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SOLUTION AND STABILITY OF A CUBIC TYPE FUNCTIONAL EQUATION: USING DIRECT AND FIXED POINT METHODS

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Abstract. In this concept, we investigate the generalized Ulam-Hyers-Rassias stability for the new type of cubic functional equation of the form

$$g\left(ax_1+bx_2+2cx_3\right)+g\left(ax_1+bx_2-2cx_3\right)+8\ a^3g(x_1)+8\ b^3g(x_2)$$
$$=2g(ax_1+bx_2)+4\left(g(ax_1+cx_3)+g(ax_1-cx_3)+g(bx_2+cx_3)+g(bx_2-cx_3)\right)$$

by using direct and fixed point alternative.

1. Introduction

Sometime in modeling applied problems there may be a degree of uncertainty in the parameters used in the model or some measurements may be imprecise. Due to such features, we are tempted to consider the study of the functional equation in the alternative settings. One of the most interesting questions in the theory of functional equations, concerning the famous Ulam [38] stability problem, is as follows: when is it true that a mapping satisfying a functional equation approximately must be close to exact solution of the given functional equation?

In 1940, S. M. Ulam [39] raised the following question. Under what conditions does there exist an additive mapping near an approximately additive linear mappings? The case of approximately additive function was solved by D. H. Hyers [15] under certain assumptions. In 1978, a generalized version of the Theorem of Hyers for approximately linear mappings was given by Th. M. Rassias [34]. A number of mathematicians were attracted by the result of Th. M. Rassias. The stability concept that was introduced and investigated by Rassias is called the Hyers-Ulam-Rassias stability. One of the

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most famous functional equation is the additive functional equation

$$f(x+y) = f(x) + f(y).$$

In 1821, it was first solved by A. L. Cauchy in the class of the continuous real valued functions. It is often called an additive Cauchy functional equation in honor of A. L. Cauchy [39]. The theory of additive functional equations in frequently applied to the development of the theories of the other functional equations. Consider the functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y).$$

The quadratic function $f(x) = cx^2$ is a solution of this functional equation, so one can usually say that the above functional equation is quadratic [3–6, 20, 21, 27, 30].

Recently, Bae, Lee and Park [32] established some stability results for the functional equation

$$kf(x+ky) + f(kx-y) = \frac{k(k^2-1)}{2}\left(f(x+y) + f(x-y)\right) + (k \pm 1)f(y),$$

where $k \geq 2$ is a fixed integer, in the setting of non-Archimedean L-fuzzy normed spaces.

The Hyers-Ulam stability problem of the quadratic functional equation was first proved by F. Skof [37] for functions between a normed space and a Banach space. After wards, the result was extended by P. W. Cholewa [11] and S. Czerwisk [12].

The cubic function $g(x) = cx^3$ satisfies the functional equation

$$g(2x+y) + g(2x-y) = 2g(x+y) + 2g(x-y) + 12g(x). \tag{1.1}$$

Hence, throughout this concept, we promise that equation (1.1) is called a cubic functional equation and every solution of equation (1.1) is said to be a cubic function. The stability result of equation (1.1) was obtained by K. W. Jun and H. M. Kim [17].

In this concept, we present the general solution and generalized Ulam-Hyers-Rassias stability of the new type of cubic functional equation of the form

$$\begin{aligned} &g(ax_1 + bx_2 + 2cx_3) + g\left(ax_1 + bx_2 - 2cx_3\right) + 8a^3 g(x_1) + 8b^3 g(x_2) \\ =&2g(ax_1 + bx_2) + 4\left(g(ax_1 + cx_3) + g(ax_1 - cx_3) + g(bx_2 + cx_3) + \right. \\ &\left. g(bx_2 - cx_3)\right). \end{aligned} \tag{1.2}$$

The main goal of this concept is to obtain the generalized Hyers-Ulam-Rassias stability result for the functional equation (1.2) by using the direct and fixed point alternative [7, 13, 18, 19, 22–26, 29, 33, 35, 36] in [1, 2, 8–10, 14, 16, 28, 31].

For completeness, we will first investigate solution of the functional equation (1.2).

Proposition 1.1. *Let X and Y be real vector spaces. A function $g : X \to Y$ satisfies the functional equation* (1.1) *if and only if $g : X \to Y$ also satisfies the functional equation* (1.2).

Proof. Substituting (x,y) by $(0,0)$ in (1.1) yields $g(0)=0$. Replacing (x,y) by $(0,x)$ in (1.1), gives $g(-x)=-g(x)$ for all $x\in X$, which implies that g is odd. Now, replacing (x,y) by $(x,0)$ in (1.1), we obtain $g(2x)=8g(x)$, and replacing (x,y) by (x,x) in (1.1), we get that $g(3x)=27g(x)$ for all $x\in X$. Substituting (x,y) by $(ax,ax+by)$ in (1.1), we have

$$\begin{aligned} g(2ax+ax+by)+g(2ax-(ax+by)) =&2g(ax+ax+by)\\ &+2g(ax-(ax+by))+12g(ax),\\ g(3ax+by)+g(ax-by) =&2g(2ax+by)+2g(-by)+12g(ax), \end{aligned}$$

$$g(3ax+by)+g(ax-by) =2g(2ax+by)-2g(by)+12g(ax), \tag{1.3}$$

for all $x,y\in X$. Replacing (x,y) by $(ax,ax-by)$ in (1.1), we get

$$\begin{aligned} g(2ax+ax-by)+g(2ax-(ax-by)) =&2g(ax+ax-by)\\ &+2g(ax-(ax-by))+12g(ax), \end{aligned}$$

$$g(3ax-by)+g(ax+by) =2g(2ax-by)+2g(by)+12g(ax), \tag{1.4}$$

for all $x,y\in X$. Adding (1.3) and (1.4) and then using (1.1), we see that

$$\begin{aligned} &g(3ax+by)+g(ax-by)+g(3ax-by)+g(ax+by)\\ =&2g(2ax+by)-2g(by)+12g(ax)+2g(2ax-by)+2g(by)+12g(ax),\\ &g(3ax+by)+g(3ax-by)+g(ax+by)+g(ax-by)\\ =&2g(2ax+by)+2g(2ax-by)+24g(ax),\\ &g(3ax+by)+g(3ax-by)+g(ax+by)+g(ax-by)\\ =&2\left(2g(ax+by)+2g(ax-by)+12g(ax)\right)+24g(ax),\\ &g(3ax+by)+g(3ax-by)+g(ax+by)+g(ax-by)\\ =&4g(ax+by)+4g(ax-by)+24g(ax)+24g(ax),\\ &g(3ax+by)+g(3ax-by)+g(ax+by)+g(ax-by)\\ =&4g(ax+by)+4g(ax-by)+48g(ax), \end{aligned}$$

