Kragujevac Journal of Mathematics
Volume 43(4) (2019), Pages 523–534.

SPECTRAL PROPERTIES OF NC-GRAPHS

M. GHORBANI[1] AND Z. GHARAVI-ALKHANSARI[1]

Abstract. Let G be a non-abelian group and $Z(G)$ be the center of G. The non-commuting graph (NC-graph) $\Gamma(G)$ of the group G is a graph with the vertex set $G \setminus Z(G)$ and two distinct vertices x and y are adjacent whenever $xy \neq yx$. The aim of this paper is to prove that for given group G, $\frac{G}{Z(G)} \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $\Gamma(G)$ is a regular $(p+1)$-partite graph. Also we consider the isomorphism of the non-commuting graph with some special graphs.

1. Introduction

All graphs considered in this paper are simple and finite, also all groups are finite. There are a number of constructions of graphs from groups or semi-groups in the literature. Let G be a non-abelian group with the center $Z(G)$. The non-commuting graph (NC-graph) $\Gamma(G)$ is a simple and undirected graph with the vertex set $G \setminus Z(G)$ and two vertices $x, y \in G \setminus Z(G)$ are adjacent whenever $xy \neq yx$. The concept of NC-graphs was first considered by Paul Erdős to answer a question on the size of the cliques of a graph in 1975, see [14]. For background materials about NC-graphs, we encourage the reader to see references [1, 3, 7, 10, 12, 13].

Here, in the next section, we give necessary definitions and some preliminary results and the third section contains the main results on complete multipartite NC-graphs. Finally, in Section 4, we determine which graphs are NC-graph.

2. Definitions and Preliminaries

Our notation is standard and mainly taken from standard books such as [6, 15]. The vertex and edge sets of graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively. The degree $\deg_\Gamma(v)$ of a vertex v in Γ is the number of edges incident to v. A graph Γ

Key words and phrases. Non-commuting graph, centralizer, p-groups.
2010 *Mathematics Subject Classification.* Primary: 05E15. Secondary: 05C25, 20D99.
Received: June 16, 2018.
Accepted: September 06, 2018.

is *regular* if the degrees of all vertices of Γ are the same. The length of the shortest cycle in a graph Γ is called the *girth* of Γ. The largest distance between all pairs of vertices of Γ is called the *diameter* of Γ. A graph which has a cycle that contains every vertex of Γ, is called a *Hamiltonian graph*. An *Eulerian graph* is a graph whose all vertices have even degree.

An *independent set* is a set of vertices in which none of them are adjacent. A *k-partite graph* is a graph whose vertices are partitioned into k disjoint sets $V = V_1 \cup \cdots \cup V_k$, where V_i, $1 \leq i \leq k$, are independent sets. In other words, the graph for which the endpoints of every edge are in different sets is called k-partite graph. A graph consisting of two rows of s paired vertices in which all vertices but the paired ones are connected with an edge is called a *hyperoctahedral graph* or *cocktail-party graph*, denoted by $CP(s)$.

The *lexicographic product* or *graph composition* $\Gamma_1 \circ \Gamma_2$ of two graphs Γ_1 and Γ_2 is a graph with the vertex set $V(\Gamma_1) \times V(\Gamma_2)$ and any two vertices (u, v) and (x, y) are adjacent in $\Gamma_1 \circ \Gamma_2$ if and only if either u is adjacent with x in Γ_1 or $u = x$ and v is adjacent with y in Γ_2.

Let Γ be a connected graph of order n, the vertex $x \in V(\Gamma)$ is called *well-connected* if $\deg_\Gamma(x) = n - 1$. In other words, the well-connected vertex is adjacent to all other vertices. Let $S \subseteq V(\Gamma)$ be a non-empty set, then $N_\Gamma[S]$ is the set of vertices in Γ which are in S or adjacent to a vertex in S. If $N_\Gamma[S] = V(\Gamma)$, then S is said to be a *dominating set* of vertices in Γ. The *domination number* of a graph Γ denoted by $\gamma(\Gamma)$ is the minimum size of a dominating set of vertices in Γ. In [1, Proposition 2.12], it is proved that if $\{x\}$ is a dominating set for $\Gamma(G)$, then $Z(G) = 1$ and $C_G(x) = \langle x \rangle$, where $x^2 = 1$.

A *p-group* is a group that the order of every element is a power of p. Let G, H be two groups with group homomorphism $\varphi : H \to \mathrm{Aut}(G)$. A *semi-direct product* of G and H with respect to φ denoted by $G \rtimes_\varphi H$ (or simply $G \rtimes H$) is a new group with set $G \times H$ and multiplication operation $(g_1; h_1)(g_2; h_2) = (g_1\varphi_{h_1}(g_2), h_1h_2)$. For a group G, we recall that $\mathrm{Cent}(G) = \{C_G(x) \mid x \in G\}$, where $C_G(x)$ is the centralizer of the element x in G, see [2, 4, 5]. A group G is called an *AC-group*, if for each $x \in G \setminus Z(G)$, $C_G(x)$ is abelian, see [1, 4].

Here, in the following examples, we determine the structures of NC-graphs of two groups $\mathbb{S}_3$ and U_{6n}.

