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# ON THE ESTRADA INDEX OF POINT ATTACHING STRICT $k$-QUASI TREE GRAPHS

MOHAMMAD A. IRANMANESH[1] AND RAZIYEH NEJATI[2]

Abstract. Let $G = (V, E)$ be a finite and simple graph with $\lambda_1, \lambda_2, \ldots, \lambda_n$ as its eigenvalues. The Estrada index of $G$ is $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$. For a positive integer $k$, a connected graph $G$ is called strict $k$-quasi tree if there exists a set $U$ of vertices of size $k$ such that $G \setminus U$ is a tree and this is as small as possible with this property. In this paper, we define point attaching strict $k$-quasi tree graphs and obtain the graph with minimum Estrada index among point attaching strict $k$-quasi tree graphs with $k$ even cycles.

## 1. Introduction

Let $G = (V(G), E(G))$ be a finite and simple graph of order $n$, where by $V(G)$ and $E(G)$ we denote the set of vertices and edges, respectively. Let $A(G)$ be the adjacency matrix of $G$, and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues. The Estrada index of $G$ is defined as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}.$$

which was first proposed by Estrada in 2000 [6]. We refer reader to [7, 8, 15, 16] for multiple applications of Estrada index in various fields, for example in network science and biochemistry. The results for trees can be found in [3, 10, 13, 19]. Gutman approximated the Estrada index of cycles and paths in [9]. The unicyclic graphs with maximum and minimum Estrada index have been determined in [5]. Recently, the Esrada index of the cactus graphs in which every block is a triangle, has been characterized in [11, 12].

A connected graph $G$ is called *quasi tree* if there exists $v_0 \in V(G)$ such that $G \setminus \{v_0\}$ is a tree. Lu in [14] has determined the Randić index of quasi trees. The Harary index

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of quasi tree graphs and generalized quasi tree graphs are presented in [18]. A *strict $k$-quasi tree graph* $G$ is a connected graph which is not a tree, and $k$ is the smallest positive integer such that there exists a $k$-element subset $U$ of vertices for which $G \setminus U$ is a tree.

Let $G$ be a connected graph constructed from pairwise disjoint connected graphs $G_1, G_2, \ldots, G_d$ as follows: select a vertex of $G_1$,, a vertex of $G_2$,, and identify these two vertices. Then continue in this manner inductively. More precisely, suppose that we have already used $G_1, G_2, \ldots, G_i$ in the construction, where $2 \leq i \leq d-1$. Then select a vertex in the already constructed graph (which may in particular be one of the already selected vertices) and a vertex $G_{i+1}$; and identify these two vertices. Note that the graph $G$ constructed in this way has a tree-like structure, the $G_i$'s being its building stones. We will briefly say that $G$ is obtained by *point attaching* from $G_1, G_2, \ldots, G_d$ and that $G_i$'s are the primary subgraphs of $G$ [4].

A graph $G$ is said to be point attaching strict $k$-quasi, if it is constructed from primary subgraphs $G_1, G_2, \ldots, G_d$ where each primary subgraph $G_i$ is a strict $k_i$-quasi tree graph for each $1 \leq i \leq d$, and $k = \sum_{i=1}^{d} k_i$.

In this paper we study the Estrada index of point attaching strict $k$-quasi graphs.

## 2. Preliminaries

For $\ell \in \mathbb{N} \cup \{0\}$, let $S_\ell(G) = \sum_{i=1}^{n} \lambda_i^\ell$ be the $\ell^{th}$ spectral moment of $G$, which is equal to the number of closed walks of length $\ell$ in $G$ [2]. For every graph $G$, we have $S_0(G) = n, S_1(G) = \mathbf{C}, S_2(G) = 2m, S_3(G) = 6\mathbf{D}$, and $S_4(G) = 2\sum_{i=1}^{n} d_i^2 - 2m + 8\mathbf{Q}$, where $n, \mathbf{C}, m, \mathbf{D}, \mathbf{Q}$ denote the number of vertices, the number of loops, the number of edges, the number of triangles and the number of quadrangles in $G$, respectively and $d_i = d_i(G)$ is the degree of vertex $v_i$ in $G$ [2]. Bearing in mind the Taylor expansion of $e^x$, we have the following equation for the Estrada index of graph $G$,

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i} = \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} \frac{\lambda_i^\ell}{\ell!} = \sum_{\ell=0}^{\infty} \frac{S_\ell(G)}{\ell!}. \tag{2.1}$$

It follows from Equation 2.1 that $EE(G)$ is a strictly monotonously increasing function of $S_\ell(G)$. Let $G_1$ and $G_2$ be two graphs. If $S_\ell(G_1) \leq S_\ell(G_2)$ holds for all positive integer $\ell$, then $EE(G_1) \leq EE(G_2)$. Moreover, if the strict inequality $S_\ell(G_1) < S_\ell(G_2)$ holds for at least one value $\ell_0 \geq 0$, then $EE(G_1) < EE(G_2)$.

Recall that a sequence $a_0, a_1, \ldots, a_n$ of numbers is said to be *unimodal* if for some $0 \leq i \leq n$ we have $a_0 \leq a_1 \leq \cdots \leq a_i \geq a_{i+1} \geq \cdots \geq a_n$, and this sequence is called *symmetric* if $a_i = a_{n-i}$ for $0 \leq i \leq n$ [17]. Thus a symmetric unimodal sequence $a_0, a_1, \ldots, a_n$ has its maximum at the middle term ($n$ even) or middle two terms ($n$ odd). Let $A$ be the adjacency matrix of the graph $G$. It is well-known that the entry$(A^\ell)_{i,j}$ represents the number of walks of length $\ell$ from vertex $v_i$ to vertex $v_j$ [1]. Obviously, $(A^\ell)_{i,j} = (A^\ell)_{j,i}$ for undirected graphs.

Throughout this paper, $\Gamma(k)$ is a point attaching strict $k$-quasi tree graph with $k$ even cycles (see Figure 1) and $M_\ell(G)$ denotes the set of closed walks of length $\ell$ in $G$, and we show that among all point attaching strict $k$-quasi tree graphs with $k$ even cycles, $\Gamma(k)$ is the graph with minimum Estrada index.

## 3. The Number of Closed Walks of Length $\ell$ in $\Gamma(k)$

Let $M_\ell(k(c-1),i)$ denote the set of closed walks of length $\ell$ starting at the vertex $v_i$ in $\Gamma(k)$ with $k$ even cycles of length $c$ and $|M_\ell(k(c-1),i)| = S_\ell(k(c-1),i)$ denote the number of closed walks of length $\ell$ starting at the vertex $v_i$ in $\Gamma(k)$ (see Figure 1).

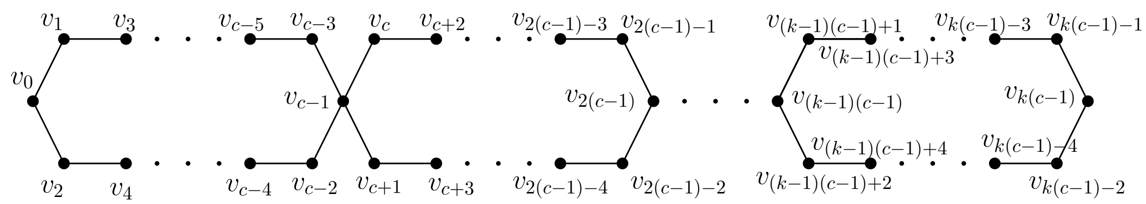

Figure 1. The graph $\Gamma(k)$.

**Lemma 3.1.** *The map $\varphi : V(\Gamma(k)) \longrightarrow V(\Gamma(k))$, given by $\varphi(v_i) = v_{k(c-1)-i}$ is an automorphism.*

*Proof.* One can easily see that $\varphi$ is bijective. Let vertices $v_i$ and $v_j$ be adjacent. Then by definition of $\varphi$, we have the following cases.

(i) $\varphi(v_0) = v_{k(c-1)}$ and $\varphi(v_{k(c-1)}) = v_0$.

(ii) $i = t(c-1)$, $0 < t < k$. In this case $v_j \in \{v_{i-1}, v_{i-2}, v_{i+1}, v_{i+2}\}$. Hence, $k(c-1) - i = k(c-1) - t(c-1) = (k-t)(c-1)$. This implies that $\varphi(v_i) = v_{(k-t)(c-1)}$.

We will only prove the case $v_j = v_{i-1}$. A similar argument can be used for other cases. If $v_j = v_{i-1}$, then $k(c-1) - j = k(c-1) - t(c-1) + 1 = (k-t)(c-1) + 1$. Hence $\varphi(v_j) = v_{(k-t)(c-1)+1}$ which is adjacent to $\varphi(v_i)$.

(iii) $i = t(c-1) + s$, $0 < t \leqslant k-1$, $1 \leqslant s \leqslant c-2$. In this case $v_j \in \{v_{i-2}, v_{i+2}\}$. Hence, $k(c-1) - j = k(c-1) - t(c-1) - s = (k-t)(c-1) - s$. This implies that $\varphi(v_i) = v_{(k-t)(c-1)-s}$.

If $v_j = v_{i-2}$, then $k(c-1) - t(c-1) - s + 2 = (k-t)(c-1) - s + 2$. Hence, $\varphi(v_j) = v_{(k-t)(c-1)-s+2}$ which is adjacent to $\varphi(v_i)$. The proof for case $v_j = v_{i+2}$ is similar. □

**Corollary 3.1.** *Let $A$ be the adjacency matrix of the point attaching strict $k$-quasi tree graph $\Gamma(k)$. Then $(A^\ell)_{i,j} = (A^\ell)_{k(c-1)-i,k(c-1)-j}$ for $0 \leqslant i, j \leqslant k(c-1)$.*

*Proof.* This is an immediate consequence of Lemma 3.1. □

**Lemma 3.2.** *If $k \geqslant 2$ and $t$ are integers and $0 \leqslant t \leqslant c-2$, then:*

$$\begin{aligned} S_\ell(k(c-1),t) &\le S_\ell(k(c-1),t+(c-1)) \\ &\le \cdots \le S_\ell\left(k(c-1), t+\left(\left[\frac{k}{2}\right]-1\right)(c-1)\right) \\ &\le S_\ell\left(k(c-1), t+\left[\frac{k}{2}\right](c-1)\right), \end{aligned}$$

*where $\ell \geq c-1$. If $\ell \geq [\frac{k}{2}]$, then strict inequalities hold.*

*Proof.* We prove every diagonal and the main diagonal of the matrix $A^\ell$ are unimodal. By Lemma 3.1, $(A^\ell)_{t,j} = (A^\ell)_{k(c-1)-t,k(c-1)-j}$. So we only need to show that the diagonals paralleling to the main diagonal increase for $t+j \leqslant k(c-1)$.

By induction on integer $\ell$, we will show that for every $j \leqslant k(c-1)$ where $t+j+2c-2 \leqslant k(c-1)$, we have:

$$(A^\ell)_{t+c-1,j+c-1} \geqslant (A^\ell)_{t,j}.$$

By the definition of $\Gamma(k)$ we have $A_{t,j}=1$ if and only if $A_{t+c-1,j+c-1}=1$. Therefore, the result is hold for $\ell=1$. Assume that the result holds for integer $\ell$. There are four cases as follows.
**Case 1:** $t, j \equiv 0 \pmod{(c-1)}$.

Since the set of walks of length $\ell+1$ from $v_t$ to $v_j$ is in bijective correspondence with the set of walks of length $\ell$ from $v_t$ to $v_h$ adjacent to $v_j$, so

$$\begin{aligned} (A^{\ell+1})_{t+c-1,j+c-1} =& (A^\ell)_{t+c-1,j+c-2} + (A^\ell)_{t+c-1,j+c-3} + (A^\ell)_{t+c-1,j+c} \\ &+ (A^\ell)_{t+c-1,j+c+1}, \\ (A^{\ell+1})_{t,j} =& (A^\ell)_{t,j-1} + (A^\ell)_{t,j-2} + (A^\ell)_{t,j+1} + (A^\ell)_{t,j+2}. \end{aligned}$$

By the induction hypothesis, we have the following results:

$$\begin{aligned} (A^\ell)_{t+c-1,j+c-2} \geq& (A^\ell)_{t,j-1}, \\ (A^\ell)_{t+c-1,j+c} \geq& (A^\ell)_{t,j+1}, \quad \text{for } t+j+2 \leq k(c-1), \\ (A^\ell)_{t+c-1,j+c-3} \geq& (A^\ell)_{t,j-2}, \\ (A^\ell)_{t+c-1,j+c+1} \geq& (A^\ell)_{t,j+2}, \quad \text{for } t+j+2 \leq k(c-1). \end{aligned}$$

Hence, we have $(A^{\ell+1})_{t+c-1,j+c-1} \geq (A^{\ell+1})_{t,j}$. In addition we will show that for $\ell \geq [k(c-1)/2]$ the strict inequalities hold.

For the strict inequality, let $1 \leq r \leq k$ be a fixed number, we consider two rows $r(c-1)$ and $(r-1)(c-1)$, $j \leq k(c-1)$. Then

$$(A^{\ell+1})_{r(c-1),c-1} = (A^\ell)_{r(c-1),c-2} + (A^\ell)_{r(c-1),c-3} + (A^\ell)_{r(c-1),c} + (A^\ell)_{r(c-1),c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1),0} = (A^\ell)_{(r-1)(c-1),1} + (A^\ell)_{(r-1)(c-1),2}.$$

Note that, since $\Gamma(k)$ is symmetric we have,

$$\begin{aligned}(A^\ell)_{r(c-1),c-2} =&(A^\ell)_{r(c-1),c-3} > 0,\\ (A^\ell)_{r(c-1),c} =&(A^\ell)_{r(c-1),c+1} > 0,\\ (A^\ell)_{r(c-1),1} =&(A^\ell)_{r(c-1),2} > 0,\end{aligned}$$

for $\ell \geq r(c-1)$. So,

$$(A^{\ell+1})_{r(c-1),c-1} = 2(A^\ell)_{r(c-1),c-2} + 2(A^\ell)_{r(c-1),c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1),0} = 2(A^\ell)_{(r-1)(c-1),2}.$$

By the induction hypothesis, the following inequality holds:

$$(A^\ell)_{r(c-1),c+1} \geq (A^\ell)_{(r-1)(c-1),2}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1),c-1} > (A^{\ell+1})_{(r-1)(c-1),0}$. This causes the chain of strict inequalities

$$\begin{aligned}&(A^{\ell+2})_{r(c-1),2(c-1)} > (A^{\ell+2})_{(r-1)(c-1),c-1},\\ &(A^{\ell+3})_{r(c-1),3(c-1)} > (A^{\ell+3})_{(r-1)(c-1),2(c-1)}.\end{aligned}$$

Finally, we have

$$(A^{\ell+(k-r+1)})_{r(c-1),(k-r+1)(c-1)} > (A^{\ell+(k-r+1)})_{(r-1)(c-1),(k-r)(c-1)}.$$

**Case 2:** $t \equiv 0 \pmod{(c-1)}$ and $j \not\equiv 0 \pmod{(c-1)}$. Let $j \equiv 1 \pmod{(c-1)}$. Then

$$\begin{aligned}(A^{\ell+1})_{t+c-1,j+c-1} =&(A^\ell)_{t+c-1,j+c-2} + (A^\ell)_{t+c-1,j+c+1},\\ (A^{\ell+1})_{t,j} =&(A^\ell)_{t,j-1} + (A^\ell)_{t,j+2}.\end{aligned}$$

Similarly, by the induction hypothesis, we have

$$\begin{aligned}&(A^\ell)_{t+c-1,j+c-2} \geq (A^\ell)_{t,j-1},\\ &(A^\ell)_{t+c-1,j+c+1} \geq (A^\ell)_{t,j+2}, \quad \text{for } t+j+2 \leq k(c-1).\end{aligned}$$

Hence, we have $(A^{\ell+1})_{t+c-1,j+c-1} \geq (A^{\ell+1})_{t,j}$.

In addition for the strict inequality, let $1 \leq r \leq k$ be a fixed number, we consider two rows $r(c-1)$ and $(r-1)(c-1)$. Then

$$\begin{aligned}(A^{\ell+1})_{r(c-1),c} =&(A^\ell)_{r(c-1),c-1} + (A^\ell)_{r(c-1),c+2} = (A^{\ell-1})_{r(c-1),c-2}\\ &+ (A^{\ell-1})_{r(c-1),c-3} + (A^{\ell-1})_{r(c-1),c} + (A^{\ell-1})_{r(c-1),c+1} + (A^\ell)_{r(c-1),c+2}\end{aligned}$$

and

$$\begin{aligned}(A^{\ell+1})_{(r-1)(c-1),1} =&(A^\ell)_{(r-1)(c-1),0} + (A^\ell)_{(r-1)(c-1),3}\\ =&(A^{\ell-1})_{(r-1)(c-1),1} + (A^{\ell-1})_{(r-1)(c-1),2} + (A^\ell)_{(r-1)(c-1),3}.\end{aligned}$$

Note that, since $\Gamma(k)$ is symmetric we have,

$$\begin{aligned}(A^{\ell-1})_{r(c-1),c-2} =&(A^{\ell-1})_{r(c-1),c-3} > 0,\\ (A^{\ell-1})_{r(c-1),c} =&(A^{\ell-1})_{r(c-1),c+1} > 0,\\ (A^{\ell-1})_{r(c-1),1} =&(A^{\ell-1})_{r(c-1),2} > 0,\end{aligned}$$

for $\ell \geq r(c-1)$.

So,

$$(A^{\ell+1})_{r(c-1),c} = 2(A^{\ell-1})_{r(c-1),c-2} + 2(A^{\ell-1})_{r(c-1),c} + (A^{\ell})_{r(c-1),c+2}$$

and

$$(A^{\ell+1})_{(r-1)(c-1),1} = 2(A^{\ell-1})_{(r-1)(c-1),1} + (A^{\ell})_{(r-1)(c-1),3}.$$

By the induction hypothesis, the following inequalities hold:

$$(A^{\ell-1})_{r(c-1),c} \geq (A^{\ell-1})_{(r-1)(c-1),1},\ (A^{\ell})_{r(c-1),c+2} \geq (A^{\ell})_{(r-1)(c-1),3}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1),c} > (A^{\ell+1})_{(r-1)(c-1),1}$. This causes the chain of strict inequalities

$$\begin{aligned}&(A^{\ell+2})_{r(c-1),2(c-1)+1} > (A^{\ell+2})_{(r-1)(c-1),c},\\ &(A^{\ell+3})_{r(c-1),3(c-1)+1} > (A^{\ell+3})_{(r-1)(c-1),2(c-1)+1}.\end{aligned}$$

Finally, we have

$$(A^{\ell+k-r})_{r(c-1),(k-r+1)(c-1)+1} > (A^{\ell+k-r})_{(r-1)(c-1),(k-r)(c-1)+1}.$$

A similar argument can be used for the cases $j \equiv \{2, 3, \ldots, c-2\} \pmod{(c-1)}$ .