$$g(3ax+by)+g(3ax-by)=3g(ax+by)+3g(ax-by)+48g(ax), \tag{1.5}$$

for all $x,y\in X$. Now, replacing (ax,by) by $(ax+by,ax-by)$ in (1.5), respectively, we have

$$\begin{aligned} &g(3(ax+by)+(ax-by))+g(3(ax+by)-(ax-by))\\ =&3g((ax+by)+(ax-by))+3g((ax+by)-(ax-by))+48g(ax+by),\\ &g(3ax+3by+ax-by)+g(3ax+3by-ax+by)\\ =&3g(2ax)+3g(2by)+48g(ax+by),\\ &g(4ax+2by)+g(2ax+4by)=3g(2ax)+3g(2by)+48g(ax+by), \end{aligned}$$

for all $x,y\in X$, which, in view of the identity $g(2x)=8g(x)$, reduces to

$$g(4ax+2by)+g(2ax+4by)$$

$$=3\,(8g(ax)) + 3\,(8g(by)) + 48g(ax+by),$$
$$8g(2ax+by) + 8g(ax+2by) = 24g(ax) + 24g(by) + 48g(ax+by),$$

and dividing by 8, we get

$$g(2ax+by) + g(ax+2by) = 3g(ax) + 3g(by) + 6g(ax+by), \tag{1.6}$$

for all $x, y \in X$. Now, replacing (ax, by) by $(ax+3by, ax-3by)$ in (1.6), we arrive to

$$\begin{aligned}
&g(2(ax+3by) + (ax-3by)) + g\,(ax+3by+2(ax-3by)) \\
=&3g(ax+3by) + 3g(ax-3by) + 6g(ax+3by+ax-3by), \\
&g(2ax+6by+ax-3by) + g\,(ax+3by+2ax-6by) \\
=&3g(ax+3by) + 3g(ax-3by) + 6g(2ax), \\
&g\,(3ax+3by) + g\,(3ax-3by) = 3g(ax+3by) + 3g(ax-3by) + (6\times 8)g(ax), \\
&27g\,(ax+by) + 27g\,(ax-by) = 3g(ax+3by) + 3g(ax-3by) + 48g(ax),
\end{aligned}$$
$$9g\,(ax+by) + 9g\,(ax-by) = g(ax+3by) + g(ax-3by) + 16g(ax), \tag{1.7}$$

for all $x, y \in X$. Let us interchange ax in by and by in ax in (1.7) to get the identities

$$9g\,(ax+by) + 9g\,(by-ax) = g(3ax+by) + g(by-3ax) + 16g(by),$$
$$9g\,(ax+by) - 9g\,(ax-by) = g(3ax+by) - g(3ax-by) + 16g(by), \tag{1.8}$$

for all $x, y \in X$. Then, by adding (1.7) and (1.8), we get

$$\begin{aligned}
&9g\,(ax+by) + 9g\,(ax-by) + 9g\,(ax+by) - 9g\,(ax-by)\,v \quad (1.9)\\
=&g(ax+3by)x + g(ax-3by) + 16g(ax) \\
&+ g(3ax+by) - g(3ax-by) + 16g(by), \\
18g(ax+by) =&g(ax+3by) + g(ax-3by) + g(3ax+by) \\
&- g(3ax-by) + 16g(ax) + 16g(by), \quad (1.10)
\end{aligned}$$

for all $x, y \in X$. Now, we interchange ax with by and by with ax in (1.5), respectively we get

$$g\,(3by+ax) + g\,(3by-ax) = 3g(by+ax) + 3g(by-ax) + 48g(by),$$
$$g\,(ax+3by) - g\,(ax-3by) = 3g(ax+by) - 3g(ax-by) + 48g(by), \tag{1.11}$$

for all $x, y \in X$. Hence, according to (1.5) and (1.11), we obtain

$$g\,(3ax+by) + g\,(3ax-by) = 3g(ax+by) + 3g(ax-by) + 48g(ax),$$
$$g\,(ax+3by) - g\,(ax-3by) = 3g(ax+by) - 3g(ax-by) + 48g(by).$$

Adding the above equations we get

$$\begin{aligned}
&g\,(3ax+by) + g\,(3ax-by) + g\,(ax+3by) - g\,(ax-3by) \\
=&6g(ax+by) + 48g(ax) + 48g(by), \\
6g(ax+by) =&g(3ax+by) + g(3ax-by) + g(ax+3by) - g(ax-3by)
\end{aligned}$$

$$-48g(ax)-48g(by), \tag{1.12}$$

for all $x, y \in X$. Again by adding (1.10) and (1.12), we get

$$\begin{aligned}
18g(ax+by) =& g(ax+3by)+g(ax-3by)+g(3ax+by)\\
&-g(3ax-by)-16g(ax)-16g(by),\\
6g(ax+by) =& g(3ax+by)+g(3ax-by)+g(ax+3by)\\
&-g(ax-3by)-48g(ax)-48g(by),\\
24g(ax+by) =& 2g(ax+3by)+2g(3ax+by)\\
&-32g(ax)-32g(by),\\
12g(ax+by) =& g(ax+3by)+g(3ax+by)-16g(ax)-16g(by),
\end{aligned}$$

$$g(ax+3by)+g(3ax+by) = 12g(ax+by)+16g(ax)+16g(by), \tag{1.13}$$

for all $x, y \in X$. Taking (1.5), we have

$$\begin{aligned}
&g(3ax+by)+g(3ax-by) = 3g(ax+by)+3g(ax-by)+48g(ax),\\
&g(3ax+cz)+g(3ax-cz) = 3g(ax+cz)+3g(ax-cz)+48g(ax),\\
&g(3by+cz)+g(3by-cz) = 3g(by+cz)+3g(by-cz)+48g(by),\\
&g(3ax+cz)+g(3ax-cz)+g(3by+cz)+g(3by-cz)\\
=&3g(ax+cz)+3g(ax-cz)+48g(ax)\\
&+3g(by+cz)+3g(by-cz)+48g(by),\\
&16g(3ax+cz)+16g(3ax-cz)+16g(3by+cz)+16g(3by-cz)\\
=&48g(ax+cz)+48g(ax-cz)+48g(by+cz) \qquad (1.14)\\
&+48g(by-cz)+768g(ax)+768g(by),
\end{aligned}$$

