $\phi$  denote a permutation on the set  $\{E_1, E_2, \dots, E_n\}$  and the corresponding permutation on  $\{B_1, B_2, \dots, B_n\}$ , i.e.,  $\phi(B_i) = B_j$  iff  $\phi(E_i) = E_j$  for all  $i$ .

If  $xy \in B_r$  for some  $r$  and

$$\mu_{B_i^\phi}(xy) = \mu_A(x) \wedge \mu_A(y) - \bigvee_{j \neq i} \mu_{\phi B_j}(xy),$$

$$\nu_{B_i^\phi}(xy) = \nu_A(x) \vee \nu_A(y) - \bigvee_{j \neq i} \nu_{\phi B_j}(xy), \quad i = 1, 2, \dots, n,$$

then  $xy \in B_m^\phi$ , while  $m$  is chosen such that  $\mu_{B_m^\phi}(xy) \geq \mu_{B_i^\phi}(xy)$  and  $\nu_{B_m^\phi}(xy) \geq \nu_{B_i^\phi}(xy)$  for all  $i$ .

Then IFGS  $(A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$  denoted by  $\check{G}_s^{\phi c}$ , is called the  $\phi$ -complement of IFGS  $\check{G}_s$ .

**Theorem 3.1.** *A  $\phi$ -complement of an intuitionistic fuzzy graph structure is always a strong IFGS. Moreover, if  $\phi(i) = r$  for  $r, i \in \{1, 2, \dots, n\}$ , then all  $B_r$ -edges in IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  become  $B_i^\phi$ -edges in  $\check{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$ .*

*Proof.* First part is obvious from the definition of  $\phi$ -complement  $\check{G}_s^{\phi c}$  of IFGS  $\check{G}_s$ , since for any  $B_i^\phi$ -edge  $xy$ ,  $\mu_{B_i^\phi}^\phi(xy)$  and  $\nu_{B_i^\phi}^\phi(xy)$  respectively have the maximum values of

$$(3.1) \quad [\mu_A(x) \wedge \mu_A(y)] - \bigvee_{j \neq i} \mu_{\phi B_j}(xy) \quad \text{and} \quad [\nu_A(x) \vee \nu_A(y)] - \bigvee_{j \neq i} \nu_{\phi B_j}(xy).$$

That is,

$$(3.2) \quad \mu_{B_i^\phi}^\phi(xy) = \mu_A(x) \wedge \mu_A(y), \quad \nu_{B_i^\phi}^\phi(xy) = \nu_A(x) \vee \nu_A(y),$$

for all edges  $xy$  in  $\check{G}_s^{\phi c}$ , hence  $\check{G}_s^{\phi c}$  is always a strong IFGS.

Now suppose on contrary that  $\phi(i) = r$  but  $xy$  is a  $B_s$ -edge in  $\check{G}_s$  with  $s \neq r$ , which implies that  $\phi B_i \neq B_s$ . Comparing expressions (3.1) and (3.2), we get

$$\bigvee_{j \neq i} \mu_{\phi B_j}(xy) = 0, \quad \bigvee_{j \neq i} \nu_{\phi B_j}(xy) = 0,$$

which is not possible because  $B_s = \phi B_j$  for some  $j \in \{1, 2, \dots, i-1, i+1, \dots, n\}$ . So our supposition is wrong and  $xy$  must be a  $B_r$ -edge. Hence we can conclude that if  $\phi(i) = r$ , then all  $B_r$ -edges in IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  become  $B_i^\phi$ -edges in  $\check{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$  for  $r, i \in \{1, 2, \dots, n\}$ .  $\square$

*Example 3.10.* Consider IFGS  $\check{G}_s = (A, B_1, B_2)$  shown in Fig. 4 and let  $\phi$  be a permutation on the set  $\{B_1, B_2\}$  such that  $\phi(B_1) = B_2$  and  $\phi(B_2) = B_1$ .

Now for  $a_1 a_2 \in B_1$ ,

$$\mu_{B_1^\phi}^\phi(a_1 a_2) = \mu_A(a_1) \wedge \mu_A(a_2) - \bigvee_{j \neq 1} [\mu_{\phi B_j}(a_1 a_2)] = 0.3 \wedge 0.5 - [\mu_{\phi B_2}(a_1 a_2)]$$

$$= 0.3 - \mu_{B_1}(a_1 a_2) = 0.3 - 0.3 = 0,$$

$$\nu_{B_1^\phi}^\phi(a_1 a_2) = \nu_A(a_1) \vee \nu_A(a_2) - \bigvee_{j \neq 1} [\nu_{\phi B_j}(a_1 a_2)] = 0.7 \vee 0.4 - [\nu_{\phi B_2}(a_1 a_2)]$$