**Case 3:** $t \not\equiv 0 \pmod{(c-1)}$ and $j \equiv 0 \pmod{(c-1)}$. Let $t \equiv 1 \pmod{(c-1)}$. Then

$$\begin{aligned}(A^{\ell+1})_{t+c-1,j+c-1} =&(A^{\ell})_{t+c-1,j+c-2} + (A^{\ell})_{t+c-1,j+c-3} + (A^{\ell})_{t+c-1,j+c} + (A^{\ell})_{t+c-1,j+c+1},\\ (A^{\ell+1})_{t,j} =&(A^{\ell})_{t,j-1} + (A^{\ell})_{t,j-2} + (A^{\ell})_{t,j+1} + (A^{\ell})_{t,j+2}.\end{aligned}$$

By the induction hypothesis, we have:

$$\begin{aligned}(A^{\ell})_{t+c-1,j+c-2} \geq&(A^{\ell})_{t,j-1},\\ (A^{\ell})_{t+c-1,j+c} \geq&(A^{\ell})_{t,j+1}, \quad \text{for } t+j+1 \leqslant k(c-1),\\ (A^{\ell})_{t+c-1,j+c-3} \geq&(A^{\ell})_{t,j-2},\\ (A^{\ell})_{t+c-1,j+c+1} \geq&(A^{\ell})_{t,j+2}, \quad \text{for } t+j+2 \leqslant k(c-1).\end{aligned}$$

Hence, we have $(A^{\ell+1})_{t+c-1,j+c-1} \geq (A^{\ell+1})_{t,j}$.

For the strict inequality, let $1 \leq r \leq k$ be a fixed number, for two rows $r(c-1)+1$ and $(r-1)(c-1)+1$ we have

$$(A^{\ell+1})_{r(c-1)+1,c-1} = (A^{\ell})_{r(c-1)+1,c-2} + (A^{\ell})_{r(c-1)+1,c-3} + (A^{\ell})_{r(c-1)+1,c} + (A^{\ell})_{r(c-1)+1,c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1)+1,0} = (A^{\ell})_{(r-1)(c-1)+1,1} + (A^{\ell})_{(r-1)(c-1)+1,2}.$$

Note that since $\Gamma(k)$ is symmetric we have

$$\begin{aligned}(A^{\ell})_{r(c-1)+1,c-2} =&(A^{\ell})_{r(c-1)+1,c-3} > 0,\\ (A^{\ell})_{r(c-1)+1,c} =&(A^{\ell})_{r(c-1)+1,c+1} > 0,\\ (A^{\ell})_{r(c-1)+1,1} =&(A^{\ell})_{r(c-1)+1,2} > 0,\end{aligned}$$

for $\ell \geq r(c-1)$.

So,

$$(A^{\ell+1})_{r(c-1)+1,c-1} = 2(A^{\ell})_{r(c-1)+1,c-2} + 2(A^{\ell})_{r(c-1)+1,c+1}$$

and

$$(A^{\ell+1})_{(r-1)(c-1)+1,0} = 2(A^{\ell})_{(r-1)(c-1)+1,2}.$$

By the induction hypothesis, the following inequality holds:

$$(A^{\ell})_{r(c-1)+1,c+1} \geq (A^{\ell})_{(r-1)(c-1)+1,2}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1)+1,c-1} > (A^{\ell+1})_{(r-1)(c-1)+1,0}$. This causes the chain of strict inequalities

$$\begin{aligned}(A^{\ell+2})_{r(c-1)+1,2(c-1)} &> (A^{\ell+2})_{(r-1)(c-1)+1,c-1},\\ (A^{\ell+3})_{r(c-1)+1,3(c-1)} &> (A^{\ell+3})_{(r-1)(c-1)+1,2(c-1)}.\end{aligned}$$

Finally, we have:

$$(A^{\ell+(k-r+1)})_{r(c-1)+1,(k-r+1)(c-1)-1} > (A^{\ell+(k-r+1)})_{(r-1)(c-1)+1,(k-r)(c-1)-1}.$$

A similar argument can be used for the cases $t \equiv \{2,3,\ldots,c-2\} \pmod{(c-1)}$.

**Case 4:** $t \not\equiv 0 \pmod{(c-1)}$ and $j \equiv 1 \pmod{(c-1)}$. Let $t \equiv 1 \pmod{(c-1)}$, we have

$$\begin{aligned}(A^{\ell+1})_{t+c-1,j+c-1} =&(A^{\ell})_{t+c-1,j+c-2} + (A^{\ell})_{t+c-1,j+c+1},\\ (A^{\ell+1})_{t,j} =&(A^{\ell})_{t,j-1} + (A^{\ell})_{t,j+2}.\end{aligned}$$

By the induction hypothesis, the following inequality holds:

$$(A^{\ell})_{t+c-1,j+c-2} \geq (A^{\ell})_{t,j-1}, \quad (A^{\ell})_{t+c-1,j+c+1} \geq (A^{\ell})_{t,j+2}.$$

Hence, we have $(A^{\ell})_{t+c-1,j+c-1} \geq (A^{\ell})_{t,j}$. For the strict inequality, let $1 \leq r \leq k$ be a fixed number, we consider two rows $r(c-1)+1$ and $(r-1)(c-1)+1$. Then

$$\begin{aligned}(A^{\ell+1})_{r(c-1)+1,c} =&(A^{\ell})_{r(c-1)+1,c-1} + (A^{\ell})_{r(c-1)+1,c+2}\\ =&(A^{\ell-1})_{r(c-1)+1,c-2} + (A^{\ell-1})_{r(c-1)+1,c-3} + (A^{\ell-1})_{r(c-1)+1,c}\\ &+ (A^{\ell-1})_{r(c-1)+1,c+1} + (A^{\ell})_{r(c-1)+1,c+2}\end{aligned}$$

and

$$\begin{aligned}(A^{\ell+1})_{(r-1)(c-1)+1,1} =&(A^{\ell})_{(r-1)(c-1)+1,0} + (A^{\ell})_{(r-1)(c-1)+1,3}\\ =&(A^{\ell-1})_{(r-1)(c-1)+1,1} + (A^{\ell-1})_{(r-1)(c-1)+1,2} + (A^{\ell})_{(r-1)(c-1)+1,3}.\end{aligned}$$

Note that since $\Gamma(k)$ is symmetric, $(A^{\ell-1})_{r(c-1)+1,c-2} = (A^{\ell-1})_{r(c-1)+1,c-3} > 0$, $(A^{\ell-1})_{r(c-1)+1,c} = (A^{\ell-1})_{r(c-1)+1,c+1} > 0$ and $(A^{\ell-1})_{r(c-1)+1,1} = (A^{\ell-1})_{r(c-1)+1,2} > 0$, for $\ell \geq r(c-1)$. So,

$$(A^{\ell+1})_{r(c-1)+1,c} = 2(A^{\ell-1})_{r(c-1)+1,c-2} + 2(A^{\ell-1})_{r(c-1)+1,c} + (A^{\ell})_{r(c-1)+1,c+2}$$

and

$$(A^{\ell+1})_{(r-1)(c-1)+1,1} = 2(A^{\ell-1})_{(r-1)(c-1)+1,1} + (A^{\ell})_{(r-1)(c-1)+1,3}.$$

By the induction hypothesis, the following inequalities hold:

$$(A^{\ell-1})_{r(c-1)+1,c} \geqslant (A^{\ell-1})_{(r-1)(c-1)+1,1}, \ (A^{\ell})_{r(c-1)+1,c+2} \geqslant (A^{\ell})_{(r-1)(c-1)+1,3}.$$

Thus, we have the strict inequality $(A^{\ell+1})_{r(c-1)+1,c} > (A^{\ell+1})_{(r-1)(c-1)+1,1}$. This causes the chain of strict inequalities

$$\begin{aligned}
&(A^{\ell+2})_{r(c-1)+1,2(c-1)+1} > (A^{\ell+2})_{(r-1)(c-1)+1,c},\\
&(A^{\ell+3})_{r(c-1)+1,3(c-1)+1} > (A^{\ell+3})_{(r-1)(c-1)+1,2(c-1)+1}.
\end{aligned}$$

Finally,

$$(A^{\ell+k-r})_{r(c-1)+1,(k-r+1)(c-1)+1} > (A^{\ell+k-r})_{(r-1)(c-1)+1,(k-r)(c-1)+1}.$$

A similar argument can be used for $t \equiv r \in \{2, 3, \ldots, c-2\} \pmod{(c-1)}$. $\square$

The number of closed walks of length $\ell$ starting at the vertex $v_t$ is equal to the entry $(t,t)$ in matrix $A^{\ell}$. Therefore,

$$S_{\ell}(k(c-1), t+(c-1)) = (A^{\ell})_{t+(c-1),t+(c-1)}.$$

By the induction hypothesis, we conclude that $S_{\ell}(k(c-1), t+(r-1)(c-1)) \leq S_{\ell}(k(c-1), t+r(c-1))$ for all $0 \leq t \leq c-1$ and $r \leq [\frac{k}{2}](c-1)$. Hence the strict inequality holds when $\ell \geq [\frac{k}{2}]$.

## 4. The Minimum Estrada Index of $\Gamma(k)$

Let $G'$ be a point attaching strict $k_1$-quasi tree graph of even length $c$ and $\delta \in V(G')$. For $k - k_1 = k_2$, let $G'(\lfloor \frac{k_2}{2} \rfloor, \lceil \frac{k_2}{2} \rceil)$ be the graph obtained from $G'$ by attaching two graphs $\Gamma(\lfloor \frac{k_2}{2} \rfloor)$ and $\Gamma(\lceil \frac{k_2}{2} \rceil)$ at $\delta$.

Let $N_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1); \delta)$ (respectively, $N_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1); \delta)$ be the set of $(\delta,\delta)$-walks of length $\ell$ in $G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1))$ (respectively, $G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1)$ starting and ending at the edges or only one edge in $G'$ and let $N'_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1)); \delta)$ (respectively, $N'_{\ell}(G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1); \delta))$ be the set of $(\delta,\delta)$-walks of length $\ell$ in $G'(\lfloor \frac{k_2}{2} \rfloor(c-1), \lceil \frac{k_2}{2} \rceil(c-1))$ (respectively, $G'(\lfloor \frac{k_2}{2} \rfloor(c-1)+c-1, \lceil \frac{k_2}{2} \rceil(c-1)-c+1)$ starting and ending at the edges or only one edge in union $\Gamma(\lfloor \frac{k_2}{2} \rfloor) \cup \Gamma(\lceil \frac{k_2}{2} \rceil)$ (respectively, $\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1))$.

In the following let $G'(\lfloor\frac{k_2}{2}\rfloor(c-1), \lceil\frac{k_2}{2}\rceil(c-1)) := G(1)$ and let $G'(\lfloor\frac{k_2}{2}\rfloor(c-1)+c-1, \lceil\frac{k_2}{2}\rceil(c-1)-c+1) := G(2)$. By our definition, both graphs $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$ and $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$ are isomorphic to $\Gamma(k_2)$, so they are denoted by $\Gamma(k_2)$.

**Lemma 4.1.** *If $\lfloor\frac{k_2}{2}\rfloor \geq 1$, then for positive integer $\ell$,*

(i) $|N_\ell(G'(2);\delta)| \leq |N_\ell(G'(1));\delta)|$;
(ii) $|N'_\ell(G'(2);\delta)| \leq |N'_\ell(G'(1));\delta)|$.

*Proof.* Let $\omega \in N_\ell(G'(2);\delta)$, we may decompose $\omega$ into maximal sections in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$ or in $G'$. Each of them is one of the following types.

(Type 1): a $(\delta,\delta)$- walk in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$.
(Type 2): a walk in $G'(2)$ with all edges in $G'$.

Similarly, we may decompose any $\omega \in N_\ell(G'(1);\delta)$ into maximal sections in $G'$ or in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$. Each of them is one of the following types.

(Type 3): a $(\delta,\delta)$- walk in union $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$.
(Type 4): a walk in $G'(1)$ with all edges in $G'$.

Next, for any $\omega \in N_\ell(G'(2);\delta)$, we can replace the even indices by the odd indices that are in front of each other see Figure 2. Hence, from now on, $\omega$ is a $(\delta,\delta)$- walk with

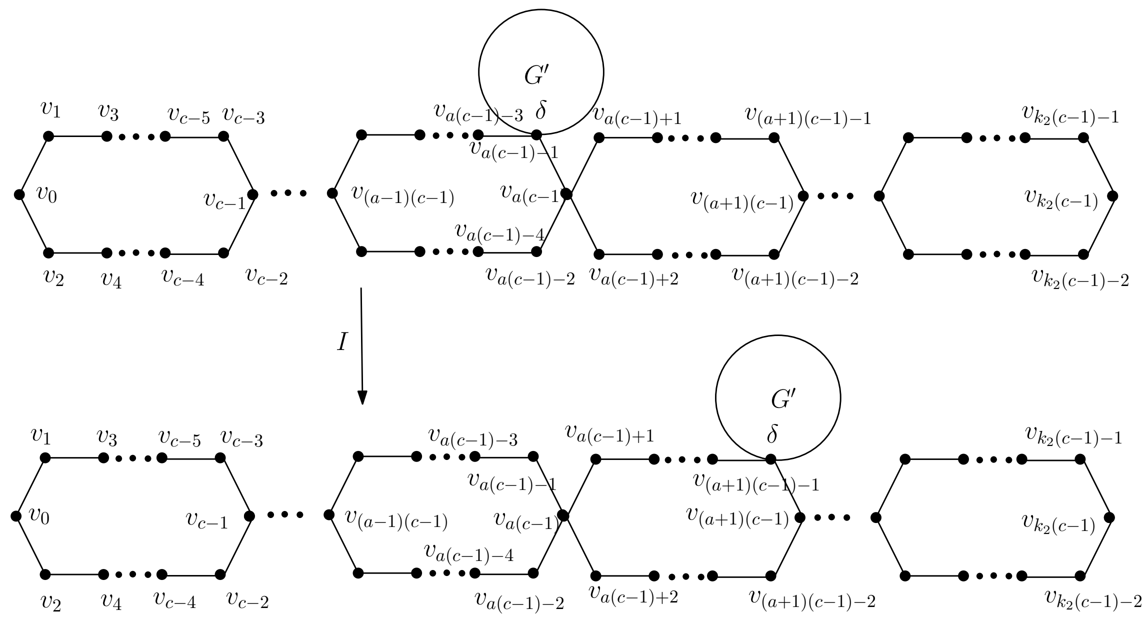


FIGURE 2. Transformation $I$.

only odd or even indices. So $\omega$ is a $(\delta,\delta)$- walk with odd indices. By Lemma 3.2 there is an injection mapping $\xi^1_{s'}$ that is a $(\delta,\delta)$- walk of length $s'$ in $\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1)$ into a $(\delta,\delta)$- walk of length $s'$ in $\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil)$.

Let $\omega' = \omega_1\omega_2\omega_3\cdots \in N_\ell(\Gamma(k_2))$, where $\omega_i$ is a walk of length $s'_i$ of type (1) or (2) for $i \geq 1$. Let $\xi^\star(\omega') = \xi^\star(\omega_1)\xi^\star(\omega_2)\cdots$, where $\xi^\star(\omega_i) = \xi^1_{s'_i}(\omega_i)$ and $\xi^\star(\omega_i) = \omega_i$ if $\omega_i$

is of type 2 so $\xi^\star(\omega_i)$ for $i \geq 1$ is of type 3 or 4 and thus $\xi^\star(\omega') \in N_\ell(G'(1))$. Thus $|N_\ell(G'(2);\delta)| \leqslant |N_\ell(G'(1);\delta)|$. This prove (i). The proof for (ii) is similar. $\square$

**Theorem 4.1.** *If $\lfloor \frac{k_2}{2} \rfloor \geq 1$, then $S_\ell(G'(2)) \leq S_\ell(G'(1))$. For $\ell \geq [\frac{k_2}{2}](c-1)$, the strict inequality holds.*

*Proof.* Let $B_1$ and $B_2$ be the sets of closed walks of length $\ell$ in $G'(1)$ and $G'(2)$ respectively, containing some edges in $G'$. Then $S_\ell(G'(2)) = S_\ell(\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)) + |B_2|$ and $S_\ell(G'(1)) = S_\ell(\Gamma(\lfloor \frac{k_2}{2} )\rfloor \cup \Gamma(\lceil \frac{k_2}{2} \rceil)) + |B_1|$. Since $\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)$ and $\Gamma(\lfloor \frac{k_2}{2} \rfloor) \cup \Gamma(\lceil \frac{k_2}{2} \rceil)$ are isomorphic to $\Gamma(k_2)$, we only need to prove that $|B_2| \leqslant |B_1|$ for all $\ell \geqslant 0$. Let $B_{21}$ and $B_{22}$ be two subsets of $B_2$ for which every closed walk starts at a vertex in $V(\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1))$ and $V(G') - \{\delta\}$, respectively. Then $|B_2| = |B_{21}| + |B_{22}|$. Let $B_{11}$ and $B_{12}$ be two subsets of $B_1$ for which every closed walk starts at a vertex in $V(\Gamma(\lfloor \frac{k_2}{2} \rfloor) \cup \Gamma(\lceil \frac{k_2}{2} \rceil))$ and $V(G') - \{\delta\}$, respectively. Then $|B_1| = |B_{11}| + |B_{12}|$.

We may decompose any $\omega \in B_{21}$ into three parts $\omega_1\omega_2\omega_3$, where $\omega_1, \omega_3$ are walks in $\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)$ and $\omega_2$ is the longest walk of $\omega$ in $G'(2)$ starting and ending at the edges or only one edge in $G'$. By the choice of $\omega_2$, we have that $\omega_2$ is a $(\delta,\delta)$-walk. Let $B_{21}(\omega,\ell) = \{\omega \in B_{21} : \omega_2 \text{ is a } (\delta,\delta)\text{- walk}\}$. Thus $|B_{21}| = |B_{21}(\omega,\ell)|$. Let $B_{11}(\omega,\ell) = \{\omega \in B_{11} : \omega_2 \text{ is a } (\delta,\delta)\text{-walk}\}$. So $|B_{11}| = |B_{11}(\omega,\ell)|$.

Let $V(\Gamma(\lfloor \frac{k_2}{2} \rfloor + 1) \cup \Gamma(\lceil \frac{k_2}{2} \rceil - 1)) := V(2)$. Then

$$\begin{aligned}
|B_{21}(\omega,\ell)| &= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell\\ \ell_1,\ell_3\geq 0,\ell_2\geq 2}} \sum_{\beta\in V(2)} S_{\ell_1}\left(\Gamma\left(\left\lfloor \frac{k_2}{2}\right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2}\right\rceil - 1\right); \beta, \delta\right)\\
&\quad \times |N_{\ell_2}(G'(2);\delta)| S_{\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2}\right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2}\right\rceil - 1\right); \delta, \beta\right)\\
&= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell\\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} |N_{\ell_2}(G'(2);\delta)|\\
&\quad \times \sum_{\beta\in V(2)} S_{\ell_1}\left(\Gamma\left(\left\lfloor \frac{k_2}{2}\right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2}\right\rceil - 1\right); \beta, \delta\right)\\
&\quad \times S_{\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2}\right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2}\right\rceil - 1\right); \delta, \beta\right)\\
&= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell\\ \ell_1,\ell_3\geq 0,\ell_2\geq 2}} |N_{\ell_2}(G'(2);\delta)|.S_{\ell_1+\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2}\right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2}\right\rceil - 1\right); \delta\right).
\end{aligned}$$

Similarly,

$$|B_{21}(\omega,\ell)| = \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell\\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} |N_{\ell_2}(G'(1);\delta)| S_{\ell_1+\ell_3}\left(\Gamma\left(\left\lfloor \frac{k_2}{2}\right\rfloor + 1\right) \cup \Gamma\left(\left\lceil \frac{k_2}{2}\right\rceil - 1\right); \delta\right).$$

By Lemma 4.1, we have $|N_{\ell_2}(G'(2);\delta)| \leq |N_{\ell_2}(G'(1);\delta)|$ for all positive integers $\ell_2$ and by Lemma 3.2, we have $S_t(\Gamma(\lfloor\frac{k_2}{2}\rfloor+1)\cup\Gamma(\lceil\frac{k_2}{2}\rceil-1));\delta) \leq S_t(\Gamma(\lfloor\frac{k_2}{2}\rfloor)\cup\Gamma(\lceil\frac{k_2}{2}\rceil);\delta)$ for all positive integers $t$. Thus $|B_{21}(\omega,\ell)| \leq |B_{11}(\omega,\ell)|$. Note that this inequality is strict for some positive integer $\ell_0 = t_0 + c - 1$ where $t_0 \geq \frac{k_2}{2}$. Also $|B_{21}| \leq |B_{11}|$ for all positive integers $\ell$, and it is strict for some positive integer $\ell_0$.