for all $x, y \in X$. Also, replacing (ax, by) by $(3ax+cz, 3by+cz)$ in (1.13), respectively we get

$$\begin{aligned}
&g(ax+3by)+g(3ax+by) = 12g(ax+by)+16g(ax)+16g(by),\\
&g(3ax+cz+3(3by+cz))+g(3(3ax+cz)+3by+cz)\\
=&12g(3ax+cz+3by+cz)+16g(3ax+cz)+g(3by+cz),\\
&g(3ax+cz+9by+3cz)+g(9ax+3cz+3by+cz)\\
=&12g(3ax+cz+3by+cz)+16g(3ax+cz)+16g(3by+cz),\\
&g(3ax+4cz+9by)+g(9ax+4cz+3by)\\
=&12g(3ax+2cz+3by)+16g(3ax+cz)+16g(3by+cz), \qquad (1.15)
\end{aligned}$$

for all $x, y \in X$. Replacing (ax, by) by $(3ax-cz, 3by-cz)$ in (1.13) we obtain

$$\begin{aligned}
&g(ax+3by)+g(3ax+by) = 12g(ax+by)+16g(ax)+16g(by),\\
&g(3ax-cz+3(3by-cz))+g(3(3ax-cz)+3by-cz)\\
=&12g(3ax-cz+3by-cz)+16g(3ax-cz)+g(3by-cz),
\end{aligned}$$

$$
\begin{aligned}
&g(3ax-cz+9by-3cz)+g(9ax-3cz+3by-cz)\\
=&12g(3ax-cz+3by-cz)+16g(3ax-cz)+16g(3by-cz),\\
&g(3ax-4cz+9by)+g(9ax-4cz+3by)\\
=&12g(3ax-2cz+3by)+16g(3ax-cz)+16g(3by-cz), \qquad (1.16)
\end{aligned}
$$

for all $x,y\in X$. Using (1.15) and (1.16), we get the following identities

$$
\begin{aligned}
&g(3ax+4cz+9by)+g(9ax+4cz+3by)+g(3ax-4cz+9by)\\
&+g(9ax-4cz+3by)\\
=&12g(3ax+2cz+3by)+16g(3ax+cz)+16g(3by+cz)\\
&+12g(3ax-2cz+3by)+16g(3ax-cz)+16g(3by-cz),\\
&g(3ax+4cz+9by)+g(9ax+4cz+3by)+g(3ax-4cz+9by)\\
&+g(9ax-4cz+3by)-12g(3ax+2cz+3by)-12g(3ax-2cz+3by)\\
=&16g(3ax+cz)+16g(3by+cz)+16g(3ax-cz)+16g(3by-cz), \qquad (1.17)
\end{aligned}
$$

for all $x,y\in X$. Using (1.5) we obtain

$$
\begin{aligned}
&g(3ax+by)+g(3ax-by)=3g(ax+by)+3g(ax-by)+48g(ax),\\
&g(3(ax+3by)+4cz)+g(3(ax+3by)-4cz)\\
=&3g(ax+3by+4cz)+3g(ax+3by-4cz)+48g(ax+3by),\\
&g(3ax+9by+4cz)+g(3ax+9by-4cz)\\
=&3g(ax+3by+4cz)+3g(ax+3by-4cz)+48g(ax+3by), \qquad (1.18)
\end{aligned}
$$

for all $x,y\in X$. Again using (1.5) it follows that

$$
\begin{aligned}
&g(3ax+by)+g(3ax-by),\\
=&3g(ax+by)+3g(ax-by)+48g(ax),\\
&g(3(3ax+by)+4cz)+g(3(3ax+by)-4cz)\\
=&3g(3ax+by+4cz)+3g(3ax+by-4cz)+48g(3ax+by),\\
&g(9ax+3by+4cz)+g(9ax+3by-4cz)\\
=&3g(3ax+by+4cz)+3g(3ax+by-4cz)+48g(3ax+by), \qquad (1.19)
\end{aligned}
$$

for all $x,y\in X$. Adding (1.18) and (1.19), we obtain

$$
\begin{aligned}
&g(3ax+9by+4cz)+g(3ax+9by-4cz)+g(9ax+3by+4cz)\\
&+g(9ax+3by-4cz)=3g(ax+3by+4cz)+3g(ax+3by-4cz)\\
&+48g(ax+3by)+3g(3ax+by+4cz)\\
&+3g(3ax+by-4cz)+48g(3ax+by), \qquad (1.20)
\end{aligned}
$$

for all $x,y\in X$. Then applying (1.20) in (1.17), we get

$$
\begin{aligned}
&16g(3ax+cz)+16g(3by+cz)+16g(3ax-cz)+16g(3by-cz)\\
=&3g(ax+3by+4cz)+3g(ax+3by-4cz)+48g(ax+3by)
\end{aligned}
$$

$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz)$$
$$- 12g(3ax + 3by + 2cz) - 12g(3ax + 3by - 2cz) + 48g(3ax + by), \quad (1.21)$$

for all $x, y \in X$. From (1.5), we obtain

$$g(3ax + by) + g(3ax - by) = 3g(ax + by) + 3g(ax - by) + 48g(ax),$$
$$g(3(ax + by) + 2cz) + g(3(ax + by) - 2cz) = 3g(ax + by + 2cz)$$
$$+ 3g(ax + by - 2cz) + 48g(ax + by),$$
$$g(3ax + 3by + 2cz) + g(3ax + 3by - 2cz)$$
$$=3g(ax + by + 2cz) + 3g(ax + by - 2cz) + 48g(ax + by), \quad (1.22)$$

for all $x, y, z \in X$. Using (1.22) in (1.21), we get

$$16g(3ax + cz) + 16g(3by + cz) + 16g(3ax - cz) + 16g(3by + cz)$$
$$=3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by)$$
$$- 36g(ax + by + 2cz) - 36g(ax + by - 2cz) - 576g(ax + by),$$
$$16g(3ax + cz) + 16g(3by + cz) + 16g(3ax - cz) + 16g(3by + cz)$$
$$=3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz)$$
$$+ 48g(3ax + by) - 36g(ax + by + 2cz) - 36g(ax + by - 2cz) - 576g(ax + by),$$

for all $x, y, z \in X$, which, by modifying of (1.14), yields to the relation

$$3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by)$$
$$- 36g(ax + by + 2cz) - 36g(ax + by - 2cz) - 576g(ax + by)$$
$$=48g(ax + cz) + 48g(ax - cz) + 768g(ax) + 48g(by + cz)$$
$$+ 48g(by - cz) + 768g(by).$$

