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ON GRADED 2-NIL-GOOD RINGS

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Abstract. In this paper we introduce and study the notion of a graded 2-nil-good ring which is graded by a group. We discuss graded group ring and graded matrix ring extensions of graded 2-nil-good rings. The question of when the 2-nil-good property of the component, which corresponds to the identity element of the grading group, implies the graded 2-nil-good property of the whole graded ring is also examined.

1. Introduction

Ever since the introduction of *clean rings* in [20] as rings in which every element can be written as a sum of an idempotent and a unit, many papers have been written discussing the ring structure depending on the various ring element properties. In particular, many results are obtained concerning *nil clean rings* introduced in [6]. For instance, study of matrix rings over nil clean rings is related to the famous Köthe's Conjecture (see [16] and references therein). Nil-cleanness of group rings has also attracted attention (see [17, 21]).

Theory of graded rings has also been studied by many authors (see [13, 19]). *Graded nil clean rings* are introduced in [10], and in this paper we continue with studying rings determined by various properties defined elementwise from the graded ring theory point of view. Namely, we introduce and study *graded 2-nil-good rings* as a graded version of the notion introduced recently in [1]. In [1], a *2-nil-good ring* is defined as a ring whose every element is a sum of two units and a nilpotent and the main results deal with the question of when the matrix rings are 2-nil-good.

Here, by a *graded 2-nil-good ring* we mean a group graded ring whose every homogeneous element can be written as a sum of two homogeneous units and a homogeneous

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nilpotent. We start by giving the basic properties of graded 2-nil-good rings which represent graded versions of results concerning 2-nil-good rings. We are also interested in the question of when the (graded) group ring is (graded) 2-nil-good, which is the content of several theorems. These yield an interesting question of how the graded 2-nil-good property of a group graded ring depends on the 2-nil-good property of the component which corresponds to the identity element of the grading group. It is shown that 2-nil-good property of the component corresponding to the identity element of the grading group does not imply the graded 2-nil-good property of the whole graded ring in general. However, under some additional assumptions, this implication does hold true. Finally, we prove that the graded matrix ring over a crossed product, which is graded 2-nil-good, is also a graded 2-nil-good ring.

2. Preliminaries

All rings are assumed to be associative with identity. If R is a ring, then, as usual, $J(R)$ denotes the Jacobson radical of R, and $U(R)$ stands for the multiplicative group of units of R.

Next we recall the notions of a group graded ring and module, and how the group ring and the matrix ring over a group graded ring can be graded. For other graded ring theory notions and further details, we refer to [13, 19].

Let R be a ring, G a group with the identity element e, and let $\{R_g\}_{g\in G}$ be a family of additive subgroups of R. R is said to be G-*graded* if $R = \bigoplus_{g\in G} R_g$ and $R_gR_h \subseteq R_{gh}$ for all $g, h \in G$. The set $H = \bigcup_{g\in G} R_g$ is called the *homogeneous part* of R, elements of H are called *homogeneous*, and subgroups R_g $(g \in G)$ are called *components*. If $a \in R_g$, then we say that a has the *degree* g.

A G-graded ring $R = \bigoplus_{g\in G} R_g$ is called a *crossed product* if $U(R) \cap R_g \neq \emptyset$ for all $g \in G$.

A right ideal (left, two-sided) I of a G-graded ring $R = \bigoplus_{g\in G} R_g$ is called *homogeneous* or *graded* if $I = \bigoplus_{g\in G} I \cap R_g$. If I is a two-sided homogeneous ideal (homogeneous ideal in the rest of the paper), then R/I is a G-graded ring with components $(R/I)_g = R_g/I \cap R_g$. A graded ring R is *graded-nil* if every homogeneous element of R is nilpotent.

Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, and observe the group ring $R[G]$. According to [18], we have that $R[G]$ is G-graded with the g-component $(R[G])_g = \sum_{h\in G} R_{gh^{-1}}h$ and with the multiplication defined via the rule $(r_g g')(r_h h') = r_g r_h (h^{-1}g'hh')$, where $g, g', h, h' \in G$ and $r_g \in R_g$, $r_h \in R_h$.

If H is a normal subgroup of G, then, according to [19], we may observe $R[H]$ as a G-graded ring $\bigoplus_{g\in G}(R[H])_g$, where $(R[H])_g = \bigoplus_{h\in H} R_{gh^{-1}}h$, and where the multiplication is given by $(r_g g')(r_h h') = r_g r_h (h^{-1}g'hh')$, where $g, h \in G$, $g', h' \in H$ and $r_g \in R_g$, $r_h \in R_h$.

All of the group rings in this paper, if observed as graded rings, are assumed to be graded in one of the above described ways.

If R is a G-graded ring and n a natural number, then the matrix ring $M_n(R)$ can be made into a G-graded ring in the following manner. Let $\overline{\sigma} = (g_1, \ldots, g_n) \in G^n$, $\lambda \in G$ and $M_n(R)_\lambda(\overline{\sigma}) = (a_{ij})_{n\times n}$, where $a_{ij} \in R_{g_i\lambda g_j^{-1}}$, $i, j \in \{1, 2, \ldots, n\}$. Then $M_n(R) = \bigoplus_{\lambda\in G} M_n(R)_\lambda(\overline{\sigma})$ is a G-graded ring with respect to the usual matrix addition and multiplication. This ring is usually denoted by $M_n(R)(\overline{\sigma})$.

If $R = \bigoplus_{g\in G} R_g$ is a G-graded ring, then a *right G-graded R-module* is a right R-module M such that $M = \bigoplus_{x\in G} M_x$, where M_x are additive subgroups of M, and such that $M_xR_g \subseteq M_{xg}$ for all $x, g \in G$. A submodule N of a G-graded R-module $M = \bigoplus_{x\in G} M_x$ is called *homogeneous* if $N = \bigoplus_{x\in G} N \cap M_x$.

A right G-graded R-module M is said to be *graded irreducible* if $MR \neq 0$ and if the only homogeneous submodules of M are trivial submodules. The *graded Jacobson radical* $J^g(R)$ of a G-graded ring R is defined to be the intersection of annihilators of all graded irreducible graded R-modules. It is known that $J^g(R)$ coincides with the intersection of all maximal homogeneous right ideals of R, and that it is left-right symmetric.

3. Graded 2-Nil-Good Rings

Let G be a group with the identity element e.

Definition 3.1. A homogeneous element of a G-graded ring is said to be *graded 2-nil-good* if it can be written as a sum of two homogeneous units and a homogeneous nilpotent. A G-graded ring is said to be *graded 2-nil-good* if every of its homogeneous elements is graded 2-nil-good.

Example 3.1. Let $p > 2$ be a prime number, $G = \{e, g\}$ a cyclic group of order 2, and $R = \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ \mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix}$. The ring $\mathbb{Z}_p$ is a 2-nil-good ring (see [1, Example 2.1]). Then $R = \begin{pmatrix} \mathbb{Z}_p & 0 \\ 0 & \mathbb{Z}_p \end{pmatrix} \oplus \begin{pmatrix} 0 & \mathbb{Z}_p \\ \mathbb{Z}_p & 0 \end{pmatrix}$ is a G-graded 2-nil-good ring.

Remark 3.1. Let us notice that if $R = \bigoplus_{g\in G} R_g$ is a G-graded ring which is graded 2-nil-good, then R_e is a 2-nil-good ring. Namely, even if $a \in R_e$ is a nilpotent, we can always write $a = 1 + (-1) + a$, and $1 \in R_e$. If $g \neq e$, for a nilpotent element $a \in R_g$, we may have a different situation, that is, it may be the case that a is written as a sum of itself and of two units which are not of degree g, for instance $a = 1 + (-1) + a$. However, if we assume that R is a crossed product, then every homogeneous element can be written as a sum of two homogeneous units and a homogeneous nilpotent, all of which are of the same degree.