By a similar argument as above, we can prove that $|B_{22}| \leq |B_{12}|$. Thus $|B_2| \leq |B_1|$ for all positive integers $\ell$, and it is strict for some positive integer $\ell_0$. □

**Lemma 4.2.** *For all integer $\ell > c$, $k \geq 2$, we have*

$$S_\ell(k(c-1),2) \leq S_\ell(k(c-1),4) \leq \cdots \leq S_\ell(k(c-1),c/2-2), S_\ell(k(c-1),c/2).$$

*Proof.* First, we show that every diagonal parallel to the main diagonal and the main diagonal are unimodal. Let $H$ be the subgraph of $\Gamma(k)$ with vertex set $\{v_0, v_1, \ldots, v_c - 1\}$. By Lemma 3.1, we only need to show that the diagonals parallel to the main diagonal increase for $s + j \leq c - 1$. Let $s$ be an even integer. For the odd integer the proof is similar. Using induction on integer $\ell$, we will prove that $(A^\ell)_{s+2,j+2} \geq (A^\ell)_{s,j}$ for all $0 \leq s, j \leq c-2$ with $s + j \leq c - 1$.

Note that by the definition of $\Gamma(k)$, two vertices $v_s$ and $v_j$ are adjacent if and only if $v_{s+2}$ and $v_{j+2}$ are adjacent.

We have the following cases.

**Case 1:** $j \equiv 0 \pmod 2$ and $j \neq 0$. Then

$$\begin{aligned}(A^{\ell+1})_{s+2,j+2} =& (A^\ell)_{s+2,j} + (A^\ell)_{s+2,j+4},\\ (A^{\ell+1})_{s,j} =& (A^\ell)_{s,j-2} + (A^\ell)_{s,j+2}.\end{aligned}$$

By the induction hypothesis, we have the following results:

$$\begin{aligned}(A^\ell)_{s+2,j} \geq& (A^\ell)_{s,j-2},\\ (A^\ell)_{s+2,j+4} \geq& (A^\ell)_{s,j+2}, \quad \text{for } s+j+4 \leq c-1.\end{aligned}$$

Hence, we have $(A^\ell)_{s+2,j+2} \geq (A^\ell)_{s,j}$. Since, there is a closed walk of length $c$ starting from $v_0$ which is not including the edge $v_c v_{c+1}$, the inequality is strict for $\ell > c$.

**Case 2:** $j \equiv 1 \pmod 2$. The proof is similar to Case 1. □

The number of closed walks of length $\ell$ starting at the even vertex $v_s$ is equal to the entry $(s,s)$ in matrix $A^\ell$,

$$S_\ell(c-1,s) = (A^\ell)_{s,s}.$$

By induction hypothesis, we can conclude that $S_\ell(c-1,s) \leq S_\ell(c-1,s+2)$ for every $0 < s < c-1$. Note that the strict inequality holds when $\ell \geq \frac{c}{2}$.

Let $G$ be a point attaching strict $k_1$-quasi tree graph of even length $c$ and $\alpha \in V(G)$ and let $C_c$ be the cycle $H$ of $\Gamma(k)$ with $k_2$ cycles where $k_1 + k_2 = k$. We decompose $C_c$ into two paths denote by $P_{\frac{c}{2}}$ and $Q_{\frac{c}{2}}$, having common vertices in initial and final. Let $G(\frac{c}{2},\frac{c}{2})$ be the graph obtained from $G$ by attaching $P_{\frac{c}{2}}$ and $Q_{\frac{c}{2}}$ at $\alpha$ in $G$.

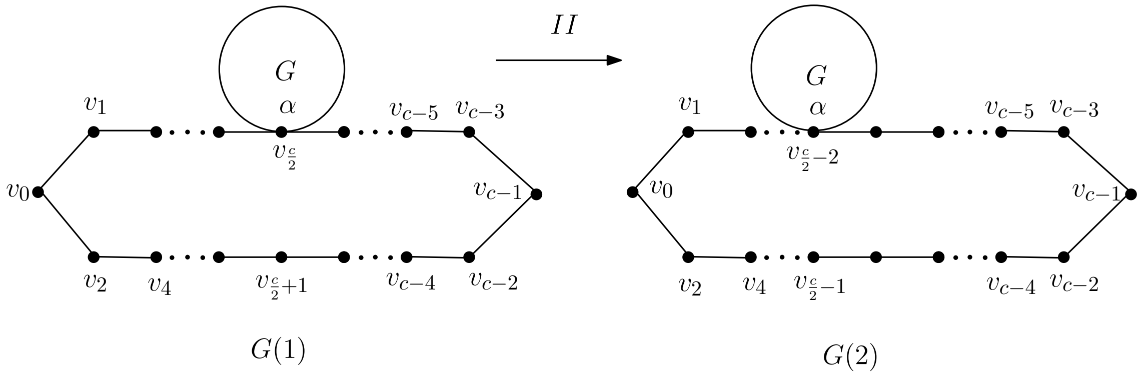


FIGURE 3. Transformation $II$.

Let $M_\ell(G(\frac{c}{2},\frac{c}{2});\alpha)$ (respectively $M_\ell(G(\frac{c}{2}+2,\frac{c}{2}-2);\alpha)$) be the set of $(\alpha,\alpha)$-walks of length $\ell$ in $G(\frac{c}{2},\frac{c}{2})$ (respectively $G(\frac{c}{2}+2,\frac{c}{2}-2)$) starting and ending at the edges or only one edge in $G$ and let $M'_\ell(G(\frac{c}{2},\frac{c}{2});\alpha)$ (respectively $M'_\ell(G(\frac{c}{2}+2,\frac{c}{2}-2);\alpha)$) be the set of $(\alpha,\alpha)$-walks of length $\ell$ in $G(\frac{c}{2},\frac{c}{2})$ (respectively $G(\frac{c}{2}+2,\frac{c}{2}-2)$), starting and ending at the edges or only one edge in $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ (respectively $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$). In the following let $G(\frac{c}{2},\frac{c}{2}):=G(1)$ and $G(\frac{c}{2}+2,\frac{c}{2}-2):=G(2)$. By definition $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ and $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ are isomorphic to $C_1$, so we denoted them by $C_1$.

**Lemma 4.3.** *Let $c$ be an even integer. If $\ell\geq\frac{c}{2}$, then*

(i) $|M_\ell(G(2);\alpha)|\leq|M_\ell(G(1));\alpha)|$;
(ii) $|M'_\ell(G(2);\alpha)|\leq|M'_\ell(G(1));\alpha)|$.

*Proof.* Let $\omega\in M_\ell(G(2);\alpha)$, we may decompose $\omega$ into maximal sections in union $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ or in $G$. Each of them is one of the following types.

(1) a $(\alpha,\alpha)$- walk in union $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$.
(2) a walk in $G(2)$ with all edges in $G$.

Similarly, we may decompose any $\omega\in M_\ell(G(1);\alpha)$ into maximal sections in union $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ or in $G$. Each of these maximal sections has one of the following types.

(3) a $(\alpha,\alpha)$-walk in union $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$.
(4) a walk in $G(1)$ with all edges in $G$.

Next, since $\Gamma(k)$ is symmetric, for any $\omega\in M_\ell(G(2);\alpha)$, we can replace the even indices with the odd indices that are in front of each other see Figure 3. Hence, from now on, $\omega$ is a $(\alpha,\alpha)$- walk with only odd or even indices. So without loss of generality $\omega$ is a $(\alpha,\alpha)$-walk with only odd indices. By definition, two unions $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$ and $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ are isomorphic to $C_1$ and by Lemma 4.2 there exists an injection mapping $\eta^1_\ell$ from a $(\alpha,\alpha)$-walk of length $\ell$ in $P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1}$ into a $(\alpha,\alpha)$- walk of length $\ell$ in $P_{\frac{c}{2}}\cup Q_{\frac{c}{2}}$. Let $\omega=\omega_1\omega_2\omega_3\cdots\in M_\ell(P_{\frac{c}{2}+1}\cup Q_{\frac{c}{2}-1})$, where $\omega_i$ is a walk of length $\ell_i$ of type (1) or (2) for $i\geq 1$. Let $\eta^\star(\omega)=\eta^\star(\omega_1)\eta^\star(\omega_2)\ldots$ where $\eta^\star(\omega_i)=\eta^1_{\ell_i}(\omega_i)$ and $\eta^\star(\omega_i)=\omega_i$ if $\omega_i$ is type (2) so $\eta^\star(\omega_i)$ for $i\geq 1$ is of type (3) or (4) and thus

$\eta^\star(\omega) \in M_\ell(G(1))$. Thus, $|M_\ell(G(2);\alpha)| \leq |M_\ell(G(1);\alpha)|$. This prove (i). The proof of (ii) is similar. □

**Theorem 4.2.** *Let $c$ be an even integer. If $\frac{c}{2} \geq 3$, then $S_\ell(G(2)) \leq S_\ell(G(1))$. For $\ell > \frac{c}{2}$, the strict inequality holds.*

*Proof.* Let $A_1$ and $A_2$ be two sets of closed walks of length $\ell$ in $G(1)$ and $G(2)$, respectively, containing some edges in $G$. Then $S_\ell(G(2)) = S_\ell(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}) + |A_2|$ and $S_\ell(G(1)) = S_\ell(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}}) + |A_1|$.

By our definition, $P_{\frac{c}{2}} \cup Q_{\frac{c}{2}}$ and $P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}$ are isomorphic to $C_1$, and we need only to prove that $|A_2| \leq |A_1|$ for all $\ell \geq 0$.

Let $A_{21}$ and $A_{22}$ be two subsets of $A_2$ for which every closed walk starts at a vertex in $V(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1})$ and in $V(G) - \{\alpha\}$, respectively. Then $|A_2| = |A_{21}| + |A_{22}|$.

Let $A_{11}$ and $A_{12}$ be two subsets of $A_1$ for which every closed walk starts at a vertex in $V(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}})$ and in $V(G) - \{\alpha\}$, respectively. Then $|A_1| = |A_{11}| + |A_{12}|$.

We may decompose any $\omega \in A_{21}$ into three sections $\omega_1\omega_2\omega_3$, where $\omega_1, \omega_3$ are walks in $P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}$ and $\omega_2$ is the longest walk of $\omega$ in $G(2)$ starting and ending at the edges in $G$. By the choice of $\omega_2$, we have that $\omega_2$ is a $(\alpha,\alpha)$-walk. Let $A_{21}(\omega,\ell) = \{\omega \in A_{21} : \omega_2 \text{ is a } (\alpha,\alpha)\text{-walk}\}$. So, we have $|A_{21}| = |A_{21}(\omega,\ell)|$.

Let $A_{11}(\omega,\ell) = \{\omega \in A_{11} : \omega_2 \text{ is a } (\alpha,\alpha)\text{-walk}\}$. So, we have $|A_{11}| = |A_{11}(\omega,\ell)|$.

Let $V(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1}) := V(1)$. Let $t = |M_{\ell_2}(G(2);\alpha)|$. From this decomposition for $\omega \in A_{21}$ and by the definition of $A_{21}(\omega,\ell)$, we have

$$\begin{aligned}
|A_{21}(\omega,\ell)| &= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} \sum_{\beta\in V(1)} S_{\ell_1}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\beta,\alpha).t.S_{\ell_3}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha,\beta) \\
&= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} t. \sum_{\beta\in V(1)} S_{\ell_1}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\beta,\alpha).S_{\ell_3}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha,\beta) \\
&= \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} .t.S_{\ell_1+\ell_3}(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha).
\end{aligned}$$

Similarly,

$$|A_{21}(\omega,\ell)| = \sum_{\substack{\ell_1+\ell_2+\ell_3=\ell \\ \ell_1,\ell_3\geqslant 0,\ell_2\geqslant 2}} |M_{\ell_2}(G(1);\alpha)|.S_{\ell_1+\ell_3}(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}};\alpha).$$

By Lemma 4.3, we have $|M_{\ell_2}(G(2);\alpha)| \leq |M_{\ell_2}(G(1);\alpha)|$ for all positive integers $\ell_2$ and by Lemma 4.2, we have $S_t(P_{\frac{c}{2}+1} \cup Q_{\frac{c}{2}-1};\alpha) \leq S_t(P_{\frac{c}{2}} \cup Q_{\frac{c}{2}};\alpha)$ for all positive integers $t$. Thus $|A_{21}(\omega,\ell)| \leq |A_{11}(\omega,\ell)|$. Note that this inequality is strict for some positive integer $\ell_0 = t_0 + c - 1$ where $t_0 \geq \frac{c}{2}$. Also $|A_{21}| \leq |A_{11}|$ for all positive integers $\ell$, and it is strict for some positive integer $\ell_0$.

By similar argument as above, we can prove that $|A_{22}| \leq |A_{12}|$. Thus $|A_2| \leq |A_1|$ for all positive integers $\ell$, and it is strict for some positive integer $\ell_0$. □

**Corollary 4.1.** *For graphs $G(1)$ and $G(2)$ we have $EE(G(1)) > EE(G(2))$.*

*Proof.* From Theorem 4.2, we have

$$EE(G(2)) = \sum_{\ell\geq 0} \frac{S_\ell(G(2))}{(\ell)!} < \sum_{\ell\geq 0} \frac{S_\ell(G(1))}{(\ell)!} = EE(G(1)). \qquad \square$$

The transformation from $G(1)$ to $G(2)$, depicted in Figure 3, is called transformation slowromancapi@ of $G(1)$.

**Corollary 4.2.** *For two graphs $G'(1)$ and $G'(2)$, we have $EE(G'(1)) > EE(G'(2))$.*

*Proof.* By Theorem 4.1, we have

$$EE(G'(2)) = \sum_{\ell\geq 0} \frac{S_\ell(G'(2))}{(\ell)!} < \sum_{\ell\geq 0} \frac{S_\ell(G'(1))}{(\ell)!} = EE(G'(1)). \qquad \square$$

The transformation from $G'(1)$ to $G'(2)$, depicted in Figure 2, is called transformation slowromancapi@ of $G'(1)$. Transformation slowromancapiii@ is similar to transformation slowromancapii@ which obtained by attaching $\alpha \in G$ at $v_0$. There is a closed walks in $M_c((c-1), 0)$ which is not including the edge $v_c v_{c+1}$. So there is a closed walk in $M_c((c-1), 1)$ not in $M_c((c-1), 0)$. Hence, transformation slowromancapiii@ strictly decreases the Estrada index for $\ell \geq c$.

Let $G$ be a point attaching strict $k$-quasi tree graph with $k$ even cycles of length $c$, obtained by attaching the subgraphs $G_1, G_2, \dots, G_{\frac{\Delta}{2}}$ at $u$ with the maximum degree $\Delta$. By using transformations slowromancapi@, slowromancapii@ and slowromancapiii@, $G_i$ s, $(1 \leq i \leq \frac{\Delta}{2})$ can be changed into the graphs $\Gamma_i$ s. These transformations change $G$ into $G^*$ which is obtained by attaching $\Gamma_i$ s at $u$. Each application of transformation strictly decreases its Estrada index. So we have $EE(G^*) < EE(G)$. Finally repeatedly applying transformation $I$, $G^*$ can be changed into the graph $\Gamma(k)$ that is obtained from $\bigcup_{i=1}^{\frac{\Delta}{2}} \Gamma(k_i)$. So we have the following result.

**Theorem 4.3.** *Let $G$ be a point attaching strict $k$-quasi tree graph with $k$ even cycles. Then $EE(\Gamma(k)) \leq EE(G)$.*

## References


[1] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge, 1993.

[2] D. M. Cvetković, M. Doob and H. Sachs, *Spectra of Graphs: Theory and Application*, Academic Press, New York, 1980.

[3] H. Deng, *A proof of a conjecture on the estrada index*, MATCH Commun. Math. Comput. Chem. **62** (2009), 599–606.

[4] E. Deutsch and S. Klavzar, *Computing the hosoya polynomial of graphs from primary subgraphs*, MATCH Commun. Math. Comput. Chem. **70** (2013), 627–644.

[5] Z. Du and B. Zhou, *The estrada index of unicyclic graphs*, Linear Algebra Appl. **436** (2012), 3149–3159.

[6] E. Estrada, *Characterization of 3d molecular structure*, Chemical Physics Letters **319** (2000), 713–718.

[7] E. Estrada, *Characterization of the folding degree of proteins*, Bioinformatics **18** (2002), 697–704.

[8] E. Estrada and J. A. Rodriguez-Velazquez, *Subgraph centrality in complex networks*, Phys. Rev. E **71** (2005), Paper ID 056103.

[9] I. Gutman and A. Graovac, *Estrada index of cycles and paths*, Chemical Physics Letters **436** (2007), 294–296.
[10] A. Ilić and D. Stevanović, *The estrada index of chemical trees*, J. Math. Chem. **47**(1) (2010), 305–314.
[11] F. Li, L. Wei, J. Cao, F. Hu and H. Zhao, *On the maximum estrada index of 3-uniform linear hypertrees*, Scientific World Journal (2014), DOI 10.1155/2014/637865.
[12] F. Li, L. Wei, H. Zhao, F. Hu and X. Ma, *On the estrada index of cactus graphs*, Discrete Appl. Math. **203** (2016), 94–105.
[13] J. Li, X. Li and L. Wang, *The minimal estrada index of trees with two maximum degree vertices*, MATCH Commun. Math. Comput. Chem **64** (2010), 799–810.
[14] M. Lu and J. Gao, *On the randić index of quasi-tree graphs*, J. Math. Chem. **42** (2007), 297–310.
[15] Y. Shang, *Biased edge failure in scale-free networks based on natural connectivity*, Indian Journal of Physics **86** (2012), 485–488.
[16] Y. Shang, *Random lifts of graphs: network robustness based on the estrada index*, Appl. Math. E-Notes **12** (2012), 53–61.
[17] R. P. Stanley, *Log-concave and unimodal sequences in algebra, combinatorics, and geometry*, Annals of the New York Academy of Sciences **576** (1989), 500–535.
[18] K. Xu, J. Wang and H. Liu, *The harary index of ordinary and generalized quasi-tree graphs*, J. Appl. Math. Comput. **45** (2014), 365–374.
[19] J. Zhang, B. Zhou and J. Li, *On estrada index of trees*, Linear Algebra Appl. **434** (2011), 215–223.

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# NEW UPPER AND LOWER BOUNDS FOR SOME DEGREE-BASED GRAPH INVARIANTS

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Abstract. For a simple graph $G$ with vertex set $V(G)$ and edge set $E(G)$, let $\deg(u)$ be the degree of the vertex $u \in V(G)$. The forgotten index of $G$ and its coindex are defined as $F(G) = \sum_{v \in V(G)} \deg^3(v)$ and $\overline{F}(G) = \sum_{uv \notin E(G)} \left[\deg^2(u) + \deg^2(v)\right]$. New bonds for the first Zagreb index $M_1(G) = \sum_{v \in V(G)} \deg(v)^2$, forgotten index, and its coindex are obtained.