Example 2.1. Consider the symmetric group $\mathbb{S}_3$ by the following presentation:

$$\mathbb{S}_3 = \langle a, b \mid a^2 = 1, b^3 = 1, a^{-1}ba = b^{-1} \rangle.$$

This group is the smallest non-abelian group and its order is 6. The center of this group is trivial and so $\mathbb{S}_3 \setminus Z(\mathbb{S}_3) = \{a, b, b^2, ab, ab^2\}$. Since, only b commutes with b^2, we have $\Gamma(\mathbb{S}_3) \cong K_5 - e$, where $K_n - e$ denotes the graph obtained from the complete graph K_n by deleting an edge.

Example 2.2. The group U_{6n} has the following presentation:

$$U_{6n} = \langle a, b \mid a^{2n} = 1, b^3 = 1, a^{-1}ba = b^{-1}\rangle.$$

The elements of this group are

$$\{1, a, \dots, a^{2n-1}, b, ba, \dots, ba^{2n-1}, b^2, b^2a, \dots, b^2a^{2n-1}\}.$$

One can see that $Z(U_{6n}) = \langle a^2\rangle$ and so $|Z(U_{6n})| = n$. This implies that

$$|V(\Gamma(U_{6n}))| = |U_{6n}| - |Z(U_{6n})| = 5n.$$

Let i, j be odd, then

$$(a^ib)(a^jb) = (a^ib)a(a^{j-1}b) = a^i(ba)a^{j-1}b = a^{i+j} = (a^jb)(a^ib).$$

Hence, $\{ab, a^3b, \dots, a^{2n-1}b\}$ is an independent set. By a similar way, we can prove that if i, j are odd, then $(a^ib^2)(a^jb^2) = (a^jb^2)(a^ib^2)$ and so $\{ab^2, \dots, a^{2n-1}b^2\}$ is an independent set. Now, we can easily prove that the following sets are independent:

$$\{a, a^3, \dots, a^{2n-1}\}, \{ab, a^3b, \dots, a^{2n-1}b\}, \{ab^2, a^3b^2, \dots, a^{2n-1}b^2\},$$
$$\{b, b^2, a^2b, a^2b^2, \dots, a^{2n-2}b, a^{2n-2}b^2\}.$$

This implies that $\Gamma(U_{6n})$ is a 4-partite graph.

3. Main Results

In [9], it is proved that there is no regular NC-graph of valency p^n, where p is an odd prime number and n is a positive integer. In general, we have the following result.

Theorem 3.1. ([9, Theorem 3.4]). *Let G be a finite non-abelian group such that $\Gamma(G)$ is 2^s-regular, for some positive integer s. Then G is a 2-group.*

Theorem 3.2. *Let G be a finite non-abelian group such that $\Gamma(G)$ is a regular graph. Then $\Gamma(G)$ is Eulerian.*

Proof. Let $\Gamma(G)$ be a k-regular graph. It is sufficient to prove that k is even. On the contrary, suppose that k is odd. Then, for any non-central element $x \in G$,

$$k = |G| - |C_G(x)| = |C_G(x)|(|G : C_G(x)| - 1),$$

from which we deduce that $|C_G(x)|$ is odd and that $|G \setminus Z(G)|$ is even. Since $|C_G(x)|$ is odd for all non-central elements $x \in G$, all non-central elements of G have odd order as does $Z(G)$. This contradicts the fact that $|G \setminus Z(G)|$ is even. □

Theorem 3.3. *There is no 2-regular NC-graph.*

Proof. Let G be a finite non-abelian group such that $\Gamma(G)$ is a 2-regular graph. Then for each non-central element $x \in G$, we have

$$2 = |G| - |C_G(x)| = |C_G(x)|(|G : C_G(x)| - 1).$$

Since x is a non-central element, $|C_G(x)| = 2$ and $|G : C_G(x)| - 1 = 1$. Hence, $|G| = |C_G(x)| + 2 = 4$. Therefore, G is abelian, which is a contradiction. □

Theorem 3.4. *Let p be a prime number. If $[G : Z(G)] = p^2$, then $\Gamma(G)$ is a regular graph. In particular, $\Gamma(G)$ is Eulerian.*

Proof. Since $[G : Z(G)] = p^2$, for every $x \in G$, we have

$$[G : Z(G)] = [G : C_G(x)][C_G(x) : Z(G)] = p^2.$$

If $x \in G \setminus Z(G)$, then we claim that $[G : C_G(x)] = p$ and so $[C_G(x) : Z(G)] = p$. If $[G : C_G(x)] = 1$, then clearly $x \in C_G(x) = Z(G)$, a contradiction. If $[G : C_G(x)] = p^2$ and $[C_G(x) : Z(G)] = 1$, then $\deg(x) = |G| - |C_G(x)| = |G| - |Z(G)| = |V(\Gamma(G)|$, i.e., the vertex x has a loop, a contradiction. Thus for every $x \in G \setminus Z(G)$, we have $[G : C_G(x)] = p$ and therefore $\deg(x) = |G| - |C_G(x)| = |G| - |G|/p = \frac{(p-1)|G|}{p}$. This implies that $\Gamma(G)$ is a regular graph. □

Theorem 3.5 ([5])**.** *Let G be a finite non-abelian group, then $|\operatorname{Cent}(G)| = 4$ if and only if $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Theorem 3.6 ([5])**.** *Let G be a finite non-abelian group and p be a prime. If $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ then $|\operatorname{Cent}(G)| = p + 2$.*