In [22], 2-*good rings* are defined as rings in which every element can be written as a sum of two units. Since 2-good rings are closely related to 2-nil-good rings, as every 2-good ring is 2-nil-good, it is natural to introduce the following notion as well.

Definition 3.2. A homogeneous element of a G-graded ring is said to be *graded 2-good* if it can be written as a sum of two homogeneous units. A G-graded ring is said to be *graded 2-good* if every of its homogeneous elements is graded 2-good.

Remark 3.2. Let us notice that all graded 2-good rings are crossed products. Also, obviously, every graded 2-good ring is graded 2-nil-good. Example 3.1 also serves as an example of a graded 2-good ring.

In [1] it is proved that R is a 2-nil-good ring if and only if R/I is 2-nil-good, whenever I is a nil ideal of R. Here we have the following result.

Theorem 3.1. *Let R be a G-graded ring and I a graded-nil ideal of R. Then R is graded 2-nil-good if and only if R/I is graded 2-nil-good.*

Proof. If R is graded 2-nil-good, then R/I is also graded 2-nil-good as a graded homomorphic image of R.

Conversely, let R/I be a graded 2-nil-good ring and let $\bar{x} = x + I \in R_g/I_g$, where $g \in G$. Then $\bar{x} = \bar{u} + \bar{v} + \bar{w}$, where $\bar{u}$, $\bar{v}$ are homogeneous units of R/I, and $\bar{w}$ is a nilpotent element of degree g in R/I. Since I is graded-nil, we have that w is a homogeneous nilpotent of degree g in R. Also, since I, as a graded-nil ideal, is contained in the graded Jacobson radical $J^g(R)$, homogeneous units lift modulo I (see [19, Proposition 2.9.1]), and the claim follows. ☐

Corollary 3.1. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, where G is a finite group, and R_e is a PI-ring. Also, let $I \subseteq J(R)$ be a homogeneous ideal of R such that I_e is nil. Then R is graded 2-nil-good if and only if R/I is graded 2-nil-good.*

Proof. Since G is finite and R_e is a PI-ring, by [12] we know that R is also a PI-ring. This and the fact that $I \subseteq J(R)$ is a homogeneous ideal with I_e nil together imply that I is nil by [14, Lemma 5]. In particular, I is graded-nil, and the claim follows by the previous theorem. ☐

Definition 3.3 ([10])**.** A homogeneous element a of a G-graded ring is said to be *graded strongly π-regular* if it can be written as a sum of a homogeneous idempotent element f and a homogeneous unit u such that $fa = af$ and faf is nilpotent.

Naturally, by a *graded strongly π-regular ring* we mean a G-graded ring whose every homogeneous element is graded strongly π-regular.

The following result represents a graded version of [1, Theorem 2.1].

Theorem 3.2. *Let $R = \bigoplus_{g\in G} R_g$ be a graded strongly π-regular ring. The following statements are equivalent:*

i) *R is graded 2-nil-good;*
ii) *$1 = u + v$ for some units u, v from R_e.*

Proof. i)⇒ii) If R is a graded 2-nil-good ring, it follows that R_e is 2-nil-good. Since $1 \in R_e$, the claim follows by [1, Theorem 2.1] applied to the ring R_e.

ii)⇒i) Again, if we apply [1, Theorem 2.1] to the ring R_e, we have that R_e is 2-nil-good. Let $0 \neq x \in R_g$, where $g \neq e$. Then, since R is graded strongly π-regular, it follows that x is a unit. Since $1 = u + v$, with $u, v \in U(R_e)$, we have that $x = 1x = ux + vx + 0$, hence x is graded 2-nil-good. ☐

By [3, Proposition 10], if R is a clean ring with $2 \in U(R)$, then R is 2-good. We end this section with a graded version of this result.