Then we obtain

$$3g(ax + 3by + 4cz) + 3g(ax + 3by - 4cz) + 48g(ax + 3by)$$
$$+ 3g(3ax + by + 4cz) + 3g(3ax + by - 4cz) + 48g(3ax + by)$$
$$=36g(ax + by + 2cz) + 36g(ax + by - 2cz) + 576g(ax + by) + 48g(ax + cz)$$
$$+ 48g(ax - cz) + 768g(ax) + 48g(by + cz) + 48g(by - cz) + 768g(by), \quad (1.23)$$

for all $x, y, z \in X$. With the concept of (1.13) and (1.5), the left side of (1.14) can be written in the form

$$g(3ax + by) + g(3ax - by) = 3g(ax + by) + 3g(ax - by) + 48g(ax),$$
$$g(3(3ax + by) + 2cz) + g(3(3ax + by) - 2cz)$$
$$=3g(3ax + by + 2cz) + 3g(3ax + by - 2cz) + 48g(3ax + by),$$

$$g(9ax+3by+2cz)+g(9ax+3by-2cz)=3g(3ax+by+2cz)$$
$$+3g(3ax+by-2cz)+48g(3ax+by), \tag{1.24}$$
$$g(3(ax+3by)+2cz)+g(3(ax+3by)-2cz)$$
$$=3g(ax+3by+2cz)+3g(ax+3by-2cz)+48g(ax+3by),$$
$$g(3ax+9by+2cz)+g(3ax+9by-2cz)=3g(ax+3by+2cz)$$
$$+3g(ax+3by-2cz)+48g(ax+3by). \tag{1.25}$$

Adding (1.24) and (1.25), we get

$$g(3ax+9by+2cz)+g(3ax+9by-2cz)+g(9ax+3by+2cz)$$
$$+g(9ax+3by-2cz)=3g(ax+3by+2cz)+3g(ax+3by-2cz)$$
$$+48g(ax+3by)+3g(3ax+by+2cz)$$
$$+3g(3ax+by-2cz)+48g(3ax+by),$$
$$g(3ax+9by+2cz)+g(3ax+9by-2cz)+g(9ax+3by+2cz)$$
$$+g(9ax+3by-2cz)-48g(3ax+by)-48g(ax+3by)$$
$$=3g(3ax+by+2cz)+3g(ax+3by-2cz)+3g(ax+3by+2cz)$$
$$+3g(ax+3by-2cz),$$
$$g(3ax+9by+2cz)+g(3ax+9by-2cz)+g(9ax+3by+2cz)$$
$$+g(9ax+3by-2cz)-12g(3ax+3by)-12g(3ax+3by)$$
$$=3g(ax+3by+2cz)+3g(ax+3by-2cz)$$
$$+3g(ax+3by+2cz)+3g(ax+3by-2cz), \tag{1.26}$$

for all $x,y,z\in X$. Using (1.26), we get the identity

$$16g(3ax+cz)+16g(3by+cz)+16g(3ax-cz)+16g(3by-cz)$$
$$=3g(ax+3by+2cz)+3g(ax+3by-2cz)+48g(ax+3by)$$
$$+3g(3ax+by+2cz)-648g(ax+by)+48g(3ax+by), \tag{1.27}$$

for all $x,y,z\in X$. Replacing z by $2z$ in (1.27) and then using (1.23), we get

$$16g(3ax+2cz)+16g(3by+2cz)+16g(3ax-2cz)+16g(3by-2cz)$$
$$=3g(ax+3by+4cz)+3g(ax+3by-4cz)+48g(ax+3by)$$
$$+3g(3ax+by+4cz)+3g(3ax+by-4cz)$$
$$-648g(ax+by)+48g(3ax+by), \tag{1.28}$$

for all $x,y\in X$. Using (1.28) in (1.23), we obtain

$$16g(3ax+2cz)+16g(3by+2cz)+16g(3ax-2cz)+16g(3by-2cz)$$
$$=48g(ax+cz)+48g(ax-cz)+768g(ax)+48g(by+cz)$$
$$+48g(by-cz)+768g(by)+36g(ax+by+2cz)$$
$$+36g(ax+by-2cz)+576g(ax+by)-648g(ax+by), \tag{1.29}$$

for all $x, y, z \in X$. Again making use of (1.13) and (1.5), we get

$$\begin{aligned}
&16g(3ax+2cz)+16g(3by+2cz)+16g(3ax-2cz)+16g(3by-2cz)\\
=&g(12ax+4cz)+g(12ax-4cz)-12g(6ax)+g(12by+4cz)\\
&+g(12by-4cz)-12g(6by)\\
=&64g(3ax+cz)+64g(3ax-cz)-2592g(ax)+64g(3by+cz)\\
&+64g(3by-cz)-2592g(by)\\
=&64(g(3ax+cz)+g(3ax-cz)+g(3by+cz)+g(3by-cz))\\
&-2592g(ax)-2592g(by)\\
=&64(3g(ax+cz)+3g(ax-cz)+48g(ax)+3g(by+cz)+3g(by-cz)\\
&+48g(by))-2592g(ax)-2592g(by),
\end{aligned} \tag{1.30}$$

for all $x, y, z \in X$. Using (1.30) we have the following reduction

$$\begin{aligned}
&16g(3ax+2cz)+16g(3by+2cz)+16g(3ax-2cz)+16g(3by-2cz)\\
=&192g(ax+cz)192g(ax-cz)-480g(ax)\\
&+192g(by+cz)+192g(by-cz)-480g(by),
\end{aligned} \tag{1.31}$$

for all $x, y, z \in X$. Finally, if we compare (1.31) with (1.29), we can conclude that