$$= 0.7 - \nu_{B_1}(a_1 a_2) = 0.7 - 0.7 = 0,$$

$$\mu_{B_2^\phi}^\phi(a_1 a_2) = \mu_A(a_1) \wedge \mu_A(a_2) - \bigvee_{j \neq 2} [\mu_{\phi B_j}(a_1 a_2)] = 0.3 \wedge 0.5 - [\mu_{\phi B_1}(a_1 a_2)]$$

$$\begin{aligned}
 &= 0.3 - \mu_{B_2}(a_1a_2) = 0.3 - 0 = 0.3, \\
 \nu_{B_2}^\phi(a_1a_2) &= \nu_A(a_1) \vee \nu_A(a_2) - \bigvee_{j \neq 2} [\nu_{\phi B_j}(a_1a_2)] = 0.7 \vee 0.4 - [\nu_{\phi B_1}(a_1a_2)] \\
 &= 0.7 - \nu_{B_2}(a_1a_2) = 0.7 - 0 = 0.7.
 \end{aligned}$$

Clearly,  $\mu_{B_2}^\phi(a_1a_2) = 0.3 > 0 = \mu_{B_1}^\phi(a_1a_2)$  and  $\nu_{B_2}^\phi(a_1a_2) = 0.7 > 0 = \nu_{B_1}^\phi(a_1a_2)$ , so  $a_1a_2 \in B_2^\phi$ .

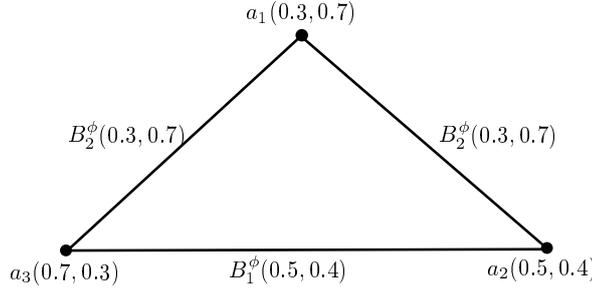


FIGURE 9. IFGS  $\check{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi)$

Similarly for  $a_1a_3 \in B_1$ ,

$$\mu_{B_1}^\phi(a_1a_3) = 0, \quad \nu_{B_1}^\phi(a_1a_3) = 0.4, \quad \mu_{B_2}^\phi(a_1a_3) = 0.3, \quad \nu_{B_2}^\phi(a_1a_3) = 0.7.$$

Clearly,  $\mu_{B_2}^\phi(a_1a_3) = 0.3 > 0 = \mu_{B_1}^\phi(a_1a_3)$  and  $\nu_{B_2}^\phi(a_1a_3) = 0.7 > 0.4 = \nu_{B_1}^\phi(a_1a_3)$ , so  $a_1a_3 \in B_2^\phi$ .

And for  $a_2a_3 \in B_2$

$$\mu_{B_1}^\phi(a_2a_3) = 0.5, \quad \nu_{B_1}^\phi(a_2a_3) = 0.4, \quad \mu_{B_2}^\phi(a_2a_3) = 0, \quad \nu_{B_2}^\phi(a_2a_3) = 0.1,$$

that is,  $\mu_{B_1}^\phi(a_2a_3) = 0.5 > 0 = \mu_{B_2}^\phi(a_2a_3)$  and  $\nu_{B_1}^\phi(a_2a_3) = 0.4 > 0.1 = \nu_{B_2}^\phi(a_2a_3)$ , so  $a_2a_3 \in B_1^\phi$ .

This implies that  $B_1^\phi = \{(a_2a_3, 0.5, 0.4)\}$ ,  $B_2^\phi = \{(a_1a_2, 0.3, 0.7), (a_1a_3, 0.3, 0.7)\}$  and  $\check{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi)$  shown in Fig. 9 is the  $\phi$ -complement of  $\check{G}_s$ .

**Definition 3.19.** Let  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  be an IFGS and  $\phi$  be a permutation on the set  $\{1, 2, \dots, n\}$ . Then

- (i)  $\check{G}_s$  is *self-complementary*, if it is isomorphic to  $\check{G}_s^{\phi c}$ , the  $\phi$ -complement of  $\check{G}_s$ .
- (ii)  $\check{G}_s$  is *strong self-complementary*, if it is identical to  $\check{G}_s^{\phi c}$ .
- (iii)  $\check{G}_s$  is *totally self-complementary*, if it is isomorphic to  $\check{G}_s^{\phi c}$ , the  $\phi$ -complement of  $\check{G}_s$  for all permutations  $\phi$  on the set  $\{1, 2, \dots, n\}$ .
- (iv)  $\check{G}_s$  is *totally strong self-complementary*, if it is identical to  $\check{G}_s^{\phi c}$ , the  $\phi$ -complement of  $\check{G}_s$  for all permutations  $\phi$  on the set  $\{1, 2, \dots, n\}$ .

