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ON THE LOCAL VERSION OF THE CHERN CONJECTURE: CMC HYPERSURFACES WITH CONSTANT SCALAR CURVATURE IN $\mathbb{S}^{n+1}$

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Abstract. After nearly 50 years of research the Chern conjecture for isoparametric hypersurfaces in spheres is still an unsolved and important problem and in particular its local version is of great interest, since here one loses the power of Stokes' Theorem as a method for proving it. Here we present a related result for CMC hypersurfaces in $\mathbb{S}^{n+1}$ with constant scalar curvature and three distinct principal curvatures.

1. Introduction

The Chern conjecture for isoparametric hypersurfaces in spheres can be stated as follows. *Let M be a closed, minimally immersed hypersurface of the $(n+1)$-dimensional sphere $\mathbb{S}^{n+1}$ with constant scalar curvature. Then M is isoparametric.*

One obvious generalization is that on non-closed manifolds, i.e., a local version of the conjecture. This has in particular been proposed by Bryant for the case $n = 3$.

Let $M \subset \mathbb{S}^4$ be a minimal hypersurface with constant scalar curvature. Then M is isoparametric.

For more details, a short history and an overview of results we would like to refer to the review article [3] by Scherfner, Weiss and Yau.

Here we will give a result related to the local version.

Let $n > 3$ and $M \subset \mathbb{S}^{n+1}$ be a hypersurface with constant mean and scalar curvatures which has three pairwise distinct principal curvatures everywhere, then M is isoparametric.

Key words and phrases. Constant mean and scalar curvature, isoparametric hypersurfaces, Chern conjecture.

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2. Preliminaries

Let M be an n-dimensional hypersurface in a unit sphere $\mathbb{S}^{n+1}(1)$. We choose a local orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ in $\mathbb{S}^{n+1}(1)$, so that restricted to M, $e_1, \ldots, e_n$ are tangent to M. Let $\omega_1, \ldots, \omega_{n+1}$ denote the dual co-frame field in $\mathbb{S}^{n+1}(1)$. We use the following convention for the indices: A, B, C, D range from 1 to $n+1$ and i, j, k from 1 to n. The structure equations of $\mathbb{S}^{n+1}(1)$ as a hypersurface of the Euclidean space $\mathbb{R}^{n+2}$ are given by

$$d\,\omega_A = -\sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0,$$

$$d\,\omega_{AB} = -\sum_C \omega_{AC} \wedge \omega_{CB} + \frac{1}{2}\sum_{C,D} \bar{R}_{ABCD}\omega_C \wedge \omega_D,$$

where $\bar{R}$ is the Riemannian curvature tensor

$$\bar{R}_{ABCD} = \delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}.$$

The contractions $\bar{R}_{AC} = \sum_B \bar{R}_{ABCB}$ and $\bar{R} = \sum_{A,B} \bar{R}_{ABAB}$ are the Ricci curvature tensor and the scalar curvature of $\mathbb{S}^{n+1}(1)$, respectively. Next, we restrict all the tensors to M. First of all, since $\omega_{n+1} = 0$ on M, $\sum_i \omega_{n+1,i} \wedge \omega_i = d\omega_{n+1} = 0$. By Cartan's lemma we can write

$$\omega_{n+1,i} = \sum_j h_{ij}\omega_i, \quad h_{ij} = h_{ji}. \tag{2.1}$$

Here $h = \sum_{i,j} h_{ij}\omega_i\omega_j$ denotes the second fundamental form of M and the principal curvatures λ_i are the eigenvalues of the matrix (h_{ij}). Furthermore the mean curvature is given by $H = \frac{1}{n}\sum_i h_{ii} = \frac{1}{n}\sum_i \lambda_i$ and $K = \det(h_{ij}) = \prod_i \lambda_i$ is the Gauss-Kronecker curvature. We also define

$$S := |h|^2 = \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2 \tag{2.2}$$

and for $r \geq 3$

$$f_r := \operatorname{tr}\left((h_{ij})^r\right). \tag{2.3}$$

Independently of the choice of the e_i we have

$$f_3 = \sum_{i,j,k} h_{ij}h_{jk}h_{ki} = \sum_i \lambda_i^3, \quad f_4 = \sum_{i,j,k,l} h_{ij}h_{jk}h_{kl}h_{li} = \sum_i \lambda_i^4, \tag{2.4}$$

and so on.