## 1. Introduction

Throughout this paper, all graphs considered are assumed to be simple, i.e., without directed, weighted, or multiple edges, without self-loops and with a finite number of vertices. Let $G$ be such a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. A graph with $n$ vertices and $m$ edges will be referred to as an $(n, m)$-graph.

By $\deg(v)$ or $\deg_G(v)$ is denoted the degree of the vertex $v \in V(G)$. Let $D(G) = \{\deg(v_1), \deg(v_2), \dots, \deg(v_n)\}$. If $D(G) = \{r\}$, then $G$ is said to be $r$-regular. If $D(G) = \{r, s\}$, then we say that $G$ is $(r, s)$-biregular. This includes the case of regular graphs if $r = s$. Analogously, if $D(G) =\{r, s, t\}$, then the graph $G$ will be said to be $(r, s, t)$-triregular. Let, in addition, $\Delta = \max_{v \in V(G)} \deg(v)$ and $\delta = \min_{v \in V(G)} \deg(v)$.

The *first Zagreb index* $M_1(G)$ is defined as [13]

$$M_1 = M_1(G) = \sum_{v \in V(G)} \deg^2(v) = \sum_{uv \in E(G)} \left[\deg(u) + \deg(v)\right].$$

It is the oldest and most studied degree-based graph invariant; details of its mathematical theory and chemical applications can be found in the surveys [5, 11, 17].

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In the paper [13], $M_1$ was used for designing approximate expressions for total $\pi$-electron energy. In the same paper, also the sum of cubes of vertex degrees ($F$) was used for the same purpose. However, whereas $M_1$ eventually gained much popularity [5, 11, 17], no attention was paid to $F$. Only more than forty years later, the invariant $F$ attracted some interest, thanks to the discovery of its applicability in physical chemistry [4]. For this reason it was named *forgotten index* and is defined as [4]:

$$F = F(G) = \sum_{v \in V(G)} \deg(v)^3 = \sum_{uv \in E(G)} \left[\deg(u)^2 + \deg(v)^2\right].$$

In the last few years, numerous mathematical studies of the forgotten index have been published, see [1–3, 6, 7, 10, 12, 16].

Some of pharmacological applications of the $F$-index were also attempted [15].

Both $M_1$ and $F$ are special cases of the so-called *first general Zagreb index*, defined as

$$M_1^\alpha = M_1^\alpha(G) = \sum_{u \in V(G)} \deg(u)^\alpha = \sum_{uv \in E(G)} \left[\deg(u)^{\alpha-1} + \deg(v)^{\alpha-1}\right],$$

where $\alpha$ is an arbitrary real number [15, 18].

The coindex of $M_1^\alpha$ is defined as [18]

$$\overline{M_1^\alpha}(G) = \sum_{\substack{uv \notin E(G) \\ u \neq v}} \left[\deg(u)^{\alpha-1} + \deg(v)^{\alpha-1}\right].$$

The special case of this expressions for $\alpha = 3$ is the coindex of the forgotten index [8, 14]

$$\overline{F}(G) = \sum_{\substack{uv \notin E(G) \\ u \neq v}} \left[\deg(u)^2 + \deg(v)^2\right].$$

## 2. Main Results

We first state results that improve those reported in [12]. Denote by $\overline{G}$ the complement of the graph $G$.

**Theorem 2.1.** *Let $G$ be an $(n, m)$-graph. Then*

$$F(G) + F(\overline{G}) = n^4 + M_1(G)(3n-3) - 2m(3n^2 - 6n + 3) - n(3n^2 - 3n + 1)$$

*and*

$$\begin{aligned} F(G) \times F(\overline{G}) =& n^4 F(G) + (3n-3)F(G)\, M_1(G) - 2m(3n^2 - 6n + 3)F(G) \\ &- n(3n^2 - 3n + 1)F(G) - F(G)^2. \end{aligned}$$

*Proof.* By definition of a graph complement, we have

$$\begin{aligned}F(\overline{G}) &= \sum_{u\in V(G)} \deg_{\overline{G}}(u)^3 = \sum_{u\in V(G)} \Big[n-1-\deg_G(u)\Big]^3\\ &= \sum_{u\in V(G)} \Big[n^3+\deg_G(u)^2(3n-3)-\deg_G(u)(3n^2-6n+3)\\ &\quad -3n^2+3n-1-\deg_G(u)^3\Big]\\ &= n^4+M_1(G)(3n-3)-2m(3n^2-6n+3)-n(3n^2-3n+1)-F(G). \qquad \square\end{aligned}$$

**Theorem 2.2.** *Let $G$ be an $(n, m)$-graph. Then*

$$F(G) \le n\Delta^3+3\Delta M_1(G)-6m\Delta^2 \text{ and } F(G) \ge n\delta^3+3\delta M_1(G)-6m\delta^2,$$

*with equalities if and only if $G$ is regular.*

*Proof.* Define an auxiliary function $Y_1(G) = \sum_{u\in V(G)} [\deg(u)-k]^3$, where $k$ is a real number. Then,

$$\begin{aligned}Y_1(G) &= \sum_{u\in V(G)} \Big[\deg(u)^3-k^3-3\deg(u)^2k+3\deg(u)k^2\Big]\\ &= F(G)-nk^3-3kM_1(G)+6mk^2.\end{aligned}$$

If $k=\Delta$, then $Y_1(G)\le 0$ and $F(G)\le n\Delta^3+3\Delta M_1(G)-6m\Delta^2$. For $k=\delta$, $Y_1(G)\ge 0$ and $F(G)\ge n\delta^3+3\delta M_1(G)-6m\delta^2$. The equalities hold if and only if $G$ is regular. $\square$

**Theorem 2.3.** *Let $G$ be an $(n, m)$-graph. Then*

$$F(G) \ge M_1(G)(\delta+2\Delta)-\Delta^2(2m-n\delta)-4m\Delta\delta$$

*and*

$$F(G) \le M_1(G)(\Delta+2\delta)-\delta^2(2m-n\Delta)-4m\delta\Delta$$

*with equalities if and only if $G$ is $(\Delta, \delta)$-biregular.*

*Proof.* Define $Y_2(G) = \sum_{u\in V(G)} [\deg(u)-k]^2 [\deg(u)-h]$, where $k$ and $h$ are real numbers. Then,

$$\begin{aligned}Y_2(G) &= \sum_{u\in V(G)} \Big[\deg(u)^2+k^2-2\deg(u)k\Big] [\deg(u)-h]\\ &= \sum_{u\in V(G)} \Big[\deg(u)^3-\deg(u)^2h+\deg(u)k^2-k^2h-2\deg(u)^2k+2\deg(u)kh\Big]\\ &= F(G)-M_1(G)(h+2k)+k^2(2m-nh)+4mkh.\end{aligned}$$

If $k=\Delta$ and $h=\delta$, then $Y_2(G)\ge 0$ and $F(G)\ge M_1(G)(\delta+2\Delta)-\Delta^2(2m-n\delta)-4m\Delta\delta$. For $k=\delta$ and $h=\Delta$, we have $Y_2(G)\le 0$ and $F(G)\le M_1(G)(\Delta+2\delta)-\delta^2(2m-n\Delta)-4m\delta\Delta$. The equalities hold if and only if $G$ is $(\Delta,\delta)$-biregular. $\square$

**Theorem 2.4.** *Let $G$ be an $(n, m)$-graph. Then $F(G) \ge 2[M_1(G)+m-n]$. If $G$ is connected, then equality holds if and only if $G \cong P_n$ or $G \cong C_n$.*

*Proof.* Define the auxiliary function $Y_3(G) = \sum_{u \in V(G)} [\deg(u)^2 - 1][\deg(u) - 2]$ and note that $Y_3(G) = 0$ if and only if $\Delta(G) \leq 2$. In case of connected graphs, this will occur if either $G \cong P_n$ or $G \cong C_n$.

Now,

$$\begin{aligned} Y_3(G) = & \sum_{u \in V(G)} \Big[\deg(u)^3 - 2\deg(u)^2 - \deg(u) + 2\Big] \\ = & F(G) - 2M_1(G) - 2m + 2n. \end{aligned}$$

Since $Y_3(G) \geq 0$, $F(G) \geq 2[M_1(G) + m - n]$ with equality for connected graphs if and only if $G \cong P_n$ or $G \cong C_n$. □

**Theorem 2.5.** *Let $G$ be an $(n, m)$-graphs. Then*

$$F(G) \leq (3\Delta - 3)M_1(G) - 2m(3\Delta^2 - 6\Delta + 2) + n\Delta(\Delta - 1)(\Delta - 2)$$

*and*

$$F(G) \geq (3\delta + 3)M_1(G) - 2m(3\delta^2 + 6\delta + 2) + n\delta(\delta + 1)(\delta + 2)\,.$$

*The equalities holds if and only if $G$ is $(\delta, \delta+1, \delta+2)$-triregular.*

*Proof.* Define $Y_4(G) = \sum_{u \in V(G)} [\deg(u) - a]\,[\deg(u) - b]\,[\deg(u) - c]$, where $a$, $b$, and $c$ are real numbers. Then,

$$\begin{aligned} Y_4(G) = & \sum_{u \in V(G)} \Big[\deg(u)^3 - \deg(u)^2(a + b + c) + \deg(u)(ab + ac + bc) - abc\Big] \\ = & F(G) - (a + b + c)M_1(G) + 2m(ab + ac + bc) - nabc. \end{aligned}$$

If $a = \Delta$, $b = \Delta - 1$ and $c = \Delta - 2$, then $Y_4(G) \leq 0$ and $F(G) \leq (3\Delta - 3)M_1(G) - 2m(3\Delta^2 - 6\Delta + 2) + n\Delta(\Delta - 1)(\Delta - 2)$. For $a = \delta$, $b = \delta + 1$ and $c = \delta + 2$, $Y_4(G) \geq 0$ and $F(G) \geq (3\delta + 3)M_1(G) - 2m(3\delta^2 + 6\delta + 2) + n\delta(\delta + 1)(\delta + 2)$. The equalities hold if and only if $G$ is $(\delta, \delta + 1, \delta + 2)$-triregular. □

For the sake of completeness, we mention here a result from [18].

**Theorem 2.6.** [18] *Let $G$ be an $(n, m)$-graph. Then for $\alpha \geq 1$,*

$$\overline{M_1^{\alpha+1}}(G) = (n - 1)M_1^{\alpha}(G) - M_1^{\alpha+1}(G)\,.$$

**Theorem 2.7.** *Let $G$ be an $(n, m)$-graph. Then*

$$\overline{F}(G) \geq 2m[2\Delta(n - 1) + 3\Delta^2] - n[(n - 1)\Delta^2 + \Delta^3] - 3\Delta\, M_1(G)\,.$$

*The equality holds if and only if $G$ is regular.*

*Proof.* Define

$$Y_5(G) = (n - 1) \sum_{u \in V(G)} [\deg(u) - \Delta]^2 - \sum_{u \in V(G)} [\deg(u) - \Delta]^3\,.$$

Then,

$$\begin{aligned}
Y_5(G) =& (n-1)\sum_{u\in V(G)}\Big[\deg(u)^2+\Delta^2-2\Delta\deg(u)\Big]\\
&-\sum_{u\in V(G)}\Big[\deg(u)^3-\Delta^3-3\Delta\deg(u)^2+3\Delta^2\deg(u)\Big]\\
=&(n-1)M_1(G)-F(G)+n\left[(n-1)\Delta^2+\Delta^3\right]\\
&-2m\left[2\Delta(n-1)+3\Delta^2\right]+3\Delta\, M_1(G).
\end{aligned}$$

Since $Y_5(G)\geq 0$, one can see that

$$(n-1)M_1(G)-F(G)\geq 2m\Big[2\Delta(n-1)+3\Delta^2\Big]-n\Big[(n-1)\Delta^2+\Delta^3\Big]-3\Delta\, M_1(G).$$

The equality holds if and only if $G$ is a regular graph. Therefore, by Theorem 2.6,

$$\overline{F}(G)\geq 2m\Big[2\Delta(n-1)+3\Delta^2\Big]-n\Big[(n-1)\Delta^2+\Delta^3\Big]-3\Delta\, M_1(G)$$

with equality if and only if $G$ is regular. □

**Theorem 2.8.** *Let $G$ be an $(n,m)$-graph. Then*

$$\begin{aligned}
\overline{F}(G)\geq& 2m\left[(n-1)(2\Delta-1)+\Delta^2+2\Delta(\Delta-1)\right]-M_1(G)(3\Delta-1)\\
&-n\left[(n-1)\Delta(\Delta-1)+\Delta^2(\Delta-1)\right].
\end{aligned}$$

*The equality holds if and only if $G$ is $(\Delta,\Delta-1)$-biregular.*

*Proof.* We define the auxiliary function

$$\begin{aligned}
Y_6(G) =&(n-1)\sum_{u\in V(G)}[\deg(u)-\Delta]\,[\deg(u)-(\Delta-1)]\\
&-\sum_{u\in V(G)}[\deg(u)-\Delta]^2\,[\deg(u)-(\Delta-1)]\,.
\end{aligned}$$

Then,

$$\begin{aligned}
Y_6(G) =&(n-1)\sum_{u\in V(G)}\Big[\deg(u)^2-\deg(u)(2\Delta-1)+\Delta(\Delta-1)\Big]\\
&-\sum_{u\in V(G)}\Big[\deg(u)^3-\deg(u)^2(3\Delta-1)+\deg(u)\Delta^2\\
&-\Delta^2(\Delta-1)+2\deg(u)\Delta(\Delta-1)\Big]\\
=&(n-1)M_1(G)-2m(n-1)(2\Delta-1)+n(n-1)\Delta(\Delta-1)\\
&-F(G)+M_1(G)(3\Delta-1)-2m\Delta^2+n\Delta^2(\Delta-1)-4m\Delta(\Delta-1)\\
=&(n-1)M_1(G)-F(G)-2m\left[(n-1)(2\Delta-1)+\Delta^2+2\Delta(\Delta-1)\right]\\
&+n\left[(n-1)\Delta(\Delta-1)+\Delta^2(\Delta-1)\right]+M_1(G)(3\Delta-1)\,.
\end{aligned}$$

Since $Y_6(G) \geq 0$,

$$\begin{aligned}(n-1)M_1(G) - F(G) \geq & 2m\left[(n-1)(2\Delta-1) + \Delta^2 + 2\Delta(\Delta-1)\right] \\ & - n\left[(n-1)\Delta(\Delta-1) + \Delta^2(\Delta-1)\right] - (3\Delta-1)M_1(G),\end{aligned}$$

with equality if and only if $G$ is a $(\Delta, \Delta-1)$-biregular graph. We now apply Theorem 2.6 to show that

$$\begin{aligned}\overline{F}(G) \geq & 2m\left[(n-1)(2\Delta-1) + \Delta^2 + 2\Delta(\Delta-1)\right] \\ & - n\left[(n-1)\Delta(\Delta-1) + \Delta^2(\Delta-1)\right] - (3\Delta-1)M_1(G)\end{aligned}$$

with equality if and only if $G$ is $(\Delta, \Delta-1)$-biregular. □

**Theorem 2.9.** *Let $G$ be an $(n, m)$-graph. Then*

$$\overline{F}(G) \leq 2m\left[(n-1)(\delta+\Delta) + \Delta^2 + 2\Delta\delta\right] - n\left[(n-1)\Delta\delta + \Delta^2\delta\right] - (\delta+2\Delta)M_1(G).$$

*The equality holds if and only if $G$ is $(\Delta, \delta)$-biregular.*

*Proof.* Define the function

$$Y_7(G) = (n-1)\sum_{u\in V(G)} [\deg(u)-\Delta][\deg(u)-\delta] - \sum_{u\in V(G)} [\deg(u)-\Delta]^2 [\deg(u)-\delta].$$

Then,

$$\begin{aligned}Y_7(G) =& (n-1)\sum_{u\in V(G)} \left[\deg(u)^2 - \deg(u)(\delta+\Delta) + \Delta\delta\right] \\ & - \sum_{u\in V(G)} \left[\deg(u)^3 - \deg(u)^2(\delta+2\Delta) + \deg(u)\Delta^2 - \Delta^2\delta + 2\deg(u)\Delta\delta\right] \\ =& (n-1)M_1(G) - 2m(n-1)(\delta+\Delta) + n(n-1)\Delta\delta \\ & - F(G) + M_1(G)(\delta+2\Delta) - 2m\Delta^2 + n\Delta^2\delta - 4m\Delta\delta \\ =& (n-1)M_1(G) - F(G) - 2m\left[(n-1)(\delta+\Delta) + \Delta^2 + 2\Delta\delta\right] \\ & + n\left[(n-1)\Delta\delta + \Delta^2\delta\right] + (\delta+2\Delta)M_1(G).\end{aligned}$$

Since $Y_7(G) \leq 0$,

$$\begin{aligned}(n-1)M_1(G) - F(G) \leq & 2m\left[(n-1)(\delta+\Delta) + \Delta^2 + 2\Delta\delta\right] \\ & - n\left[(n-1)\Delta\delta + \Delta^2\delta\right] - (\delta+2\Delta)M_1(G),\end{aligned}$$

and the equality holds if and only if $G$ is a $(\Delta, \delta)$-biregular graph. We now apply Theorem 2.6 to show that,

$$\overline{F}(G) \leq 2m\left[(n-1)(\delta+\Delta) + \Delta^2 + 2\Delta\delta\right] - n\left[(n-1)\Delta\delta + \Delta^2\delta\right] - (\delta+2\Delta)M_1(G),$$

with equality holding if and only if $G$ is $(\Delta, \delta)$-biregular. □

**Theorem 2.10.** *Let $G$ be an $(n, m)$-graph. Then the following holds.*

(a) $M_1(G) \leq 2m(\delta + \Delta) - n\Delta\delta$, *with equality if and only if* $G$ *is* $(\Delta, \delta)$*-biregular.*
(b) $M_1(G) \geq 2m(2\Delta - 1) - n\Delta(\Delta - 1)$ *and* $M_1(G) \geq 2m(2\delta + 1) - n\delta(\delta + 1)$. *The equalities holds if and only if* $G$ *is* $(\delta, \delta + 1)$*-biregular.*
(c) *Let* $r$ *be a real number. Then* $M_1(G) \geq 4ma - n\,r^2$, *with equality if and only if* $G$ *is an* $r$*-regular graph.*

*Proof.* Consider the function $Y_8(G) = \sum_{u \in V(G)} \Big[\deg(u) - a\Big]\Big[\deg(u) - b\Big]$, where $a$ and $b$ are real numbers. Then we have,

$$\begin{aligned} Y_8(G) = &\sum_{u \in V(G)} \Big[\deg(u)^2 - \deg(u)b - \deg(u)a + ab\Big] \\ =&M_1(G) - 2m(a + b) + nab. \end{aligned}$$

If $a = \Delta$ and $b = \delta$, then $Y_8(G) \leq 0$ and $M_1(G) \leq 2m(\delta+\Delta) - n\Delta\delta$. Now the equality holds if and only if $G$ is a $(\Delta, \delta)$-biregular graph. This completes the part (a).

Suppose that $a = \Delta$ and $b = \Delta - 1$. Then $Y_8(G) \geq 0$ and $M_1(G) \geq 2m(2\Delta - 1) - n\Delta(\Delta - 1)$. For $a = \delta$ and $b = \delta + 1$, $Y_8(G) \geq 0$ and $M_1(G) \geq 2m(2\delta + 1) - n\delta(\delta + 1)$. The equalities hold if and only if $G$ is $(\delta, \delta + 1)$-biregular, which completes the proof of part (b).