Theorem 3.3. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. If R is graded clean and $2 \in U(R)$, then R is graded 2-good.*

Proof. By assumption, R is graded clean, which means that R_e is clean (see [10]). Since $2 \in R_e$, we have that R_e is 2-good by [3, Proposition 10]. Now, let $0 \neq x \in R_g$, where $g \neq e$. Since R is by assumption graded clean, we have that x is a unit. Therefore $x/2 \neq 0$ is a homogeneous unit u of degree g. Hence $x = 2u = u + u$, and so, R is graded 2-good. □

4. Extensions of Graded 2-Nil-Good Rings

4.1. **Group rings.** In this subsection we investigate graded 2-nil-good property of graded group rings. However, we first establish some sufficient conditions for a group ring to be 2-nil-good.

Theorem 4.1. *Let R be a 2-nil-good ring, and let p be a prime number which is nilpotent in R. If G is a locally finite p-group, then $R[G]$ is a 2-nil-good ring.*

Proof. As in the proof of [21, Theorem 2.3], we may assume that G is a finite p-group. Since p is nilpotent, by [5, Theorem 9], we have that the augmentation ideal $\Delta(R[G])$ is nilpotent. Since $R[G]/\Delta(R[G])$ and R are isomorphic as rings, by [1, Theorem 2.2], we then have that $R[G]$ is a 2-nil-good ring. □

Remark 4.1. One example of a 2-nil-good ring satisfying the assumptions of the previous theorem is $\mathbb{Z}_p$, where $p > 2$ is a prime number.

Theorem 4.2. *Let R be a clean ring with $2 \in U(R)$. If $p > 2$ is a prime number belonging to $J(R)$, and G a locally finite p-group, then $R[G]$ is 2-nil-good.*

Proof. Since R is clean, G a locally finite p-group and $p \in J(R)$, according to [24, Theorem 4], we have that $R[G]$ is clean. Also, since 2 is a unit in R, it is also a unit in $R[G]$. By [3, Proposition 10], $R[G]$ is 2-good, and therefore, 2-nil-good. □

Theorem 4.3. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, where G is a finite group. If R is a semilocal ring with $2 \in U(R)$, then $R[G]$ is a 2-nil-good ring.*

Proof. If R is semilocal with $2 \in U(R)$, then by [23, Proposition 2.10] we have that R is 2-good, and therefore 2-nil-good. Now, by [18, Proposition 2.1(4)], we have that $(R[G])_e$ and R are isomorphic as rings. Therefore $(R[G])_e$ is a semilocal ring. According to [2], we have that $R[G]$ is semilocal too. Also, as 2 is a unit in R it is also a unit in $R[G]$. Hence, $R[G]$ is 2-good, and therefore 2-nil-good. □

Next we deal with the graded 2-nil-good property of graded group rings. It is convenient now to recall that if G is a group, and H a normal subgroup of G, then a G-graded ring $R = \bigoplus_{g\in G} R_g$ can be viewed as a G/H-graded ring $R = \bigoplus_{C\in G/H} R_C$, where $R_C = \bigoplus_{x\in C} R_x$ (see, for instance, [13, 19]).

Theorem 4.4. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, where G is a locally finite p-group, and let H be a normal subgroup of G. Also, let us assume that p is nilpotent in R. If R is graded 2-nil-good as a G/H-graded ring, then $R[H]$ is graded 2-nil-good as a G/H-graded ring.*

Proof. Again, as in the proof of [21, Theorem 2.3], we may assume that H is a finite p-group. We know from [19], page 180, that $R[H]/\Delta(R[H])$ and R are graded isomorphic as G/H-graded rings. Since p is nilpotent, according to [5, Theorem 9], we have that $\Delta(R[H])$ is nilpotent, and in particular, graded-nil. Hence, by Theorem 3.1, $R[H]$ is graded 2-nil-good as a G/H-graded ring. □