On M we have

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.5}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k \wedge \omega_l, \tag{2.6}$$

where R is the Riemannian curvature tensor on M with components satisfying

$$0 = R_{ijkl} + R_{ijlk}.$$

These structure equations imply the following integrability condition (Gauss equation):

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.7}$$

For the scalar curvature we have

$$\kappa = n(n-1) + n^2H^2 - S.$$

If we consider minimal hypersurfaces, the Ricci curvature and scalar curvature are given by, respectively,

$$R_{ij} = (n-1)\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.8}$$

$$\kappa = n(n-1) - S. \tag{2.9}$$

It follows from (2.9) that κ is constant if and only if S is constant. The covariant derivative ∇h with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{jk}\omega_{ik} + \sum_k h_{ik}\omega_{jk}. \tag{2.10}$$

Then the exterior derivative of (2.8) together with the structure equations yields the following Codazzi equation

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{2.11}$$

In addition we have

$$h_{ijk} = (h_{ij})_k + \sum_l h_{jl}\omega_{il}(e_k) + \sum_l h_{il}\omega_{jl}(e_k), \tag{2.12}$$

$$h_{ijkl} = (h_{ijk})_l + \sum_m h_{mjk}\omega_{im}(e_l) + \sum_m h_{imk}\omega_{jm}(e_l) + \sum_m h_{ijm}\omega_{km}(e_l), \tag{2.13}$$

$$h_{ijkl} = h_{ijlk} + \sum_m h_{mj}R_{mikl} + \sum_m h_{mi}R_{mjkl}, \tag{2.14}$$

$$\sum_{ijk} h_{ijk}^2 = (S-n)S - nHf_3 + n^2H^2. \tag{2.15}$$

We will use the following result by Otsuki given in [2].

Lemma 2.1. *Let M be a hypersurface in a $(n+1)$-dimensional Riemannian manifold of constant curvature such that the multiplicities of the principal curvatures are all constant. Then the distribution of the space of principal vectors corresponding to each principal curvature is completely integrable. If the multiplicity of a principal curvature is greater than 1, then this principal curvature is constant on each integral submanifold of the corresponding distribution of the space of principal vectors.*

3. Proof of the Theorem

Theorem 3.1. *Let $n > 3$ and $M \subset \mathbb{S}^{n+1}$ be a hypersurface with constant mean and scalar curvatures which has three pairwise distinct principal curvatures everywhere, then M is isoparametric.*

Proof. Let λ, μ und ν be the distinct principal curvatures with corresponding multiplicities r_1, r_2 and r_3. From $r_1 + r_2 + r_3 = n$ and the definitions of H and S one has a system of equations with continuous coefficients which the r_i solve uniquely. Thus the r_i are continuous functions and therefore constant.

Locally we choose the e_i such that h is diagonal in every point. For the directional derivatives of the principal curvatures one has

$$r_1\lambda_k + r_2\mu_k + r_3\nu_k = r_1\lambda\lambda_k + r_2\mu\mu_k + r_3\nu\nu_k = 0. \tag{3.1}$$

Let the principal curvature directions corresponding to the three principal curvatures be called e_A, e_a and e_α. Then (2.12) implies

$$h_{ijk} = \delta_{ij}(\lambda_i)_k, \tag{3.2}$$

for $\lambda_i = \lambda_j$ and

$$\omega_{ij}(e_k) = \frac{1}{\lambda_j - \lambda_i} h_{ijk}, \tag{3.3}$$

for $\lambda_i \neq \lambda_j$.

We consider different cases for the multiplicities of the principal curvatures. Without loss of generality, let $r_1 \geq r_2 \geq r_3$.

Case 1: $r_1, r_2, r_3 > 1$. Then Lemma 2.1 implies $\lambda_A = \mu_a = \nu_\alpha = 0$, and with (3.1) it follows that all derivatives of the principal curvatures vanish.

