ISSN 1450-9628

KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 44, Number 1, 2020

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

51

KRAGUJEVAC Journal of Mathematics / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Kragujevac : InterPrint). - 24 cm

Tromesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2001

Published By:	Faculty of Science University of Kragujevac Radoja Domanovića 12 34000 Kragujevac Serbia Tel.: +381 (0)34 336223 Fax: +381 (0)34 335040 Email: `krag_j_math@kg.ac.rs` Website: `http://kjm.pmf.kg.ac.rs`
Designed By:	Thomas Lampert
Front Cover:	Željko Mališić
Printed By:	InterPrint, Kragujevac, Serbia From 2018 the journal appears in one volume and four issues per annum.

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Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 7–26.

SOLUTION AND STABILITY OF A CUBIC TYPE FUNCTIONAL EQUATION: USING DIRECT AND FIXED POINT METHODS

V. GOVINDAN[1], S. MURTHY[2], AND M. SARAVANAN[3]

Abstract. In this concept, we investigate the generalized Ulam-Hyers-Rassias stability for the new type of cubic functional equation of the form

$$g\left(ax_1+bx_2+2cx_3\right)+g\left(ax_1+bx_2-2cx_3\right)+8\ a^3g(x_1)+8\ b^3g(x_2)$$
$$=2g(ax_1+bx_2)+4\left(g(ax_1+cx_3)+g(ax_1-cx_3)+g(bx_2+cx_3)+g(bx_2-cx_3)\right)$$

by using direct and fixed point alternative.

1. Introduction

Sometime in modeling applied problems there may be a degree of uncertainty in the parameters used in the model or some measurements may be imprecise. Due to such features, we are tempted to consider the study of the functional equation in the alternative settings. One of the most interesting questions in the theory of functional equations, concerning the famous Ulam [38] stability problem, is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to exact solution of the given functional equation?

In 1940, S. M. Ulam [39] raised the following question. Under what conditions does there exist an additive mapping near an approximately additive linear mappings? The case of approximately additive function was solved by D. H. Hyers [15] under certain assumptions. In 1978, a generalized version of the Theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [34]. A number of mathematicians were attracted by the result of Th. M. Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers-Ulam-Rassias stability. One of the

Key words and phrases. Cubic functional equation, generalized Hyers-Ulam stability, fixed point.
2010 *Mathematics Subject Classification.* Primary: 39B52. Secondary: 32B72, 32B82.
DOI 10.46793/KgJMat2001.007G
Received: November 15, 2016.
Accepted: January 30, 2018.

most famous functional equation is the additive functional equation

$$f(x+y)=f(x)+f(y).$$

In 1821, it was first solved by A. L. Cauchy in the class of the continuous real valued functions. It is often called an additive Cauchy functional equation in honor of A. L. Cauchy [39]. The theory of additive functional equations in frequently applied to the development of the theories of the other functional equations. Consider the functional equation

$$f(x+y)+f(x-y)=2f(x)+2f(y).$$

The quadratic function $f(x)=cx^2$ is a solution of this functional equation, so one can usually say that the above functional equation is quadratic [3–6, 20, 21, 27, 30].

Recently, Bae, Lee and Park [32] established some stability results for the functional equation

$$kf(x+ky)+f(kx-y)=\frac{k(k^2-1)}{2}\left(f(x+y)+f(x-y)\right)+(k\pm 1)f(y),$$

where $k\geq 2$ is a fixed integer, in the setting of non-Archimedean L-fuzzy normed spaces.

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [37] for functions between a normed space and a Banach space. After wards, the result was extended by P. W. Cholewa [11] and S. Czerwisk [12].

The cubic function $g(x)=cx^3$ satisfies the functional equation

$$g(2x+y)+g(2x-y)=2g(x+y)+2g(x-y)+12g(x). \tag{1.1}$$

Hence, throughout this concept, we promise that equation (1.1) is called a cubic functional equation and every solution of equation (1.1) is said to be a cubic function. The stability result of equation (1.1) was obtained by K. W. Jun and H. M. Kim [17].

In this concept, we present the general solution and generalized Ulam-Hyers-Rassias stability of the new type of cubic functional equation of the form

$$\begin{aligned} &g(ax_1+bx_2+2cx_3)+g\left(ax_1+bx_2-2cx_3\right)+8a^3g(x_1)+8b^3g(x_2)\\ &=2g(ax_1+bx_2)+4\left(g(ax_1+cx_3)+g(ax_1-cx_3)+g(bx_2+cx_3)+\right.\\ &\left.g(bx_2-cx_3)\right). \end{aligned} \tag{1.2}$$

The main goal of this concept is to obtain the generalized Hyers-Ulam-Rassias stability result for the functional equation (1.2) by using the direct and fixed point alternative [7, 13, 18, 19, 22–26, 29, 33, 35, 36] in [1, 2, 8–10, 14, 16, 28, 31].

For completeness, we will first investigate solution of the functional equation (1.2).

Proposition 1.1. *Let X and Y be real vector spaces. A function $g:X\to Y$ satisfies the functional equation* (1.1) *if and only if $g:X\to Y$ also satisfies the functional equation* (1.2).

Proof. Substituting (x, y) by $(0, 0)$ in (1.1) yields $g(0) = 0$. Replacing (x, y) by $(0, x)$ in (1.1), gives $g(-x) = -g(x)$ for all $x \in X$, which implies that g is odd. Now, replacing (x, y) by $(x, 0)$ in (1.1), we obtain $g(2x) = 8g(x)$, and replacing (x, y) by (x, x) in (1.1), we get that $g(3x) = 27g(x)$ for all $x \in X$. Substituting (x, y) by $(ax, ax + by)$ in (1.1), we have

$$\begin{aligned}
g(2ax + ax + by) + g(2ax - (ax + by)) =& 2g(ax + ax + by) \\
&+ 2g(ax - (ax + by)) + 12g(ax), \\
g(3ax + by) + g(ax - by) =& 2g(2ax + by) + 2g(-by) + 12g(ax), \\
g(3ax + by) + g(ax - by) =& 2g(2ax + by) - 2g(by) + 12g(ax), \qquad (1.3)
\end{aligned}$$

for all $x, y \in X$. Replacing (x, y) by $(ax, ax - by)$ in (1.1), we get

$$\begin{aligned}
g(2ax + ax - by) + g(2ax - (ax - by)) =& 2g(ax + ax - by) \\
&+ 2g(ax - (ax - by)) + 12g(ax), \\
g(3ax - by) + g(ax + by) =& 2g(2ax - by) + 2g(by) + 12g(ax), \qquad (1.4)
\end{aligned}$$

for all $x, y \in X$. Adding (1.3) and (1.4) and then using (1.1), we see that

$$\begin{aligned}
&g(3ax + by) + g(ax - by) + g(3ax - by) + g(ax + by) \\
=& 2g(2ax + by) - 2g(by) + 12g(ax) + 2g(2ax - by) + 2g(by) + 12g(ax), \\
&g(3ax + by) + g(3ax - by) + g(ax + by) + g(ax - by) \\
=& 2g(2ax + by) + 2g(2ax - by) + 24g(ax), \\
&g(3ax + by) + g(3ax - by) + g(ax + by) + g(ax - by) \\
=& 2\left(2g(ax + by) + 2g(ax - by) + 12g(ax)\right) + 24g(ax), \\
&g(3ax + by) + g(3ax - by) + g(ax + by) + g(ax - by) \\
=& 4g(ax + by) + 4g(ax - by) + 24g(ax) + 24g(ax), \\
&g(3ax + by) + g(3ax - by) + g(ax + by) + g(ax - by) \\
=& 4g(ax + by) + 4g(ax - by) + 48g(ax),
\end{aligned}$$

$$g(3ax + by) + g(3ax - by) = 3g(ax + by) + 3g(ax - by) + 48g(ax), \qquad (1.5)$$

for all $x, y \in X$. Now, replacing (ax, by) by $(ax + by, ax - by)$ in (1.5), respectively, we have

$$\begin{aligned}
&g(3(ax + by) + (ax - by)) + g(3(ax + by) - (ax - by)) \\
=& 3g((ax + by) + (ax - by)) + 3g((ax + by) - (ax - by)) + 48g(ax + by), \\
&g(3ax + 3by + ax - by) + g(3ax + 3by - ax + by) \\
=& 3g(2ax) + 3g(2by) + 48g(ax + by), \\
&g(4ax + 2by) + g(2ax + 4by) = 3g(2ax) + 3g(2by) + 48g(ax + by),
\end{aligned}$$

for all $x, y \in X$, which, in view of the identity $g(2x) = 8g(x)$, reduces to

$$g(4ax + 2by) + g(2ax + 4by)$$

$$
\begin{aligned}
&= 3\left(8g(ax)\right) + 3\left(8g(by)\right) + 48g(ax+by),\\
&8g(2ax+by) + 8g(ax+2by) = 24g(ax) + 24g(by) + 48g(ax+by),
\end{aligned}
$$

and dividing by 8, we get

$$
g(2ax+by) + g(ax+2by) = 3g(ax) + 3g(by) + 6g(ax+by), \tag{1.6}
$$

for all $x, y \in X$. Now, replacing (ax, by) by $(ax+3by, ax-3by)$ in (1.6), we arrive to

$$
\begin{aligned}
&g(2(ax+3by) + (ax-3by)) + g\left(ax+3by+2(ax-3by)\right)\\
&= 3g(ax+3by) + 3g(ax-3by) + 6g(ax+3by+ax-3by),\\
&g(2ax+6by+ax-3by) + g\left(ax+3by+2ax-6by\right)\\
&= 3g(ax+3by) + 3g(ax-3by) + 6g(2ax),\\
&g\left(3ax+3by\right) + g\left(3ax-3by\right) = 3g(ax+3by) + 3g(ax-3by) + (6\times 8)g(ax),\\
&27g\left(ax+by\right) + 27g\left(ax-by\right) = 3g(ax+3by) + 3g(ax-3by) + 48g(ax),
\end{aligned}
$$

$$
9g\left(ax+by\right) + 9g\left(ax-by\right) = g(ax+3by) + g(ax-3by) + 16g(ax), \tag{1.7}
$$

for all $x, y \in X$. Let us interchange ax in by and by in ax in (1.7) to get the identities

$$
9g\left(ax+by\right) + 9g\left(by-ax\right) = g(3ax+by) + g(by-3ax) + 16g(by),
$$

$$
9g\left(ax+by\right) - 9g\left(ax-by\right) = g(3ax+by) - g(3ax-by) + 16g(by), \tag{1.8}
$$

for all $x, y \in X$. Then, by adding (1.7) and (1.8), we get

$$
\begin{aligned}
&9g\left(ax+by\right) + 9g\left(ax-by\right) + 9g\left(ax+by\right) - 9g\left(ax-by\right)v \\
&= g(ax+3by)x + g(ax-3by) + 16g(ax)\\
&\quad + g(3ax+by) - g(3ax-by) + 16g(by),
\end{aligned} \tag{1.9}
$$

$$
\begin{aligned}
18g(ax+by) =& g(ax+3by) + g(ax-3by) + g(3ax+by)\\
&- g(3ax-by) + 16g(ax) + 16g(by),
\end{aligned} \tag{1.10}
$$

for all $x, y \in X$. Now, we interchange ax with by and by with ax in (1.5), respectively we get

$$
g\left(3by+ax\right) + g\left(3by-ax\right) = 3g(by+ax) + 3g(by-ax) + 48g(by),
$$

$$
g\left(ax+3by\right) - g\left(ax-3by\right) = 3g(ax+by) - 3g(ax-by) + 48g(by), \tag{1.11}
$$

for all $x, y \in X$. Hence, according to (1.5) and (1.11), we obtain

$$
\begin{aligned}
&g\left(3ax+by\right) + g\left(3ax-by\right) = 3g(ax+by) + 3g(ax-by) + 48g(ax),\\
&g\left(ax+3by\right) - g\left(ax-3by\right) = 3g(ax+by) - 3g(ax-by) + 48g(by).
\end{aligned}
$$

Adding the above equations we get

$$
\begin{aligned}
&g\left(3ax+by\right) + g\left(3ax-by\right) + g\left(ax+3by\right) - g\left(ax-3by\right)\\
&= 6g(ax+by) + 48g(ax) + 48g(by),\\
6g(ax+by) =& g(3ax+by) + g(3ax-by) + g(ax+3by) - g(ax-3by)
\end{aligned}
$$

$$- 48g(ax) - 48g(by), \tag{1.12}$$

for all $x, y \in X$. Again by adding (1.10) and (1.12), we get

$$\begin{aligned}
18g(ax+by) =& g(ax+3by) + g(ax-3by) + g(3ax+by) \\
& - g(3ax-by) - 16g(ax) - 16g(by), \\
6g(ax+by) =& g(3ax+by) + g(3ax-by) + g(ax+3by) \\
& - g(ax-3by) - 48g(ax) - 48g(by), \\
24g(ax+by) =& 2g(ax+3by) + 2g(3ax+by) \\
& - 32g(ax) - 32g(by), \\
12g(ax+by) =& g(ax+3by) + g(3ax+by) - 16g(ax) - 16g(by),
\end{aligned}$$

$$g(ax+3by) + g(3ax+by) = 12g(ax+by) + 16g(ax) + 16g(by), \tag{1.13}$$

for all $x, y \in X$. Taking (1.5), we have

$$\begin{aligned}
& g(3ax+by) + g(3ax-by) = 3g(ax+by) + 3g(ax-by) + 48g(ax), \\
& g(3ax+cz) + g(3ax-cz) = 3g(ax+cz) + 3g(ax-cz) + 48g(ax), \\
& g(3by+cz) + g(3by-cz) = 3g(by+cz) + 3g(by-cz) + 48g(by), \\
& g(3ax+cz) + g(3ax-cz) + g(3by+cz) + g(3by-cz) \\
=& 3g(ax+cz) + 3g(ax-cz) + 48g(ax) \\
& + 3g(by+cz) + 3g(by-cz) + 48g(by), \\
& 16g(3ax+cz) + 16g(3ax-cz) + 16g(3by+cz) + 16g(3by-cz) \\
=& 48g(ax+cz) + 48g(ax-cz) + 48g(by+cz) \\
& + 48g(by-cz) + 768g(ax) + 768g(by),
\end{aligned} \tag{1.14}$$

for all $x, y \in X$. Also, replacing (ax, by) by $(3ax+cz, 3by+cz)$ in (1.13), respectively we get

$$\begin{aligned}
& g(ax+3by) + g(3ax+by) = 12g(ax+by) + 16g(ax) + 16g(by), \\
& g(3ax+cz+3(3by+cz)) + g(3(3ax+cz)+3by+cz) \\
=& 12g(3ax+cz+3by+cz) + 16g(3ax+cz) + g(3by+cz), \\
& g(3ax+cz+9by+3cz) + g(9ax+3cz+3by+cz) \\
=& 12g(3ax+cz+3by+cz) + 16g(3ax+cz) + 16g(3by+cz), \\
& g(3ax+4cz+9by) + g(9ax+4cz+3by) \\
=& 12g(3ax+2cz+3by) + 16g(3ax+cz) + 16g(3by+cz),
\end{aligned} \tag{1.15}$$

for all $x, y \in X$. Replacing (ax, by) by $(3ax-cz, 3by-cz)$ in (1.13) we obtain

$$\begin{aligned}
& g(ax+3by) + g(3ax+by) = 12g(ax+by) + 16g(ax) + 16g(by), \\
& g(3ax-cz+3(3by-cz)) + g(3(3ax-cz)+3by-cz) \\
=& 12g(3ax-cz+3by-cz) + 16g(3ax-cz) + g(3by-cz),
\end{aligned}$$

$$
\begin{aligned}
&g(3ax - cz + 9by - 3cz) + g(9ax - 3cz + 3by - cz) \\
=&12g(3ax - cz + 3by - cz) + 16g(3ax - cz) + 16g(3by - cz), \\
&g(3ax - 4cz + 9by) + g(9ax - 4cz + 3by) \\
=&12g(3ax - 2cz + 3by) + 16g(3ax - cz) + 16g(3by - cz), \qquad (1.16)
\end{aligned}
$$

for all $x, y \in X$. Using (1.15) and (1.16), we get the following identities

$$
\begin{aligned}
&g(3ax + 4cz + 9by) + g(9ax + 4cz + 3by) + g(3ax - 4cz + 9by) \\
&+ g(9ax - 4cz + 3by) \\
=&12g(3ax + 2cz + 3by) + 16g(3ax + cz) + 16g(3by + cz) \\
&+ 12g(3ax - 2cz + 3by) + 16g(3ax - cz) + 16g(3by - cz), \\
&g(3ax + 4cz + 9by) + g(9ax + 4cz + 3by) + g(3ax - 4cz + 9by) \\
&+ g(9ax - 4cz + 3by) - 12g(3ax + 2cz + 3by) - 12g(3ax - 2cz + 3by) \\
=&16g(3ax + cz) + 16g(3by + cz) + 16g(3ax - cz) + 16g(3by - cz), \qquad (1.17)
\end{aligned}
$$

for all $x, y \in X$. Using (1.5) we obtain

$$
\begin{aligned}
&g(3ax + by) + g(3ax - by) = 3g(ax + by) + 3g(ax - by) + 48g(ax), \\
&g(3(ax + 3by) + 4cz) + g(3(ax + 3by) - 4cz) \\
=&3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by), \\
&g(3ax + 9by + 4cz) + g(3ax + 9by - 4cz) \\
=&3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by), \qquad (1.18)
\end{aligned}
$$

for all $x, y \in X$. Again using (1.5) it follows that

$$
\begin{aligned}
&g(3ax + by) + g(3ax - by), \\
=&3g(ax + by) + 3g(ax - by) + 48g(ax), \\
&g(3(3ax + by) + 4cz) + g(3(3ax + by) - 4cz) \\
=&3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by), \\
&g(9ax + 3by + 4cz) + g(9ax + 3by - 4cz) \\
=&3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by), \qquad (1.19)
\end{aligned}
$$

for all $x, y \in X$. Adding (1.18) and (1.19), we obtain

$$
\begin{aligned}
&g(3ax + 9by + 4cz) + g(3ax + 9by - 4cz) + g(9ax + 3by + 4cz) \\
&+ g(9ax + 3by - 4cz) = 3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) \\
&+ 48g(ax + 3by) + 3g(3ax + by + 4cz) \\
&+ 3g(3ax + by - 4cz) + 48g(3ax + by), \qquad (1.20)
\end{aligned}
$$

for all $x, y \in X$. Then applying (1.20) in (1.17), we get

$$
\begin{aligned}
&16g(3ax + cz) + 16g(3by + cz) + 16g(3ax - cz) + 16g(3by - cz) \\
=&3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)
\end{aligned}
$$

$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz)$$
$$- 12g(3ax + 3by + 2cz) - 12g(3ax + 3by - 2cz) + 48g(3ax + by), \tag{1.21}$$

for all $x, y \in X$. From (1.5), we obtain

$$g(3ax + by) + g(3ax - by) = 3g(ax + by) + 3g(ax - by) + 48g(ax),$$
$$g(3(ax + by) + 2cz) + g(3(ax + by) - 2cz) = 3g(ax + by + 2cz)$$
$$+ 3g(ax + by - 2cz) + 48g(ax + by),$$
$$g(3ax + 3by + 2cz) + g(3ax + 3by - 2cz)$$
$$=3g(ax + by + 2cz) + 3g(ax + by - 2cz) + 48g(ax + by), \tag{1.22}$$

for all $x, y, z \in X$. Using (1.22) in (1.21), we get

$$16g(3ax + cz) + 16g(3by + cz) + 16g(3ax - cz) + 16g(3by + cz)$$
$$=3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by)$$
$$- 36g(ax + by + 2cz) - 36g(ax + by - 2cz) - 576g(ax + by),$$
$$16g(3ax + cz) + 16g(3by + cz) + 16g(3ax - cz) + 16g(3by + cz)$$
$$=3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz)$$
$$+ 48 g(3ax + by) - 36g(ax + by + 2cz) - 36g(ax + by - 2cz) - 576g(ax + by),$$

for all $x, y, z \in X$, which, by modifying of (1.14), yields to the relation

$$3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by)$$
$$- 36g(ax + by + 2cz) - 36g(ax + by - 2cz) - 576g(ax + by)$$
$$=48g(ax + cz) + 48g(ax - cz) + 768g(ax) + 48g(by + cz)$$
$$+ 48g(by - cz) + 768g(by).$$

Then we obtain

$$3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by)$$
$$=36g(ax + by + 2cz) + 36g(ax + by - 2cz) + 576g(ax + by) + 48g(ax + cz)$$
$$+ 48g(ax - cz) + 768g(ax) + 48g(by + cz) + 48g(by - cz) + 768g(by), \tag{1.23}$$

for all $x, y, z \in X$. With the concept of (1.13) and (1.5), the left side of (1.14) can be written in the form

$$g(3ax + by) + g(3ax - by) = 3g(ax + by) + 3g(ax - by) + 48g(ax),$$
$$g(3(3ax + by) + 2cz) + g(3(3ax + by) - 2cz)$$
$$=3g(3ax + by + 2cz) + 3g(3ax + by - 2cz) + 48g(3ax + by),$$

$$
\begin{aligned}
&g(9ax+3by+2cz)+g(9ax+3by-2cz)=3g(3ax+by+2cz)\\
&\quad+3g(3ax+by-2cz)+48g(3ax+by), \qquad (1.24)\\
&g(3(ax+3by)+2cz)+g(3(ax+3by)-2cz)\\
=&3g(ax+3by+2cz)+3g(ax+3by-2cz)+48g(ax+3by),\\
&g(3ax+9by+2cz)+g(3ax+9by-2cz)=3g(ax+3by+2cz)\\
&\quad+3g(ax+3by-2cz)+48g(ax+3by). \qquad (1.25)
\end{aligned}
$$

Adding (1.24) and (1.25), we get

$$
\begin{aligned}
&g(3ax+9by+2cz)+g(3ax+9by-2cz)+g(9ax+3by+2cz)\\
&\quad+g(9ax+3by-2cz)=3g(ax+3by+2cz)+3g(ax+3by-2cz)\\
&\quad+48g(ax+3by)+3g(3ax+by+2cz)\\
&\quad+3g(3ax+by-2cz)+48g(3ax+by),\\
&g(3ax+9by+2cz)+g(3ax+9by-2cz)+g(9ax+3by+2cz)\\
&\quad+g(9ax+3by-2cz)-48g(3ax+by)-48g(ax+3by)\\
=&3g(3ax+by+2cz)+3g(ax+3by-2cz)+3g(ax+3by+2cz)\\
&\quad+3g(ax+3by-2cz),\\
&g(3ax+9by+2cz)+g(3ax+9by-2cz)+g(9ax+3by+2cz)\\
&\quad+g(9ax+3by-2cz)-12g(3ax+3by)-12g(3ax+3by)\\
=&3g(ax+3by+2cz)+3g(ax+3by-2cz)\\
&\quad+3g(ax+3by+2cz)+3g(ax+3by-2cz), \qquad (1.26)
\end{aligned}
$$

for all $x, y, z \in X$. Using (1.26), we get the identity

$$
\begin{aligned}
&16g(3ax+cz)+16g(3by+cz)+16g(3ax-cz)+16g(3by-cz)\\
=&3g(ax+3by+2cz)+3g(ax+3by-2cz)+48g(ax+3by)\\
&\quad+3g(3ax+by+2cz)-648g(ax+by)+48g(3ax+by), \qquad (1.27)
\end{aligned}
$$

for all $x, y, z \in X$. Replacing z by $2z$ in (1.27) and then using (1.23), we get

$$
\begin{aligned}
&16g(3ax+2cz)+16g(3by+2cz)+16g(3ax-2cz)+16g(3by-2cz)\\
=&3g(ax+3by+4cz)+3g(ax+3by-4cz)+48g(ax+3by)\\
&\quad+3g(3ax+by+4cz)+3g(3ax+by-4cz)\\
&\quad-648g(ax+by)+48g(3ax+by), \qquad (1.28)
\end{aligned}
$$

for all $x, y \in X$. Using (1.28) in (1.23), we obtain

$$
\begin{aligned}
&16g(3ax+2cz)+16g(3by+2cz)+16g(3ax-2cz)+16g(3by-2cz)\\
=&48g(ax+cz)+48g(ax-cz)+768g(ax)+48g(by+cz)\\
&\quad+48g(by-cz)+768g(by)+36g(ax+by+2cz)\\
&\quad+36g(ax+by-2cz)+576g(ax+by)-648g(ax+by), \qquad (1.29)
\end{aligned}
$$

for all $x, y, z \in X$. Again making use of (1.13) and (1.5), we get

$$\begin{aligned}
&16g(3ax + 2cz) + 16g(3by + 2cz) + 16g(3ax - 2cz) + 16g(3by - 2cz) \\
=&g(12ax + 4cz) + g(12ax - 4cz) - 12g(6ax) + g(12by + 4cz) \\
&+ g(12by - 4cz) - 12g(6by) \\
=&64g(3ax + cz) + 64g(3ax - cz) - 2592g(ax) + 64g(3by + cz) \\
&+ 64g(3by - cz) - 2592g(by) \\
=&64(g(3ax + cz) + g(3ax - cz) + g(3by + cz) + g(3by - cz)) \\
&- 2592g(ax) - 2592g(by) \\
=&64(3g(ax + cz) + 3g(ax - cz) + 48g(ax) + 3g(by + cz) + 3g(by - cz) \\
&+ 48g(by)) - 2592g(ax) - 2592g(by),
\end{aligned} \tag{1.30}$$

for all $x, y, z \in X$. Using (1.30) we have the following reduction

$$\begin{aligned}
&16g(3ax + 2cz) + 16g(3by + 2cz) + 16g(3ax - 2cz) + 16g(3by - 2cz) \\
=&192g(ax + cz)192g(ax - cz) - 480g(ax) \\
&+ 192g(by + cz) + 192g(by - cz) - 480g(by),
\end{aligned} \tag{1.31}$$

for all $x, y, z \in X$. Finally, if we compare (1.31) with (1.29), we can conclude that

$$\begin{aligned}
&48g(ax + cz) + 48g(by + cz) + 48g(ax - cz) + 48g(by - cz) + 768g(ax) + 768g(by) \\
&+ 36g(ax + by + 2cz) + 36g(ax + by - 2cz) + 576g(ax + by) - 648g(ax + by) \\
=&192g(ax + cz) + 192g(ax - cz) - 480g(ax) + 192g(by + cz) + 192g(by - cz) \\
&- 480g(by), \\
&36g(ax + by + 2cz) + 36g(ax + by - 2cz) = 192g(ax + cz) - 48g(ax + cz) \\
&+ 192g(ax - cz) - 48g(ax - cz) + 480g(ax) - 768g(ax) - 192g(by + cz) \\
&- 48g(by + cz) + 192g(by - cz) - 48g(by - cz) + 480g(by) - 768g(by) \\
&+ 72g(ax + by), \\
&36g(ax + by + 2cz) + 36g(ax + by - 2cz) = 144g(ax + cz) + 144g(ax - cz) \\
&+ 144g(by + cz) + 144g(by - cz) + 72g(ax + by) - 288g(ax) - 288g(by), \\
&g(ax + by + 2cz) + g(ax + by - 2cz) = 2g(ax + by) + 4g(ax + cz) \\
&+ 4g(ax - cz) + 4g(by + cz) + 4g(by - cz) - 8g(ax) - 8g(by),
\end{aligned}$$

for all $x, y, z \in X$. By considering $g(ax) = a^3 g(x)$, we get

$$\begin{aligned}
&g(ax + by + 2cz) + g(ax + by - 2cz) = 2g(ax + by) \\
&+ 4(g(ax + cz) + g(ax - cz) + g(by + cz) + g(by - cz)) - 8a^3 g(x) - 8b^3 g(y),
\end{aligned}$$

for all $x, y, z \in X$, which implies that g is cubic. Conversely, suppose that $g : X \to Y$ satisfies the functional equation (1.1). Putting $x = y = z = 0$ in (1.2) we get $g(0) = 0$. Changing (x, y, z) by $\left(\frac{-x}{a}, \frac{x}{b}, \frac{xc}{2}\right)$ in the result we get $g(-x) = -g(x)$, which implies that g is odd. Replacing $y = 0$ in (1.2) and employing the fact that g is odd, we obtain that

$$\begin{aligned}
&g(ax+2cz)+g(ax-2cz)=2g(ax)+4g(ax+cz)+g(ax-cz)\\
&+4g(cz)+4g(-cz)-8a^3g(x),\\
&g(ax+cz)+g(ax-2cz)=-6a^3g(x)+4g(ax+cz)+4g(ax-cz),\\
&g(ax+2cz)+g(ax-2cz)=-6a^3g(x)+4g(ax+cz)+4g(ax-cz), \qquad (1.32)
\end{aligned}$$

for all $x, y, z \in X$. Replacing (x, y, z) by $(x, 0, 0)$ in (1.6), we get

$$g(ax) = a^3 g(x),$$

since

$$\begin{aligned}
g(ax)+g(ax)&=2g(ax)+4g(ax)+4g(ax)-8a^3g(x),\\
2g(ax)&=10g(ax)-8a^3g(x),\\
2g(ax)-10g(ax)&=-8a^3g(x),\\
-8g(ax)&=-8a^3g(x),\\
g(ax)&=a^3g(x).
\end{aligned}$$

So we replace x by $2x$ in (1.32) and we get

$$\begin{aligned}
g(2ax+2cz)+g(2ax-2cz)&=-6a^3g(2x)+4g(2ax+cz)+4g(2ax-cz),\\
8g(ax+cz)+8g(ax-cz)&=-48a^3g(x)+4g(2ax+cz)+4g(2ax-cz),\\
2g(ax+cz)+2g(ax-cz)&=-12a^3g(x)+g(2ax+cz)+g(2ax-cz),\\
g(2ax+cz)+g(2ax-cz)&=12a^3g(x)+2g(ax+cz)+2g(ax-cz),
\end{aligned}$$

and

$$g(2x+y)+g(2x-y)=12g(x)+2g(x+y)+2g(x-y),$$

for all $x, y, z \in X$, which implies that g is cubic. This completes the proof. □

In this section, we present the generalized Hyers-Ulam-Rassias stability of the function (1.6).

Theorem 1.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(a^{kj}x, a^{kj}y, a^{kj}z\right)}{a^{3kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\alpha\left(a^{kj}x, a^{kj}y, a^{kj}z\right)}{a^{3kj}} = 0, \qquad (1.33)$$

for all $x, y \in X$. Let $g : X \to Y$ be an odd function satisfying the inequality

$$\|D_g(x, y, z)\| \leq \alpha(x, y, z), \qquad (1.34)$$

for all $x, y, z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the functional equation (1.6) *and*

$$\|f(x) - C(x)\| \leq \frac{1}{8a^3} \sum_{k=\frac{i-j}{2}}^{\infty} \frac{\alpha\left(a^{kj}x, 0, 0\right)}{a^{3kj}}, \tag{1.35}$$

for all $x \in X$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{n\to\infty} \frac{g\left(a^{kj}x\right)}{a^{3kj}},$$

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing (x, y, z) by $(x, 0, 0)$ in (1.34), we get

$$\left\|8g(ax) - 8a^3 g(x)\right\| \leq \alpha(x, 0, 0), \tag{1.36}$$

for all $x \in X$. From (1.36) it follows that

$$\left\|\frac{g(ax)}{a^3} - g(x)\right\| \leq \frac{1}{8\ a^3}\alpha(x, 0, 0), \tag{1.37}$$

for all $x \in X$. Replacing x by ax in (1.37) and dividing by a^3, we obtain

$$\left\|\frac{g(a(ax))}{a^6} - \frac{g(ax)}{a^3}\right\| \leq \frac{1}{8\ a^6}\alpha(ax, 0, 0),$$

$$\left\|\frac{g(a^2x)}{a^6} - \frac{g(ax)}{a^3}\right\| \leq \frac{1}{8\ a^6}\alpha(ax, 0, 0), \tag{1.38}$$

for all $x \in X$. From the identity (1.37) and (1.38), it follows that

$$\begin{aligned}
\left\|\frac{g(a^2x)}{a^6} - g(x)\right\| &\leq \frac{1}{8a^3}\alpha(x, 0, 0) + \frac{1}{8a^6}\alpha(ax, 0, 0) \\
&\leq \frac{1}{8\ a^3}\left\{\alpha(x, 0, 0) + \frac{1}{a^3}\alpha(ax, 0, 0)\right\} \\
&\leq \sum_{k=0}^{n-1} \frac{1}{8a^3}\left\{\frac{\alpha(a^k x, 0, 0)}{a^{3k}}\right\},
\end{aligned}$$

$$\left\|\frac{g(a^n x)}{a^{3n}} - g(ax)\right\| \leq \frac{1}{8a^3}\sum_{k=0}^{n-1}\left\{\frac{\alpha(a^k x, 0, 0)}{a^{3k}}\right\}, \tag{1.39}$$

for all $x \in X$. We prove the convergence of the sequence $\left\{\frac{g\left(a^k x\right)}{a^{3k}}\right\}$ for all $x \in X$. Replacing x by $a^m x$ and dividing by a^m in (1.39), we obtain

$$\left\|\frac{g(a^m x)}{a^{3m}} - \frac{g(a^{m+n}x)}{a^{3(m+n)}}\right\| \leq \frac{1}{8a^3}\sum_{k=0}^{n-1}\left\{\frac{\alpha(a^{m+n}x, 0, 0)}{a^{3(m+n)}}\right\},$$

for all $x \in X$. Hence, the sequence $\left\{\frac{g(a^m x)}{a^{3m}}\right\}$ is a Cauchy sequence. Since Y is complete normed space, there exists a mapping $C : X \to Y$ such that

$$C(x) = \lim_{n\to\infty} \frac{g\left(a^{kj}x\right)}{a^{3kj}},$$

for all $x \in X$. Letting $k \to \infty$ in (1.39), we see that (1.34) holds for $x \in X$. To prove that C satisfies (1.6), we replace (x, y, z) by $(a^n x, a^n y, a^n z)$ and divide (1.34) by a^{3n}, which gives that

$$\frac{1}{a^{3n}} \ \|D_g\left(a^n x, a^n y, a^n z\right)\| \ \leq \ \frac{1}{a^{3n}} \ \alpha\left(a^n x, a^n y, a^n z\right),$$

for all $x, y, z \in X$. As n approaches to ∞ in the above inequality and using the definition of $C(x)$, we have $DC(x, y, z) = 0$. Hence, C satisfies (1.6) for all $x, y, z \in X$.

We will show that C is unique. Let $B(x)$ be another cubic mapping satisfying (1.6) and (1.35), such that

$$\begin{aligned}
\|C(x) - B(x)\| &= \frac{1}{a^{3n}} \|c(a^n x) - B(a^n x)\| \\
&\leq \frac{1}{a^{3n}} \left\{\|C(a^n x) - g(a^n x)\| + \|g(a^n x) - B(a^n x)\|\right\} \\
&\leq \ \frac{1}{8 \ a^{3n}} \sum_{k=0}^{\infty} \frac{\alpha\left(a^{m+n}x, 0, 0, \right)}{a^{3(m+n)}} \ \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

for all $x \in X$. Hence, C is unique. Now, replacing x by $\frac{x}{a}$ in (1.34), we get

$$\begin{aligned}
\left\|8g(ax) - 8a^3 g(x)\right\| &\leq \alpha\left(\frac{x}{a}, 0, 0\right), \\
\left\|g(ax) - a^3 g(x)\right\| &\leq \frac{1}{8}\alpha\left(\frac{x}{a}, 0, 0\right),
\end{aligned}$$

for all $x \in X$. The remaining part of the proof of this theorem for $j = 1$ with replacing x by $\frac{x}{a}$ in (1.37) is similar. Also, we can prove the theorem for $j = -1$ in the same manner. This completes the proof of the theorem. □

Corollary 1.1. *Let λ and q be a non-negative real numbers. Let an odd function $g : X \to Y$ satisfying the inequality*

$$\|Dg(x, y, z)\| \leq \begin{cases} \lambda, \\ \lambda\left\{\|x\|^q + \|y\|^q + \|z\|^q\right\}, \\ \lambda\left\{\|x\|^q \|y\|^q \|z\|^q + \|x\|^{3q} + \|y\|^{3q} + \|z\|^{3q}\right\}, \end{cases}$$

for all $x, y, z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|g(x) - C(x)\| \leq \begin{cases} \dfrac{1}{8} \cdot \dfrac{1}{a^3 - 1}, \\ \dfrac{\lambda}{8} \cdot \dfrac{\|x\|^q}{|a^3 - a^q|}, \\ \dfrac{\lambda}{8} \cdot \dfrac{\|x\|^{3q}}{|a^3 - a^{3q}|}, \end{cases}$$

for all $x \in X$.

Proof. Setting

$$\alpha(x, y, z) \ \leq \ \begin{cases} \lambda, \\ \lambda \ \{\|x\|^q + \|y\|^q + \|z\|^q\}, \\ \lambda \ \left\{\|x\|^q \|y\|^q \|z\|^q + \|x\|^{3q} + \|y\|^{3q} + \|z\|^{3q}\right\}, \end{cases}$$

for all $x \in X$ and using $\alpha(x, y, z)$ in Theorem 1.1, we obtain desired result. $\square$

In this section, we investigate the generalized Ulam-Hyers-Rassias stability of the functional equation (1.6).

Theorem 1.2 (The Alternative of Fixed Point, [29])**.** *Suppose that complete generalized metric space (τ, d) and a strictly contractive mapping $T : \tau \to \tau$ with Lipchitz constant L are given. Then for each given $x \in \tau$, either*

$$d\left(T^n x, T^{n+1} x\right) = \infty,$$

for all $n \geq 0$ or there exists a natural number n_0 such that

(a) *$d(T^n x, T^{n+1} x) < \infty$ for all $n \geq 0$;*
(b) *the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;*
(c) *y^* is the unique fixed point of T in the set*

$$Y = \{y \in Y : d(P^{n_0} x, y) < \infty\};$$

(d) *$d(y^*, y) \leq \frac{1}{1-L}\ d(y, Py)$ for all $y \in Y$.*

Utilizing the above mentioned fixed point alternative, we now obtain our main results, that is the generalized Hyers-Ulam-Rassias stability of the functional equation (1.6).

From now on, let X be a real vector space and Y be a real Banach space. For given mapping $g : X \to Y$, we get

$$\begin{aligned} Dg(x, y, z) =& g(ax + by + 2cz) + g(ax + by - 2cz) - 2g(ax + by) - 4g(ax + cz) \\ & - 4g(ax - cz) - 4g(by + cz) - 4g(by - cz) + 8a^3 g(x) + 8b^3 g(y), \end{aligned}$$

for all $x, y, z \in X$. Let $\psi : X \times X \times X \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\psi\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)}{\mu_i^{3k}} = 0, \tag{1.40}$$

for all $x, y, z \in X$, where

$$\mu_i = \begin{cases} 2, & i = 0, \\ \frac{1}{2}, & i = 1. \end{cases}$$

Theorem 1.3. *Suppose that function $g : X \to Y$ satisfies the functional inequality*

$$\|Dg(x, y, z)\| \le \psi(x, y, z), \tag{1.41}$$

for all $x, y, z \in X$. If there exists $L = L(i)$ such that function

$$x \mapsto \beta(x) = \frac{1}{2}\ \alpha\left(\frac{x}{a}, 0, 0\right),$$

has the property

$$\frac{1}{\mu_i^3}\beta(\mu_i x) = L\beta(x), \tag{1.42}$$

for all $x \in X$, then there exists a unique cubic function $c : X \to Y$ that satisfies the functional equation (1.6) *and*

$$\|g(x) - c(x)\| \le \frac{L^{1-i}}{1 - L}\beta(x),$$

for all $x \in X$.

Proof. Consider the set $W = \{p/P : X \to \beta, p(0) = 0\}$ and introduced generalized metric on X

$$d(p, q) = \inf\left\{k \in (0, \infty) : \|p(x) - q(x)\| \le k\beta(x), x \in X\right\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$ by

$$Tp(x) = \frac{1}{\mu_i^3}p(\mu_i),$$

for all $x \in X$. Now, for $p, q \in X$, we have

$$\begin{aligned} d(p, q) &\le k, \quad x \in W, \\ |\, \|p(x) - q(x)\| &\le k\beta(x), \quad x \in W, \\ \left\|\frac{1}{\mu_i^3}p(\mu_i x) - \frac{1}{\mu_i^3}q(\mu_i x)\right\| &\le \frac{1}{\mu_i^3}k\beta(\mu_i x), \\ \|Tp(x) - Tq(x)\| &\le L\ k\ \beta(x), \quad x \in W, \\ d(Tp, Tq) &\le Lk, \quad x \in W. \end{aligned}$$

This implies that $d(Tp, Tq) \le Ld(p, q)$ for all $p, q \in X$.That is, T is strictly contractive mapping on X with Lipschitz constant L. From (1.36) it follows that

$$\left\|8g(ax) - 8a^3g(x)\right\| \le \alpha(x, 0, 0), \tag{1.43}$$

for all $x \in X$. From (1.43) it follows that

$$\left\| \frac{g(ax)}{a^3} - g(x) \right\| \leq \frac{1}{8a^3} \alpha(x, 0, 0),$$

for all $x \in X$. Using (1.42), for the case $i = 0$, this reduces to

$$\left\| g(x) - \frac{g(ax)}{a^3} \right\| \leq \frac{1}{4} \beta(x),$$

for all $x \in X$, that is

$$d(g_a, Tg_a) \leq \frac{1}{4} = L = L^1 < \infty.$$

Again replacing x by $\frac{x}{a}$ in (1.43), we get

$$\left\| g(x) - a^3 g\left(\frac{x}{a}\right) \right\| \leq \frac{1}{8} \alpha\left(\frac{x}{a}, 0, 0\right),$$

for all $x \in X$. Using (1.42) for the case $i = 1$, this reduces to

$$\left\| g(x) - a^3 g\left(\frac{x}{a}\right) \right\| \leq \frac{1}{4} \beta(x),$$

for all $x \in X$, that is $d(g_a, Tg_a) \leq 1$. This implies that $d(g_a, Tg_a) \leq 1 = L^0 < \infty$. In the above case, we write $d(g_a, Tg_a) \leq L^{1-i}$. Therefore, the first two conditions (a) and (b) of the Alternative fixed point theorem holds for T, and it follows that there exists a fixed point C of T in X such that

$$C(x) = \lim_{k \to \infty} \frac{g\left(\mu_i^k x\right)}{\mu_i^{3k}}, \quad \text{for all } x \in X. \tag{1.44}$$

In order to prove that $C : X \to Y$ is cubic we replace (x, y, z) by $(\mu_i^k x, \mu_i^k y, \mu_i^k z)$ and divide (1.41) by μ_i^{3k}. From that, using (1.40) and (1.44), we see that C satisfies (1.6) for all $x, y, z \in X$. Hence, C satisfies the functional equation (1.6).

By fixed point condition (2), C is the unique fixed point of T in the set

$$Y = \{g_a \in X : d(Tg_a, C) < \infty\}.$$

Using the fixed point alternative result, C is the unique function such that

$$\|g_a(x) - C(x)\| \leq k \ \beta(x),$$

for all $x \in X$ and $k > 0$. Finally, by (4), we obtain

$$d(g_a, C) \leq \frac{1}{1 - L} \ d(g_a, Tg_a).$$

That is, we have

$$d(g_a, C) \leq \frac{L^{1-i}}{1 - L}.$$

Hence, we conclude that

$$\|g(x) - C(x)\| \leq \frac{L^{1-i}}{1 - L} \ \beta(x),$$

for all $x \in X$. This completes the proof of the theorem. $\square$

Corollary 1.2. *Let $g : X \to Y$ be a mapping and let $t\ \gamma$ and p be real numbers such that*

$$\|Dg(x,y,z)\| \leq \begin{cases} \gamma, \\ \gamma \left\{\|x\|^p + \|y\|^p + \|z\|^p\right\}, \\ \gamma \left\{\|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right)\right\}, \end{cases}$$

for all $x, y, z \in X$. Then there exist a unique cubic mapping $C : X \to Y$ such that

$$\|g(x) - C(x)\| \leq \begin{cases} \dfrac{\gamma}{8} \cdot \dfrac{1}{a^3 - 1}, & \\ \dfrac{\gamma}{8} \cdot \dfrac{\|x\|^p}{a^3 - a^p}, & p \neq 3, \\ \dfrac{\gamma}{8} \cdot \dfrac{\|x\|^{3q}}{a^3 - a^{3q}}, & p \neq 1, \end{cases}$$

for all $x \in X$.

Proof. Set

$$\alpha(x,y,z) = \begin{cases} \gamma, \\ \gamma \left\{\|x\|^p + \|y\|^p + \|z\|^p\right\}, \\ \gamma \left\{\|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right)\right\}, \end{cases},$$

for all $x \in X$. Now,

$$\begin{aligned} &\frac{\alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)}{\mu_i^{3k}} \\ &= \begin{cases} \dfrac{\gamma}{\mu_i^{3k}}, \\ \dfrac{\gamma}{\mu_i^{3k}} \left\{\left\|\mu_i^k x\right\|^p + \left\|\mu_i^k y\right\|^p + \left\|\mu_i^k z\right\|^p\right\}, \\ \dfrac{\gamma}{\mu_i^{3k}} \left\{\left\|\mu_i^k x\right\|^p \left\|\mu_i^k y\right\|^p \left\|\mu_i^k z\right\|^p + \left(\left\|\mu_i^k x\right\|^{3p} + \left\|\mu_i^k y\right\|^{3p} + \left\|\mu_i^k z\right\|^{3p}\right)\right\}, \end{cases} \\ &\to \begin{cases} 0, & k \to \infty, \\ 0, & k \to \infty, \\ 0, & k \to \infty. \end{cases} \end{aligned}$$

That is, (1.40) holds. But we have $\beta(x) = \frac{1}{2}\ \alpha\left(\frac{x}{a}, 0, 0\right)$. Hence,

$$\beta(x) = \frac{1}{2}\ \alpha\left(\frac{x}{a}, 0, 0\right) = \begin{cases} \dfrac{\gamma}{8}, \\ \dfrac{\gamma}{2^3\ a^p}\ \|x\|^p, \\ \dfrac{\gamma}{2^3\ a^{3p}}\ \|x\|^{3p}. \end{cases}$$

Also,

$$\frac{1}{\mu_i^3}\beta(\mu, x) = \begin{cases} \dfrac{\gamma}{8\ \mu_i^3}, \\ \dfrac{\gamma}{8\ \mu_i^3}\ \ \|\mu_i\|^p, \\ \dfrac{\gamma}{8\ \mu_i^3}\ \ \|\mu_i\|^{3p}, \end{cases} = \begin{cases} \mu_i^{-3}\ \beta(x), \\ \mu_i^{p-3}\ \beta(x), \\ \mu_i^{3p-3}\ \beta(x). \end{cases}$$

Hence, the inequality (1.42) holds.

