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CHAIN CONNECTED SETS IN A TOPOLOGICAL SPACE

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Abstract. Shekutkovski's paper [5] compares two definitions of connectedness: the standard one and a definition using coverings. The second definition seems to be an effective description of quasicomponents. In our paper rather than as a space, we generalize the notion of connectedness as a set in a topological space called chain connected set. We also introduce a notion of two chain separated sets in a space and using this notion of chain, we study the properties of chain connected and chain separated sets in a topological space. Moreover, we prove the properties of connected spaces using chain connectedness. Chain connectedness of two points in a topological space is an equivalence relation. Chain connected components of a set in a topological space are a union of quasicomponents of the set, and if the set agrees with the space, chain connected components match with quasicomponents.

1. Introduction

In [5] a definition of connectedness of a topological space using coverings is given. In our paper rather than as a space, we generalize the notion of connectedness as a set in a topological space that we call chain connected set.

In Section 2 we introduce the definition of chain connectedness. Every connected subset of a topological space is chain connected in the space, and the converse claim holds, if the set agrees with the space. In Section 3 we define a pair of chain separated sets in a space. In these sections we also study the properties of the chain connected sets and the pairs of chain separated sets and the connection of these with connected and separated sets. As a result we prove the properties of connected spaces by using chain connectedness.

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In Section 4 we give criteria for chain connected and connected spaces by using a star of coverings. Then we show that two chain separated sets in their union are functionally separated sets. At the end of the section, we define the chain connectedness relation of two points in a space, which is an equivalence relation, and we give criteria for chain connectedness, connectedness and chain separatedness, using that relation.

We show that chain connected components of a set in a space, introduced in Section 5, divides the space into classes. All chain connected subsets of the set in a topological space are chain connected components with their subsets. Chain connected components of a set in a space are a union of quasicomponents of the set, and if the set agrees with the space, chain connected components match with quasicomponents.

2. Chain Connectedness

By a covering we understand a covering consisting of open sets.

Suppose X is a set, $\mathcal{U}$ is a family of subsets of X and $x, y \in X$. **A chain in $\mathcal{U}$ that connects x and y** (from x to y, from y to x) is a finite sequence $U_1, U_2, \ldots, U_n$ in $\mathcal{U}$, such that $U_i \cap U_{i+1} \neq \emptyset$, $i = 1, 2, \ldots, n-1$, and $x \in U_1$, $y \in U_n$. In this paper a set is a topological subspace and a chain is a chain that consists of open sets.

Using the notion of chain we define the notion of chain connected set, which is closely related to the notion of connected set.

Definition 2.1. Let X be a topological space and $C \subseteq X$. The set C is **chain connected** in X, if for every covering $\mathcal{U}$ of X in X and every $x, y \in C$, there exists a chain in $\mathcal{U}$ that connects x and y.

Let X be a topological space and $C \subseteq Y \subseteq X$.

The first property of a chain connected set, shown in the next theorem, is an implication of chain connectedness from a space to each of its super spaces (X is super space of C if C is a subspace of X).

Theorem 2.1. *If C is chain connected in Y, then C is chain connected in X.*

Proof. Let C be chain connected in Y and $\mathcal{U}$ be a covering of X. Then $\mathcal{U}_y = \mathcal{U} \cap Y = \{U \cap Y \mid U \in \mathcal{U}\}$ is a covering of Y in Y. Since C is chain connected in Y, it follows that for every two points $x, y \in X$, there exists a chain $U_1 \cap Y, U_2 \cap Y, \ldots, U_n \cap Y$ of elements of $\mathcal{U}_y$. Then $U_1, U_2, \ldots, U_n$ is a chain in X of elements of $\mathcal{U}$, that connects x and y. It follows that C is chain connected in X. □

Remark 2.1. The most important case of the previous theorem is when $Y = C$.

The following two examples show that the converse statement does not hold in general.

Example 2.1. Let

$$X = \left\{ \left(x, \frac{1}{n}\right) \;\middle|\; x \in [-1, 1], n \in \mathbb{N} \right\} \cup \{(x, 0) \mid x \in [-1, 0) \cup (0, 1]\}$$

be a topological space (this space is given in [2]).