**Theorem 3.2.** An IFGS  $\check{G}_s$  is strong if and only if  $\check{G}_s$  is totally self-complementary.

*Proof.* Let  $\check{G}_s$  be a strong IFGS and  $\phi$  be any permutation on the set  $\{1, 2, \dots, n\}$ .

If  $\phi^{-1}(i) = j$ , then by Theorem 3.1, all  $B_i$ -edges in  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  become  $B_j^\phi$ -edges in  $\check{G}_s^{\phi c} = (A, B_1^\phi, B_2^\phi, \dots, B_n^\phi)$ . Also  $\check{G}_s^{\phi c}$  is strong, so

$$\begin{aligned}\mu_{B_i}(a_1 a_2) &= \mu_A(a_1) \wedge \mu_A(a_2) = \mu_{B_j^\phi}(a_1 a_2), \\ \nu_{B_i}(a_1 a_2) &= \nu_A(a_1) \vee \nu_A(a_2) = \nu_{B_j^\phi}(a_1 a_2).\end{aligned}$$

Then  $\check{G}_s$  is isomorphic to  $\check{G}_s^{\phi c}$ , under the identity mapping  $f : U \rightarrow U$  and a permutation  $\phi [\phi^{-1}(i) = j, i, j = 1, 2, \dots, n]$ , such that

$$\mu_A(a) = \mu_A(f(a)), \quad \nu_A(a) = \nu_A(f(a)), \quad \text{for all } a \in U$$

and

$$\begin{aligned}\mu_{B_i}(a_1 a_2) &= \mu_{B_j^\phi}(a_1 a_2) = \mu_{B_j^\phi}(f(a_1) f(a_2)), \\ \nu_{B_i}(a_1 a_2) &= \nu_{B_j^\phi}(a_1 a_2) = \nu_{B_j^\phi}(f(a_1) f(a_2)), \quad \text{for all } a_1 a_2 \in E_i.\end{aligned}$$

This holds for all permutations on the set  $\{1, 2, \dots, n\}$ . Hence  $\check{G}_s$  is totally self-complementary.

Conversely, let  $\phi$  be any permutation on the set  $\{1, 2, \dots, n\}$  and  $\check{G}_s$  and  $\check{G}_s^{\phi c}$  be isomorphic. From the definition of  $\phi$ -complement and isomorphism of IFGSs, we have

$$\begin{aligned}\mu_{B_i}(a_1 a_2) &= \mu_{B_j^\phi}(f(a_1) f(a_2)) = \mu_A(f(a_1)) \wedge \mu_A(f(a_2)) = \mu_A(a_1) \wedge \mu_A(a_2), \\ \nu_{B_i}(a_1 a_2) &= \nu_{B_j^\phi}(f(a_1) f(a_2)) = \nu_A(f(a_1)) \vee \nu_A(f(a_2)) = \nu_A(a_1) \vee \nu_A(a_2),\end{aligned}$$

for all  $a_1 a_2 \in E_i, i = 1, 2, \dots, n$ .

Hence,  $\check{G}_s$  is a strong IFGS. □

*Remark 3.2.* Every self-complementary IFGS is necessarily totally self-complementary.

**Theorem 3.3.** *If graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$  is totally strong self-complementary and  $A$  is an IFS of  $U$  with constant fuzzy mappings  $\mu_A$  and  $\nu_A$  then a strong IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  of  $G^*$  is totally strong self-complementary.*

*Proof.* Consider a strong IFGS  $\check{G}_s = (A, B_1, B_2, \dots, B_n)$  of a graph structure  $G^* = (U, E_1, E_2, \dots, E_n)$ . Suppose that  $G^*$  is totally strong self-complementary and that for some constants  $s, t \in [0, 1]$ ,  $A = (\mu_A, \nu_A)$  is an IFS of  $U$  such that  $\mu_A(u) = s, \nu_A(u) = t$ , for all  $u \in U$ . Then we have to prove that  $\check{G}_s$  is totally strong self-complementary.

Let  $\phi$  be an arbitrary permutation on the set  $\{1, 2, \dots, n\}$  and  $\phi^{-1}(j) = i$ . Since  $G^*$  is totally strong self-complementary, so there exists a bijection  $f : U \rightarrow U$ , such that for every  $E_i$ -edge  $a_1 a_2$  in  $G^*$ ,  $f(a_1) f(a_2)$  (an  $E_j$ -edge in  $G^*$ ) is an  $E_i$ -edge in  $(G^*)^{\phi^{-1}c}$ . Consequently, for every  $B_i$ -edge  $a_1 a_2$  in  $\check{G}_s$ ,  $f(a_1) f(a_2)$  (a  $B_j$ -edge in  $\check{G}_s$ ) is a  $B_i^\phi$ -edge in  $\check{G}_s^{\phi c}$ .