Finally, assume that $a = b = r$. Then $Y_8(G) \geq 0$ and $M_1(G) \geq 4ma - n\,r^2$. The equality holds if and only if $G$ is $r$-regular. □

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## References

[1] H. Abdo, D. Dimitrov and I. Gutman, *On extremal trees with respect to the $F$-index*, Kuwait J. Sci. **44**(4) (2017), 1–8.
[2] S. Akhter and M. Imran, *Computing the forgotten topological index of four operations on graphs*, AKCE Int. J. Graphs Comb. **14** (2017), 70–79.
[3] S. Akhter, M. Imran and M. R. Farahani, *Extremal unicyclic and bicyclic graphs with respect to the $F$-index*, AKCE Int. J. Graphs Comb. **14** (2017), 80–91.
[4] B. Furtula and I. Gutman, *A forgotten topological index*, J. Math. Chem. **53** (2015), 1184–1190.
[5] B. Borovićanin, K. C. Das, B. Furtula and I. Gutman, *Bounds for Zagreb indices*, MATCH Commun. Math. Comput. Chem. **78** (2017), 17–100.
[6] Z. Che and Z. Chen, *Lower and upper bounds of the forgotten topological index*, MATCH Commun. Math. Comput. Chem. **76** (2016), 635–648.
[7] N. De, S. M. Abu Nayeem and A. Pal, *$F$-index of some graph operations*, Discrete Math. Algorithms Appl. **8** (2016), ID 1650025.
[8] N. De, S. M. Abu Nayeem and A. Pal, *The $F$ coindex of some graph operations*, Springer Plus **5** (2016), Paper ID 221.
[9] W. Gao, M. K. Siddiqui, M. Imran, M. K. Jamil and M. R. Farahani, *Forgotten topological index of chemical structure in drugs*, Saudi Pharma. J. **24** (2016), 258–264.
[10] S. Ghobadi and M. Ghorbaninejad, *The forgotten topological index of four operations on some special graphs*, Bulletin of Mathematical Sciences and Applications **16** (2016), 89–95.

[11] I. Gutman and K. C. Das, *The first Zagreb index 30 years after*, MATCH Commun. Math. Comput. Chem. **50** (2004), 83–92.

[12] I. Gutman, A. Ghalavand, T. Dehghan-Zadeh and A. R. Ashrafi, *Graphs with smallest forgotten index*, Iranian Journal of Mathematical Chemistry **8** (2017), 259–273.

[13] I. Gutman and N. Trinajstić, *Graph theory and molecular orbitals. Total $\pi$-electron energy of alternant hydrocarbons*, Chemical Physics Letters **17** (1972), 535–538.

[14] A. Khaksari and M. Ghorbani, *On the forgotten topological index*, Iranian Journal of Mathematical Chemistry **8** (2017), 327–338.

[15] X. Li and H. Zhao, *Trees with the first three smallest and largest generalized topological indices*, MATCH Commun. Math. Comput. Chem. **50** (2004), 57–62.

[16] I. Ž. Milovanović, E. I. Milovanović, I. Gutman and B. Furtula, *Some inequalities for the forgotten topological index*, International Journal of Applied Graph Theory **1** (2017), 1–15.

[17] S. Nikolić, G. Kovačević, A. Miličević and N. Trinajstić, *The Zagreb indices 30 years after*, Croatica Chemica Acta **76** (2003), 113–124.

[18] S. Zhang and H. Zhang, *Unicyclic graphs with the first three smallest and largest first general Zagreb index*, MATCH Commun. Math. Comput. Chem. **55** (2006), 427–438.

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# EXISTENCE, UNIQUENESS AND STABILITY OF PERIODIC SOLUTIONS FOR NONLINEAR NEUTRAL DYNAMIC EQUATIONS

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Abstract. The nonlinear neutral dynamic equation with periodic coefficients

$$\begin{aligned}&[u(t)-g(u(t-\tau(t)))]^{\Delta}\\ =&p(t)-a(t)u^{\sigma}(t)-a(t)g(u^{\sigma}(t-\tau(t)))-h(u(t),u(t-\tau(t)))\end{aligned}$$

is considered in this work. By using Krasnoselskii's fixed point theorem we obtain the existence of periodic and positive periodic solutions and by contraction mapping principle we obtain the uniqueness. Stability results of this equation are analyzed. The results obtained here extend the work of Mesmouli, Ardjouni and Djoudi [14].

## 1. Introduction

In 1988, Stephan Hilger [10] introduced the theory of time scales (measure chains) as a means of unifying discrete and continuum calculi. Since Hilger's initial work there has been significant growth in the theory of dynamic equations on time scales, covering a variety of different problems (see [7, 8, 13] and references therein).

Let $\mathbb{T}$ be a periodic time scale such that $0 \in \mathbb{T}$. In this article, we are interested in the analysis of qualitative theory of periodic and positive periodic solutions of neutral dynamic equations. Motivated by the papers [1–6, 11, 12, 14, 15, 17] and the references therein, we consider the following nonlinear neutral dynamic equation

$$\begin{aligned}&[u(t)-g(u(t-\tau(t)))]^{\Delta}\\ =&p(t)-a(t)u^{\sigma}(t)-a(t)g(u^{\sigma}(t-\tau(t)))-h(u(t),u(t-\tau(t))).\end{aligned} \tag{1.1}$$

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Throughout this paper we assume that $a$, $p$ and $\tau$ are real valued rd-continuous functions with $a$ and $\tau$ are positive functions, $id - \tau : \mathbb{T} \to \mathbb{T}$ is increasing so that the function $u(t - \tau(t))$ is well defined over $\mathbb{T}$. The functions $g$ and $h$ are continuous in their respective arguments. To reach our desired end we have to transform (1.1) into an integral equation written as a sum of two mapping, one is a contraction and the other is continuous and compact. After that, we use Krasnoselskii's fixed point theorem, to show the existence of periodic and positive periodic solutions. We also obtain the existence of a unique periodic solution by employing the contraction mapping principle. In addition to the study of existence and uniqueness, in this research we obtain sufficient conditions for the stability of the periodic solution by using the contraction mapping principle.

The organization of this paper is as follows. In Section 2, we introduce some notations and definitions, and state some preliminary material needed in later sections. We will state some facts about the exponential function on a time scale as well as the fixed point theorems. For details on fixed point theorems we refer the reader to [16]. In Section 3, we establish the existence and uniqueness of periodic solutions. In Section 4, we give sufficient conditions to ensure the existence of positive periodic solutions. The stability of the periodic solution is the topic of Section 5. The results presented in this paper extend the main results in [14].

## 2. Preliminaries

A time scale is an arbitrary nonempty closed subset of real numbers. The study of dynamic equations on time scales is a fairly new subject, and research in this area is rapidly growing (see [1–6, 11, 12, 15] and papers therein). The theory of dynamic equations unifies the theories of differential equations and difference equations. We suppose that the reader is familiar with the basic concepts concerning the calculus on time scales for dynamic equations. Otherwise one can find in Bohner and Peterson books [7,8,13] most of the material needed to read this paper. We start by giving some definitions necessary for our work. The notion of periodic time scales is introduced in Kaufmann and Raffoul [11]. The following two definitions are borrowed from [11].

**Definition 2.1.** We say that a time scale $\mathbb{T}$ is periodic if there exist a $\omega > 0$ such that if $t \in \mathbb{T}$ then $t \pm \omega \in \mathbb{T}$. For $\mathbb{T} \neq \mathbb{R}$, the smallest positive $\omega$ is called the period of the time scale.

*Example* 2.1. The following time scales are periodic.

(a) $\mathbb{T} = \bigcup_{i=-\infty}^{\infty} [2(i-1)h, 2ih]$, $h > 0$, has period $\omega = 2h$.
(b) $\mathbb{T} = hZ$ has period $\omega = h$.
(c) $\mathbb{T} = \mathbb{R}$.
(d) $\mathbb{T} = \{t = k - q^m : k \in Z,\ m \in N_0\}$, where $0 < q < 1$ has period $\omega = 1$.

*Remark* 2.1 ([11]). All periodic time scales are unbounded above and below.

**Definition 2.2.** Let $\mathbb{T} \neq \mathbb{R}$ be a periodic time scale with period $\omega$. We say that the function $f : \mathbb{T} \to \mathbb{R}$ is periodic with period $T$ if there exists a natural number $n$ such that $T = n\omega$, $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$ and $T$ is the smallest number such that $f(t \pm T) = f(t)$.

If $\mathbb{T} = \mathbb{R}$, we say that $f$ is periodic with period $T > 0$ if $T$ is the smallest positive number such that $f(t \pm T) = f(t)$ for all $t \in \mathbb{T}$.

*Remark* 2.2 ([11]). If $\mathbb{T}$ is a periodic time scale with period $\omega$, then $\sigma(t \pm n\omega) = \sigma(t) \pm n\omega$. Consequently, the graininess function $\mu$ satisfies $\mu(t \pm n\omega) = \sigma(t \pm n\omega) - (t \pm n\omega) = \sigma(t) - t = \mu(t)$ and so, is a periodic function with period $\omega$.

**Definition 2.3** ([7])**.** A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at every right-dense point $t \in \mathbb{T}$ and its left-sided limits exist, and is finite at every left-dense point $t \in \mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by

$$C_{rd} = C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R}).$$

**Definition 2.4** ([7])**.** For $f : \mathbb{T} \to \mathbb{R}$, we define $f^{\Delta}(t)$ to be the number (if it exists) with the property that for any given $\varepsilon > 0$, there exists a neighborhood $U$ of $t$ such that

$$\left|(f(\sigma(t)) - f(s)) - f^{\Delta}(t)\,(\sigma(t) - s)\right| < \varepsilon\,|\sigma(t) - s|\,, \quad \text{for all } s \in U.$$

The function $f^{\Delta} : \mathbb{T}^k \to \mathbb{R}$ is called the delta (or Hilger) derivative of $f$ on $\mathbb{T}^k$.

**Definition 2.5** ([7])**.** A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1+\mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \text{ for all } t \in \mathbb{T}\}.$$

**Definition 2.6** ([7])**.** Let $p \in \mathcal{R}$, then the generalized exponential function $e_p$ is defined as the unique solution of the initial value problem

$$x^{\Delta}(t) = p(t)x(t),\ x(s) = 1, \quad \text{where } s \in \mathbb{T}.$$

An explicit formula for $e_p(t, s)$ is given by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(v)}(p(v))\Delta v\right), \quad \text{for all } s, t \in \mathbb{T},$$

with

$$\xi_h(v) = \begin{cases} \dfrac{\log(1 + hv)}{h}, & \text{if } h \neq 0, \\ v, & \text{if } h = 0, \end{cases}$$

where log is the principal logarithm function.

**Lemma 2.1** ([7])**.** *Let $p, q \in \mathcal{R}$. Then*

(i) $e_0(t, s) \equiv 1$ *and* $e_p(t, t) \equiv 1$*;*
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$*;*

(iii) $\frac{1}{e_p(t,s)} = e_{\ominus p}(t,s)$, *where* $\ominus p(t) = -\frac{p(t)}{1+\mu(t)p(t)}$;
(iv) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $\left(\frac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\frac{p(t)}{e_p^{\sigma}(\cdot,s)}$.

**Lemma 2.2** ([1]). *If* $p \in \mathcal{R}^+$, *then*

$$0 < e_p(t,s) \leq \exp\left(\int_s^t p(v)\Delta v\right), \quad \textit{for all } t \in \mathbb{T}.$$

We end this section by stating the fixed point theorems that we employ to help us show the existence, uniqueness and stability of periodic solutions to (1.1) (see [9,16]).

**Theorem 2.1** (Contraction Mapping Principle). *Let* $(\chi, \rho)$ *a complete metric space and let* $\mathcal{P} : \chi \to \chi$. *If there is a constant* $\alpha < 1$ *such that for any* $x, y \in \chi$ *we have*

$$\rho(\mathcal{P}x, \mathcal{P}y) \leq \alpha\rho(x,y),$$

*then there is one and only one point* $z \in \chi$ *with* $\mathcal{P}z = z$.

**Theorem 2.2** (Krasnoselskii). *Let* $\mathcal{M}$ *be a closed bounded convex nonempty subset of a Banach space* $(\chi, \|.\|)$. *Suppose that* $\mathcal{A}$ *and* $\mathcal{B}$ *map* $\mathcal{M}$ *into* $\chi$ *such that*

(i) $\mathcal{A}$ *is compact and continuous;*
(ii) $\mathcal{B}$ *is a contraction mapping;*
(iii) $x, y \in \mathcal{M}$, *implies* $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$.

*Then there exists* $z \in \mathcal{M}$ *with* $z = \mathcal{A}z + \mathcal{B}z$.

## 3. Existence and Uniqueness of Periodic Solutions

Let $T > 0$, $T \in \mathbb{T}$ be fixed and if $\mathbb{T} \neq \mathbb{R}$, $T = n\omega$ for some $n \in \mathbb{N}$. By the notation $[a,b]$ we mean

$$[a,b] = \{t \in \mathbb{T},\ a \leq t \leq b\},$$

unless otherwise specified. The intervals $[a,b)$, $(a,b]$ and $(a,b)$ are defined similarly.

Define $C_T = \{\varphi \in C(\mathbb{T},\mathbb{R}) : \varphi(t+T) = \varphi(t)\}$ where $C(\mathbb{T},\mathbb{R})$ is the space of all real-valued rd-continuous functions. Then $(C_T, \|.\|)$ is a Banach space when it is endowed with the supremum norm

$$\|\varphi\| = \max_{t\in[0,T]} |\varphi(t)|.$$

We will need the following lemma whose proof can be found in [11].

**Lemma 3.1.** *Let* $x \in C_T$. *Then* $\|x^{\sigma}\| = \|x \circ \sigma\|$ *exists and* $\|x^{\sigma}\| = \|x\|$.

In this paper we assume that $a \in \mathcal{R}^+$, $a(t) > 0$ for all $t \in \mathbb{T}$ and

$$a(t+T) = a(t), \quad p(t+T) = p(t), \quad (id-\tau)(t+T) = (id-\tau)(t), \tag{3.1}$$

with $\tau(t) \geq \tau^* > 0$ and

$$e_a(T,0) > 1. \tag{3.2}$$

The functions $g(x)$, $h(x,y)$ are also globally Lipschitz continuous in $x$ and in $x$ and $y$, respectively. That, there are positive constants $k_1$, $k_2$ and $k_3$ such that

$$|g(x) - g(y)| \leq k_1 \|x - y\| \text{ and } k_1 < 1 \tag{3.3}$$

and

$$|h(x,y) - h(z,w)| \leq k_2 \|x - z\| + k_3 \|y - w\| . \tag{3.4}$$

**Lemma 3.2.** *Suppose* (3.1) *and* (3.2) *hold. If $u \in C_T$, then $u$ is a solution of* (1.1) *if and only if*

$$\begin{aligned} u(t) =& g\left(u\left(t-\tau\left(t\right)\right)\right) \\ &+ \gamma \int_t^{t+T} \left[p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h\left(u\left(s\right), u\left(s-\tau\left(s\right)\right)\right)\right] e_{\ominus a}(t,s)\Delta s, \end{aligned} \tag{3.5}$$

*where*

$$\gamma = (e_a(T,0) - 1)^{-1}.$$

*Proof.* Let $u \in C_T$ be a solution of (1.1). Multiply both sides of (1.1) by $e_a(t,0)$ and then integrate from $t$ to $t+T$, to obtain

$$\begin{aligned} &\int_t^{t+T} \left[ \left(u(s) - g\left(u(s-\tau\left(s\right))\right)\right)^\Delta e_a(s,0) \right] \Delta s \\ &= - \int_t^{t+T} a(s) \left[u^\sigma(s) - g\left(u^\sigma(s-\tau\left(s\right)\right)\right] e_a(s,0)\Delta s \\ &\quad + \int_t^{t+T} \left[p(s) - 2a(s) g\left(u^\sigma(s-\tau\left(s\right)) - h\left(u(s), u(s-\tau(s))\right)\right)\right] e_a(s,0)\Delta s. \end{aligned}$$

Performing an integration by part, we obtain

$$\begin{aligned} &\left[u(t) - g(u(t-\tau(t)))\right] e_a(t,0) \left(e_a(T,0) - 1\right) \\ &- \int_t^{t+T} a(s) \left[u^\sigma(s) - g(u^\sigma(s-\tau(s)))\right] e_a(s,0)\Delta s \\ &= - \int_t^{t+T} a(s) \left[u^\sigma(s) - g\left(u^\sigma(s-\tau\left(s\right)\right)\right] e_a(s,0)\Delta s \\ &\quad + \int_t^{t+T} \left[p(s) - 2a(s) g\left(u^\sigma(s-\tau\left(s\right)) - h\left(u(s), u(s-\tau(s))\right)\right)\right] e_a(s,0)\Delta s. \end{aligned}$$

By dividing both sides of the above equation by $e_a(t,0)\left(e_a(T,0) - 1\right)$, we arrive at

$$\begin{aligned} u(t) =& g(u(t-\tau(t))) + \left(e_a(T,0) - 1\right)^{-1} \\ &\times \int_t^{t+T} \left[p(s) - 2a(s)g(u^\sigma(s-\tau(s))) - h(u(s), u(t-\tau(s)))\right] e_{\ominus a}(t,s)\Delta s. \end{aligned}$$

The converse implication is easily obtained and the proof is complete. □

By applying Theorems 2.1 and 2.2, we obtain in this Section the existence and the uniqueness of periodic solution of (1.1). So, let a Banach space $(C_T, \|.\|)$, a closed bounded convex subset of $C_T$,

$$\mathcal{M} = \{\varphi \in C_T, \ \ \|\varphi\| \leq L\}, \tag{3.6}$$

with $L > 0$, and by the Lemma 3.2, we define the mapping $\mathcal{P}$ given by

$$\begin{aligned}(\mathcal{P}\varphi)(t) =& g(\varphi(t-\tau(t))) \\ &+ \gamma\int_t^{t+T}[p(s) - 2a(s)\,g(\varphi^\sigma(s-\tau(s))) - h(\varphi(s),\varphi(s-\tau(s)))]\,e_{\ominus a}(t,s)\Delta s. \end{aligned} \tag{3.7}$$

Therefore, we express (3.7) as

$$\mathcal{P}\varphi = \mathcal{A}\varphi + \mathcal{B}\varphi,$$

where $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned}&(\mathcal{A}\varphi)(t) \\ =&\gamma\int_t^{t+T}[p(s) - 2a(s)\,g(\varphi^\sigma(s-\tau(s))) - h(\varphi(s),\varphi(s-\tau(s)))]\,e_{\ominus a}(t,s)\Delta s\end{aligned} \tag{3.8}$$

and

$$(\mathcal{B}\varphi)(t) = g(\varphi(t-\tau(t))). \tag{3.9}$$

Since $\varphi \in C_T$ and (3.1) holds, we have for any $\varphi \in \mathcal{M}$

$$\begin{aligned}&(\mathcal{A}\varphi)(t+T) \\ =&\gamma\int_{t+T}^{t+T+T}[p(s) - 2a(s)\,g(\varphi^\sigma(s-\tau(s))) - h(\varphi(s),\varphi(s-\tau(s)))]\,e_{\ominus a}(t+T,s)\Delta s \\ =&\gamma\int_t^{t+T}[p(s+T) - 2a(s+T)\,g(\varphi^\sigma(s+T-\tau(s+T))) \\ &-h(\varphi(s+T),\varphi(s+T-\tau(s+T)))]\,e_{\ominus a}(t+T,s+T)\Delta s \\ =&(\mathcal{A}\varphi)(t),\end{aligned}$$

and

$$(\mathcal{B}\varphi)(t+T) = g(\varphi(t+T-\tau(t+T))) = g(\varphi(t-\tau(t))) = (\mathcal{B}\varphi)(t).$$