Let $A = [-1, 0) \times \{0\}$ and $B = (0, 1] \times \{0\}$. Then $Y = A \cup B$ is chain connected in X, but is not chain connected in Y.

The converse claim does not hold for a simpler space either, for example an interval.

Example 2.2. The space $X = [-1, 1]$ is compact and metric. Let $A = [-1, 0)$, $B = (0, 1]$ and $Y = A \cup B$. Then Y is chain connected in X, but it is not chain connected in Y. Moreover, Y is not connected.

The next claim, which directly follows from the definition, shows that each subset of a chain connected set in a topological space is chain connected in the same space.

Remark 2.2. If the set C is chain connected in X, then each subset of C is chain connected in X.

We briefly recall the well-known definition of connectedness. Two nonempty subsets A and B of a topological space X are separated if $\bar{A} \cap B = A \cap \bar{B} = \emptyset$. Sets A and B are separated if and only if A and B are open in $A \cup B$, if and only if A and B are closed in $A \cup B$. A subset Y of a topological space X is connected if it cannot be represented as a union of two nonempty separated sets. For more details about connectedness, see [1, 2, 4].

Let X be a topological space, $\mathcal{U}$ be a covering of X in X and $x \in C \subseteq X$.

We denote by $A_{CX}(x,\mathcal{U})$ the set that consists of all elements $y \in C$ such that there exists a chain in $\mathcal{U}$ that connects x and y. If $C = X$, $A_X(x,\mathcal{U}) = A_{XX}(x,\mathcal{U})$.

Clearly, $A_{CX}(x,\mathcal{U}) = C \cap A_X(x,\mathcal{U})$.

Theorem 2.2. *The set $A = A_{CX}(x,\mathcal{U})$ is nonempty, open and closed in C.*

Proof. Let $A = A_{CX}(x,\mathcal{U})$. Clearly set A is nonempty.

Let $y \in A$. Then there exists a chain $U_1, U_2, \ldots, U_n$ in $\mathcal{U}$ from x to y. Then $U_n \cap C$ is an open set in C that contains y. Notice that if y is an arbitrary element from any chain that connects x with some point in C, then y is contained in A, so $U_n \cap C \subseteq A$. It follows that A is open in C.

Let $y \in \overline{A}$ in C. Then there exists a neighbourhood $U_C = U \cap C$ of y such that $U \in \mathcal{U}$ and there exists a point $z \in A \cap U_C$. Therefore, there exists a chain $U_1, U_2, \ldots, U_n$ in $\mathcal{U}$ from x to z, and $U_1, U_2, \ldots, U_n, U$ is a chain from x to y, i.e., $y \in A$. It follows that A is closed in C. □

As a consequence of the theorem, it follows that $A_X(x,\mathcal{U})$ is nonempty, open and closed.

If the set A is nonempty, open and closed, then there exists a covering $\mathcal{U} = \{A, X \backslash A\}$ of X such that $A = A_X(x,\mathcal{U})$.

Moreover C, such that $x \in C$, is chain connected in X if and only if $C \subseteq A_{CX}(x,\mathcal{U})$ for every covering $\mathcal{U}$ of X.

Example 2.3. For the topological spaces A, B, Y, X from Example 2.2, $A_X(x,\mathcal{U}) = X$ and $A_{YX}(x,\mathcal{U}) = Y$.

The following theorem gives the relationship between a connected and a chain connected set.

Theorem 2.3. *A set C is connected if and only if C is chain connected in C.*

Proof. If C is an empty set or a singleton, then C is connected and chain connected in C. Let C be composed of at least two elements and let $\mathcal{U}$ be a covering of C in C.

($\Rightarrow$) If C is connected then $C = A$, where $A = A_C(x, \mathcal{U})$. Otherwise A and $C \setminus A$ would be nonempty, open, and closed sets whose union is C, i.e., C would not be connected. It follows that C is chain connected in C.