Case (i). $L = a^{-3}$, $i = 0$,

$$\begin{aligned} \|g(x) - C(x)\| &\leq \frac{L^{1-i}}{1-L}\beta(x) \leq \frac{(a^{-3})^{1-0}}{1-a^{-3}}\ \frac{\gamma}{8} \\ &\leq \frac{a^{-3}}{1-\frac{1}{a^3}} \cdot \frac{\gamma}{8} \leq \frac{\gamma}{8} \cdot \frac{1}{a^3-1}. \end{aligned}$$

Case (ii). $L = \left(\frac{1}{a^3}\right)^{-1}$, $i = 1$,

$$\begin{aligned} \|g(x) - C(x)\| &\leq \frac{L^{1-i}}{L}\beta(x) \leq \frac{(a^3)^{1-1}}{1-a^3}\frac{\gamma}{8} \\ &\leq \frac{1}{1-a^3} \cdot \frac{\gamma}{8} \leq \frac{\gamma}{8} \cdot \frac{1}{1-a^3}. \end{aligned}$$

Case (iii). $L = a^{p-3}$, $p < 3$, $i = 0$,

$$\begin{aligned} \|g(x) - C(x)\| &\leq \frac{(a^{p-3})^{1-0}}{1-a^{p-3}} \cdot \frac{\gamma}{2^3 a^p}\ \ \|x\|^p \\ &\leq \frac{a^p a^{-3}}{1-\frac{a^p}{a^3}} \cdot \frac{\gamma}{2^3 a^p}\ \ \|x\|^p, \\ \|g(x) - C(x)\| &\leq \frac{\|x\|^p \gamma}{8} \cdot \frac{1}{a^3 - a^p}. \end{aligned}$$

Case (iv). $L = \left(\frac{1}{a}\right)^{p-3}$, $p > 3$, $i = 1$, $L = a^{3-p}$, $p > 3$, $i = 1$,

$$\begin{aligned} \|g(x) - C(x)\| &\leq \frac{L^{1-i}}{1-L}\beta(x) \\ &\leq \frac{(a^{3-p})^{1-1}}{1-a^{3-p}} \cdot \frac{\gamma}{2^3 a^p}\ \|x\|^p, \\ \|g(x) - C(x)\| &\leq \frac{a^p}{a^p - a^3} \cdot \frac{\gamma}{8a^p}\ \ \|x\|^p \leq \frac{\|x\|^p \cdot \gamma}{8} \cdot \frac{1}{a^p - a^3}. \end{aligned}$$

Case (v). $L = a^{3p-3}$, $p < 1$, $i = 0$,

$$\|g(x) - C(x)\| \leq \frac{L^{1-i}}{L}\beta(x) \leq \frac{(a^{3p-3})^{1-0}}{1-a^{3p-3}} \cdot \frac{\gamma}{2^3 a^{3p}}\ \ \|x\|^{3p},$$

$$\|g(x)-C(x)\| \leq \frac{a^{3p-3}}{1-a^{3p-3}} \frac{\gamma}{8a^{3p}} \|x\|^{3p} \leq \frac{\|x\|^{3p} \cdot \gamma}{8} \cdot \frac{1}{a^3 - a^{3p}}.$$

Case (vi). $L = a^{3-3p}$, $p > 1$, $i = 1$,

$$\|g(x)-C(x)\| \leq \frac{L^{1-i}}{L} \beta(x) \leq \frac{(a^{3-3p})^{1-1}}{1-a^{3-3p}} \frac{\gamma}{2^3 a^{3p}} \|x\|^{3p},$$

$$\|g(x)-C(x)\| \leq \frac{1}{1-a^{3-3p}} \frac{\gamma}{8a^{3p}} \|x\|^{3p} \leq \frac{\|x\|^{3p} \cdot \gamma}{8\ a^{3p}} \cdot \frac{1}{a^{3p} - a^3}.$$

Hence, the proof is completed. □

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[1]Department of Mathematics,
Sri Vidya Mandir Arts and Science College,
Uthangarai - 636 902, Tamil Nadu, India
Email address: govindoviya@gmail.com.

[2]Department of Mathematics,
Government Arts and Science College (For Men),
Krishnagiri - 635 001, Tamil Nadu, India
Email address: smurthy07@yahoo.co.in.

[3]Department of Mathematics,
Adhiyamaan College of Engineering,
Hosur - 635 109, Tamil Nadu, India
Email address: saravanan040683@gmail.com.

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 27–39.

NEW STRONG DIFFERENTIAL SUBORDINATION AND SUPERORDINATION OF MEROMORPHIC MULTIVALENT QUASI-CONVEX FUNCTIONS

ABBAS KAREEM WANAS[1] AND ABDULRAHMAN H. MAJEED[2]

Abstract. New strong differential subordination and superordination results are obtained for meromorphic multivalent quasi-convex functions in the punctured unit disk by investigating appropriate classes of admissible functions. Strong differential sandwich results are also obtained.

1. Introduction and Preliminaries

Let Σ_p denote the class of all functions f of the form:

$$f(z)=z^{-p}+\sum_{k=1-p}^{\infty}a_kz^k \quad (p\in\mathbb{N}=\{1,2,\ldots\}),$$

which are analytic in the punctured unit disk $U^*=\{z\in\mathbb{C}:0<|z|<1\}$.

A function $f\in\Sigma_p$ is meromorphic multivalent starlike if $f(z)\neq 0$ and

$$-\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0 \quad (z\in U^*).$$

Similarly, $f\in\Sigma_p$ is meromorphic multivalent convex if $f'(z)\neq 0$ and

$$-\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \quad (z\in U^*).$$

Moreover, a function $f\in\Sigma_p$ is called meromorphic multivalent quasi-convex function if there exists a meromorphic multivalent convex function g such that $g'(z)\neq 0$

Key words and phrases. Strong differential subordination, strong differential superordination, meromorphic functions , quasi-convex functions, admissible functions.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 30A20, 34A40.

DOI 10.46793/KgJMat2001.027W

Received: August 26, 2017.

Accepted: February 13, 2018.

and

$$-\mathrm{Re}\left\{\frac{(zf'(z))'}{g'(z)}\right\} > 0 \quad (z \in U^*).$$

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. For a positive integer n and $a \in \mathbb{C}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots ,$$

with $\mathcal{H} = \mathcal{H}[1, 1]$.

Let f and g be members of $\mathcal{H}(U)$. The function f is said to be subordinate to g, or (equivalently) g is said to be superordinate to f, if there exists a Schwarz function w which is analytic in U with $w(0) = 0$ and $|w(z)| < 1 (z \in U)$ such that $f(z) = g(w(z))$. In such a case, we write $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalent (see [5])

$$f(z) \prec g(z) \;\Leftrightarrow\; f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $G(z, \zeta)$ be analytic in $U \times \bar{U}$ and let $f(z)$ be analytic and univalent in U. Then the function $G(z, \zeta)$ is said to be strongly subordinate to $f(z)$ or $f(z)$ is said to be strongly superordinate to $G(z, \zeta)$, written as $G(z, \zeta) \prec\prec f(z)$, if for $\zeta \in \bar{U} = \{z \in \mathbb{C} : |z| \le 1\}$, $G(z, \zeta)$ as a function of z is subordinate to $f(z)$. We note that

$$G(z, \zeta) \prec\prec f(z) \;\Leftrightarrow\; G(0, \zeta) = f(0) \text{ and } G(U \times \bar{U}) \subset f(U).$$

Definition 1.1. [6] Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ and let h be a univalent function in U. If F is analytic in U and satisfies the following (second-order) strong differential subordination:

$$\phi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right) \prec\prec h(z), \tag{1.1}$$

then F is called a solution of the strong differential subordination (1.1). The univalent function q is called a dominant of the solutions of the strong differential subordination or more simply a dominant if $F(z) \prec q(z)$ for all F satisfying (1.1). A dominant $\check{q}$ that satisfies $\check{q}(z) \prec q(z)$ for all dominants q of (1.1) is said to be the best dominant.

Definition 1.2. [7] Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ and let h be analytic function in U. If F and $\phi(F(z), zF'(z), z^2F''(z); z, \zeta)$ are univalent in U for $\zeta \in \bar{U}$ and satisfy the following (second-order) strong differential superordination:

$$h(z) \prec\prec \phi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right), \tag{1.2}$$

then F is called a solution of the strong differential superordination (1.2). An analytic function q is called a subordinant of the solutions of the strong differential superordination or more simply a subordinant if $q(z) \prec F(z)$ for all F satisfying (1.2). A univalent subordinant $\check{q}$ that satisfies $q(z) \prec \check{q}(z)$ for all subordinants q of (1.2) is said to be the best subordinant.

Definition 1.3. [6] Denote by Q the set consisting of all functions q that are analytic and injective on $\bar{U}\backslash E(q)$, where

$$E(q) = \left\{ \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty \right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial U \backslash E(q)$.

Furthermore, let the subclass of Q for which $q(0) = a$ be denoted by $Q(a)$, $Q(0) \equiv Q_0$, and $Q(1) \equiv Q_1$.

Definition 1.4. [9] Let Ω be a set in $\mathbb{C}$, $q \in Q$, and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n [\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; z, \zeta) \notin \Omega$, whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \mathrm{Re}\left\{ \frac{t}{s} + 1 \right\} \geq k \mathrm{Re}\left\{ \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right\},$$

$z \in U$, $\xi \in \partial U \backslash E(q)$, $\zeta \in \bar{U}$, and $k \geq n$.

We simply write $\Psi_1 [\Omega, q] = \Psi [\Omega, q]$.

Definition 1.5. [8] Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H} [a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n [\Omega, q]$ consists of those functions $\psi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the following admissibility condition: $\psi(r, s, t; \xi, \zeta) \in \Omega$, whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \mathrm{Re}\left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \mathrm{Re}\left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},$$

$z \in U$, $\xi \in \partial U$, $\zeta \in \bar{U}$, and $m \geq n \geq 1$.

In particular, we write $\Psi'_1 [\Omega, q] = \Psi' [\Omega, q]$.

In our investigations, we will need the following lemmas.

Lemma 1.1. [9] *Let $\psi \in \Psi_n [\Omega, q]$ with $q(0) = a$. If $F \in \mathcal{H} [a, n]$ satisfies*

$$\psi \left(F(z), zF'(z), z^2 F''(z); z, \zeta \right) \in \Omega,$$

then $F(z) \prec q(z)$.

Lemma 1.2. [8] *Let $\psi \in \Psi'_n [\Omega, q]$ with $q(0) = a$. If $F \in Q(a)$ and $\psi (F(z), zF'(z), z^2 F''(z); z, \zeta)$ is univalent in U for $\zeta \in \bar{U}$, then*

$$\Omega \subset \left\{ \psi \left(F(z), zF'(z), z^2 F''(z); z, \zeta \right) : z \in U, \zeta \in \bar{U} \right\}$$

implies $q(z) \prec F(z)$.

In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1–4]. In this present investigation, by making use of the strong differential subordination results and strong differential superordination results of Oros and Oros [8, 9], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and superordination properties of meromorphic multivalent quasi-convex functions.

2. Strong Subordination Results

Definition 2.1. Let Ω be a set in $\mathbb{C}$ and $q \in Q_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{\mathcal{H}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; z, \zeta) \notin \Omega$, whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi)}{q(\xi)}, \quad q(\xi) \neq 0 \quad \text{and} \quad \mathrm{Re}\left\{\frac{w + v^2}{v}\right\} \geq k\mathrm{Re}\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

where $z \in U$, $\zeta \in \bar{U}$, $\xi \in \partial U \backslash E(q)$, and $k \geq 1$.

Theorem 2.1. *Let $\phi \in \Phi_{\mathcal{H}}[\Omega, q]$. If $f \in \Sigma_p$ satisfies*

(2.1)
$$\left\{\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right.\right.$$
$$\left.\left.\times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) : z \in U, \zeta \in \bar{U}\right\} \subset \Omega,$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

Proof. Let the analytic function F in U be defined by

$$F(z) = -\frac{(z^p f'(z))'}{g'(z)}. \tag{2.2}$$

After some calculation, we have

$$\frac{zF'(z)}{F(z)} = \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}. \tag{2.3}$$

Further computations show that

$$\begin{aligned}\frac{z^2F''(z)}{F(z)} + \frac{zF'(z)}{F(z)} - \left(\frac{zF'(z)}{F(z)}\right)^2 &= z\left[\frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}\right]' \\ =\frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} &+ \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} \\ &+ \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right).\end{aligned} \tag{2.4}$$

Define the transforms from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s}{r}, \quad w = \frac{r(t+s) - s^2}{r^2}.$$

Let

$$\psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi\left(r, \frac{s}{r}, \frac{r(t+s) - s^2}{r^2}; z, \zeta\right). \tag{2.5}$$

The proof will make use of Lemma 1.1. Using equations (2.2), (2.3) and (2.4), it follows from (2.5) that

$$\begin{aligned}(2.6)\qquad &\psi\left(F(z), zF'(z), z^2F''(z); z,\zeta\right)\\ =&\phi\left(-\frac{(z^pf'(z))'}{g'(z)}, \frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}-\frac{zg''(z)}{g'(z)}, \frac{z^2\left(z^pf'(z)\right)'''}{(z^pf'(z))'}+\frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right.\\ &\left.\times\left(1-\frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right)-\frac{z^2g'''(z)}{g'(z)}+\frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)}-1\right); z,\zeta\right).\end{aligned}$$

Therefore, (2.1) becomes $\psi\left(F(z), zF'(z), z^2F''(z); z,\zeta\right)\in\Omega$.

To complete the proof, we next show that the admissibility condition for $\phi\in\Phi_{\mathcal{H}}[\Omega,q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.4. Note that

$$\frac{t}{s}+1=\frac{w+v^2}{v}.$$

Hence $\psi\in\Psi[\Omega,q]$. By Lemma 1.1, $F(z)\prec q(z)$ or equivalently

$$-\frac{(z^pf'(z))'}{g'(z)}\prec q(z).\qquad\square$$

We consider the special situation when $\Omega\neq\mathbb{C}$ is a simply connected domain. In this case $\Omega=h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi_{\mathcal{H}}[h(U),q]$ is written as $\Phi_{\mathcal{H}}[h,q]$. The following result is an immediate consequence of Theorem 2.1.

Theorem 2.2. *Let $\phi\in\Phi_{\mathcal{H}}[h,q]$. If $f\in\Sigma_p$ satisfies*

$$\begin{aligned}(2.7)\qquad &\phi\left(-\frac{(z^pf'(z))'}{g'(z)}, \frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}-\frac{zg''(z)}{g'(z)}, \frac{z^2\left(z^pf'(z)\right)'''}{(z^pf'(z))'}+\frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right.\\ &\left.\times\left(1-\frac{z\left(z^pf'(z)\right)''}{(z^pf'(z))'}\right)-\frac{z^2g'''(z)}{g'(z)}+\frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)}-1\right); z,\zeta\right)\prec\prec h(z),\end{aligned}$$

then

$$-\frac{(z^pf'(z))'}{g'(z)}\prec q(z).$$

By taking $\phi(u,v,w;z,\zeta)=u+\frac{v}{\beta u+\gamma}$, $\beta,\gamma\in\mathbb{C}$, in Theorem 2.2, we state the following corollary.

Corollary 2.1. *Let $\beta,\gamma\in\mathbb{C}$ and let h be convex in U with $h(0)=1$ and* $\operatorname{Re}\{\beta h(z)+\gamma\}>0$. *If $f\in\Sigma_p$ satisfies*

$$-\frac{(z^pf'(z))'}{g'(z)}+\frac{\frac{z(z^pf'(z))''}{(z^pf'(z))'}g'(z)-zg''(z)}{\gamma g'(z)-\beta\left(z^pf'(z)\right)'}\prec\prec h(z),$$

then

$$-\frac{(z^pf'(z))'}{g'(z)}\prec q(z).$$

The next result is an extension of Theorem 2.1 to the case where the behavior of q on ∂U is not known.

Corollary 2.2. *Let $\Omega \in \mathbb{C}$ and q be univalent in U with $q(0) = 1$. Let $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$ for some $\rho \in (0,1)$, where $q_\rho(z) = q(\rho z)$. If $f \in \Sigma_p$ satisfies*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right.$$
$$\left.\times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) \in \Omega,$$

then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

Proof. Theorem 2.1 yields $-\frac{(z^p f'(z))'}{g'(z)} \prec q_\rho(z)$. The result is now deduced from the fact that $q_\rho(z) \prec q(z)$. $\square$

Theorem 2.3. *Let h and q be univalent in U with $q(0) = 1$ and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ satisfy one of the following conditions:*

(1) *$\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$ for some $\rho \in (0,1)$;*
(2) *there exists $\rho_0 \in (0,1)$ such that $\phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

Proof. (1) By applying Theorem 2.1, we obtain $-\frac{(z^p f'(z))'}{g'(z)} \prec q_\rho(z)$, since $q_\rho(z) \prec q(z)$, we deduce

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z).$$

(2) Let $F(z) = -\frac{(z^p f'(z))'}{g'(z)}$ and $F_\rho(z) = F(\rho z)$. Then

$$\phi\left(F_\rho(z), zF'_\rho(z), z^2F''_\rho(z); \rho z, \zeta\right) = \phi\left(F(\rho z), zF'(\rho z), z^2F''(\rho z); \rho z, \zeta\right) \in h_\rho(U).$$

By using Theorem 2.1 and the comment associated with

$$\phi\left(F(z), zF'(z), z^2F''(z); w(z), \zeta\right) \in \Omega,$$

where w is any function mapping U into U, with $w(z) = \rho z$, we obtain $F_\rho(z) \prec q_\rho(z)$ for $\rho \in (\rho_0, 1)$. By letting $\rho \to 1^-$, we get $F(z) \prec q(z)$. Therefore,

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z). \qquad \square$$

The next result gives the best dominant of the strong differential subordination (2.7).

Theorem 2.4. *Let h be univalent in U and $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z)}{q(z)}, \frac{z^2 q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left(\frac{zq'(z)}{q(z)}\right)^2; z, \zeta\right) = h(z) \tag{2.8}$$

has a solution q with $q(0) = 1$ and satisfies one of the following conditions:

(1) *$q \in Q_1$ and $\phi \in \Phi_{\mathcal{H}}[h, q]$;*
(2) *q is univalent in U and $\phi \in \Phi_{\mathcal{H}}[h, q_\rho]$ for some $\rho \in (0, 1)$;*
(3) *q is univalent in U and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathcal{H}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \Sigma_p$ satisfies (2.7), then

$$-\frac{(z^p f'(z))'}{g'(z)} \prec q(z)$$

and q is the best dominant.

Proof. By applying Theorem 2.2 and Theorem 2.3, we deduce that q is a dominant of (2.7). Since q satisfies (2.8), it is also a solution of (2.7) and therefore q will be dominated by all dominants. Hence, q is the best dominant of (2.7). $\square$

In the particular case $q(z) = 1 + Mz, M > 0$ and in view of Definition 2.1, the class of admissible functions $\Phi_{\mathcal{H}}[\Omega, q]$ denoted by $\Phi_{\mathcal{H}}[\Omega, M]$ can be expressed in the following form.

Definition 2.2. Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible function $\Phi_{\mathcal{H}}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ such that

$$\phi\left(1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left(\frac{kM}{M + e^{-i\theta}}\right)^2; z, \zeta\right) \notin \Omega, \tag{2.9}$$

whenever $z \in U$, $\zeta \in \bar{U}$, $\theta \in \mathbb{R}$, $\operatorname{Re}\left\{Le^{-i\theta}\right\} \geq k(k-1)M$, for all θ and $k \geq 1$.

Corollary 2.3. *Let $\phi \in \Phi_{\mathcal{H}}[\Omega, M]$. If $f \in \Sigma_p$ satisfies*

$$\begin{aligned}\phi\Bigg(&-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'} \\ &\times \left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\Bigg) \in \Omega,\end{aligned}$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

When $\Omega = q(U) = \{w : |w - 1| < M\}$, the class $\Phi_{\mathcal{H}}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{H}}[M]$, then Corollary 2.3 takes the following form.

Corollary 2.4. *Let $\phi \in \Phi_{\mathcal{H}}[M]$. If $f \in \Sigma_p$ satisfies*

$$\left|\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right.\right.$$
$$\left.\left.\times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) - 1\right| < M,$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

Example 2.1. If $M > 0$ and $f \in \Sigma_p$ satisfies

$$\left|\frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} - \left(\frac{z(z^p f'(z))''}{(z^p f'(z))'}\right)^2 - \frac{z^2 g'''(z)}{g'(z)} + \left(\frac{zg''(z)}{g'(z)}\right)^2\right| < M,$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

This implication follows from Corollary 2.4 by taking $\phi(u, v, w; z, \zeta) = w - v + 1$.

Example 2.2. If $M > 0$ and $f \in \Sigma_p$ satisfies

$$\left|\frac{z(z^p f'(z))''}{(z^p f'(z))'} - \left(\frac{zg''(z)}{g'(z)} + 1\right)\right| < \frac{M}{M+1},$$

then

$$\left|\frac{(z^p f'(z))'}{g'(z)} + 1\right| < M.$$

This implication follows from Corollary 2.3 by taking $\phi(u, v, w; z, \zeta) = v$ and $\Omega = h(U)$, where $h(z) = \frac{M}{M+1}z, M > 0$. To apply Corollary 2.3, we need to show that $\phi \in \Phi_{\mathcal{H}}[\Omega, M]$, that is the admissibility condition (2.9) is satisfied follows from

$$\left|\phi\left(1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left(\frac{kM}{M + e^{-i\theta}}\right)^2; z, \zeta\right)\right| = \frac{kM}{M+1} \geq \frac{M}{M+1},$$

for $z \in U$, $\zeta \in \bar{U}$, $\theta \in \mathbb{R}$, and $k \geq 1$.

3. Strong Superordination Results

In this section, we obtain strong differential superordination. For this purpose the class of admissible functions given in the following definition will be required.

Definition 3.1. Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{\mathcal{H}}[\Omega, q]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$ that satisfy the admissibility condition: $\phi(u, v, w; \xi, \zeta) \in \Omega$, whenever

$$u = q(z), \quad v = \frac{zq'(z)}{mq(z)}, \quad q(z) \neq 0 \quad \text{and} \quad \operatorname{Re}\left\{\frac{w + v^2}{v}\right\} \leq \frac{1}{m}\operatorname{Re}\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

where $z \in U$, $\zeta \in \bar{U}$, $\xi \in \partial U$, and $m \geq 1$.

Theorem 3.1. *Let $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$, and*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'}\right.$$
$$\left.+\frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U, then

(3.1)
$$\Omega \subset \left\{\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'} + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right.\right.$$
$$\left.\left.\times\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right) : z \in U, \zeta \in \bar{U}\right\}$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

Proof. Let F defined by (2.2) and $\psi(F(z), zF'(z), z^2F''(z); z, \zeta)$ defined by (2.6). Since $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$, from (2.6) and (3.1), we have

$$\Omega \subset \left\{\psi\left(F(z), zF'(z), z^2F''(z); z, \zeta\right) : z \in U, \zeta \in \bar{U}\right\}.$$

From (2.5), we see that the admissibility condition for $\phi \in \Phi'_{\mathcal{H}}[\Omega, q]$ is equivalent to the admissibility condition for ψ as given in Definition 1.5. Hence $\psi \in \Psi'[\Omega, q]$ and by Lemma 1.2, $q(z) \prec F(z)$ or equivalently

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}. \qquad \square$$

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping h of U onto Ω and the class $\Phi'_{\mathcal{H}}[h(U), q]$ is written as $\Phi'_{\mathcal{H}}[h, q]$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *Let $\phi \in \Phi'_{\mathcal{H}}[h, q]$, $q \in \mathcal{H}$, and h be analytic in U. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$, and*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2(z^p f'(z))'''}{(z^p f'(z))'}\right.$$
$$\left.+\frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U*, then*

(3.2)

$$h(z) \prec\prec \phi \left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z (z^p f'(z))''}{(z^p f'(z))'} - \frac{z g''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} \right.$$
$$\left. + \frac{z (z^p f'(z))''}{(z^p f'(z))'} \left(1 - \frac{z (z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left(\frac{z g''(z)}{g'(z)} - 1 \right); z, \zeta \right)$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

By taking $\phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}$, $\beta, \gamma \in \mathbb{C}$, in Theorem 3.2, we state the following corollary.

Corollary 3.1. *Let* $\beta, \gamma \in \mathbb{C}$ *and let* h *be convex in* U *with* $h(0) = 1$*. Suppose that the differential equation* $q(z) + \frac{z q'(z)}{\beta q(z) + \gamma} = h(z)$ *has a univalent solution* q *that satisfies* $q(0) = 1$ *and* $q(z) \prec h(z)$*. If* $f \in \Sigma_p$*,* $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$*, and*

$$-\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z (z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta (z^p f'(z))'}$$

is univalent in U*, then*

$$h(z) \prec\prec -\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z (z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta (z^p f'(z))'}$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

The next result gives the best subordinant of the strong differential superordination (3.2).

Theorem 3.3. *Let* h *be analytic in* U *and* $\phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C}$*. Suppose that the differential equation*

$$\phi \left(q(z), \frac{z q'(z)}{q(z)}, \frac{z^2 q''(z)}{q(z)} + \frac{z q'(z)}{q(z)} - \left(\frac{z q'(z)}{q(z)} \right)^2; z, \zeta \right) = h(z)$$

has a solution $q \in Q_1$*. If* $\phi \in \Phi'_{\mathcal{H}} [h, q]$*,* $f \in \Sigma_p$*,* $-\frac{(z^p f'(z))'}{g'(z)} \in Q_1$*, and*

$$\phi \left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z (z^p f'(z))''}{(z^p f'(z))'} - \frac{z g''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'} \right.$$
$$\left. + \frac{z (z^p f'(z))''}{(z^p f'(z))'} \left(1 - \frac{z (z^p f'(z))''}{(z^p f'(z))'} \right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{z g''(z)}{g'(z)} \left(\frac{z g''(z)}{g'(z)} - 1 \right); z, \zeta \right)$$

is univalent in U, then

$$h(z) \prec\prec \phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'}\right.$$
$$\left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

implies

$$q(z) \prec -\frac{(z^p f'(z))'}{g'(z)}.$$

and q is the best subordinant.

Proof. The proof is similar to that of Theorem 2.4 and is omitted. □

4. Sandwich Results

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich theorem.

Theorem 4.1. *Let h_1 and q_1 be analytic functions in U, h_2 be univalent in U, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{\mathcal{H}}[h_2, q_2] \cap \Phi'_{\mathcal{H}}[h_1, q_1]$. If $f \in \Sigma_p$, $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$ and*

$$\phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'}\right.$$
$$\left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$

is univalent in U, then

$$h_1(z) \prec\prec \phi\left(-\frac{(z^p f'(z))'}{g'(z)}, \frac{z(z^p f'(z))''}{(z^p f'(z))'} - \frac{zg''(z)}{g'(z)}, \frac{z^2 (z^p f'(z))'''}{(z^p f'(z))'}\right.$$
$$\left. + \frac{z(z^p f'(z))''}{(z^p f'(z))'}\left(1 - \frac{z(z^p f'(z))''}{(z^p f'(z))'}\right) - \frac{z^2 g'''(z)}{g'(z)} + \frac{zg''(z)}{g'(z)}\left(\frac{zg''(z)}{g'(z)} - 1\right); z, \zeta\right)$$
$$\prec\prec h_2(z)$$

implies

$$q_1(z) \prec -\frac{(z^p f'(z))'}{g'(z)} \prec q_2(z).$$

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich corollary.

Corollary 4.1. *Let $\beta, \gamma \in \mathbb{C}$ and let h_1, h_2 be convex in U with $h_1(0) = h_2(0) = 1$. Suppose that the differential equations $q_1(z) + \frac{zq_1'(z)}{\beta q_1(z)+\gamma} = h_1(z)$, $q_2(z) + \frac{zq_2'(z)}{\beta q_2(z)+\gamma} = h_2(z)$*

have a univalent solutions q_1 *and* q_2*, respectively, that satisfy* $q_1(0) = q_2(0) = 1$ *and* $q_1(z) \prec h_1(z)$*,* $q_2(z) \prec h_2(z)$*. If* $f \in \Sigma_p$*,* $-\frac{(z^p f'(z))'}{g'(z)} \in \mathcal{H} \cap Q_1$*, and*

$$-\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta \left(z^p f'(z)\right)'}$$

is univalent in U*, then*

$$h_1(z) \prec\prec -\frac{(z^p f'(z))'}{g'(z)} + \frac{\frac{z(z^p f'(z))''}{(z^p f'(z))'} g'(z) - z g''(z)}{\gamma g'(z) - \beta \left(z^p f'(z)\right)'} \prec\prec h_2(z)$$

implies

$$q_1(z) \prec -\frac{(z^p f'(z))'}{g'(z)} \prec q_2(z).$$

Acknowledgements. The authors would like to thank the referee(s) for their helpful comments and suggestions.

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[1]Department of Mathematics,
College of Computer Science and Information Technology,
University of Al-Qadisiyah,
Diwaniya, Iraq
Email address: abbas.kareem.w@qu.edu.iq

[2]Department of Mathematics,
College of Science, University of Baghdad,
Baghdad, Iraq
Email address: ahmajeed6@yahoo.com

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 41–63.

ON THE TRANSMISSION-BASED GRAPH TOPOLOGICAL INDICES

R. SHARAFDINI[1] AND T. RÉTI[2]

Abstract. The distance $d(u, v)$ between vertices u and v of a connected graph G is equal to the number of edges in a minimal path connecting them. The transmission of a vertex v is defined by $\sigma(v) = \sum_{u \in V(G)} d(v, u)$. A topological index is said to be a transmission-based topological index (TT index) if it includes the transmissions $\sigma(u)$ of vertices of G. Because $\sigma(u)$ can be derived from the distance matrix of G, it follows that transmission-based topological indices form a subset of distance-based topological indices. So far, relatively limited attention has been paid to TT indices, and very little systematic studies have been done. In this paper our aim was i) to define various types of transmission-based topological indices ii) establish lower and upper bounds for them, and iii) determine a family of graphs for which these bounds are best possible. Additionally, it has been shown in examples that using a group theoretical approach the transmission-based topological indices can be easily computed for a particular set of regular, vertex-transitive, and edge-transitive graphs. Finally, it is demonstrated that there exist TT indices which can be successfully applied to predict various physicochemical properties of different organic compounds. Some of them give better results and have a better discriminatory power than the most popular degree-based and distance-based indices (Randić, Wiener, Balaban indices).

1. Introduction and Preliminaries

Let G be a simple connected graph with the finite vertex set $V(G)$ and the edge set $E(G)$, and denote by $n = |V(G)|$ and $m = |E(G)|$ the number of vertices and edges, respectively. Using the standard terminology in graph theory, we refer the reader to [44]. The degree $d(u)$ of the vertex $u \in V(G)$ is the number of the edges incident to u. The edge of the graph G connecting the vertices u and v is denoted by uv.

Key words and phrases. Graph distance, topological index, transmission.
2010 *Mathematics Subject Classification.* Primary: 05C12. Secondary: 05C05, 05C07, 05C90.
DOI 10.46793/KgJMat2001.041S
Received: July 29, 2017.
Accepted: February 14, 2018.

The role of molecular descriptors (especially topological descriptors) is remarkable in mathematical chemistry especially in QSPR/QSAR investigations. In mathematical chemistry, the *first Zagreb* index $M_1(G)$ and the *second Zagreb* index $M_2(G)$ belong to the family of the most important degree-based molecular descriptors. They are defined as [22,23,25,31,36]

$$M_1(G) = \sum_{uv\in E(G)} d(u) + d(v) = \sum_{u\in V(G)} d^2(u), \quad M_2(G) = \sum_{uv\in E(G)} d(u)d(v).$$

Similarly, the *first variable Zagreb* index and the *second variable Zagreb* index are defined as [33,36,46]

$$M_1^\lambda(G) = \sum_{u\in V(G)} d(u)^{2\lambda}, \quad M_2^\lambda(G) = \sum_{uv\in E(G)} d(u)^\lambda d(v)^\lambda,$$

where λ is a real number.

The *Randić* index $R(G)$, the *ordinary sum-connectivity* index $X(G)$, the *harmonic* index $H(G)$ and *geometric-arithmetic* index $GA(G)$ are also widely used degree-based topological indices [17,40,45,48–50]. By definition,

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)d(v)}}, \quad X(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d(u)+d(v)}},$$

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}, \quad GA(G) = \sum_{uv\in E(G)} \frac{2\sqrt{d(u)d(v)}}{d(u)+d(v)}.$$

Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ be the maximum and the minimum degrees, respectively, of vertices of G. The average degree of G is $\frac{2m}{n}$. A connected graph G is said to be *bidegreed* with degrees Δ and δ, $\Delta > \delta \geq 1$, if at least one vertex of G has degree Δ and at least one vertex has degree δ, and if no vertex of G has degree different from Δ or δ. A connected bidegreed bipartite graph is called *semi-regular* if each vertex in the same part of a bipartition has the same degree. A graph G is called *regular* if all its vertices have the same degree, otherwise it is said to be *irregular*. In many applications and problems in theoretical chemistry, it is important to know how a given graph is irregular. The (vertex) regularity of a graph is defined in several approaches. Two most frequently used graph topological indices that measure how irregular a graph is, are the *irregularity* and *variance of degrees*. Let $\mathrm{imb}(e) = |d(u) - d(v)|$ be the *imbalance* of an edge $e = uv \in E(G)$. In [1], the *irregularity* of G, which is a measure of irregularity of graph G, defined as

$$\mathrm{irr}(G) = \sum_{e\in E(G)} \mathrm{imb}(e) = \sum_{uv\in E(G)} |d_G(u) - d_G(v)|. \tag{1.1}$$

The *variance of degrees* of graph G is defined as [7]

$$\mathrm{Var}(G) = \frac{1}{n} \sum_{u\in V(G)} \left(d(u) - \frac{2m}{n}\right)^2 = \frac{M_1(G)}{n} - \frac{4m^2}{n^2}. \tag{1.2}$$

Another measure of irregularity, which is called *degree deviation*, defined as [37]

$$s(G) = \sum_{u \in V(G)} \left| d(u) - \frac{2m}{n} \right|.$$

It is worth mentioning that $\frac{s(G)}{n}$ is nothing but the *mean deviation* of the data set $(d(u) \mid u \in V(G))$.

The *distance* between the vertices u and v in graph G is denoted by $d(u,v)$ and it is defined as the number of edges in a minimal path connecting them. The *eccentricity* $\varepsilon(v)$ of a vertex v is the maximum distance from v to any other vertex. The *diameter* $\mathrm{diam}(G)$ of G is the maximum eccentricity among the vertices of G. The *transmission* (or *status*) of a vertex v of G is defined as $\sigma(v) = \sigma_G(v) = \sum_{u \in V(G)} d(v,u)$. A graph G is said to be *transmission regular* [3] if $\sigma(u) = \sigma(v)$ for any vertex u and v of G. A transmission regular graph G is called *k-transmission regular* if there exists a positive integer k, for which $\sigma(v) = k$ for any vertex v of G. In K_n, the complete graph of order n, each vertex has transmission $n-1$. So it is $(n-1)$-transmission regular. The the cycle C_n and the complete bipartite graph $K_{a,a}$ are transmission regular. It has been verified that there exist regular and non-regular transmission regular graphs [3]. Consider the polyhedron depicted in Figure 1. It is the rhombic dodecahedron that contains 14 vertices, (8 vertices of degree 3 and 6 vertices of degree 4), 24 edges and 12 faces, all of them are congruent rhombi. The graph G_{RD} of the rhombic

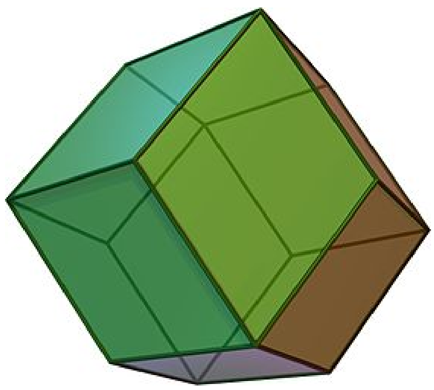

FIGURE 1. The rhombic dodecahedron

dodecahedron is a bidegreed, semi-regular 28-transmission regular graph (see Figure 2). An interesting observation is that the 14-vertex polyhedral graph G_{RD} depicted in Figure 2 is identical to the semi-regular graph published earlier in an alternative form in [3]. It is conjectured that G_{RD} is the smallest non-regular, bipartite, polyhedral (3-connected) and transmission regular graph.

If ω is a vertex weight of graph G, then one can see that

$$\sum_{\{u,v\} \subseteq V(G)} (\omega(u) + \omega(v))\, d(u,v) = \sum_{v \in V(G)} \omega(v)\sigma(v). \tag{1.3}$$

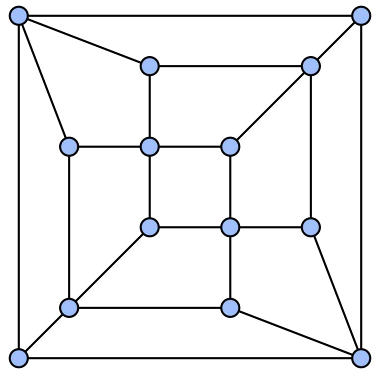

FIGURE 2. The edge-graph of the rhombic dodecahedron which is a 28-transmission regular graph but not regular

It is easy to construct various transmission-based indices having the same structure as the known degree-based topological indices. Based on this analogy-concept, the corresponding transmission-based indices are defined.

Let us define the *transmission Randić* index $RS(G)$, the *transmission ordinary sum-connectivity* index $XS(G)$, the *transmission harmonic* index $HS(G)$ and the *transmission geometric-arithmetic* index $GAS(G)$ as follows:

$$RS(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\sigma(u)\sigma(v)}}, \quad XS(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{\sigma(u)+\sigma(v)}},$$

$$HS(G) = \sum_{uv\in E(G)} \frac{2}{\sigma(u)+\sigma(v)}, \quad GAS(G) = \frac{n}{2m} \sum_{uv\in E(G)} \frac{2\sqrt{\sigma(u)\sigma(v)}}{\sigma(u)+\sigma(v)}.$$

It follows that $GAS(G) \leq \frac{n}{2}$, with equality if and only if G is a transmission regular graph.

The *Wiener* index $W(G)$, the *Balaban* index $J(G)$ and the *sum-Balaban* index $SJ(G)$ represent a particular class of transmission-based topological indices. They are defined as [4–6, 9, 10, 16, 21, 51]

$$\begin{aligned} W(G) =& \frac{1}{2} \sum_{u\in VG)} \sum_{v\in V(G)} d(u,v) = \frac{1}{2} \sum_{u\in VG)} \sigma(u), \\ J(G) =& \frac{m}{m-n+2} \sum_{uv\in E(G)} \frac{1}{\sqrt{\sigma(u)\sigma(v)}} = \frac{m}{m-n+2} RS(G), \\ SJ(G) =& \frac{m}{m-n+2} \sum_{uv\in E(G)} \frac{1}{\sqrt{\sigma(u)+\sigma(v)}} = \frac{m}{m-n+2} XS(G). \end{aligned}$$

In [39] the *first transmission Zagreb* index $MS_1(G)$ and the *second transmission Zagreb* index $MS_2(G)$ are defined as

$$MS_1(G) = \sum_{uv\in E(G)} \sigma(u)+\sigma(v) = \sum_{u\in V(G)} d(u)\sigma(u), \quad MS_2(G) = \sum_{uv\in E(G)} \sigma(u)\sigma(v).$$

It is important to note that $MS_1(G)$ coincides with the *degree distance* $DD(G)$ that was introduced in [11, 24] and [43].

In fact by (1.3),

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} (d(u)+d(v))d(u,v) = \sum_{v\in V(G)} d(v)\sigma(v) = MS_1(G). \tag{1.4}$$

Consequently, if G is a k-transmission regular graph with m vertices, then $DD(G) = MS_1(G) = 2mk$.

Let us propose the *variable degree transmission Zagreb* index $MSD^\lambda(G)$ and the *variable transmission Zagreb* index $MS^\lambda(G)$ as follows

$$MSD^\lambda(G) = \sum_{u\in V(G)} d(u)\sigma(u)^{2\lambda-1}, \quad MS^\lambda(G) = \sum_{u\in V(G)} \sigma(u)^{2\lambda},$$

where λ is a real number.

The *eccentric distance sum* of a graph G, denoted by $\xi^d(G)$, defined as [20]

$$\xi^d(G) = \sum_{u\in V(G)} \varepsilon(u)\sigma(u).$$

It follows from (1.3) that

$$\xi^d(G) = \sum_{\{u,v\}\subseteq V(G)} (\varepsilon(u)+\varepsilon(v))d(u,v) = \sum_{v\in V(G)} \varepsilon(v)\sigma(v). \tag{1.5}$$

Starting with (1.6) and (1.7) we introduce two transmission-based irregularity indices defined as follows. Let G be a connected graph with n vertices and m edges. The *transmission imbalance* of an edge $e = uv \in E(G)$ is defined as $\mathrm{imb}_{\mathrm{Tr}}(e) = |\sigma_G(u) - \sigma_G(v)|$. Let us define the *transmission irregularity* $\mathrm{irr}_{\mathrm{Tr}}(G)$ and the *transmission variance* $\mathrm{Var}_{\mathrm{Tr}}(G)$ of G as follows:

$$\mathrm{irr}_{\mathrm{Tr}}(G) = \sum_{e\in E(G)} \mathrm{imb}_{\mathrm{Tr}}(e) = \sum_{uv\in E(G)} |\sigma_G(u) - \sigma_G(v)|, \tag{1.6}$$

$$\begin{aligned}\mathrm{Var}_{\mathrm{Tr}}(G) &= \frac{1}{n}\sum_{u\in V(G)} \left(\sigma_G(u) - \frac{2W(G)}{n}\right)^2 = \frac{1}{n}\sum_{u\in V(G)} \sigma_G(u)^2 - \frac{4W(G)^2}{n^2} \\ &= \frac{MS^1(G)}{n} - \frac{4W(G)^2}{n^2} \geq 0,\end{aligned} \tag{1.7}$$

where $\frac{2W(G)}{n}$ is the average vertex transmission of G. It is obvious that $\mathrm{Var}_{\mathrm{Tr}}(G)$ is equal to zero if and only if G is transmission regular.

Let us also define the transmission-based topological indices $QS_e(G)$ and $QS_{v,e}(G)$ as follows

$$QS_e(G) = \frac{1}{m}\mathrm{irr}_{\mathrm{Tr}}(G), \quad QS_{v,e}(G) = \frac{n}{2}\left(1 + \frac{1}{m}\mathrm{irr}_{\mathrm{Tr}}(G)\right) = \frac{n}{2}\left(1 + QS_e(G)\right).$$

Remark 1.1. Let G be an n-vertex graph. Comparing topological indices $GAS(G)$ and $QS_{v,e}(G)$, we get

$$GAS(G) \leq \frac{n}{2} \leq QS_{v,e}(G).$$

Equalities hold in both sides simultaneously if and only if G is transmission regular.

2. Establishing Lower and Upper Bounds

Lemma 2.1. *Let G be a connected graph with $n \geq 2$ vertices and m edges. Then*

$$\begin{aligned} 0 \leq & \mathrm{irr}_{\mathrm{Tr}}(G) \leq m(n-2), \\ 0 \leq & \sum_{uv \in E(G)} (\sigma(u) - \sigma(v))^2 \leq m(n-2)^2. \end{aligned}$$

The equality on the right-hand sides holds if and only if G is isomorphic to S_n. The equality on the left-hand sides holds if and only if G is transmission regular.

Proof. For an arbitrary edge uv of G, we have $|\sigma(u) - \sigma(v)| \leq n - 2$. Therefore,

$$\mathrm{irr}_{\mathrm{Tr}}(G) = \sum_{uv \in E(G)} |\sigma(u) - \sigma(v)| \leq \sum_{uv \in E(G)} (n-2) = m(n-2).$$

It is trivial that in both formulas the equality on the right-hand side holds if and only if G isomorphic to S_n, since the star is the only graph where equality holds for each edge. □

Corollary 2.1. *Let T be a tree with $n \geq 2$ vertices. Then*

$$\begin{aligned} 0 \leq & \mathrm{irr}_{\mathrm{Tr}}(T) \leq (n-1)(n-2), \\ 0 \leq & \sum_{uv \in E(T)} (\sigma(u) - \sigma(v))^2 \leq (n-1)(n-2)^2. \end{aligned}$$

The equality on the right-hand sides holds if and only if G is isomorphic to S_n. The equality on the left-hand sides holds if and only if G is transmission regular.

Proof. It is a consequence of Lemma 2.1 and the fact that a tree with n vertices has exactly $n - 1$ edges. □

Corollary 2.2. *Let G be a connected graph with $n \geq 2$ vertices. Then*

$$(n-2) \geq QS_e(G) \geq 0$$

and

$$\frac{n(n-1)}{2} \geq QS_{v,e}(G) \geq \frac{n}{2}.$$

The upper bounds are achieved if and only if G is isomorphic to S_n and the lower bounds are achieved if and only if G is transmission regular.

Proof. It is a direct consequence of Lemma 2.1. □

Lemma 2.2. *Let G be a connected graph with $n \geq 3$ vertices and with maximum vertex degree Δ. Then for each arbitrary vertex u of G*

$$\sigma(u) \geqslant 2(n-1) - d(u) \geqslant 2(n-1) - \Delta \geqslant n-1.$$

Proof. Because $n - 1 \geq \Delta \geq d(u)$ one obtains that

$$\begin{aligned}\sigma(u) &= \sum_{(w \in V | d(u,w)=1)} d(u,w) + \sum_{(w \in V | d(u,w)>1)} d(u,w) = d(u) + \sum_{(w \in V | d(u,w)>1)} d(u,w) \\ &\geqslant d(u) + 2(n-1-d(u)) = 2n-2-d(u) \geqslant 2(n-1) - \Delta \geqslant n-1.\end{aligned}$$ □

Remark 2.1. There are several graphs containing a vertex u for which $\sigma(u) = n-1$. For example, $\sigma(u) = d(u) = n-1$ for any vertex u of a complete graph K_n.

Remark 2.2. Let G be a connected graph. It is easy to see that for any $u \in V(G)$, $\sigma(u) \geqslant 2(n-1) - d(u)$, with equality if and only if $\varepsilon(u) \leq 2$. This implies that

(i) $\sigma(u) = 2(n-1) - d(u)$ for any vertex u of a connected graph G if and only if $\operatorname{diam}(G) \leq 2$;
(ii) if G is a connected graph with $\operatorname{diam}(G) \leq 2$, then G is transmission regular if and only if G is regular.

Proposition 2.1. *Let G be a connected graph with n vertices. Then*

$$MSD^{\frac{3}{2}}(G) \geq 2(n-1)MS^1(G) - MS^{\frac{3}{2}}(G),$$

with equality if and only if $\operatorname{diam}(G) \leq 2$.

Proof. It follows from Lemma 2.2 that

$$\sum_{u \in V(G)} d(u)\sigma^2(u) \geq \sum_{u \in V(G)} (2n-2-\sigma(u))\,\sigma^2(u) = 2(n-1) \sum_{u \in V(G)} \sigma^2(u) - \sum_{u \in V(G)} \sigma^3(u),$$

and by Remark 2.2, the equality holds if and only if $\operatorname{diam}(G) \leq 2$. □

Proposition 2.2. *Let G be a connected graph with n vertices. Then*

$$MS_1(G) \geq 4(n-1)W(G) - MS^1(G),$$

with equality if and only if $\operatorname{diam}(G) \leq 2$.