Then

$$\mathcal{A}\mathcal{M}, \mathcal{B}\mathcal{M} \subset C_T. \tag{3.10}$$

**Theorem 3.1.** *Assume that* (3.1)–(3.4) *hold. Let a constant $L > 0$ defined in $\mathcal{M}$ such that*

$$k_1L + |g(0)| + \gamma\beta T\left(\mu + 2\lambda k_1 L + |g(0)| + k_2L + k_3L + |h(0,0)|\right) \le L, \tag{3.11}$$

*where*

$$\beta = e_a(T,0), \quad \lambda = \sup_{t\in[0,T]}\{a(t)\}, \quad \mu = \sup_{t\in[0,T]}|p(t)|.$$

*Then* (1.1) *has a $T$-periodic solution.*

*Proof.* First, let $\mathcal{A}$ defined by (3.8), we show that $\mathcal{A}$ is continuous in the supremum norm and the image of $\mathcal{A}$ is contained in a compact set. Let $\varphi_n \in \mathcal{M}$ where $n$ is a positive integer such that $\varphi_n \to \varphi$ as $n \to \infty$. Then

$$\begin{aligned}
&\left|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)\right| \\
\leq & 2\gamma \int_t^{t+T} a(s)\left|g\left(\varphi_n^\sigma(s-\tau(s))\right) - g\left(\varphi^\sigma(s-\tau(s))\right)\right| e_{\ominus a}(t,s)\Delta s \\
&+ \gamma \int_t^{t+T} \left|h\left(\varphi_n(s), \varphi_n(s-\tau(s))\right) - h\left(\varphi(s), \varphi(s-\tau(s))\right)\right| e_{\ominus a}(t,s)\Delta s.
\end{aligned}$$

Since $g$ and $h$ are continuous, the dominated convergence theorem implies,

$$\lim_{n\to\infty} \left|(\mathcal{A}\varphi_n)(t) - (\mathcal{A}\varphi)(t)\right| = 0,$$

then $\mathcal{A}$ is continuous. Now, by (3.3) and (3.4), we obtain

$$\begin{aligned}
|g(y)| &\leq k_1|y| + |g(0)|, \\
|h(x,y)| &\leq k_2|x| + k_3|y| + |h(0,0)|.
\end{aligned}$$

Then, let $\varphi_n \in \mathcal{M}$ where $n$ is a positive integer, we have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi_n)(t)\right| \\
\leq & \gamma \int_t^{t+T} \left[|p(s)| + 2a(s)\left|g\left(\varphi_n^\sigma(s-\tau(s))\right)\right| + \left|h\left(\varphi_n(s), \varphi_n(s-\tau(s))\right)\right|\right] e_{\ominus a}(t,s)\Delta s \\
\leq & \gamma \int_t^{t+T} \left[p(s) + 2a(s)\left(k_1\|\varphi_n^\sigma\| + |g(0)|\right) + k_2\|\varphi_n\| + k_3\|\varphi_n\| + |h(0,0)|\right] e_{\ominus a}(t,s)\Delta s \\
\leq & \gamma\beta T\left(\mu + 2\lambda\left(k_1 L + |g(0)|\right) + k_2 L + k_3 L + |h(0,0)|\right) \leq L,
\end{aligned}$$

by (3.11). Next, we calculate $(\mathcal{A}\varphi_n)^\Delta(t)$ and show that it is uniformly bounded. By making use of (3.1) we obtain by taking the derivative in (3.8) that

$$(\mathcal{A}\varphi_n)^\Delta(t) = -a(t)(\mathcal{A}\varphi_n)^\sigma(t) + p(t) - 2a(t)g(\varphi_n^\sigma(t-\tau(t))) - h(\varphi_n(t), \varphi_n(t-\tau(t))).$$

Then, by (3.4) and (3.11) we have

$$\left|(\mathcal{A}\varphi_n)^\Delta(t)\right| \leq \lambda L + \mu + 2\lambda\left(k_1 L + |g(0)|\right) + k_2 L + k_3 L + |h(0,0)| = Q.$$

Thus the sequence $(\mathcal{A}\varphi_n)$ is uniformly bounded and equicontinuous. Hence, by Ascoli–Arzela's theorem $\mathcal{A}\mathcal{M}$ is compact.

Second, let $\mathcal{B}$ be defined by (3.9). Then for $\varphi_1, \varphi_2 \in \mathcal{M}$ we have by (3.3)

$$\begin{aligned}
|(\mathcal{B}\varphi_1)(t) - (\mathcal{B}\varphi_2)(t)| &= \left|g\left(\varphi_1(t-\tau(t))\right) - g\left(\varphi_2(t-\tau(t))\right)\right| \\
&\leq k_1\|\varphi_1 - \varphi_2\|.
\end{aligned}$$

Hence, $\mathcal{B}$ is contraction because $k_1 < 1$.

Finally, we show that if $\varphi, \phi \in \mathcal{M}$, then$\|\mathcal{A}\varphi + \mathcal{B}\phi\| \leq L$. Let $\varphi, \phi \in \mathcal{M}$ with $\|\varphi\|, \|\phi\| \leq L$, then

$$\begin{aligned}
&\|\mathcal{A}\varphi + \mathcal{B}\phi\| \leq k_1 \|\phi\| + |g(0)| \\
&+ \gamma \int_t^{t+T} [p(s) + 2a(s)(k_1 \|\varphi^\sigma\| + |g(0)|) + k_2 \|\varphi\| + k_3 \|\varphi\| + |h(0,0)|] \, e_{\ominus a}(t,s)\Delta s \\
\leq& k_1 L + |g(0)| + \gamma\beta T(\mu + 2\lambda(k_1 L + |g(0)|) + k_2 L + k_3 L + |h(0,0)|) \leq L,
\end{aligned}$$

by (3.11). Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 3.2 this fixed point is a solution of (1.1). Hence, (1.1) has a $T$-periodic solution. ☐

**Theorem 3.2.** *Suppose* (3.1)–(3.4) *hold. If*

$$k_1 + \gamma\beta T(2\lambda k_1 + k_2 + k_3) < 1, \tag{3.12}$$

*then* (1.1) *has a unique $T$-periodic solution.*

*Proof.* Let the mapping $\mathcal{P}$ be given by (3.7). For any $\varphi_1, \varphi_2 \in C_T$, we have

$$\begin{aligned}
&|(\mathcal{P}\varphi_1)(t) - (\mathcal{P}\varphi_2)(t)| \\
\leq& |g(\varphi_1(t-\tau(t))) - g(\varphi_2(t-\tau(t)))| \\
&+ 2\gamma \int_t^{t+T} a(s) |g(\varphi_1^\sigma(s-\tau(s))) - g(\varphi_2^\sigma(s-\tau(s)))| \, e_{\ominus a}(t,s)\Delta s \\
&+ \gamma \int_t^{t+T} |h(\varphi_1(s), \varphi_1(s-\tau(s))) - h(\varphi_2(s), \varphi_2(s-\tau(s)))| \, e_{\ominus a}(t,s)\Delta s \\
\leq& k_1 \|\varphi_1 - \varphi_2\| + \gamma \int_t^{t+T} (2\lambda k_1 + k_2 + k_3) \|\varphi_1 - \varphi_2\| \, e_{\ominus a}(t,s)\Delta s \\
\leq& [k_1 + \gamma\beta T(2\lambda k_1 + k_2 + k_3)] \|\varphi_1 - \varphi_2\|.
\end{aligned}$$

Since (3.12) hold, the contraction mapping principle completes the proof. ☐

**Corollary 3.1.** *Suppose* (3.1)–(3.4) *hold and let $\beta$, $\lambda$ and $\mu$ be constants defined in Theorem* 3.1. *Let $\mathcal{M}$ defined by* (3.6). *Suppose there are positive constants $k_1^*$, $k_2^*$ and $k_3^*$ such that for any $x, y, z, w \in \mathcal{M}$, we have*

$$|g(x) - g(y)| \leq k_1^* \|x - y\| \ \text{ and } \ k_1^* < 1, \tag{3.13}$$

$$|h(x,y) - h(z,w)| \leq k_2^* \|x - z\| + k_3^* \|y - w\| \tag{3.14}$$

*and*

$$k_1^* L + |g(0)| + \gamma\beta T(\mu + 2\lambda(k_1^* L + |g(0)|) + k_2^* L + k_3^* L + |h(0,0)|) \leq L. \tag{3.15}$$

*Then* (1.1) *has a $T$-periodic solution in $\mathcal{M}$. Moreover, if*

$$k_1^* + \gamma\beta T(2\lambda k_1^* + k_2^* + k_3^*) < 1, \tag{3.16}$$

*then* (1.1) *has a unique $T$-periodic solution in $\mathcal{M}$.*

*Proof.* Let the mapping $\mathcal{P}$ defined by (3.7). Then the proof follow immediately from Theorem 3.1 and Theorem 3.2. □

Notice that the constants $k_1^*$, $k_2^*$ and $k_3^*$ may depend on $L$.

## 4. Existence of Positive Periodic Solutions

It is for sure that existence of positive solutions is important for many applied problems. In this Section, by applying the Krasnoselskii's fixed point theorem and some techniques, to establish a set of sufficient conditions which guarantee the existence of positive periodic solutions of (1.1). So, we let $(\chi, \|.\|) = (C_T, \|.\|)$ and $\mathcal{M}(E,K) = \{\varphi \in C_T : E \le \varphi(t) \le K \text{ for all } t \in [0,T]\}$, for any $0 < E < K$. We assume that, there exist constants $a_1$, $a_2$, $g_1$ and $g_2$ such that for all $(t,(x,y,z)) \in [0,T] \times [E,K]^3$ we have

$$0 \le g_1, \quad 0 \le g_2 < 1, \quad -g_1 y \le g(y) \le g_2 y, \tag{4.1}$$

$$0 < a_1 \le a(t) \le a_2, \tag{4.2}$$

$$(E + g_1 K)\, a_2 \le p(t) - 2a(t)\, g(z) - h(x,y) \le (1-g_2)\, K a_1. \tag{4.3}$$

**Theorem 4.1.** *Assume that* (3.1)–(3.4) *and* (4.1)–(4.3) *hold. Then* (1.1) *has at least one positive $T$-periodic solution in* $\mathcal{M}(E,K)$.

*Proof.* By Lemma 3.2, it is obvious that (1.1) has a solution $\varphi$ if and only if $\mathcal{P}\varphi = \varphi$ has a solution $\varphi$. Let $\mathcal{A}$, $\mathcal{B}$ defined by (3.8), (3.9) respectively. A change of variable $t \mapsto t+T$ in (3.8) and (3.9) show that for any $\varphi \in \mathcal{M}(E,K)$ and $t \in \mathbb{R}$

$$\mathcal{A}(\mathcal{M}(E,K)) \subseteq C_T, \quad \mathcal{B}(\mathcal{M}(E,K)) \subseteq C_T. \tag{4.4}$$

Arguing as in the Theorem 3.1, the operator $\mathcal{A}$ is continuous. Next, we claim that $\mathcal{A}$ is compact. It is sufficient to show that $\mathcal{A}(\mathcal{M}(E,K))$ is uniformly bounded and equicontinuous in $[0,T]$. Notice that (4.2) and (4.3) ensure that

$$\begin{aligned}
&\|\mathcal{A}\varphi\| \\
&\le \sup_{t\in[0,T]} \left| \gamma \int_t^{t+T} [p(s) - 2a(s)\, g(\varphi^\sigma(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))]\, e_{\ominus a}(t,s)\Delta s \right| \\
&\le (1-g_2)\, K\gamma a_1 \sup_{t\in[0,T]} \int_t^{t+T} e_{\ominus a}(t,s)\Delta s \\
&\le (1-g_2)\, K, \quad \text{for all } \varphi \in [E,K]
\end{aligned}$$

and

$$\begin{aligned}
&\left| (\mathcal{A}\varphi)^\Delta(t) \right| \\
&\le a(t)\, |(\mathcal{A}\varphi)^\sigma(t)| + |p(t) - 2a(t)\, g(\varphi^\sigma(t-\tau(t))) - h(\varphi(t), \varphi(s-\tau(t)))| \\
&\le a_2(1-g_1)\, K + (1-g_1)\, a_1 K \\
&= (a_2+a_1)(1-g_1)\, k, \quad \text{for all } (t,\varphi) \in [0,T] \times [E,K],
\end{aligned}$$

which give that $\mathcal{A}(\mathcal{M}(E,K))$ is uniformly bounded and equicontinuous in $[0,T]$. Hence by Ascoli–Arzela's theorem $\mathcal{A}$ is compact. Next, let $\mathcal{B}$ defined by (3.9), for all $\varphi_1, \varphi_2 \in \mathcal{M}(E,K)$ and $t \in \mathbb{R}$, we obtain by (3.3)

$$\|\mathcal{B}\varphi_1 - \mathcal{B}\varphi_2\| \le k_1 \|\varphi_1 - \varphi_2\| .$$

Thus $\mathcal{B}$ is a contraction. Moreover, by (4.1)–(4.3), we infer that for all $\varphi, \phi \in \mathcal{M}(E,K)$ and $t \in \mathbb{R}$

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t) \\
=&g(\phi(t-\tau(t))) \\
&+\gamma \int_t^{t+T} [p(s) - 2a(s)g(\varphi^\sigma(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))] e_{\ominus a}(t,s)\Delta s \\
\le& g_2 K + (1-g_2) K\gamma \int_t^{t+T} a(s) e_{\ominus a}(t,s)\Delta s = K.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&(\mathcal{A}\varphi)(t) + (\mathcal{B}\phi)(t) \\
=&g(\phi(t-\tau(t))) \\
&+\gamma \int_t^{t+T} [p(s) - 2a(s)g(\varphi^\sigma(s-\tau(s))) - h(\varphi(s), \varphi(s-\tau(s)))] e_{\ominus a}(t,s)\Delta s \\
\ge& -g_1 K + (E + g_1 K)\gamma \int_t^{t+T} a(s) e_{\ominus a}(t,s)\Delta s = E,
\end{aligned}$$

which imply that

$$\mathcal{A}\varphi + \mathcal{B}\phi \in \mathcal{M}(E,K), \quad \text{for all } \varphi, \phi \in \mathcal{M}(E,K) \text{ and } t \in \mathbb{R}. \tag{4.5}$$

Clearly, all the hypotheses of the Krasnoselskii's theorem are satisfied. Thus there exists a fixed point $z \in \mathcal{M}(E,K)$ such that $z = \mathcal{A}z + \mathcal{B}z$. By Lemma 3.2 this fixed point is a solution of (1.1). Hence, (1.1) has a positive $T$-periodic solution. This completes the proof. □

**Theorem 4.2.** *Assume that* (3.1)–(3.4) *hold. Suppose that there exist constants $E$, $K$, $a_1$, $a_2$, $g_1$, $g_2$ and $t_0 \in [0,T]$ satisfying* (4.1)–(4.3) *with*

$$0 \le E < K, \tag{4.6}$$

*and either*

$$(E + g_1 K) a_2 < p(t_0) - 2a(t_0) g(z) - h(x,y), \quad \text{for all } x, y, z \in [E,K], \tag{4.7}$$

*or*

$$a(t_0) < a_2. \tag{4.8}$$

*Then* (1.1) *has at least one positive $T$-periodic solution in $\mathcal{M}(E,K)$, with $E < u \le K$ for each $t \in [0,T]$.*

*Proof.* As in the proof of Theorem 4.1, we conclude similarly that (1.1) has an $T$-periodic solution $u \in \mathcal{M}(E,K)$. Now we assert that $u(t) > E$ for all $t \in [0,T]$. Otherwise, there exists $t^* \in [0,T]$ satisfying $u(t^*) = E$. In view of (3.5), (3.7), (4.1) and (4.6), we have

$$\begin{aligned}
E =& g\left(u\left(t^*-\tau\left(t^*\right)\right)\right) \\
&+\gamma \int_{t^*}^{t^*+T}\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right]e_{\ominus a}(t^*,s)\Delta s \\
\geq& \gamma \int_{t^*}^{t^*+T}\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right]e_{\ominus a}(t^*,s)\Delta s \\
&-g_1K,
\end{aligned}$$

which implies that

$$\begin{aligned}
0 \geq& \gamma \int_{t^*}^{t^*+T}\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right]e_{\ominus a}(t^*,s)\Delta s \\
&-\left(E+g_1K\right) \\
=& \gamma \int_{t^*}^{t^*+T}\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right. \\
&\left.-(E+g_1K)a(s)\right]e_{\ominus a}(t^*,s)\Delta s. \qquad (4.9)
\end{aligned}$$

Assume that (4.7) holds. By means of (4.2), (4.3), (4.7) and the continuity of $h$, $g$, $a$, $p$, $\tau$ and $u$, we get that

$$\begin{aligned}
&\gamma \int_{t^*}^{t^*+T} e_{\ominus a}(t^*,s)\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right. \\
&\left.-(E+g_1K)a(s)\right]\Delta s \\
\geq& \int_{t^*}^{t^*+T} e_{\ominus a}(t^*,s)\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right. \\
&\left.-(E+g_1K)a_2\right]\Delta s \\
>&0,
\end{aligned}$$

which contradicts (4.9).

Assume that (4.8) holds. In light of (4.2), (4.3), (4.8) and the continuity of $h$, $g$, $a$, $p$, $\tau$ and $u$, we get that

$$\begin{aligned}
&\gamma \int_{t^*}^{t^*+T}\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)\right. \\
&\left.-(E+g_1K)a(s)\right]e_{\ominus a}(t^*,s)\Delta s \\
>& \int_{t^*}^{t^*+T} e_{\ominus a}(t^*,s)\Delta s \int_{t^*}^{t^*+T} e_{\ominus a}(t^*,s)\left[p(s)-2a(s)g(u^\sigma(s-\tau(s)))\right. \\
&\left.-h\left(u\left(s\right),u\left(s-\tau\left(s\right)\right)\right)-(E+g_1K)a_2\right]\Delta s \\
>&0,
\end{aligned}$$

which contradicts (4.9). This completes the proof. $\square$

*Example* 4.1. Consider (1.1), where

$$\mathbb{T}=\mathbb{R},\quad p(t)=3+\frac{\sin t}{5},\quad a(t)=1+\frac{\cos t}{4},\quad \tau(t)=2\cos^2 t,$$

$$g(x)=-\frac{x\sin x}{20},\quad \text{for all } x\in\mathbb{R},$$

$$h(x,y)=1+\sin^2 x+\cos^2 y,\quad \text{for all } (x,y)\in\mathbb{R}^2.$$

Let $T=2\pi$, $K=10$, $E=1$, $g_1=g_2=\frac{1}{20}$, $a_1=\frac{3}{4}$, $a_2=\frac{5}{4}$, $k_1=\frac{11}{20}$. It is easy to see that (3.3), (3.4) hold. Notice that

$$\begin{aligned}(E+g_1K)\,a_2=&\frac{15}{8}<\frac{195}{40}=3-\frac{1}{5}+2\left(1-\frac{1}{4}\right)\frac{1}{20}+2\\ \le& p\,(t)-2a\,(t)\,g\,(z)-h\,(x,y)\\ \le& 3+\frac{1}{5}+2\cdot\frac{5}{4}\cdot\frac{1}{20}+3=\frac{253}{40}\\ <&\frac{285}{40}=(1-g_2)\,Ka_1,\quad \text{for all } (t,x,y,z)\in\mathbb{R}^4.\end{aligned}$$

That is, (4.3) is satisfied. Thus Theorem 4.1 yields that (1.1) has a positive $2\pi$-periodic solution in $\mathcal{M}(1,10)$.