($\Leftarrow$) Let C be a chain connected set in C. If C is not connected, then C can be represented as a union of two open and closed sets M and N such that $x \in M$ and $y \in N$ for some $x, y \in C$. Then for the covering $\mathcal{U} = \{M, N\}$ does not exist a chain in $\mathcal{U}$ that connects x and y. The last claim contradicts the chain connectedness of C in C. $\square$

Notice that, as a consequence, it follows that if C is connected set, then C is chain connected set in every super space X. In addition, a topological space C is chain connected in C, if and only if it cannot be represented as a union of two separated sets.

3. Chain Separatedness

Chain separated sets will be defined in such a way that the notions of chain connectedness and chain separatedness are analogous to the notions of connectedness and separatedness. In this section we will discuss the relationship of chain separated sets and chain connected sets.

Definition 3.1. Let X be a topological space, and let A and B be nonempty subsets of X. The sets A and B are **chain separated in** X, if there exists a covering $\mathcal{U}$ of X in X such that for every point $x \in A$ and every $y \in B$, there is no chain in $\mathcal{U}$ that connects x and y.

From the definition, it follows that if A and B are chain separated sets in a topological space X, then any two sets C and D, where $C \subseteq A$ and $D \subseteq B$, are chain separated in X.

Let X be a topological space, let $Y \subseteq X$ and let A and B be nonempty subsets of X.

The following proposition will show that two sets, which are chain separated in a space, are also chain separated in every subspace, and its proof is trivial

Proposition 3.1. *If A and B are chain separated in X, then A and B are chain separated in Y.*

Remark 3.1. The most important case of the previous theorem is when $Y = A \cup B$.

Example 3.1. If X and Y are topological spaces from Example 2.2, then A and B are chain separated in $Y = A \cup B$, but they are not chain separated in X.

The next theorem gives the relationship of chain separated sets and separated sets.

Theorem 3.1. *Nonempty sets A and B are chain separated in $A \cup B$ if and only if A and B are separated.*

Proof. ($\Rightarrow$) Let nonempty sets A and B not be separated. It follows that $\bar{A} \cap B \neq \emptyset$ or $A \cap \bar{B} \neq \emptyset$. It is enough to assume that $\bar{A} \cap B \neq \emptyset$, i.e., there exists $x \in \bar{A} \cap B$ (when $A \cap \bar{B} \neq \emptyset$, the proof is analogous).

Let $\mathcal{U}$ be an arbitrary covering of $A \cup B$ in $A \cup B$. Let $U \in \mathcal{U}$ be a neighbourhood of x in $A \cup B$. Since $x \in \bar{A}$, there exists a point $y \in A$ that lies in U. It follows that U is a chain in $\mathcal{U}$ that connects $x \in B$ and $y \in A$. Therefore, the sets A and B are not chain separated in $A \cup B$.

($\Leftarrow$) Now, let A and B be separated sets. It follows that A and B are nonempty, open, and closed in $A \cup B$. So if for the covering $\mathcal{U} = \{A, B\}$ of $A \cup B$ in $A \cup B$ there is no chain that connects a point from A with a point from B. □

From the theorem it follows that if A and B are chain separated in X, then A and B are separated.

Now, let us consider two statements that give criteria for connected and chain connected sets by using chain separatedness. Let X be a topological space and $C \subseteq X$.

Theorem 3.2. *A set C is chain connected in X if and only if C cannot be represented as a union of two chain separated sets A and B in X.*

Proof. ($\Rightarrow$) If C can be represented as a union of chain separated sets A and B in X, then C is not chain connected in X.

($\Leftarrow$) Let C not be chain connected in X. It follows that there exist a covering $\mathcal{U}$ of X and two elements x and y of C for which there is no chain in $\mathcal{U}$ that connects the elements x and y. We consider the set $A = A_{CX}(x, \mathcal{U})$. Sets A and $C \setminus A$ are nonempty, open an closed in C. It follows that C is represented as a union of two chain separated sets A and $C \setminus A$. □

Corollary 3.1. *A set C is connected if and only if it can not be represented as a union of two chain separated sets A and B in C.*

The next theorem and its corollary determine the location of chain connected and connected sets, each in a topological space that is a union of two separated sets.