Proof. It follows from Lemma 2.2 that

$$\sum_{u \in V(G)} d(u)\sigma(u) \geq \sum_{u \in V(G)} (2n-2-\sigma(u))\,\sigma(u) = 2(n-1) \sum_{u \in V(G)} \sigma(u) - \sum_{u \in V(G)} \sigma^2(u).$$

It follows from Remark 2.2 that the equality holds if and only if $\operatorname{diam}(G) \leq 2$. □

Lemma 2.3. *Let G be a connected graph with n vertices and m edges. If* $\operatorname{diam}(G) \leq 2$, *then*

(i) $\operatorname{irr}_{\mathrm{Tr}}(G) = \operatorname{irr}(G) \geq 0$;
(ii) $QS_{v,e}(G) = \frac{n}{2}\left(1 + \frac{1}{m}\operatorname{irr}(G)\right) \geq \frac{n}{2}$.

In particular, in both cases equality holds if and only if G is regular.

Proof. (i) It is a direct consequence of Lemma 2.1 and Remark 2.2.
(ii) It follows directly from part (i). □

Corollary 2.3. *Let $K_{p,q}$ be the complete bipartite graph with $p+q$ vertices and with parts of size p and q. Then*
(i) $\mathrm{irr}_{\mathrm{Tr}}(K_{p,q}) = pq\,|p-q| \geq 0$;
(ii) $QS_{v,e}(K_{p,q}) = \frac{p+q}{2}\,(1+|p-q|) \geq \frac{p+q}{2}$, *specially* $QS_{v,e}(S_n) = \frac{n(n-1)}{2}$.

In particular, the equalities in (i) *and* (ii) *hold if and only if $p=q$.*

Proof. (i) Since $\mathrm{diam}(K_{p,q}) = 2$ and $|E(K_{p,q})| = pq$, it follows from Lemma 2.3 (i) that $\mathrm{irr}_{\mathrm{Tr}}(K_{p,q}) = \mathrm{irr}(K_{p,q}) = \sum_{uv\in E(K_{p,q})} |p-q| = pq\,|p-q|$.
(ii) Since $\mathrm{diam}(K_{p,q}) = 2$ and $|V(K_{p,q})| = p+q$, it follows from Lemma 2.3 (ii) that
$$QS_{v,e}(K_{p,q}) = \frac{p+q}{2}\,(1+|p-q|) \geq \frac{p+q}{2}.$$
Specially, let $n \geq 2$ and $p=1$ and $q=n-1$. Then $K_{p,q}$ is isomorphic to the star S_n, $(n=p+q)$. Consequently, we obtain that
$$QS_{v,e}(S_n) = \frac{n}{2}(1+|2-n|) = \frac{n(n-1)}{2}.$$
It follows from Lemma 2.3 that the equalities in (i) and (ii) hold if and only if $K_{p,q}$ is regular if and only if $p=q$. □

An edge uv of a connected graph G is said to be a *strong edge* of G, if $|d(u)-d(v)| > 0$. Denote by $es(G)$ the number of strong edges of G. It is obvious that if G is a connected graphs, then $es(G)=0$ if and only if G is regular. From this observation it follows that the topological invariant $es(G)$ can be considered as a graph irregularity index. There are several graphs in which each edge is strong, that is $es(G) = |E(G)|$. For example, $es(K_{p,q}) = |E(K_{p,q})| = pq$ if p is not equal to q. It can be easily constructed a tree graph T with an arbitrary large edge number $m(T)$, for which $es(T) = m(T)$. Consider the $(n \geq 5)$-vertex windmill graph denoted by $Wd(n)$. It is a graph with diameter 2, with the vertex number $n = 2k+1$ and with the edge number $m = 3k$, where $k \geq 2$ is an arbitrary positive integer. Note that $es(Wd(n)) = 2k = \frac{2}{3}m = n-1$.

Proposition 2.3. *For the windmill graph $Wd(n)$ we have*
(i) $\mathrm{irr}_{\mathrm{Tr}}(Wd(n)) = es(Wd(n))(n-3) = \frac{2}{3}m(n-3) = (n-1)(n-3)$;
(ii) $QS_{v,e}(Wd(n)) = \frac{n}{2}\left(1+\frac{2}{3}(n-3)\right)$.

Proof. (i) Let E_0 be the set of strong edges of $Wd(n)$. It is easy to see that
$$E_0 = \Big\{uv \in E(Wd(n)) \mid d(u)=2, d(v)=n-1\Big\}, \quad es(Wd(n)) = |E_0|.$$

Since $\mathrm{diam}(Wd(n)) = 2$, it follows from Lemma 2.3 (i) that

$$\begin{aligned}\mathrm{irr}_{\mathrm{Tr}}(Wd(n)) = \mathrm{irr}(Wd(n)) &= \sum_{uv\in E_0} |d(u) - d(v)| = \sum_{uv\in E_0} |2-(n-1)| \\ &= es(Wd(n))\,|2-(n-1)| \\ &= \frac{2}{3}m(n-3) = (n-1)(n-3).\end{aligned}$$

(ii) It follows from part (i) that

$$\begin{aligned}QS_{v,e}(Wd(n)) &= \frac{n}{2}\left(1+\frac{1}{m}\mathrm{irr}_{\mathrm{Tr}}(Wd(n))\right) = \frac{n}{2}\left(1+\frac{2}{3}(n-3)\right) \\ &= \frac{n}{2}\left(1+\frac{1}{m}(n-1)(n-3)\right).\end{aligned}$$

□

Lemma 2.4. [32] *Let P_n be a path of order n such that $V(P_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(P_n) = \{v_iv_{i+1} \mid i = 0, \ldots, n-2\}$. Then for each $0 \le i \le n-1$*

$$\sigma_{P_n}(v_i) = \frac{1}{2}\Big(2i^2 - 2(n-1)i + (n-1)^2 + (n-1)\Big).$$

The following is a direct consequence of Lemma 2.4.

Proposition 2.4. *The transmission irregularity index of P_n is given by*

$$\mathrm{irr}_{\mathrm{Tr}}(P_n) = \begin{cases}\frac{n(n-2)}{2}, & \text{if } n \text{ is even,}\\ \frac{(n-1)^2}{2}, & \text{if } n \text{ is odd.}\end{cases}$$

For an edge uv of a connected graph G, define the positive integers N_u and N_v where N_u is the number of vertices of G whose distance to vertex u is smaller than distance to vertex v, and analogously, N_v is the number of vertices of G whose distance to the vertex v is smaller than to u. The number of vertices equidistant from u and v is denoted by N_{uv}. An edge uv of G is called a distance-balanced edge if $N_u = N_v$. A graph G is said to be *distance-balanced* [26] if its each edge is distance-balanced. It is known that a connected graph G is transmission regular if and only if G is distance balanced [3, 26].

The *Szeged* index $Sz(G)$ and the *revised Szeged* index $Sz^*(G)$ of a connected graph G are defined as [29, 35, 38]

$$Sz(G) = \sum_{uv\in E(G)} N_uN_v, \qquad Sz^*(G) = \sum_{uv\in E(G)} \left(N_u + \frac{N_{uv}}{2}\right)\left(N_v + \frac{N_{uv}}{2}\right).$$

Remark 2.3. For any connected graph G with n vertices, the following known relations are fulfilled [3, 12, 13, 16, 28, 29, 35, 38, 47].

(i) For any edge uv of G, $n = N_u + N_v + N_{uv}$. This implies that a graph G is bipartite if and only if $n = N_u + N_v$ holds for any edge uv of G.

(ii) The inequality $Sz(G) \ge W(G)$ is fulfilled.

(iii) $Sz(G) \le Sz^*(G)$ with equality if and only if G is bipartite.

(iv) For an n-vertex tree T, $W(S_n) \leq W(T) \leq W(P_n)$.
(v) For a tree graph T, $Sz^*(T) = Sz(T) = W(T)$.

The fundamental properties of Wiener index and their extremal graphs are summarized in [9, 12, 13, 16, 21]. Transmission regular graphs are characterized by the following property.

Lemma 2.5. [3,26,29] *Let G be a connected graph with n vertices and m edges. Then*

$$Sz^*(G) \leq \frac{n^2 m}{4},$$

with equality if and only if G is transmission regular.

Lemma 2.6 ([3, 12])**.** *Let G be a connected graph and let uv be an edge of G. Then*

$$\sigma(u) - \sigma(v) = N_v - N_u.$$

Lemma 2.7. *Let G be a connected graph. Then the following hold:*

(i) $\operatorname{irr}_{\mathrm{Tr}}(G) = \sum_{uv \in E(G)} |N_u - N_v| \geq 0$;

(ii) $\sum_{uv \in E(G)} (N_u - N_v)^2 = MSD^{\frac{3}{2}}(G) - 2MS_2(G) \geq 0$;

(iii) $\sum_{uv \in E(G)} (\sigma(u) - \sigma(v))^2 = \sum_{uv \in E(G)} \left(N_u^2 + N_v^2\right) - 2Sz(G) \geq 0$;

(iv) $\sum_{uv \in E(G)} \left(N_u^2 + N_v^2\right) = MSD^{\frac{3}{2}}(G) + 2Sz(G) - 2MS_2(G)$.

In (i), (ii), *and* (iii) *the equality holds if and only if G is transmission regular.*

Proof. (i) This is a direct consequence of Lemma 2.6.

(ii)

$$\begin{aligned} 0 \leq \sum_{uv \in E(G)} (N_u - N_v)^2 &= \sum_{uv \in E(G)} (\sigma(u) - \sigma(v))^2 \\ &= \sum_{uv \in E(G)} \left(\sigma^2(u) + \sigma^2(v)\right) - 2 \sum_{uv \in E(G)} \sigma(u)\sigma(v) \\ &= \sum_{u \in V(G)} d(u)\sigma^2(u) - 2MS_2(G) \\ &= MSD^{\frac{3}{2}}(G) - 2MS_2(G). \end{aligned}$$

(iii)

$$\begin{aligned} 0 \leq \sum_{uv \in E(G)} (\sigma(u) - \sigma(v))^2 &= \sum_{uv \in E(G)} (N_u - N_v)^2 \\ &= \sum_{uv \in E(G)} \left(N_u^2 + N_v^2\right) - 2Sz(G). \end{aligned}$$

(iv) It follows from the proof of the part (ii) and (iii) that

$$\begin{aligned}\sum_{uv\in E(G)}\left(N_u^2+N_v^2\right) &= \sum_{uv\in E(G)}(\sigma(u)-\sigma(v))^2+2Sz(G)\\ &= MSD^{\frac{3}{2}}(G)-2MS_2(G)+2Sz(G).\end{aligned}$$ ☐

Remark 2.4. Based on Lemma 2.7, the transmission-based topological index $QS_{v,e}(G)$ can be represented in the following alternative form:

$$QS_{v,e}(G)=\frac{n}{2}\left(1+\frac{1}{m}\sum_{uv\in E(G)}|\sigma(u)-\sigma(v)|\right)=\frac{n}{2}\left(1+\frac{1}{m}\sum_{uv\in E(G)}|N_u-N_v|\right).$$

Proposition 2.5. *Let G be a connected graph with n vertices and m edges. Then*

$$n^2m\geq MSD^{\frac{3}{2}}(G)+4Sz(G)-2MS_2(G),$$

with equality if and only if G is a bipartite graph.

Proof. Let G be a connected graph with n vertices. It follows from Remark 2.3 (i) that for any edge uv of G, $N_u+N_v\leq n$, with equality if and only if G is bipartite. This implies that

$$n^2\geq(N_u+N_v)^2=\left(N_u^2+N_v^2\right)+2N_uN_v,$$

with equality if and only if G is bipartite. Consequently, by Lemma 2.7 (iv) we have

$$\begin{aligned}n^2m=\sum_{uv\in E(G)}n^2 &\geq \sum_{uv\in E(G)}\left(N_u^2+N_v^2\right)+2\sum_{uv\in E(G)}N_uN_v\\ &= \sum_{uv\in E(G)}\left(N_u^2+N_v^2\right)+2Sz(G)\\ &= MSD^{\frac{3}{2}}(G)+4Sz(G)-2MS_2(G),\end{aligned}$$

with equality if and only if G is bipartite. ☐

Proposition 2.6. *Let G be a connected graph with n vertices. Then*

$$\mathrm{irr}_{\mathrm{Tr}}(G)=\sum_{uv\in E(G)}|N_u-N_v|\geq\frac{1}{n}\sum_{uv\in E(G)}\left|N_u^2-N_v^2\right|, \tag{2.1}$$

with equality if and only if G is a bipartite graph.

Proof. Let G be a connected graph with n vertices. It follows from Remark 2.3 (i) that for any edge uv of G, $N_u+N_v\leq n$, with equality if and only if G is bipartite. Therefore, it follows from Lemma 2.6 and

$$\left|N_u^2-N_v^2\right|=(N_u+N_v)\,|N_u-N_v|\leq n\,|N_u-N_v|=n\,|\sigma(u)-\sigma(v)|,$$

with equality if and only if G is bipartite. This implies that (2.1) holds with equality if and only if G is bipartite. ☐

Corollary 2.4. *Let T_n be an n vertex tree. Then*

$$\begin{aligned} MS_2(T_n) =& 2W(T_n) + \frac{1}{2} MSD^{\frac{3}{2}}(T_n) - \frac{n^2(n-1)}{2}, \\ \mathrm{irr}_{\mathrm{Tr}}(T_n) =& \frac{1}{n} \sum_{uv \in E(T_n)} \left| N_u^2 - N_v^2 \right|. \end{aligned}$$

Proof. It is a consequence of Proposition 2.5, Proposition 2.6 and Remark 2.3, since a tree with n vertices is bipartite and has exactly $n-1$ edges. ☐

Proposition 2.7. [12] *Let G_B be a connected bipartite graph with n vertices and m edges. Then*

$$Sz^*(G_B) = Sz(G_B) = \frac{n^2 m}{4} - \frac{1}{4} \sum_{uv \in E(G_B)} (\sigma(u) - \sigma(v))^2 \le \frac{n^2 m}{4},$$

with equality if and only if G is transmission regular.

Corollary 2.5. *Let G_B be a connected bipartite graph with n vertices and m edges. Then*

$$QS_{v,e}(G_B) \le \sqrt{n^2 - \frac{4}{m} Sz(G_B)},$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any edge $uv \in G_B$.

Proof. Using Cauchy-Schwartz inequality and Proposition 2.7 one obtains for G_B that

$$\left(\frac{1}{m} \sum_{uv \in E(G_B)} |\sigma(u) - \sigma(v)| \right)^2 \le \frac{1}{m} \sum_{uv \in E(G_B)} (\sigma(u) - \sigma(v))^2 = n^2 - \frac{4}{m} Sz(G_B),$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any edge $uv \in G_B$. Consequently,

$$\frac{1}{m} \sum_{uv \in E(G_B)} |\sigma(u) - \sigma(v)| \le \sqrt{n^2 - \frac{4}{m} Sz(G_B)},$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any edge $uv \in G_B$. Because

$$QS_{v,e}(G_B) - \frac{n}{2} = \frac{n}{2m} \sum_{uv \in E(G_B)} |\sigma(u) - \sigma(v)|,$$

we have

$$QS_{v,e}(G_B) - \frac{n}{2} \le \frac{n}{2} \sqrt{n^2 - \frac{4}{m} Sz(G_B)},$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any edge $uv \in G_B$. ☐

Lemma 2.8. [12] *Let T_n be an n-vertex tree. Then*

$$Sz(T_n) = W(T_n) = \frac{1}{4} \left(n(n-1) + MS_1(T_n) \right).$$

The following proposition demonstrates that the Wiener index and the first transmission Zagreb index are closely related.

Proposition 2.8. *Let T_n be an n-vertex tree. Then*

$$MS_1(T_n) = 4W(T_n) - n(n-1) = 4Sz(T_n) - n(n-1). \tag{2.2}$$

Proof. For any connected graph G we have

$$MS_1(G) = \sum_{uv \in E(G)} (\sigma(u) + \sigma(v)) = \sum_{u \in V(G)} d(u)\sigma(u).$$

Therefore, by Lemma 2.8 the result follows. □

Remark 2.5. As a consequence of (2.2), we conclude that in the family of n-vertex trees there is a linear correspondence (a perfect linear correlation) between the topological indices $W(T_n)$ and $MS_1(T_n)$.

In [39] it is reported that for a connected graph G, $W(G) < MS_1(G)$. This relation can be strengthened as follows.

Proposition 2.9. *Let G be a connected graph with minimum degree δ and maximum degree Δ. Then*

$$2\delta W(G) \leq MS_1(G) \leq 2\Delta W(G),$$

and equalities hold in both sides if and only if G is a regular graph.

Proof. Because for any connected graph G, $MS_1(G) = \sum_{uv \in E(G)} (\sigma(u) + \sigma(v)) = \sum_{u \in V(G)} d(u)\sigma(u)$, and for any vertex u of G, $\delta \leq d(u) \leq \Delta$, we have that

$$2\delta WG) \leq \sum_{uv \in E(G)} (\sigma(u) + \sigma(v)) = \sum_{u \in V(G)} d(u)\sigma(u) \leq 2\Delta WG).$$

Consequently, if G is an r-regular graph, we have $MS_1(G) = 2rW(G)$. □

Corollary 2.6. *Let T_n be an n-vertex tree. Then*

$$(n-1)(3n-4) \leq MS_1(T_n) \leq \frac{1}{3}n(n-1)(2n-1),$$

where

(i) *the right-hand side equality holds if and only if T_n is the path P_n;*
(ii) *the left-hand side equality holds if and only if T_n is the star S_n.*

Proof. For an n-vertex tree T_n we have $W(S_n) \leq W(T_n) \leq W(P_n)$, where $W(S_n) = (n-1)^2$ and $W(P_n) = \frac{(n^3-n)}{6}$. Therefore, from Proposition 2.8, we have the following inequalities:

$$MS_1(T_n) \leq \frac{4n(n-1)(n+1)}{6} - n(n-1) = \frac{1}{3}n(n-1)(2n-1),$$

with equality if and only if T_n is the path P_n, and

$$MS_1(T_n) \geq 4(n-1)^2 - n(n-1) = (n-1)(3n-4),$$

with equality if and only if T_n is the star S_n. □

The following is a direct consequence of Proposition 2.9.

Corollary 2.7. *If G_{be} is a benzenoid graph with $\Delta = 3$ and $\delta = 2$, then*

$$4W(G_{be}) \leq MS_1(G_{be}) \leq 6W(G_{be}).$$

It is easy to show that the inequality represented by

$$MS_2(G) = \sum_{uv \in E(G)} \sigma(u)\sigma(v) \leq \frac{1}{2} MSD^{\frac{3}{2}}(G),$$

can be sharpened in the following form.

Proposition 2.10. *Let G be a connected graph with m edges. Then*

$$MS_2(G) \leq \frac{1}{2} MSD^{\frac{3}{2}}(G) - \frac{1}{2m} \mathrm{irr}_{\mathrm{Tr}}(G)^2,$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any $uv \in E(G)$.

Proof. Using Cauchy-Schwartz inequality we have

$$\begin{aligned}\left(\frac{1}{m} \sum_{uv \in E(G)} |\sigma(u) - \sigma(v)|\right)^2 &\leq \frac{1}{m} \sum_{uv \in E(G)} (\sigma(u) - \sigma(v))^2, \\ &= \frac{1}{m} \sum_{uv \in E(G)} (\sigma^2(u) + \sigma^2(v)) - \frac{2}{m} \sum_{uv \in E(G)} \sigma(u)\sigma(v),\end{aligned}$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any $uv \in E(G)$. It follows that

$$MS_2(G) \leq \frac{1}{2} MSD^{\frac{3}{2}}(G) - \frac{1}{2m} \mathrm{irr}_{\mathrm{Tr}}(G)^2,$$

with equality if and only if $|\sigma(u) - \sigma(v)|$ is constant for any $uv \in E(G)$. □

Corollary 2.8. *Let G be a connected graph with m edges. If* $\mathrm{diam}(G) \leq 2$*, then*

$$MS_2(G) \leq \frac{1}{2} MSD^{\frac{3}{2}}(G) - \frac{1}{2m} \mathrm{irr}(G)^2,$$

with equality if and only if $|d(u) - d(v)|$ is constant for any $uv \in E(G)$.

Proof. Let G be a connected graph with m edges. It follows from Remark 2.2 that for any $uv \in E(G)$, $|d(u) - d(v)|$ is constant if $\mathrm{diam}(G) \leq 2$. Now the result follows from Lemma 2.3 and Proposition 2.10. □

Lemma 2.9. [14, 43] *Let G be a connected graph with n vertices and m edges. Then*

$$W(G) \geq n(n-1) - m,$$

with equality if and only if $\mathrm{diam}(G) \leq 2$*.* (*For example, the equality holds for complete graphs, complete bipartite and complete multipartite graphs, moreover wheel graphs and windmill graphs composed of triangles.*)

Proposition 2.11. *Let G be a connected k-transmission regular with n vertices and m edges. Then*

$$k = \frac{2W(G)}{n} \geq 2(n-1) - \frac{2m}{n},$$

with equality if and only if $\operatorname{diam}(G) \leq 2$.

Proof. Since G is k-transmission regular, $W(G) = \frac{nk}{2}$ holds. Now the result follows from Lemma 2.9. □

Proposition 2.12. *Let G be a connected graph with n vertices and m edges. Then*

$$MS^1(G) \geq 4(n-1)W(G) - MS_1(G) \geq 4(n-1)\left(n^2 - n - m\right) - MS_1(G),$$

and equalities hold in both sides simultaneously if $\operatorname{diam}(G) \leq 2$.

Proof. The result follows directly, using Lemma 2.9 and Proposition 2.2. □

Proposition 2.13. *Let G be a connected graph with n vertices and m edges. Then*

$$MS_1(G) \leq \sqrt{m\left(MSD^{\frac{3}{2}}(G) + 2MS_2(G)\right)}, \tag{2.3}$$

with equality if and only if $\sigma(u) + \sigma(v)$ is constant for each edge $uv \in E(G)$.

Proof. Using the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}\left(\frac{1}{m}\sum_{uv\in E(G)}(\sigma(u)+\sigma(v))\right)^2 &\leq \frac{1}{m}\sum_{uv\in E(G)}(\sigma(u)+\sigma(v))^2 \\ &= \frac{1}{m}\left(\sum_{uv\in E(G)}\left(\sigma^2(u)+\sigma^2(v)\right) + 2\sum_{uv\in E(G)}\sigma(u)\sigma(v)\right),\end{aligned}$$

with equality if and only if $\sigma(u) + \sigma(v)$ is constant for each edge $uv \in E(G)$. This implies that

$$\left(\frac{1}{m}MS_1(G)\right)^2 \leq \frac{1}{m}\left(MSD^{\frac{3}{2}}(G) + 2MS_2(G)\right),$$

with equality if and only if $\sigma(u) + \sigma(v)$ is constant for each edge $uv \in E(G)$. Consequently, we have

$$MS_1(G) \leq \sqrt{m\left(MSD^{\frac{3}{2}}(G) + 2MS_2(G)\right)}.$$ □

Let G be a connected graph with n vertices. Let us define the topological invariant $\Phi(G)$ as follows

$$\Phi(G) = \frac{\left(\sum\limits_{u\in V(G)}\sigma(u)\right)^2}{n\sum\limits_{u\in V(G)}\sigma^2(u)} = \frac{4W(G)^2}{nMS^1(G)}.$$

The following theorem shows that $\Phi(G)$ quantifies the degree of transmission regularity of a connected graph G.

Theorem 2.1. *Let G be a connected graph with n vertices. Then $\Phi(G) \leq 1$, with equality if and only if G is transmission regular.*

Proof. Using Cauchy-Schwartz inequality, we obtain

$$\left(\sum_{u\in V(G)} \sigma(u)\right)^2 \leq n \sum_{u\in V(G)} \sigma(u)^2,$$

with equality if and only if $\sigma(u) = \sigma(v)$ for each $u, v \in V(G)$. This completes the proof. □

Proposition 2.14. *Let G be a connected graph with n vertices and m-edges. If $\rho_D(G)$ denotes the distance spectral radius of G, then*

$$2(n-1) - \frac{2m}{n} \leq \frac{2W(G)}{n} \leq \rho_D(G).$$

The left-hand side equality holds if and only if $\mathrm{diam}(G) \leq 2$. *The right-hand side equality holds if and only if G is transmission regular.*

Proof. The left-hand side inequality is nothing but Lemma 2.9. From Theorem 2.1 and [2, Theorem 5.5] one obtains that $\frac{2W(G)}{n} \leqslant \sqrt{\frac{1}{n} MS^1(G)} \leqslant \rho_D(G)$, with equality if and only if G is transmission regular. □

Let us finish this section with following result showing how $W(G)$, $MS_1(G)$, and $\xi^d(G)$ relates to each other.

Theorem 2.2. [27] *Let G be a connected graph on $n \geqslant 3$ vertices. Then*

$$MS_1(G) \leqslant 2nW(G) - \xi^d(G),$$

with equality if and only if $G \cong P_4$, or $G \cong K_n - ke$, for $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, where ke denotes a matching of size k.

3. Vertex and Edge Transitive Graphs

In this section, following Darafsheh [8, 34], we aim to use a method which applies group theory to graph theory. For more details regarding the theory of groups and graph theory one can see [15] and [19], respectively.

Let Γ be a group acting on a set X. We shall denote the action of $\alpha \in \Gamma$ on $x \in X$ by x^α. Then $U \subseteq X$ is call an *orbit* of Γ on X if for every $x, y \in U$ there exists $\alpha \in \Gamma$ such that $x^\alpha = y$. The action of group Γ on X is called *transitive* if X is itself an orbit of Γ on X.

Let G be a graph. A bijection α on $V(G)$ is called an *automorphism* of G if it preserves $E(G)$. In other words, α is an automorphism if for each $u, v \in V(G)$, $e = uv \in E(G)$ if and only if $u^\alpha v^\alpha \in E(G)$. Let us denote by $\mathrm{Aut}(G)$ the set of all automorphisms of G.

It is known that $\mathrm{Aut}(G)$ forms a group under the composition of mappings. This is a subgroup of the symmetric group on $V(G)$. Note that $\mathrm{Aut}(G)$ acts on $V(G)$

naturally, i.e., for each $\alpha \in \mathrm{Aut}(G)$ and $v \in V(G)$ the action of α on v, v^α, is defined as $\alpha(v)$. The action of $\mathrm{Aut}(G)$ on $V(G)$ induces an action on $E(G)$. In fact, for $\alpha \in \mathrm{Aut}(G)$ and $e = uv \in E(G)$, the action of α on $e = uv$, e^α, is defined as $u^\alpha v^\alpha$.

A graph G is called *vertex-transitive* (*edge-transitive*) if the action of $\mathrm{Aut}(G)$ on $V(G)$ $(E(G))$ is transitive.

Let G be a graph, $V_1, V_2, \ldots, V_t$ be the orbits of $\mathrm{Aut}(G)$ under its natural action on $V(G)$. Then for each $1 \leq i \leq t$ and for $u, v \in V_i$, $\sigma(u) = \sigma(v)$. In particular, if G is vertex transitive ($t = 1$), then for each $u, v \in V(G)$, $\sigma(u) = \sigma(v)$. Therefore vertex-transitive graphs are transmission regular. It is known that any vertex-transitive graph is (vertex degree) regular [19] and transmission regular [8], but note vice versa.

Lemma 3.1. *Let G be a connected k-transmission regular graph with n vertices and m edges. Then*

$$\begin{aligned} SJ(G) =& \frac{m^2}{(m-n+2)\sqrt{2k}}, \quad GAS(G) = \frac{n}{2}, \quad HS(G) = \frac{m}{k}, \\ J(G) =& \frac{m^2}{(m-n+2)k}. \end{aligned}$$

Lemma 3.2. *Let G be a connected vertex-transitive graph with n vertices and m edges and the valency r. Then*

$$\begin{aligned} SJ(G) =& \frac{m^2\sqrt{n}}{2(m-n+2)\sqrt{W(G)}}, \quad GAS(G) = \frac{2W(G)}{n}, \\ HS(G) =& \frac{nm}{2W(G)} = \frac{n^2 r}{4W(G)}, \\ J(G) =& \frac{m^2 n}{2(m-n+2)W(G)} = \frac{mn^2 r}{4(m-n+2)W(G)}. \end{aligned}$$

Proof. If G is a connected vertex-transitive graph with n vertices and m edges, then G is of valency r (r-regular) and k-transmission regular, for some natural numbers r and k. It follows that $2m = nr$ and $2W(G) = nk$. □

Lemma 3.3. *Let G be a connected k-transmission regular with n vertices and m edges. Then*

$$HS(G) \leq \frac{m}{2(n-1) - \frac{2m}{n}},$$

with equality if and only if $\mathrm{diam}(G) \leq 2$.

Proof. Follows from Proposition 2.11 and the fact that for a k-transmission regular graph G with n vertices and m edges, $HS(G) = \frac{m}{k}$. □

Theorem 3.1. *Let G be a connected graph with n vertices and m edges. Let us denote the orbits of the action* $\mathrm{Aut}(G)$ *on* $E(G)$ *by* $E_1, E_2, \ldots, E_l$. *Suppose that for* *each $1 \leq i \leq t$, $e_i = u_i v_i$ is a fixed edge in the orbit E_i. Then*

$$
\begin{aligned}
HS(G) =& \sum_{i=1}^{l} \frac{2|E_i|}{\sigma(u_i) + \sigma(v_i)}, \quad SJ(G) = \frac{m}{m-n+2} \sum_{i=1}^{l} \frac{|E_i|}{\sqrt{\sigma(u_i) + \sigma(v_i)}}, \\
GAS(G) =& \frac{n}{2m} \sum_{i=1}^{l} \frac{|E_i| \sqrt{\sigma(u_i)\sigma(v_i)}}{\sigma(u_i) + \sigma(v_i)}, \quad \mathrm{irr}_{\mathrm{Tr}}(G) = \sum_{i=1}^{l} |E_i| \left| \sigma(u_i) - \sigma(v_i) \right|, \\
MS_1(G) =& \sum_{i=1}^{l} |E_i| (\sigma(u_i) + \sigma(v_i)), \quad MS_2(G) = \sum_{i=1}^{l} |E_i| \sigma(u_i)\sigma(v_i).
\end{aligned}
$$

Corollary 3.1. *Let G be a connected graph with n vertices and m edges. If G is edge-transitive and uv is a fixed edge of G, then*

$$
\begin{aligned}
HS(G) =& \frac{2m}{\sigma(u) + \sigma(v)}, \quad SJ(G) = \frac{m^2}{(m-n+2)\sqrt{\sigma(u) + \sigma(v)}}, \\
GAS(G) =& \frac{n}{2} \frac{\sqrt{\sigma(v)\sigma(v)}}{\sigma(u) + \sigma(v)}, \quad MS_2(G) = m\sigma(u)\sigma(v), \\
\mathrm{irr}_{\mathrm{Tr}}(G) =& m \left| \sigma(u) - \sigma(v) \right|, \quad QS_e(G) = \left| \sigma(u) - \sigma(v) \right|, \\
QS_{v,e}(G) =& \frac{n}{2} \left(1 + \left| \sigma(u) - \sigma(v) \right| \right), \quad MS_1(G) = m(\sigma(u) + \sigma(v)).
\end{aligned}
$$

Fullerenes are zero-dimensional nanostructures, discovered experimentally in 1985 [30]. Fullerenes C_n can be drawn for $n = 20$ and for all even $n \geq 24$. They have n carbon atoms, $\frac{3n}{2}$ bonds, 12 pentagonal and $\frac{n}{2} - 10$ hexagonal faces. The most important member of the family of fullerenes is C_{60} [30]. The smallest fullerene is C_{20}. It is a well-known fact that among all fullerene graphs only C_{20} and C_{60} (see Figure 3) are vertex-transitive [18]. Since for every vertex of $v \in V(C_{20})$, $\sigma(v) = 50$ and for every $v \in V(C_{60})$, $\sigma(v) = 278$, then

$$
\begin{aligned}
SJ(C_{20}) =& 7.5, \quad GAS(C_{20}) = 50, \quad HS(C_{20}) = 0.6, \\
J(C_{20}) =& 1.5, \quad SJ(C_{60}) = 10.73, \quad GAS(C_{60}) = 278, \\
HS(C_{60}) =& 0.32, \quad J(C_{60}) = 0.9.
\end{aligned}
$$

A nanostructure called *achiral polyhex nanotorus* (or *toroidal fullerenes* of parameter p and length q, denoted by $T = T[p, q]$ is depicted in Figure 4 and its 2-dimensional molecular graph is in Figure 5. It is regular of valency 3 and has pq vertices and $\frac{3pq}{2}$ edges. It follows the following proposition.

Proposition 3.1.

$$
\begin{aligned}
SJ(T) =& \frac{9(pq)^2 \sqrt{pq}}{8(pq+2)\sqrt{W(T)}}, \quad GAS(T) = \frac{2W(T)}{pq}, \\
HS(T) =& \frac{3(pq)^2}{4W(T)}, \quad J(T) = \frac{9(pq)^3}{8(pq+2)W(T)}.
\end{aligned}
$$

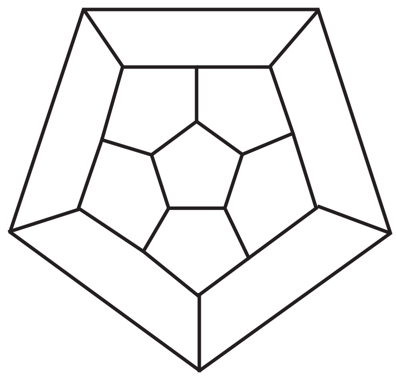

FIGURE 3. 2-dimensional graph of fullerene C_{20}

FIGURE 4. A achiral polyhex nanotorus (or toroidal fullerene) $T[p, q]$

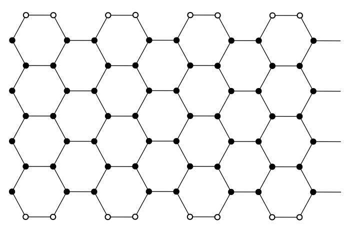

FIGURE 5. A 2-dimensional lattice for an achiral polyhex nanotorus $T[p, q]$

The vertex set of the hypercube H_n consists of all n-tuples $(b_1, b_2, \dots, b_n)$ with $b_i \in \{0, 1\}$. Two vertices are adjacent if the corresponding tuples differ in precisely one place. Moreover, H_n has exactly $2n$ vertices and $n2^{n-1}$ edges.

Lemma 3.4. [8] *The hypercube H_n is $(n2^{n-1})$-transmission regular which is vertex- and edge-transitive.*

Therefore from Lemma 3.1 and Lemma 3.4 we have the following result.

Corollary 3.2.

$$SJ(H_n) = \frac{n^2 2^{2(n-1)}}{(n2^{n-1} - 2n + 2)\sqrt{n2^n}}, \quad GAS(H_n) = n, \quad HS(H_n) = 2n^2 2^{2(n-1)},$$
$$J(H_n) = \frac{n^2 2^{2(n-1)}}{(n2^{n-1} - 2n + 2)n2^{n-1}}.$$

4. On the Application Possibilities of Transmission-Based Topological Indices in QSPR Studies

As we have already mentioned, the transmission-based topological indices represent a particular family of distance-based topological invariants. In what follows we demonstrate in examples the promising applications of TT indices in QSPR studies. In the literature we found some TT indices used successfully for predicting physico-chemical properties of unbranched alkanes. Ren [41] introduced a topological index denoted by Xu, it is defined for an n-vertex connected graph G as follows:

$$Xu(G) = \sqrt{n} \log \left(\frac{\sum_{u \in V(G)} d(u)\sigma^2(u)}{\sum_{u \in V(G)} d(u)\sigma(u)} \right).$$

Analyzing the mono-parametric correlations with different degree-based and distance-based indices (Randić connectivity index, Balaban's J index), the linear prediction model based on $Xu(G)$ index gives the best results.

Shamsipur et al. [42] proposed a family of bond-additive TT topological indices, called as shamsipur indices (Sh_1–Sh_{10} indices) and used them for prediction of different physico-chemical properties of a large number of alkanes and alkane isomers. In [42] for 379 organic compounds ten different versions of Sh indices were calculated and their ability were evaluated in QSPR studies. The resulting regression data were compared with the results based on several known topological indices, and in most cases, betters results were obtained by the Sh_1–Sh_{10} indices. For example, using the Sh_1 index defined as

$$Sh_1(G) = \log \left(\sum_{uv \in E(G)} \frac{\sigma(u)\sigma(v)}{d(u)d(v)} \right),$$

a correlation coefficient of 0,983 between the boiling point (BP) and Sh_1 was obtained.

Acknowledgements. The first author would like to thank Persian Gulf University (PGU)'s research council for its support.

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[1]Department of Mathematics,
Persian Gulf University,
Bushehr 7516913817, Iran
Email address: sharafidi@pgu.ac.ir

[2]Óbuda University
Bécsiút 96/B, H-1034 Budapest, Hungary
Email address: reti.tamas@bgk.uni-obuda.hu

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 65–73.

ON A NEW CLASS OF UNIFIED REDUCTION FORMULAS FOR SRIVASTAVA'S GENERAL TRIPLE HYPERGEOMETRIC FUNCTION $F^{(3)}[x,y,z]$

YONG SUP KIM[1], ADEM KILICMAN[2], AND ARJUN K. RATHIE[3,4]

Abstract. Very recently, by applying the so called Beta integral method to the well-known hypergeometric identities due to Bailey and Ramanujan, Choi et al. [*Reduction formula for Srivastava's triple hypergeometric series* $F^{(3)}[x,y,z]$, Kyungpook Math. J. **55** (2015), 439–447] have obtained three interesting reduction formulas for the Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$.

The aim of this paper is to provide three unified reduction formulas for the Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$ from which as many as reduction formulas desired (including those obtained by *Choi et al.*) can be deduced.

In the end, three unified relationships between Srivastava's triple hypergeometric series and Kampé de Fériet function have also been given.

1. Introduction

The generalized hypergeometric function ${}_pF_q$ with p numerator parameters $\alpha_1,\ldots,\alpha_p$ such that $\alpha_j\in\mathbb{C}$, $j=1,\ldots,p$, and q denominator parameters $\beta_1,\ldots,\beta_q$, $j=1,\ldots,q$, such that $\beta_j\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $j=1,\ldots,q$, $\mathbb{Z}_0^-:=\mathbb{Z}\cup 0=\{0,-1,,-2,\cdots\}$, is defined by (see, for example [14, Chapter 4], see also [19, pp. 71–72]

$$\begin{aligned}{}_pF_q[\alpha_1,\,\ldots,\,\alpha_p;\,\beta_1,\,\ldots,\,\beta_q;\,z] &= {}_p\mathrm{F}_q\left[\begin{array}{c}\alpha_1,\,\ldots,\,\alpha_p;\\ \beta_1,\,\ldots,\,\beta_q;\end{array}\ z\right]\\ &=\sum_{n=0}^{\infty}\frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\frac{z^n}{n!},\end{aligned}\tag{1.1}$$

Key words and phrases. Generalized hypergeometric function ${}_pF_q$, gamma function, Pochhammer symbol, beta integral, Kampé de Fériet function, Srivastava's triple hypergeometric series $F^{(3)}[x,y,z]$.

2010 *Mathematics Subject Classification* Primary 33C20, 33B20. Secondary 33C90; 33C05.

DOI 10.46793/KgJMat2001.065K

Received: November 09, 2017.

Accepted: February 16, 2018.

$p \leq q$ and $|z| < \infty$, $p = q+1$ and $|z| < 1$, $p = q+1$, $|z| = 1$ and $\mathrm{Re}(\omega) > 0$, where

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

and $(\alpha)_n$ denotes the Pochhammer symbol defined in terms of Gamma functions by

$$(\alpha)_n := \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \begin{cases} 1, & n = 0, \alpha \in \mathbb{C} \setminus \{0\}, \\ \alpha(\alpha+1)\cdots(\alpha+n-1), & n \in \mathbb{N}, \alpha \in \mathbb{C}. \end{cases} \tag{1.2}$$

It is to be noted here that whenever the hypergeometric function ${}_2F_1$ and the generalized hypergeometric function ${}_pF_q$ reduce to be expressed in terms of Gamma functions, the results are very important in view of applications as well as themselves. Thus the well known classical summation theorems [3] such as those of Gauss, Gauss second, Kummer and Bailey for the series ${}_2F_1$, and those of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$, and others have played important roles.

Moreover, it is well known that, if the product of two generalized hypergeometric series can be expressed as a generalized hypergeometric series with argument x, the coefficient of x^n in the product should be expressed in terms of gamma functions. Following this technique and using above mentioned classical summation theorems, in a well known, popular and very interesting paper [2], Bailey derived a large number of new as well as known results involving products of generalized hypergeometric series. Here, in our present investigation, we choose to recall some of those results in [2]:

$$e^{-x}\, {}_1\mathrm{F}_1(\alpha\,;\,\rho\,;x) = {}_1\mathrm{F}_1(\rho-\alpha\,;\,\rho\,;-x), \tag{1.3}$$

$${}_1\mathrm{F}_1(\alpha\,;\,2\alpha\,;x)\,{}_1\mathrm{F}_1(\beta\,;\,2\beta\,;-x) = {}_2\mathrm{F}_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta),\, \frac{1}{2}(\alpha+\beta+1) & ; & \dfrac{x^2}{4} \\ \alpha+\frac{1}{2},\, \beta+\frac{1}{2},\, \alpha+\beta & ; & \end{matrix}\right], \tag{1.4}$$

$${}_1\mathrm{F}_1(\alpha\,;\,\rho\,;x)\,{}_1F_1(\alpha\,;\,\rho\,;-x) = {}_2\mathrm{F}_3\left[\begin{matrix} \alpha,\, \rho-\alpha & ; & \dfrac{x^2}{4} \\ \rho,\, \frac{\rho}{2},\, \frac{\rho}{2}+\frac{1}{2} & ; & \end{matrix}\right], \tag{1.5}$$

$$\begin{aligned} &{}_1\mathrm{F}_1\left[\begin{matrix} \alpha & \\ \rho & ;x \end{matrix}\right] {}_1F_1\left[\begin{matrix} \alpha-\rho+1 & \\ 2-\rho & ;-x \end{matrix}\right] \\ &= {}_2\mathrm{F}_3\left[\begin{matrix} \alpha-\frac{\rho}{2}+\frac{1}{2},\, \frac{\rho}{2}-\alpha+\frac{1}{2} & ; & \dfrac{x^2}{4} \\ \frac{1}{2},\, \frac{\rho}{2}+\frac{1}{2},\, \frac{3}{2}-\frac{\rho}{2} & ; & \end{matrix}\right] \\ &\quad + \frac{(2\alpha-\rho)\,(1-\rho)}{\rho\,(2-\rho)}\, x\, {}_2\mathrm{F}_3\left[\begin{matrix} \alpha-\frac{\rho}{2}+1,\, \frac{\rho}{2}-\alpha+1 & ; & \dfrac{x^2}{4} \\ \frac{3}{2},\, \frac{\rho}{2}+1,\, 2-\frac{\rho}{2} & ; & \end{matrix}\right]. \end{aligned} \tag{1.6}$$

It is interesting to note here that if we use the result (1.3) in (1.4), (1.5) and (1.6), we get, respectively, the following alternative forms, that we will use in our present investigations:

$$e^{-x}\, {}_1\mathrm{F}_1(\alpha\,;\,2\alpha\,;x)\,{}_1F_1(\beta\,;\,2\beta\,;x) = {}_2\mathrm{F}_3\left[\begin{matrix} \frac{1}{2}(\alpha+\beta),\, \frac{1}{2}(\alpha+\beta+1) & ; & \dfrac{x^2}{4} \\ \alpha+\frac{1}{2},\, \beta+\frac{1}{2},\, \alpha+\beta & ; & \end{matrix}\right], \tag{1.7}$$

$$e^{-x}\,{}_1\mathrm{F}_1(\alpha\,;\,\rho\,;x)\,{}_1F_1(\rho-\alpha\,;\,\rho\,;x)={}_2\mathrm{F}_3\left[\begin{array}{c}\alpha,\,\rho-\alpha\\ \rho,\,\frac{\rho}{2},\,\frac{\rho}{2}+\frac{1}{2}\end{array};\;\begin{array}{c}\frac{x^2}{4}\end{array}\right], \tag{1.8}$$

$$\begin{aligned}&e^{-x}\,{}_1\mathrm{F}_1\left[\begin{array}{c}\alpha\\ \rho\end{array};\,x\right]{}_1\mathrm{F}_1\left[\begin{array}{c}1-\alpha\\ 2-\rho\end{array};\,x\right]\\ &={}_2\mathrm{F}_3\left[\begin{array}{c}\alpha-\frac{\rho}{2}+\frac{1}{2},\,\frac{\rho}{2}-\alpha+\frac{1}{2}\\ \frac{1}{2},\,\frac{\rho}{2}+\frac{1}{2},\,\frac{3}{2}-\frac{\rho}{2}\end{array};\,\frac{x^2}{4}\right]\\ &\quad+\frac{(2\alpha-\rho)\,(1-\rho)}{\rho\,(2-\rho)}\,x\,{}_2\mathrm{F}_3\left[\begin{array}{c}\alpha-\frac{\rho}{2}+1,\,\frac{\rho}{2}-\alpha+1\\ \frac{3}{2},\,\frac{\rho}{2}+1,\,2-\frac{\rho}{2}\end{array};\,\frac{x^2}{4}\right].\end{aligned} \tag{1.9}$$

Also in 1987, Henrici [10] gave the following elegant result for a product of three generalized hypergeometric functions:

$$\begin{aligned}&{}_0\mathrm{F}_1\left[\begin{array}{c}-\\ 6c\end{array};\,x\right]{}_0\mathrm{F}_1\left[\begin{array}{c}-\\ 6c\end{array};\,\omega x\right]{}_0\mathrm{F}_1\left[\begin{array}{c}-\\ 6c\end{array};\,\omega^2 x\right]\\ &={}_2\mathrm{F}_7\left[\begin{array}{c}3c-\frac{1}{4},\,3c+\frac{1}{4}\\ 6c,\,2c,\,2c+\frac{1}{3},\,2c+\frac{2}{3},\,4c-\frac{1}{3},\,4c,\,4c+\frac{1}{3}\end{array};\,\left(\frac{4x}{9}\right)^3\right],\end{aligned} \tag{1.10}$$

where $\omega=\exp(\frac{2\pi i}{3})$.

It is interesting to mention here that by making use of certain known transformations in the theory of generalized hypergeometric functions, in 1990, Karlsson and Srivastava [11] established a general triple series identity which readily yields the Henrici's identity (1.10).

On the other hand, just as the Gauss function ${}_2F_1$ was extended to ${}_pF_q$ by increasing the number of parameters in the numerator as well as in the denominator, the four Appell functions were introduced and generalized by Appell and Kampé de Fériet [1] who defined a general hypergeometric function in two variables. The notation defined and introduced by Kampé de Fériet for his double hypergeometric function of superior order was subsequently abbreviated by Burchnall and Chaundy [4, 5]. We, however, recall here the definition of a more general double hypergeometric function (than the one defined by Kampé de Fériet) in a slightly modified notation given by Srivastava and Panda [22, pp. 423, (26)]. For this, let (H_h) denote the sequence of parameters $(H_1,\,H_2,\,\ldots,\,H_h)$ and for nonnegative integers define the Pochhammer symbols $((H_h))_n=(H_1)_n\,(H_2)_n\cdots(H_h)_n$, where, when $n=0$, the product is understood to reduce to unity. Therefore, the convenient generalization of the Kampé de Fériet function is defined as follows:

$$\begin{aligned}&\mathrm{F}^{h:a;b}_{g:c;d}\left[\begin{array}{c}(H_h):\;(A_a)\,;\;(B_b)\,;\\ (G_g):\;(C_c)\,;\;(D_d)\,;\end{array}\;x,\,y\right]\\ &=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((H_h))_{m+n}\,((A_a))_m\,((B_b))_n}{((G_g))_{m+n}\,((C_c))_m\,((D_d))_n}\,\frac{x^m}{m!}\,\frac{y^n}{n!}.\end{aligned} \tag{1.11}$$

The symbol (H) is a convenient contraction for the sequence of the parameters $H_1,\ H_2,\ \ldots,\ H_h$ and the Pochhammer symbol $(H)_n$ is defined by in (1.2). For more details about the convergence for this function, we refer to [21].