Case 2: $r_1, r_2 > 1$, $r_3 = 1$. Without loss of generality let $\alpha = n$. Then Lemma 2.1 and (3.1) imply that the derivatives of the principal curvatures in directions e_A and e_a vanish. From (3.2), (3.3) and (2.13) one has

$$\begin{aligned} h_{AaBa} =& (h_{AaB})_a + \sum_m h_{maB}\omega_{Am}(e_a) + \sum_m h_{AmB}\omega_{am}(e_a) + \sum_m h_{Aam}\omega_{Bm}(e_a) \\ =& \frac{2}{\nu - \lambda} h_{aAn}h_{aBn} + \delta_{AB}\frac{\lambda_n\mu_n}{\nu - \mu}, \\ h_{AaaB} =& \frac{2}{\nu - \mu} h_{aAn}h_{aBn} + \delta_{AB}\frac{\lambda_n\mu_n}{\nu - \lambda}. \end{aligned}$$

From (2.14) one has

$$h_{AaBa} - h_{AaaB} = (\lambda - \mu)R_{AaBa} = \delta_{AB}(\lambda - \mu)(1 + \mu\lambda)$$

and thus

$$h_{aAn}h_{aBn} = \frac{z_1}{2}\delta_{AB}, \tag{3.4}$$

where

$$z_1 := (\nu - \lambda)(\nu - \mu)(1 + \lambda\mu) + \lambda_n\mu_n.$$

Let v_a be the column vector of the h_{aAn} for a given a, then this can be expressed in the matrix equation

$$v_a v_a^t = \frac{z_1}{2} id.$$

Since the left hand side can only have rank 0 or 1, it follows that $z_1 = 0$ and therefore $h_{aAn} = 0$ for all a und A. From (3.1) it follows that

$$\lambda_n = \frac{1}{r_1}\frac{\nu-\mu}{\mu-\lambda}\nu_n, \quad \mu_n = \frac{1}{r_2}\frac{\nu-\lambda}{\lambda-\mu}\nu_n$$

and thus

$$\begin{aligned}\sum_{ijk} h_{ijk}^2 =& 3\sum_A h_{AAn}^2 + 3\sum_a h_{aan}^2 + h_{nnn}^2 = 3r_1\lambda_n^2 + 3r_2\mu_n^2 + \nu_n^2 \\ =& \left(\frac{3}{r_1}\frac{(\nu-\mu)^2}{(\mu-\lambda)^2} + \frac{3}{r_2}\frac{(\nu-\lambda)^2}{(\lambda-\mu)^2} + 1\right)\nu_n^2 \\ =& (3r_2(\nu-\mu)^2 + 3r_1(\nu-\lambda)^2 + r_1r_2(\lambda-\mu)^2)\frac{1}{r_1r_2}\frac{1}{(\lambda-\mu)^2}\nu_n^2.\end{aligned}$$

On the other hand, $z_1 = 0$ implies

$$\frac{1}{r_1r_2}\frac{1}{(\lambda-\mu)^2}\nu_n^2 = -\frac{\lambda_n\mu_n}{(\nu-\mu)(\nu-\lambda)} = 1 + \lambda\mu$$

and one has

$$r_2(\nu-\mu)^2 + r_1(\nu-\lambda)^2 + r_1r_2(\lambda-\mu)^2 = \frac{1}{2}\sum_{ij}(\lambda_i-\lambda_j)^2 = nS - n^2H^2.$$

Then (2.15) is of the form

$$(S-n)S + n^2H^2 = (1+\lambda\mu)(3nS - 3n^2H^2 - 2r_1r_2(\lambda-\mu)^2) + nHf_3. \tag{3.5}$$

From

$$r_1\lambda + r_2\mu + \nu = nH, \quad r_1\lambda^2 + r_2\mu^2 + \nu^2 = S,$$

one has

$$r_1(1+r_1)\lambda^2 + r_2(1+r_2)\mu^2 + n^2H^2 - 2nHr_1\lambda - 2nHr_2\mu + 2r_1r_2\lambda\mu - S = 0.$$

Solving for λ yields

$$\lambda = \frac{nH - r_2\mu}{1+r_1} + w,$$

where

$$w := \pm\sqrt{\frac{-nr_2\mu^2 + 2nr_2H\mu + (1+r_1)S - n^2H^2}{r_1(1+r_1)^2}}.$$