Theorem 3.3. *Let $X = A \cup B$, where A and B are chain separated sets in X and C is a chain connected set in X. Then $C \subseteq A$ or $C \subseteq B$.*

Proof. If there exists $x, y \in C$ such that $x \in A$ and $y \in B$, because C is a chain connected set, it follows that for every covering $\mathcal{U}$ of $X = A \cup B$ there exists chain in $\mathcal{U}$ that connects x and y. The last claim contradicts the claim that sets A and B are chain separated in X. So, $C \subseteq A$ or $C \subseteq B$. □

A direct consequence of the theorem is the following corollary.

Corollary 3.2. *Let $X = A \cup B$, where A and B are separated. If C is a connected set, then $C \subseteq A$ or $C \subseteq B$.*

The next theorem and its remark show that chain connectedness and connectedness are invariant with respect to continuous function.

Theorem 3.4. *If C is chain connected in X and $f : X \to Y$ is a continuous function, then $f(C)$ is chain connected in $f(X)$.*

Proof. The function $f : X \to Y$ is continuous if and only if $f : X \to f(X)$ is continuous.

Let $f(x), f(y) \in f(C)$ and $\mathcal{V}$ be a covering of $f(X)$. Then $\mathcal{U} = f^{-1}(\mathcal{V})$ is an open covering of X. Since C is chain connected in X, there exists a chain $f^{-1}(V_1), f^{-1}(V_2), \ldots, f^{-1}(V_n)$ in $\mathcal{U}$ that connects x and y. Since

$$f^{-1}(V_i) \cap f^{-1}(V_{i+1}) \neq \emptyset, \quad i = 1, 2, \ldots, n-1,$$

it follows that $\emptyset \neq f(f^{-1}(V_i)) \cap f(f^{-1}(V_{i+1})) = V_i \cap V_{i+1}$, $i = 1, 2, \ldots, n$, i.e., $V_1, V_2, \ldots, V_n$ is a chain in $\mathcal{V}$ that connects $f(x)$ and $f(y)$. □

The next well-known result is a consequence of Theorem 3.4.

Remark 3.2. Let $C \subseteq X$. If C is a connected set and $f : X \to Y$ is a continuous function then $f(C)$ is connected.

Now we shall turn to a theorem for the equivalence of chain connectedness between a set and its super set that contains boundary points of the set.

Theorem 3.5. *Let $C \subseteq D \subseteq \bar{C} \subseteq X$. The set C is chain connected in X if and only if D is chain connected in X.*

Proof. Let C be a chain connected set in X and $x, y \in C$. It follows that for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y. Let $x, y \in D$. There exist neighbourhoods $U \in \mathcal{U}$ of x and $V \in \mathcal{U}$ of y. Since $D \subseteq \bar{C}$, it follows that $x, y \in \bar{C}$, so there exist points $z \in U \cap C$ and $t \in V \cap C$. It follows firstly that there exists a chain $U_1, U_2, \ldots, U_n$ in $\mathcal{U}$, which connects z and t, and secondly that $U, U_1, U_2, \ldots, U_n, V$ is a chain in $\mathcal{U}$ that connects x and y. It follows that D is chain connected in X.

Conversely, considering that each subset of a chain connected set is chain connected, if D is chain connected in X, then C is chain connected in X. □

Remark 3.3. The most important case of the theorem is when $D = \bar{C}$.

As a consequence, an analogous statement for connected space holds, but only in one direction.

Corollary 3.3. *Let X be a topological space and $C \subseteq X$. If C is a connected set and $C \subseteq D \subseteq \bar{C}$, then D is connected.*

Proof. If C is a connected set, it follows that C is chain connected in D. According to the previous theorem, D is chain connected in D, from which it follows that D is connected. □

The next example shows that the reverse claim does not have to be valid.

Example 3.2. Let $X = [-1, 1]$. Set $C = [-1, 0) \cup (0, 1]$ is not connected, but $\bar{C} = [-1, 1]$ is connected. Let us notice that C is chain connected in X.

At the end of this section we turn to a union of chain connected sets in a topological space.