Theorem 4.5. *Let R be a G-graded ring and H a normal subgroup of G. Also, let R be graded clean as a G/H-graded ring with $2 \in U(R)$. If $p > 2$ is a prime number belonging to the H-component of the graded Jacobson radical $J^{G/H}(R)$ of R, regarded as a G/H-graded ring, and G a locally finite p-group, then $R[H]$ is graded 2-nil-good as a G/H-graded ring.*

Proof. According to our assumptions, since R is graded clean as a G/H-graded ring, we have that $R[H]$ is graded clean as a G/H-graded ring. This follows by Theorem 4.1 in [E. Ilić-Georgijević, *On graded clean group rings*, preprint]. We give here a short proof for readers' convenience. We may assume that G is a finite p-group. Since $R[H]/\Delta(R[H])$ and R are graded isomorphic as G/H-graded rings, we have that $R[H]/\Delta(R[H])$ is graded clean. In particular, $(R[H]/\Delta(R[H]))_H$ is clean, that is, $R[H]_H \cong R_H$ is clean by [19, Proposition 6.2.1]. Since G is finite, $J^{G/H}(R)_H \subseteq J^g(R) \subseteq J(R)$, by [4, Theorem 4.4]. Now, by [24, Lemma 2], we have that $\Delta(R[H])$ is contained in $J(R[H])$. By [4, Theorem 4.4], we have that $\Delta(R[H]) \subseteq J^{G/H}(R[H])$. This, together with the fact that $R[H]_H \cong R_H$, and [19, Proposition 2.9.1vi)], implies that $R[H]$ is a graded clean G/H-graded ring. Now, as 2 is a unit in R, we have that 2 is also a unit in $R[H]$. Therefore by Theorem 3.3, $R[H]$ is graded 2-nil-good as a G/H-graded ring. □

Theorem 4.6. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring, where G is a locally finite 2-group, and let H be a normal subgroup of G. Also, let us assume that R_e is a nil clean ring. If R is graded 2-nil-good as a G/H-graded ring, then $R[H]$ is graded 2-nil-good as a G/H-graded ring.*

Proof. We again can assume that H is finite. We know from [19], page 180, that $R[H]/\Delta(R[H])$ and R are graded isomorphic as G/H-graded rings. Since R_e is by assumption nil clean, we have that 2 is nilpotent by [6, Proposition 3.14] applied to R_e. Hence, according to [5, Theorem 9], $\Delta(R[H])$ is nilpotent, and in particular, graded-nil. Now, by Theorem 3.1, it follows that $R[H]$ is graded 2-nil-good ring as a G/H-graded ring. □

Let us return to Theorem 4.1 for a moment. Since R and $(R[G])_e$ are isomorphic as rings, we have that $(R[G])_e$ is 2-nil-good. If we moreover assume that the units and nilpotents of $R[G]$ are all homogeneous, then we of course get that $R[G]$ is graded 2-nil-good. Also, let us take a look at the following example.

Example 4.1. Let S be a 2-nil-good ring, $G = \{e, g\}$ a cyclic group of order 2, and $R = \begin{pmatrix} S & S \\ 0 & S \end{pmatrix}$. Then $R = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix} \oplus \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ is a G-graded ring whose e-component $R_e = \begin{pmatrix} S & 0 \\ 0 & S \end{pmatrix}$ is a 2-nil-good ring, and also R is a graded 2-nil-good ring since elements of $R_g = \begin{pmatrix} 0 & S \\ 0 & 0 \end{pmatrix}$ are nilpotent and therefore, graded 2-nil-good, as every $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ can be written as $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$.

These observations lead to the question of when the following implication holds true

$$R_e \text{ is 2-nil-good} \Rightarrow R = \bigoplus_{g \in G} R_g \text{ is graded 2-nil-good.} \tag{4.1}$$

The following example proves that the above implication does not hold in general.