$$\begin{aligned}
&48g(ax+cz)+48g(by+cz)+48g(ax-cz)+48g(by-cz)+768g(ax)+768g(by)\\
&+36g(ax+by+2cz)+36g(ax+by-2cz)+576g(ax+by)-648g(ax+by)\\
=&192g(ax+cz)+192g(ax-cz)-480g(ax)+192g(by+cz)+192g(by-cz)\\
&-480g(by),\\
&36g(ax+by+2cz)+36g(ax+by-2cz)=192g(ax+cz)-48g(ax+cz)\\
&+192g(ax-cz)-48g(ax-cz)+480g(ax)-768g(ax)-192g(by+cz)\\
&-48g(by+cz)+192g(by-cz)-48g(by-cz)+480g(by)-768g(by)\\
&+72g(ax+by),\\
&36g(ax+by+2cz)+36g(ax+by-2cz)=144g(ax+cz)+144g(ax-cz)\\
&+144g(by+cz)+144g(by-cz)+72g(ax+by)-288g(ax)-288g(by),\\
&g(ax+by+2cz)+g(ax+by-2cz)=2g(ax+by)+4g(ax+cz)\\
&+4g(ax-cz)+4g(by+cz)+4g(by-cz)-8g(ax)-8g(by),
\end{aligned}$$

for all $x, y, z \in X$. By considering $g(ax) = a^3 g(x)$, we get

$$\begin{aligned}
&g(ax+by+2cz)+g(ax+by-2cz)=2g(ax+by)\\
&+4(g(ax+cz)+g(ax-cz)+g(by+cz)+g(by-cz))-8a^3g(x)-8b^3g(y),
\end{aligned}$$

for all $x, y, z \in X$, which implies that g is cubic. Conversely, suppose that $g : X \to Y$ satisfies the functional equation (1.1). Putting $x = y = z = 0$ in (1.2) we get $g(0) = 0$. Changing (x, y, z) by $\left(\frac{-x}{a}, \frac{x}{b}, \frac{xc}{2}\right)$ in the result we get $g(-x) = -g(x)$, which implies that g is odd. Replacing $y = 0$ in (1.2) and employing the fact that g is odd, we obtain that

$$g(ax+2cz)+g(ax-2cz)=2g(ax)+4g(ax+cz)+g(ax-cz)$$
$$+4g(cz)+4g(-cz)-8a^3g(x),$$
$$g(ax+cz)+g(ax-2cz)=-6a^3g(x)+4g(ax+cz)+4g(ax-cz),$$
$$g(ax+2cz)+g(ax-2cz)=-6a^3g(x)+4g(ax+cz)+4g(ax-cz), \tag{1.32}$$

for all $x, y, z \in X$. Replacing (x, y, z) by $(x, 0, 0)$ in (1.6), we get

$$g(ax)=a^3g(x),$$

since

$$g(ax)+g(ax)=2g(ax)+4g(ax)+4g(ax)-8a^3g(x),$$
$$2g(ax)=10g(ax)-8a^3g(x),$$
$$2g(ax)-10g(ax)=-8a^3g(x),$$
$$-8g(ax)=-8a^3g(x),$$
$$g(ax)=a^3g(x).$$

So we replace x by $2x$ in (1.32) and we get

$$g(2ax+2cz)+g(2ax-2cz)=-6a^3g(2x)+4g(2ax+cz)+4g(2ax-cz),$$
$$8g(ax+cz)+8g(ax-cz)=-48a^3g(x)+4g(2ax+cz)+4g(2ax-cz),$$
$$2g(ax+cz)+2g(ax-cz)=-12a^3g(x)+g(2ax+cz)+g(2ax-cz),$$
$$g(2ax+cz)+g(2ax-cz)=12a^3g(x)+2g(ax+cz)+2g(ax-cz),$$

and

$$g(2x+y)+g(2x-y)=12g(x)+2g(x+y)+2g(x-y),$$

for all $x, y, z \in X$, which implies that g is cubic. This completes the proof. □

In this section, we present the generalized Hyers-Ulam-Rassias stability of the function (1.6).

Theorem 1.1. *Let $j \in \{-1, 1\}$ and $\alpha : X^3 \to [0, \infty)$ be a function such that*

$$\sum_{k=0}^{\infty} \frac{\alpha\left(a^{kj}x, a^{kj}y, a^{kj}z\right)}{a^{3kj}}$$

converges in $\mathbb{R}$ and

$$\sum_{k=0}^{\infty} \frac{\alpha\left(a^{kj}x, a^{kj}y, a^{kj}z\right)}{a^{3kj}} = 0, \tag{1.33}$$

for all $x, y \in X$. Let $g : X \to Y$ be an odd function satisfying the inequality

$$\|D_g(x,y,z)\| \le \alpha(x,y,z), \tag{1.34}$$

for all $x, y, z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ which satisfies the functional equation (1.6) *and*

$$\|f(x) - C(x)\| \leq \frac{1}{8a^3} \sum_{k=\frac{i-j}{2}}^{\infty} \frac{\alpha\left(a^{kj}x, 0, 0\right)}{a^{3kj}}, \tag{1.35}$$

for all $x \in X$. The mapping $C(x)$ is defined by

$$C(x) = \lim_{n\to\infty} \frac{g\left(a^{kj}x\right)}{a^{3kj}},$$

for all $x \in X$.

Proof. Assume that $j = 1$. Replacing (x, y, z) by $(x, 0, 0)$ in (1.34), we get

$$\left\|8g(ax) - 8a^3 g(x)\right\| \leq \alpha(x, 0, 0), \tag{1.36}$$

for all $x \in X$. From (1.36) it follows that

$$\left\|\frac{g(ax)}{a^3} - g(x)\right\| \leq \frac{1}{8\,a^3}\alpha(x, 0, 0), \tag{1.37}$$

for all $x \in X$. Replacing x by ax in (1.37) and dividing by a^3, we obtain

$$\left\|\frac{g(a(ax))}{a^6} - \frac{g(ax)}{a^3}\right\| \leq \frac{1}{8\,a^6}\alpha(ax, 0, 0),$$

$$\left\|\frac{g(a^2x)}{a^6} - \frac{g(ax)}{a^3}\right\| \leq \frac{1}{8\,a^6}\alpha(ax, 0, 0), \tag{1.38}$$

for all $x \in X$. From the identity (1.37) and (1.38), it follows that

$$\begin{aligned}\left\|\frac{g(a^2x)}{a^6} - g(x)\right\| &\leq \frac{1}{8a^3}\alpha(x, 0, 0) + \frac{1}{8a^6}\alpha(ax, 0, 0)\\ &\leq \frac{1}{8\,a^3}\left\{\alpha(x, 0, 0) + \frac{1}{a^3}\alpha(ax, 0, 0)\right\}\\ &\leq \sum_{k=0}^{n-1} \frac{1}{8a^3}\left\{\frac{\alpha(a^kx, 0, 0)}{a^{3k}}\right\},\end{aligned}$$