Lemma 3.1. *Let $C, D \subseteq X$. If C and D are chain connected in X and $\bar{C} \cap \bar{D} \neq \emptyset$, then the union $\bar{C} \cup \bar{D}$ is chain connected in X.*

Proof. Let $\mathcal{U}$ be a covering of X and $x, y \in \bar{C} \cup \bar{D}$. If $x, y \in \bar{C}$ or $x, y \in \bar{D}$, then since $\bar{C}$ and $\bar{D}$ are chain connected, there exists a chain in $\mathcal{U}$ that connects x and y. If $x \in \bar{C}$ and $y \in \bar{D}$ it follows firstly that there exists $z \in \bar{C} \cap \bar{D}$, and secondly that there exist chains in $\mathcal{U}$ that connect x with z, and z with y, from which it follows that there is a chain in $\mathcal{U}$ that connects x and y.

So, $\bar{C} \cup \bar{D}$ is chain connected in X. □

Theorem 3.6. *Let C_i, $i \in I$, be a family of chain connected subspaces of X. If there exists $i_0 \in I$ such that for every $i \in I$, $\bar{C}_{i_0} \cap \bar{C}_i \neq \emptyset$, then the union $\bigcup_{i \in I} \bar{C}_i$ is chain connected in X.*

Proof. Let $\mathcal{U}$ be a covering of X and C_i, $i \in I$, be a family of chain connected subspaces of X. Let $x, y \in \bigcup_{i \in I} \bar{C}_i$, i.e., $x \in \bar{C}_x$ and $y \in \bar{C}_y$ for some $x, y \in I$.

Because $\bar{C}_{i_0} \cap \bar{C}_x \neq \emptyset$ from the previous lemma, it follows that $\bar{C}_{i_0} \cup \bar{C}_x$ is chain connected in X. Similarly, $\bar{C}_{i_0} \cup \bar{C}_y$ is chain connected in X. Then because $C_{i_0} \neq \emptyset$, it follows that $\bar{C}_{i_0} \cup \bar{C}_x \cup \bar{C}_y$ is chain connected in X, i.e., for every covering $\mathcal{U}$ of X, there exists a chain in $\mathcal{U}$ that connects x and y. So, $\bigcup_{i \in I} \bar{C}_i$ is chain connected in X. □

From the theorem it follows that if C_i, $i \in I$, is a family of chain connected subspaces of $\bigcup_{i \in I} C_i$, then the union $\bigcup_{i \in I} C_i$ is chain connected. A direct consequence of the last statement is the following statement.

Corollary 3.4. *Let $C_i \subseteq X$, $i \in I$, be a family of connected sets. If there exists $i_0 \in I$ such that for every $i \in I$, $C_{i_0} \cap C_i \neq \emptyset$, then the union $\bigcup_{i \in I} C_i$ is connected.*

It is clear that if every two points x and y of X are in a chain connected set C_{xy} in X, then X is chain connected. The next corollary follows.

Corollary 3.5. *If every two points x and y of X are in a connected set C_{xy} in X, then X is connected.*

Theorem 3.7. *If A and B are chain separated sets in X, then there exist:*

(a) *open neighbourhoods* U *and* V *that separate them, i.e.,* $A \subseteq U$*,* $B \subseteq V$ *and* $U \cap V = \emptyset$*;*
(b) *closed neighbourhoods* F *and* G *that separate them, i.e.,* $A \subseteq F$*,* $B \subseteq G$ *and* $F \cap G = \emptyset$*.*

Proof. Let $\mathcal{U}$ be a covering of X such that for every $x \in A$ and $y \in B$ there is no chain in $\mathcal{U}$ that connects x and y.

(a) If U is a union of all elements of chains in $\mathcal{U}$ that connect elements in A and if V is a union of all elements of chains in $\mathcal{U}$ that connect elements in B, then $A \subseteq U$, $B \subseteq V$, and $U \cap V = \emptyset$. Namely if there exists $z \in U \cap V$, then z is connected by a chain in $\mathcal{U}$ with an element from $a \in A$, and by a chain in $\mathcal{U}$ with an element from $b \in B$. Therefore, there exists a chain from a to b, which contradicts the chain separability of A and B.