Example 4.2. Let R be a commutative 2-good ring which is moreover reduced, and let $R[x]$ be a polynomial ring with indeterminate x. Then $R[x]$ is $\mathbb{Z}$-graded with i-component Rx^i if $i \geq 0$ and 0 if $i < 0$ (see for instance [19]). Then $R_0 = R$ is a 2-nil-good ring. Also, since R is reduced, we have that $U(R[x]) = U(R)$ by [11, Corollary 1.7]. Hence if x is graded 2-nil-good, then $x = u + v + w$, where $u, v \in U(R)$ and $w \in N(R[x])$. In other words, $x - u - v$ is nilpotent, which is impossible. Therefore $R[x]$ is not graded 2-nil-good.

Theorem 4.7. *Let $R = \bigoplus_{g \in G} R_g$ be a G-graded PI-ring which is graded local, that is, it has a unique maximal homogeneous right ideal, and let G be a finite group such that the order of G is a unit in R. Also, let $R_g R_{g^{-1}} = 0$ for every $g \in G \setminus \{e\}$. If R_e is 2-nil-good ring with nil Jacobson radical $J(R_e)$, then R is graded 2-nil-good.*

Proof. Assumptions on R_e imply that $R_e/J(R_e)$ is a 2-nil-good ring. Further, [4, Corollary 4.2, Theorem 4.4] and [14, Theorem 3] together imply that $J(R)$ is a graded-nil ideal of R. According to [10, Theorem 3.27] (see also the proof of [9, Theorem 3.2]), we have that every homogeneous element of $R/J(R)$ is a 2-nil-good element of $R_e/J(R_e)$. Hence $R/J(R)$ is graded 2-nil-good, and thus by Theorem 3.1, R is graded 2-nil-good. □

Theorem 4.8. *Let $R = \bigoplus_{g \in G} R_g$ be a G-graded ring of finite support, where G is a torsion free group. Also, let R be a semiprimary ring with R_e local and $2 \in U(R)$. Then R is graded 2-nil-good.*

Proof. Since $2 \in R_e$, and R_e is local, by [23, Proposition 2.10] we know that R_e is 2-nil-good. It follows that $R_e/J(R_e)$ is 2-nil-good. By [19, Proposition 9.6.4], we have that $J^g(R) = J(R)$ and that $R/J(R) = R_e/J(R_e)$. Since R is semiprimary, $J(R)$ is nil and so the claim follows by Theorem 3.1. □

4.2. **Matrix rings.** Since graded 2-good rings are graded 2-nil-good, let us start with the question of whether the graded matrix ring over a graded 2-good ring is also a graded 2-good ring. In [23] this is answered in affirmative for the classical, that is, ungraded case. Their proof relies on the technique which can be seen in the proof of [8, Lemma], that is, they prove (see [23, Proposition 3.6]) that a ring R is 2-good if the corner rings, with respect to some idempotent of a ring, are 2-good. The following theorem represents a graded version of that result.

Theorem 4.9. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring and $f \in R_e$ an idempotent. Let us write $\bar{f} = 1 - f$. If fRf and $\bar{f}R\bar{f}$ are graded 2-good rings, then R is also a graded 2-good ring.*

Proof. Since graded 2-good rings are crossed products, the proof of [23, Proposition 3.6] can be easily modified to our setting. We give a sketch of the proof. Let $R = \begin{pmatrix} fRf & fR\bar{f} \\ \bar{f}Rf & \bar{f}R\bar{f} \end{pmatrix}$ be the Pierce decomposition of R, and let $A = \begin{pmatrix} a & x \\ y & b \end{pmatrix} \in R_g$, where $g \in G$. Since fRf is by assumption graded 2-good ring, and graded 2-good rings are crossed products, there exist $u_1, u_2 \in U(fRf) \cap R_g$ such that $a = u_1 + u_2$. Now, $b - yu_2^{-1}x \in \bar{f}R\bar{f}$. Again, by assumption, $\bar{f}R\bar{f}$ is a graded 2-good ring, and as it is a crossed product, there exist $v_1, v_2 \in U(\bar{f}R\bar{f}) \cap R_g$ such that $b - yu_2^{-1}x = v_1 + v_2$. The rest of the proof goes as in the proof of [23, Proposition 3.6] (see also [8, Lemma]). □

This theorem by mathematical induction implies the following corollaries.