Theorem 3.7. *Let G be a finite non-abelian group. Then $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ if and only if $\Gamma(G)$ is a complete tripartite graph.*

Proof. Suppose that $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Theorem 3.5, $|\operatorname{Cent}(G)| = 4$ and hence $G = C_G(x_1) \cup C_G(x_2) \cup C_G(x_3)$, where $x_1, x_2, x_3 \in G \setminus Z(G)$. According to the proof of [5, Theorem 2] all $C_G(x_i)$'s are abelian. Set $X_i := C_G(x_i) \setminus Z(G)$. Clearly the sets X_1, X_2, X_3 are independent sets and thus are the all three parts of $\Gamma(G)$. According to Theorem 3.4, $\Gamma(G)$ is regular, so $|C_G(x_i)| = |C_G(x_j)|$, $1 \leq i, j \leq 3$, and thus $|X_i| = |X_j|$. Since every proper centralizer of G is abelian, by [2, Remark 2.1], we conclude that X_i's are disjoint sets with $G \setminus Z(G) = X_1 \cup X_2 \cup X_3$. On the other hand, for every pair of vertices $x \in X_i$, $y \in X_j$, $i \neq j$, $xy \neq yx$ and so x and y are adjacent. Hence $\Gamma(G)$ is a complete tripartite graph with parts X_1, X_2, X_3. Conversely, assume that $\Gamma(G)$ is a complete tripartite graph with parts X_1, X_2, X_3. Thus each part X_i of $\Gamma(G)$ is abelian. For every vertex $x \in X_i$, x commutes with all other elements of X_i and does not commute with elements of X_j, $j \neq i$. For every vertex $x_i \in X_i$, we have $X_i = C_G(x_i) \setminus Z(G)$ and thus

$$V(\Gamma(G)) = X_1 \cup X_2 \cup X_3 = (C_G(x_1) \cup C_G(x_2) \cup C_G(x_3)) \setminus Z(G).$$

This implies that $G = C_G(x_1) \cup C_G(x_2) \cup C_G(x_3)$ and so $|\operatorname{Cent}(G)| = 4$. Therefore by Theorem 3.5, $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and this completes the proof. □

Theorem 3.8. *Let G be a finite non-abelian group and p be a prime number. If $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\Gamma(G)$ is a complete $(p+1)$-partite graph such that each part has cardinality $(p-1)|Z(G)|$.*

Proof. By Theorem 3.6, $|\operatorname{Cent}(G)| = p + 2$. Let

$$\operatorname{Cent}(G) = \{G, C_G(x_1), \ldots, C_G(x_{p+1})\}.$$

According to the proof of [5, Theorem 5] all of these proper centralizers of G, i.e., $C_G(x_i)$'s are abelian. By [2, Remark 2.1], the set $\{C_G(x)|x \in G \setminus Z(G)\}$ is a partition for G and has $Z(G)$ as its kernel. Hence $C_G(x_i) \setminus Z(G)$, $1 \leq i \leq p+1$, are the parts of the graph $\Gamma(G)$. Therefore $\Gamma(G)$ is a complete $(p+1)$-partite graph. Now let x be the cardinality of each part of the graph $\Gamma(G)$. Then the number of vertices of the graph is $(p+1)x$. Thus $|G| - |Z(G)| = (p+1)x$. Therefore,

$$x = \frac{|G| - |Z(G)|}{p+1} = \frac{(p^2-1)|Z(G)|}{p+1} = (p-1)|Z(G)|.$$

Hence, the proof is complate. □

Remark 3.1. Let $G \cong P \times \mathbb{Z}_q$ where p, q are prime numbers and P be a p-group. Hence we have $G/Z(G) \cong P/Z(P)$. Thus, $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

Proposition 3.1 ([1]). *Let G be a finite non-abelian group such that $\Gamma(G)$ is a regular graph. Then G is nilpotent of class at most 3 and $G = P \times A$, where A is an abelian group, P is a p-group (p is a prime) and furthermore $\Gamma(P)$ is a regular graph.*

Proposition 3.2. *Let p, q be prime numbers and P be a p-group. Then $\Gamma(P)$ is k-regular if and only if $\Gamma(P \times \mathbb{Z}_q)$ is kq-regular.*

Proof. Suppose that $\Gamma(P)$ is k-regular and let $G = P \times \mathbb{Z}_q$. Then for every vertex $x \in P \setminus Z(P)$, we have $\deg_{\Gamma(P)}(x) = |P| - |C_P(x)| = k$. Thus for every $(x, y) \in G \setminus Z(G)$,

$$\begin{aligned}\deg_{\Gamma(G)}(x, y) &= \deg_{\Gamma(P)}(x)|\mathbb{Z}_q| \\ &= |P||\mathbb{Z}_q| - |C_P(x)||\mathbb{Z}_q| \\ &= (|P| - |C_P(x)|)q = kq.\end{aligned}$$

Conversely, if $G = P \times \mathbb{Z}_q$ and $\Gamma(G)$ is kq-regular, then $\deg_{\Gamma(G)}(x, y) = \deg_{\Gamma(P)}(x).q$ and so

$$\deg_{\Gamma(P)}(x) = \frac{\deg_{\Gamma(P \times \mathbb{Z}_q)}(x, y)}{q} = \frac{kq}{q} = k.$$

Thus the proof is complete. □

In the following, by $\overline{K}_n$ we mean the complement of graph K_n.