If $w = 0$ on an open set, then μ and consequently λ and ν are constant there. Therefore it is sufficient to show the proposition under the assumption that the sign of w remains the same. One calculates

$$1 + \lambda\mu = \frac{nH\mu - r_2\mu^2}{1+r_1} + 1 + w\mu,$$

$$\begin{aligned}(\lambda-\mu)^2 =&\left(n\frac{H-\mu}{1+r_1}+w\right)^2\\ =&n^2\frac{\mu^2-2H\mu+H^2}{(1+r_1)^2}+2n\frac{H-\mu}{1+r_1}w+w^2\\ =&\frac{n^2r_1-nr_2}{r_1(1+r_1)^2}\mu^2+\frac{2nH(r_2-nr_1)}{r_1(1+r_1)^2}\mu+\frac{S}{r_1(1+r_1)}\\ &+\frac{n^2H^2(r_1-1)}{r_1(1+r_1)^2}+2n\frac{H-\mu}{1+r_1}w,\\ f_3 =&r_1\lambda^3+r_2\mu^3+\nu^3\\ =&r_1(1-r_1^2)\lambda^3+r_2(1-r_2^2)\mu^3+n^3H^3-3n^2H^2r_1\lambda-3n^2H^2r_2\mu\\ &+3nHr_1^2\lambda^2+3nHr_2^2\mu^2-3r_1^2r_2\lambda^2\mu-3r_1r_2^2\lambda\mu^2+6nHr_1r_2\lambda\mu\\ =&\dots\mu^3+\dots\mu^2+\dots\mu+\dots+w\mu(\dots\mu+\dots),\end{aligned}$$

that is

$$(1+\lambda\mu)(3nS-3H^2-2r_1r_2(\lambda-\mu)^2)+nHf_3=P_1(\mu)+P_2(\mu)w,$$

where P_1 and P_2 are polynomials of constant coefficients. For P_2 one has

$$P_2(t)=\dots t^3+\dots t^2+\dots t-\frac{4nr_1r_2H}{1+r_1},$$

therefore, it is not identically zero if $H\neq 0$. For the case $H=0$ the same follows from

$$P_2(t)=\dots t^3+\frac{(3n+3nr_1-2r_2)S+4nr_1r_2}{1+r_1}t,$$

with

$$3n+3nr_1-2r_2\geq 3n-2r_2\geq n>0.$$

It follows that $w-R(\mu)=0$ for a rational function R. The function

$$F(t):=\pm\sqrt{\frac{-nr_2t^2+2nr_2Ht+(1+r_1)S-n^2H^2}{r_1(1+r_1)^2}}-R(t)$$

is analytical and not constant. $F(\mu)=0$ then implies that μ is constant. Consequently λ and ν are also constant and the proposition follows.

Case 3: $r_1=:r=n-2>1$, $r_2=r_3=1$.

Without loss of generality let $a=1$ and $\alpha=n$. Then the derivatives of the principal curvatures in e_A direction vanish, and analogously to case 2 one has

$$\begin{aligned}h_{AnBn} =&\delta_{AB}\left(\lambda_{nn}+\frac{\lambda_1\nu_1}{\mu-\nu}\right)+\frac{2}{\mu-\lambda}h_{1An}h_{1Bn},\\ h_{AnnB} =&\delta_{AB}\left(\frac{\nu_1\lambda_1}{\mu-\lambda}+\frac{\nu_n\lambda_n}{\nu-\lambda}+\frac{2\lambda_n^2}{\lambda-\nu}\right)+\frac{2}{\mu-\nu}h_{1An}h_{1Bn}\end{aligned}$$

and thus

$$h_{1An}h_{1Bn} = \frac{z_2}{2}\delta_{AB}, \tag{3.6}$$

where

$$z_2 := \lambda_1\nu_1 + \frac{(\mu-\lambda)(\mu-\nu)}{\lambda-\nu}\left((\lambda-\nu)(1+\lambda\nu) - \lambda_{nn} + \frac{\lambda_n\nu_n}{\nu-\lambda} + \frac{2\lambda_n^2}{\lambda-\nu}\right).$$