From the definition of  $A$  and the definition of strong IFGS  $\check{G}_s$

$$\mu_A(a) = s = \mu_A(f(a)), \quad \nu_A(a) = t = \nu_A(f(a)), \quad \text{for all } a, f(a) \in U,$$

and

$$\begin{aligned} \mu_{B_i}(a_1a_2) &= \mu_A(a_1) \wedge \mu_A(a_2) = \mu_A(f(a_1)) \wedge \mu_A(f(a_2)) = \mu_{B_i^\phi}(f(a_1)f(a_2)), \\ \nu_{B_i}(a_1a_2) &= \nu_A(a_1) \vee \nu_A(a_2) = \nu_A(f(a_1)) \vee \nu_A(f(a_2)) = \nu_{B_i^\phi}(f(a_1)f(a_2)), \end{aligned}$$

for all  $a_1a_2 \in B_i, i = 1, 2, \dots, n$ , which shows  $\check{G}_s$  is strong self-complementary. Hence  $\check{G}_s$  is totally strong self-complementary, since  $\phi$  is arbitrary.  $\square$

*Remark 3.3.* Converse of Theorem 3.3 is not necessary, since a totally strong self-complementary and strong IFGS  $\check{G}_s = (A, B_1, B_2, B_3)$  as shown in Fig. 11, has a totally strong self-complementary underlying graph structure but  $\mu_A$  and  $\nu_A$  are not constant fuzzy functions.

*Example 3.11.* The IFGS shown in Fig. 8 is self-complementary, i.e., it is isomorphic to its  $\phi$ -complement, where  $\phi = (1\ 2)$ . Also, it is totally self-complementary because  $\phi$  is the only non-identity permutation on set  $\{1, 2\}$ .

*Example 3.12.* The IFGS  $\check{G}_s = (A_1, B_1, B_2, B_3, B_4)$  shown in Fig. 10, is strong self-complementary, i.e., it is identical to its  $\phi$ -complement where the permutation  $\phi$  is  $(1\ 2)(3\ 4)$ . It is not totally strong self-complementary.

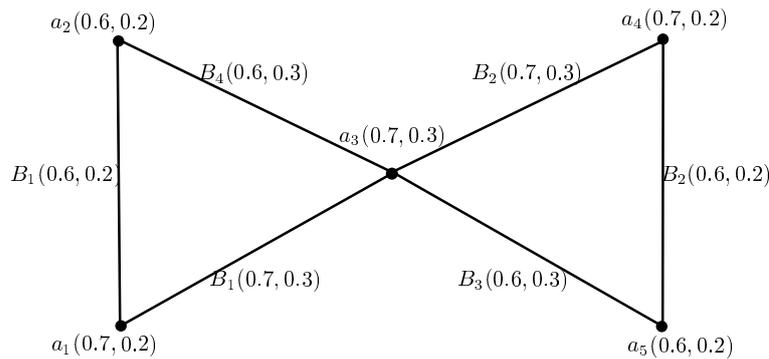
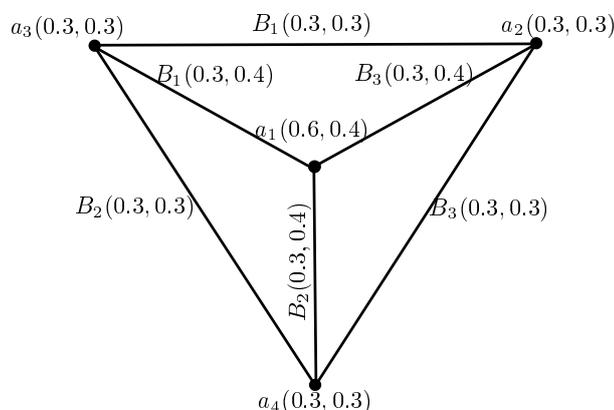


FIGURE 10. IFGS  $\check{G}_s = (A_1, B_1, B_2, B_3, B_4)$

*Example 3.13.* The IFGS  $\check{G}_s = (A_1, B_1, B_2, B_3)$ , shown in Fig. 11, is totally strong self-complementary because it is identical to its  $\phi$ -complement for all the permutations  $\phi$  on the set  $\{1, 2, 3\}$ .

FIGURE 11. IFGS  $\check{G}_s = (A_1, B_1, B_2, B_3)$ 

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