Later on, a unification of Lauricella's 14 triple hypergeometric series $F_1, \ldots, F_{14}$ [20] and the additional three triple hypergeometric series H_A, H_B and H_C was introduced by Srivastava [18] who defined the following general triple hypergeometric series $F^{(3)}[x, y, z]$ (see, e.g., [20, pp. 44, (14) and (15)]:

$$\begin{aligned}\mathrm{F}^{(3)}[x,\ y,\ z] &\equiv F^{(3)}\left[\begin{array}{llllllll}(a) :: & (b); & (b'); & (b''): & (c); & (c'); & (c''); & \\ (e) :: & (g); & (g'); & (g''): & (h); & (h'); & (h''); & \end{array} x,\ y,\ z\right]\\ &= \sum_{m,\,n,\,p=0}^{\infty} \Lambda(m,\ n,\ p)\,\frac{x^m}{m!}\,\frac{y^n}{n!}\,\frac{z^p}{p!},\end{aligned} \tag{1.12}$$

where, for convenience,

$$\begin{aligned}\Lambda(m,\ n,\ p) =& \frac{\prod_{j=1}^{A}(a_j)_{m+n+p}\prod_{j=1}^{B}(b_j)_{m+n}\prod_{j=1}^{B'}(b'_j)_{n+p}\prod_{j=1}^{B''}(b''_j)_{p+m}}{\prod_{j=1}^{E}(e_j)_{m+n+p}\prod_{j=1}^{G}(g_j)_{m+n}\prod_{j=1}^{G'}(g'_j)_{n+p}\prod_{j=1}^{G''}(g''_j)_{p+m}}\\ &\times \frac{\prod_{j=1}^{C}(c_j)_m\prod_{j=1}^{C'}(c'_j)_n\prod_{j=1}^{C''}(c''_j)_p}{\prod_{j=1}^{H}(h_j)_m\prod_{j=1}^{H'}(h'_j)_n\prod_{j=1}^{H''}(h''_j)_p},\end{aligned} \tag{1.13}$$

and (a) abbreviates the array of A parameters $a_1, \ldots, a_A$, with similar interpretations for (b), (b'), (b''), and so on.

Recently, Choi et al. [8, 9] have obtained the following very interesting reduction formulas for the Srivastava's triple hypergeometric series $F^{(3)}[x,\ y,\ z]$ by applying the so-called Beta integral method (see [12], see also [7]) to the Henrici's triple product formula (1.10) and using (1.7) to (1.9):

$$\begin{aligned}&\mathrm{F}^{(3)}\left[\begin{array}{lllllll} e :: & -; & -; & -: & -; & -; & -; \\ d :: & -; & -; & -: & 6c; & 6c; & 6c; \end{array} 1,\ \omega,\ \omega^2\right]\\ &= {}_5\mathrm{F}_{10}\left[\begin{array}{c} 3c-\frac{1}{4},\ 3c+\frac{1}{4},\ \frac{e}{3},\ \frac{e}{3}+\frac{1}{3},\ \frac{e}{3}+\frac{2}{3} \\ 6c,\ 2c,\ 2c+\frac{1}{3},\ 2c+\frac{2}{3},\ 4c-\frac{1}{3},\ 4c,\ 4c+\frac{1}{3},\ \frac{d}{3},\ \frac{d}{3}+\frac{1}{3},\ \frac{d}{3}+\frac{2}{3} \end{array} ;\ \left(\frac{4}{9}\right)^3\right],\end{aligned} \tag{1.14}$$

where $\omega = \exp(\frac{2\pi i}{3})$,

$$\mathrm{F}^{(3)}\left[\begin{array}{lllllll} d :: & -; & -; & -: & -; & \alpha; & \beta; \\ e :: & -; & -; & -: & -; & 2\alpha; & 2\beta; \end{array} -1,\ 1,\ 1\right] \tag{1.15}$$

$$= {}_4\mathrm{F}_5\left[\begin{array}{c} \frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1),\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2};\ \frac{1}{4} \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta,\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{array}{lllllll} d :: & -; & -; & -: & -; & \alpha; & \rho-\alpha; \\ e :: & -; & -; & -: & -; & \rho; & \rho; \end{array} -1,\ 1,\ 1\right] \tag{1.16}$$

$$= {}_4\mathrm{F}_5\left[\begin{array}{c} \alpha,\ \rho-\alpha,\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2};\ \frac{1}{4} \\ \rho,\ \frac{\rho}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{array}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & -; & -; & -: & -; & \alpha; & 1-\alpha; \\ e:: & -; & -; & -: & -; & \rho; & 2-\rho; \end{matrix}\; -1,\, 1,\, 1\right] \tag{1.17}$$

$$={}_4\mathrm{F}_5\left[\begin{matrix} \alpha-\frac{\rho}{2}+\frac{1}{2},\ \frac{\rho}{2}-\alpha+\frac{1}{2},\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \frac{1}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{3}{2}-\frac{\rho}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{matrix}\ \frac{1}{4}\right]$$

$$+\frac{d\,(2\alpha-\rho)(1-\rho)}{e\,\rho(2-\rho)}\,{}_4\mathrm{F}_5\left[\begin{matrix} \alpha-\frac{\rho}{2}+1,\ \frac{\rho}{2}-\alpha+1,\ \frac{d}{2}+\frac{1}{2},\ \frac{d}{2}+1; \\ \frac{3}{2},\ \frac{\rho}{2}+1,\ 2-\frac{\rho}{2},\ \frac{e}{2}+\frac{1}{2},\ \frac{e}{2}+1; \end{matrix}\ \frac{1}{4}\right]. \tag{1.18}$$

In this sequel, motivated essentially by the results (1.14) to (1.17), we establish three unified reduction formulas for the function $F^{(3)}(x,\, y,\, z)$ by using the identities (1.7), (1.8) and (1.9) due to Bailey [2] and Ramanujan [15] which will be given in the next section.

2. Main Results

In this section, we shall establish three unified reduction formulas asserted by the following theorem.

Theorem 2.1. *For all finite x, each of the following reduction formulas holds true:*

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & -; & -; & -: & -; & \alpha; & \beta; \\ e:: & -; & -; & -: & -; & 2\alpha; & 2\beta; \end{matrix}\; -x,\, x,\, x\right] \tag{2.1}$$

$$={}_4\mathrm{F}_5\left[\begin{matrix} \frac{1}{2}(\alpha+\beta),\ \frac{1}{2}(\alpha+\beta+1),\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \alpha+\frac{1}{2},\ \beta+\frac{1}{2},\ \alpha+\beta,\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{matrix}\ \frac{x^2}{4}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & -; & -; & -: & -; & \alpha; & \rho-\alpha; \\ e:: & -; & -; & -: & -; & \rho; & \rho; \end{matrix}\; -x,\, x,\, x\right] \tag{2.2}$$

$$={}_4\mathrm{F}_5\left[\begin{matrix} \alpha,\ \rho-\alpha,\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \rho,\ \frac{\rho}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{matrix}\ \frac{x^2}{4}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & -; & -; & -: & -; & \alpha; & 1-\alpha; \\ e:: & -; & -; & -: & -; & \rho; & 2-\rho; \end{matrix}\; -x,\, x,\, x\right] \tag{2.3}$$

$$={}_4\mathrm{F}_5\left[\begin{matrix} \alpha-\frac{\rho}{2}+\frac{1}{2},\ \frac{\rho}{2}-\alpha+\frac{1}{2},\ \frac{d}{2},\ \frac{d}{2}+\frac{1}{2}; \\ \frac{1}{2},\ \frac{\rho}{2}+\frac{1}{2},\ \frac{3}{2}-\frac{\rho}{2},\ \frac{e}{2},\ \frac{e}{2}+\frac{1}{2}; \end{matrix}\ \frac{x^2}{4}\right]$$

$$+\,x\frac{d\,(2\alpha-\rho)(1-\rho)}{e\,\rho(2-\rho)}\,{}_4\mathrm{F}_5\left[\begin{matrix} \alpha-\frac{\rho}{2}+1,\ \frac{\rho}{2}-\alpha+1,\ \frac{d}{2}+\frac{1}{2},\ \frac{d}{2}+1; \\ \frac{3}{2},\ \frac{\rho}{2}+1,\ 2-\frac{\rho}{2},\ \frac{e}{2}+\frac{1}{2},\ \frac{e}{2}+1; \end{matrix}\ \frac{x^2}{4}\right]. \tag{2.4}$$

3. Outline of Proof of Theorem 2.1

The proofs of the above results (2.1) to (2.3) are quite straightforward. In order to prove (2.1), we first replace e^{-x} by ${}_0F_0[-;-;\,-x]$ in (1.7), then multiply each side of the resulting identity by $x^{d-1}\,(1-x)^{e-d-1}$ and expand the involved generalized hypergeometric functions as series. Now integrate both sides of the present resulting identity with respect to x between 0 to 1 and then change the order of integration and summation, which is easily seen to be justified due to the uniform convergence of the involved series. The integrations are easily evaluated to be expressed in terms of Gamma functions Γ by just recalling the well known relationship between the Beta function $B(\alpha,\beta)$ and the Gamma function (see, e.g., [19, pp. 8, (42)]. After some simplification, the left-hand side of the last resulting identity becomes

$$\frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)}\sum_{m,n,p=0}^{\infty}\frac{(d)_{m+n+p}}{(e)_{m+n+p}}\frac{(-1)^m}{m!}\frac{(\alpha)_n}{(2\alpha)_n\,n!}\frac{(\beta)_p}{(2\beta)_p\,p!}x^{m+n+p},$$

which, except for the Gamma fraction in front of the triple summations, in view of (1.12), is easily seen to correspond with the left-hand side of (2.1).

On the other hand, applying the Legendre's duplication formula for the Gamma function (see, e.g., [19, p. 6, (29)]):

$$\sqrt{\pi}\,\Gamma(2z)=2^{2z-1}\,\Gamma(z)\,\Gamma\left(z+\frac{1}{2}\right),\quad z\neq 0,\,-\frac{1}{2},\,-1,\,-\frac{3}{2},\ldots, \tag{3.1}$$

to the right-hand side of the above last resulting identity, we obtain

$$\frac{\Gamma(d)\,\Gamma(e-d)}{\Gamma(e)}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}(\alpha+\beta))_n\,(\frac{1}{2}(\alpha+\beta+1))_n\,(\frac{d}{2})_n\,(\frac{d}{2}+\frac{1}{2})_n}{(\alpha+\frac{1}{2})_n\,(\beta+\frac{1}{2})_n\,(\alpha+\beta)_n\,(\frac{e}{2})_n\,(\frac{e}{2}+\frac{1}{2})_n}\frac{1}{4^n\,n!}x^{2n},$$

which, except for the head-located Gamma fraction, is easily seen to correspond with the right-hand side of (2.1). This completes the proof of (2.1). A similar argument will establish the results (2.2) and (2.3).

We conclude this section by mentioning some special cases of our main results. The special case of (2.1) when $\beta=\alpha$ is equal to that of (2.2) when $\rho=2\,\alpha$. The special case of (2.3) when $\rho=2\alpha$ is also equal to that of (2.1) when $\beta=1-\alpha$.

4. Special Cases

In our unified reduction formulas (2.1), (2.2) and (2.3), if we take $x=1$, we at once get the known results (1.5), (1.6) and (1.7) respectively due to Choi et al. [9]. Further since our results are valid for all finite x, so from our results, we can obtain as many reduction formulas as desired new and interesting results.

5. Further Observations

On the other hand, if we apply beta integral method to (1.4) to (1.6), we, after some simplification, obtain the following transformation formulas between Kampé de Fériet functions and generalized hypergeometric functions:

$$\begin{aligned}\mathrm{F}^{1:1;1}_{1:1;1}&\left[\begin{array}{lll} d: & \alpha; & \beta; \\ e: & 2\alpha; & 2\beta;\end{array}\; x,\,-x\right]\\ &={}_4\mathrm{F}_5\left[\begin{array}{l} \frac{1}{2}(\alpha+\beta),\,\frac{1}{2}(\alpha+\beta+1),\,\frac{1}{2}d,\,\frac{1}{2}d+\frac{1}{2} \\ \alpha+\frac{1}{2},\,\beta+\frac{1}{2},\,\alpha+\beta,\,\frac{1}{2}e,\,\frac{1}{2}e+\frac{1}{2}\end{array}\;;\;\frac{x^2}{4}\right],\end{aligned} \tag{5.1}$$

$$\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{array}{lll} d: & \alpha; & \alpha; \\ e: & \rho; & \rho;\end{array}\; x,\,-x\right] \tag{5.2}$$

$$={}_4\mathrm{F}_5\left[\begin{matrix}\alpha,\, e-\alpha,\, \frac{1}{2}d,\, \frac{1}{2}d+\frac{1}{2} & ; & \dfrac{x^2}{4}\\ \rho,\, \frac{1}{2}\rho,\, \frac{1}{2}\rho+\frac{1}{2},\, \frac{1}{2}e,\, \frac{1}{2}e+\frac{1}{2} & ; & \end{matrix}\right],$$

$$\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & \alpha-e+1; & \\ e: & \rho; & 2-\rho; & x,\,-x\end{matrix}\right] \tag{5.3}$$

$$={}_4\mathrm{F}_5\left[\begin{matrix}\alpha-\frac{1}{2}\rho+\frac{1}{2},\, \frac{1}{2}\rho-\alpha+\frac{1}{2},\, \frac{1}{2}d,\, \frac{1}{2}d+\frac{1}{2} & ; & \dfrac{x^2}{4}\\ \frac{1}{2},\, \frac{1}{2}\rho+\frac{1}{2},\, \frac{3}{2}-\frac{1}{2}\rho,\, \frac{1}{2}e,\, \frac{1}{2}e+\frac{1}{2} & ; & \end{matrix}\right]$$
$$+\frac{d(2\alpha-\rho)(1-\rho)}{e\rho(2-\rho)}\, x\, {}_4\mathrm{F}_5\left[\begin{matrix}\alpha-\frac{1}{2}\rho+1,\, \frac{1}{2}\rho-\alpha+1,\, \frac{1}{2}d+\frac{1}{2},\, \frac{1}{2}d+1 & ; & \dfrac{x^2}{4}\\ \frac{3}{2},\, \frac{1}{2}\rho+1,\, 2-\frac{1}{2}\rho,\, \frac{1}{2}e+\frac{1}{2},\, \frac{1}{2}e+1 & ; & \end{matrix}\right].$$

It is noted that the results (5.1) and (5.2) are seen to be special cases when $p=q=1$ of the more general results [21, pp. 31, (47) and (46)], while (5.3) is a special case of a more general result in [13, pp. 21, (2.6)].

Finally, comparing (1.15), (1.16) and (1.17) with (5.1), (5.2) and (5.3), respectively, we get the following transformation formulas between Srivastava's triple hypergeometric series $F^{(3)}(x,\, y,\, z)$ and Kampé de Fériet double series:

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & \text{—}; & \text{—}; & \text{—}; & \alpha; & \beta; & \\ e:: & \text{—}; & \text{—}; & \text{—}; & 2\alpha; & 2\beta; & -x,\, x,\, x\end{matrix}\right] \tag{5.4}$$
$$=\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & \beta; & \\ e: & 2\alpha; & 2\beta; & x,\,-x\end{matrix}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & \text{—}; & \text{—}; & \text{—}; & \alpha; & \rho-\alpha; & \\ e:: & \text{—}; & \text{—}; & \text{—}; & \rho; & \rho; & -x,\, x,\, x\end{matrix}\right] \tag{5.5}$$
$$=\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & \alpha; & \\ e: & \rho; & \rho; & x,\,-x\end{matrix}\right],$$

$$\mathrm{F}^{(3)}\left[\begin{matrix} d:: & \text{—}; & \text{—}; & \text{—}; & \alpha; & 1-\alpha; & \\ e:: & \text{—}; & \text{—}; & \text{—}; & \rho; & 2-\rho; & -x,\, x,\, x\end{matrix}\right] \tag{5.6}$$
$$=\mathrm{F}^{1:1;1}_{1:1;1}\left[\begin{matrix} d: & \alpha; & \alpha-\rho+1; & \\ e: & \rho; & 2-\rho; & x,\,-x\end{matrix}\right].$$

Similarly other results can also be obtained.

Remark 5.1. For recent works in this area, we refer [6, 16, 17].

Acknowledgements. The research work of Yong Sup Kim is supported by the Wonkwang University Research Fund (2017). The second and third authors are very grateful to Universiti Putra Malaysia for the partial support under reseach grant Grant Putra having vot number 9543000.

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[1]Department of Mathematics Education,
Wonkwang University,
Iksan, Korea
Email address: yspkim@wonkwang.ac.kr

[2]Department of Mathematics and Institute for Mathematical Research,
University Putra Malaysia (UPM),
Serdang, Selangor 43400, Malaysia
Email address: akilic@upm.edu.my

[3]Department of Mathematics, School of Physical Sciences,
Central University of Kerala,
Kasaragod-671316, Kerala State, India
Email address: akrathie@cukerala.ac.in

[4]Department of Mathematics, Vedant College of Engineering and Technology,
Rajasthan Technical University,
Tulsi, Bundi, India
Email address: akrathie@gmail.com

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 75–88.

CONSERVATION LAWS OF THE TIME-FRACTIONAL ZAKHAROV-KUZNETSOV-BURGERS EQUATION

AZADEH NADERIFARD[1], S. REZA HEJAZI[2], AND ELHAM DASTRANJ[3]

Abstract. An important application of Lie group theory of differential equations is applied to study conservation laws of time-fractional Zakharov-Kuznetsov-Burgers (ZKB) equation with Riemann-Liouville and Caputo derivatives. This analysis is based on a modified version of Noether's theorem provided by Ibragimov to construct the conserved vectors of the equation. This is done by non-linearly self-adjointness of the equation which will be stated via a formal Lagrangian in the sequel.

1. Introduction

Fractional order differential equations (FDEs) are important concepts in physic, mathematics and engineering. The theory of derivatives and integrals of fractional order illustrate the previous time history in the mathematical models of natural phenomena.

In the recent years, FDEs have been widely used and have numerous applications in various fields of sciences, as example probability and statistics, engineering, chemistry, electro-chemistry, biology, economics, modeling, astrophysics, electronics, dynamics, thermodynamics, vibration, viscoelasticity, control theory, electromagnetic theory, signal processing, arheology, geology, polymer and systems identification [2, 6, 9–14, 16–19, 25–27, 29, 30] and [37].

Conservation laws can be used in the analysis of the essential properties of the solutions, particularly, investigation of existence, uniqueness and stability of the solutions [22]. There are some methods for constructing of conservation laws for

Key words and phrases. Generalized Zakharov-Kuznetsov-Burgers equation, Riemann Liouviile derivative, Caputo fractional derivative, Lie point symmetry, fractional conservation laws.

2010 *Mathematics Subject Classification.* Primary: 34K37. Secondary: 76M60.

DOI 10.46793/KgJMat2001.075N

Received: December 03, 2017.

Accepted: February 20, 2018.

PDEs, for example the Noether's theorem [20] and Ibragimov's theorem [7]. Almost all of these methods can be used for differential equations with fractional derivatives.

Lukashchuk, considered the fractional generalizations of the Noether's operators without Lagrangian and derived conservation laws for an arbitrary time-fractional FPDEs by formal Lagrangian [15].

One of the most important PDE which has a vast application in solitary wave's theory is the ZKB equation, also it makes an important role in electromagnetic and describes the propagation of Langmuir waves in an ionized plasma. Some of its modified forms illustrate the interactions of small amplitude, high frequency waves with acoustic waves. There are useful articles for finding the solitary waves solutions (specially for ZKB equation), see [5, 32, 34–36]. In this article, we focus on the time-fractional ZKB equation by omitting the details of derivation in the following form:

$$\partial_t^\alpha u + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy} = 0, \tag{1.1}$$

where $\partial_t^\alpha u$ is the fractional derivative of order α and α $(1 < \alpha \leqslant 2)$ is the order of the time-fractional. Taking $\alpha = 1$, Zakharov and Kuznetsov established non-linear evolution equation which is related to nonlinear ion-acoustic waves in magnetized plasma including cold ions and hot isothermal electrons. We can see some useful papers in the literature to study the applications of this equation, see [33, 38] for more details. This equation by omitting the details of derivatives can be written as

$$u_t + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy} = 0, \tag{1.2}$$

where a, b, c, d and e are constant quantities which involve the physical quantities and x, y, t are independent variables where $u(t, x, y)$ is the dependent variable indicates the wave profile. El-Bedwehy and Moslem acquired the ZKB equation from an electron-positron-ion plasma [1].

This paper is organized as follows. Section 2 describes some basic properties of time-fractional derivatives and four particular cases of time-fractional of ZKB equation. In Section 3 Lie symmetry analysis of the fractional ZKB equation is investigated. In Section 4, the concept of non-linear self-adjointness of ZKB equation is studied and conservation laws of (1.1) are obtained by using the Noether's operators. Some conclusions are given in the last section.

2. Notations of Time-Fractional Generalizations

There are several types of definitions for fractional derivatives, such as Riemann-Liouville derivative, Caputo derivative, the modified Riemann-Liouville derivative, Riesz derivative and etc. [28, 31].

Functions that have no first-order derivative could have Riemann-Liouville derivative but could not have Caputo fractional derivative and on the other hand Caputo fractional derivative is related to physical models.

In this paper we adopt the fractional derivatives in Riemann-Liouville derivatives as $\mathcal{D}_t^\alpha$ and Caputo derivative as ${}^C\mathcal{D}_t^\alpha$.

Definition 2.1. Let $f(t) \in L^1(a,b)$, be the set of all integrable functions, the time-fractional integrals and left-sided and right-sided time-fractional integrals of order α are defined respectively as follow:

$$\begin{aligned} J_t^{\alpha} f(t) :=& \frac{1}{\Gamma(\alpha)} \int_a^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \\ {}_0J_t^{\alpha} f(t) :=& \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau)(t-\tau)^{\alpha-1} d\tau, \\ {}_tJ_T^{\alpha} f(t) :=& \frac{1}{\Gamma(\alpha)} \int_t^T f(\tau)(\tau-t)^{\alpha-1} d\tau, \end{aligned} \tag{2.1}$$

where $t > 0$ and $J_t^0 f(t) = f(t)$.

Definition 2.2. For $\alpha > 0$, the Riemann-Liouville time-fractional is defined as

$$\mathcal{D}_t^{\alpha} f(t,x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \dfrac{\partial^n}{\partial t^n} \displaystyle\int_0^t \dfrac{f(\xi,x)}{(t-\xi)^{\alpha+1-n}} d\xi, & n-1 < \alpha < n, \\ \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

Definition 2.3. The Caputo derivative of order α is defined as

$${}^C\mathcal{D}_t^{\alpha} f(t,x) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t \dfrac{1}{(t-\xi)^{\alpha+1-n}} \dfrac{\partial^n f(\xi,x)}{\partial \xi^n} d\xi, & n-1 < \alpha < n, \\ \\ \dfrac{d^n}{dt^n} f(t), & \alpha = n \in \mathbb{N}. \end{cases}$$

Now we should introduce some notations. Let

$$\begin{aligned} & J_t^{\alpha} \left(\lambda f(t) + g(t)\right) = \lambda J_t^{\alpha} f(t) + J_t^{\alpha} g(t), \\ & J_t^{\alpha} \left(J_t^{\beta} f(t)\right) = J_t^{\beta} \left(J_t^{\alpha} f(t)\right) = J_t^{\alpha+\beta} f(t), \\ & \mathcal{D}_t^{\alpha} f(t) = \mathcal{D}_t^{n} \left(J_t^{n-\alpha} f(t)\right) = \mathcal{D}_t^{n} \left(D_t^{-(n-\alpha)} f(t)\right), \\ & \mathcal{D}_t^{\alpha} \left(\mathcal{D}_t^{-\alpha} f(t)\right) = f(t), \\ & {}^C\mathcal{D}_t^{\alpha} c = 0, \quad c \text{ is constant}, \\ & {}^C\mathcal{D}_t^{\alpha} \mathcal{D}_t^{-\alpha} f(t) = f(t), \end{aligned} \tag{2.2}$$

where $\alpha \in \mathbb{R}$ such that $n-1 < \alpha < n$ and $n \in \mathbb{N}$. The classical ZKB equation can be written as follows: $u_t = C[u]$,

$$C[u] = -auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}.$$

In this paper we consider four forms of time-fractional generalization of ZKB equation as

$$u_t = J_t^\alpha C[u], \tag{2.3}$$

$$u_t = \mathcal{D}_t^{1-\alpha} C[u], \tag{2.4}$$

$$u_t = C[J_t^\alpha f], \tag{2.5}$$

$$u_t = C[\mathcal{D}_t^{1-\alpha} u], \tag{2.6}$$

where $\mathcal{D}_t^{1-\alpha}$ and J_t^α are left-sided fractional Riemann-Liouville derivative of order $1-\alpha$ and Riemann-Liouville integral of order α, respectively.

One can rewrite (2.3)-(2.6), so that their right-hand sides are exactly the right-hand side of (1.1). For this, we act on each of (2.3)-(2.6) by different operators.

Now, by acting the operator $\mathcal{D}_t^\alpha$ on (2.3) and denoting the dependent variable u by v, and using formula (2.2), we can rewrite (2.3) as:

$$\mathcal{D}_t^\alpha v_t = -avv_x - bv_{xxx} - cv_{xyy} + dv_{xx} + ev_{yy}. \tag{2.7}$$

By actting classical integral operator on (2.4) with respect to t, we have

$$u(t,x) - u(0,x) = J_t^\alpha \left(-auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}\right).$$

Now we act the operator ${}^C D_t^\alpha$ on the above equation and denote the dependent variable u by v. We get

$${}^C\mathcal{D}_t^\alpha v = -avv_x - bv_{xxx} - cv_{xyy} + dv_{xx} + ev_{yy}. \tag{2.8}$$

In (2.5), we denote a new non-local dependent variable v by $J_t^\alpha u$, then we have $u = \mathcal{D}_t^\alpha v$. Hence this equation can be rewritten as

$$\mathcal{D}_t^{\alpha+1} v = -avv_x - bv_{xxx} - cv_{xyy} + dv_{xx} + ev_{yy}. \tag{2.9}$$

In (2.6), by taking $u = J_t^{1-\alpha} v$, we obtain $u_x = J_t^{1-\alpha} v_x$ and other expressions. Finally this equation can be rewritten as:

$$\mathcal{D}_t^\alpha v = -avv_x - bv_{xxx} - cv_{xyy} + dv_{xx} + ev_{yy}. \tag{2.10}$$

Thus, four different time-fractional (2.7)-(2.10) are different forms of time-fractional generalization of ZKB equation. After replacing v by u, we can formally rewrite the (2.7)-(2.10) as

$$\mathcal{D}_t^\alpha u_t = -auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}, \tag{2.11}$$

$${}^C\mathcal{D}_t^\alpha u = -auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}, \tag{2.12}$$

$$\mathcal{D}_t^{\alpha+1} u = -auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}, \tag{2.13}$$

$$\mathcal{D}_t^\alpha u = -auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}. \tag{2.14}$$

Clearly these equations coincide with the classical ZKB (1.2) in the limiting case of $\alpha = 1$. In this paper, the order of time-fractional differential, in all of equations belongs to $(1,2)$. So $1<\alpha<2$. By using of summary mode of (2.11)-(2.14), we have

$$F(t,x,y,u,\mathcal{D}_t^{\mu(\alpha)}u,u_x,\dots,u_{xyy}) = -auu_x - bu_{xxx} - cu_{xyy} + du_{xx} + eu_{yy}, \tag{2.15}$$

where $\mathcal{D}_t^{\mu(\alpha)}$ denotes Riemann-Liouville operator or Caputo operator in (2.11)-(2.14).

3. Lie Symmetry Analysis of the Time-Fractional Generalized ZKB Equation

In this paper we consider Lie symmetry method in order to find conservation laws of the ZKB equation [4, 22, 23, 39]. Consider one-parameter Lie group of infinitesimal transformations for that (2.15)

$$\begin{aligned}
\bar{t} =& t + \varepsilon\tau(t,x,y,u) + O(\varepsilon^2),\\
\bar{x} =& x + \varepsilon\xi(t,x,y,u) + O(\varepsilon^2),\\
\bar{y} =& y + \varepsilon\rho(t,x,y,u) + O(\varepsilon^2),\\
\bar{u} =& u + \varepsilon\eta(t,x,y,u) + O(\varepsilon^2),\\
\bar{u}_{\bar{t}}^{\alpha} =& u_t^{\alpha} + \varepsilon\eta_t^{\alpha}(t,x,y,u) + O(\varepsilon^2),\\
\bar{u}_{\bar{x}} =& u_x + \varepsilon\eta^{x}(t,x,y,u) + O(\varepsilon^2),\\
\bar{u}_{\bar{x}\bar{x}} =& u_{xx} + \varepsilon\eta^{xx}(t,x,y,u) + O(\varepsilon^2),\\
\bar{u}_{\bar{x}\bar{x}\bar{x}} =& u_{xxx} + \varepsilon\eta^{xxx}(t,x,y,u) + O(\varepsilon^2),\\
\bar{u}_{\bar{y}\bar{y}} =& u_{yy} + \varepsilon\eta^{yy}(t,x,y,u) + O(\varepsilon^2),\\
\bar{u}_{\bar{y}\bar{y}\bar{x}} =& u_{yyx} + \varepsilon\eta^{yyx}(t,x,y,u) + O(\varepsilon^2),
\end{aligned} \tag{3.1}$$

where ε is the group parameter, then the associated Lie algebra of symmetries is the set of vector fields of the form

$$X = \tau(t,x,y,u)\frac{\partial}{\partial t} + \xi(t,x,y,u)\frac{\partial}{\partial x} + \rho(t,x,y,u)\frac{\partial}{\partial y} + \eta(t,x,y,u)\frac{\partial}{\partial u}, \tag{3.2}$$

where

$$\begin{aligned}
\left.\frac{d\bar{t}}{d\varepsilon}\right|_{\varepsilon=0} &= \tau(t,x,y,u), & \left.\frac{d\bar{x}}{d\varepsilon}\right|_{\varepsilon=0} &= \xi(t,x,y,u),\\
\left.\frac{d\bar{y}}{d\varepsilon}\right|_{\varepsilon=0} &= \rho(t,x,y,u), & \left.\frac{d\bar{u}}{d\varepsilon}\right|_{\varepsilon=0} &= \eta(t,x,y,u).
\end{aligned}$$

The third order prolongation of (3.2) leaves invariant (2.15). In other words

$$X^{(\alpha,3)}\left(F(t,x,y,u,\mathcal{D}_t^{\mu(\alpha)}u,u_x,\dots,u_{xyy})\right)|_{(F=0)} = 0, \tag{3.3}$$

satisfied on solutions of (2.15), where $X^{(\alpha,3)}$ is the third prolongation of the generator (3.2). By keeping the essential terms we have

$$X^{(\alpha,3)} = X + \eta^{x}\frac{\partial}{\partial u_x} + \eta^{xx}\frac{\partial}{\partial u_{xx}} + \eta^{xxx}\frac{\partial}{\partial u_{xxx}} + \eta^{yy}\frac{\partial}{\partial u_{yy}} + \eta^{yyx}\frac{\partial}{\partial u_{yyx}} + \eta_t^{\alpha}\frac{\partial}{\partial u_t^{\alpha}}. \tag{3.4}$$

Expanding the invariance condition (3.3) yields

$$\eta_t^\alpha + a\eta u_x + au\eta^x + b\eta^{xxx} + c\eta^{yyx} - d\eta^{xx} - e\eta^{yy} = 0. \tag{3.5}$$

The prolongation coefficients are

$$\begin{aligned}
\eta_t^\alpha &= \mathcal{D}_t^\alpha\left(\eta - \tau u_t - \xi u_x - \rho u_y\right) + \tau \mathcal{D}_t^\alpha(u_t) + \xi \mathcal{D}_t^\alpha(u_x) + \rho \mathcal{D}_t^\alpha(u_y),\\
\eta^x &= D_x\left(\eta - \tau u_t - \xi u_x - \rho u_y\right) + \tau u_{tx} + \xi u_{xx} + \rho u_{yx},\\
\eta^{xx} &= D_{xx}\left(\eta - \tau u_t - \xi u_x - \rho u_y\right) + \tau u_{txx} + \xi u_{xxx} + \rho u_{yxx},\\
\eta^{xxx} &= D_{xxx}\left(\eta - \tau u_t - \xi u_x - \rho u_y\right) + \tau u_{txxx} + \xi u_{xxxx} + \rho u_{yxxx},\\
\eta^{yy} &= D_{yy}\left(\eta - \tau u_t - \xi u_x - \rho u_y\right) + \tau u_{tyy} + \xi u_{xyy} + \rho u_{yyy},\\
\eta^{xyy} &= D_{yyx}\left(\eta - \tau u_t - \xi u_x - \rho u_y\right) + \tau u_{tyyx} + \xi u_{yyxx} + \rho u_{yyyx}.
\end{aligned}$$

By using the generalized chain rule for a composite function and the generalized Leibnitz rule, we have the explicit form of η_t^α (see [18, 21, 24]),

$$\begin{aligned}
\eta_t^\alpha =& \sum_{n=1}^{\infty}\left[\binom{\alpha}{n}\partial_t^n \eta_u - \binom{\alpha}{n+1} D_t^{n+1}(\tau)\right]\partial_t^{\alpha-n}u + \partial_t^\alpha \eta - u\partial_t^\alpha \eta_u\\
&- \sum_{n=1}^{\infty}\binom{\alpha}{n} D_t^n(\xi)\partial_t^{\alpha-n}(u_x) + \left(\eta_u - \alpha D_t(\tau)\right)\partial_t^\alpha u + \mu,
\end{aligned}$$

where μ is

$$\mu = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{l=0}^{k-1}\binom{\alpha}{n}\binom{n}{m}\binom{k}{l}\frac{t^{n-\alpha}}{k!\Gamma(n+1-\alpha)}(-u)^l\frac{\partial^m}{\partial t^m}(u^{k-l})\frac{\partial^{n-m+k}\eta}{\partial t^{n-m}\partial u^k}.$$

After substituting the values of $\eta^x, \eta^{xx}, \eta^{xxx}, \eta^{yy}, \eta^{xyy}$ and η_t^α into (3.5) and equating the coefficients of derivatives u to zero, the determining equations are obtained.

The solutions of this system are

$$\xi = C_1, \quad \rho = C_2, \quad \tau = C_3,$$

where C_i, $i = 1, 2, 3$, are arbitrary constants.

The lower limit of the integral in Riemann-Liouville derivative and Caputo derivative is fixed. So the condition $t = 0$ should be invariant with respect to transformation (3.1) and therefore we have $\tau(t, x, u)\mid_{t=0} = 0$. So for $C_3 = 0$ vector field $\frac{\partial}{\partial t}$ is not a symmetry for (2.15).

Consequently, (2.15) admits two Lie point symmetries:

$$X_1 = \frac{\partial}{\partial x} \quad \text{and} \quad X_2 = \frac{\partial}{\partial y}.$$

4. Conservation Laws

The theory of finding conservation laws for PDEs have a lot applications. This theory can describe some physically measures. Until three decades ago, all paper about conservation laws refer to problems with integer derivatives.

Let us define components of a conservation law $C = (C^t, C^x, C^y)$ for (2.15) in the same manner that is defined for PDEs. Let

$$\begin{aligned} C^t =& C^t\,(t,x,y,u,\ldots)\,,\\ C^x =& C^x\,(t,x,y,u,\ldots)\,,\\ C^y =& C^y\,(t,x,y,u,\ldots)\,. \end{aligned}$$

These components satisfy in

$$\mathcal{D}_t C^t + \mathcal{D}_x C^x + \mathcal{D}_y C^y = 0, \tag{4.1}$$

on all solutions of (2.15).

Many definitions and concepts for constructing conservation laws of FDEs as the formal Lagrangian, the adjoint equation and Euler-Lagrangian operator are similar to PDEs. Emmy Noether illustrated symmetry and conservation law are connected for all linear and non-linear equations. Using Noether's theorem, the equation must be derived from the variational principle and have a Lagrangian in classical sense. Finding Lagrangian is not easy. In the other hand, there are equations that do not have classical Lagrangian.

In this paper, we construct the conservation laws of the ZKB fractional (2.15) via Ibragimove's method [3, 8].

The formal Lagrangian can be written

$$\mathcal{L} = vF(t,x,y,u,\mathcal{D}_t^{\mu(\alpha)}u,u_x,\ldots,u_{xyy}),$$

where v is new dependent variable. We can define the formal Lagrangian for (2.15) by:

$$\mathcal{L} = v\mathcal{D}_t^{\mu(\alpha)} + avuu_x + bvu_{xxx} + cvu_{xyy} - dvu_{xx} - evu_{yy}, \quad v = v(t,x,y). \tag{4.2}$$

The Euler-Lagrange operator with respect to u for a finite time interval $t \in [0,T]$ is

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + (\mathcal{D}_t^{\mu(\alpha)})^* \frac{\partial}{\partial(\mathcal{D}_t^{\mu(\alpha)})} + \sum_{m=1}^{\infty}(-1)^m D_{i_1}\cdots D_{i_m}\frac{\partial}{\partial u_{i_1,\ldots,i_m}},$$

where $(\mathcal{D}_t^{\mu(\alpha)})^*$ will be adjoint operator of $(\mathcal{D}_t^{\mu(\alpha)})$. The adjoint operator is different for Riemann-Liouville derivative and Caputo fractional derivatives.

$(\mathcal{D}_t^{\alpha})^*$ is adjoint operator for Riemman-Liouville derivative and $({}^C\mathcal{D}_t^{\alpha})^*$ is adjoint operator for Caputo fractional derivatives that are defined as follows (see [15])

$$\begin{aligned} ({}_0\mathcal{D}_t^{\alpha})^* &= (-1)^n\,{}_tJ_T^{n-\alpha}(\mathcal{D}_t^n) \equiv {}_t^C\mathcal{D}_T^{\alpha},\\ ({}_0^C\mathcal{D}_t^{\alpha})^* &= (-1)^n\mathcal{D}_t^n({}_tI_T^{n-\alpha}) \equiv {}_t\mathcal{D}_T^{\alpha}, \end{aligned}$$

where ${}_tJ_T^{n-\alpha}$ is the right-sided fractional integral (2.1), ${}_t\mathcal{D}_T^{\alpha}$ and ${}_t^C\mathcal{D}_T^{\alpha}$ are the right-sided Riemann-Liouville and Caputo fractional derivative of order α.

The adjoint operator F^* of (2.15) is

$$F^* = \frac{\delta\mathcal{L}}{\delta u} = (\mathcal{D}_t^{\mu(\alpha)})^* v - av_x u - bv_{xxx} - cv_{xyy} - dv_{xx} - ev_{yy}. \tag{4.3}$$

Then adjoint operator $(\mathcal{D}_t^{\mu(\alpha)})^*$, for each of (2.11)-(2.14) is

$$\text{for (2.11)} \quad (\mathcal{D}_t^{\mu(\alpha)})^* \equiv (\mathcal{D}_t^{\alpha}\mathcal{D}_t)^* = {}_t\mathcal{D}_T^{\alpha}\mathcal{D}_t, \tag{4.4}$$

$$\text{for (2.12)} \quad (\mathcal{D}_t^{\mu(\alpha)})^* \equiv ({}^C\mathcal{D}_t^{\alpha})^* = {}_t\mathcal{D}_T^{\alpha}, \tag{4.5}$$

$$\text{for (2.13)} \quad (\mathcal{D}_t^{\mu(\alpha)})^* \equiv (\mathcal{D}_t^{\alpha+1})^* = {}_t^C\mathcal{D}_T^{\alpha+1}, \tag{4.6}$$

$$\text{for (2.14)} \quad (\mathcal{D}_t^{\mu(\alpha)})^* \equiv (\mathcal{D}_t^{\alpha})^* = {}_t^C\mathcal{D}_T^{\alpha}. \tag{4.7}$$

Similar to PDEs, the fractional (2.15) is non-linearly self-adjoint, if there exists function $v = v(t, x, y)$ that solve the adjoint (4.3) for all solutions $u(x)$ of (2.15) and $v \neq 0$ [7].

Substituting $v = v(t, x, y) = \phi(t)\psi(x)\eta(y)$ into (4.3), yields:

$$\begin{aligned} &\psi^{'}(x)\eta(y)u + d\psi^{''}(x)\eta(y) + b\psi^{'''}(x)\eta(y) + e\psi(x)\eta^{''}(y) + c\psi^{'}(x)\eta^{''}(y) = 0,\\ &(D_t^{\mu(\alpha)})^*(\phi(t)) = 0. \end{aligned} \tag{4.8}$$

The first equation in the above system is the second order PDE, which one of its solution is: $\psi(x)\eta(y) = \psi$, where $\psi \neq 0$ and is constant functions.

The second equation in system (4.8) depends on the type of fractional differential operator $\mathcal{D}_t^{\mu(\alpha)}u$, then (2.15) must be solved separately via each of equations (2.11)-(2.14).

For (2.11) we have $(\mathcal{D}_t^{\alpha}\mathcal{D}_t(\Phi(t)))^* = 0$, so by (4.4): $\Phi(t) = \phi_1(T-t)^{\alpha} + \phi_2$, for (2.12) we have $({}^C\mathcal{D}_t^{\alpha}(\Phi(t)))^* = 0$, so by (4.5): $\Phi(t) = \phi_1(T-t)^{\alpha}$, for Eq. (2.13) we have $(\mathcal{D}_t^{\alpha+1}(\Phi(t)))^* = 0$, so by (4.6): $\Phi(t) = \phi_1 t^2 + \phi_2 t + \phi_3$, for (2.14) we have $(\mathcal{D}_t^{\alpha}(\Phi(t)))^* = 0$, so by (4.7): $\Phi(t) = \phi_1 t + \phi_2$, where ϕ_1, ϕ_2 and ϕ_3 are arbitrary constants. Note that for solving all of above equations we have used properties Riemann-Liouville and Caputo time-fractional derivatives.

In the Ibragimove's method, the components of conserved vector are obtained with effect the Noether's operators on the Lagrangian. Noether operators can be found from the fundamental operator identity, whose formula depends on the number of variables. The fundamental identity for ZKB equation with three independent variables t, x, y and a dependent variable $u(t, x, y)$ can be written as follows:

$$\bar{X} + \mathcal{D}_t(\tau)\mathcal{I} + \mathcal{D}_x(\xi)\mathcal{I} + \mathcal{D}_y(\rho)\mathcal{I} = W\frac{\delta}{\delta u} + \mathcal{D}_t\mathcal{N}^t + \mathcal{D}_x\mathcal{N}^x + \mathcal{D}_y\mathcal{N}^y, \tag{4.9}$$

where $\bar{X}$ is prolongation operator (3.4), $\mathcal{I}$ is identity operator, $\frac{\delta}{\delta u}$ is the Euler-Lagrangiane operator and W is characteristic for Lie point group generator (3.2),

$$W = \eta - \tau u_t - \xi u_x - \rho u_y.$$

Finally $\mathcal{N}^t$, $\mathcal{N}^x$ and $\mathcal{N}^y$ are Noether operators. Because (2.15) do not have the fractional derivatives with respect to x and y, definitions for them are exactly the same as general formula that are given for each of symmetries as follows (see [7]):

$$\begin{aligned}\mathcal{N}^x =&\xi\mathcal{I} + W\left(\frac{\partial}{\partial u_x} - \mathcal{D}_i\frac{\partial}{\partial u_{xi}} + \mathcal{D}_i\mathcal{D}_k\frac{\partial}{\partial u_{xik}} - \cdots\right)\\ &+ \mathcal{D}_i(W)\left(\frac{\partial}{\partial u_{xi}} - \mathcal{D}_k\frac{\partial}{\partial u_{xik}} - \cdots\right) + \mathcal{D}_i\mathcal{D}_k(W)\left(\frac{\partial}{\partial u_{xik}} - \dots\right),\end{aligned} \tag{4.10}$$

$$\begin{aligned}\mathcal{N}^y =&\rho\mathcal{I} + W\left(\frac{\partial}{\partial u_y} - \mathcal{D}_i\frac{\partial}{\partial u_{yi}} + \mathcal{D}_i\mathcal{D}_k\frac{\partial}{\partial u_{yik}} - \cdots\right)\\ &+ \mathcal{D}_i(W)\left(\frac{\partial}{\partial u_{yi}} - \mathcal{D}_k\frac{\partial}{\partial u_{yik}} - \cdots\right) + \mathcal{D}_i\mathcal{D}_k(W)\left(\frac{\partial}{\partial u_{yik}} - \cdots\right),\end{aligned} \tag{4.11}$$

where i and k are x or y.

Since (2.15) has fractional derivatives respect to t, Nother's operator $\mathcal{N}^t$ for the case with the Riemann-Liouville time-fractional derivative is:

$$\mathcal{N}^t = \tau\mathcal{I} + \sum_{k=0}^{n-1}(-1)^k\mathcal{D}_t^{\alpha-1-k}(W)\mathcal{D}_t^k\frac{\partial}{\partial(\mathcal{D}_t^\alpha u)} - (1)^n J\left(W, \mathcal{D}_t^n\frac{\partial}{\partial(\mathcal{D}_t^\alpha u)}\right). \tag{4.12}$$

For the another case, with the Caputo time-fractional derivative $\mathcal{N}^t$ is

$$\mathcal{N}^t = \tau\mathcal{I} + \sum_{k=0}^{n-1}\mathcal{D}_t^k(W)\mathcal{D}_T^{\alpha-1-k}\frac{\partial}{\partial(\mathcal{D}_t^\alpha u)} - J\left(\mathcal{D}_t^n(W), \frac{\partial}{\partial(\mathcal{D}_t^\alpha u)}\right). \tag{4.13}$$

In (4.12) and (4.13),

$$J(f,g) = \frac{1}{\Gamma(n-\alpha)}\int_0^t\int_t^T \frac{f(\tau,x,y)g(\mu,x,y)}{(\mu-\tau)^{n-\alpha}}d\mu d\tau.$$

This integral has the following property:

$$\mathcal{D}_t J(f,g) = f_t\mathcal{J}_T^{n-\alpha}g - g_0\mathcal{J}_t^{n-\alpha}f.$$

The (2.15) is non-linearly self-adjoint, because there exists a non-unique function $v = v(t,x,y)$ such that (4.3) is satisfied for any solution of (2.15). We act on both sides of (4.9) by formal Lagrangian (4.2). Because formal Lagrangian $\mathcal{L}$ vanishes on the solutions of (2.15), the left-hand side of equallity (4.9) is equal to zero:

$$\begin{aligned}\bar{X}\mathcal{L} + \mathcal{D}_t(\tau)\mathcal{I}(\mathcal{L}) + \mathcal{D}_x(\xi)\mathcal{I}(\mathcal{L}) + \mathcal{D}_y(\rho)\mathcal{I}(\mathcal{L}) =&\bar{X}\mathcal{L} + \mathcal{D}_t(\tau)\mathcal{L} + \mathcal{D}_x(\xi)\mathcal{L} + \mathcal{D}_y(\rho)\mathcal{L}\\ =&0,\end{aligned}$$

and by considering (4.9),

$$W\frac{\delta\mathcal{L}}{\delta u} + \mathcal{D}_t\left(\mathcal{N}^t\mathcal{L}\right) + \mathcal{D}_x\left(\mathcal{N}^x\mathcal{L}\right) + \mathcal{D}_y\left(\mathcal{N}^y\mathcal{L}\right) = 0.$$

Since for non-linearly self-adjoint equation this condition is valid, i.e., $\frac{\delta\mathcal{L}}{\delta u} = 0$, so

$$\mathcal{D}_t\left(\mathcal{N}^t\mathcal{L}\right) + \mathcal{D}_x\left(\mathcal{N}^x\mathcal{L}\right) + \mathcal{D}_y\left(\mathcal{N}^y\mathcal{L}\right)\Big|_{(2.15)} = 0. \tag{4.14}$$

By comparing (4.1) and (4.14), we have

$$C^t = \mathcal{N}^t(\mathcal{L}), \quad C^x = \mathcal{N}^x(\mathcal{L}), \quad C^y = \mathcal{N}^y(\mathcal{L}).$$

In the sequel, conserved vectors associated with different symmetries and different terms of (2.15) are constructed.