Remark 3.2. Let p be a prime number, P be a p-group and $G = P \times A$, where A is an abelian group. Then the graph $\Gamma(G)$ is lexicographic product of $\Gamma(P)$ around $\overline{K}_{|A|}$, i.e., $\Gamma(G) \cong \Gamma(P) \circ \overline{K}_{|A|}$.

Theorem 3.9. *Let p be a prime number and P be a non-abelian p-group. Then $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $\Gamma(P)$ is a regular complete $(p+1)$-partite graph.*

Proof. If $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then according to Theorem 3.8, $\Gamma(P)$ is a regular complete $(p+1)$-partite graph. Conversely, assume that $\Gamma(P)$ is a regular complete

$(p+1)$-partite graph and let $n = |P| - |Z(P)| = p^t - p^k$, for some $t, k \in \mathbb{N}$. This yields that the size of each part is $\frac{n}{p+1}$. Hence for every $x \in P \setminus Z(P)$, we have

$$\frac{n}{p+1} = |C_P(x)| - |Z(P)| = p^u - p^k, \quad u \in \mathbb{N}.$$

This implies that $\frac{p^t - p^k}{p+1} = p^u - p^k$ and so $p^{t-k} - 1 = p^{u-k+1} - p + p^{u-k} - 1$. Since $u < t$, we deduce that $t - k = u - k + 1$ and $u - k = 1$. Consequently, $t = k + 2$, $u = k + 1$ and thus $|P|/|Z(P)| = p^t/p^k = p^2$. But as $P/Z(P)$ is non-abelian, it can not be cyclic. Hence, $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and the proof is complete. □

Corollary 3.1. *Let G be a finite group and p be a prime number. Then $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ if and only if $\Gamma(G)$ is a regular complete $(p+1)$-partite graph.*

Proof. By Theorem 3.8, if $G/Z(G) \cong \mathbb{Z}_p \times \mathbb{Z}_p$, then $\Gamma(G)$ is a regular complete $(p+1)$-partite graph. Conversely, assume that $\Gamma(G)$ is a k-regular complete $(p+1)$-partite graph. By Proposition 3.1, $G = P \times A$, where A is abelian and P is a non-abelian p-group for some prime p. Clearly $G/Z(G) \cong P/Z(P)$. Since $\Gamma(G)$ is a k-regular and $(p+1)$-partite graph, $\Gamma(P)$ is a $\frac{k}{|A|}$-regular and $(p+1)$-partite graph. Hence, by Theorem 3.9, $P/Z(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$ and the proof is complete. □

Notice that, the regularity of $\Gamma(G)$ in Corollary 3.1 is necessary. For example, the center of group U_{6n} introduced in Example 2.2 is of order n. Hence $|U_{6n}/Z(U_{6n})| = 6$. Since, U_{6n} is non-abelian, $U_{6n}/Z(U_{6n})$ is isomorphic with symmetric group $\mathbb{S}_3$. In other words, $\Gamma(U_{6n})$ is a 4-partite graph while $U_{6n}/Z(U_{6n}) \ncong \mathbb{Z}_3 \times \mathbb{Z}_3$.

Lemma 3.1. *Let G be a non-abelian group and $|\operatorname{Cent}(G)| = n + 1$, where n is a positive integer. Then G is an AC-group if and only if $\Gamma(G)$ is a complete n-partite graph.*

Proof. Let G be an AC-group. Thus, for all $x \in G \setminus Z(G)$, $C_G(x)$ is abelian. [2, Remark 2.1] implies that for all $x \in G \setminus Z(G)$, $C_G(x) \setminus Z(G)$ is a part of the graph $\Gamma(G)$. Therefore for two distinct parts such as V_1 and V_2, all vertices of V_1 are adjacent with all vertices of V_2. Hence $\Gamma(G)$ is a complete n-partite graph. Conversely, let $\Gamma(G)$ be a complete n-partite graph with parts V_i, $i = 1, \ldots, n$. If $x \in V_i$ then x is not adjacent with other vertices of V_i and so x commutes with them. Therefore $V_i = C_G(x) \setminus Z(G)$. On the other hand, since V_i is an independent set, all of its elemnts commute with each other and therefore $C_G(x)$ is abelian. This yeilds that G is an AC-group. □

Theorem 3.10. *Let p be the smallest prime dividing $|G|$. If $[G : Z(G)] = p^3$, then $\Gamma(G)$ is a complete $(p^2 + 1)$ or $(p^2 + p + 1)$-partite graph.*

Proof. According to [4, Lemma 2.1], G is an AC-group. On the other hand, by [4, Proposition 2.2], $|\operatorname{Cent}(G)| = p^2 + 2$ or $p^2 + p + 2$. Now, by Lemma 3.1 the result follows. □

Remark 3.3. The graph considered in Theorem 3.10 is not necessarily regular. Notice to the following example.