As in case 2 it follows that $h_{1An} = 0$ for all A. From $z_2 = 0$ one has

$$\begin{aligned}\lambda_{nn} =&(\lambda-\nu)(1+\lambda\nu) + \frac{\lambda_n\nu_n}{\nu-\lambda} + \frac{2\lambda_n^2}{\lambda-\nu} + \frac{\lambda-\nu}{(\mu-\lambda)(\mu-\nu)}\lambda_1\nu_1\\ =&(\lambda-\nu)(1+\lambda\nu) - (n-2)\frac{\lambda-\nu}{(\mu-\nu)^2}\lambda_1^2 + \frac{(n+1)\mu-\nu-nH}{(\lambda-\nu)(\mu-\nu)}\lambda_n^2\end{aligned} \tag{3.7}$$

and in the same way it follows that

$$\lambda_{11} = (\lambda-\mu)(1+\lambda\mu) + \frac{(n+1)\nu-\mu-nH}{(\lambda-\mu)(\nu-\mu)}\lambda_1^2 - (n-2)\frac{\lambda-\mu}{(\nu-\mu)^2}\lambda_n^2. \tag{3.8}$$

From $h_{a1an} - h_{a1na} = 0$ one has

$$\lambda_{1n} = \frac{(n-2)(\lambda-\mu)^2(\lambda-\nu) + n(n-1)(\mu-\nu)^2(\lambda-H)}{(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu)}\lambda_1\lambda_n \tag{3.9}$$

and again the same holds true for reversed indices:

$$\lambda_{n1} = \frac{(n-2)(\lambda-\nu)^2(\lambda-\mu) + n(n-1)(\mu-\nu)^2(\lambda-H)}{(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu)}\lambda_1\lambda_n. \tag{3.10}$$

(2.15) is of the form

$$\begin{aligned}|\nabla h|^2 =&3(n-2)\lambda_1^2 + \mu_1^2 + 3\nu_1^2 + 3(n-2)\lambda_n^2 + 3\mu_n^2 + \nu_n^2\\ =&\left(3(n-2) + (n-2)^2\frac{(\lambda-\nu)^2}{(\nu-\mu)^2} + 3(n-2)^2\frac{(\lambda-\mu)^2}{(\mu-\nu)^2}\right)\lambda_1^2\\ &+\left(3(n-2) + 3(n-2)^2\frac{(\lambda-\nu)^2}{(\nu-\mu)^2} + (n-2)^2\frac{(\lambda-\mu)^2}{(\mu-\nu)^2}\right)\lambda_n^2,\end{aligned}$$

that is

$$\begin{aligned}(\nu-\mu)^2|\nabla h|^2 =&(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\nu)^2)\lambda_1^2\\ &+(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\mu)^2)\lambda_n^2.\end{aligned} \tag{3.11}$$

If $\lambda_1 = 0$ on an open set, (3.8) and (3.11) imply

$$|\nabla h|^2 = (3nS - 3H^2 - 2(n-2)(\lambda-\mu)^2)(1+\lambda\mu)$$

and as in case 2 it follows that the principal curvatures are constant. The same holds true for $\lambda_n = 0$, therefore we can presume in the following that $\lambda_1 \neq 0$ and $\lambda_n \neq 0$. Deriving (3.11) in direction e_1 yields

$$2(\nu-\mu)(\nu_1-\mu_1)|\nabla h|^2 + (\nu-\mu)^2(|\nabla h|^2)_1$$

$$
\begin{aligned}
&= -4(n-2)^2(\lambda-\nu)(\lambda_1-\nu_1)\lambda_1^2 \\
&\quad + 2(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\nu)^2)\lambda_1\lambda_{11} \\
&\quad - 4(n-2)^2(\lambda-\mu)(\lambda_1-\mu_1)\lambda_n^2 \\
&\quad + 2(3(n-2)(nS-H^2) - 2(n-2)^2(\lambda-\mu)^2)\lambda_n\lambda_{n1}
\end{aligned}
$$

and with

$$
\begin{aligned}
\nu_1 - \mu_1 &= n(n-2)\frac{\lambda-H}{\mu-\nu}\lambda_1, \\
(|\nabla h|^2)_1 &= -nH(f_3)_1 = -3n(n-2)H(\lambda-\mu)(\lambda-\nu)\lambda_1
\end{aligned}
$$

one has

$$
\begin{aligned}
&n|\nabla h|^2(H-\lambda) - \frac{3n}{2}H(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu) \\
&= -2n(n-2)(\mu-H)\frac{\lambda-\nu}{\mu-\nu}\lambda_1^2 - 2n(n-2)(\nu-H)\frac{\lambda-\mu}{\nu-\mu}\lambda_n^2 \\
&\quad + (3nS - 3H^2 - 2(n-2)(\lambda-\nu)^2)\lambda_{11} \\
&\quad + (3nS - 3H^2 - 2(n-2)(\lambda-\mu)^2)\frac{\lambda_n}{\lambda_1}\lambda_{n1}. \qquad (3.12)
\end{aligned}
$$