Now we will find the conservation laws of the (2.11). The formal Lagrangian for (2.11) after substitution acceptable v is defined by

$$\mathcal{L} = (\psi\phi_1(T-t)^\alpha + \psi\phi_2)\left(\mathcal{D}_t^{\mu(\alpha)} + auu_x + bu_{xxx} + cu_{xyy} - du_{xx} - eu_{yy}\right).$$

In this case, using (4.10), (4.11) and (4.13), one can get the components of conserved vectors:

$$\begin{aligned}
C^x =& a\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)uW - d\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)W_x \\
&+ b\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)W_{xx} + c\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)W_{yy}, \\
C^y =& e\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)W_y + c\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)W_{yy}, \\
C^t =& \mathcal{J}_t^{1-\alpha}W_t\psi\phi_1(T-t)^\alpha + \mathcal{J}_t^{1-\alpha}W_t\psi\phi_2 + \mathcal{J}_T^{1-\alpha}\left(\alpha\psi\phi_1(T-t)^{\alpha-1}\right)W \\
&+ J\left(W_t, \alpha\psi\phi_1(T-t)^{\alpha-1}\right).
\end{aligned}$$

By applying above equations and considering $W = -u_x$ coordinate with X_1 the following components are obtained:

$$\begin{aligned}
C^x =& -au_x\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)u + du_{xx}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right) \\
&- bu_{xx}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right) - cu_{xyy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right), \\
C^y =& -eu_{xy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right) - cu_{xyy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right), \\
C^t =& -\mathcal{J}_t^{1-\alpha}u_{xt}\psi\phi_1(T-t)^\alpha - \mathcal{J}_t^{1-\alpha}u_{xt}\psi\phi_2 - u_x\Gamma(\alpha+1)\psi\phi_1 \\
&+ \alpha\psi\phi_1 J\left(u_{xt}, (T-t)^{\alpha-1}\right).
\end{aligned}$$

Similarly, by considering X_2, the conserved vectors are:

$$\begin{aligned}
C^x =& -au_y\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right)u + du_{xy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right) \\
&- bu_{xxy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right) - cu_{yyy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right), \\
C^y =& -eu_{yy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right) - cu_{yyy}\left(\psi\phi_1(T-t)^\alpha + \psi\phi_2\right), \\
C^t =& -u_{yt}\mathcal{J}_t^{1-\alpha}\psi\phi_1(T-t)^\alpha - \mathcal{J}_t^{1-\alpha}u_{yt}\psi\phi_2 - \mathcal{J}_T^{1-\alpha}\left(\alpha\psi\phi_1(T-t)^{\alpha-1}\right)u_y \\
&- J\left(u_{ty}, \alpha\psi\phi_1(T-t)^{\alpha-1}\right).
\end{aligned}$$

The corresponding conserved vectors for (2.12)-(2.14) are presented in Tables 1 and 2.

TABLE 1. Components of conservation laws for (2.12) and (2.13)

X_i	W_i	$C^{x,y,t}$	Components of conservation laws for (2.12)
X_1	$-u_x$	C^x	$-a\left(\psi\phi_1(T-t)^{\alpha}\right)u_xu+du_{xx}\left(\psi\phi_1(T-t)^{\alpha}\right)$ $-bu_{xxx}\left(\psi\phi_1(T-t)^{\alpha}\right)-cu_{xyy}\left(\psi\phi_1(T-t)^{\alpha}\right)$
		C^y	$-eu_{xy}\left(\psi\phi_1(T-t)^{\alpha}\right)-cu_{xyy}\left(\psi\phi_1(T-t)^{\alpha}\right)$
		C^t	$-\phi_1\psi u_x\Gamma(\alpha)-\phi_1\psi u_{xt}\Gamma(\alpha)(T-t)+\phi_1\psi J\left(u_{xtt},(T-t)^{\alpha-1}\right)$
X_2	$-u_y$	C^x	$-a\left(\psi\phi_1(T-t)^{\alpha}\right)u_yu+du_{xy}\left(\psi\phi_1(T-t)^{\alpha}\right)$ $-bu_{xxy}\left(\psi\phi_1(T-t)^{\alpha}\right)-cu_{yyy}\left(\psi\phi_1(T-t)^{\alpha}\right)$
		C^y	$-eu_{yy}\left(\psi\phi_1(T-t)^{\alpha}\right)-cu_{yyy}\left(\psi\phi_1(T-t)^{\alpha}\right)$
		C^t	$-\phi_1\psi u_y\Gamma(\alpha)-\phi_1\psi u_{yt}\Gamma(\alpha)(T-t)+\phi_1\psi J\left(u_{ytt},(T-t)^{\alpha-1}\right)$
X_i	W_i	$C^{x,y,t}$	Components of conservation laws for (2.13)
X_1	$-u_x$	C^x	$-a\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)u_xu+du_{xx}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)$ $-bu_{xxx}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)-cu_{xyy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)$
		C^y	$-eu_{xy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)-cu_{xyy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)$
		C^t	$\mathcal{D}_t^{\alpha}(-u_x)\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)+\mathcal{D}_t^{\alpha-1}(u_x)\left(2\psi\phi_1t+\psi\phi_2\right)$ $+J\left(u_x,2\psi\phi_1\right)$
X_2	$-u_y$	C^x	$-a\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)u_yu+du_{xy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)$ $-bu_{xxy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)-cu_{yyy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)$
		C^y	$-eu_{yy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)-cu_{yyy}\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)$
		C^t	$\mathcal{D}_t^{\alpha}(-u_y)\left(\psi\phi_1t^2+\psi\phi_2t+\psi\phi_3\right)+\mathcal{D}_t^{\alpha-1}(u_y)\left(2\psi\phi_1t+\psi\phi_2\right)$ $+J\left(u_y,2\psi\phi_1\right)$

Table 2. Components of conservation laws for (2.14)

X_i	W_i	$C^{x,y,t}$	Component of conservation laws for (2.14)
X_1	$-u_x$	C^x	$-a\left(\psi\phi_1 t+\psi\phi_2\right)u_x u+du_{xx}\left(\psi\phi_1 t+\psi\phi_2\right)$ $-bu_{xxx}\left(\psi\phi_1 t+\psi\phi_2\right)-cu_{xyy}\left(\psi\phi_1 t+\psi\phi_2\right)$
		C^y	$-eu_{xy}\left(\psi\phi_1 t+\psi\phi_2\right)-cu_{xyy}\left(\psi\phi_1 t+\psi\phi_2\right)$
		C^t	$\mathcal{D}_t^{\alpha}(-u_x)\left(\psi\phi_1 t+\psi\phi_2\right)+\mathcal{D}_t^{\alpha-2}(u_x)\phi_1\psi$
X_2	$-u_y$	C^x	$-a\left(\psi\phi_1 t+\psi\phi_2\right)u_y u+du_{xy}\left(\psi\phi_1 t+\psi\phi_2\right)$ $-bu_{xxy}\left(\psi\phi_1 t+\psi\phi_2\right)-cu_{yyy}\left(\psi\phi_1 t+\psi\phi_2\right)$
		C^y	$-eu_{yy}\left(\psi\phi_1 t+\psi\phi_2\right)-cu_{yyy}\left(\psi\phi_1 t+\psi\phi_2\right)$
		C^t	$\mathcal{D}_t^{\alpha}(-u_y)\left(\psi\phi_1 t+\psi\phi_2\right)+\mathcal{D}_t^{\alpha-2}(u_y)\phi_1\psi$

5. Conclusion

In this paper the time-fractional generalizations of the Zakharov-Kuznetsov-Burgers equation is studied. This is an important topic in investigation of nonlinear cold-ion-acoustic waves and hot-isothermal electrons in magnetized plasma. The conservation laws of the equation is found via a modified version of Noether's theorem. This version is provided by Ibragimov and stated by considering a formal Lagrangian for a given PDE or FDE. Consequently, a generalized fractional version of Ibragomov's theorem between fractional symmetries and conservation laws are presented.

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[1]Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Semnan, Iran
Email address: `a.naderifard1384@gmail.com`

[2]Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Semnan, Iran
Email address: `ra.hejazi@gmail.com`

[3]Faculty of Mathematical Sciences,
Shahrood University of Technology,
Shahrood, Semnan, Iran
Email address: `dastranj.e@gmail.com`

KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 44(1) (2020), PAGES 89–100.

A NEW CLASS OF LAGUERRE-BASED GENERALIZED HERMITE-EULER POLYNOMIALS AND ITS PROPERTIES

N. U. KHAN[1], T. USMAN[2], AND W. A. KHAN[3]

ABSTRACT. The special polynomials of more than one variable provide new means of analysis for the solutions of a wide class of partial differential equations often encountered in physical problems. Motivated by their importance and potential for applications in a variety of research fields, recently, numerous polynomials and their extensions have been introduced and investigated. In this paper, we introduce a new family of Laguerre-based generalized Hermite-Euler polynomials, which are related to the Hermite, Laguerre and Euler polynomials and numbers. The results presented in this paper are based upon the theory of the generating functions. We derive summation formulas and related bilateral series associated with the newly introduced generating function. We also point out that the results presented here, being very general, can be specialized to give many known and new identities and formulas involving relatively simple numbers and polynomials.

1. INTRODUCTION

The generating function of the two variable Laguerre polynomials (2-VLP) $Ł_n(x,y)$ [5] is defined by

$$\frac{1}{(1-yt)}\exp\left(\frac{-xt}{1-yt}\right)=\sum_{n=0}^{\infty}Ł_n(x,y)t^n,\quad |yt|<1,$$

which is equivalently [6] given by

$$\exp(yt)C_0(xt)=\sum_{n=0}^{\infty}Ł_n(x,y)\frac{t^n}{n!}, \tag{1.1}$$

Key words and phrases. Hermite polynomials, Laguerre polynomials, generalized Euler polynomials, Laguerre-based generalized Hermite-Euler polynomials, summation formulae, bilateral series.
2010 *Mathematics Subject Classification.* Primary: 11B68, 33C45, 33C90. Secondary: 33C05.
DOI 10.46793/KgJMat2001.089K
Received: December 17, 2017.
Accepted: February 02, 2018.

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined as:

$$C_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{r!(n+r)!}, \quad n \in \mathbb{N}_0, \tag{1.2}$$

with the following generating function:

$$\exp\left(t - \frac{x}{t}\right) = \sum_{n=-\infty}^{\infty} C_n(x) t^n,$$

for $t \neq 0$ and for all finite x.

The Tricomi functions $C_n(x)$ are characterized by the following link with the Bessel function $J_n(x)$:

$$C_n(x) = x^{-\frac{n}{2}} J_n(2\sqrt{x}).$$

From equations (1.1) and (1.2), we obtain

$$L_n(x,y) = n! \sum_{s=0}^{n} \frac{(-1)^s x^s y^{n-s}}{(s!)^2 (n-s)!} = y^n L_n\left(\frac{x}{y}\right).$$

Thus, we have

$$\text{Ł}_n(x,0) = \frac{(-1)^n x^n}{n!}, \quad \text{Ł}_n(0,y) = y^n, \quad \text{Ł}_n(x,1) = \text{Ł}_n(x),$$

where $L_n(x)$ are the ordinary Laguerre polynomials [1].

The 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [2,4] are defined as:

$$H_n(x,y) = n! \sum_{r=0}^{[\frac{n}{2}]} \frac{y^r x^{n-2r}}{r!(n-2r)!},$$

and is supported by the following generating function:

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y) \frac{t^n}{n!}. \tag{1.3}$$

When $y = -1$ and x is replaced by $2x$, (1.3) reduce to the ordinary Hermite polynomials $H_n(x)$ (see [2]).

Currently, Dattoli et al. ([8], p. 241) introduced the 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ which is defined as:

$${}_LH_n(x,y,z) = n! \sum_{k=0}^{[n/2]} \frac{z^k L_{n-2k}(x,y)}{k!(n-2k)!}.$$

The 3-variable Laguerre-Hermite polynomials (3VLHP) ${}_LH_n(x,y,z)$ of the following generating function:

$$\frac{1}{(1-zt)} \exp\left(\frac{-xt}{1-zt} + \frac{yt^2}{1-zt^2}\right) = \sum_{n=0}^{\infty} {}_LH_n(x,y,z) t^n,$$

equivalent to

$$\exp(yt+zt^2)C_0(xt)=\sum_{n=0}^{\infty}{}_LH_n(x,y,z)\frac{t^n}{n!}.$$

It is clear that

$${}_LH_n\left(x,y,-\frac{1}{2}\right)={}_LH_n(x,y),$$

$${}_LH_n(x,1,-1)={}_LH_n(x),$$

where ${}_LH_n(x,y)$ denotes the 2-variable Laguerre-Hermite polynomials (2VLHP) (see [6]) and ${}_LH_n(x)$ denotes the Laguerre-Hermite polynomials (LHP) (see [7]), respectively.

The generalized Bernoulli $B_n^{(\alpha)}(x)$, Euler $E_n^{(\alpha)}(x)$ and Genocchi $G_n^{(\alpha)}(x)$ polynomials of order $\alpha\in\mathbb{C}$, each of degree n are defined respectively, by the following generating functions (see [3, 5, 10, 18, 20]):

$$\begin{aligned}
&\left(\frac{t}{e^t-1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}B_n^{(\alpha)}(x)\frac{t^n}{n!},\quad |t|<2\pi,1^{\alpha}=1,\\
&\left(\frac{2}{e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}E_n^{(\alpha)}(x)\frac{t^n}{n!},\quad |t|<\pi,1^{\alpha}=1,\\
&\left(\frac{2t}{e^t+1}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}G_n^{(\alpha)}(x)\frac{t^n}{n!},\quad |t|<\pi,1^{\alpha}=1.
\end{aligned}\tag{1.4}$$

It is easy to see that

$$B_n^{(1)}(x)=B_n(x),\quad E_n^{(1)}(x)=E_n(x),\quad G_n^{(1)}(x)=G_n(x).$$

Recently, Kurt [17] introduced and investigated the generalized Euler polynomials $E^{[\alpha,m-1]}(x)$, $m\in\mathbb{N}$ defined in a suitable neighborhood of $t=0$, by means of the generating function:

$$\left(\frac{2^m}{e^t+\sum_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{xt}=\sum_{n=0}^{\infty}E_n^{[\alpha,m-1]}(x)\frac{t^n}{n!}.\tag{1.5}$$

Furthermore, we recall here an interesting (partly bilateral and partly unilateral) generating function for $L_n^{(\alpha)}(x)$ due to Exton [9] in the following modified form (see Pathan and Yasmeen [19], Pathan and Bin-Saad [22]):

$$\exp\left(y+z-\frac{xz}{y}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\frac{L_n^{(m)}(x)y^mz^n}{(m+n)!},$$

where $m^*=\max\{0,-m\}$, $m\in\mathbb{Z}:=\{0,\pm1,\dots\}$.

The reason of interest for this family of Laguerre-based generalized Hermite-Euler polynomials is due to their intrinsic mathematical importance and to the fact that these polynomials are shown to be natural solutions of a particular set of partial differential equations which often appears in the treatment of radiation physics problems such as the electromagnetic wave propagation and quantum beam life-time in storage rings. Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis and other field of applied mathematics, a number of certain number and polynomials, and their generalizations have recently been extensively investigated.

The organization of this paper is given as follows. In Section 2, we introduce a new class of generalized Laguerre-based Hermite-Euler polynomials ${}_LH^{E_n}{}^{[\alpha,m-1]}(x,y,z)$ and develop some elementary properties by using generating functions for the numbers. In Section 3, we derive the summation formulae for these generalized polynomials by using different analytical means on their respective generating functions. In Section 4, we establish generating function for Laguerre-based Hermite-Euler polynomials involving bilateral series, some of whose special cases are also presented. Relevant connections of some results presented here with those involving simpler known partly unilateral and partly bilateral representations are also indicated.

2. A New Class of Laguerre-Based Generalized Hermite-Euler Polynomials

In this section, we introduce the Laguerre-based generalized Hermite-Euler polynomials ${}_LH^{E_n}{}^{[\alpha,m-1]}(x,y,z)$, for a real or complex parameter α defined by means of the generating function in a suitable neighborhood of $t=0$:

$$\left(\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha} e^{yt+zt^2}C_0(xt) =\sum_{n=0}^{\infty}{}_LH^{E_n}{}^{[\alpha,m-1]}(x,y,z)\frac{t^n}{n!}, \tag{2.1}$$

so that

$${}_LH^{E_n}{}^{[\alpha,m-1]}(x,y,z)=\sum_{r=0}^{n}\binom{n}{r}E_{n-r}^{[\alpha,m-1]}{}_LH_r(x,y,z).$$

It contain as its special cases not only generalized Euler polynomials (1.5), $E_n^{[\alpha,m-1]}(x)$ (see (1.4)), but also generalization of Laguerre-Hermite polynomials (see (1.5)).

Setting $m=10$, $z=0$, $y\to x$, $z\to y$ in (2.1), the result reduces to known result of Khan et al. [11]. Again setting $x=0$, $y\to x$, $z\to y$, the result reduces to known result of Pathan and Khan [20].

For $m=1$, $x=0$, $y\to x$, $z\to y$, we obtain from (2.1):

$$\left(\frac{2}{e^t+1}\right)^{\alpha} e^{yt+zt^2}=\sum_{n=0}^{\infty}{}_HE_n^{(\alpha)}(y,z)\frac{t^n}{n!},$$

which is a generalization of the generating function of Dattoli et al. ([4], p. 386 (1.6)) in the form:

$$\left(\frac{2}{e^t+1}\right) e^{xt+yt^2}=\sum_{n=0}^{\infty}{}_HE_n(x,y)\frac{t^n}{n!}.$$

Now, here we investigate the connection between Laguerre-Hermite polynomials ${}_LH_n(x,y,z)$ and generalized Euler numbers $E_n^{[\alpha,m-1]}$ which are great importance in the following theorems.

Theorem 2.1. *The following formula involving Laguerre-Hermite polynomials* ${}_LH_n(x,y,z)$ *and Laguerre-based generalized Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$$ {}_LH_n(x,y,z) = \frac{1}{2}\left({}_LH^{E_n{}^{[1,1]}}(x,y+1,z) + {}_LH^{E_n{}^{[1,1]}}(x,y,z)\right). \tag{2.2} $$

Proof. Consider equation (2.1), we have

$$ \begin{aligned} e^{yt+zt^2}C_0(xt) =& \frac{e^t+1}{2}\left(\frac{2}{e^t+1}\right)e^{yt+zt^2}C_0(xt) \\ =& \frac{1}{2}\left(\left(\frac{2}{e^t+1}\right)e^{(y+1)t+zt^2}C_0(xt) + \left(\frac{2}{e^t+1}\right)e^{yt+zt^2}C_0(xt)\right). \end{aligned} $$

Then by using the definition of Kampé de Fériet generalization of the Laguerre-Hermite polynomials ${}_LH_n(x,y)$ and Laguerre-based Hermite-Bernoulli polynomials ${}_LH^{B_n{}^{[\alpha,m-1]}}(x,y,z)$, we have

$$ \sum_{n=0}^{\infty} {}_LH_n(x,y,z)\frac{t^n}{n!} = \frac{1}{2}\sum_{n=0}^{\infty}\left({}_LH^{E_n{}^{[1,1]}}(x,y+1,z) + {}_LH^{E_n{}^{[1,1]}}(x,y,z)\right)\frac{t^n}{n!}. $$

Finally, comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we get (2.2). □

Theorem 2.2. *The following formula involving Laguerre-based generalized Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$$ {}_LH^{E_n{}^{[\alpha+\beta,m-1]}}(w,x,y,z) = \sum_{r=0}^{n}\binom{n}{r}E_{n-r}^{[\alpha,m-1]}(w)\,{}_LH^{E_n{}^{[\beta,m-1]}}(x,y,z). \tag{2.3} $$

Proof. By analyzing definition (2.1), we have

$$ \begin{aligned} &\left(\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha+\beta}\left(\exp\left(y+w\right)t+zt^2\right)C_0(xt) = \sum_{n=0}^{\infty} {}_LH^{E_n{}^{[\alpha+\beta,m-1]}}(x,y,z)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}E_n^{[\alpha,m-1]}(w)\frac{t^n}{n!}\sum_{r=0}^{\infty} {}_LH^{E_r{}^{[\beta,m-1]}}(x,y,z)\frac{t^r}{r!}. \end{aligned} $$

Now replacing n by $n-r$ in the r.h.s. of above equation and comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we obtain the result (2.3). □

Theorem 2.3. *The following formula involving Laguerre-based generalized Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$$ {}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z) = \sum_{r=0}^{n}E_{n-r}^{[m-1]}\,{}_LH^{E_n{}^{[\alpha-1,m-1]}}(x,y,z). \tag{2.4} $$

Proof. Definition (2.1) can be written as

$$\sum_{n=0}^{\infty} {}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)\frac{t^n}{n!} = \left(\frac{2^m}{e^t + \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}\right)\left(\left(\frac{2^m}{e^t + \sum\limits_{h=0}^{m-1} \frac{t^h}{h!}}\right)^{\alpha-1} \exp(yt+zt^2)C_0(xt)\right)$$

$$= \sum_{n=0}^{\infty} E_n^{[m-1]}\frac{t^n}{n!} \sum_{r=0}^{\infty} {}_LH^{E_n{}^{[\alpha-1,m-1]}}(x,y,z)\frac{t^r}{r!}.$$

On replacing n by $n-r$ in the r.h.s. of above equation and comparing the coefficients of $\frac{t^n}{n!}$ in both sides, we arrive at the desired result (2.4). □

3. Summation Formulae for Laguerre-Based Generalized Hermite-Euler Polynomials

For the derivation of implicit summation formulae involving the LHEP ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ the same consideration as developed for the Hermite-Bernoulli polynomials in Pathan [20] and Khan et al. [12–16] holds as well. First, we prove the following result involving the LHEP ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ by using series rearrangement techniques and considered its special case.

Theorem 3.1. *The following summation formula for Laguerre-based generalized Hermite-Euler polynomials ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ holds true:*

$$ {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,w,z) = \sum_{n,p=0}^{q,l} \binom{q}{n}\binom{l}{p}(w-y)^{n+p}{}_LH^{E_{q+l-n-p}{}^{[\alpha,m-1]}}(x,y,z). \tag{3.1}$$

Proof. Replacing t by $t+u$ in (2.1) and then using the formula ([21], p. 52(2)):

$$\sum_{N=0}^{\infty} f(N)\frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m)\frac{x^n}{n!}\frac{y^m}{m!}, \tag{3.2}$$

in the resultant equation, we find the following generating function for the Laguerre-based Hermite-Euler polynomials ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$:

$$\left(\frac{2^m}{e^{t+u} + \sum\limits_{h=0}^{m-1} \frac{(t+u)^h}{h!}}\right)^{\alpha} e^{z(t+u)^2}C_0(x(t+u)) = e^{-y(t+u)}\sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!}.$$

Replacing y by w in the above equation and equating the resultant equation to the above equation, we find

$$\exp((w-y)(t+u))\sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!} = \sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.3}$$

On expanding exponential function (3.3) gives

$$\sum_{N=0}^{\infty} \frac{[(w-y)(t+u)]^N}{N!} \sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$= \sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!},$$

which on using formula (3.2) in the first summation on the left hand side becomes

$$\sum_{n,p=0}^{\infty} \frac{(w-y)^{n+p}t^n u^p}{n!p!} \sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,y,z)\frac{t^q}{q!}\frac{u^l}{l!}$$
$$= \sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}. \tag{3.4}$$

Now replacing q by $q-n$, l by $l-p$ and using the lemma ([19, p. 100(1)]):

$$\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} A(n,k) = \sum_{k=0}^{\infty}\sum_{n=0}^{k} A(n,k-n), \tag{3.5}$$

in the l.h.s. of (3.4), we find

$$\sum_{q,l=0}^{\infty}\sum_{n,p=0}^{q,l} \frac{(w-y)^{n+p}}{n!p!} {}_LH^{E_{q+l-n-p}{}^{[\alpha,m-1]}}(x,y,z)\frac{t^q}{(q-n)!}\frac{u^l}{(l-p)!}$$
$$= \sum_{q,l=0}^{\infty} {}_LH^{E_{q+l}{}^{[\alpha,m-1]}}(x,w,z)\frac{t^q}{q!}\frac{u^l}{l!}.$$

Finally, on equating the coefficients of the like powers of t and u in the above equation, we get the assertion (3.1) of Theorem 3.1. □

Remark 3.1. Taking $l=0$ in assertion (3.1) of Theorem 3.1, we deduce the following consequence of Theorem 3.1.

Corollary 3.1. *The following summation formula for Laguerre-based Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$${}_LH^{E_q{}^{[\alpha,m-1]}}(x,w,z) = \sum_{n=0}^{q}\binom{q}{n}(w-y)^n {}_LH^{E_{q-n}{}^{[\alpha,m-1]}}(x,y,z). \tag{3.6}$$

Remark 3.2. Replacing w by $w+y$ in (3.6), we obtain

$${}_LH^{E_q{}^{[\alpha,m-1]}}(x,w+y,z) = \sum_{n=0}^{q}\binom{q}{n} w^n {}_LH^{E_{q-n}{}^{[\alpha,m-1]}}(x,y,z).$$

Next, we prove the following result involving the product of the Laguerre-based Hermite-Euler polynomials ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ by using series rearrangement techniques.

Theorem 3.2. *The following summation formula involving the product of Laguerre-based Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$$
{}_LH^{E_n{}^{[\alpha,m-1]}}(x,w,u)\,{}_LH^{E_s{}^{[\alpha,m-1]}}(X,W,U)=\sum_{r,k=0}^{n,s}\binom{n}{r}\binom{s}{k}H_r(w-y,u-z)
$$

$$
\times\,{}_LH^{E_{n-r}{}^{[\alpha,m-1]}}(x,y,z)H_k(W-Y,U-Z)\,{}_LH^{E_{s-k}{}^{[\alpha,m-1]}}(X,Y,Z). \tag{3.7}
$$

Proof. Consider the product of the Laguerre-based Hermite-Euler polynomials generating function (2.1) in the following form:

$$
\left(\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}\exp(yt+zt^2)C_0(xt)\left(\frac{2^m}{e^T+\sum\limits_{h=0}^{m-1}\frac{T^h}{h!}}\right)^{\alpha}\exp(YT+ZT^2)C_0(XT)
$$

$$
=\sum_{n=0}^{\infty}{}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)\frac{t^n}{n!}\sum_{s=0}^{\infty}{}_LH^{E_s{}^{[\alpha,m-1]}}(X,Y,Z)\frac{T^s}{s!}. \tag{3.8}
$$

Replacing y by w, z by u, Y by W and Z by U in (3.8) and equating the resultant equation to itself, we find

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}{}_LH^{E_n{}^{[\alpha,m-1]}}(x,w,u)\;{}_LH^{E_s{}^{[\alpha,m-1]}}(X,W,U)\frac{t^n}{n!}\frac{T^s}{s!}\\
&=\exp((w-y)t+(u-z)t^2)\;\exp((W-Y)T+(U-Z)T^2)\\
&\times\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}{}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)\;{}_LH^{E_s{}^{[\alpha,m-1]}}(X,Y,Z)\frac{t^n}{n!}\frac{T^s}{s!},
\end{aligned}
$$

which on using the generating function (3.5) in the exponential on the r.h.s, becomes

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}{}_LH^{E_n{}^{[\alpha,m-1]}}(x,w,u)\,{}_LH^{E_s{}^{[\alpha,m-1]}}(X,W,U)\frac{t^n}{n!}\frac{T^s}{s!}\\
&=\sum_{n,r=0}^{\infty}\sum_{s,k=0}^{\infty}H_r(w-y,u-z)\,{}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)\frac{t^{n+r}}{n!r!}\\
&\times H_k(W-Y,U-Z)\,{}_LH^{E_s{}^{[\alpha,m-1]}}(X,Y,Z)\frac{T^{s+k}}{s!k!}.
\end{aligned}
$$

Finally, replacing n by $n-r$ and s by $s-k$ and using equation (3.5) in the r.h.s. of the above equation and then equating the coefficients of like powers of t and T, we get assertion (3.7) of Theorem 3.2. □

Remark 3.3. Replacing u by z and U by Z in assertion (3.7) of Theorem 3.2, we deduce the the following consequence of Theorem 3.2.

Corollary 3.2. *The following summation formula involving the product of Laguerre-based Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$$\begin{aligned}&{}_LH^{E_n{}^{[\alpha,m-1]}}(x,w,z)\,{}_LH^{E_s{}^{[\alpha,m-1]}}(X,W,Z)=\sum_{r,k=0}^{n,s}\binom{n}{r}\binom{s}{k}(w-y)^r\\&\times{}_LH^{E_{n-r}{}^{[\alpha,m-1]}}(x,y,z)(W-Y)^k\,{}_LH^{E_{s-k}{}^{[\alpha,m-1]}}(X,Y,Z).\end{aligned}$$

Further, we prove the following results concerning the Laguerre-based Hermite-Euler polynomials ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ with 2VgLP $L_n(x,y)$ and the generalized Hermite Euler polynomials ${}_HE_n^{[\alpha,m-1]}$ by using operational techniques.

Theorem 3.3. *The following summation formula for Laguerre-based Hermite-Euler polynomials* ${}_LH^{E_n{}^{[\alpha,m-1]}}(x,y,z)$ *holds true:*

$$(3.9)\qquad {}_LH^{E_n{}^{[\alpha,m-1]}}(z,w,y)=\sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}\,{}_HE_{l+k-n-p}^{[\alpha,m-1]}(x,y)_qL_{n+r}(w,z-x).$$

Proof. We start by a recently derived summation formula for the generalized Hermite-Euler polynomials ${}_HE_n^{[\alpha,m-1]}$ (see [18]):

$$(3.10)\qquad {}_HE_{k+l}^{[\alpha,m-1]}(z,y)=\sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}(z-x)^{n+p}\,{}_HE_{l+k-n-p}^{[\alpha,m-1]}(x,y).$$

Operating $\exp\left(D_w^{-1}\frac{\delta^q}{\delta z^q}\right)$ on both sides of equation (3.10), we have

$$\begin{aligned}&\exp\left(D_w^{-1}\frac{\delta^q}{\delta z^q}\right){}_HE_{k+l}^{[\alpha,m-1]}(z,y)\\(3.11)\qquad &=\sum_{n,p=0}^{k,l}\binom{k}{n}\binom{l}{p}\,{}_HE_{l+k-n-p}^{[\alpha,m-1]}(x,y)\exp\left(D_w^{-1}\frac{\delta^q}{\delta z^q}\right)(z-x)^{n+p}.\end{aligned}$$

Using the operational definitions (see [12]) in the l.h.s. and r.h.s. respectively of equation (3.11), we get assertion (3.9) of Theorem 3.3. □

4. Generating Functions for the Laguerre-Based Hermite-Euler Polynomials Involving Bilateral Series

Let us consider the following a function:

$$V^{(\alpha,m)}=V^{(\alpha,m)}(x,y,z,w;s,t)=\left(\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{s-\frac{wt}{s}+yt+zt^2}C_0(xt).$$

Expanding $\exp(s-\frac{wt}{s})$ in series form and then by using (2.1), we get

$$(4.1)\qquad V^{(\alpha,m)}=\sum_{M=0}^{\infty}\frac{s^M}{M!}\sum_{K=0}^{\infty}\left(\frac{-wt}{s}\right)^K\frac{1}{K!}\sum_{N=0}^{\infty}{}_LH^{E_N{}^{[\alpha,m-1]}}(x,y,z)\frac{t^N}{N!}.$$

Upon replacing the summation indices M and N in (4.1) by $K+N=n$ and $M-K=m$, respectively and rearranging the summation series:

$$V^{(\alpha,m)}=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH^{E_{n-K}{}^{[\alpha,m-1]}}(x,y,z), \tag{4.2}$$

(which can be justified by absolute convergence of the series involved), we are led to the generating function:

$$e^{s-\frac{wt}{s}+yt+zt^2}C_0(xt)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH_{n-K}(x,y,z).$$

Some special cases of the result (4.2) are as follows.

(i) Setting $x=0$, $y=1$ and using $L_n(0,1)=1$, (4.2) reduces to

$$\left(\frac{2^m}{e^t+\sum\limits_{h=0}^{m-1}\frac{t^h}{h!}}\right)^{\alpha}e^{s-\frac{wt}{s}+zt^2}$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH^{E_{n-K}{}^{[\alpha,m-1]}}(0,1,z).$$

(ii) Setting $s=t=\frac{w}{2}$, $\alpha=x=0$ and $y=1$ in (4.2), we get

$$e^{w^2z/4}=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\left(\frac{w}{2}\right)^{m+n}\times\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH^{E_{n-K}{}^{[\alpha,m-1]}}(0,1,z).$$

(iii) Setting $s=t=\frac{w}{2}$, $x=1$ and $\alpha=y=0$, $z=\frac{2}{w}$ in (3.2), we get a new representation of Tricomi function:

$$C_0\left(\frac{2}{w}\right)=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}\left(\frac{w}{2}\right)^{m+n}\times\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH^{E_{n-K}{}^{[\alpha,m-1]}}\left(1,0,\frac{2}{w}\right).$$

(iv) Taking $m=1, x=1$ and $y=0$ in (4.2), we obtain

$$\left(\frac{2}{e^t+1}\right)^{\alpha}e^{s-\frac{wt}{s}+zt^2}C_0(t)$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH^{E_{n-K}{}^{[\alpha]}}(1,0,z).$$

(v) Letting $\alpha=m=1$ and $y=0$ in (4.2), we obtain

$$\left(\frac{2}{e^t+1}\right)e^{s-\frac{wt}{s}+zt^2}C_0(xt)$$
$$=\sum_{m=-\infty}^{\infty}\sum_{n=m^*}^{\infty}s^m t^n\sum_{K=0}^{n}\frac{(-w)^K}{K!(m+K)!(n-K)!}{}_LH^{E_{n-K}}(1,0,z).$$

Acknowledgements. The authors gratefully acknowledge the reviewers for their helpful comments.

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[1]Department of Applied Mathematics,
Faculty of Engineering and Technology,
Aligarh Muslim University, Aligarh-202002, India
Email address: nukhanmath@gmail.com

[2]Department of Applied Sciences and Humanities,
Faculty of Engineering and Technology,
Jamia Millia Islamia (A Central University), New Delhi-110025, India
Email address: talhausman.maths@gmail.com

[3]Department of Mathematics,
Faculty of Science,
Integral University, Lucknow-226026, India
Email address: waseem08_khan@rediffmail.com

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 101–111.

ON THE LOCAL VERSION OF THE CHERN CONJECTURE: CMC HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN $\mathbb{S}^{n+1}$

S. C. DE ALMEIDA[1], F. G. B. BRITO[2], M. SCHERFNER[3], AND S. WEISS[4]

Abstract. After nearly 50 years of research the Chern conjecture for isoparametric hypersurfaces in spheres is still an unsolved and important problem and in particular its local version is of great interest, since here one loses the power of Stokes' Theorem as a method for proving it. Here we present a related result for CMC hypersurfaces in $\mathbb{S}^{n+1}$ with constant scalar curvature and three distinct principal curvatures.

1. Introduction

The Chern conjecture for isoparametric hypersurfaces in spheres can be stated as follows. *Let M be a closed, minimally immersed hypersurface of the $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. Then M is isoparametric.*

One obvious generalization is that on non-closed manifolds, i.e., a local version of the conjecture. This has in particular been proposed by Bryant for the case $n = 3$.

Let $M \subset \mathbb{S}^4$ be a minimal hypersurface with constant scalar curvature. Then M is isoparametric.

For more details, a short history and an overview of results we would like to refer to the review article [3] by Scherfner, Weiss and Yau.

Here we will give a result related to the local version.

Let $n > 3$ and $M \subset \mathbb{S}^{n+1}$ be a hypersurface with constant mean and scalar curvatures which has three pairwise distinct principal curvatures everywhere, then M is isoparametric.

Key words and phrases. Constant mean and scalar curvature, isoparametric hypersurfaces, Chern conjecture.

2010 *Mathematics Subject Classification.* Primary: 53C42. Secondary: 53C21.

DOI 10.46793/KgJMat2001.101A

Received: December 25, 2017.

Accepted: February 24, 2018.

2. Preliminaries

Let M be an n-dimensional hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. We choose a local orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ in $\mathbb{S}^{n+1}(1)$, so that restricted to M, $e_1, \ldots, e_n$ are tangent to M. Let $\omega_1, \ldots, \omega_{n+1}$ denote the dual co-frame field in $\mathbb{S}^{n+1}(1)$. We use the following convention for the indices: A, B, C, D range from 1 to $n+1$ and i, j, k from 1 to n. The structure equations of $\mathbb{S}^{n+1}(1)$ as a hypersurface of the Euclidean space $\mathbb{R}^{n+2}$ are given by

$$d\,\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\,\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} \bar{R}_{ABCD}\omega_C \wedge \omega_D,$$

where $\bar{R}$ is the Riemannian curvature tensor

$$\bar{R}_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}.$$

The contractions $\bar{R}_{AC} = \sum_B \bar{R}_{ABCB}$ and $\bar{R} = \sum_{A,B} \bar{R}_{ABAB}$ are the Ricci curvature tensor and the scalar curvature of $\mathbb{S}^{n+1}(1)$, respectively. Next, we restrict all the tensors to M. First of all, since $\omega_{n+1} = 0$ on M, $\sum_i \omega_{n+1,i} \wedge \omega_i = d\omega_{n+1} = 0$. By Cartan's lemma we can write

$$\omega_{n+1,i} = \sum_j h_{ij}\omega_i, \quad h_{ij} = h_{ji}. \tag{2.1}$$

Here $h = \sum_{i,j} h_{ij}\omega_i\omega_j$ denotes the second fundamental form of M and the principal curvatures λ_i are the eigenvalues of the matrix (h_{ij}). Furthermore the mean curvature is given by $H = \frac{1}{n}\sum_i h_{ii} = \frac{1}{n}\sum_i \lambda_i$ and $K = \det(h_{ij}) = \prod_i \lambda_i$ is the Gauss-Kronecker curvature. We also define

$$S := |h|^2 = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2 \tag{2.2}$$

and for $r \geq 3$

$$f_r := \operatorname{tr}\left((h_{ij})^r\right). \tag{2.3}$$

Independently of the choice of the e_i we have

$$f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_i \lambda_i^3, \quad f_4 = \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li} = \sum_i \lambda_i^4, \tag{2.4}$$

and so on.

On M we have

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.6}$$

where R is the Riemannian curvature tensor on M with components satisfying

$$0 = R_{ijkl} + R_{ijlk}.$$

These structure equations imply the following integrability condition (Gauss equation):

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.7}$$

For the scalar curvature we have

$$\kappa = n(n-1) + n^2H^2 - S.$$

If we consider minimal hypersurfaces, the Ricci curvature and scalar curvature are given by, respectively,

$$R_{ij} = (n-1)\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.8}$$

$$\kappa = n(n-1) - S. \tag{2.9}$$

It follows from (2.9) that κ is constant if and only if S is constant. The covariant derivative ∇h with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{jk}\omega_{ik} + \sum_k h_{ik}\omega_{jk}. \tag{2.10}$$

Then the exterior derivative of (2.8) together with the structure equations yields the following Codazzi equation

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{2.11}$$

In addition we have

$$h_{ijk} = (h_{ij})_k + \sum_l h_{jl}\omega_{il}(e_k) + \sum_l h_{il}\omega_{jl}(e_k), \tag{2.12}$$

$$h_{ijkl} = (h_{ijk})_l + \sum_m h_{mjk}\omega_{im}(e_l) + \sum_m h_{imk}\omega_{jm}(e_l) + \sum_m h_{ijm}\omega_{km}(e_l), \tag{2.13}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{mi}R_{mjkl}, \tag{2.14}$$

$$\sum_{ijk} h_{ijk}^2 = (S-n)S - nHf_3 + n^2H^2. \tag{2.15}$$

We will use the following result by Otsuki given in [2].

Lemma 2.1. *Let M be a hypersurface in a $(n+1)$-dimensional Riemannian manifold of constant curvature such that the multiplicities of the principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. If the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

3. Proof of the Theorem

Theorem 3.1. *Let $n > 3$ and $M \subset \mathbb{S}^{n+1}$ be a hypersurface with constant mean and scalar curvatures which has three pairwise distinct principal curvatures everywhere, then M is isoparametric.*

Proof. Let λ, μ und ν be the distinct principal curvatures with corresponding multiplicities r_1, r_2 and r_3. From $r_1 + r_2 + r_3 = n$ and the definitions of H and S one has a system of equations with continuous coefficients which the r_i solve uniquely. Thus the r_i are continuous functions and therefore constant.

Locally we choose the e_i such that h is diagonal in every point. For the directional derivatives of the principal curvatures one has

$$r_1\lambda_k + r_2\mu_k + r_3\nu_k = r_1\lambda\lambda_k + r_2\mu\mu_k + r_3\nu\nu_k = 0. \tag{3.1}$$

Let the principal curvature directions corresponding to the three principal curvatures be called e_A, e_a and e_α. Then (2.12) implies

$$h_{ijk} = \delta_{ij}(\lambda_i)_k, \tag{3.2}$$

for $\lambda_i = \lambda_j$ and

$$\omega_{ij}(e_k) = \frac{1}{\lambda_j - \lambda_i} h_{ijk}, \tag{3.3}$$

for $\lambda_i \neq \lambda_j$.

We consider different cases for the multiplicities of the principal curvatures. Without loss of generality, let $r_1 \geq r_2 \geq r_3$.

Case 1: $r_1, r_2, r_3 > 1$. Then Lemma 2.1 implies $\lambda_A = \mu_a = \nu_\alpha = 0$, and with (3.1) it follows that all derivatives of the principal curvatures vanish.

Case 2: $r_1, r_2 > 1$, $r_3 = 1$. Without loss of generality let $\alpha = n$. Then Lemma 2.1 and (3.1) imply that the derivatives of the principal curvatures in directions e_A and e_a vanish. From (3.2), (3.3) and (2.13) one has

$$\begin{aligned} h_{AaBa} =& (h_{AaB})_a + \sum_m h_{maB}\omega_{Am}(e_a) + \sum_m h_{AmB}\omega_{am}(e_a) + \sum_m h_{Aam}\omega_{Bm}(e_a) \\ =& \frac{2}{\nu - \lambda} h_{aAn}h_{aBn} + \delta_{AB}\frac{\lambda_n\mu_n}{\nu - \mu}, \\ h_{AaaB} =& \frac{2}{\nu - \mu} h_{aAn}h_{aBn} + \delta_{AB}\frac{\lambda_n\mu_n}{\nu - \lambda}. \end{aligned}$$

From (2.14) one has

$$h_{AaBa} - h_{AaaB} = (\lambda - \mu)R_{AaBa} = \delta_{AB}(\lambda - \mu)(1 + \mu\lambda)$$

and thus

$$h_{aAn}h_{aBn} = \frac{z_1}{2}\delta_{AB}, \tag{3.4}$$

where

$$z_1 := (\nu - \lambda)(\nu - \mu)(1 + \lambda\mu) + \lambda_n\mu_n.$$

Let v_a be the column vector of the h_{aAn} for a given a, then this can be expressed in the matrix equation

$$v_a v_a^t = \frac{z_1}{2} id.$$

Since the left hand side can only have rank 0 or 1, it follows that $z_1 = 0$ and therefore $h_{aAn} = 0$ for all a und A. From (3.1) it follows that

$$\lambda_n = \frac{1}{r_1}\frac{\nu-\mu}{\mu-\lambda}\nu_n, \quad \mu_n = \frac{1}{r_2}\frac{\nu-\lambda}{\lambda-\mu}\nu_n$$

and thus

$$\begin{aligned}\sum_{ijk} h_{ijk}^2 =& 3\sum_A h_{AAn}^2 + 3\sum_a h_{aan}^2 + h_{nnn}^2 = 3r_1\lambda_n^2 + 3r_2\mu_n^2 + \nu_n^2\\ =& \left(\frac{3}{r_1}\frac{(\nu-\mu)^2}{(\mu-\lambda)^2} + \frac{3}{r_2}\frac{(\nu-\lambda)^2}{(\lambda-\mu)^2} + 1\right)\nu_n^2\\ =& (3r_2(\nu-\mu)^2 + 3r_1(\nu-\lambda)^2 + r_1r_2(\lambda-\mu)^2)\frac{1}{r_1r_2}\frac{1}{(\lambda-\mu)^2}\nu_n^2.\end{aligned}$$

On the other hand, $z_1 = 0$ implies

$$\frac{1}{r_1r_2}\frac{1}{(\lambda-\mu)^2}\nu_n^2 = -\frac{\lambda_n\mu_n}{(\nu-\mu)(\nu-\lambda)} = 1+\lambda\mu$$

and one has

$$r_2(\nu-\mu)^2 + r_1(\nu-\lambda)^2 + r_1r_2(\lambda-\mu)^2 = \frac{1}{2}\sum_{ij}(\lambda_i-\lambda_j)^2 = nS - n^2H^2.$$

Then (2.15) is of the form

$$(3.5)\quad (S-n)S + n^2H^2 = (1+\lambda\mu)(3nS - 3n^2H^2 - 2r_1r_2(\lambda-\mu)^2) + nHf_3.$$

From

$$r_1\lambda + r_2\mu + \nu = nH, \quad r_1\lambda^2 + r_2\mu^2 + \nu^2 = S,$$

one has

$$r_1(1+r_1)\lambda^2 + r_2(1+r_2)\mu^2 + n^2H^2 - 2nHr_1\lambda - 2nHr_2\mu + 2r_1r_2\lambda\mu - S = 0.$$

Solving for λ yields

$$\lambda = \frac{nH - r_2\mu}{1+r_1} + w,$$

where

$$w := \pm\sqrt{\frac{-nr_2\mu^2 + 2nr_2H\mu + (1+r_1)S - n^2H^2}{r_1(1+r_1)^2}}.$$

If $w = 0$ on an open set, then μ and consequently λ and ν are constant there. Therefore it is sufficient to show the proposition under the assumption that the sign of w remains the same. One calculates

$$1+\lambda\mu = \frac{nH\mu - r_2\mu^2}{1+r_1} + 1 + w\mu,$$

$$
\begin{aligned}
(\lambda-\mu)^2 =& \left(n\frac{H-\mu}{1+r_1}+w\right)^2\\
=&n^2\frac{\mu^2-2H\mu+H^2}{(1+r_1)^2}+2n\frac{H-\mu}{1+r_1}w+w^2\\
=&\frac{n^2r_1-nr_2}{r_1(1+r_1)^2}\mu^2+\frac{2nH(r_2-nr_1)}{r_1(1+r_1)^2}\mu+\frac{S}{r_1(1+r_1)}\\
&+\frac{n^2H^2(r_1-1)}{r_1(1+r_1)^2}+2n\frac{H-\mu}{1+r_1}w,\\
f_3 =&r_1\lambda^3+r_2\mu^3+\nu^3\\
=&r_1(1-r_1^2)\lambda^3+r_2(1-r_2^2)\mu^3+n^3H^3-3n^2H^2r_1\lambda-3n^2H^2r_2\mu\\
&+3nHr_1^2\lambda^2+3nHr_2^2\mu^2-3r_1^2r_2\lambda^2\mu-3r_1r_2^2\lambda\mu^2+6nHr_1r_2\lambda\mu\\
=&\ldots\mu^3+\ldots\mu^2+\ldots\mu+\cdots+w\mu(\ldots\mu+\ldots),
\end{aligned}
$$

that is

$$
(1+\lambda\mu)(3nS-3H^2-2r_1r_2(\lambda-\mu)^2)+nHf_3=P_1(\mu)+P_2(\mu)w,
$$

where P_1 and P_2 are polynomials of constant coefficients. For P_2 one has

$$
P_2(t)=\ldots t^3+\ldots t^2+\ldots t-\frac{4nr_1r_2H}{1+r_1},
$$

therefore, it is not identically zero if $H\neq 0$. For the case $H=0$ the same follows from

$$
P_2(t)=\ldots t^3+\frac{(3n+3nr_1-2r_2)S+4nr_1r_2}{1+r_1}t,
$$

with

$$
3n+3nr_1-2r_2\geq 3n-2r_2\geq n>0.
$$

It follows that $w-R(\mu)=0$ for a rational function R. The function

$$
F(t):=\pm\sqrt{\frac{-nr_2t^2+2nr_2Ht+(1+r_1)S-n^2H^2}{r_1(1+r_1)^2}}-R(t)
$$

is analytical and not constant. $F(\mu)=0$ then implies that μ is constant. Consequently λ and ν are also constant and the proposition follows.