Corollary 4.1. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. If $1 = f_1 + \cdots + f_n$ in R, where $f_i \in R_e$ are orthogonal idempotents and each f_iRf_i is graded 2-good, then R is graded 2-good.*

Corollary 4.2. *Let $R = \bigoplus_{g\in G} R_g$ be a G-graded ring. If R is graded 2-good and n a natural number, then $M_n(R)(\overline{\sigma})$ is graded 2-good for every $\overline{\sigma} \in G^n$.*

In order to obtain a similar result for graded 2-nil-good rings, we first give a graded version of [1, Theorem 4.1].

However, let us first recall from [7] what a *G-graded Morita context* is. So, let $A = \bigoplus_{g\in G} A_g$ and $B = \bigoplus_{g\in G} B_g$ be G-graded rings, and let $V = \bigoplus_{g\in G} V_g$ and $W = \bigoplus_{g\in G} W_g$ be G-graded $A-B$ and $B-A$-bimodules, respectively. Then a quadruple (A, V, W, B) is a G-graded Morita context if (A, V, W, B) is a Morita context and if $V_gW_h \subseteq A_{gh}$ and $W_hV_g \subseteq B_{hg}$ for all $g, h \in G$ (see [7]). The ring $R = \begin{pmatrix} A & V \\ W & B \end{pmatrix}$ can be G-graded with respect to any $\overline{\sigma} \in G^2$ as it is described in Preliminaries (see [19]) and then it will be denoted by $R(\overline{\sigma}) = \bigoplus_{\lambda\in G} R_\lambda(\overline{\sigma})$.

Theorem 4.10. *Let (A, V, W, B) be a G-graded Morita context. If A and B are graded 2-nil-good rings which are crossed products, then $R(\overline{\sigma})$ is a graded 2-nil-good ring for every $\overline{\sigma} = (g_1, g_2) \in G \times G$.*

Proof. First, let us notice that, since A and B are crossed products, that $R(\overline{\sigma})$ is also a crossed product. Let $M \in R_\lambda(\overline{\sigma})$, where $\lambda \in G$. Then $M = \begin{pmatrix} a & x \\ y & b \end{pmatrix}$, where $a \in A_{g_1\lambda g_1^{-1}}$, $x \in V_{g_1\lambda g_2^{-1}}$, $y \in W_{g_2\lambda g_1^{-1}}$, $b \in B_{g_2\lambda g_2^{-1}}$. Since A and B are moreover crossed products, there exist $u_1^a, u_2^a \in U(A) \cap A_{g_1\lambda g_1^{-1}}$, and $u_1^b, u_2^b \in U(B) \cap B_{g_2\lambda g_2^{-1}}$ such that $a = u_1^a + u_2^a + n^a$ and $b = u_1^b + u_2^b + n^b$ for some nilpotents $n^a \in A_{g_1\lambda g_1^{-1}}$, $n^b \in B_{g_2\lambda g_2^{-1}}$. Therefore $M = \begin{pmatrix} u_1^a & x \\ 0 & u_1^b \end{pmatrix} + \begin{pmatrix} u_2^a & 0 \\ y & u_2^b \end{pmatrix} + \begin{pmatrix} n^a & 0 \\ 0 & n^b \end{pmatrix}$ is a graded 2-nil-good element. □

Corollary 4.3. *Let $R = \bigoplus_{g\in G} R_g$ be a crossed product. If R is graded 2-nil-good and n a natural number, then $M_n(R)(\overline{\sigma})$ is graded 2-nil-good for every $\overline{\sigma} \in G^n$.*

Proof. This follows by the previous theorem by using mathematical induction (cf. [1, Corollary 4.2]). □

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References

[1] M. S. Abdolyousefi, N. Ashrafi and H. Chen, *On 2-nil-good rings*, J. Algebra Appl. **17**(6) (2018), Paper ID 1850110, 13 pages.