To simplify notation, we set

$$A_1(x,y) = 3(nS-H^2) - 2(n-2)(x-y)^2.$$

Putting (3.8) and (3.10) into (3.12) we have

$$
\begin{aligned}
&n|\nabla h|^2(H-\lambda) - \frac{3n}{2}H(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu) - A_1(\lambda,\nu)(\lambda-\mu)(1+\lambda\mu) \\
&= \left(-2n(n-2)(\mu-H)\frac{\lambda-\nu}{\mu-\nu} + A_1(\lambda,\nu)\frac{(n+1)\nu-\mu-nH}{(\lambda-\mu)(\nu-\mu)}\right)\lambda_1^2 \\
&\quad - \left(2n(\nu-H)\frac{\lambda-\mu}{\nu-\mu} + \frac{3nS-3H^2+2(n-2)(\lambda-\mu)(\lambda-\nu)}{\mu-\nu}\right. \\
&\quad \left. - \frac{n(n-1)(\lambda-H)}{(n-2)\lambda-\mu)(\lambda-\nu)}A_1(\lambda,\mu)\right)(n-2)\lambda_n^2. \qquad (3.13)
\end{aligned}
$$

Analogously we have

$$
\begin{aligned}
&n|\nabla h|^2(H-\lambda) - \frac{3n}{2}H(\mu-\nu)^2(\lambda-\mu)(\lambda-\nu) - A_1(\lambda,\mu)(\lambda-\nu)(1+\lambda\nu) \\
&= \left(-2n(n-2)(\nu-H)\frac{\lambda-\mu}{\nu-\mu} + A_1(\lambda,\mu)\frac{(n+1)\mu-\nu-nH}{(\lambda-\nu)(\mu-\nu)}\right)\lambda_n^2 \\
&\quad - \left(2n(\mu-H)\frac{\lambda-\nu}{\mu-\nu} + \frac{3nS-3H^2+2(n-2)(\lambda-\mu)(\lambda-\nu)}{\nu-\mu}\right. \\
&\quad \left. - \frac{n(n-1)(\lambda-H)}{(n-2)(\lambda-\mu)(\lambda-\nu)}A_1(\lambda,\nu)\right)(n-2)\lambda_1^2. \qquad (3.14)
\end{aligned}
$$

From (3.11) one has

$$(n-2)\lambda_n^2 = \frac{(\nu-\mu)^2|\nabla h|^2}{A_1(\lambda,\mu)} - \frac{(n-2)\varepsilon(\nu)}{A_1(\lambda,\mu)}\lambda_1^2. \tag{3.15}$$

As in case 2

$$(n-2)\lambda+\mu+\nu = nH, \quad (n-2)\lambda^2+\mu^2+\nu^2 = S,$$

yield

$$\lambda = \frac{1}{n-1}(nH-\nu) - \frac{1}{n-2}w, \quad \mu = \frac{1}{n-1}(nH-\nu)+w, \tag{3.16}$$

with

$$w := \pm\frac{\sqrt{n-2}}{n-1}\sqrt{-n\nu^2+2nH\nu+(n-1)S-n^2H^2}, \tag{3.17}$$

where again the sign of w can be assumed to remain the same. We set

$$\varepsilon(\lambda,\mu,\nu) = A_1(\lambda,\mu)A_2(\lambda,\mu,\nu)A_3(\lambda,\mu,\nu) + A_1(\lambda,\nu)A_2(\lambda,\nu,\mu)A_3(\lambda,\nu,\mu),$$