$$\left\|\frac{g(a^nx)}{a^{3n}} - g(ax)\right\| \leq \frac{1}{8a^3} \sum_{k=0}^{n-1}\left\{\frac{\alpha(a^kx, 0, 0)}{a^{3k}}\right\}, \tag{1.39}$$

for all $x \in X$. We prove the convergence of the sequence $\left\{\frac{g\left(a^kx\right)}{a^{3k}}\right\}$ for all $x \in X$. Replacing x by $a^m x$ and dividing by a^m in (1.39), we obtain

$$\left\|\frac{g(a^mx)}{a^{3m}} - \frac{g(a^{m+n}x)}{a^{3(m+n)}}\right\| \leq \frac{1}{8a^3} \sum_{k=0}^{n-1}\left\{\frac{\alpha(a^{m+n}x, 0, 0)}{a^{3(m+n)}}\right\},$$

for all $x \in X$. Hence, the sequence $\left\{\frac{g(a^m x)}{a^{3m}}\right\}$ is a Cauchy sequence. Since Y is complete normed space, there exists a mapping $C : X \to Y$ such that

$$C(x) = \lim_{n\to\infty} \frac{g\left(a^{kj}x\right)}{a^{3kj}},$$

for all $x \in X$. Letting $k \to \infty$ in (1.39), we see that (1.34) holds for $x \in X$. To prove that C satisfies (1.6), we replace (x, y, z) by $(a^n x, a^n y, a^n z)$ and divide (1.34) by a^{3n}, which gives that

$$\frac{1}{a^{3n}} \ \|D_g\left(a^n x, a^n y, a^n z\right)\| \ \leq \ \frac{1}{a^{3n}} \ \alpha\left(a^n x, a^n y, a^n z\right),$$

for all $x, y, z \in X$. As n approaches to ∞ in the above inequality and using the definition of $C(x)$, we have $DC(x, y, z) = 0$. Hence, C satisfies (1.6) for all $x, y, z \in X$.

We will show that C is unique. Let $B(x)$ be another cubic mapping satisfying (1.6) and (1.35), such that

$$\begin{aligned}
\|C(x) - B(x)\| &= \frac{1}{a^{3n}} \|c(a^n x) - B(a^n x)\| \\
&\leq \frac{1}{a^{3n}} \left\{\|C(a^n x) - g(a^n x)\| + \|g(a^n x) - B(a^n x)\|\right\} \\
&\leq \ \frac{1}{8 \ a^{3n}} \sum_{k=0}^{\infty} \frac{\alpha\left(a^{m+n}x, 0, 0, \right)}{a^{3(m+n)}} \ \to 0 \quad \text{as } n \to \infty,
\end{aligned}$$

for all $x \in X$. Hence, C is unique. Now, replacing x by $\frac{x}{a}$ in (1.34), we get

$$\begin{aligned}
\left\|8g(ax) - 8a^3 g(x)\right\| &\leq \alpha\left(\frac{x}{a}, 0, 0\right), \\
\left\|g(ax) - a^3 g(x)\right\| &\leq \frac{1}{8}\alpha\left(\frac{x}{a}, 0, 0\right),
\end{aligned}$$

for all $x \in X$. The remaining part of the proof of this theorem for $j = 1$ with replacing x by $\frac{x}{a}$ in (1.37) is similar. Also, we can prove the theorem for $j = -1$ in the same manner. This completes the proof of the theorem. □

Corollary 1.1. *Let λ and q be a non-negative real numbers. Let an odd function $g : X \to Y$ satisfying the inequality*

$$\|Dg(x, y, z)\| \leq \begin{cases} \lambda, \\ \lambda\left\{\|x\|^q + \|y\|^q + \|z\|^q\right\}, \\ \lambda\left\{\|x\|^q \|y\|^q \|z\|^q + \|x\|^{3q} + \|y\|^{3q} + \|z\|^{3q}\right\}, \end{cases}$$

for all $x, y, z \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|g(x) - C(x)\| \leq \begin{cases} \frac{1}{8} \cdot \frac{1}{a^3 - 1}, \\ \frac{\lambda}{8} \cdot \frac{\|x\|^q}{|a^3 - a^q|}, \\ \frac{\lambda}{8} \cdot \frac{\|x\|^{3q}}{|a^3 - a^{3q}|}, \end{cases}$$

for all $x \in X$.

Proof. Setting

$$\alpha(x, y, z) \ \leq \ \begin{cases} \lambda, \\ \lambda \ \{\|x\|^q + \|y\|^q + \|z\|^q\}, \\ \lambda \ \left\{\|x\|^q \|y\|^q \|z\|^q + \|x\|^{3q} + \|y\|^{3q} + \|z\|^{3q}\right\}, \end{cases}$$

for all $x \in X$ and using $\alpha(x, y, z)$ in Theorem 1.1, we obtain desired result. □

In this section, we investigate the generalized Ulam-Hyers-Rassias stability of the functional equation (1.6).

Theorem 1.2 (The Alternative of Fixed Point, [29]). *Suppose that complete generalized metric space (τ, d) and a strictly contractive mapping $T : \tau \to \tau$ with Lipchitz constant L are given. Then for each given $x \in \tau$, either*

$$d\left(T^n x, T^{n+1} x\right) = \infty,$$

for all $n \geq 0$ or there exists a natural number n_0 such that

(a) *$d(T^n x, T^{n+1} x) < \infty$ for all $n \geq 0$;*
(b) *the sequence $\{T^n x\}$ is convergent to a fixed point y^* of T;*
(c) *y^* is the unique fixed point of T in the set*

$$Y = \{y \in Y : d(P^{n_0} x, y) < \infty\};$$

(d) *$d(y^*, y) \leq \frac{1}{1-L}\ d(y, Py)$ for all $y \in Y$.*

Utilizing the above mentioned fixed point alternative, we now obtain our main results, that is the generalized Hyers-Ulam-Rassias stability of the functional equation (1.6).