## 5. Stability of Periodic Solutions

This Section concerned with the stability of a $T$-periodic solution $u^*$ of (1.1). Let $v=u-u^*$ then (1.1) is transformed as

$$\begin{aligned}&(v\,(t)-G\,(v\,(t-\tau\,(t))))^{\Delta}\\ &=-\,a\,(t)\,v^{\sigma}\,(t)-a\,(t)\,G\,(v^{\sigma}\,(t-\tau\,(t)))-H\,(v\,(t)\,,v\,(t-\tau\,(t)))\,,\end{aligned}\tag{5.1}$$

where

$$G\,(v\,(t-\tau\,(t)))=g\,(u^*\,(t-\tau\,(t))+v\,(t-\tau\,(t)))-g\,(u^*(t-\tau\,(t))$$

and

$$\begin{aligned}&H\,(v\,(t)\,,v\,(t-\tau\,(t)))\\ =&h\,(u^*\,(t)+v\,(t))\,,u^*\,(t-\tau\,(t))+v\,(t-\tau\,(t))-h\,(u^*\,(t)\,,u^*\,(t-\tau\,(t)))\,.\end{aligned}$$

Clearly, (5.1) has trivial solution $v\equiv 0$, and the conditions (3.3) and (3.4) hold for $G$ and $H$ respectively. To arrive at the Lemma 3.2, as in the proof of this Lemma, multiply both sides of (5.1) by $e_a(t,0)$ and then integrate from 0 to $t$, to obtain

$$\begin{aligned}v(t)=&\,(v(0)-G\,(v\,(-\tau\,(0))))\,e_{\ominus a}(t,0)+G\,(v\,(t-\tau\,(t)))\\ &-\int_0^t\left[2a(s)G(v^{\sigma}(s-\tau(s)))+H\,(v\,(s)\,,v\,(s-\tau\,(s)))\right]e_{\ominus a}(t,s)\Delta s.\end{aligned}\tag{5.2}$$

Thus, we see that $v$ is a solution of (5.1) if and only if it satisfies (5.2). Assumed initial function

$$v(t)=\psi(t),\quad t\in[m_0,0]\,,$$

with $\psi \in C([m_0, 0], R)$, $[m_0, 0] = \{s \leq 0 \mid s = t - \tau(t),\ t \geq 0\}$. For the stability definition we refer the reader to the book [9].

Define the set $S_\psi$ by

$$S_\psi = \{\varphi \in C_T,\ \|\varphi\| \leq R,\ \varphi(t) = \psi(t) \text{ if } t \in [m_0, 0],\ \varphi(t) \to 0 \text{ as } t \to \infty\}, \tag{5.3}$$

for some positive constant $R$. Then, $(S_\psi, \|.\|)$ is a complete metric space where $\|.\|$ is the supremum norm.

**Theorem 5.1.** *If* (3.1)*,* (3.3)*,* (3.4) *and*

$$e_{\ominus a}(t, 0) \to 0 \ \text{as } t \to \infty, \tag{5.4}$$

$$t - \tau(t) \to \infty \ \text{as } t \to \infty, \tag{5.5}$$

$$k_1 + \int_0^t (2\lambda k_1 + k_2 + k_3)\, e_{\ominus a}(t, s)\Delta s \leq \alpha < 1, \tag{5.6}$$

*hold. Then every solution* $v(t, 0, \psi)$ *of* (5.1) *with small continuous initial function* $\psi$*, is bounded and asymptotically stable.*

*Proof.* Let the mapping $\mathcal{F}$ defined by $\psi(t)$ if $t \in [m_0, 0]$ and

$$\begin{aligned}(\mathcal{F}\varphi)(t) = {} & (\psi(0) - G(\psi(-\tau(0))))\, e_{\ominus a}(t, 0) + G(\varphi(t - \tau(t))) \\ & - \int_0^t [2a(s)G(\varphi^\sigma(s - \tau(s))) + H(\varphi(s), \varphi(s - \tau(s)))]\, e_{\ominus a}(t, s)\Delta s,\end{aligned} \tag{5.7}$$

if $t \geq 0$. Since $G$ and $H$ are continuous, it is easy to show that $\mathcal{F}\varphi$ is continuous. Let $\psi$ be a small given continuous initial function with $\|\psi\| < \delta$ ($\delta > 0$). Then using the condition (5.6) and the definition of $\mathcal{F}$ in (5.7), we have for $\varphi \in S_\psi$

$$\begin{aligned}|(\mathcal{F}\varphi)(t)| \leq {} & |\psi(0) - G(\psi(-\tau(0)))|\, e_{\ominus a}(t, 0) + k_1 R \\ & + R\int_0^t (2\lambda k_1 + k_2 + k_3)\, e_{\ominus a}(t, s)\Delta s \\ \leq {} & (1 + k_1)\delta + k_1 R + R\int_0^t (2\lambda k_1 + k_2 + k_3)\, e_{\ominus a}(t, s)\Delta s \\ \leq {} & (1 + k_1)\delta + \alpha R \leq R,\end{aligned}$$

which implies $\|\mathcal{F}\varphi\| \leq R$, for the right $\delta$. Next we show that $(\mathcal{F}\varphi)(t) \to 0$ as $t \to \infty$. The first term on the right side of (5.7) tends to zero, by condition (5.4). Also, the second term on the right side tends to zero, because of (5.5) and the fact that $\varphi \in S_\psi$. Let $\epsilon > 0$ be given, then there exists a $t_1 > 0$ such that for $t > t_1$, $\varphi(t - \tau(t)) < \epsilon$. By the condition (5.4), there exists a $t_2 > t_1$ such that for $t > t_2$ implies that

$$e_{\ominus a}(t, t_2) < \frac{\epsilon}{\alpha R}.$$

Thus for $t > t_2$, we have

$$\begin{aligned}
&\left|\int_0^t \left[2a(s)G(\varphi^\sigma(s-\tau(s))) + H\left(\varphi\left(s\right), \varphi\left(s-\tau\left(s\right)\right)\right)\right] e_{\ominus a}(t,s)\Delta s\right| \\
\leq & R\int_0^{t_1} \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t,s)\Delta s + \epsilon\int_0^t \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t,s)\Delta s \\
\leq & Re_{\ominus a}(t,t_2)\int_0^t \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t_2,s)\Delta s + \alpha\epsilon \\
\leq & \alpha Re_{\ominus a}(t,t_2)\alpha + \alpha\epsilon < \alpha\epsilon + \epsilon.
\end{aligned}$$

Hence, $\left(\mathcal{F}\varphi\right)(t) \to 0$ as $t \to \infty$. It is natural now to prove that $\mathcal{F}$ is contraction under the supremum norm. Let $\varphi_1, \varphi_2 \in S_\psi$. Then

$$\begin{aligned}
&\left|\left(\mathcal{F}\varphi_1\right)(t) - \left(\mathcal{F}\varphi_2\right)(t)\right| \\
\leq & \left|G\left(\varphi_1\left(t-\tau\left(t\right)\right)\right) - G\left(\varphi_2\left(t-\tau\left(t\right)\right)\right)\right| \\
& + 2\lambda\int_0^t \left|G(\varphi_1^\sigma(s-\tau(s))) - G(\varphi_2^\sigma(s-\tau(s)))\right| e_{\ominus a}(t,s)\Delta s \\
& + \int_0^t \left|H\left(\varphi_1\left(s\right), \varphi_1\left(s-\tau\left(s\right)\right)\right) - H\left(\varphi_2\left(s\right), \varphi_2\left(s-\tau\left(s\right)\right)\right)\right| e_{\ominus a}(t,s)\Delta s \\
\leq & k_1\left\|\varphi_1 - \varphi_2\right\| + \int_0^t \left(2\lambda k_1 + k_2 + k_3\right)\left\|\varphi_1 - \varphi_2\right\| e_{\ominus a}(t_2,s)\Delta s \\
\leq & \left[k_1 + \int_0^t \left(2\lambda k_1 + k_2 + k_3\right) e_{\ominus a}(t_2,s)\Delta s\right]\left\|\varphi_1 - \varphi_2\right\| \\
\leq & \alpha\left\|\varphi_1 - \varphi_2\right\|.
\end{aligned}$$

Hence, the contraction mapping principle implies, $\mathcal{F}$ has a unique fixed point in $S_\psi$ which solves (5.1), bounded and asymptotically stable. ☐

**Theorem 5.2.** *If* (3.1), (3.3), (3.4) *and* (5.6) *hold. Then, the zero solution is stable.*

*Proof.* The stability of the zero solution of (5.1) follows simply by replacing $R$ by $\epsilon$ in the above theorem. ☐

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## References


[1] M. Adivar and Y. N. Raffoul, *Existence of periodic solutions in totally nonlinear delay dynamic equations*, Electron. J. Qual. Theory Differ. Equ. **2009**(1) (2009), 1–20.

[2] A. Ardjouni and A. Djoudi, *Existence of positive periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale*, Malaya Journal of Matematik **2**(1) (2013), 60–67.

[3] A. Ardjouni and A. Djoudi, *Existence of periodic solutions for nonlinear neutral dynamic equations with functional delay on a time scale*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **52**(1) (2013), 5–19.

[4] A. Ardjouni and A. Djoudi, *Existence of periodic solutions for nonlinear neutral dynamic equations with variable delay on a time scale*, Commun. Nonlinear Sci. Numer. Simul. **17** (2012), 3061–3069.

[5] A. Ardjouni and A Djoudi, *Stability in neutral nonlinear dynamic equations on time scale with unbounded delay*, Stud. Univ. Babeç-Bolyai Math. **57**(4) (2012), 481–496.
[6] A. Ardjouni and A. Djoudi, *Periodic solutions in totally nonlinear dynamic equations with functional delay on a time scale*, Rend. Semin. Mat. Univ. Politec. Torino **68**(4) (2010), 349–359.
[7] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales, An Introduction with Applications*, Birkhäuser, Boston, 2001.
[8] M. Bohner and A. Peterson, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.
[9] T. A. Burton, *Stability by Fixed Point Theory for Functional Differential Equations*, Dover Publications, New York, 2006.
[10] S. Hilger, *Ein Masskettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten*, PhD thesis, Universität Würzburg, 1988.
[11] E. R. Kaufmann and Y. N. Raffoul, *Periodic solutions for a neutral nonlinear dynamical equation on a time scale*, J. Math. Anal. Appl. **319** (2006) 315–325.
[12] E. R. Kaufmann and Y. N. Raffoul, *Periodicity and stability in neutral nonlinear dynamic equation with functional delay on a time scale*, Electron. J. Differential Equations **2007**(27) (2007), 1–12.
[13] V. Lakshmikantham, S. Sivasundaram and B. Kaymarkcalan, *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers, Dordrecht, 1996.
[14] M. B. Mesmouli, A. Ardjouni and A. Djoudi, *Existence and stability of periodic solutions for nonlinear neutral differential equations with variable delay using fixed point technique*, Acta Univ. Palack. Olomuc. Fac. Rerum Natur. Math. **54**(1) (2015), 95–108.
[15] R. Olach, *Positive periodic solutions of delay differential equations*, Appl. Math. Lett. **26** (2013), 1141–1145.
[16] D. R. Smart, *Fixed Points Theorems*, Cambridge University Press, Cambridge, UK, 1980.
[17] Y. Yuan and Z. Guo, *On the existence and stability of periodic solutions for a nonlinear neutral functional differential equation*, Abstr. Appl. Anal. **2013** (2013), 1–8.

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# SOME COMMUTATIVITY THEOREMS FOR NEAR-RINGS WITH LEFT MULTIPLIERS

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Abstract. Let $\mathcal{N}$ be a 3-prime near-ring with the center $Z(\mathcal{N})$, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. In the present paper it is shown that a 3-prime near-ring $\mathcal{N}$ is a commutative ring if and only if it admits left multipliers $\mathcal{F}$ and $G$ satisfying any one of the following properties: (i) $\mathcal{F}(x)G(y) \pm [x,y] \in Z(\mathcal{N})$; (ii) $\mathcal{F}(x)G(y) \pm x \circ y \in Z(\mathcal{N})$; (iii) $\mathcal{F}(x)G(y) \pm yx \in Z(\mathcal{N})$; (iv) $\mathcal{F}(x)G(y) \pm xy \in Z(\mathcal{N})$ and (v) $\mathcal{F}([x,y]) \pm G(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in U$.

## 1. Introduction

In the present paper, $\mathcal{N}$ will denote a right near-ring with center $Z(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $0x = 0$). A non empty subset $U$ of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$ ) and if $U$ is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. As usual for all $x, y$ in $\mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a near-ring, $-(x + y) = -y - x$. Recall that $\mathcal{N}$ is 3-prime, that is, for all $a, b$ in $\mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies that $a = 0$ or $b = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [15], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The concept of derivation in rings has been generalized in several ways by various authors. Generalized derivation has been introduced already in rings by M. Brešar [7]. Also the notions of generalized derivation has been introduced in

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near-rings by Öznur Gölbasi [11]. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is called a right generalized derivation with associated derivation $d$ if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in \mathcal{N}$ and $\mathcal{F}$ is called a left generalized derivation with associated derivation $d$ if $\mathcal{F}(xy) = d(x)y + x\mathcal{F}(y)$ for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is called a generalized derivation with associated derivation $d$ if it is both a left as well as a right generalized derivation with associated derivation $d$. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is said to be a left (resp. right) multiplier (or centralizer) if $\mathcal{F}(xy) = \mathcal{F}(x)y$ (resp. $\mathcal{F}(xy) = x\mathcal{F}(y)$) holds for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is said to be a multiplier if it is both left as well as right multiplier. Notice that a right (resp. left) generalized derivation with associated derivation $d = 0$ is a left (resp. right) multiplier. Several authors investigated the commutativity in prime and semiprime rings admitting derivations and generalized derivations which satisfy appropriate algebraic conditions on suitable subset of the rings. For example, we refer the reader to [1, 2, 4–6, 8, 10, 12–14], where further references can be found. In [2] the authors proved that the prime ring $\mathcal{R}$ must be commutative if $\mathcal{R}$ is equipped with a generalized derivation $F$ associated with a nonzero derivation $d$ satisfying any one of the following conditions:

(i) $\mathcal{F}(x)\mathcal{F}(y) - xy \in Z(\mathcal{R})$ for all $x, y \in I$;
(ii) $\mathcal{F}(x)\mathcal{F}(y) + xy \in Z(\mathcal{R})$ for all $x, y \in I$, where $I$ is a nonzero two sided ideal of $\mathcal{R}$.

From these identities, it is natural to consider the situations

(iii) $\mathcal{F}(x)\mathcal{F}(y) - yx \in Z(\mathcal{R})$ and
(iv) $\mathcal{F}(x)\mathcal{F}(y)+yx \in Z(\mathcal{R})$ for all $x, y$ in some suitable subset of $\mathcal{R}$, which is studied by Dhara et al. in [9].

Further, A. Ali et al. [10] proved that the prime ring $\mathcal{R}$ must be commutative if $\mathcal{R}$ is equipped with a generalized derivation $F$ associated with a nonzero derivation $d$ satisfying any one of the following conditions: $(i)$ $\mathcal{F}(x)\mathcal{F}(y) \pm [x, y] \in Z(\mathcal{R})$ for all $x, y \in I$, $(ii)$ $\mathcal{F}(x)\mathcal{F}(y) \pm x \circ y \in Z(\mathcal{R})$ for all $x, y \in I$. In this line of investigation, it is more interesting to study the identities in two directions replacing ring by near-ring and the generalized derivation by left multiplier. Motivated by the above results, here we continue this line of investigation by considering more general situations. More precisely, we explore the commutativity of a 3-prime ring provided with left multipliers $\mathcal{F}$, $G$ satisfying any one of the following identities:

(i) $\mathcal{F}(x)G(y) \pm [x, y] \in Z(\mathcal{N})$;
(ii) $\mathcal{F}(x)G(y) \pm x \circ y \in Z(\mathcal{N})$;
(iii) $\mathcal{F}(x)G(y) \pm yx \in Z(\mathcal{N})$;
(iv) $\mathcal{F}(x)G(y) \pm xy \in Z(\mathcal{N}) \in Z(\mathcal{N})$ and
(v) $\mathcal{F}([x, y]) \pm G(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.

## 2. Some Preliminaries

In this section, we give some well known results of near-rings which will be used extensively in the forthcoming sections.

**Lemma 2.1.** ([3, Lemma 1.3]). *Let $\mathcal{N}$ be a 3-prime near-ring.*

(i) *If $z \in Z(\mathcal{N}) - \{0\}$ and $xz \in Z(\mathcal{N})$, then $x \in Z(\mathcal{N})$.*

(ii) *If $U$ is a nonzero semigroup right ideal (resp. semigroup left ideal) and $x$ is an element of $\mathcal{N}$ such that $Ux = \{0\}$ (resp. $xU = \{0\}$), then $x = 0$.*

**Lemma 2.2.** ([3, Lemma 1.5]). *Let $\mathcal{N}$ be a 3-prime near-ring, such that $Z(\mathcal{N})$ contains a nonzero semigroup left ideal or semigroup right ideal. Then $\mathcal{N}$ is a commutative ring.*

**Lemma 2.3.** ([3, Theorem 2.1]). *Let $\mathcal{N}$ be a 3-prime near-ring, $U$ a nonzero semigroup left ideal or semigroup right ideal. If $\mathcal{N}$ admits a nonzero derivation $d$ such that $d(U) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

**Lemma 2.4.** ([3, Lemma 1.4]). *Let $\mathcal{N}$ be a 3-prime near-ring and $U$ a nonzero semigroup ideal of $\mathcal{N}$. If $x, y \in \mathcal{N}$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*

## 3. Main Result

**Proposition 3.1.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits nonzero left multiplier $\mathcal{F}$ and nonzero derivation $d$, then the following assertions are equivalent:*

(i) *$\mathcal{F}([x, y]) \in Z(\mathcal{N})$ for all $x, y \in U$;*

(ii) *$\mathcal{F}([d(x), y]) \in Z(\mathcal{N})$ for all $x, y \in U$;*

(iii) *$\mathcal{N}$ is a commutative ring.*

*Proof.* It is obvious that (iii) implies (i) and (iii) implies (ii). So we need to prove that (i)⇒(iii) and (ii)⇒(iii).
(i)⇒(iii) Suppose that

$$\mathcal{F}([x, y]) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.1}$$

Replacing $y$ by $yx$ in (3.1), we get

$$(\mathcal{F}([x, y]))x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

Using Lemma 2.1 (i) together with (3.1), we obtain

$$\mathcal{F}([x, y]) = 0 \text{ or } x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

Which implies that

$$\mathcal{F}([x, y]) = 0, \quad \text{for all } x, y \in U. \tag{3.2}$$

Since $\mathcal{F}$ is left multiplier, (3.2) gives $\mathcal{F}(x)y = \mathcal{F}(y)x$ for all $x, y \in U$. Replacing $y$ by $[u, v]y$ and invoking (3.2), we get $\mathcal{F}(x)[u, v]y = 0$ for all $u, v, x, y \in U$. Taking $x = xr$ where $r \in \mathcal{N}$ in the last expression, we arrive at $\mathcal{F}(x)\mathcal{N}[u, v]y = \{0\}$ for all $u, v, x, y \in U$. Using the 3-primeness of $\mathcal{N}$ with the fact that $\mathcal{F} \neq 0$, we obtain $[u, v]U = \{0\}$ for all $u, v \in U$ and Lemma 2.1 (ii) gives $[u, v] = 0$ for all $u, v \in U$

which forces that $\mathcal{N}$ is a commutative ring.
(ii)⇒(iii) Assume that

$$\mathcal{F}([d(x), y]) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.3}$$

Substituting $y$ with $yd(x)$ in the pervious equation we obtain

$$\mathcal{F}([d(x), y])d(x) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.4}$$

So by using Lemma 2.1(i) and (3.3), we get

$$\mathcal{F}([d(x), y]) = 0 \text{ or } d(x) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

Which implies that

$$\mathcal{F}([d(x), y]) = 0, \quad \text{for all } x, y \in U. \tag{3.5}$$

But $\mathcal{F}$ is left multiplier, then (3.5) implies that $\mathcal{F}(d(x))y = \mathcal{F}(y)d(x)$ for all $x, y \in U$. Replacing $y$ by $[d(u), v]y$ and invoking (3.5), we get $\mathcal{F}(d(x))[d(u), v]y = 0$ for all $u, v, x, y \in U$. Putting $x = xr$ where $r \in \mathcal{N}$ in the latter equation, we arrive at $\mathcal{F}(x)\mathcal{N}[d(u), v]y = \{0\}$ for all $u, v, x, y \in U$. By using the 3-primeness of $\mathcal{N}$ and the fact that $\mathcal{F} \neq 0$, we get $[d(u), v]U = \{0\}$ for all $u, v \in U$. Hence, by Lemma 2.1 (ii) we obtain $[d(u), v] = 0$ for all $u, v \in U$, which forces that $\mathcal{N}$ is a commutative ring by [4, Theorem 2.9]. □

It is clear that $id_{\mathcal{N}}$ is a left multiplier of $\mathcal{N}$. If we replace $\mathcal{F}$ by $id_{\mathcal{N}}$, we find a result similar to [4, Theorem 2.9] in the case of right near-rings.