Case 3: $r_1=:r=n-2>1$, $r_2=r_3=1$.

Without loss of generality let $a=1$ and $\alpha=n$. Then the derivatives of the principal curvatures in e_A direction vanish, and analogously to case 2 one has

$$
\begin{aligned}
h_{AnBn} =&\delta_{AB}\left(\lambda_{nn}+\frac{\lambda_1\nu_1}{\mu-\nu}\right)+\frac{2}{\mu-\lambda}h_{1An}h_{1Bn},\\
h_{AnnB} =&\delta_{AB}\left(\frac{\nu_1\lambda_1}{\mu-\lambda}+\frac{\nu_n\lambda_n}{\nu-\lambda}+\frac{2\lambda_n^2}{\lambda-\nu}\right)+\frac{2}{\mu-\nu}h_{1An}h_{1Bn}
\end{aligned}
$$

and thus

$$h_{1An}h_{1Bn} = \frac{z_2}{2}\delta_{AB}, \tag{3.6}$$

where

$$z_2 := \lambda_1\nu_1 + \frac{(\mu-\lambda)(\mu-\nu)}{\lambda-\nu}\left((\lambda-\nu)(1+\lambda\nu) - \lambda_{nn} + \frac{\lambda_n\nu_n}{\nu-\lambda} + \frac{2\lambda_n^2}{\lambda-\nu}\right).$$

As in case 2 it follows that $h_{1An} = 0$ for all A. From $z_2 = 0$ one has

$$\begin{aligned}\lambda_{nn} =&(\lambda-\nu)(1+\lambda\nu) + \frac{\lambda_n\nu_n}{\nu-\lambda} + \frac{2\lambda_n^2}{\lambda-\nu} + \frac{\lambda-\nu}{(\mu-\lambda)(\mu-\nu)}\lambda_1\nu_1\\ =&(\lambda-\nu)(1+\lambda\nu) - (n-2)\frac{\lambda-\nu}{(\mu-\nu)^2}\lambda_1^2 + \frac{(n+1)\mu-\nu-nH}{(\lambda-\nu)(\mu-\nu)}\lambda_n^2\end{aligned} \tag{3.7}$$

and in the same way it follows that

$$\lambda_{11} = (\lambda-\mu)(1+\lambda\mu) + \frac{(n+1)\nu-\mu-nH}{(\lambda-\mu)(\nu-\mu)}\lambda_1^2 - (n-2)\frac{\lambda-\mu}{(\nu-\mu)^2}\lambda_n^2. \tag{3.8}$$

From $h_{a1an} - h_{a1na} = 0$ one has

$$\lambda_{1n} = \frac{(n-2)(\lambda-\mu)^2(\lambda-\nu) + n(n-1)(\mu-\nu)^2(\lambda-H)}{(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu)}\lambda_1\lambda_n \tag{3.9}$$

and again the same holds true for reversed indices:

$$\lambda_{n1} = \frac{(n-2)(\lambda-\nu)^2(\lambda-\mu) + n(n-1)(\mu-\nu)^2(\lambda-H)}{(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu)}\lambda_1\lambda_n. \tag{3.10}$$

(2.15) is of the form

$$\begin{aligned}|\nabla h|^2 =&3(n-2)\lambda_1^2 + \mu_1^2 + 3\nu_1^2 + 3(n-2)\lambda_n^2 + 3\mu_n^2 + \nu_n^2\\ =&\left(3(n-2) + (n-2)^2\frac{(\lambda-\nu)^2}{(\nu-\mu)^2} + 3(n-2)^2\frac{(\lambda-\mu)^2}{(\mu-\nu)^2}\right)\lambda_1^2\\ &+\left(3(n-2) + 3(n-2)^2\frac{(\lambda-\nu)^2}{(\nu-\mu)^2} + (n-2)^2\frac{(\lambda-\mu)^2}{(\mu-\nu)^2}\right)\lambda_n^2,\end{aligned}$$

that is

$$\begin{aligned}(\nu-\mu)^2|\nabla h|^2 =&(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\nu)^2)\lambda_1^2\\ &+ (3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\mu)^2)\lambda_n^2.\end{aligned} \tag{3.11}$$

If $\lambda_1 = 0$ on an open set, (3.8) and (3.11) imply

$$|\nabla h|^2 = (3nS - 3H^2 - 2(n-2)(\lambda-\mu)^2)(1+\lambda\mu)$$

and as in case 2 it follows that the principal curvatures are constant. The same holds true for $\lambda_n = 0$, therefore we can presume in the following that $\lambda_1 \neq 0$ and $\lambda_n \neq 0$. Deriving (3.11) in direction e_1 yields

$$2(\nu-\mu)(\nu_1-\mu_1)|\nabla h|^2 + (\nu-\mu)^2(|\nabla h|^2)_1$$

$$
\begin{aligned}
=& - 4(n-2)^2(\lambda-\nu)(\lambda_1-\nu_1)\lambda_1^2 \\
&+ 2(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\nu)^2)\lambda_1\lambda_{11} \\
&- 4(n-2)^2(\lambda-\mu)(\lambda_1-\mu_1)\lambda_n^2 \\
&+ 2(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\mu)^2)\lambda_n\lambda_{n1}
\end{aligned}
$$

and with

$$
\begin{aligned}
\nu_1-\mu_1 =& n(n-2)\frac{\lambda-H}{\mu-\nu}\lambda_1, \\
(|\nabla h|^2)_1 =& - nH(f_3)_1 = -3n(n-2)H(\lambda-\mu)(\lambda-\nu)\lambda_1
\end{aligned}
$$

one has

$$
\begin{aligned}
& n|\nabla h|^2(H-\lambda) - \frac{3n}{2}H(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu) \\
=& - 2n(n-2)(\mu-H)\frac{\lambda-\nu}{\mu-\nu}\lambda_1^2 - 2n(n-2)(\nu-H)\frac{\lambda-\mu}{\nu-\mu}\lambda_n^2 \\
&+ (3nS - 3H^2 - 2(n-2)(\lambda-\nu)^2)\lambda_{11} \\
&+ (3nS - 3H^2 - 2(n-2)(\lambda-\mu)^2)\frac{\lambda_n}{\lambda_1}\lambda_{n1}.
\end{aligned} \tag{3.12}
$$

To simplify notation, we set

$$
A_1(x,y) = 3(nS-H^2) - 2(n-2)(x-y)^2.
$$

Putting (3.8) and (3.10) into (3.12) we have

$$
\begin{aligned}
& n|\nabla h|^2(H-\lambda) - \frac{3n}{2}H(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu) - A_1(\lambda,\nu)(\lambda-\mu)(1+\lambda\mu) \\
=& \left(-2n(n-2)(\mu-H)\frac{\lambda-\nu}{\mu-\nu} + A_1(\lambda,\nu)\frac{(n+1)\nu-\mu-nH}{(\lambda-\mu)(\nu-\mu)}\right)\lambda_1^2 \\
&- \left(2n(\nu-H)\frac{\lambda-\mu}{\nu-\mu} + \frac{3nS-3H^2+2(n-2)(\lambda-\mu)(\lambda-\nu)}{\mu-\nu}\right. \\
&\left. - \frac{n(n-1)(\lambda-H)}{(n-2)\lambda-\mu)(\lambda-\nu)}A_1(\lambda,\mu)\right)(n-2)\lambda_n^2.
\end{aligned} \tag{3.13}
$$

Analogously we have

$$
\begin{aligned}
& n|\nabla h|^2(H-\lambda) - \frac{3n}{2}H(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu) - A_1(\lambda,\mu)(\lambda-\nu)(1+\lambda\nu) \\
=& \left(-2n(n-2)(\nu-H)\frac{\lambda-\mu}{\nu-\mu} + A_1(\lambda,\mu)\frac{(n+1)\mu-\nu-nH}{(\lambda-\nu)(\mu-\nu)}\right)\lambda_n^2 \\
&- \left(2n(\mu-H)\frac{\lambda-\nu}{\mu-\nu} + \frac{3nS-3H^2+2(n-2)(\lambda-\mu)(\lambda-\nu)}{\nu-\mu}\right. \\
&\left. - \frac{n(n-1)(\lambda-H)}{(n-2)(\lambda-\mu)(\lambda-\nu)}A_1(\lambda,\nu)\right)(n-2)\lambda_1^2.
\end{aligned} \tag{3.14}
$$

From (3.11) one has

$$(n-2)\lambda_n^2 = \frac{(\nu-\mu)^2|\nabla h|^2}{A_1(\lambda,\mu)} - \frac{(n-2)\varepsilon(\nu)}{A_1(\lambda,\mu)}\lambda_1^2. \tag{3.15}$$

As in case 2

$$(n-2)\lambda+\mu+\nu = nH, \quad (n-2)\lambda^2+\mu^2+\nu^2 = S,$$

yield

$$\lambda = \frac{1}{n-1}(nH-\nu) - \frac{1}{n-2}w, \quad \mu = \frac{1}{n-1}(nH-\nu)+w, \tag{3.16}$$

with

$$w := \pm\frac{\sqrt{n-2}}{n-1}\sqrt{-n\nu^2+2nH\nu+(n-1)S-n^2H^2}, \tag{3.17}$$

where again the sign of w can be assumed to remain the same. We set

$$\varepsilon(\lambda,\mu,\nu) = A_1(\lambda,\mu)A_2(\lambda,\mu,\nu)A_3(\lambda,\mu,\nu)+A_1(\lambda,\nu)A_2(\lambda,\nu,\mu)A_3(\lambda,\nu,\mu),$$

where A_1 was already defined by

$$A_1(x,y) := 3nS-3H^2-2(n-2)(x-y)^2,$$

and A_2 and A_3 given by

$$\begin{aligned}
A_2(x,y,z) :=&(y-z)(x-z)\Big(n(n-2)|\nabla h|^2(x-H)(x-y)A_1(x,z)\\
&+\frac{3}{2}n(n-2)H(y-z)^2(x-y)^2(x-z)A_1(x,z)\\
&+(n-2)(1+xy)(x-y)^2A_1(x,z)^2\\
&+2n(n-2)|\nabla h|^2(y-H)(x-y)(x-z)(z-y)\\
&+|\nabla h|^2((n+1)z-y-nH)(z-y)A_1(x,z)\Big),\\
A_3(x,y,z) :=&-2(n-2)^2n(y-H)(x-y)(x-z)^2A_1(x,y)\\
&+(n-2)^2(x-y)(x-z)(3nS-3H^2+2(n-2)(x-y)(x-z))A_1(x,y)\\
&+n(n-1)(n-2)(x-H)(y-z)A_1(x,y)A_1(x,z)\\
&-2n(n-2)^2(z-H)(x-y)^2(x-z)A_1(x,z)\\
&-(n-2)(x-y)((n+1)y-z-nH)A_1(x,y)A_1(x,z).
\end{aligned}$$

From (3.13), (3.14) and (3.15) one has the following condition for λ, μ and ν:

$$\varepsilon(\lambda,\mu,\nu) = |\nabla h|^2(\mu-\nu)^2A_3(\lambda,\mu,\nu)A_3(\lambda,\nu,\mu). \tag{3.18}$$

Using (3.16) and (3.17) the terms in (3.18) can be written as polynomials in ν and w whose leading coefficients are given by

$$\begin{aligned}
A_1(\lambda,\mu) =&2n\nu^2-4nH\nu+(n+2)S+(2n^2-3)H^2,\\
A_1(\lambda,\nu) =&-2\frac{(n-2)n^2-n}{(n-1)^2}\nu^2+\cdots-\left(\frac{4n}{n-1}\nu+\cdots\right)w,
\end{aligned}$$

$$
\begin{aligned}
A_2(\lambda,\mu,\nu) =& -\frac{8n^4(7n^6-42n^5+57n^4+44n^3-79n^2-18n-1)}{(n-2)(n-1)^8}\nu^{10}+\cdots \\
&+\left(\frac{8n^4(n^6-6n^5-9n^4+68n^3-41n^2-46n+1)}{(n-2)(n-1)^7}\nu^9+\cdots\right)w, \\
A_2(\lambda,\nu,\mu) =& \frac{8n^4(3n^2-6n+1)}{(n-1)^4}\nu^{10}+\cdots+\left(\frac{8n^4(n^2-2n+3)}{(n-1)^3}\nu^9+\cdots\right)w, \\
A_3(\lambda,\mu,\nu) =& -\frac{2(n-2)n^3(7n^3-17n^2-19n+1)}{(n-1)^3}\nu^6+\cdots \\
&+\left(-\frac{2n^2(12n^3-20n^2+3n+1)}{(n-1)^2}\nu^5+\cdots\right)w, \\
A_3(\lambda,\nu,\mu) =& \frac{8(n-2)n^3(n^4-4n^3-2n^2+12n+1)}{(n-1)^4}\nu^6+\cdots \\
&+\left(\frac{4n^3(6n^4-9n^3-25n^2+29n+15)}{(n-1)^3}\nu^5+\cdots\right)w.
\end{aligned}
$$

(3.18) is then of the form

$$Q_1(\nu)+Q_2(\nu)w=0, \tag{3.19}$$

for polynomials Q_1 and Q_2 with constant coefficients. The leading coefficient of Q_1 is given by

$$
\begin{aligned}
Q_1(t) =& \frac{32(n-2)n^8}{(n-1)^{11}}(73n^{10}-709n^9+2273n^8-1255n^7-7101n^6+12067n^5 \\
&-1089n^4-6461n^3+1048n^2+134n-4)t^{18}+\cdots,
\end{aligned}
$$

therefore, Q_1 is not identically zero. One then has from (3.19) that $w=R(\nu)$ for a rational function R or that $Q_1(\nu)=0$; in both cases the proposition follows. ☐

Acknowledgements. This work was partially supported by UFC and Universidade Federal do ABC.

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[1]CAEN, Universidade Federal do Ceará,
Av. da Universidade, 2700 - 2º andar - Benfica,
60020-181 Fortaleza-CE,
Brazil
Email address: sebastc@caen.ufc.br

[2]Centro de Matemática, Computação e Cognição,
Universidade Federal do ABC,
09.210-170 Santo André,
Brazil
Email address: fabiano.brito@ufabc.edu.br

[3]Anhalt University of Applied Sciences,
Computer Science and Languages,
Lohmannstr. 23,
06366 Köthen, Germany
Email address: mike.scherfner@hs-anhalt.de

[4]Berlin,
Germany

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 113–125.

TWO-SIDED LIMIT SHADOWING PROPERTY ON ITERATED FUNCTION SYSTEMS

M. MOHTASHAMIPOUR[1] AND A. ZAMANI BAHABADI[1*]

Abstract. In this article, we introduce the two-sided limit shadowing property on an iterated function system (IFS) and attain some results such as totally transitivity, and shadowing property. Also, by means of the strong shadowing property, we achieve topologically mixing for this IFS. Then, we study the strong two-sided limit shadowing property and obtain the topologically mixing property, immediately. Moreover, we find a criterion to obtain the two-sided limit shadowing property.

1. Introduction and Definitions

To find real trajectories close to approximate trajectories, usually, the shadowing property and its various cases are used. What we want to study on iterated function systems, the systems with several generators, is two-sided limit shadowing property. Some mathematicians, like Oprocha, Carvalho and Kwietniak worked on the systems with just one generator, ordinary dynamical systems, that have this property and obtained remarkable results. For example, in [4], authors showed that systems having two-sided limit shadowing property are transitive and have the shadowing property. The relationship between two-sided limit shadowing property and another kinds of shadowing was studied in [3] and [6].

Let us mention some notations and necessary definitions on ordinary dynamical systems and iterated function systems. One can see these definitions in [2], [5], [8], and [10] for dynamical systems with one generator.

Let (X,d) be a compact metric space and $f : X \to X$ be a homeomorphism. Assume that ε and δ are positive integer numbers. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be a

Key words and phrases. Iterated function system, two-sided limit shadowing property, totally transitive, topologically mixing, skew product.

2010 *Mathematics Subject Classification.* Primary: 37B05. Secondary: 54H20.

DOI 10.46793/KgJMat2001.113M

Received: August 29, 2017.

Accepted: February 25, 2018.

δ-pseudo trajectory, if $d(f(x_i), x_{i+1}) < \delta$, for all $i \in \mathbb{Z}$. This sequence is *ε-shadowed* whenever there exists $x \in X$ such that $d(f^i(x),x_i) < \varepsilon$, for all $i \in \mathbb{Z}$. We say that f has the *shadowing property* if for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ-pseudo trajectory is ε-shadowed by some point of X.

A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is a *limit pseudo trajectory*, provided that $d(f(x_i), x_{i+1}) \to 0$ as $i \to +\infty$, and it is *limit shadowed* if there is $x \in X$ such that $d(f^i(x),x_i) \to 0$ as $i \to +\infty$. Also, a sequence $\{x_i\}_{i\in\mathbb{Z}}$ is a *negative limit pseudo trajectory* if $d(f(x_i),x_{i+1}) \to 0$ as $i \to -\infty$, and it is *negative limit shadowed* whenever there exists $x \in X$ such that $d(f^i(x),x_i) \to 0$ as $i \to -\infty$.

If $d(f(x_i), x_{i+1}) \to 0$ as $|i| \to \infty$, then $\{x_i\}_{i\in\mathbb{Z}}$ is called a *two-sided limit pseudo trajectory.* The sequence $\{x_i\}_{i\in\mathbb{Z}}$ is said to be *two-sided limit shadowed* if $d(f^i(x),x_i) \to 0$ as $|i| \to \infty$, for some $x \in X$. Principally, f has the *two-sided limit shadowing property* while every two-sided limit pseudo trajectory is two-sided limit shadowed. Analogous definitions can be presented for limit shadowing and negative limit shadowing properties.

There is a weaker case, namely, the two-sided limit shadowing property with a gap (see [4]). A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is *two-sided limit shadowed with gap K* if $d(f^i(x),x_i) \to 0$ as $i \to -\infty$ and $d(f^{i+K}(x),x_i) \to 0$ as $i \to +\infty$, for some x in X. f is said to have the *two-sided limit shadowing property with gap N* if every two-sided limit pseudo trajectory is two-sided limit shadowed with gap K for $K \in \mathbb{Z}$ and $|K| \le N$. Generally, f has the *two-sided limit shadowing property with a gap* if there exists such N.

A homeomorphism f is *transitive* whenever for every two nonempty open subsets U and V, there is a non-negative integer n such that $f^n(U) \cap V \neq \emptyset$. Also it is *topologically mixing* if for every two nonempty open subset U and V there is $m \in \mathbb{N}$ such that for all $n \ge m$, $f^n(U) \cap V \neq \emptyset$.

The following theorems were proved by Carvalho and Kwietniak in [4].

Theorem 1.1. *If a homeomorphism f of a compact metric space X has the two-sided limit shadowing property with a gap, then it is transitive and has the shadowing property.*

Theorem 1.2. *If a homeomorphism of a compact metric space has the two-sided limit shadowing property then it is topologically mixing.*

They also borrowed expansivity and specification properties as tools to obtain the two-sided limit shadowing property in [3]. For definitions of expansivity and specification properties, see [3] and [7].

Theorem 1.3. *Every expansive homeomorphism $f : X \to X$ with the shadowing and specification properties has the two-sided limit shadowing property.*

Now, we extend some of these definitions to iterated function systems.

Let $F = \{f_1, f_2, \ldots, f_k\}$ be a finite sequence of homeomorphisms on compact metric space X.

An *iterated function system* is the semigroup action generated by F, and denoted by $IFS(F)$. The elements of F are called generators of $IFS(F)$.

Let $\omega = (\ldots,\omega_{-1},\omega_0,\omega_1,\ldots) \in \{1,\ldots,k\}^{\mathbb{Z}}$. We set $f_\omega^0 = id$, and for all $n > 0$, $f_\omega^n = f_{\omega_{n-1}} o f_{\omega_{n-2}} o \ldots o f_{\omega_0}$ and $f_\omega^{-n} = f_{\omega_{-n}}^{-1} o \ldots o f_{\omega_{-1}}^{-1}$. Consider $\sigma : \{1,\cdots,k\}^{\mathbb{Z}} \to \{1,\cdots,k\}^{\mathbb{Z}}$, $\sigma(\cdots,\omega_{-1},\omega_0^*,\omega_1,\omega_2,\cdots) = (\cdots,\omega_{-1},\omega_0,\omega_1^*,\omega_2,\cdots)$, be the shift map. The map

$$\theta : \{1,\cdots,k\}^{\mathbb{Z}} \times X \to \{1,\cdots,k\}^{\mathbb{Z}} \times X, \quad \theta(\omega,x) = \Big(\sigma\omega, f_{\omega_0}(x)\Big),$$

is called the *skew product of* $IFS(F)$, which $F = \{f_1,\cdots,f_k\}$.

Assume that $\varepsilon > 0$ and $\delta > 0$ are given. A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a *δ-pseudo trajectory for $IFS(F)$* if there exists $\omega \in \{1,\ldots,k\}^{\mathbb{Z}}$ such that

$$d(f_{\omega_i}(x_i), x_{i+1}) < \delta,$$

for every $i \in \mathbb{Z}$. A δ-pseudo trajectory is said to be *ε-shadowed* whenever there are $y \in X$ and $\varphi \in \{1,\ldots,k\}^{\mathbb{Z}}$ such that

$$d(f_\varphi^i(y), x_i) < \varepsilon,$$

for every $i \in \mathbb{Z}$. We say that $IFS(F)$ has *the shadowing property*, if for every $\varepsilon > 0$, there is $\delta > 0$ such that every δ-pseudo trajectory for $IFS(F)$ is ε-shadowed and it has the *strong shadowing property* whenever φ, in the definition of the shadowing property, is equal to ω, in definition of δ-pseudo trajectory for $IFS(F)$.

Remark 1.1. If $IFS(F)$ has the strong shadowing property, then every generator has the shadowing property but its converse is not true (see Example 1.5 in [11]).

In continuation of the previous definitions, we define limit, negative limit and two-sided limit shadowing properties on iterated function systems.

A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called a (*negative*) *limit pseudo trajectory for $IFS(F)$* if there exists $\omega \in \{1,\ldots,k\}^{\mathbb{Z}}$ such that

$$d(f_{\omega_i}(x_i), x_{i+1}) \to 0,$$

as $(i \to -\infty)$ $i \to +\infty$. This sequence is said to be (*negative*) *limit-shadowed* if there are $y \in X$ and $\varphi \in \{1,\ldots,k\}^{\mathbb{Z}}$ such that

$$d(f_\varphi^i(y), x_i) \to 0,$$

as $(i \to -\infty)$ $i \to +\infty$. We say that $IFS(F)$ has the (*negative*) *limit shadowing property* whenever every (negative) limit pseudo trajectory for $IFS(F)$ is (negative) limit shadowed by some point of X.

A sequence $\{x_i\}_{i\in\mathbb{Z}}$ is called *two-sided limit pseudo trajectory for $IFS(F)$* if there is $\omega \in \{1,\ldots,k\}^{\mathbb{Z}}$ such that

$$d(f_{\omega_i}(x_i), x_{i+1}) \to 0,$$

as $|i| \to \infty$. This pseudo trajectory is said to be *two-sided limit shadowed* while there are $y \in X$ and $\varphi \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that

$$d(f_{\varphi}^{i}(y), x_i) \to 0,$$

as $|i| \to \infty$. Also it is *strong two-sided limit shadowed* whenever $\varphi = \omega$. We say that $IFS(F)$ has the (*strong*) *two-sided limit shadowing property* whenever every two-sided limit pseudo trajectory is (strong) two-sided limit shadowed.

Remark 1.2. If an iterated function system has the strong two-sided limit shadowing property, then one can see that every its generator has the two-sided limit shadowing property.

An iterated function system is called *chain transitive* while for all $\delta > 0$, and every $(x, y) \in X \times X$, there exist $n \in \mathbb{N}$ and a finite δ-pseudo trajectory $\{r_i\}_{i=0}^{n}$ such that $r_0 = x$ and $r_n = y$. We say that $IFS(F)$ is *transitive* if for every two nonempty open subsets U and V, there are $\omega \in \{1, \ldots, k\}^{\mathbb{Z}}$ and a positive integer n such that $f_{\omega}^{n}(U) \cap V \neq \emptyset$. Let $\{1, \ldots, k\}^{n}$ be the set of all words of length n. We say that $IFS(F)$ is *totally transitive* whenever $IFS(F^n)$ is transitive, for all $n \in \mathbb{N}$, where $F^n := \{f_{\omega}^{n} | \omega \in \{1, \ldots, k\}^{n}\}$.

$IFS(F)$ is called *topologically mixing* if for every two nonempty open subsets U and V in X, there is $m \in \mathbb{N}$ such that for all $n \geq m$ there exists $\omega^n \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $f_{\omega^n}^{n}(U) \cap V \neq \emptyset$. We say that $IFS(F)$ is *uniformly contracting*, whenever

$$\sup_{i \in \{1,\ldots,k\}} \sup_{y \neq x} \frac{d(f_i(x), f_i(y))}{d(x, y)}$$

exists and is smaller than 1.

In the next section, we show that the iterated function systems equipped by the two-sided limit shadowing property are totally transitive and if these IFS's have the strong shadowing property then they also are topollogically mixing. Moreover, we find a relation between two-sided limit shadowing and shadowing properties as in the following.

Theorem A. If $IFS(F)$ has the two-sided limit shadowing property, then it is totally transitive and has the shadowing property.

Theorem B. Let $IFS(F)$ has the two-sided limit shadowing and the strong shadowing properties.Then it is topologically mixing.

Moreover, we study the strong two-sided limit shadowing property on iterated function systems and obtain the topologically mixing property, immediately.

Theorem C. If $IFS(F)$ has the strong two-sided limit shadowing property, then it is topologically mixing.

Also, we find a relation between iterated function system and its skew product, whenever, they have the two-sided limit shadowing property.

Theorem D. $IFS(F)$ has the strong two-sided limit shadowing property if and only if its corresponding skew product has the two-sided limit shadowing property.

After that, in Section 3, we introduce a criterion to obtain the two-sided limit shadowing property by the following theorem.
Theorem E. Every uniformly contracting iterated function system with one to one continuous generators has the strong two-side limit shadowing property.

2. Transitivity and Mixing Property

Along to this section, we assume that X is a compact metric space and $F = \{f_1, f_2, \ldots, f_k\}$ is a finite sequence of homeomorphisms on X.

Proposition 2.1. *Let $IFS(F)$ has the two-sided limit shadowing property and $F^{-1}:= \left\{f_1^{-1}, \ldots, f_k^{-1}\right\}$. Then $IFS(F)$ and $IFS(F^{-1})$ have the limit shadowing property.*

Proof. Let $\{x_i\}_{i\in\mathbb{Z}}$ be a limit pseudo trajectory for $IFS(F)$ and let $\{z_i\}_{i\in\mathbb{Z}}$ be a limit pseudo trajectory for $IFS(F^{-1})$. There exist $\omega, t \in \{1, \ldots k\}^{\mathbb{Z}}$ such that $d(f_{\omega_i}(x_i), x_{i+1}) \to 0$ as $i \to +\infty$ and $d({f^{-1}}_{t_i}(z_i), z_{i+1}) \to 0$ as $i \to +\infty$.

Since f_i , $i = 1, \ldots, k$, is a homeomorphism, we have $d(z_i, f_{t_i}(z_{i+1})) \to 0$ as $i \to +\infty$. Set $z_{i+1}:=x_{-i}$, for all $i > 0$. It is easily seen that $\{x_i\}_{i\in\mathbb{Z}}$ is a two-sided limit pseudo trajectory for $IFS(F)$ with $s = \begin{cases} \omega_i, & i \geq 0, \\ t_{-i}, & i < 0. \end{cases}$

$IFS(F)$ has the two-sided limit shadowing property, so there are $y \in X$ and $\varphi \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that

$$d(f^i_\varphi(y), x_i) \to 0 \text{ as } i \to -\infty \tag{2.1}$$

and

$$d(f^i_\varphi(y), x_i) \to 0 \text{ as } i \to +\infty. \tag{2.2}$$

(2.1) implies that $IFS(F^{-1})$ has the limit shadowing property and by (2.2) $IFS(F)$ has the limit shadowing property. ☐

In the following, we prove the chain transitivity of iterated function systems equipped to the two-sided limit shadowing property.

Proposition 2.2. *If $IFS(F)$ has the two-sided limit shadowing property, then it is chain transitive.*

Proof. Let $(x, y) \in X \times X$. Denote the ω-limit set of x for f_1 and α-limit set of y for f_1 by $\omega_{f_1}(x)$ and $\alpha_{f_1}(y)$, respectively.

Assume $z \in \omega_{f_1}(x), w \in \alpha_{f_1}(y)$ and $p_n = \begin{cases} f_1^n(z), & n < 0, \\ f_1^n(w), & n \geq 0. \end{cases}$

The sequence $\{p_n\}$ is a two-sided limit pseudo trajectory, so there are $p \in X$ and $\omega \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that

$$\begin{cases} d(f^i_\omega(p), p_i) \to 0 \text{ as } i \to -\infty, \\ d(f^i_\omega(p), p_i) \to 0 \text{ as } i \to +\infty. \end{cases}$$

Infact,

$$\begin{cases} d\left(f_{\omega}^{i}(p), f_{1}^{i}(z)\right) \to 0 \text{ as } i \to -\infty, \\ d\left(f_{\omega}^{i}(p), f_{1}^{i}(w)\right) \to 0 \text{ as } i \to +\infty. \end{cases}$$

For $\delta > 0$, there exists $M \in \mathbb{N}$ such that

$$\begin{cases} d\left(f_{\omega}^{-M}(p), f_{1}^{-M}(z)\right) < \delta/2, \\ d\left(f_{\omega}^{M}(p), f_{1}^{M}(w)\right) < \delta/2. \end{cases}$$

Since $f_{1}^{-M}(z) \in \omega_{f_1}(x)$ and $f_{1}^{M}(w) \in \alpha_{f_1}(y)$, there are $M_1 > 0$ and $M_2 > 0$ such that

$$\begin{cases} d\left(f_{1}^{M_1}(x), f_{1}^{-M}(z)\right) < \delta/2, \\ d\left(f_{1}^{-M_2}(y), f_{1}^{M}(w)\right) < \delta/2. \end{cases}$$

So,

$$\begin{cases} d\left(f_{\omega}^{-M}(p), f_{1}^{M_1}(x)\right) < \delta, \\ d\left(f_{\omega}^{M}(p), f_{1}^{-M_2}(y)\right) < \delta. \end{cases}$$

Hence, we have the chain $x, f_1(x), \ldots, f_1^{M_1-1}(x), f_{\omega}^{-M}(p), f_{\omega}^{-M+1}(p), \ldots, f_{\omega}^{M-1}(p)$, $f_1^{-M_2}(y), f_1^{-M_2+1}(y), \ldots, y$. It shows chain transitivity of $IFS(F)$. □

Now, we are ready to approximate pseudo trajectories by real trajectories.

Proposition 2.3. *If $IFS(F)$ is chain transitive and has the limit shadowing property, then it has the shadowing property.*

Proof. Consider $IFS(F)$ does not have the shadowing property. Therefore, there exists $\varepsilon > 0$ such that for every $n > 0$ there is $\omega^n \in \{1, \ldots, k\}^{\mathbb{Z}}$ and there exists a finite $\frac{1}{n}$-pseudo trajectory $A_n^{\omega^n}$ that it cannot be ε-shadowed by any points of X. By assumption, $IFS(F)$ is chain transitive, so for all $n > 0$ there exist $\gamma^n \in \{1, \ldots, k\}^{\mathbb{Z}}$ and a $\frac{1}{n}$-pseudo trajectory $B_n^{\gamma^n}$ from the end member of $A_n^{\omega^n}$ to the first member of $A_{n+1}^{\omega^{n+1}}$ and hence a finite $\frac{1}{n}$-pseudo trajectory $A_n^{\omega^n} B_n^{\gamma^n} A_{n+1}^{\omega^{n+1}}$. The sequence $\{y_i\} = A_1^{\omega^1} B_1^{\gamma^1} A_2^{\omega^2} B_2^{\gamma^2} \ldots$ is an infinite limit pseudo trajectory. $IFS(F)$ has the limit shadowing property, so the sequence $\{y_i\}$ is limit shadowed by a point $y \in X$ and $\omega \in \{1, \ldots, k\}^{\mathbb{Z}}$; that is, we have $d(f_{\omega}^{n}(y), y_n) \to 0$ as $n \to \infty$. Hence, for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(f_{\omega}^{n}(y), y_n) < \varepsilon$, for every $n \geq N$. This means that there exists a finite pseudo trajectory $A_k^{\omega^k}$ which is ε-shadowed by some point in X. It is a contradiction. □

At present, we obtain the transitivity by the previous results.

Proposition 2.4. *Let $IFS(F)$ has the shadowing property and let be chain transitive. Then it is transitive.*

Proof. Let U and V be two nonempty open subsets of X, $x \in U$, $y \in V$ and $\varepsilon > 0$. By the shadowing property, there is $\delta > 0$ such that every δ-pseudo trajectory, specially the δ-pseudo trajectory from x to y, is ε-shadowed by a point $z \in X$. Let $x = z_0, z_1, \ldots, z_n = y$ be a δ-pseudo trajectory for $IFS(F)$ from x to y. So $d\left(f_{\varphi}^{i}(z), z_i\right) < \varepsilon$, for all i, some $z \in X$ and some $\varphi \in \{1, \ldots, k\}^{\mathbb{Z}}$. If we choose ε small enough such that $B(x, \varepsilon) \subset U$, then $f_{\varphi}^{n}(U) \cap V \neq \emptyset$. □

Proposition 2.5. *If $IFS(F)$ has the two-sided limit shadowing property, then $IFS(F^n)$ also has the two-sided limit shadowing property, for all $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. Let $n \in \mathbb{Z} \setminus \{0\}$ and $\{x_i\}_{i \in \mathbb{Z}}$ be a two-sided limit pseudo trajectory for $IFS(F^n)$. There is a sequence $\omega = \{\omega^i\}_{i \in \mathbb{Z}}, \omega^i \in \{1, \ldots, k\}^n$, such that

$$d(f_{\omega^i}^{n}(x_i), x_{i+1}) \to 0,$$

as $|i| \to \infty$. It is easily seen that the sequence

$$y_m = \begin{cases} x_i, & \text{if there is } i \in \mathbb{Z} \text{ such that } m = in, \\ f_{\omega^i}^{m-in}(x_i), & \text{if there is } i \in \mathbb{Z} \text{ such that } in < m < (i+1)n, \end{cases}$$

is a two-sided limit pseudo trajectory for $IFS(F)$. The right elements of this sequence are

$$x_0, f_{\omega^0}^{1}(x_0), f_{\omega^0}^{2}(x_0), \ldots, f_{\omega^0}^{n-1}(x_0), x_1, f_{\omega^1}^{1}(x_1), \ldots, f_{\omega^1}^{n-1}(x_1), \ldots, x_k,$$
$$f_{\omega^k}^{1}(x_k), \ldots, f_{\omega^k}^{n-1}(x_k), \ldots.$$

We can write the rest of this sequence, similarly. $IFS(F)$ has the two-sided limit shadowing property so the sequence $\{y_m\}$ is two-sided limit shadowed by a point $y \in X$ and $\gamma \in \mathbb{Z}$ such that

$$d(f_{\gamma}^{m}(y), y_m) \to 0,$$

as $|i| \to \infty$, that it implies $d(f_{\gamma}^{in}(y), x_i) \to 0$ as $|i| \to \infty$. □

Proof of Theorem A. With regard to the Propositions 2.1, 2.2, 2.3 and 2.4, if $IFS(F)$ has the two-sided limit shadowing property, then it is transitive and has the shadowing property. Then the Proposition 2.5 conclude the proof of theorem A.

Proposition 2.6. *If $IFS(F)$ is transitive and has the strong shadowing property, then for every nonempty open subset U there exist a closed subset $B \subset U$, $m \in \mathbb{N} \cup \{0\}$ and $\varphi \in \{1, \cdots, k\}^{\mathbb{Z}}$ such that $f_{\varphi}^{m}(B) = B$.*

Proof. Assume U is a nonempty open subset of X. Choose $\varepsilon > 0$ such that for some $u \in U, B(u, 3\varepsilon) \subset U$. By shadowing property for every $\varepsilon > 0$ there is δ, $0 < \delta < \varepsilon$, such that every δ-pseudo trajectory is ε-shadowed. Transitivity of $IFS(F)$ follows that there exist $n > 0$, $x \in B(u,\varepsilon)$ and $\omega \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $d(f_{\omega}^{n}(x), x) < \delta$. The sequence $\left\{z_m = f_{\omega}^{m \pmod n}(x)\right\}$ is a δ-pseudo trajectory.

So there is $z \in B(x, \varepsilon) \subset B(u, 2\varepsilon)$ such that $d(f_{\varphi}^{mn}(z), x) < \varepsilon$ for all $m \geq 0$, when $\varphi = (\ldots, \omega_0, \omega_1, \ldots, \omega_{n-1}, \omega_0, \ldots, \omega_{n-1}, \ldots) \in \{1, \ldots, k\}^{\mathbb{Z}}$.

Indeed, $\{z_m\}_{m\in\mathbb{Z}}$ is the following sequence

$$\{..., x, f_\omega^1(x), f_\omega^2(x), \ldots, f_\omega^{n-1}(x), x, f_\omega^1(x), f_\omega^2(x), \ldots, f_\omega^{n-1}(x), \ldots\}.$$

The set of all limits of subsequents of $\{f_\varphi^{mn}(z) : m \geq 0\}$ is the subset of closure of $B(u, 2\varepsilon)$. We denote this set by C. Therefore, $C \subset U$. Since, here and for this φ, $f_\varphi^{mn} = (f_\varphi^n)^m$ and ω-limit set of f_φ^n is f_φ^n-invariant, $f_\varphi^n(C) = C$. □

Now, it is prepared some qualifications to obtain topologically mixing IFS's.

Proposition 2.7. *If $IFS(F)$ has the strong shadowing property and is totally transitive, then it is topologically mixing.*

Proof. Let U and V are two nonempty open subsets of X. Choose $\varepsilon > 0$ so that $U_1 = B(u, 2\varepsilon) \subset U$ and $V_1 = B(v, 2\varepsilon) \subset V$, for some $u \in U$ and $v \in V$.

By the shadowing property, there is $\delta < \varepsilon/2$ such that every δ-pseudo trajectory is ε-shadowed.

$IFS(F)$ is transitive so for $U_2 = B(u, \delta/2)$ and V_1 there exist $n \in \mathbb{N} \cup \{0\}$ and $w \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $f_w^n(U_2) \cap V_1 \neq \varnothing$. By Proposition 2.6, there exist a closed subset $B \subset U_2 \cap (f_w^n)^{-1}V_1$, $m \in \mathbb{N} \cup \{0\}$ and $\varphi \in \{1, \ldots, k\}^{\mathbb{Z}}$ that $f_\varphi^m(B) = B$. Also, $f_w^n(f_\varphi^m)^j(U_2) \cap V_1 \neq \varnothing$, for all $j \in \mathbb{N} \cup \{0\}$. So, for every $j \in \mathbb{N} \cup \{0\}$, there is $r^j \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $f_{r^j}^{n+mj}(U_2) \cap V_1 \neq \varnothing$.

Set $G := F^m$. Transitivity of $IFS(G)$ implies that for given $\gamma \in \{1, \ldots, k\}^{\mathbb{Z}}$ and integer $s \geq 0$, there is $j_s \geq 0$ and $\xi^s \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $f_{xi^s}^{mj_s}\Big(f_\gamma^s(U_2)\Big) \cap (f_w^n)^{-1}V_1 \neq \varnothing$. Hence, there exists $\eta^s \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $f_{\eta^s}^{n+mj_s+s}(U_2) \cap V_1 \neq \varnothing$.

Set $J_s := \min\{j_s \mid f_{\eta^s}^{n+mj_s+s}(U_2) \cap V_1 \neq \varnothing$, for some $\eta^s \in \{1, \ldots, k\}^{\mathbb{Z}}\}$ and $M := \max\{n + mJ_s \mid 0 \leq s \leq m-1\}$.

We claim that for all $l \geq M$ there is $\theta^l \in \{1, \ldots, k\}^{\mathbb{Z}}$ such that $f_{\theta^l}^l(U_2) \cap V_1 \neq \varnothing$. For this aim, consider $l \geq M$. So there exist $j \geq 0$ and $0 \leq s \leq m-1$ such that $l = n + mj + s$. Since $l \geq M$, $j \geq J_s$. Therefore, $l - mp = n + mJ_s + s$ for some $p \geq 0$. The sequence

$$y_{l,t} = \begin{cases} f_\varphi^{t \pmod m}(b), & 0 \leq t \leq mp - 1, \\ f_{\eta^s}^{t-mp}(y^s), & mp \leq t, \end{cases}$$

where $b \in B$ and $y^s \in U_2 \cap (f_{\eta^s}^{n+mJ_s+s})^{-1}V_1 \neq \varnothing$, is a δ-pseudo trajectory. So it is ε-shadowed with a point $y^l \in X$ and $\theta^l \in \{1, \ldots, k\}^{\mathbb{Z}}$. In fact, $d\Big(f_{\theta^l}^t(y^l), y_{l,t}\Big) < \varepsilon$ for all $t \geq 0$.

When $t = 0$, we have $d(y^l, b) < \varepsilon$ so $y^l \in U$.

If $t = l$, then $d\Big(f_{\theta^l}^l(y^l), f_{\eta^s}^{n+mJ_s+s}(y^s)\Big) < \varepsilon$ and since $f_{\eta^s}^{n+mJ_s+s}(y^s) \in V_1$, $f_{\theta^l}^l(U_2) \cap V_1 \neq \varnothing$. Therefore $IFS(F)$ is topologically mixing. □

We are now ready to prove Theorem B.

Proof of Theorem B. By Theorem A, $IFS(F)$ is totally transitive and because it has the strong shadowing property, Proposition 2.7 implies it is topologically mixing.

Here, we want to study about the strong two-sided limit shadowing property on iterated function systems as a stronger property than the two-sided limit shadowing property.

Similar to proof of the Proposition 2.1, we have the following proposition.

Proposition 2.8. *If $IFS(F)$ has the strong two-sided limit shadowing property, Then it has the strong limit shadowing property.*

We can use a similar proof of the Proposition 2.3 to obtain the following result.

Proposition 2.9. *If $IFS(F)$ is chain transitive and has the strong limit shadowing property, then it has the strong shadowing property.*

Proposition 2.10. *If $IFS(F)$ has the strong two-sided limit shadowing property, Then it is totally transitive and has the strong shadowing property.*

Proof. Since an iterated function system with the strong two-sided limit shadowing property has the two-sided limit shadowing property, by Theorem A, it is totally transitive and obviousely chain transitive; and, by Propositions 2.8 and 2.9, it has the strong shadowing property. □

Now, by using the strong two-sided limit shadowing, we can prove Theorem C.

Proof of Theorem C. By Proposition 2.10, $IFS(F)$ has the strong shadowing property. Theorem B completes the proof.

In the following, we prove Theorem D to show a relation between iterated function systems having the strong two-sided limit shadowing property and their skew product.

Proof of Theorem D. First, assume that $IFS(F)$ has the strong two-sided limit shadowing property and $\{(\omega^i, x^i)\}_{i\in\mathbb{Z}}$ is a two-sided limit pseudo trajectory for θ where

$$\theta : \{1, \dots, k\}^{\mathbb{Z}} \times X \to \{1, \dots, k\}^{\mathbb{Z}} \times X, \quad \theta(\omega, x) = \Big(\sigma\omega, f_{\omega_0}(x)\Big),$$

is the skew product of $IFS(F)$.