(b) If A and B are chain separated sets in X, then $\bar{A}$ and $\bar{B}$ are chain separated in X. Let $F = \bar{A}$ and $G = \bar{B}$. Then $A \subseteq F$, $B \subseteq G$ and $F \cap G = \emptyset$. $\square$

4. Criteria for Chain Connectedness and Connectedness Using Notions of Continuous Function or Star of a Covering. Chain Connectedness Relation

We will give criteria for chain connectedness and connectedness in topological spaces, using the notion of star of a covering.

Let X be a topological space, $x \in X$, and U be an open set in X. According to [3], the star of the set A with respect to the covering $\mathcal{U}$ of X in X, is the set

$$\mathrm{st}\,(A,\mathcal{U}) = \bigcup \{U \in \mathcal{U} \mid U \cap A \neq \emptyset\},$$

the star of degree n, for $n > 1$, of A and $\mathcal{U}$ in X is $\mathrm{st}^n\,(A,\mathcal{U}) = \mathrm{st}\,(st^{n-1}\,(A,\mathcal{U}))$, and the infinite star of A and $\mathcal{U}$ in X is $\mathrm{st}^\infty\,(A,\mathcal{U}) = \bigcup_{n=1}^{\infty} \mathrm{st}^n\,(A,\mathcal{U})$. If $A = \{x\}$ we will denote $\mathrm{st}\,(\{x\}\,,U)$ with $\mathrm{st}\,(x,U)$.

The definition of a star of A with respect to $\mathcal{U}$ in X, the definition of chain connectedness, and a criterion for chain connectedness using connectedness lead to the following two statements.

Let X be a topological space and $C \subseteq X$.

Theorem 4.1. *The set* C *is chain connected in* X *if and only if for every* $x \in C$ *and every covering* $\mathcal{U}$ *of* X*,* $C \subseteq \mathrm{st}^\infty\,(x,\mathcal{U})$*.*

Corollary 4.1. *The space* X *is connected if and only if for every* $x \in X$ *and every covering* $\mathcal{U}$ *of* X*,* $X = \mathrm{st}^\infty\,(x,\mathcal{U})$*.*

In the next theorem we will give a chain connectedness criterion using continuous function.

Theorem 4.2. X *is chain connected if and only if every continuous function* $f : X \to \{0,1\}$ *is constant.*

As a consequence, it follows that a topological space X is chain connected if and only if there is no continuous function $f : X \to [0,1]$, such that $f(A) = \{0\}$ and $f(B) = \{1\}$ for every nonempty pair of sets A and B such that $A \cup B = X$.

Corollary 4.2. *If $f : X \to [0,1]$ is a continuous function such that $f(A) = \{0\}$ and $f(B) = \{1\}$, then sets A and B are separated.*

The reverse claim does not have to be valid.

Example 4.1. The space $X = [-1,1]$ is connected, and hence, chain connected. Let $A = [-1,0)$ and $B = (0,1]$. Then sets A and B are separated, but there is no continuous function $f : X \to [0,1]$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Namely, if such a continuous function were to exist, we would get $f\left(\bar{A}\right) = f([-1,0]) = 0$, from which it follows that $f(0) = 0$ and $f\left(\bar{B}\right) = f([0,1]) = 1$, from which it follows that $f(0) = 1$, which contradicts that f is a function.

Corollary 4.3. *The sets A and B are chain separated in $A \cup B$ if and only if the function $f : A \cup B \to \{0,1\}$ ($f : A \cup B \to [0,1]$), such that $f(A) = \{0\}$ and $f(B) = \{1\}$ is continuous.*

According to [6], two nonempty sets A and B are functionally separated in X if there exists a continuous function $f : X \to \{0,1\}$ such that $f(A) = \{0\}$ and $f(B) = \{1\}$. Therefore, A and B are chain separated sets in $A \cup B$ if and only if A and B are functionally separated in $A \cup B$.

Now, we will define a relation of chain connectedness in X.