Example 3.1. Let $G = \text{SmallGroup}(32, 9)$, by using GAP [8], one can see that $|Z(G)| = 4$ and so $[G : Z(G)] = 2^3 = 8$. Each vertex of $\Gamma(G)$ is of degree 24 or 16. The number of vertices of this graph is $|G| - |Z(G)| = 32 - 4 = 28$. According to Theorem 3.10, this graph is 5 or 7-partite. We can see that every vertex of degree 24 commutes with exactly four vertices and every vertex of degree 16 commutes with 12 vertices. This implies that each part of graph $\Gamma(G)$ is of order 4 or 12. The possible decompositions of 28 into parts with 12 and 4 vertices are two partitions $28 = 4+4+4+4+12$ and $28 = 12+12+4$ which yields $\Gamma(G)$ is a complete non-regular 5-partite graph.

Theorem 3.11. *Let $p \geq q$ be prime numbers. If $[G : Z(G)] = pq$, then $\Gamma(G)$ is a complete $(p+1)$-partite graph.*

Proof. By [4, Corollary 2.5], $|\operatorname{Cent}(G)| = p + 2$ and for all $x \in G \setminus Z(G)$, $C_G(x)$ is abelian. Thus G is an *AC*-group. Now, by Lemma 3.1 the result follows. □

Example 3.2. Let $G = \text{SmallGroup}(48, 4)$, then $|Z(G)| = 8$ and so $[G : Z(G)] = 3 \times 2$, is a product of two primes. By using a GAP program, we can see that each vertex of $\Gamma(G)$ which is of degree 32 commutes with exactly 8 vertices and each vertex of degree 24 commutes with 16 vertices. The only decomposition of 40 into parts with 8 and 16 vertices is $40 = 8 + 8 + 8 + 16$. Thus $\Gamma(G)$ is a complete 4-partite graph.

Consider the dihedral group $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. This group has $2n$ elements as $\{1, a, \ldots, a^{n-1}, b, ba, \ldots, ba^{n-1}\}$.

Theorem 3.12. *If n is odd, then $\Gamma(D_{2n})$ is a complete $(n+1)$-partite graph and if n is even, then $\Gamma(D_{2n})$ is a complete $(n/2+1)$-partite graph.*

Proof. By [4, Corollary 2.4], if n is odd, then $|\operatorname{Cent}(D_{2n})| = n + 2$ and if n is even, then $|\operatorname{Cent}(D_{2n})| = n/2 + 2$. On the other hand, by [4, Theorem 2.3] D_{2n} is an *AC*-group. Now, by Lemma 3.1 the result follows. □

Theorem 3.13. *Let G be a finite nonabelian group. Then $\Gamma(G)$ is a complete 6-partite graph if and only if $G/Z(G) \cong \mathbb{Z}_5 \times \mathbb{Z}_5$, D_{10} or $\langle x, y \mid x^5 = y^4 = 1, y^{-1}xy = x^3\rangle$.*

Proof. By [2, Theorem A], $|\operatorname{Cent}(G)| = 7$ and G is an *AC*-group. Now, by Lemma 3.1 the result follows. □

Theorem 3.14. *Let G be a finite nonabelian group. If $\Gamma(G)$ is a complete 7-partite graph, then $G/Z(G) \cong A_4$, D_{12} or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.*

Proof. By [2, Theorem B], $|\operatorname{Cent}(G)| = 8$ and G is an *AC*-group. Now, by Lemma 3.1 the result follows. □

Theorem 3.15. *Let G be a finite non-abelian group of odd order. Then $\Gamma(G)$ is a complete 8-partite graph if and only if $G/Z(G) \cong \mathbb{Z}_7 \times \mathbb{Z}_7$ or $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$.*

Proof. By [4, Theorem 3.3], $|\operatorname{Cent}(G)| = 9$ and by [2, Lemma 2.6], G is an AC-group. Now, by Lemma 3.1 the result follows. ☐

Theorem 3.16. *Let G be a finite non-abelian group. If $G/Z(G) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ then $\Gamma(G)$ is a complete 7-partite or 5-partite graph.*

Proof. By [2, Proposition 3.4], $|\operatorname{Cent}(G)| = 8$ or 6. ☐

4. Which Graphs are NC-Graph?

In this section, we determine whether some special graphs can be NC-graphs of some groups. In [9], it was shown that there is no regular NC-graph of odd valancy and therefore there is not a group for which its NC-graph is the Petersen graph. By Theorem 3.3, the cycle graph C_n cannot be a NC-graph. Also by [1, Proposition 2.2], all NC-graphs are Hamiltonian, thus we do not consider non-Hamiltonian graphs. In addition, if a regular graph Γ_1 is not Eulerian, then there is no group G for which $\Gamma(G) \cong \Gamma_1$. Hence, for a given regular graph Γ_1, a necessary condition to exist a group G with $\Gamma(G_1) \cong \Gamma_1$ is that Γ_1 be Eulerian.

Darafsheh in [7] proved that there is no group G for which $\Gamma(G)$ is a complete graph or a bipartite graph. Here, we continue his method for the graphs $K_n - e$, $K_n - 2e$ and $K_n - 3e$, where $K_n - te$ denotes the graph obtained from the complete graph K_n by deleting t edges and $t \in \mathbb{N}$. In the following, the set of all Sylow p-subgroups of G is denoted by $Syl_p(G)$.