From now on, let X be a real vector space and Y be a real Banach space. For given mapping $g : X \to Y$, we get

$$\begin{aligned} Dg(x, y, z) =& g(ax + by + 2cz) + g(ax + by - 2cz) - 2g(ax + by) - 4g(ax + cz) \\ & - 4g(ax - cz) - 4g(by + cz) - 4g(by - cz) + 8a^3 g(x) + 8b^3 g(y), \end{aligned}$$

for all $x, y, z \in X$. Let $\psi : X \times X \times X \to [0, \infty)$ be a function such that

$$\lim_{n\to\infty} \frac{\psi\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)}{\mu_i^{3k}} = 0, \tag{1.40}$$

for all $x, y, z \in X$, where

$$\mu_i = \begin{cases} 2, & i = 0, \\ \frac{1}{2}, & i = 1. \end{cases}$$

Theorem 1.3. *Suppose that function $g : X \to Y$ satisfies the functional inequality*

$$\|Dg(x, y, z)\| \leq \psi(x, y, z), \tag{1.41}$$

for all $x, y, z \in X$. If there exists $L = L(i)$ such that function

$$x \mapsto \beta(x) = \frac{1}{2}\ \alpha\left(\frac{x}{a}, 0, 0\right),$$

has the property

$$\frac{1}{\mu_i^3}\beta(\mu_i x) = L\beta(x), \tag{1.42}$$

for all $x \in X$, then there exists a unique cubic function $c : X \to Y$ that satisfies the functional equation (1.6) *and*

$$\|g(x) - c(x)\| \leq \frac{L^{1-i}}{1-L}\beta(x),$$

for all $x \in X$.

Proof. Consider the set $W = \{p/P : X \to \beta, p(0) = 0\}$ and introduced generalized metric on X

$$d(p, q) = \inf\left\{k \in (0, \infty) : \|p(x) - q(x)\| \leq k\beta(x), x \in X\right\}.$$

It is easy to see that (X, d) is complete. Define $T : X \to X$ by

$$Tp(x) = \frac{1}{\mu_i^3}p(\mu_i),$$

for all $x \in X$. Now, for $p, q \in X$, we have

$$\begin{aligned} d(p, q) &\leq k, \quad x \in W, \\ |\, \|p(x) - q(x)\| &\leq k\beta(x), \quad x \in W, \\ \left\|\frac{1}{\mu_i^3}p(\mu_i x) - \frac{1}{\mu_i^3}q(\mu_i x)\right\| &\leq \frac{1}{\mu_i^3}k\beta(\mu_i x), \\ \|Tp(x) - Tq(x)\| &\leq L\ k\ \beta(x), \quad x \in W, \\ d(Tp, Tq) &\leq Lk, \quad x \in W. \end{aligned}$$

This implies that $d(Tp, Tq) \leq Ld(p, q)$ for all $p, q \in X$.That is, T is strictly contractive mapping on X with Lipschitz constant L. From (1.36) it follows that

$$\left\|8g(ax) - 8a^3g(x)\right\| \leq \alpha(x, 0, 0), \tag{1.43}$$

for all $x \in X$. From (1.43) it follows that

$$\left\|\frac{g(ax)}{a^3} - g(x)\right\| \leq \frac{1}{8a^3}\alpha(x,0,0),$$

for all $x \in X$. Using (1.42), for the case $i = 0$, this reduces to

$$\left\|g(x) - \frac{g(ax)}{a^3}\right\| \leq \frac{1}{4}\beta(x),$$

for all $x \in X$, that is

$$d(g_a, Tg_a) \leq \frac{1}{4} = L = L^1 < \infty.$$

Again replacing x by $\frac{x}{a}$ in (1.43), we get

$$\left\|g(x) - a^3 g\left(\frac{x}{a}\right)\right\| \leq \frac{1}{8}\alpha\left(\frac{x}{a}, 0, 0\right),$$

for all $x \in X$. Using (1.42) for the case $i = 1$, this reduces to

$$\left\|g(x) - a^3 g\left(\frac{x}{a}\right)\right\| \leq \frac{1}{4}\beta(x),$$

for all $x \in X$, that is $d(g_a, Tg_a) \leq 1$. This implies that $d(g_a, Tg_a) \leq 1 = L^0 < \infty$. In the above case, we write $d(g_a, Tg_a) \leq L^{1-i}$. Therefore, the first two conditions (a) and (b) of the Alternative fixed point theorem holds for T, and it follows that there exists a fixed point C of T in X such that

$$C(x) = \lim_{k\to\infty} \frac{g\left(\mu_i^k x\right)}{\mu_i^{3k}}, \quad \text{for all } x \in X. \tag{1.44}$$

In order to prove that $C : X \to Y$ is cubic we replace (x, y, z) by $(\mu_i^k x, \mu_i^k y, \mu_i^k z)$ and divide (1.41) by μ_i^{3k}. From that, using (1.40) and (1.44), we see that C satisfies (1.6) for all $x, y, z \in X$. Hence, C satisfies the functional equation (1.6).

By fixed point condition (2), C is the unique fixed point of T in the set

$$Y = \{g_a \in X : d(Tg_a, C) < \infty\}.$$

Using the fixed point alternative result, C is the unique function such that

$$\|g_a(x) - C(x)\| \leq k\ \beta(x),$$

for all $x \in X$ and $k > 0$. Finally, by (4), we obtain

$$d(g_a, C) \leq \frac{1}{1-L}\ d(g_a, Tg_a).$$

That is, we have

$$d(g_a, C) \leq \frac{L^{1-i}}{1-L}.$$

Hence, we conclude that

$$\|g(x) - C(x)\| \leq \frac{L^{1-i}}{1-L}\ \beta(x),$$

for all $x \in X$. This completes the proof of the theorem. □

Corollary 1.2. *Let $g : X \to Y$ be a mapping and let t γ and p be real numbers such that*

$$\|Dg(x,y,z)\| \leq \begin{cases} \gamma, \\ \gamma \{\|x\|^p + \|y\|^p + \|z\|^p\}, \\ \gamma \left\{\|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right)\right\}, \end{cases}$$

for all $x, y, z \in X$. Then there exist a unique cubic mapping $C : X \to Y$ such that

$$\|g(x) - C(x)\| \leq \begin{cases} \frac{\gamma}{8} \cdot \frac{1}{a^3 - 1}, \\ \frac{\gamma}{8} \cdot \frac{\|x\|^p}{a^3 - a^p}, & p \neq 3, \\ \frac{\gamma}{8} \cdot \frac{\|x\|^{3q}}{a^3 - a^{3q}}, & p \neq 1, \end{cases}$$

for all $x \in X$.