Let $\mathcal{U}$ be a covering of X in X. Let $x, y \in X$. Then x is $\mathcal{U}$-chain related to y in X, and we denote it by $x \underset{X,\mathcal{U}}{\sim} y$ or just $x \underset{\mathcal{U}}{\sim} y$, if there exists a chain in $\mathcal{U}$ from x to y. It is obvious that the $\mathcal{U}$-chain relation in a topological space is an equivalence relation, and that it depends on the set X, the covering $\mathcal{U}$ of X, and the topology τ of X.

Definition 4.1. Let X be a topological space and $x, y \in X$. Element x is chain related to y in X, and we denote it by $x \underset{X}{\sim} y$ or just $x \sim y$, if for every covering $\mathcal{U}$ of X in X there exists a chain in $\mathcal{U}$ that connects x and y.

The chain relation in a topological space is an equivalence relation, and it depends on the set X and the topology τ of X.

Let X be a topological space and $C \subseteq X$.

The accuracy of the next claims follows from the last definition and the definitions of chain connectedness and chain separatedness (criteria for chain connectedness, connectedness and chain separatedness by using the chain relation).

Remark 4.1. A set C is chain connected in X if and only if for every $x, y \in C$, $x \underset{X}{\sim} y$.

Therefore, C is not chain connected in X if and only if there exist $x, y \in C$ such that $x \underset{X}{\nsim} y$.

Remark 4.2. If a set C is connected then for every $x, y \in C$, $x \underset{X}{\sim} y$.

Remark 4.3. A space X is connected (chain connected in X) if and only if for every $x, y \in X$, $x \underset{X}{\sim} y$.

Remark 4.4. If two sets A and B are chain separated in X, then for every $x \in A$ and $y \in B$, $x \underset{X}{\nsim} y$.

Therefore, if A and B are separated, then for every $x \in A$ and $y \in B$, $x \underset{A \cup B}{\nsim} y$.

Proposition 4.1. *Let $x, y \in X$. Then $x \underset{X}{\sim} y$ if and only if X cannot be represented as a union of two nonempty separated sets A and B that contain x and y, respectively.*

5. Chain Connected Components

Analogous to the connected component, we will define a chain connected component.

Definition 5.1. Let X be a topological space and $x \in Y \subseteq X$. **The chain connected component** of the point x of Y in X, denote by $V_{YX}(x)$, is the biggest chain connected subset of Y in X that contains x.

Let X be a topological space and $x \in Y \subseteq X$.

Theorem 5.1. *The chain connected component $V_{YX}(x)$ of the point x of Y in X is the set of all points $y \in Y$ such that for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y.*

Proof. $A = A_{YX}(x)$ denotes the set of all points $y \in Y$ such that for every covering $\mathcal{U}$ of X there exists a chain in $\mathcal{U}$ that connects x and y. The set A is chain connected in X and is a subset of Y. Namely, for every two points $y, z \in A$ and for every covering $\mathcal{U}$ of X there exist chains in $\mathcal{U}$ from y to x and from x to z, from which it follows that their union is a chain in $\mathcal{U}$ from y to z. Because $V = V_{YX}(x)$ is the biggest chain connected set that contains x and is a subset of Y, it is true that $A \subseteq V$.

There can be no point $y \in V \setminus A$, because if $y \notin A$ there will be a covering $\mathcal{U}$ of X for which there is no chain in $\mathcal{U}$ that connects x and y, which contradicts $y \in V$.

So, $A = V$, i.e., $A_{YX}(x) = V_{YX}(x)$. □

The following statement explicitly determines all chain connected sets in a topological space.

Proposition 5.1. *The set of all chain connected subsets of Y in X consist of all of chain connected components of Y in X and their subsets.*

The chain relation decomposes the space into classes. A set is chain connected if and only if all its elements are pairwise disjoint in a chain relation. It follows that the classes are chain connected components. The conclusion is also directly obtained from the next two results.