Theorem 4.1. *Let $\Gamma(G) \cong K_n - e$ for some n, then G is isomorphic with the symmetric group $\mathbb{S}_3$.*

Proof. Let $x, y \in G \setminus Z(G)$ be two non-adjacent vertices of $\Gamma(G)$, then $xy = yx$. For every vertex $u \in (G \setminus Z(G)) \setminus \{x, y\}$, $\{u\}$ is a dominating set for $\Gamma(G)$. Hence, according to [1, Proposition 2.12], $Z(G) = 1$ and we can consider two following cases.

Case 1. $y \neq x^{-1}$, in this case every element $w \in V(\Gamma(G))$ has order 2, because if $o(w) \geq 3$, then w is not adjacent to w^{-1} which is impossible, since w is a well-connected vertex. On the other hand, $y \neq x$ yields that $x = x^{-1}$ and $y = y^{-1}$. Thus for all $w \in V(\Gamma(G))$, we have $o(w) = 2$. In other words, $o(x) = 2$, $o(y) = 2$ and $o(xy) = 2$. But x and y commute with non-central element xy which contradicts the fact that only x and y are non-adjacent.

Case 2. $y = x^{-1}$, similar to the last case, we can deduce that all vertices of $V(\Gamma(G)) \setminus \{x, y\}$ are of order two and $o(x) \geq 3$, because if $o(x) = 2$, then $x = x^{-1}$ which is a contradiction. If x has order $k \geq 4$, then x commutes with both x^2, x^3 which contradicts the fact that x has only one non-adjacent vertex in $\Gamma(G)$. Thus, x has order 3 and $y = x^{-1}$ does as well. In other words, G is a $\{2, 3\}$-group and so we can verify that $|G| = 2^r.3^s$ for some positive integers r and s. Let $P \in Syl_3(G)$ and $Q \in Syl_2(G)$. Then $P \cap Q = 1$ and $G = PQ$. There are only two elements of order three and this means that P is a normal subgroup of G and thus $G = P \rtimes Q$. We have $|P| = 3$, $|Q| = 2^r$ and we claim that $r = 1$. Suppose on the contrary that $r \geq 2$, then there are two distinct elements such as $a, b \in Q$ for which we have $o(a) = o(b) = 2$ and therefore $o(ab) = 2$. So $ab = b^{-1}a^{-1} = ba$ and this means that a and b commute with each other, a contradiction. So $r = 1$ and therefore $G \cong \mathbb{Z}_3 \rtimes \mathbb{Z}_2$. In other words, up to isomorphism there is only one non-abelian group of order six, namely $\mathbb{S}_3$ and so $G \cong \mathbb{S}_3$. This completes the proof. □

Theorem 4.2. *There is no group G such that $\Gamma(G) \cong K_n - 2e$.*

Proof. One of the two following cases hold.

Case 1. Two removed edges share a common vertex. Note that $K_3 - 2e$ is disconnected, hence it cannot be a NC-graph. Thus, we may assume $n \geq 4$, in which case $\gamma(G) = 1$ and $Z(G) = 1$. Suppose two removed edges share the common vertex y and x, z be two vertices non-adjacent to y. If $y \neq x^{-1}$ and $y \neq z^{-1}$, then similar to the proof of Theorem 4.1, for every element $u \in V(\Gamma(G))$ we have $o(u) = 2$. Thus $x = x^{-1}$, $y = y^{-1}$ and $z = z^{-1}$. Since x commutes with y, it commutes with xy, too. But $xy \neq 1$ and so $xy \notin Z(G)$. Since y is the only vertex non-adjacent with x, we come to a contradiction. Now assume that $y \neq x^{-1}$ and $y = z^{-1}$. So we conclude that $o(x) = 2$ and x commutes with xy, a contradiction. Thus, this case does not hold.

Case 2. Two removed edges do not share a common vertex. Clearly, $n \geq 4$. The girth of $K_4 - 2e$ is 4, thus it is not a NC-graph, see [1, Proposition 2.1]. Hence $n \geq 5$, which clearly implies $Z(G) = 1$ since every such graph has domination number 1. Assume that the two removed edges from K_n are $\{x, w\}$ and $\{y, z\}$. If $x \neq w^{-1}$, since x commutes with xw, then similar to the last discussion, we have a contradiction. If $y \neq z^{-1}$, then again we have a contradiction. Thus assume that $x = w^{-1}$ and $y = z^{-1}$. Then necessarily $o(x) = o(w) = 3$, $o(z) = o(y) = 3$ and for every element $u \in V(\Gamma(G)) \setminus \{x, w, y, z\}$, we have $o(u) = 2$. So, we may assume that $|G| = 2^t.3^s$, where $t, s \geq 1$. According to Sylow theorem $|Syl_3(G)| = 1 + 3k (k \geq 0)$. On the other hand, only the element x, w, y, z are of order three which is a contradiction. These contradictions show that there is no group G with $\Gamma(G) \cong K_n - 2e$ and the proof is complete. □

Theorem 4.3. *If there is a group G such that $\Gamma(G) \cong K_n - 3e$, then $n = 6$ and $G \cong D_8$ or Q_8.*

Proof. Let $\Gamma(G) \cong K_n - 3e$. Then there are five cases for the removed edges as the following:

Case 1. Three removed edges share a common vertex, namely w and thus $n \geq 4$. Note that in this case, $K_4 - 3e$ is disconnected and hence it cannot be a NC-graph. Thus, we may assume $n \geq 5$, in which case $\gamma(G) = 1$ and $Z(G) = 1$. Suppose x_1, x_2 and x_3 be the other end of these edges. Then w is the inverse of each of x_i, $i = 1, 2, 3$, which is a contradiction, since x_i, $i = 1, 2, 3$, are distinct. Thus, this case does not hold.