Consider metrics D, d_1 and d on $\{1, \dots, k\}^{\mathbb{Z}} \times X$, $\{1, \dots, k\}^{\mathbb{Z}}$ and X respectively and

$$D((w, x), (\varphi, y)) = \max\{d_1(w, \varphi), d(x, y)\}.$$

We have

$$D\Big(\theta(\omega^i, x^i), (\omega^{i+1}, x^{i+1})\Big) \to 0 \text{ as } |i| \to \infty,$$

that is, $D\Big((\sigma\omega^i, f_{w_0^i}(x^i)),\ (\omega^{i+1}, x^{i+1})\Big) \to 0$ as $|i| \to \infty$. It is equivalent to

$$d_1(\sigma\omega^i, \omega^{i+1}) \to 0 \text{ as } |i| \to \infty, \tag{2.3}$$

$$d\Big(f_{\omega_0^i}(x^i), x^{i+1}\Big) \to 0 \text{ as } |i| \to \infty. \tag{2.4}$$

The expression (2.3) says that the sequence $\{\omega^i\}_{i\in\mathbb{Z}}$ is a two-sided limit pseudo trajectory. This sequence is two-sided limit shadowed by $\varphi = (\dots, \omega_0^{-1}, \omega_0^{*0}, \omega_0^1, \dots)$, (see Theorem 5.1 in [4]). Infact, we have

$$d_1\Big(\sigma^i(\varphi), \omega^i\Big) \to 0 \text{ as } |i| \to \infty. \tag{2.5}$$

Equation (2.4) implies that $\{x^i\}$ is a two-sided limit pseudo trajectory for $IFS(F)$. Since $IFS(F)$ has the strong two-sided limit shadowing property, there exists a point $x \in X$ such that that

$$d\Big(f^i_{\varphi}(x), x^i\Big) \to 0 \text{ as } |i| \to \infty. \tag{2.6}$$

The statements (2.5) and (2.6) imply that

$$D\Big(\theta^i(\varphi, x), (\omega^i, x^i)\Big) \to 0 \text{ as } |i| \to \infty.$$

Conversly, assume $\{x^i\}$ is a two-sided limit pseudo trajectory for $IFS(F)$. There is $\omega \in \{1, \dots, k\}^{\mathbb{Z}}$ such that

$$d\Big(f_{\omega_i}(x^i), x^{i+1}\Big) \to 0 \text{ as } |i| \to \infty.$$

It is obvious that the sequence $\{\varphi^i = \sigma^i\omega\}_{i\in\mathbb{Z}}$ is a two-sided limit pseudo trajectory and it is two-sided limit shadowed by ω. Therefore, $\{(\varphi^i, x^i)\}_{i\in\mathbb{Z}}$ is a two-sided limit pseudo trajectory for θ and is two-sided limit shadowed by some $x \in X$ and $\gamma = (\dots, \varphi_0^{-1}, \varphi_0^{*0}, \varphi_0^1, \dots)$ $=\omega$. So we have $d\Big(f^i_{\omega}(x), x^i\Big) \to 0$ as $|i| \to \infty$, and it means $IFS(F)$ has the strong two-sided limit shadowing property.

As an application of the Theorem D, we present the following example to show that the inverse of Theorem B and Theorem C are not true.

Example 2.1. Suppose that $X = [0, 1]$ and $f_0 : X \to X$, $f_0(x) = 0$ and $f_1 : X \to X$, $f_1(x) = 1 - |1 - 2x|$. Let θ be the skew product of $IFS(\{f_0, f_1\})$. $IFS(\{f_0, f_1\})$ is topologically mixing but θ is not topologically mixing (see Example 1 in [9]). Theorem B in [4] implies that θ does not have the two-sided limit shadowing property. By Theorem D, $IFS(\{f_0, f_1\})$ does not have the strong two-sided limit shadowing property. Therefore, we have an iterated function system that is topologically mixing but does not have the strong two-sided limit shadowing property. This implies the inverse of Theorem C is not true. Moreover, $IFS(\{f_0, f_1\})$ dose not have the strong shadowing property (see Example 1.3 in [11]). So, the inverse of Theorem B does not hold.

3. A Criterion

Assume that X is a compact metric space and $F = \{f_1, f_2, \dots, f_k\}$ is a finite sequence of one to one continuous functions from X to itself.

Proof of Theorem E. Suppose that $IFS(F)$ is uniformly contracting,

$$\beta = \sup_{i\in\{1,\dots,k\}} \sup_{x\neq y} \frac{d(f_i(x), f_i(y))}{d(x, y)} < 1,$$

and the sequence $\{x_i\}_{i\in\mathbb{Z}}$ is a two-sided limit pseudo trajectory for $IFS(F)$. There is $w \in \{1, \dots, k\}^{\mathbb{Z}}$ such that $d(f_{w_n}(x_n), x_{n+1}) \to 0$ as $|n| \to \infty$.

Choose the sequence $\{y_i\}_{i\in\mathbb{Z}}$ so that $y_0 = x_0$ and $y_{i+1} = f_{w_i}(y_i)$, $i \in \mathbb{Z}$. In Theorem 3.2 in [1], it is proved that every uniformly contracting IFS has the limit shadowing property, and also $d(f_w^n(y_0), x_n) \to 0$ as $n \to \infty$. Here, we show that $d(f_w^n(y_0), x_n) \to 0$ as $n \to -\infty$.

Set $\theta_i := d(f_{w_i}(x_i), x_{i+1})$, $i \in \mathbb{Z}$. We have

$$\begin{aligned} d(x_{-1}, y_{-1}) &\leq d(f_{w_{-2}}(y_{-2}), f_{w_{-2}}(x_{-2})) + d(f_{w_{-2}}(x_{-2}), x_{-1}) \\ &\leq \beta d(y_{-2}, x_{-2}) + \theta_{-2} \end{aligned}$$

and

$$\begin{aligned} d(x_{-2}, y_{-2}) &\leq d(f_{w_{-3}}(y_{-3}), f_{w_{-3}}(x_{-3})) + d(f_{w_{-3}}(x_{-3}), x_{-2}) \\ &\leq \beta d(x_{-3}, y_{-3}) + \theta_{-3}. \end{aligned}$$

By induction,

$$d(x_{-n}, y_{-n}) \leq \beta d(x_{-(n+1)}, y_{-(n+1)}) + \theta_{-(n+1)}.$$

Set $\gamma_n := \theta_{-n}$, for $n \geq 0$. $\gamma_n \to 0$ as $n \to \infty$. So, for given $\varepsilon > 0$, there is $T \in \mathbb{N}$ such that for all $n \geq T$, we have $\gamma_n < \frac{\varepsilon(1-\beta)}{2}$. Assume that $a_n := d(x_{-n}, y_{-n})$, for all $n > 0$. Since X is compact, the sequence $\{a_n\}$has a convergent subsequence $\{a_{n_k}\}_{k\geq 0}$. This fact and $\beta < 1$ imply that for all $n_k \geq T$, the following inequalities hold:

$$\begin{aligned} a_{n_k} &\leq \beta^{n_{k+1}-n_k} a_{n_{k+1}} + \beta^{n_{k+1}-n_k-1}\gamma_{n_{k+1}} + \cdots + \gamma_{n_k+1} \\ &\leq \beta^{n_{k+1}-n_k} a_{n_{k+1}} + \frac{\varepsilon}{2}(1-\beta)(1 + \cdots + \beta^{n_{k+1}-n_k-1}) \\ &\leq \beta a_{n_{k+1}} + \frac{\varepsilon}{2}(1 - \beta^{n_{k+1}-n_k-1}) \\ &\leq \beta a_{n_{k+1}} + \varepsilon. \end{aligned}$$

As $k \to \infty$, we have $a \leq \beta a + \frac{\varepsilon}{2}$. ε is arbitrary so $a(1-\beta) \leq 0$ and therefore $a = 0$.

Now, we claim that every subsequence of the sequence $\{a_n\}$ is convergent to zero. Consider a subsequence $\{a_{n_t}\}_{t\geq 0}$ such that $a_{n_t} \nrightarrow 0$ as $t \to 0$. This subsequence has a subsequence $\{a_{n_{t_l}}\}_{l\geq 0}$ such that $a_{n_{t_l}} > \varepsilon$ for all l $(*)$.

X is compact, so $\{a_{n_{t_l}}\}_{l\geq 0}$ has a convergent subsequence $\{a_{n_{t_{l_k}}}\}_{k\geq 0}$. Similar to before the claim, we have $a_{n_{t_{l_k}}} \to 0$ as $k \to \infty$. This contradicts $(*)$. Hence, $a_n \to 0$ as $n \to \infty$. Namely, $\lim_{n\to\infty} d(x_{-n}, y_{-n}) = 0$. It means that $\lim_{n\to-\infty} d(f_w^n(y_0), x_n) = 0$. Therefore, $d(f_w^n(y_0), x_n) \to 0$ as $|n| \to \infty$.

Example 3.1. Let $d_1 : \{0,1\}^{\mathbb{Z}} \times \{0,1\}^{\mathbb{Z}} \to \mathbb{R}$

$$d_1(w, \varphi) = \sup_{i\in\mathbb{Z}} \frac{\delta(w_i, \varphi_i)}{2^{|i|}},$$

be a metric for $\{0,1\}^{\mathbb{Z}}$, where

$$w = (\ldots, w_{-1}, w_0^*, w_1, \ldots), \quad \varphi = (\ldots, \varphi_{-1}, \varphi_0^*, \varphi_1, \ldots), \quad \delta(w_i, \varphi_i) = \begin{cases} 1, & w_i \neq \varphi_i, \\ 0, & w_i = \varphi_i. \end{cases}$$

Assume that $f_0 : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$,

$$f_0(\ldots, w_{-1}, w_0^*, w_1, \ldots) = (\ldots, \varphi_{-1}, \varphi_0^*, \varphi_1, \ldots),$$

such that

$$\varphi_i = \begin{cases} w_{i-1}, & i \geq 1, \\ 1, & i = 0, -1, \\ w_{i+1}, & i \leq -1, \end{cases}$$

and $f_1 : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$

$$f_1(\ldots, w_{-1}, w_0^*, w_1 \ldots) = (\ldots, \gamma_{-1}, \gamma_0^*, \gamma_1, \ldots),$$

such that

$$\gamma_i = \begin{cases} w_{i-1}, & i \geq 1, \\ 0, & i = 0, -1, \\ w_{i+1}, & i < 1. \end{cases}$$

One can see $IFS(f_0, f_1)$ is uniformly contracting and by Theorem A, it has the strong two-sided limit shadowing property. Moreover, Proposition 3.1 implies that the skew product of $IFS(f_0, f_1)$ has the two-sided limit shadowing property.

Let Y be a compact metric space, $G = \{g_1, \ldots, g_k\}$ and let $g_i : Y \to Y$ be a homeomorphism, for $i = 1, \ldots, k$.

We say that $IFS(F)$ and $IFS(G)$ are *conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f_i = g_i \circ h$, for $i = 1, \ldots, k$. It can be proved easily that conjugacy preserves the two-sided limit shadowing property. So if $IFS(F)$ has the two-sided limit shadowing property and it is conjugate to $IFS(G)$, then $IFS(G)$ has the two-sided limit shadowing property, too. The proof of this subject is straightforward, so we eliminate it.

Acknowledgements. The authors would like to thank the respectful referee for his/her comments on the manuscript. Also we would like to thank F.H. Ghane for her useful conversations.

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[1]Department of Pure Mathematics,
Ferdowsi University of Mashhad,
Mashhad, Iran
Email address: ma_mo149@mail.um.ac.ir
Email address: zamany@um.ac.ir

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 127–143.

OSTROWSKI–GRÜSS TYPE INEQUALITIES AND A 2D OSTROWSKI TYPE INEQUALITY ON TIME SCALES INVOLVING A COMBINATION OF Δ-INTEGRAL MEANS

SETH KERMAUSUOR[1] AND EZE R. NWAEZE[2*]

Abstract. In this paper, we derived two Ostrowski–Grüss type inequalities on time scales involving a combination of Δ-integral means. One of the inequalities is sharp. We also obtained 2-dimensional Ostrowski type inequality involving a combination of Δ-integral means. Our results extend some known results in the literature. Furthermore, we apply our results to the continuous, discrete and quantum calculus to obtain some interesting inequalities in these directions.

1. Introduction

In 1938, Alexander Ostrowski [23] provided a bound for the deviation of a function from its integral mean. The inequality, which is today known in the literature as Ostrowski inequality, states as follows.

Theorem 1.1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable in (a,b) and its derivative $f':(a,b)\to\mathbb{R}$ is bounded in (a,b). If $|f'(t)|\leq M$ for all $t\in[a,b]$, then we have*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leq\left(\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right)(b-a)M,$$

for all $x\in[a,b]$. The inequality is sharp in the sense that the constant $1/4$ cannot be replaced by a smaller one.

Key words and phrases. Montgomery identity, Ostrowski's inequality, Ostrowski–Grüss inequality, Δ-integral means, double integrals, time scales.
2010 *Mathematics Subject Classification.* Primary: 35A23. Secondary: 26E70, 34N05.
DOI 10.46793/KgJMat2001.127K
Received: November 11, 2017.
Accepted: February 27, 2018.

This inequality has received considerable attention over the past years (see for example [9, 10, 19] and the references therein). In 1997, Dragomir and Wang [9] obtained the following Ostrowski–Grüss type integral inequality.

Theorem 1.2. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$, $a < b$. If $f : I \to \mathbb{R}$ is a differentiable function such that there exist constants γ, $\Gamma \in \mathbb{R}$, with $\gamma \leq f'(x) \leq \Gamma$ for all $x \in [a, b]$, then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt - \frac{f(b) - f(a)}{b-a} \Big(x - \frac{a+b}{2} \Big) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma),$$

for all $x \in [a, b]$.

In 1988, the German mathematician Stefan Hilger [11] introduced the theory of time scales to unify the continuous and discrete calculus in a consistent manner. Since then many authors have studied several integral inequalities on time scales for functions of a single variable (see [15, 19, 22, 26] and the references therein) as well as for functions of two independent variables (see [12, 13, 17, 18, 24, 25] and the references therein). In 2008, Bohner and Matthews [2] extended Theorem 1.1 to an arbitrary time scale $\mathbb{T}$ as follows.

Theorem 1.3. *Let a, b, s, $t \in \mathbb{T}$, $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable. Then*

$$\left| f(t) - \frac{1}{b-a} \int_a^b f(\sigma(s))\Delta s \right| \leq \frac{M}{b-a} [h_2(t, a) + h_2(t, b)], \tag{1.1}$$

where $h_2(\cdot, \cdot)$ is defined by Definition 2.8 in Section 2 and $M = \sup_{a<t<b} |f^{\Delta}(t)| < \infty$. Inequality (1.1) is sharp in the sense that the right-hand side cannot be replaced by a smaller one.

In 2009, Liu and Ngô [20] used the Grüss inequality obtained by Bohner and Matthews [2] to extend Theorem 1.2 to an arbitrary time scale as follows.

Theorem 1.4. *Suppose $a, b, x, t \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ is differentiable. Suppose $f^{\Delta} \in C_{rd}$ and $\gamma \leq f^{\Delta}(x) \leq \Gamma$ for all $x \in [a, b]$ and some γ, $\Gamma \in \mathbb{R}$. Then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(\sigma(t))\Delta t - \frac{f(b) - f(a)}{(b-a)^2} \Big(h_2(x, a) - h_2(x, b) \Big) \right| \leq \frac{1}{4}(b-a)(\Gamma - \gamma),$$

for all $x \in [a, b]$.

The same authors in [21] obtained a sharp bound for the inequality in Theorem 1.4. Specifically, they proved the next theorem.

Theorem 1.5. *Suppose $a, b, x, t \in \mathbb{T}$ and $f : [a, b] \to \mathbb{R}$ is differentiable. Suppose also $f^{\Delta} \in C_{rd}$ and $\gamma \leq f^{\Delta}(x) \leq \Gamma$ for all $x \in [a, b]$ and some γ, $\Gamma \in \mathbb{R}$. Then we have*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(\sigma(t))\Delta t - \frac{f(b) - f(a)}{(b-a)^2} \Big(h_2(x, a) - h_2(x, b) \Big) \right|$$

$$\leq \frac{\Gamma - \gamma}{2(b-a)} \int_a^b \left| K(x,t) - \frac{h_2(x,a) - h_2(x,b)}{b-a} \right| \Delta t,$$

for all $x \in [a,b]$*, where*

$$K(x,t) = \begin{cases} t-a, & a \leq t < x, \\ t-b, & x \leq t \leq b. \end{cases}$$

Motivated by the above works and the paper [17], we obtain two Ostrowski-Grüss type inequalities on time scales involving a combination of Δ-integral means. The results above then become particular case of our results. Also, we provide a 2D Ostrowski type inequality for double integrals involving a combination of Δ-integral means. The result in [17] then becomes a particular case of our result.

This paper is arranged in the following order: first, we present some time scale essentials in Section 2. In Section 3, our first two results are formulated and proved. Finally, we provide a 2D Ostrowski-type inequality in Section 4.

2. Some Basic Notions of Time Scales

In this section, we briefly recall some fundamental facts about the time scale theory. For further details and proofs we invite the interested reader to Hilger's Ph.D. thesis [11], the books [4, 5, 16], and the survey [1].

Definition 2.1. A time scale is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

Throughout this work we assume $\mathbb{T}$ is a time scale and $\mathbb{T}$ has the topology that is inherited from the standard topology on $\mathbb{R}$. It is also assumed throughout that in $\mathbb{T}$ the interval $[a,b]$ means the set $\{t \in \mathbb{T}\colon\ a \leq t \leq b\}$ for the points $a < b$ in $\mathbb{T}$. Since a time scale may not be connected, we need the following concept of jump operators.

Definition 2.2. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, respectively.

The jump operators σ and ρ allow the classification of points in $\mathbb{T}$ as follows.

Definition 2.3. If $\sigma(t) > t$, then we say that t is right-scattered, while if $\rho(t) < t$ then we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. If $\sigma(t) = t$, then t is called right-dense, and if $\rho(t) = t$ then t is called left-dense. Points that are both right-dense and left-dense are called dense.

Definition 2.4. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. The set $\mathbb{T}^k$ is defined as follows: if $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T} - \{m\}$, otherwise, $\mathbb{T}^k = \mathbb{T}$.

If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and when $\mathbb{T} = \mathbb{Z}$, we have $\mu(t) = 1$.

Definition 2.5. Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that for any given $\epsilon > 0$ there exists a neighborhood U of t such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t) \left[\sigma(t) - s \right] \right| \leq \epsilon \left| \sigma(t) - s \right|, \quad \text{for all } s \in U.$$

We call $f^{\Delta}(t)$ the delta derivative of f at t. Moreover, we say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}^k$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$. The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is then called the delta derivative of f on $\mathbb{T}^k$.

In the case $\mathbb{T} = \mathbb{R}$, $f^{\Delta}(t) = \frac{df(t)}{dt}$. In the case $\mathbb{T} = \mathbb{Z}$, $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$, which is the usual forward difference operator. If $\mathbb{T} = q^{\mathbb{N}_0}, q > 1$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then $f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}$.

Theorem 2.1. *Assume f, $g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$. Then the product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$(fg)^{\Delta}(t) = f^{\Delta}(t) g(t) + f(\sigma(t)) g^{\Delta}(t).$$

Definition 2.6. The function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous on $\mathbb{T}$ provided it is continuous at all right-dense points $t \in \mathbb{T}$ and its left-sided limits exist at all left-dense points $t \in \mathbb{T}$. The set of all rd-continuous function $f : \mathbb{T} \to \mathbb{R}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$. Also, the set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $C^1_{rd}(\mathbb{T},\mathbb{R})$.

It follows from [2, Theorem 1.74] that every rd-continuous function has an antiderivative.

Definition 2.7. Let $F : \mathbb{T} \to \mathbb{R}$ be a function. Then $F : \mathbb{T} \to \mathbb{R}$ is called the antiderivative of f on $\mathbb{T}$ if it satisfies $F^{\Delta}(t) = f(t)$ for any $t \in \mathbb{T}^k$. In this case, the Cauchy integral

$$\int_a^b f(t) \Delta t = F(b) - F(a), \quad a, b \in \mathbb{T}.$$

Theorem 2.2. *If $a, b, c \in \mathbb{T}$ with $a < c < b$, $\alpha \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$, then*

(i) $\int_a^b [f(t) + g(t)] \Delta t = \int_a^b f(t) \Delta t + \int_a^b g(t) \Delta t$;
(ii) $\int_a^b \alpha f(t) \Delta t = \alpha \int_a^b f(t) \Delta t$;
(iii) $\int_a^b f(t) \Delta t = - \int_b^a f(t) \Delta t$;
(iv) $\int_a^b f(t) \Delta t = \int_a^c f(t) \Delta t + \int_c^b f(t) \Delta t$;
(v) $\left| \int_a^b f(t) \Delta t \right| \leq \int_a^b |f(t)| \Delta t$;
(vi) $\int_a^b f(t) g^{\Delta}(t) \Delta t = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t) g^{\sigma}(t) \Delta t$.

Definition 2.8. Let $h_k : \mathbb{T}^2 \to \mathbb{R}$, $k \in \mathbb{N}_0$ be defined by $h_0(t, s) = 1$ for all s,$t \in \mathbb{T}$ and then recursively by $h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta \tau$ for all s,$t \in \mathbb{T}$.

If $\mathbb{T}=\mathbb{R}$, then $h_k(t,s)=\frac{(t-s)^k}{k!}$ for all $s,t\in\mathbb{R}$. If $\mathbb{T}=\mathbb{Z}$, then $h_k(t,s)=\binom{t-s}{k}$ for all s,$t\in\mathbb{Z}$. If $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, then $h_k(t,s)=\Pi_{\nu=0}^{k-1}\dfrac{t-q^\nu s}{\sum_{\mu=0}^{\nu}q^\mu}$ for all s,$t\in q^{\mathbb{N}_0}$.

3. Ostrowski-Grüss Type Inequality Involving a Combination of Δ-integral Means

To prove our theorems, we need the following lemmas. The first lemma was first provided in [8] for the case $\mathbb{T}=\mathbb{R}$ and extended to any arbitrary time scale in [14].

Lemma 3.1 (Montgomery identity involving a combination of Δ-integral means)**.** *Let $a,b,t\in\mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Then for all $x\in[a,b]$, we have*

(3.1)

$$\int_a^b P(x,t)f^\Delta(t)\Delta t=f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right],$$

where α, $\beta\in\mathbb{R}$ are nonnegative and not both zero, and

$$P(x,t)=\begin{cases}\dfrac{\alpha}{\alpha+\beta}\left(\dfrac{t-a}{x-a}\right), & a\le t<x,\\[2mm] \dfrac{-\beta}{\alpha+\beta}\left(\dfrac{b-t}{b-x}\right), & x\le t\le b.\end{cases}$$

The next lemma is the Grüss inequality on time scales obtained by Bohner and Matthews [2].

Lemma 3.2. [2] *Let $a,b,s\in\mathbb{T}$, f, $g\in C_{rd}$ and f, $g:[a,b]\to\mathbb{R}$. Then for*

$$m_1\le f(s)\le M_1,\quad m_2\le g(s)\le M_2,$$

we have

$$\left|\frac{1}{b-a}\int_a^b f^\sigma(s)g^\sigma(s)\Delta s-\frac{1}{(b-a)^2}\int_a^b f^\sigma(s)\Delta s\int_a^b g^\sigma(s)\Delta s\right|$$
$$\le\frac{1}{4}(M_1-m_1)(M_2-m_2).$$

We now state and prove our first theorem.

Theorem 3.1. *Let a, b, $t\in\mathbb{T}$, $a<b$ and $f:[a,b]\to\mathbb{R}$ be differentiable. Suppose $f^\Delta\in C_{rd}$ and $\gamma\le f^\Delta(t)\le\Gamma$ for all $t\in[a,b]$. Then we have*

$$\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t+\frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\right]\right.$$
$$\left.-\frac{f(b)-f(a)}{(b-a)}\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}h_2(x,a)-\frac{\beta}{b-x}h_2(x,b)\right]\right|$$
$$\le\frac{1}{4}(b-a)(\Gamma-\gamma),$$

for all $x \in [a, b]$.

Proof. Let $M_1 = \sup_{a<t<b} P(x,t)$ and $m_1 = \inf_{a<t<b} P(x,t)$. By the definition of $P(x,t)$ we have that $M_1 = \frac{\alpha}{\alpha+\beta}$ and $m_1 = \frac{-\beta}{\alpha+\beta}$. Thus, $M_1 - m_1 = 1$ and

$$m_1 \leq P(x,t) \leq M_1.$$

Now by applying Lemma 3.2 to the functions $f(t) := P(x,t)$ and $g(t) := f^{\Delta}(t)$, we get

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b P(x,t)f^{\Delta}(t)\Delta t - \frac{1}{(b-a)^2}\int_a^b f^{\Delta}(t)\Delta t \int_a^b P(x,t)\Delta t\right| \\ \leq & \frac{1}{4}(M_1 - m_1)(\Gamma - \gamma).\end{aligned}$$

Thus,

$$\begin{aligned}&\left|\int_a^b P(x,t)f^{\Delta}(t)\Delta t - \frac{1}{b-a}\int_a^b f^{\Delta}(t)\Delta t \int_a^b P(x,t)\Delta t\right| \\ \leq & \frac{1}{4}(b-a)(\Gamma - \gamma). \end{aligned} \tag{3.2}$$

By a simple computation, we have

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a) \tag{3.3}$$

and

$$\int_a^b P(x,t)\Delta t = \frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(b,x)\Big]. \tag{3.4}$$

The desired inequality is obtained by substituting (3.1), (3.3) and (3.4) into (3.2). $\square$

Remark 3.1. We note that the inequality in Theorem 3.1 is not sharp. We will provide the sharp version in our next theorem.

Remark 3.2. If we set $\alpha = x - a$ and $\beta = b - x$ in Theorem 3.1, then we recapture Theorem 1.4.

If we apply Theorem 3.1 to the continuous, discrete and quantum calculus, we obtain some interesting inequalities which generalize the results in [20].

Corollary 3.1 (Continuous case)**.** *If we let* $\mathbb{T} = \mathbb{R}$ *in Theorem* 3.1*, then we have the inequality*

$$\begin{aligned}&\Bigg|f(x) - \frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\Big] \\ &- \frac{f(b)-f(a)}{2(b-a)}\frac{1}{\alpha+\beta}\Big[\alpha(x-a) - \beta(b-x)\Big]\Bigg| \\ \leq & \frac{1}{4}(b-a)(\Gamma - \gamma),\end{aligned}$$

for all $x \in [a,b]$.

Corollary 3.2 (Discrete case)**.** *If we let* $\mathbb{T} = \mathbb{Z}$ *in Theorem* 3.1*, then we have the inequality*

$$\begin{aligned}&\Bigg|f(x) - \frac{1}{\alpha+\beta}\Bigg[\frac{\alpha}{x-a}\sum_{t=a}^{x-1} f(t+1) + \frac{\beta}{b-x}\sum_{t=x}^{b-1} f(t+1)\Bigg]\\ &- \frac{f(b)-f(a)}{2(b-a)}\frac{1}{\alpha+\beta}\Big[\alpha(x-a-1) - \beta(b-x+1)\Big]\Bigg|\\ \leq&\frac{1}{4}(b-a)(\Gamma-\gamma),\end{aligned}$$

for all $x \in \{a, a+1, \ldots, b-1, b\}$.

Corollary 3.3 (Quantum case)**.** *If we let* $\mathbb{T} = q^{\mathbb{N}_0}$*,* $q > 1$ *in Theorem* 3.1*, then we have the inequality*

$$\begin{aligned}&\Bigg|f(x) - \frac{1}{\alpha+\beta}\Bigg[\frac{\alpha}{x-a}\int_a^x f(qt)d_qt + \frac{\beta}{b-x}\int_x^b f(qt)d_qt\Bigg]\\ &- \frac{f(b)-f(a)}{b-a}\frac{1}{(\alpha+\beta)(1+q)}\Big[\alpha(x-qa) - \beta(qb-x)\Big]\Bigg|\\ \leq&\frac{1}{4}(b-a)(\Gamma-\gamma),\end{aligned}$$

for all $x \in [a,b]$.

In our next theorem, we provide a sharp bound for the inequality in Theorem 3.1. To do this, we need the following lemma which can be found in [21].

Lemma 3.3. ([21]) *Let* $a, b, x \in \mathbb{T}$*,* $f, g \in C_{rd}$ *and* $f, g : [a,b] \to \mathbb{R}$*. Then if* $\gamma \leq g(x) \leq \Gamma$ *for all* $x \in [a,b]$ *and some* $\gamma, \Gamma \in \mathbb{R}$*, we have*

$$\begin{aligned}&\left|\int_a^b f(t)g(t)\Delta t - \frac{1}{b-a}\int_a^b f(t)\Delta t\int_a^b g(t)\Delta t\right|\\ \leq&\frac{\Gamma-\gamma}{2}\int_a^b\left|f(t) - \frac{1}{b-a}\int_a^b f(s)\Delta s\right|\Delta t. \qquad (3.5)\end{aligned}$$

Moreover, the inequality in (3.5) *is sharp.*

Theorem 3.2. *Under the conditions of Lemma* 3.1*, we have the inequality*

$$\begin{aligned}&\Bigg|f(x) - \frac{1}{\alpha+\beta}\Bigg[\frac{\alpha}{x-a}\int_a^x f(\sigma(t))\Delta t + \frac{\beta}{b-x}\int_x^b f(\sigma(t))\Delta t\Bigg]\\ &- \frac{f(b)-f(a)}{b-a}\frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\Big]\Bigg|\\ \leq&\frac{\Gamma-\gamma}{2}\int_a^b\left|P(x,t) - \frac{1}{b-a}\frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(x,b)\Big]\right|\Delta t, \qquad (3.6)\end{aligned}$$

for all $x \in [a,b]$. Moreover, the inequality in (3.6) *is sharp in the sense that the constant $\frac{1}{2}$ cannot be replaced by a smaller one.*

Proof. By applying Lemma 3.3 to the functions $f(t) := P(x,t)$ and $g(t) := f^{\Delta}(t)$, we have

$$\left|\int_a^b P(x,t)f^{\Delta}(t)\Delta t - \frac{1}{b-a}\int_a^b P(x,t)\Delta t \int_a^b f^{\Delta}(t)\Delta t\right|$$

$$\leq \frac{\Gamma-\gamma}{2}\int_a^b \left|P(x,t) - \frac{1}{b-a}\int_a^b P(x,s)\Delta s\right|\Delta t. \tag{3.7}$$

Now, we observe that

$$\int_a^b f^{\Delta}(t)\Delta t = f(b) - f(a) \tag{3.8}$$

and

$$\int_a^b P(x,t)\Delta t = \frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}h_2(x,a) - \frac{\beta}{b-x}h_2(b,x)\Big]. \tag{3.9}$$

The desired inequality is obtained by substituting (3.1), (3.8) and (3.9) in (3.7). ☐

Remark 3.3. Let $\alpha = x-a$ and $\beta = b-x$ in Theorem 3.2. Then we recapture Theorem 1.5. Note that in this case $P(x,t) = \frac{K(x,t)}{b-a}$.

We now apply Theorem 3.2 to the continuous, discrete and quantum time scales to obtain some interesting inequalities which generalize the results in [21].

Corollary 3.4 (Continuous case)**.** *If we let $\mathbb{T} = \mathbb{R}$ in Theorem* 3.2*, then we have the inequality*

$$\Bigg|f(x) - \frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}\int_a^x f(t)dt + \frac{\beta}{b-x}\int_x^b f(t)dt\Big]$$
$$- \frac{f(b)-f(a)}{2(b-a)}\frac{1}{\alpha+\beta}\Big[\alpha(x-a) - \beta(b-x)\Big]\Bigg|$$
$$\leq \frac{\Gamma-\gamma}{2}\int_a^b \left|P(x,t) - \frac{1}{2(b-a)}\frac{1}{\alpha+\beta}\Big[\alpha(x-a) - \beta(b-x)\Big]\right|dt, \tag{3.10}$$

for all $x \in [a,b]$. Moreover, the inequality in (3.10) *is sharp in the sense that the constant $\frac{1}{2}$ cannot be replaced by a smaller one.*

Corollary 3.5 (Discrete case)**.** *If we let $\mathbb{T} = \mathbb{Z}$ in Theorem* 3.2*, then we have the inequality*

$$\Bigg|f(x) - \frac{1}{\alpha+\beta}\Big[\frac{\alpha}{x-a}\sum_{t=a}^{x-1} f(t+1) + \frac{\beta}{b-x}\sum_{t=x}^{b-1} f(t+1)\Big]$$
$$- \frac{f(b)-f(a)}{2(b-a)}\frac{1}{\alpha+\beta}\Big[\alpha(x-a-1) - \beta(b-x+1)\Big]\Bigg|$$

$$\leq \frac{\Gamma-\gamma}{2}\sum_{t=a}^{b-1}\left|P(x,t)-\frac{1}{2(b-a)}\frac{1}{\alpha+\beta}\Big[\alpha(x-a-1)-\beta(b-x+1)\Big]\right|, \tag{3.11}$$

for all $x \in \{a, a+1, \ldots, b-1, b\}$. *Moreover, the inequality in* (3.11) *is sharp in the sense that the constant* $\frac{1}{2}$ *cannot be replaced by a smaller one.*

Corollary 3.6 (Quantum case)**.** *If we let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, *in Theorem* 3.2, *then we have the inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{\alpha+\beta}\left[\frac{\alpha}{x-a}\int_a^x f(qt)d_qt+\frac{\beta}{b-x}\int_x^b f(qt)d_qt\right]\right.\\&\left.\quad-\frac{f(b)-f(a)}{b-a}\frac{1}{(\alpha+\beta)(1+q)}\Big[\alpha(x-qa)-\beta(qb-x)\Big]\right|\end{aligned}$$

$$\leq \frac{\Gamma-\gamma}{2}\int_a^b\left|P(x,t)-\frac{1}{b-a}\frac{1}{(\alpha+\beta)(1+q)}\Big[\alpha(x-qa)-\beta(qb-x)\Big]\right|d_qt, \tag{3.12}$$

for all $x \in [a,b]$. *Moreover, the inequality in* (3.12) *is sharp in the sense that the constant* $1/2$ *cannot be replaced by a smaller one.*

4. A 2-Dimensional Ostrowski Inequality on Time Scales Involving a Combination of Δ-Integral Means

In what follows, we will let $\mathbb{T}_1$ and $\mathbb{T}_2$ denote two arbitrary time scales, with forward jump orperators σ_1 and σ_2 respectively. For an interval $[a,b]$, $[a,b]_{\mathbb{T}_i} := [a,b] \cap \mathbb{T}_i$, i =1, 2. For $a < b$ and $c < d$, we define the rectangle $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2}$ as follows: $[a,b]_{\mathbb{T}_1} \times [c,d]_{\mathbb{T}_2} = \{(x,y) : x \in [a,b]_{\mathbb{T}_1}, y \in [c,d]_{\mathbb{T}_2}\}$. For the sake of brevity, we will simply write $[a,b]$ instead of $[a,b]_{\mathbb{T}_1}$ and $[c,d]$ instead of $[c,d]_{\mathbb{T}_2}$. For more on the two-variable time scale calculus, we invite the interested reader to the papers [6,7] and the references therein.

To prove our next theorem, we need the following lemma.

Lemma 4.1. *Let* $a, b, x, s \in \mathbb{T}_1$, $a < b$, $c, d, y, t \in \mathbb{T}_2$, $c < d$ *and let* $f : [a,b] \times [c,d] \to \mathbb{R}$ *be such that the partial derivatives* $\frac{\partial f(s,t)}{\Delta_1 s}$, $\frac{\partial f(s,t)}{\Delta_2 t}$, $\frac{\partial^2 f(s,t)}{\Delta_2 t \Delta_1 s}$ *exist and are continuous on* $[a,b] \times [c,d]$. *Then we have*

$$\begin{aligned}&f(x,y)-\frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\left[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2t\Delta_1s\right.\\&+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2t\Delta_1s\\&+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2t\Delta_1s\\&\left.+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2t\Delta_1s\right]\end{aligned}$$

$$
\begin{aligned}
=&\frac{\alpha_1}{(\alpha_1+\beta_1)(x-a)}\int_a^x\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Delta_1 s\\
&+\frac{\beta_1}{(\alpha_1+\beta_1)(b-x)}\int_x^b\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Delta_1 s\\
&+\frac{\alpha_2}{(\alpha_2+\beta_2)(y-c)}\int_c^y\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t\\
&+\frac{\beta_2}{(\alpha_2+\beta_2)(d-y)}\int_y^d\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t\\
&+\int_c^d\int_a^b P_1(x,s)P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_1 s\Delta_2 t, \qquad (4.1)
\end{aligned}
$$

for all $x\in[a,b]$ and $y\in[c,d]$, where

$$
P_1(x,s)=\begin{cases}\dfrac{\alpha_1}{\alpha_1+\beta_1}\left(\dfrac{s-a}{x-a}\right), & s\in[a,x),\\[2mm] \dfrac{-\beta_1}{\alpha_1+\beta_1}\left(\dfrac{b-s}{b-x}\right), & s\in[x,b],\end{cases}
$$

$$
P_2(y,t)=\begin{cases}\dfrac{\alpha_2}{\alpha_2+\beta_2}\left(\dfrac{t-c}{y-c}\right), & t\in[c,y),\\[2mm] \dfrac{-\beta_2}{\alpha_2+\beta_2}\left(\dfrac{d-t}{d-y}\right), & t\in[y,d],\end{cases}
$$

α_1, β_1, α_2 and β_2 are nonnegative numbers with $\alpha_1+\beta_1>0$ and $\alpha_2+\beta_2>0$.

Proof. By applying Lemma 3.1 to the partial map $f(\cdot,y)$, $y\in[c,d]$, we have for $x\in[a,b]$

$$
\begin{aligned}
f(x,y)=&\frac{1}{\alpha_1+\beta_1}\left[\frac{\alpha_1}{x-a}\int_a^x f(\sigma_1(s),y)\Delta_1 s+\frac{\beta_1}{b-x}\int_x^b f(\sigma_1(s),y)\Delta_1 s\right]\\
&+\int_a^b P_1(x,s)\frac{\partial f(s,y)}{\Delta_1 s}\Delta_1 s. \qquad (4.2)
\end{aligned}
$$

If we apply Lemma 3.1 to the maps $f(\sigma_1(s),\cdot)$ and $\frac{\partial f(s,\cdot)}{\Delta_1 s}$, we have

$$
\begin{aligned}
f(\sigma_1(s),y)=&\frac{1}{\alpha_2+\beta_2}\left[\frac{\alpha_2}{y-c}\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t+\frac{\beta_2}{d-y}\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\right]\\
&+\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t \qquad (4.3)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial f(s,y)}{\Delta_1 s}=&\frac{1}{\alpha_2+\beta_2}\left[\frac{\alpha_2}{y-c}\int_c^y \frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_2 t+\frac{\beta_2}{d-y}\int_y^d \frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_2 t\right]\\
&+\int_c^d P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t. \qquad (4.4)
\end{aligned}
$$

By substituting (4.3) and (4.4) into (4.2) we obtain,

$$
\begin{aligned}
f(x,y) =& \frac{1}{\alpha_1+\beta_1}\Bigg[\frac{\alpha_1}{x-a}\int_a^x \Bigg(\frac{1}{\alpha_2+\beta_2}\Bigg[\frac{\alpha_2}{y-c}\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t \\
&+ \frac{\beta_2}{d-y}\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Bigg] + \int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Bigg)\Delta_1 s \\
&+ \frac{\beta_1}{b-x}\int_x^b \Bigg(\frac{1}{\alpha_2+\beta_2}\Bigg[\frac{\alpha_2}{y-c}\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t \\
&+ \frac{\beta_2}{d-y}\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Bigg] + \int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Bigg)\Delta_1 s\Bigg] \\
&+ \int_a^b P_1(x,s)\Bigg(\frac{1}{\alpha_2+\beta_2}\Bigg[\frac{\alpha_2}{y-c}\int_c^y \frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_2 t \\
&+ \frac{\beta_2}{d-y}\int_y^d \frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_2 t\Bigg] + \int_c^d P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_2 t\Bigg)\Delta_1 s.
\end{aligned}
$$

By rearranging the terms we get

$$
\begin{aligned}
f(x,y) =& \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s \\
&+ \frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s \\
&+ \frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s \\
&+ \frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\Bigg] \\
&+ \frac{\alpha_1}{(\alpha_1+\beta_1)(x-a)}\int_a^x\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Delta_1 s \\
&+ \frac{\beta_1}{(\alpha_1+\beta_1)(b-x)}\int_x^b\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Delta_1 s \\
&+ \frac{\alpha_2}{(\alpha_2+\beta_2)(y-c)}\int_c^y\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t \\
&+ \frac{\beta_2}{(\alpha_2+\beta_2)(d-y)}\int_y^d\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t \\
&+ \int_c^d\int_a^b P_1(x,s)P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_1 s\Delta_2 t. \qquad (4.5)
\end{aligned}
$$

The identity in (4.1) follows directly from (4.5). ◻

Theorem 4.1. *Under the conditions of Lemma* 4.1*, we have the inequality*

$$\begin{aligned}
&\Bigg|f(x,y) - \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\Bigg]\Bigg|\\
\le&\frac{M_2}{\alpha_2+\beta_2}\left[\frac{\alpha_2}{y-c}h_2(y,c)+\frac{\beta_2}{d-y}h_2(y,d)\right]\\
&+\frac{M_1}{\alpha_1+\beta_1}\left[\frac{\alpha_1}{x-a}h_2(x,a)+\frac{\beta_1}{b-x}h_2(x,b)\right]\\
&+\frac{M_3}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\left[\frac{\alpha_1}{x-a}h_2(x,a)+\frac{\beta_1}{b-x}h_2(x,b)\right]\\
&\times\left[\frac{\alpha_2}{y-c}h_2(y,c)+\frac{\beta_2}{d-y}h_2(y,d)\right],
\end{aligned}$$

for all $x\in[a,b]$ *and* $y\in[c,d]$*, where*

$$M_1=\sup_{a<s<b}\left|\frac{\partial f(s,t)}{\Delta_1 s}\right|<\infty,\quad M_2=\sup_{c<t<d}\left|\frac{\partial f(s,t)}{\Delta_2 t}\right|<\infty\quad \text{and}$$

$$M_3=\sup_{a<s<b,c<t<d}\left|\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\right|<\infty.$$

Proof. From Lemma 4.1, we have that

$$\begin{aligned}
&f(x,y) - \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\Bigg]\\
=&\frac{\alpha_1}{(\alpha_1+\beta_1)(x-a)}\int_a^x\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Delta_1 s\\
&+\frac{\beta_1}{(\alpha_1+\beta_1)(b-x)}\int_x^b\int_c^d P_2(y,t)\frac{\partial f(\sigma_1(s),t)}{\Delta_2 t}\Delta_2 t\Delta_1 s
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\alpha_2}{(\alpha_2+\beta_2)(y-c)}\int_c^y\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t\\
&+\frac{\beta_2}{(\alpha_2+\beta_2)(d-y)}\int_y^d\int_a^b P_1(x,s)\frac{\partial f(s,\sigma_2(t))}{\Delta_1 s}\Delta_1 s\Delta_2 t\\
&+\int_c^d\int_a^b P_1(x,s)P_2(y,t)\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\Delta_1 s\Delta_2 t. \qquad (4.6)
\end{aligned}
$$

By taking the absolute values on both sides of (4.6) and applying item (v) of Theorem 2.2, we obtain

$$
\begin{aligned}
&\Bigg|f(x,y)-\frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\Bigg]\Bigg|\\
\le&\frac{\alpha_1 M_2}{(\alpha_1+\beta_1)(x-a)}\int_a^x\int_c^d |P_2(y,t)|\Delta_2 t\Delta_1 s\\
&+\frac{\beta_1 M_2}{(\alpha_1+\beta_1)(b-x)}\int_x^b\int_c^d |P_2(y,t)|\Delta_2 t\Delta_1 s\\
&+\frac{\alpha_2 M_1}{(\alpha_2+\beta_2)(y-c)}\int_c^y\int_a^b |P_1(x,s)|\Delta_1 s\Delta_2 t\\
&+\frac{\beta_2 M_1}{(\alpha_2+\beta_2)(d-y)}\int_y^d\int_a^b |P_1(x,s)|\Delta_1 s\Delta_2 t\\
&+M_3\int_c^d\int_a^b |P_1(x,s)||P_2(y,t)|\Delta_1 s\Delta_2 t\\
=&M_2\int_c^d |P_2(y,t)|\Delta_2 t+M_1\int_a^b |P_1(x,s)|\Delta_1 s\\
&+M_3\int_c^d\int_a^b |P_1(x,s)||P_2(y,t)|\Delta_1 s\Delta_2 t.
\end{aligned}
$$

That is,

(4.7)

$$
\begin{aligned}
&\Bigg|f(x,y)-\frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\\
&+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s
\end{aligned}
$$

$$+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s$$
$$\left.\left.+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\right]\right|$$
$$\leq M_1\int_a^b |P_1(x,s)|\Delta_1 s+M_2\int_c^d |P_2(y,t)|\Delta_2 t+M_3\int_c^d\int_a^b |P_1(x,s)||P_2(y,t)|\Delta_1 s\Delta_2 t.$$

The desired inequality follows from (4.7) by using the fact that

$$\int_a^b |P_1(x,s)|\Delta_1 s=\frac{1}{\alpha_1+\beta_1}\left[\frac{\alpha_1}{x-a}h_2(x,a)+\frac{\beta_1}{b-x}h_2(x,b)\right]$$

and

$$\int_c^d |P_2(y,t)|\Delta_2 t=\frac{1}{\alpha_2+\beta_2}\left[\frac{\alpha_2}{y-c}h_2(y,c)+\frac{\beta_2}{d-y}h_2(y,d)\right]. \qquad \square$$

The following corollary is Theorem 4 in [17] but we state it here for completion.

Corollary 4.1 ([17]). *Let $a,b,x,s\in\mathbb{T}_1$, $a<b$, $c,d,y,t\in\mathbb{T}_2$, $c<d$ and let $f:[a,b]\times[c,d]\to\mathbb{R}$ be such that the partial derivatives $\frac{\partial f(s,t)}{\Delta_1 s}$, $\frac{\partial f(s,t)}{\Delta_2 t}$, $\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}$ exist and are continuous on $[a,b]\times[c,d]$. Then we have the inequality*

$$\left|f(x,y)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(\sigma_1(s),\sigma_2(t))\Delta_2 t\Delta_1 s\right|$$
$$\leq\frac{M_2}{d-c}[h_2(y,c)+h_2(y,d)]+\frac{M_1}{b-a}[h_2(x,a)+h_2(x,b)]$$
$$+\frac{M_3}{(b-a)(d-c)}[h_2(x,a)+h_2(x,b)][h_2(y,c)+h_2(y,d)],$$

for all $x\in[a,b]$ and $y\in[c,d]$, where

$$M_1=\sup_{a<s<b}\left|\frac{\partial f(s,t)}{\Delta_1 s}\right|<\infty,\quad M_2=\sup_{c<t<d}\left|\frac{\partial f(s,t)}{\Delta_2 t}\right|<\infty\quad\text{and}$$
$$M_3=\sup_{a<s<b,c<t<d}\left|\frac{\partial^2 f(s,t)}{\Delta_2 t\Delta_1 s}\right|<\infty.$$

Proof. Let $\alpha_1=x-a$, $\beta_1=b-x$, $\alpha_2=y-c$ and $\beta_2=d-y$ in Theorem 4.1. $\square$

Now, we apply Theorem 4.1 to the continuous, discrete and quantum time scales to obtain some interesting inequalities which generalize the results in [17].