[2] M. Beattie and E. Jespers, *On perfect graded rings*, Comm. Algebra **19** (1991), 2363–2371.

[3] V. P. Camillo and H. P. Yu, *Exchange rings, units and idempotents*, Comm. Algebra **22**(12) (1994), 4737–4749.

[4] M. Cohen and S. Montgomery, *Group-graded rings, smash products, and group actions*, Trans. Amer. Math. Soc. **282** (1984), 237–258.

[5] I. G. Connell, *On the group ring*, Canad. J. Math. **15** (1963), 650–685.

[6] A. J. Diesl, *Nil clean rings*, J. Algebra **383** (2013), 197–211.

[7] H. J. Fang and P. Stewart, *Radical theory for graded rings*, J. Aust. Math. Soc. **52**(2) (1992), 143–153.

[8] J. Han and W. K. Nicholson, *Extensions of clean rings*, Comm. Algebra **29**(6) (2001), 2589–2595.

[9] E. Ilić-Georgijević, *On graded special radicals of graded rings*, J. Algebra Appl. **17**(6) (2018), Paper ID 1850109, 10 pages.

[10] E. Ilić-Georgijević, S. Şahinkaya, *On graded nil clean rings*, Comm. Algebra **46**(9) (2018), 4079–4089.

[11] P. Kanwar, A. Leroy and J. Matczuk, *Clean elements in polynomial rings*, Contemp. Math. **634** (2015), 197–204.

[12] A. V. Kelarev, *On semigroup graded PI-algebras*, Semigroup Forum **47** (1993), 294–298.

[13] A. V. Kelarev, *Ring Constructions and Applications*, Series in Algebra **9**, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002.

[14] A. V. Kelarev and J. Okniński, *On group graded rings satisfying polynomial identities*, Glasgow Math. J. **37** (1995), 205–210.
[15] T. Y. Lam, *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics **131**, Springer-Verlag, New York, 1991.
[16] J. Matczuk, *Conjugate* (*nil*) *clean rings and Köthe's problem*, J. Algebra Appl. **16**(2) (2017), Paper ID 1750073, 14 pages.
[17] W. Wm. McGovern, S. Raja and A. Sharp, *Commutative nil clean group rings*, J. Algebra Appl. **14**(6) (2015), Paper ID 1550094, 5 pages.
[18] C. Năstăsescu, *Group rings of graded rings. Applications*, J. Pure Appl. Algebra **33** (1984), 313–335.
[19] C. Năstăsescu and F. Van Oystaeyen, *Methods of Graded Rings*, Lecture Notes in Mathematics **1836**, Springer, Berlin, Heidelberg, 2004.
[20] W. K. Nicholson, *Lifting idempotents and exchange rings*, Trans. Amer. Math. Soc. **229** (1977), 269–278.
[21] S. Şahinkaya, G. Tang and Y. Zhou, *Nil-clean group rings*, J. Algebra Appl. **16**(5) (2017), Paper ID 1750135, 7 pages.
[22] P. Vámos, 2-*good rings*, Q. J. Math. **56**(3) (2005), 417–430.
[23] Y. Wang and Y. Ren, 2-*good rings and their extensions*, Bull. Korean Math. Soc. **50**(5) (2013), 1711–1723.
[24] Y. Zhou, *On Clean Group Rings, Advances in Ring Theory*, Bürkhauser Verlag Basel, Switzerland, 2010, 335–345.

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