Case 2. Three removed edges make the complete graph K_3. Hence, $n \geq 4$ and therefore $Z(G) = 1$. Let x, y and z be the vertices of the removed edges. Then the centralizer of x, y and z is the same and has cardinality 4. Thus, the order of x, y and z is 2 or 4. But we know that the order of each element of $V(\Gamma(G)) \setminus \{x, y, z\}$ is 2. Therefore G is a 2-group, which contradicts with $Z(G) = 1$. Thus, this case does not hold.

Case 3. Let x_1, x_2, x_3, w_1, w_2 be distinct vertices of the graph K_n and the edges $\{x_1, w_1\}$, $\{x_2, w_1\}$ and $\{x_3, w_2\}$ are removed, thus $n \geq 5$. If $n \geq 6$, then $Z(G) = 1$ and therefore w_1 is the inverse of both x_1 and x_2, which is a contradiction. If $n = 5$, then $C_G(x_1) = Z(G) \cup \{x_1, w_1\}$ and $C_G(x_1) \cap C_G(x_2) = Z(G) \cup \{w_1\}$. But $|C_G(x_1) \cap C_G(x_2)| = |Z(G)| + 1$ which clearly is not a divisor of $|C_G(x_1)| = |Z(G)| + 2$ and this is a contradiction. Thus, this case does not hold.

Case 4. Three removed edges make the path $\{x_1, w_1, x_2, w_2\}$ and hence $n \geq 4$. If $n = 4$, then the gaph $K_4 - 3e$ is the path P_4, which is not Hamiltonian. Thus $n \geq 5$ and therefore $Z(G) = 1$. In this case, $w_1 = x_1^{-1}$. On the other hand, $w_1 \in C_G(x_2)$ and therefore $x_1 \in C_G(x_2)$, which contradicts with the adjacency of x_1 and x_2. Thus, this case does not hold.

Case 5. Let $x_1, x_2, x_3, w_1, w_2, w_3$ be distinct vertices of K_n and the edges $\{x_1, w_1\}$, $\{x_2, w_2\}$ and $\{x_3, w_3\}$ are removed, thus $n \geq 6$. If $n = 6$, then the degree of each vertex in $K_6 - 3e$ is 4, which is a 4-regulr graph with 6 vertices and therefore by [12, Corollary 4], $G \cong D_8$ or Q_8. If $n \geq 7$, then $Z(G) = 1$. In this case, the elements $x_1, x_2, x_3, w_1, w_2, w_3$ have order 3 and the other vertices of $\Gamma(G)$ have order 2. Thus, G is a $\{2, 3\}-$group and therefore $|G| = 2^r 3^s$, for some positive integers r and s. So all Sylow 3-subgroups of G have order 3^s. Since there are only 6 vertices of order 3, we have $s = 1$. It is easy to see that $C_G(x_1)$, $C_G(x_2)$ and $C_G(x_3)$ are the only Sylow 3-subgroups of G. But, if n_3 is the number of Sylow 3-subgroups, then $n_3 \equiv 1 \pmod 3$, which contradicts $n_3 = 3$ and this completes the proof. □

Theorem 4.4. *Let H_s be the graph obtained by removing s disjoint edges from K_{2s}. If there is a group G such that $\Gamma(G) \cong H_s$, then $s = 3$ and $G \cong D_8$ or Q_8.*

Proof. Let G be the group such that $\Gamma(G) \cong H_s$. In fact, the graph H_s is a hyperoctahedral graph or cocktail-party graph, $CP(s)$. In this graph, for each $x \in V(H_s)$, $\deg(x) = 2s - 2 = |G| - |Z(G)| - 2$. So, for each $x \in G \setminus Z(G)$, $|C_G(x)| = |Z(G)| + 2$. Thus, $|Z(G)|([C_G(x) : Z(G)] - 1) = 2$ and therefore $|Z(G)| = 1$ or $|Z(G)| = 2$. On the other hand, x is not adjacent with only one vertex, say y and hence $C_G(x) = \{x, y\} \cup Z(G)$. If $|Z(G)| = 1$ then for every $x \in G \setminus Z(G)$, $|C_G(x)| = 3$ and therefore, for every $x \in G \setminus Z(G)$, the order of x is three. Hence G is a 3-group, which contradicts the fact that $Z(G) = 1$. Thus, $|Z(G)| = 2$ and therefore, for every $x \in G \setminus Z(G)$, $|C_G(x)| = 4$. So each element of G has order 2 or 4, which implies that G is a 2-group. By [11, Proposition 3.3], if G is a p-group and for each $x \in G \setminus Z(G)$, $[G : C_G(x)] = p^a$ (a is a positive integer) and $Z(G)$ is cyclic, then $a = 1$. Hence, $[G : C_G(x)] = 2$ and therefore $|G| = 8$. Hence $G \cong D_8$ or Q_8 and $\Gamma(G)$ is a complete tripartite graph which yeilds that $s = 3$. □

Now, in the following, we consider the general case, i.e., the graph $K_n - te$, where $t \geq 4$.