Proposition 5.2. *Let $x, y \in Y$. If $y \in V_{YX}(x)$, then $V_{YX}(x) = V_{YX}(y)$.*

Theorem 5.2. *Let $x, y \in Y$. If $V_{YX}(x) \neq V_{YX}(y)$, then $V_{YX}(x) \cap V_{YX}(y) = \emptyset$.*

Proof. Suppose the contrary, let $V_{YX}(x) \cap V_{YX}(y) \neq \emptyset$, i.e., there exists $z \in V_{YX}(x) \cap V_{YX}(y)$. Because $z \in V_{YX}(x)$ and $z \in V_{YX}(y)$, according to the previous theorem, we get $V_{YX}(x) = V_{YX}(z)$ and $V_{YX}(y) = V_{YX}(z)$.

It follows that $V_{YX}(x) = V_{YX}(y)$, which contradicts the condition in the theorem. □

Proposition 5.3. *For every $x \in Y$, one have $V_{YY}(x) \subseteq V_{YX}(x) = \bigcup_{y \in V_{YX}(x)} V_{YY}(y)$.*

Proof. Set $V_{YY}(x)$ is chain connected in Y that contains x. It follows that $V_{YY}(x)$ is chain connected in X, but because $V_{YX}(x)$ is the biggest chain connected set in X that contains x, it follows that $V_{YY}(x) \subseteq V_{YX}(x)$.

Let $x \in Y$ and $y \in V_{YX}(x)$, i.e., $V_{YX}(y) = V_{YX}(x)$. It follows that $V_{YY}(y) \subseteq V_{YX}(y) = V_{YX}(x)$.

From the arbitrariness of $y \in V_{YX}(x)$, it follows that $V_{YX}(x) = \bigcup_{y \in V_{YX}(x)} V_{YY}(y)$. □

The proposition shows that every chain connected component of Y in X is a union of chain connected components of Y in Y.

Proposition 5.4. *For every $x \in Y$, $V_{YX}(x) = Y \cap V_{XX}(x)$. Each chain connected component of X in X contains at most one chain connected component of Y in X.*

Proposition 5.5. *The chain connected components of X are closed sets, i.e., for every $x \in X$, $V(x) = \overline{V(x)}$.*

The connected component of the element x of a topological space X, denoted with $C(x)$, is the biggest connected set in X that contains x.

Proposition 5.6. *Let $x \in X$ and $C(x)$ be a connected component of X. Then $C(x) \subseteq V(x)$.*

Quasicomponent of the element x of a topological space X, denoted with $Q_X(x)$, is the intersection of all clopen (closed and open) sets in X that contain x.

The next theorem is a reformulation of Theorem 1.1 from [5].

Theorem 5.3. *Quasicomponents and chain connected components in a topological space X coincide, i.e., for every $x \in X$, $Q_X(x) = V_{XX}(x)$.*

Proof. ($\Rightarrow$) Let $x \in X$ and $y \in Q_X(x)$. If $y \notin V_{XX}(x)$, it follows that there exists a covering $\mathcal{U}$ of X such that there is no chain in $\mathcal{U}$ that connects x and y.

Let $A = A_X(x, \mathcal{U})$ be the set that contains all z such that there exists a chain in $\mathcal{U}$ from x to z. Then $x \in A$ and $y \in X \setminus A$. Set A is open and closed in X, from which it follows that $X \setminus A$ is open and closed, so x and y belong to different quasicomponents contained in A and $X \setminus A$, respectively. The last claim contradicts the assumption $y \in Q_X(x)$. It follows that $y \in V_{XX}(x)$.

($\Leftarrow$) Let $y \in V_{XX}(x)$. It follows that for every covering of X there exists a chain of the members of the covering that connects x and y.

If $y \notin Q_X(x)$, there will be an open and closed set A such that $x \in A$ and $y \notin A$, i.e., $y \in X \setminus A$. Then for the covering $\{A, X \setminus A\}$ of X there is no chain that connects x and y. □

From the last theorem it follows that quasicomponent of the point x is the biggest chain connected set in X that contains x.

The next proposition is the summary of Proposition 5.3, 5.4, and Theorem 5.3.

Proposition 5.7. *For every $x \in Y$, we have*

$$Q_Y(x) = V_{YY}(x) \subseteq \bigcup_{y \in V_{YX}(x)} V_{YY}(y) = V_{YX}(x) \subseteq V_{XX}(x) = Q_X(x).$$

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