where A_1 was already defined by

$$A_1(x,y) := 3nS - 3H^2 - 2(n-2)(x-y)^2,$$

and A_2 and A_3 given by

$$\begin{aligned}
A_2(x,y,z) :=& (y-z)(x-z)\Big(n(n-2)|\nabla h|^2(x-H)(x-y)A_1(x,z)\\
&+\frac{3}{2}n(n-2)H(y-z)^2(x-y)^2(x-z)A_1(x,z)\\
&+(n-2)(1+xy)(x-y)^2A_1(x,z)^2\\
&+2n(n-2)|\nabla h|^2(y-H)(x-y)(x-z)(z-y)\\
&+|\nabla h|^2((n+1)z-y-nH)(z-y)A_1(x,z)\Big),\\
A_3(x,y,z) :=& -2(n-2)^2n(y-H)(x-y)(x-z)^2A_1(x,y)\\
&+(n-2)^2(x-y)(x-z)(3nS-3H^2+2(n-2)(x-y)(x-z))A_1(x,y)\\
&+n(n-1)(n-2)(x-H)(y-z)A_1(x,y)A_1(x,z)\\
&-2n(n-2)^2(z-H)(x-y)^2(x-z)A_1(x,z)\\
&-(n-2)(x-y)((n+1)y-z-nH)A_1(x,y)A_1(x,z).
\end{aligned}$$

From (3.13), (3.14) and (3.15) one has the following condition for λ, μ and ν:

$$\varepsilon(\lambda,\mu,\nu) = |\nabla h|^2(\mu-\nu)^2A_3(\lambda,\mu,\nu)A_3(\lambda,\nu,\mu). \tag{3.18}$$

Using (3.16) and (3.17) the terms in (3.18) can be written as polynomials in ν and w whose leading coefficients are given by

$$\begin{aligned}
A_1(\lambda,\mu) =& 2n\nu^2 - 4nH\nu + (n+2)S + (2n^2-3)H^2,\\
A_1(\lambda,\nu) =& -2\frac{(n-2)n^2-n}{(n-1)^2}\nu^2 + \cdots - \left(\frac{4n}{n-1}\nu+\cdots\right)w,
\end{aligned}$$

$$
\begin{aligned}
A_2(\lambda,\mu,\nu) =& -\frac{8n^4(7n^6-42n^5+57n^4+44n^3-79n^2-18n-1)}{(n-2)(n-1)^8}\nu^{10}+\cdots\\
&+\left(\frac{8n^4(n^6-6n^5-9n^4+68n^3-41n^2-46n+1)}{(n-2)(n-1)^7}\nu^9+\cdots\right)w,\\
A_2(\lambda,\nu,\mu) =& \frac{8n^4(3n^2-6n+1)}{(n-1)^4}\nu^{10}+\cdots+\left(\frac{8n^4(n^2-2n+3)}{(n-1)^3}\nu^9+\cdots\right)w,\\
A_3(\lambda,\mu,\nu) =& -\frac{2(n-2)n^3(7n^3-17n^2-19n+1)}{(n-1)^3}\nu^6+\cdots\\
&+\left(-\frac{2n^2(12n^3-20n^2+3n+1)}{(n-1)^2}\nu^5+\cdots\right)w,\\
A_3(\lambda,\nu,\mu) =& \frac{8(n-2)n^3(n^4-4n^3-2n^2+12n+1)}{(n-1)^4}\nu^6+\cdots\\
&+\left(\frac{4n^3(6n^4-9n^3-25n^2+29n+15)}{(n-1)^3}\nu^5+\cdots\right)w.
\end{aligned}
$$

(3.18) is then of the form

$$Q_1(\nu)+Q_2(\nu)w=0, \tag{3.19}$$

for polynomials Q_1 and Q_2 with constant coefficients. The leading coefficient of Q_1 is given by

$$
\begin{aligned}
Q_1(t) =& \frac{32(n-2)n^8}{(n-1)^{11}}(73n^{10}-709n^9+2273n^8-1255n^7-7101n^6+12067n^5\\
&-1089n^4-6461n^3+1048n^2+134n-4)t^{18}+\cdots,
\end{aligned}
$$

therefore, Q_1 is not identically zero. One then has from (3.19) that $w=R(\nu)$ for a rational function R or that $Q_1(\nu)=0$; in both cases the proposition follows. □

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