Corollary 4.2 (Continuous case). *If we let $\mathbb{T}_1=\mathbb{T}_2=\mathbb{R}$ in Theorem 4.1, then we have*

$$\left|f(x,y)-\frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\right[\frac{\alpha_2\alpha_1}{(y-c)(x-a)}\int_a^x\int_c^y f(s,t)dtds$$
$$+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(s,t)dtds+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(s,t)dtds$$

$$
\begin{aligned}
&+ \frac{\beta_2\beta_1}{(d-y)(b-x)} \int_x^b \int_y^d f(s,t)dtds\Bigg]\Bigg| \\
\leq& \frac{M_2}{2(\alpha_2+\beta_2)}[\alpha_2(y-c)+\beta_2(d-y)] + \frac{M_1}{2(\alpha_1+\beta_1)}[\alpha_1(x-a)+\beta_1(b-x)] \\
&+ \frac{M_3}{4(\alpha_1+\beta_1)(\alpha_2+\beta_2)}[\alpha_1(x-a)+\beta_1(b-x)][\alpha_2(y-c)+\beta_2(d-y)],
\end{aligned}
$$

for all $x \in [a,b]$ *and* $y \in [c,d]$, *where*

$$
M_1 = \sup_{a<s<b} \left|\frac{\partial f(s,t)}{\partial s}\right| < \infty, \quad M_2 = \sup_{c<t<d} \left|\frac{\partial f(s,t)}{\partial t}\right| < \infty \quad \text{and}
$$

$$
M_3 = \sup_{a<s<b, c<t<d} \left|\frac{\partial^2 f(s,t)}{\partial t \partial s}\right| < \infty.
$$

Corollary 4.3 (Discrete case). *If we let* $\mathbb{T}_1 = \mathbb{T}_2 = \mathbb{Z}$ *in Theorem* 4.1, *then we have*

$$
\begin{aligned}
&\Bigg|f(x,y) - \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)} \sum_{s=a}^{x-1}\sum_{t=c}^{y-1} f(s+1,t+1) \\
&+ \frac{\beta_2\alpha_1}{(d-y)(x-a)} \sum_{s=a}^{x-1}\sum_{t=y}^{d-1} f(s+1,t+1) + \frac{\alpha_2\beta_1}{(y-c)(b-x)} \sum_{s=x}^{b-1}\sum_{t=c}^{y-1} f(s+1,t+1) \\
&+ \frac{\beta_2\beta_1}{(d-y)(b-x)} \sum_{s=x}^{b-1}\sum_{t=y}^{d-1} f(s+1,t+1)\Bigg]\Bigg| \\
\leq& \frac{M_2}{2(\alpha_2+\beta_2)}[\alpha_2(y-c-1)+\beta_2(d-y+1)] \\
&+ \frac{M_1}{2(\alpha_1+\beta_1)}[\alpha_1(x-a-1)+\beta_1(b-x+1)] \\
&+ \frac{M_3}{4(\alpha_1+\beta_1)(\alpha_2+\beta_2)}[\alpha_1(x-a-1)+\beta_1(b-x+1)] \\
&\times [\alpha_2(y-c-1)+\beta_2(d-y+1)],
\end{aligned}
$$

for all $x \in \{a, a+1, \ldots, b-1, b\}$ *and* $y \in \{c, c+1, \ldots, d-1, d\}$, *where*

$$
M_1 = \sup_{a<s<b} |f(s+1,t) - f(s,t)| < \infty, \quad M_2 = \sup_{c<t<d} |f(s,t+1) - f(s,t)| < \infty
$$

and

$$
M_3 = \sup_{a<s<b, c<t<d} |f(s+1,t+1) - f(s+1,t) - f(s,t+1) + f(s,t)| < \infty.
$$

Corollary 4.4 (Quantum case). *If we let* $\mathbb{T}_1 = q_1^{\mathbb{N}_0}$, $q_1 > 1$ *and* $\mathbb{T}_2 = q_2^{\mathbb{N}_0}$, $q_2 > 1$ *in Theorem* 4.1, *then we have*

$$
\Bigg|f(x,y) - \frac{1}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\Bigg[\frac{\alpha_2\alpha_1}{(y-c)(x-a)} \int_a^x \int_c^y f(q_1 s, q_2 t) d_{q_2}t d_{q_1}s
$$

$$
\begin{aligned}
&+\frac{\beta_2\alpha_1}{(d-y)(x-a)}\int_a^x\int_y^d f(q_1s,q_2t)d_{q_2}td_{q_1}s\\
&+\frac{\alpha_2\beta_1}{(y-c)(b-x)}\int_x^b\int_c^y f(q_1s,q_2t)d_{q_2}td_{q_1}s\\
&+\frac{\beta_2\beta_1}{(d-y)(b-x)}\int_x^b\int_y^d f(q_1s,q_2t)d_{q_2}td_{q_1}s\Bigg]\Bigg|\\
\le&\frac{M_2}{\alpha_2+\beta_2}\left[\frac{\alpha_2(y-q_2c)+\beta_2(q_2d-y)}{1+q_2}\right]+\frac{M_1}{\alpha_1+\beta_1}\left[\frac{\alpha_1(x-q_1a)+\beta_1(q_1b-x)}{1+q_2}\right]\\
&+\frac{M_3}{(\alpha_1+\beta_1)(\alpha_2+\beta_2)}\left[\frac{\alpha_1(x-q_1a)+\beta_1(q_1b-x)}{1+q_2}\right]\left[\frac{\alpha_2(y-q_2c)+\beta_2(q_2d-y)}{1+q_2}\right],
\end{aligned}
$$

for all $x\in[a,b]$ and $y\in[c,d]$ with

$$
M_1=\sup_{a<s<b}\left|\frac{f(q_1s,t)-f(s,t)}{(q_1-1)s}\right|<\infty,\quad M_2=\sup_{c<t<d}\left|\frac{f(s,q_2t)-f(s,t)}{(q_2-1)t}\right|<\infty
$$

and $M_3=\displaystyle\sup_{a<s<b,c<t<d}\left|\frac{f(q_1s,q_2t)-f(q_1s,t)-f(s,q_2t)+f(s,t)}{(q_1-1)(q_2-1)st}\right|<\infty.$

5. Conclusion

In this work, we established some new Ostrowski-Grüss and 2D Ostrowski type inequalities on time scales involving a combination of Δ-integral means. In addition, we apply our results to the continuous, discrete and quantum calculus to obtain some novel inequalites in this direction.

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[1]Department of Mathematics and Computer Science,
Alabama State University,
Montgomery, AL 36101, USA
Email address: skermausour@alasu.edu

[2]Department of Mathematics,
Tuskegee University,
Tuskegee, AL 36088, USA
Email address: enwaeze@tuskegee.edu

*Corresponding Author

Kragujevac Journal of Mathematics
Volume 44(1) (2020), Pages 145–156.

TOPOLOGICAL HOCHSCHILD (σ,τ)-COHOMOLOGY GROUPS AND (σ,τ)-SUPER WEAK AMENABILITY OF BANACH ALGEBRAS

ABOLFAZL NIAZI MOTLAGH[1], MARYAM KHOSRAVI[2], AND ABASALT BODAGHI[3]

Abstract. In this work, we introduce the new cohomology groups depended on homomorphisms which are extensions of the topological Hochschild cohomology groups and investigate some of their properties that are analogue to the Hochschild cohomology groups. In addition, we use some homomorphisms on Banach algebras to define a new concept of amenability, namely, (σ,τ)-super weak amenability which is a generalization of the cyclic amenability. Finally, we show that this new notion on a commutative Banach algebra $\mathcal{A}$ is equivalent to the (σ,τ)-weak amenability, where σ and τ are some continuous homomorphisms on $\mathcal{A}$.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. Let the products of $a\in\mathcal{A}$ and $x\in\mathcal{X}$ be denoted by $a\cdot x$ and $x\cdot a$ which are both actions $\mathcal{A}$ over $\mathcal{X}$. A derivation $D:\mathcal{A}\to\mathcal{X}$ is a linear map which satisfies $D(ab)=a\cdot D(b)+D(a)\cdot b$ for all $a,b\in\mathcal{A}$. The derivation δ is said to be *inner* if there exists $x\in\mathcal{X}$ such that $\delta(a)=\delta_x(a)=a\cdot x-x\cdot a$ for all $a\in\mathcal{A}$. The linear space of all bounded (continuous) derivations and the linear subspace of inner derivations from $\mathcal{A}$ into $\mathcal{X}$ are denoted by $Z^1(\mathcal{A},\mathcal{X})$ and $N^1(\mathcal{A},\mathcal{X})$, respectively. We consider the quotient space $H^1(\mathcal{A},\mathcal{X})=Z^1(\mathcal{A},\mathcal{X})/N^1(\mathcal{A},\mathcal{X})$ which is called the *first Hochschild cohomology group* of $\mathcal{A}$ with coefficients in $\mathcal{X}$. A Banach algebra $\mathcal{A}$ is called *amenable* if every continuous derivation from $\mathcal{A}$ into every dual Banach $\mathcal{A}$-module is inner or equivalently $H^1(\mathcal{A},\mathcal{X}^*)=\{0\}$ for every Banach $\mathcal{A}$-bimodule $\mathcal{X}$ [8]. Also, $\mathcal{A}$ is said to be weakly amenable if $H^1(\mathcal{A},\mathcal{A}^*)=\{0\}$. Recall that a bounded derivation $D:\mathcal{A}\to\mathcal{A}^*$ is

Key words and phrases. (σ,τ)-derivation, (σ,τ)-inner derivation, (σ,τ)-amenability, (σ,τ)-contractibility, approximate identity, Banach algebra, Banach module.
2010 *Mathematics Subject Classification.* Primary: 46H25. Secondary: 47B47.
DOI 10.46793/KgJMat2001.145M
Received: November 14, 2017.
Accepted: March 07, 2018.

called *cyclic* if $\langle D(a), b\rangle + \langle D(b), a\rangle = 0$ for all $a, b \in \mathcal{A}$. A Banach algebra $\mathcal{A}$ is called *cyclically amenable* if every continuous derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner [9].

Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras. We denote by $\mathrm{Hom}(\mathcal{A}, \mathcal{B})$ the metric space of all bounded homomorphisms from $\mathcal{A}$ into $\mathcal{B}$, with the metric derived from the bounded linear operators from $\mathcal{A}$ into $\mathcal{B}$, and denote $\mathrm{Hom}(\mathcal{A}, \mathcal{A})$ by $\mathrm{Hom}(\mathcal{A})$. Let $\mathcal{X}$ be an $\mathcal{A}$-bimodule, and let $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. A bounded linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a (σ, τ)-*derivation* if

$$d(ab) = d(a) \cdot \sigma(b) + \tau(a) \cdot d(b) \quad (a, b \in \mathcal{A}).$$

Also, a bounded linear mapping $d : \mathcal{A} \to \mathcal{X}$ is called a (σ, τ)-*inner derivation* if there exists $x \in \mathcal{X}$ such that

$$d(a) = x \cdot \sigma(a) - \tau(a) \cdot x \quad (a \in \mathcal{A}).$$

Then, $\mathcal{A}$ is called (σ, τ)-*amenable* if every (σ, τ)-derivation $d : \mathcal{A} \to \mathcal{X}$ is (σ, τ)-inner. We denote the space of continuous (σ, τ)-derivations from $\mathcal{A}$ into $\mathcal{X}$ by $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ and the space of inner (σ, τ)-derivations by $B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$. Consider the space $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ as the quotient space $Z^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})/B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$. The space $H^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ is called the first (σ, τ)-cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}$.

Let $\sigma, \tau \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Then, $\mathcal{B}$ is a Banach $\mathcal{A}$-bimodule by the following module actions:

$$a \cdot b = \tau(a)b, \quad b \cdot a = b\sigma(a) \quad (a \in \mathcal{A}, b \in \mathcal{B}).$$

We denote the above $\mathcal{A}$-bimodule by $\mathcal{B}_{\sigma,\tau}$ and denote $\mathcal{B}_{\sigma,\tau}$ by $\mathcal{B}_\sigma$ if $\sigma = \tau$. A Banach algebra $\mathcal{A}$ is called (σ, τ)-*weakly amenable* if $H^1(\mathcal{A}, (\mathcal{A}_{(\sigma,\tau)})^*) = \{0\}$. These concepts are introduced and investigated in [3, 10, 11] and [12] (for the generalization of n-weak amenability refer to [4]). The (σ, τ)-weak amenability on the measure algebra $M(G)$, the group algebra $L^1(G)$ and the segal algebra $S^1(G)$, where G is a locally compact group are studied in [7]. For the module versions of these notions refer to [1] and [2].

In this work, we define the new cohomology groups which are the extensions of topological Hochschild cohomology groups and study some of their properties. In other words, we show that under which conditions $H^n_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*)$ can be vanishes, where $H^n_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}^*)$ is the n-th (σ, τ)-cohomology group of $\mathcal{A}$ with coefficients in $\mathcal{X}^*$. In last section, we define a notion of amenability related to homomorphisms and find some equivalent results to the (σ, τ)-weak amenability for Banach algebras. Finally, we bring a concrete example for this new notion on a special semigroup algebra.

2. (σ, τ)-Cohomology of Banach Algebras

Throughout this paper, all mapping are assumed to be bounded. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. From now on, we denote $\overbrace{\mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}}^{n-\text{times}}$ by $\mathcal{A}^n$. Suppose that $C^0(\mathcal{A}, \mathcal{X}) = \mathcal{X}$ and for $n \in \mathbb{N}$, define $C^n(\mathcal{A}, \mathcal{X})$ by the Banach space of all bounded n-linear mappings from $\mathcal{A}^n$ into $\mathcal{X}$ together with the multi-linear operator norm $\|f\| = \sup\{\|f(a_1,\dots,a_n)\| : a_i \in \mathcal{A}, \|a_i\| \le 1\}$. Consider the sequence of linear maps as follows:

$$0 \to C^0(\mathcal{A},\mathfrak{X}) \xrightarrow{\delta^0} C^1(\mathcal{A},\mathfrak{X}) \xrightarrow{\delta^1} C^2(\mathcal{A},\mathfrak{X}) \xrightarrow{\delta^2} \cdots,$$

where $(\delta^0 x)(a) = \tau(a)\cdot x - x \cdot \sigma(a)$ and for $n \in \mathbb{N}$,

$$\begin{aligned}(\delta^n T)(a_1,\dots,a_{n+1}) =& \tau(a_1)\cdot T(a_2,\dots,a_{n+1})\\ &+ \sum_{k=1}^{n}(-1)^k T(a_1,\dots,a_k a_{k+1},\dots,a_{n+1})\\ &+ (-1)^{n+1} T(a_1,\dots,a_n)\cdot\sigma(a_{n+1}),\end{aligned}$$

in which $T \in C^n(\mathcal{A},\mathfrak{X})$. The following result indicates the relation between δ^n's.

Lemma 2.1. *Suppose that $\mathcal{A}$ is a Banach algebra and $\mathfrak{X}$ is a Banach $\mathcal{A}$-bimodule. Then, for each n, $\delta^{n+1}\circ\delta^n = 0$.*

Proof. Considering $b^i_j = \begin{cases} a_j, & j < i, \\ a_j a_{j+1}, & j = i, \\ a_{j+1}, & j > i, \end{cases}$ we have

$$\begin{aligned}
&\delta^{n+1}\circ\delta^n T(a_1,\dots,a_{n+2})\\
=&\tau(a_1)\cdot\delta^n T(a_2,\dots,a_{n+2}) + \sum_{i=1}^{n}(-1)^i\delta^n T(b^i_1,\dots,b^i_{n+1})\\
&+(-1)^{n+2}\delta^n T(a_1,\dots,a_{n+1})\cdot\sigma(a_{n+2})\\
=&\tau(a_1)\cdot[\tau(a_2)\cdot T(a_3,\dots,a_{n+2}) + \sum_{i=2}^{n+1}(-1)^{i-1}T(b^i_2,\dots,b^i_{n+1})\\
&+(-1)^{n+1}T(a_2,\dots,a_{n+1})\cdot\sigma(a_{n+2})] + \sum_{i=1}^{n+1}(-1)^i[\tau(b^i_1)T(b^i_2,\dots,b^i_{n+1})\\
&+\sum_{j=1}^{n}(-1)^j T(b^i_1,\dots,b^i_j b^i_{j+1},\dots,b^i_{n+1}) + (-1)^{n+1}T(b^i_1,\dots,b^i_n)\cdot\sigma(b^i_{n+1})]\\
&+(-1)^{n+2}[\tau(a_1)\cdot T(a_2,\dots,a_{n+1}) + \sum_{j=1}^{n}(-1)^j T(b^j_1,\dots,b^j_n)\\
&+(-1)^{n+1}T(a_1,\dots,a_n)\cdot\sigma(a_{n+1})]\cdot\sigma(a_{n+2})\\
=&-\sum_{i=1}^{n+1}(-1)^i\tau(b^i_1)\cdot T(b^i_2,\dots,b^i_{n+1}) + (-1)^{n+1}\tau(a_1)\cdot T(a_2,\dots,a_{n+1})\cdot\sigma(a_{n+2})\\
&+\sum_{i=1}^{n+1}(-1)^i\tau(b^i_1)\cdot T(b^i_2,\dots,b^i_{n+1}) + \sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j}T(b^i_1,\dots,b^i_j b^i_{j+1},\dots,b^i_{n+1})\\
&+\sum_{i=1}^{n+1}[(-1)^{i+n+1}T(b^i_1,\dots,b^i_n)\cdot\sigma(b^i_{n+1}) + (-1)^{n+2}\tau(a_1)\cdot T(a_2,\dots,a_{n+1})\cdot\sigma(a_{n+2})]
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n+1}(-1)^{j+n+2}T(b_1^j,\dots,b_n^j)\cdot\sigma(b_{n+1}^j)\\
=&\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j}T(b_1^i,\dots,b_j^ib_{j+1}^i,\dots,b_{n+1}^i).
\end{aligned}
$$

With a combinatorial discussion, it can be concluded that the last summation is zero. ☐

We denote the kernel of δ^n and the image of δ^{n-1} by $Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ and $B^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ respectively. It follows from Lemma 2.1 that $B^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ is a subspace of $Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$. In other words, we can introduce the space $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ as the quotient space

$$Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})/B^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}).$$

Note that the elements of $Z^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ are continuous (σ,τ)-derivations and the elements of $B^1_{(\sigma,\tau)}(\mathcal{A},\mathcal{X})$ are inner (σ,τ)-derivations. In the upcoming result we show that under some conditions $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)$ can be zero.

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra with a left bounded approximate identity, $\mathcal{X}$ be a right annihilating Banach $\mathcal{A}$-bimodule. Then, for all $n>0$, $H^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)=\{0\}$.*

Proof. Let $(e_\nu)_{\nu\in\Lambda}$ be a left bounded approximate identity for $\mathcal{A}$. Assume that $f\in Z^n_{(\sigma,\tau)}(\mathcal{A},\mathcal{X}^*)$. Define $g_\nu\in C^{n-1}(\mathcal{A},\mathcal{X}^*)$ via

$$g_\nu(a_1,\dots,a_{n-1})=f(e_\nu,a_1,\dots,a_{n-1})\quad(a_1,\dots,a_{n-1}\in\mathcal{A},\ \nu\in\Lambda).$$

Note that $C^{n-1}(\mathcal{A},\mathcal{X}^*)$ is the dual of $C_{n-1}(\mathcal{A},\mathcal{X})=\underbrace{\mathcal{A}\otimes_p\cdots\otimes_p\mathcal{A}}_{n-1}\otimes_p\mathcal{X}$. Indeed, the mapping $C^{n-1}(\mathcal{A},\mathcal{X}^*)\to(C_{n-1}(\mathcal{A},\mathcal{X}))^*$ defined through

$$\phi\mapsto\tilde{\phi},\quad\tilde{\phi}(a_1\otimes\cdots\otimes a_{n-1}\otimes x)=\langle\phi(a_1,\dots,a_{n-1}),x\rangle$$

is an isometrical isomorphism of Banach spaces. Since $\|g_\nu\|\le\|f\|\|e_\nu\|$, the net $(g_\nu)_{\nu\in\Lambda}$ is bounded and so by the Banach-Alaoglu theorem, it has a subnet $(g_\nu)_{\nu\in\Omega}$ which is weak*-converging to a cochain g. Hence, for every $a_1,\dots,a_n\in\mathcal{A}$ and $x\in\mathcal{X}$, we obtain

$$\lim_\nu\langle g_\nu(a_1,\dots,a_{n-1}),x\rangle=\langle g(a_1,\dots,a_{n-1}),x\rangle.$$

Since $\mathcal{X}^*$ is a left annihilating $\mathcal{A}$-bimodule, we have

$$
\begin{aligned}
\delta^{n-1}g_\nu(a_1,\dots,a_n)&=\sum_{k=1}^{n-1}(-1)^kg_\nu(a_1,\dots,a_ka_{k+1},\dots,a_n)\\
&\quad+(-1)^ng_\nu(a_1,\dots,a_{n-1})\cdot\sigma(a_n)\\
&=\sum_{k=1}^{n-1}(-1)^kf(e_\nu,a_1,\dots,a_ka_{k+1},\dots,a_n)\\
&\quad+(-1)^nf(e_\nu,a_1,\dots,a_{n-1})\cdot\sigma(a_n)
\end{aligned}
$$

$$
\begin{aligned}
&+ f(e_\nu a_1, a_2, \ldots, a_n) - f(e_\nu a_1, a_2, \ldots, a_n)\\
&= -\, \delta^n f(e_\nu, a_1, \ldots, a_n) - f(e_\nu a_1, a_2, \ldots, a_n)\\
&= -\, f(e_\nu a_1, a_2, \ldots, a_n).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\lim_\nu \langle \delta^{n-1} g_\nu(a_1, \ldots, a_n), x\rangle =& \lim_\nu \langle \tau(a_1)\cdot g_\nu(a_2, \ldots, a_n)\\
&+ \sum_{k=1}^{n-1} (-1)^k g_\nu(a_1, \ldots, a_k a_{k+1}, \ldots, a_n)\\
&+ (-1)^n g_\nu(a_1, \ldots, a_{n-1})\cdot \sigma(a_n), x\rangle\\
=&\langle \tau\cdot(a_1) g_\nu(a_2, \ldots, a_n), x\rangle\\
=&\langle \delta^{n-1} g(a_1, \ldots, a_n), x\rangle,
\end{aligned}
$$

for all $x \in \mathfrak{X}$. That is, $\lim_\nu \delta^{n-1} g_\nu = \delta^{n-1} g$ in the weak*-topology on the space $C^n(\mathcal{A}, \mathfrak{X}^*)$. Also, $\lim_\nu e_\nu a_1 = a_1$ and hence

$$
\begin{aligned}
\langle f(a_1, \cdots, a_n), x\rangle &= \lim_\nu \langle f(e_\nu a_1, \ldots, a_n), x\rangle\\
&= -\lim_\nu \langle \delta^{n-1} g_\nu(a_1, \ldots, a_n), x)\rangle\\
&= \langle -\delta^{n-1} g(a_1, \ldots, a_n), x\rangle.
\end{aligned}
$$

Therefore, $f = \delta^{n-1}(-g) \in B^n_{(\sigma,\tau)}(\mathcal{A}, \mathfrak{X}^*)$. This completes the proof. $\square$

Let $\mathcal{A}$ be a Banach algebra and $\mathfrak{X}$ be a Banach $\mathcal{A}$-bimodule. Then, the Banach space $C^k(\mathcal{A}, \mathfrak{X})$ is an $\mathcal{A}$-bimodule with the following actions.

$$(a\cdot\phi)(a_1, \ldots, a_k) = a\cdot\phi(a_1, \ldots, a_k) \tag{2.1}$$

and

$$
\begin{aligned}
(\phi\cdot a)(a_1, \ldots, a_k) =& \phi(\tau^{-1}(a) a_1, a_2, \ldots, a_k)\\
&+ \sum_{i=1}^{k-1} (-1)^i \phi(\tau^{-1}(a), a_1, \ldots, a_i a_{i+1}, \ldots, a_k)\\
&+ (-1)^k \phi(\tau^{-1}(a), a_1, \ldots, a_{k-1})\tau(a_k) \quad (a \in \mathcal{A}\ \phi \in C^k(\mathcal{A}, \mathfrak{X})).
\end{aligned} \tag{2.2}
$$

Theorem 2.2. *Let $\mathcal{A}$ and $\mathfrak{X}$ be as the above, and $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. Then, for all $n \geq 1$, $H^{n+k}_{(\sigma,\tau)}(\mathcal{A}, \mathfrak{X}) = H^n_{(\sigma,\tau)}(\mathcal{A}, C^k(\mathcal{A}, \mathfrak{X}))$, where the module actions $\mathcal{A}$ over $C^k(\mathcal{A}, \mathfrak{X})$ are defined in* (2.1) *and* (2.2).

Proof. Let T^n be the canonical mapping from $C^{n+k}(\mathcal{A}, \mathfrak{X})$ into $C^n(\mathcal{A}, C^k(\mathcal{A}, \mathfrak{X}))$ defined by

$$((T^n\phi)(a_1, \ldots, a_n))(a_{n+1}, \ldots, a_{n+k}) = \phi(a_1, \ldots, a_{n+k}),$$

where $\phi \in C^k(\mathcal{A}, \mathfrak{X})$. It is easy to check that T^n is a linear isometry. Let Δ^n be the multilinear mapping corresponding to δ^n when the $\mathcal{A}$-bimodule $\mathfrak{X}$ is replaced by the

$\mathcal{A}$-bimodule $C^k(\mathcal{A},\mathfrak{X})$. Consider the following commutative diagram:

$$\begin{array}{ccccc} C^{n-1}(\mathcal{A},C^k(\mathcal{A},\mathfrak{X})) & \overset{\Delta^{n-1}}{\Longrightarrow} & C^{n}(\mathcal{A},C^k(\mathcal{A},\mathfrak{X})) & \overset{\Delta^{n}}{\Longrightarrow} & C^{n+1}(\mathcal{A},C^k(\mathcal{A},\mathfrak{X})) \\ \Downarrow & & \Downarrow & & \Downarrow \\ C^{n+k-1}(\mathcal{A},\mathfrak{X}) & \overset{\delta^{n+k-1}}{\Longrightarrow} & C^{n+k}(\mathcal{A},\mathfrak{X}) & \overset{\delta^{n+k}}{\Longrightarrow} & C^{n+k+1}(\mathcal{A},\mathfrak{X}). \end{array}$$

The above diagram necessitates that $H^{n+k}_{(\sigma,\tau)}(\mathcal{A},\mathfrak{X}) = H^n_{(\sigma,\tau)}(\mathcal{A},C^k(\mathcal{A},\mathfrak{X}))$. □

In analogy with Theorem 2.1, we have the next consequence, shows that (σ,τ)-amenability of a Banach algebra $\mathcal{A}$ implies that $H^n_{(\sigma,\tau)}(\mathcal{A},\mathfrak{X}^*) = \{0\}$.

Theorem 2.3. *Let $\mathcal{A}$ be (σ,τ)-amenable Banach algebra, where $\sigma,\tau \in \mathrm{Hom}(\mathcal{A})$. Then, $H^n_{(\sigma,\tau)}(\mathcal{A},\mathfrak{X}^*) = \{0\}$, for every Banach $\mathcal{A}$-bimodule $\mathfrak{X}$ and for every $n \in \mathbb{N}$.*

Proof. Set $Y = \underbrace{\mathcal{A}\otimes_p \cdots \otimes_p \mathcal{A}}_{(n-1)-\text{times}} \otimes_p \mathfrak{X}$. Then, Y is a Banach $\mathcal{A}$-bimodule under the following module multiplications:

$$\begin{aligned} (a_1\otimes\cdots\otimes a_n\otimes x)\cdot a = {} & a_1\otimes\cdots\otimes a_n\otimes x\cdot a, \\ a\cdot(a_1\otimes\cdots\otimes a_n\otimes x) = {} & (\tau^{-1}(a)a_1)\otimes\cdots\otimes a_n\otimes x \\ & +\sum_{j=1}^{n-1}(-1)^j\tau^{-1}(a)\otimes a_1\otimes\cdots\otimes a_ja_{j+1}\otimes\cdots\otimes a_n\otimes x \\ & +(-1)^n\tau^{-1}(a)\otimes a_1\otimes\cdots\otimes a_{n-1}\otimes\tau(a_n)x. \end{aligned}$$

Also, there exists an isometric $\mathcal{A}$-bimodule isomorphism from Y^* onto $C^n(\mathcal{A},\mathfrak{X}^*)$. Therefore, $H^{n+1}_{(\sigma,\tau)}(\mathcal{A},\mathfrak{X}^*) \approx H^1_{(\sigma,\tau)}(\mathcal{A},C^n(\mathcal{A},\mathfrak{X}^*)) \approx H^1_{(\sigma,\tau)}(\mathcal{A},Y^*) = \{0\}$. □

Here and subsequently, for a Banach algebra $\mathcal{A}$ we set $\mathcal{A}^2 = \{ab : a,b\in\mathcal{A}\}$. Suppose that $\mathfrak{J}$ is a closed ideal of a Banach algebra $\mathcal{A}$. The quotient $\frac{\mathcal{A}}{\mathfrak{J}}$ is again a Banach algebra under the usual product and quotient norm. We also suppose that $\mathfrak{J}^2 = \mathfrak{J}$. Let $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$ such that $\sigma(\mathfrak{J})\subseteq\mathfrak{J}$ and $\tau(\mathfrak{J})\subseteq\mathfrak{J}$ and d be a (σ,τ)-derivation on $\mathcal{A}$. It is easy to check that $d(\mathfrak{J})\subseteq\mathfrak{J}$. Assume that $\sigma^*,\tau^* : \frac{\mathcal{A}}{\mathfrak{J}} \to \frac{\mathcal{A}}{\mathfrak{J}}$ are the natural homomorphisms correspond to σ and τ, respectively. Then, the mapping

$$\begin{aligned} d_0 : \frac{\mathcal{A}}{\mathfrak{J}} &\to \frac{\mathcal{A}}{\mathfrak{J}}, \\ a+\mathfrak{J} &\mapsto d(a)+\mathfrak{J} \end{aligned}$$

is a well-defined and a (σ^*,τ^*)-derivation. We have the following diagram.

$$\begin{array}{ccc} \mathcal{A} & \overset{d}{\to} & \mathcal{A} \\ p\downarrow & & \downarrow p \\ \frac{\mathcal{A}}{\mathfrak{J}} & \overset{d_0}{\to} & \frac{\mathcal{A}}{\mathfrak{J}} \end{array}$$

where p is the natural projection from A onto $\frac{\mathcal{A}}{\mathfrak{J}}$. The preceding discussion leads us to this challenge: With the above notations in which d_0 is an arbitrary (σ^*,τ^*)-derivation of $\frac{\mathcal{A}}{\mathfrak{J}}$, is there any (σ,τ)-derivation d of $\mathcal{A}$ which makes the above diagram commute? In other words, does d_0 *lift* to a derivation of $\mathcal{A}$? Considering $\mathcal{A}$ as a linear space (and ignoring topology), the subspace $\mathfrak{J}$ has a complementary subspace $\mathfrak{I}$. The restriction of p to $\mathfrak{I}$ is a linear isomorphism from $\mathfrak{I}$ onto $\frac{\mathcal{A}}{\mathfrak{J}}$, and so has an inverse q. We can consider q as a linear mapping from $\frac{\mathcal{A}}{\mathfrak{J}}$ into $\mathcal{A}$. Set $\xi = q\circ d_0\circ p$. Then, the mapping ξ is a linear from $\mathcal{A}$ into $\mathcal{A}$ so that lifts d_0. Putting $\rho = \delta^1\xi$ in $B^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}) \subseteq Z^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})$, we have

$$\begin{aligned} p(\rho(a,b)) &= p(\tau(a)\xi(b) - \xi(ab) + \xi(a)\sigma(b)) \\ &= p\tau(a)p\xi(b) - p\xi(ab) + p\xi(a)p\sigma(b) \\ &= \tau^* p(a)d_0p(b) - d_0p(ab) + d_0p(a)\sigma^* p(b) \\ &= 0, \end{aligned}$$

for all $a,b\in\mathcal{A}$. So, ρ takes all its values in the kernel $\mathfrak{J}$ of p and hence $\rho\in Z^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})$. Now, let $\eta\in C^1(\mathcal{A},\mathcal{A})$. We claim that η lifts d_0 if and only if $\eta-\xi\in C^1(\mathcal{A},\mathfrak{J})$. To prove this, note that η lifts d_0 if and only if $p\circ\eta = d_0\circ p = p\circ\xi$ which is equivalent to $p(\eta-\xi)=0$ and it means that $\eta-\xi\in C^1(\mathcal{A},\mathfrak{J})$. Thus, we can conclude that d_0 lifts to a (σ,τ)-derivation of $\mathcal{A}$ if and only if $\delta^1\xi\in B^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})$. Summing up we have the following theorem.

Theorem 2.4. *Let $\mathfrak{J}$ be a complemented closed ideal in a Banach algebra $\mathcal{A}$ and $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$ which leaves $\mathfrak{J}$ invariant. If $H^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})=0$, then every (σ^*,τ^*)-derivation of the quotient Banach algebra $\frac{\mathcal{A}}{\mathfrak{J}}$ lifts to a (σ,τ)-derivation of $\mathcal{A}$.*

We also have the partial converse of Theorem 2.4 as follows.

Theorem 2.5. *Let $\mathcal{A}$ be a Banach algebra and $\sigma,\tau\in\mathrm{Hom}(\mathcal{A})$ such that $H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})=0$. If $\mathfrak{J}$ is a closed ideal which is invariant under σ and τ, $\mathfrak{J}^2=\mathfrak{J}$, and every (σ^*,τ^*)-derivation of $\frac{\mathcal{A}}{\mathfrak{J}}$ lifts to a (σ,τ)-derivation of $\mathcal{A}$, then $H^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})=0$.*

Proof. Let $\rho\in Z^2_{(\sigma,\tau)}(\mathcal{A},\mathfrak{J})$ $(\subseteq Z^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A}))$. Then, from $H^2_{(\sigma,\tau)}(\mathcal{A},\mathcal{A})=0$, it deduces that $\rho=\delta^1\xi$ for some ξ in $C^1(\mathcal{A},\mathcal{A})$. Since $\delta^1\xi$ takes all its values in $\mathfrak{J}$, we have

$$\xi(j_1j_2) = \tau(j_1)\xi(j_2) + \xi(j_1)\sigma(j_2) - (\delta^1\xi)(j_1,j_2)\in\mathfrak{J},$$

for all $j_1,j_2\in\mathfrak{J}$. It follows that $\xi(\mathfrak{J})\subseteq\mathfrak{J}$. So, ξ induces a linear mapping

$$\begin{aligned} d_0 : \frac{\mathcal{A}}{\mathfrak{J}} &\;\to\; \frac{\mathcal{A}}{\mathfrak{J}}, \\ a+\mathfrak{J} &\;\mapsto\; \xi(a)+\mathfrak{J}. \end{aligned}$$

Thus, for all $a,b\in\mathcal{A}$, we get

$$\tau(a)\xi(b) - \xi(ab) + \xi(a)\sigma(b) = (\delta^1\xi)(a,b)\in\mathfrak{J}$$

and

$$p(\tau(a))p(\xi(b)) - p(\xi(ab)) + p(\xi(a))p(\sigma(b)) = 0,$$

$$\tau^*(p(a))d_0(p(b)) - d_0(p(ab)) + d_0(p(a))\sigma^*(p(b)) = 0.$$

This shows that d_0 is a (σ^*, τ^*)-derivation of $\frac{\mathcal{A}}{\mathcal{J}}$. By hypothesis, d_0 lifts to a derivation η on $\mathcal{A}$. So, $p \circ \eta = d_0 \circ p = p \circ \xi$ and hence $\eta - \xi \in C^1(\mathcal{A}, \mathcal{J})$. Therefore, $\rho = \delta^1\xi - \delta^1\eta \in B^1_{(\sigma,\tau)}(\mathcal{A}, \mathcal{J})$ and $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{J}) = 0$. □

An *extension* of a Banach algebra $\mathcal{B}$ is a short-exact sequence of the form

$$\{0\} \to \ker\psi \to \mathcal{A} \xrightarrow{\psi} \mathcal{B} \to \{0\},$$

where $\mathcal{A}$ is a Banach algebra and $\psi : \mathcal{A} \to \mathcal{B}$ is a continuous, surjective algebra homomorphism. The extension is called *singular* if $\ker\psi$ has the trivial product, that is, $ab = 0$ for each $a, b \in\ker\psi$. We say that the extension *splits strongly* (resp. *admissible*) if ψ has a right inverse which is a continuous algebra homomorphism (resp. is bounded and linear).

Let $\mathcal{A}$ be a Banach algebra, $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule, and $T \in Z^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$. Put $\mathcal{U}_T = \mathcal{A} \oplus_T \mathcal{X} = \{(a, x) : a \in \mathcal{A}, x \in \mathcal{X}\}$. Then, $\mathcal{U}_T$ equipped with the following product and norm is a Banach algebra:

$$\|(a, x)\| = \|a\| + \|x\|,$$

$$(a, x)(b, y) = (ab, \tau(a) \cdot y + x \cdot \sigma(b) + T(a, b)).$$

Further, $\sum(\mathcal{U}_T : \mathcal{X})$ is a singular, admissible Banach extension of $\mathcal{A}$.

The method of proof for the next consequence is similar the way for $H^2(\mathcal{A}, \mathcal{X})$ which was proved in [6, Theorem 2.8.12], so is omitted.

Theorem 2.6. *The map $T \mapsto \sum(\mathcal{U}_T, \mathcal{X})$ from $Z^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ induces a map from $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$ to the family of equivalence classes of singular, admissible Banach extension of $\mathcal{A}$ by $\mathcal{X}$ with respect to strong equivalence.*

Theorem 2.7. $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = \{0\}$*, when* $H^2(\mathcal{A}, \mathcal{X}) = \{0\}$*.*

Proof. If $H^2(\mathcal{A}, \mathcal{X}) = \{0\}$, then each singular, admissible Banach extension of $\mathcal{A}$ by $\mathcal{X}$ splits strongly. Take $T \in Z^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X})$. Then, $\sum(\mathcal{U}_T, \mathcal{X})$ splits. So, there is a homomorphism $\theta : \mathcal{A} \to \mathcal{U}_T$ such that $\theta(a) = (a, -Sa)$ $(a \in \mathcal{A})$ for some $S \in C(\mathcal{A}, \mathcal{X})$ and hence $T = \delta^1 S$. Therefore, $H^2_{(\sigma,\tau)}(\mathcal{A}, \mathcal{X}) = \{0\}$. □

3. (σ, τ)-Super Weak Amenability of Banach Algebras

In this section, we introduce a concept of amenability which is a generalization of cyclic amenability on Banach algebras that help us to investigate the (σ, τ)-weak amenability of Banach algebras in more details.

Definition 3.1. Let $\mathcal{A}$ be a Banach algebra, and $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$. Then, $\mathcal{A}$ is called (σ, τ)-supper weakly amenable if for every Banach algebra $\mathcal{B}$ and every $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$, whenever $D : \mathcal{A} \to \mathcal{B}^*_\varphi$ is a (σ, τ)-derivation, then the equality $\langle D(a), \varphi(a)\rangle = 0$ holds for all $a \in \mathcal{A}$.

It is easily verified that $\mathcal{A}$ is (σ,τ)-super weakly amenable if and only if for every Banach algebra $\mathcal{B}$ and every $\varphi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$ and each (σ,τ)-derivation $D : \mathcal{A} \to \mathcal{B}^*_\varphi$ with the following property is (σ,τ)-inner

$$\langle D(a), \varphi(b)\rangle + \langle D(b), \varphi(a)\rangle = 0 \quad (a, b \in \mathcal{A}).$$

It is obvious that:

- every (σ,τ)-supper weakly amenable Banach algebra is cyclically amenable;
- every (σ,τ)-weakly amenable Banach algebra $\mathcal{A}$ is (σ,τ)-supper weakly amenable when $\mathcal{B} = \mathcal{A}$ and φ is the identity map on $\mathcal{A}$. The converse is not true in general even for the special cases weak amenabiliy and cyclic amenability. In fact, any singly generated Banach algebra is cyclically amenable, as can be seen by looking at the values a continuous cyclic derivation must take on powers of the generator, while there are many examples of singly generated Banach algebras (even finite dimensional ones) that support bounded, non-zero point derivations, and hence are not weakly amenable [5].

However, we shall to show that two notions (σ,τ)-weak amenability and (σ,τ)-supper weak amenability coincide on commutative Banach algebras (Corollary 3.1). Before proceeding to the main results in this section, we bring the following lemma which is useful to achieve our purpose.

Lemma 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be Banach algebras, $\sigma, \tau \in \mathrm{Hom}(\mathcal{A})$ and $\varphi \in \mathrm{Hom}(\mathcal{A},\mathcal{B})$. If $d : \mathcal{A} \to \mathcal{B}^*_\varphi$ is a (σ,τ)-derivation, then $D : \mathcal{A} \to \mathcal{A}^*$ is a bounded (σ,τ)-derivation which is defined through*

$$\langle D(a), b\rangle := \langle d(a), \varphi(b)\rangle \quad (a, b \in \mathcal{A}).$$

Proof. Obviously, $D : \mathcal{A} \to \mathcal{A}^*$ is a bounded linear map. Also,

$$\begin{aligned}
\langle D(ab), c\rangle &= \langle d(ab), \varphi(c)\rangle \\
&= \langle d(a)\cdot\sigma(b) + \tau(a)\cdot d(b), \varphi(c)\rangle \\
&= \langle d(a)\cdot\varphi(\sigma(b)), \varphi(c)\rangle + \langle \varphi(\tau(a))\cdot d(b), \varphi(c)\rangle \\
&= \langle d(a), \varphi(\sigma(b)c)\rangle + \langle d(b), \varphi(c\tau(a))\rangle \\
&= \langle D(a), \sigma(b)c\rangle + \langle D(b), c\tau(a)\rangle \\
&= \langle D(a)\cdot\sigma(b), c\rangle + \langle \tau(a)\cdot D(b), c\rangle \\
&= \langle D(a)\cdot\sigma(b) + \tau(a)\cdot D(b), c\rangle \quad (a, b, c \in \mathcal{A}).
\end{aligned}$$

Therefore, D is a (σ,τ)-derivation. □

In [12], the authors have used from the Banach algebra introduced by Yong Zhang [13] to introduce a Banach algebra which is (σ,τ)-weak amenable for all homomorphisms σ,τ but not (σ,τ)-amenable for some homomorphisms σ and τ. In the oncoming example, we show that the menioned Banach algebra is a (σ,τ)-supper weakly amenable Banach algebra in which τ is the identity map.

Example 3.1. Firstly, we consider a product on Banach algebra $\ell^1 = l^1(\mathbb{N})$ as follows:

$$a \cdot b = a(1)b \qquad (a, b \in \ell^1).$$

Note that ℓ^1 has a left identity e_1 defined by

$$e_1(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}$$

The dual space $(\ell^1)^* = \ell^\infty$ is a ℓ^1-bimodule via the ordinary actions as follows:

$$a \cdot f = f(a)e_1, \quad f \cdot a = a(1)f \qquad (a \in \ell(S), f \in \ell^\infty),$$

where e_1 is regarded as an element of ℓ^∞. Next let $\sigma : \ell^1 \to \ell^1$ be a bounded homomorphism. We have $a(1)\sigma(b) = \sigma(a \cdot b) = \sigma(a) \cdot \sigma(b) = \sigma(a)(1)\sigma(b)$ and so $\sigma(b)(a(1) - \sigma(a)(1)) = 0$ for all $a, b \in \mathbb{N}$. Since $\sigma \neq 0$, we get

$$\sigma(a)(1) = a(1) \quad (a \in \ell^1).$$

Let $\mathcal{B}$ be an arbitrary Banach algebra and $\varphi \in \mathrm{Hom}(\ell^1, \mathcal{B})$. If $d : \ell^1 \to \mathcal{B}^*_\varphi$ is a bounded (σ, τ)-derivation, then by Lemma 3.1 the linear map $D : \ell^1 \to (\ell^1)^*$ defined through $\langle D(a), b\rangle = \langle d(a), \varphi(b)\rangle$, $a, b \in \ell^1$ is a (σ, τ)-derivation. Due to the (σ, τ)-weak amenability of ℓ^1 [12], there exists $f \in (\ell^1)^*$ such that $D(a) = f \cdot \sigma(a) - \tau(a) \cdot f$, $a \in \ell^1$. Hence,

$$\begin{aligned} \langle d(a), \varphi(a)\rangle &= \langle D(a), a\rangle \\ &= \langle f \cdot \sigma(a) - \tau(a) \cdot f, a\rangle \\ &= \langle f, \sigma(a) \cdot a - a \cdot \tau(a)\rangle \\ &= \langle f(\sigma(a))(1)a - a(1)\tau(a)\rangle \\ &= \langle f, a(1)a - a(1)\tau(a)\rangle \\ &= a(1)\langle f, a - \tau(a)\rangle \\ &= 0 \quad (a \in \ell^1). \end{aligned}$$

Here, we state the relationship between (σ, σ)-weak amenability and (σ, σ)-supper weak amenability on Banach algebras.

Proposition 3.1. *Let $\mathcal{A}$ be a Banach algebra and $\sigma \in \mathrm{Hom}(\mathcal{A})$ such that the range of σ commute with $\mathcal{A}$. Then, $\mathcal{A}$ is (σ, σ)-weakly amenable if and only if $\mathcal{A}$ is (σ, σ)-supper weakly amenable.*

Proof. We firstly assume that $\mathcal{A}$ is (σ, σ)-supper weakly amenable. Set $\mathcal{B} = \mathcal{A}$ and $\varphi = id$ (the identity map). Let $D : \mathcal{A} \to (\mathcal{A}_{id})^*$ be a bounded derivation. It follows from the (σ, σ)-super weak amenability of $\mathcal{A}$ that $\langle D(a), a\rangle = 0\, (a \in \mathcal{A})$ and hence D is (σ, σ)-inner.

Conversely, suppose that $\mathcal{A}$ is (σ, σ)-weakly amenable. Consider an arbitrary Banach algebra $\mathcal{B}$ and a $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathcal{B})$. Let $d : \mathcal{A} \to \mathcal{B}^*_\varphi$ be a (σ, σ)-derivation. By Lemma 3.1, the linear map $D : \mathcal{A} \to \mathcal{A}^*$ defined via $\langle D(a), b\rangle := \langle d(a), \varphi(b)\rangle$ is a (σ,σ)-derivation and so it is (σ,σ)-inner. Thus, there exists $f \in \mathcal{A}^*$ such that $D(a) = f \cdot \sigma(a) - \tau(a) \cdot f$. Hence, we have

$$\begin{aligned}\langle d(a), \varphi(a)\rangle &= \langle D(a), a\rangle \\ &= \langle f \cdot \sigma(a) - \sigma(a) \cdot f, a\rangle \\ &= \langle f, \sigma(a)a - a\sigma(a)\rangle = 0,\end{aligned}$$

for all $a, b \in \mathcal{A}$. This finishes the proof. □

One should remember that a commutative Banach algebra is weakly amenable if and only if it is cyclically amenable. We generalize this result as follows.

Corollary 3.1. *Let $\mathcal{A}$ be a commutative Banach algebra and $\sigma \in \mathrm{Hom}(\mathcal{A})$. Then, $\mathcal{A}$ is (σ,σ)-weakly amenable if and only if $\mathcal{A}$ is (σ,σ)-supper weakly amenable.*

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[1]Department of Mathematics,
Faculty of basic sciences,
University of Bojnord,
P. O. Box 1339, Bojnord, Iran
Email address: a.niazi@ub.ac.ir and niazimotlagh@gmail.com

[2]Department of Pure Mathematics,
Faculty of Mathematics and Computer,
Shahid Bahonar University of Kerman, Kerman, Iran
Email address: Khosravi_m@uk.ac.ir

[3]Department of Mathematics,
Garmsar Branch,
Islamic Azad University, Garmsar, Iran
Email address: abasalt.bodaghi@gmail.com

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