Proof. Set

$$\alpha(x,y,z) = \begin{cases} \gamma, \\ \gamma \{\|x\|^p + \|y\|^p + \|z\|^p\}, \\ \gamma \left\{\|x\|^p \|y\|^p \|z\|^p + \left(\|x\|^{3p} + \|y\|^{3p} + \|z\|^{3p}\right)\right\}, \end{cases},$$

for all $x \in X$. Now,

$$\frac{\alpha\left(\mu_i^k x, \mu_i^k y, \mu_i^k z\right)}{\mu_i^{3k}}$$
$$= \begin{cases} \frac{\gamma}{\mu_i^{3k}}, \\ \frac{\gamma}{\mu_i^{3k}} \left\{\left\|\mu_i^k x\right\|^p + \left\|\mu_i^k y\right\|^p + \left\|\mu_i^k z\right\|^p\right\}, \\ \frac{\gamma}{\mu_i^{3k}} \left\{\left\|\mu_i^k x\right\|^p \left\|\mu_i^k y\right\|^p \left\|\mu_i^k z\right\|^p + \left(\left\|\mu_i^k x\right\|^{3p} + \left\|\mu_i^k y\right\|^{3p} + \left\|\mu_i^k z\right\|^{3p}\right)\right\}, \end{cases}$$
$$\to \begin{cases} 0, & k \to \infty, \\ 0, & k \to \infty, \\ 0, & k \to \infty. \end{cases}$$

That is, (1.40) holds. But we have $\beta(x) = \frac{1}{2}\, \alpha\left(\frac{x}{a}, 0, 0\right)$. Hence,

$$\beta(x) = \frac{1}{2}\, \alpha\left(\frac{x}{a}, 0, 0\right) = \begin{cases} \frac{\gamma}{8}, \\ \frac{\gamma}{2^3\, a^p}\ \|x\|^p, \\ \frac{\gamma}{2^3\, a^{3p}}\ \|x\|^{3p}. \end{cases}$$

Also,

$$\frac{1}{\mu_i^3}\beta(\mu,x)=\begin{cases}\dfrac{\gamma}{8\ \mu_i^3},\\ \dfrac{\gamma}{8\ \mu_i^3}\ \ \|\mu_i\|^p\,, \\ \dfrac{\gamma}{8\ \mu_i^3}\ \ \|\mu_i\|^{3p}\,,\end{cases}=\begin{cases}\mu_i^{-3}\ \beta(x),\\ \mu_i^{p-3}\ \beta(x),\\ \mu_i^{3p-3}\ \beta(x).\end{cases}$$

Hence, the inequality (1.42) holds.

Case (i). $L=a^{-3}$, $i=0$,

$$\begin{aligned}\|g(x)-C(x)\| &\leq \frac{L^{1-i}}{1-L}\beta(x)\leq \frac{\left(a^{-3}\right)^{1-0}}{1-a^{-3}}\ \frac{\gamma}{8}\\ &\leq \frac{a^{-3}}{1-\frac{1}{a^3}}\cdot\frac{\gamma}{8}\leq\frac{\gamma}{8}\cdot\frac{1}{a^3-1}.\end{aligned}$$

Case (ii). $L=\left(\frac{1}{a^3}\right)^{-1}$, $i=1$,

$$\begin{aligned}\|g(x)-C(x)\| &\leq \frac{L^{1-i}}{L}\beta(x)\leq \frac{\left(a^{3}\right)^{1-1}}{1-a^{3}}\frac{\gamma}{8}\\ &\leq \frac{1}{1-a^3}\cdot\frac{\gamma}{8}\leq\frac{\gamma}{8}\cdot\frac{1}{1-a^3}.\end{aligned}$$

Case (iii). $L=a^{p-3}$, $p<3$, $i=0$,

$$\begin{aligned}\|g(x)-C(x)\| &\leq \frac{\left(a^{p-3}\right)^{1-0}}{1-a^{p-3}}\cdot\frac{\gamma}{2^3a^p}\ \ \|x\|^p\\ &\leq \frac{a^pa^{-3}}{1-\frac{a^p}{a^3}}\cdot\frac{\gamma}{2^3a^p}\ \ \|x\|^p\,,\\ \|g(x)-C(x)\| &\leq \ \frac{\|x\|^p\,\gamma}{8}\cdot\frac{1}{a^3-a^p}.\end{aligned}$$

Case (iv). $L=\left(\frac{1}{a}\right)^{p-3}$, $p>3$, $i=1$, $L=a^{3-p}$, $p>3$, $i=1$,

$$\begin{aligned}\|g(x)-C(x)\| &\leq \ \frac{L^{1-i}}{1-L}\beta(x)\\ &\leq \frac{\left(a^{3-p}\right)^{1-1}}{1-a^{3-p}}\cdot\frac{\gamma}{2^3a^p}\,\|x\|^p\,,\\ \|g(x)-C(x)\| &\leq \frac{a^p}{a^p-a^3}\cdot\frac{\gamma}{8a^p}\ \ \|x\|^p\leq\frac{\|x\|^p\cdot\gamma}{8}\cdot\frac{1}{a^p-a^3}.\end{aligned}$$

Case (v). $L=a^{3p-3}$, $p<1$, $i=0$,

$$\|g(x)-C(x)\|\leq\ \frac{L^{1-i}}{L}\beta(x)\leq\frac{\left(a^{3p-3}\right)^{1-0}}{1-a^{3p-3}}\cdot\frac{\gamma}{2^3a^{3p}}\ \ \|x\|^{3p}\,,$$

$$\|g(x) - C(x)\| \leq \frac{a^{3p-3}}{1-a^{3p-3}} \frac{\gamma}{8a^{3p}} \|x\|^{3p} \leq \frac{\|x\|^{3p} \cdot \gamma}{8} \cdot \frac{1}{a^3 - a^{3p}}.$$

Case (vi). $L = a^{3-3p}$, $p > 1$, $i = 1$,

$$\|g(x) - C(x)\| \leq \frac{L^{1-i}}{L}\beta(x) \leq \frac{(a^{3-3p})^{1-1}}{1-a^{3-3p}} \frac{\gamma}{2^3 a^{3p}} \|x\|^{3p},$$

$$\|g(x) - C(x)\| \leq \frac{1}{1-a^{3-3p}} \frac{\gamma}{8a^{3p}} \|x\|^{3p} \leq \frac{\|x\|^{3p} \cdot \gamma}{8\ a^{3p}} \cdot \frac{1}{a^{3p} - a^3}.$$

Hence, the proof is completed. □

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