Let $\gamma(K_n - te) = 1$, $t \geq 4$. In the following cases, we show that there is no group G such that $\Gamma(G) \cong K_n - te$. Let $\Gamma(G) \cong K_n - te$. Then $\gamma(\Gamma(G)) = 1$ and therefore $Z(G) = 1$ and $|G| = n + 1$. Let S be the induced subgraph of K_n by these t edges. We divide the vertex set of the graph S into two sudivisions W and X in such a way that $W = \{w_1, \ldots, w_u\}$ is a set of minmum order such that each edge of S has at least one end in W i.e. the set W dominates all edges of S and has minimum size with this property. Set $X := V(S) \setminus W$. Thus, the vertices of X are exactly one end of some edges of S and are not adjacent in S. We consider the following adjacency cases in the graph S.

Case 1. There are at least two vertices like x_1 and x_2 in X such that they are adjacent with exactly one vertex of W like w_i. In this case, the order of x_1 and x_2 is 3 and w_i is the inverse of both x_1 and x_2, which contradicts the fact that x_1 and x_2 are distinct.

Case 2. There is a vertex like w_i in W such that it is adjacent with more than one vertex of X and at least one of these vertices like x_j has degree one in S. In this case, $w_i = x_j^{-1}$. Now if $x_{j'}$ be another vertex of X which is adjacent with w_i, then if $\deg_S(x_{j'}) = 1$, then by Case 1 we come into a contradiction. Thus, let $\deg_S(x_{j'}) \geq 2$. Then $w_i \in C_G(x_{j'})$ and hence $x_j = w_i^{-1} \in C_G(x_{j'})$, which is a contradiction, since x_j and $x_{j'}$ are adjacent in $K_n - te$.

Case 3. Suppose that there are at least two vertices in X such as x_i and x_j, which differ in their l, $l \geq 0$, neighbours in W. In other words, let

$$\begin{aligned} C_G(x_i) =& \{1, x_i, w_{i_1}, \ldots, w_{i_r}\}, \\ C_G(x_j) =& \{1, x_j, w_{j_1}, \ldots, w_{j_{r'}}\}, \end{aligned}$$

where $l < \frac{2+r}{2}$ or $l < \frac{2+r'}{2}$ and

$$C_G(x_i) \cap C_G(x_j) = \{1, w_{k_1}, \ldots, w_{k_{r''}}\}.$$

Then $|C_G(x_i) \cap C_G(x_j)| = 1 + r''$ and $1 + r'' > \frac{2+r}{2}$ or $\frac{2+r'}{2}$. Therefore, $|C_G(x_i) \cap C_G(x_j)|$ does not divide at least one of $|C_G(x_i)|$ and $|C_G(x_j)|$, a contradiction.

References

[1] A. Abdollahi, S. Akbari and H. R. Maimani, *Non-commuting graph of a group*, J. Algebra **298** (2006), 468–492.

[2] A. Abdollahi, S. M. J. Amiri and A. M. Hassanabadi, *Groups with specific number of centralizers*, Houston J. Math. **33**(1) (2007), 43–57.

[3] A. Abdollahi and H. Shahverdi, *Characterization of the alternating group by its non-commuting graph*, J. Algebra **357** (2012), 203–207.

[4] S. J. Baishya, *On finite groups with specific number of centralizers*, Int. Electron. J. Algebra **13** (2013), 53–62.

[5] S. M. Belcastro and G. J. Sherma, *Counting centralizers in finite groups*, Math. Mag. **67**(5) (1994), 366–374.

[6] J. A. Bondy and J. S. Murty, *Graph Theory with Aplication*, Elsevier, New York, 1977.
[7] M. R. Darafsheh, *Groups with the same non-commuting graph*, Discrete Appl. Math. **157** (2009), 833–837.
[8] GAP, Groups, *Algorithms and Programming*, Lehrstuhl De fur Mathematik, RWTH, Aachen, 1992.
[9] M. Ghorbani and Z. Gharavi-Alkhansari, *A note on integral non-commuting graphs*, Filomat **31**(3) (2017), 663–669.
[10] M. Ghorbani, Z. Gharavi-Alkhansari and A. Zaeem-Bashi, *On the eigenvalues of non-commuting graphs*, Algebraic Structures and Their Applications **4** (2017), 27–38.
[11] N. Ito, *On finite groups with given conjugate types I*, Nagoya Math. J. **6** (1953), 17-28.
[12] A. R. Moghaddamfar, W. J. Shi, W. Zhou and A. R. Zokayi, *On the non-commuting graph associated with a finite group*, Sib. Math. J. **46** (2005), 325–332.
[13] G. L. Morgan and C. W. Parker, *The diameter of the commuting graph of a finite group with trivial centre*, J. Algebra **393** (2013), 41–59.
[14] B. H. Neumann, *A problem of Paul Erdős on groups*, J. Aust. Math. Soc. Ser. A **21** (1976), 467–472.
[15] D. B. West, *Introduction to Graph Theory*, Prentice Hall, Upper Saddle River, New Jersey, 1996.

[1]Department of Mathematics,
Faculty of Science,
Shahid Rajaee Teacher Training University,
Tehran, 16785-136, I. R. Iran
Email address: mghorbani@sru.ac.ir
Email address: hg.paper@gmail.com