**Corollary 3.1.** *Let $\mathcal{N}$ be a 3-prime near-ring. If $U$ is a nonzero semigroup ideal of $\mathcal{N}$ and $d : \mathcal{N} \to \mathcal{N}$ be a derivation, then the following assertions are equivalent:*

(i) $[x, y] \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*
(ii) $[d(x), y] \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*
(ii) $\mathcal{N}$ *is a commutative ring.*

**Proposition 3.2.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits nonzero left multiplier $G$ and nonzero derivations $d$, then the following assertions are equivalent:*

(i) $G(x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*
(ii) $G(d(x) \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*
(iii) $\mathcal{N}$ *is a commutative ring.*

*Proof.* It is obvious that (iii) implies (i) and (iii) implies (ii). So it remains to prove that (i)⇒(iii) and (ii)⇒(iii).
(i)⇒(ii) Suppose that

$$G(x \circ y) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.6}$$

Replacing $y$ by $yx$ in (3.6), we get

$$G(x \circ y)x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

Using Lemma 2.1 (i) together with (3.6), we obtain

$$G(x \circ y) = 0 \text{ or } x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.7}$$

Suppose there exists $x_0 \in U$ such that $G(x_0 \circ y) = 0$ for all $y \in U$. For $y = x_0$, we obtain $G(x_0)x_0 = 0$. Also, we have $G(x_0)y = -G(y)x_0$ for all $y \in U$. Putting $x_0y$ in place of $y$, we arrive at $G(x_0)Ux_0 = \{0\}$ and by Lemma 2.4, we get $G(x_0) = 0$ or $x_0 = 0$ which implies that $G(x_0) = 0$. In this case, our assumption gives $G(y)x_0 = 0$ for all $y \in U$. Replacing $y$ by $yt$, where $t \in N$, we get $G(y)Nx_0 = \{0\}$. By 3-primeness of $N$ with the fact that $G \neq 0$, we conclude that $x_0 = 0$. In this case, (3.7) implies that $U \subseteq Z(N)$ and Lemma 2.2 forces that $N$ is a commutative ring.
(ii)$\Rightarrow$(iii) Assume that

$$G(d(x) \circ y) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.8}$$

Replacing $y$ by $yd(x)$ in (3.8), we obtain

$$G(d(x) \circ y)d(x) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.9}$$

Using Lemma 2.1 (i) together with (3.8), we obtain

$$G(d(x) \circ y) = 0 \text{ or } d(x) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

Suppose there exists $x_0 \in U$ such that $G(d(x_0) \circ y) = 0$ for all $y \in U$. For $y = d(x_0)$, by 2-torsion we get $G(d(x_0))d(x_0) = 0$. Also, we have $G(d(x_0))y = -G(y)d(x_0)$ for all $y \in U$. Replacing $d(x_0)y$ in place of $y$, we arrive at $G(d(x_0))Ud(x_0) = \{0\}$ and by Lemma 2.4, we obtain $G(d(x_0)) = 0$ or $d(x_0) = 0$ which implies that $G(d(x_0)) = 0$. Thus, our assumption gives $G(y)d(x_0) = 0$ for all $y \in U$. Substituting $y$ by $yt$, where $t \in \mathcal{N}$, we get $G(y)\mathcal{N}d(x_0) = \{0\}$. By 3-primeness of $\mathcal{N}$ with the fact that $G \neq 0$, we conclude that $d(x_0) = 0$. In this case, (3.9) implies that $d(U) \subseteq Z(\mathcal{N})$ and Lemma 2.3 forces that $\mathcal{N}$ is a commutative ring. $\square$

When $G = id_{\mathcal{N}}$, we find a result similar to [4, Theorem 2.10] in the case of right near-rings.

**Corollary 3.2.** *Let $\mathcal{N}$ be a 2-torsion 3-prime near-ring. If $U$ is a nonzero semigroup ideal of $\mathcal{N}$ and $d : \mathcal{N} \to \mathcal{N}$ be a derivation, then the following assertions are equivalent:*

(i) $x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(ii) $d(x) \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(iii) $\mathcal{N}$ *is a commutative ring.*

**Theorem 3.1.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits nonzero left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}([x, y]) + G(x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(ii) $\mathcal{F}([x, y]) - G(x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(iii) $\mathcal{N}$ *is a commutative ring.*

*Proof.* It is obvious that (iii) implies (i) and (iii) implies (ii). So we need to prove that (i)⇒(iii) and (ii)⇒(iii).
(i)⇒(iii) Suppose that

$$\mathcal{F}([x,y]) + G(x \circ y) \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.10}$$

Replacing $y$ by $yx$ in (3.10), we get

$$(\mathcal{F}([x,y]) + G(x \circ y))x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

Using Lemma 2.1(i) together with (3.10), we obtain

$$\mathcal{F}([x,y]) + G(x \circ y) = 0 \text{ or } x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.11}$$

Suppose there exists $x_0 \in Z(\mathcal{N})$, then (3.10) implies that $G(t+t)x_0 \in Z(\mathcal{N})$ for all $t \in U$. By Lemma 2.1(i), we arrive at $x_0 = 0$ or $G(t+t) \in Z(\mathcal{N})$ for all $t \in U$, in this case, (3.11) becomes

$$\mathcal{F}([x,y]) + G(x \circ y) = 0 \text{ or } G(t+t) \in Z(\mathcal{N}), \quad \text{for all } x, y, t \in U. \tag{3.12}$$

Assume that $\mathcal{F}([x,y]) + G(x \circ y) = 0$ for all $x, y \in U$. For $x = y$, we obtain $G(x^2) = G(x)x = 0$ for all $x \in U$ and replacing $x$ by $xG(y)$ in our assumption, we have $\mathcal{F}([xG(y), y]) + G(xG(y) \circ y) = 0$ for all $x, y \in U$ and developing this equation, we find that $(G(y) - \mathcal{F}(y))UG(y) = \{0\}$ for all $y \in U$. Using Lemma 2.4, we find that either $G(y) - \mathcal{F}(y) = 0$ or $G(y) = 0$ for all $y \in U$. Suppose there exists $y_0 \in U$ such that $G(y_0) - \mathcal{F}(y_0) = 0$ and replacing $y$ by $y_0$ in $\mathcal{F}([x,y]) + G(x \circ y) = 0$, we arrive at $(\mathcal{F}(x) + G(x))y_0 = 0$ for all $x \in U$. Taking $xt$ in place of $x$, where $t \in \mathcal{N}$ the last expression becomes $(\mathcal{F}(x) + G(x))\mathcal{N}y_0 = \{0\}$ for all $x \in U$. By 3-primeness of $\mathcal{N}$, we get $\mathcal{F} = -G$ or $y_0 = 0$. Since $G \neq 0$, in all the cases, we obtain that $\mathcal{F} = -G$ which forces that $\mathcal{F}([x,y]) = \mathcal{F}(x \circ y)$ for all $x, y \in U$ and developing this expression, we find that $\mathcal{F}(y)x = 0$ for all $x, y \in U$. Using Lemma 2.1(ii), we conclude that $\mathcal{F}(U) = \{0\}$. Since $\mathcal{N}U \subseteq U$, then $\mathcal{F}(\mathcal{N})U = \{0\}$. By Lemma 2.1(ii), we obtain $\mathcal{F} = 0$, a contradiction.

Assume that $G(t+t) \in Z(\mathcal{N})$ for all $t \in U$. Putting $t = tr$, where $r \in \mathcal{N}$, then $G(tr + tr) = G(t+t)r \in Z(\mathcal{N})$ for all $t \in U$, $r \in \mathcal{N}$. Using Lemma 2.1(i), we obtain $G(t+t) = 0$ or $r \in Z(\mathcal{N})$ for all $t \in U$, $r \in \mathcal{N}$ and using 2-torsion freeness of $\mathcal{N}$ together with $G \neq 0$, we conclude that $\mathcal{N}$ is a commutative ring.

The proof of (ii)⇒(iii) is similar to (i)⇒(iii). □

In particular, when $G = id_{\mathcal{N}}$, then we have the following corollary.

**Corollary 3.3.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) $\mathcal{F}([x,y]) + x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*
(ii) $\mathcal{F}([x,y]) - x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*
(iii) $\mathcal{N}$ *is a commutative ring.*

In particular, when $\mathcal{F}$ is an identity map, then we have the following result.

**Corollary 3.4.** *Let $\mathcal{N}$ be a 2-torsion free 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a nonzero left multiplier $G$, then the following assertions are equivalent:*

(i) $[x,y] + G(x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(ii) $[x,y] - G(x \circ y) \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(iii) $\mathcal{N}$ *is a commutative ring.*

In a ring $\mathcal{R}$, if $\mathcal{F}$ is a left multiplier, then $\mathcal{F} \pm id_{\mathcal{R}}$ is also a left multiplier, where $id_{\mathcal{R}}$ denotes the identity mapping on $\mathcal{R}$. By substituting $\mathcal{F} \pm id_{\mathcal{R}}$ in place of $\mathcal{F}$ in Theorem 3.1, we get the following result.

**Corollary 3.5.** *Let $\mathcal{R}$ be a prime ring of characteristic not 2, and $U$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits nonzero left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}([x,y]) + G(x \circ y) \pm [x,y] \in Z(\mathcal{R})$ *for all* $x, y \in U$;
(ii) $\mathcal{F}([x,y]) - G(x \circ y) \pm [x,y] \in Z(\mathcal{R})$ *for all* $x, y \in U$;
(iii) $\mathcal{R}$ *is commutative.*

In particular, when $G$ is replaced by $G \pm id_{\mathcal{R}}$, then we have the following.

**Corollary 3.6.** *Let $\mathcal{R}$ be a prime ring of characteristic not 2, and $U$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits nonzero left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}([x,y]) + G(x \circ y) \pm x \circ y \in Z(\mathcal{R})$ *for all* $x, y \in U$;
(ii) $\mathcal{F}([x,y]) - G(x \circ y) \pm x \circ y \in Z(\mathcal{R})$ *for all* $x, y \in U$;
(iii) $\mathcal{R}$ *is commutative.*

In particular, when $\mathcal{F}$ and $G$ are replaced by $\mathcal{F} \pm id_{\mathcal{R}}$ and $G \pm id_{\mathcal{R}}$ respectively, then we have the following corollary.

**Corollary 3.7.** *Let $\mathcal{R}$ be a prime ring of characteristic not 2, and $U$ be a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits nonzero left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}([x,y]) + G(x \circ y) \pm 2xy \in Z(\mathcal{R})$ *for all* $x, y \in U$;
(ii) $\mathcal{F}([x,y]) - G(x \circ y) \pm 2yx \in Z(\mathcal{R})$ *for all* $x, y \in U$;
(iii) $\mathcal{R}$ *is commutative.*

**Theorem 3.2.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}(x)G(y) - [x,y] \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(ii) $\mathcal{F}(x)G(y) - x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(iii) $\mathcal{N}$ *is a commutative ring.*

*Proof.* It is obvious that (iii) implies (i) and (iii) implies (ii). So we need to prove that (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii).

(i)⇒(iii) If $\mathcal{F} = 0$ or $G = 0$, then $-[x, y] \in Z(\mathcal{N})$ for all $x, y \in U$. Replacing $y$ by $yx$, we get $(-[x, y])x \in Z(\mathcal{N})$ for all $x, y \in U$ and using Lemma 2.1(i), we obtain $[x, y] = 0$ or $x \in Z(\mathcal{N})$ for all $x, y \in U$ which implies that $[x, y] = 0$ for all $x, y \in U$. Using Corollary 3.1, we conclude that $\mathcal{N}$ is a commutative ring.

Suppose that $\mathcal{F} \neq 0$ and $G \neq 0$, we have

$$\mathcal{F}(x)G(y) - [x, y] \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.13}$$

Replacing $y$ by $yx$ in (3.13) and using the fact that $[x, yx] = ([x, y])x$, we get

$$(\mathcal{F}(x)G(y) - [x, y])x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

By Lemma 2.1 (i) together with (3.13), we obtain

$$\mathcal{F}(x)G(y) = [x, y] \text{ or } x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.14}$$

Suppose there exists $x_0 \in U$ such that $\mathcal{F}(x_0)G(y) = [x_0, y]$ for all $y \in U$. Taking $yr$ instead of $y$, where $r \in \mathcal{N}$, we get $\mathcal{F}(x_0)G(y)r = [x_0, yr]$ for all $y \in U$, $r \in \mathcal{N}$ which implies that $[x_0, y]r = [x_0, yr]$ for all $y \in U$, $r \in \mathcal{N}$ and developing this expression, we arrive at $y[x_0, r] = 0$ for all $y \in U$, $r \in \mathcal{N}$. Using Lemma 2.1 (ii), we obtain that $x_0 \in Z(\mathcal{N})$ in this case, (3.2) becomes $x \in Z(\mathcal{N})$ for all $x \in U$ which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.2.

(ii)⇒(iii) If $\mathcal{F} = 0$ or $G = 0$, then $-(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in U$. Replacing $y$ by $yx$ we get $(-x \circ y)x \in Z(\mathcal{N})$ for all $x, y \in U$ and using Lemma 2.1 (i), we obtain $x \circ y = 0$ or $x \in Z(\mathcal{N})$ for all $x, y \in U$. Using the same techniques as used in the proof of [4, Theorem 2.10], we conclude that $\mathcal{N}$ is a commutative ring.

Now assume that $\mathcal{F} \neq 0$ and $G \neq 0$, we have

$$\mathcal{F}(x)G(y) - x \circ y \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.15}$$

Putting $yx$ instead of $y$ in (3.15) and using the fact that $x \circ yx = (x \circ y)x$, we get

$$(\mathcal{F}(x)G(y) - x \circ y)x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

By Lemma 2.1(i) and using (3.15), we obtain

$$\mathcal{F}(x)G(y) = x \circ y \text{ or } x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.16}$$

If there exists $x_0 \in U$ such that $\mathcal{F}(x_0)G(y) = x_0 \circ y$ for all $y \in U$. Taking $yr$ instead of $y$, where $r \in \mathcal{N}$, we obtain $\mathcal{F}(x_0)G(y)r = x_0 \circ yr$ for all $y \in U$, $r \in \mathcal{N}$ this reduces to $(x_0 \circ y)r = x_0 \circ yr$ for all $y \in U$, $r \in \mathcal{N}$, so $y[x_0, r] = 0$ for all $y \in U$, $r \in \mathcal{N}$. By Lemma 2.1 (ii), we conclude that $x_0 \in Z(\mathcal{N})$, in this case, (3.16) becomes $x \in Z(\mathcal{N})$ for all $x \in U$. By Lemma 2.2, we conclude that $\mathcal{N}$ is a commutative ring. □

Using similar techniques with necessary variations, we get the following Theorem. We skip the details of the proof just to avoid repetition.

**Theorem 3.3.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}(x)G(y) + [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in U$;
(ii) $\mathcal{F}(x)G(y) + x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$;

(iii) $\mathcal{N}$ *is a commutative ring.*

If we put $F = G$ in Theorem 3.2 and Theorem 3.3, we obtain the following result.

**Corollary 3.8.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) $\mathcal{F}(x)\mathcal{F}(y) \pm [x, y] \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*

(ii) $\mathcal{F}(x)\mathcal{F}(y) \pm x \circ y \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*

(iii) $\mathcal{N}$ *is a commutative ring.*

**Theorem 3.4.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits left multipliers $\mathcal{F}$ and $G$, then the following assertions are equivalent:*

(i) $\mathcal{F}(x)G(y) - yx \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*

(ii) $\mathcal{F}(x)G(y) + yx \in Z(\mathcal{N})$ *for all* $x, y \in U$*;*

(iii) $\mathcal{N}$ *is a commutative ring.*

*Proof.* It is clear that (iii) implies (i) and (iii) implies (ii). It remains to show that (i)$\Rightarrow$(iii) and (ii)$\Rightarrow$(iii).
(i)$\Rightarrow$(iii) If $\mathcal{F} = 0$ or $G = 0$, then $-yx \in Z(\mathcal{N})$ for all $x, y \in U$, so $-yxz \in Z(\mathcal{N})$ for all $x, y, z \in U$. Using Lemma 2.1 (i), we obtain $-yx = 0$ or $z \in Z(\mathcal{N})$ for all $x, y, z \in U$. Using Lemma 2.1 (ii) and Lemma 2.2 with the fact that $U \neq \{0\}$, we conclude that $\mathcal{N}$ is a commutative ring.

Suppose that $\mathcal{F} \neq 0$ and $G \neq 0$, we have

$$\mathcal{F}(x)G(y) - yx \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.17}$$

Replacing $y$ by $yx$ in (3.17), we get

$$(\mathcal{F}(x)G(y) - yx)x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U.$$

By Lemma 2.1 (i), the last expression becomes

$$\mathcal{F}(x)G(y) = yx \text{ or } x \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.18}$$

Suppose there exists $x_0 \in U$ such that $\mathcal{F}(x_0)G(y) = yx_0$ for all $y \in U$. Taking $yr$ instead of $y$, where $r \in \mathcal{N}$, we find that $\mathcal{F}(x_0)G(y)r = yrx_0$ for all $y \in U$, $r \in \mathcal{N}$ which implies that $yx_0r = yrx_0$ for all $y \in U$, $r \in \mathcal{N}$ and therefore $y[x_0, r] = 0$ for all $y \in U$, $r \in \mathcal{N}$. Using Lemma 2.1 (ii), we obtain that $x_0 \in Z(\mathcal{N})$ in this case, (3.18) becomes $x \in Z(\mathcal{N})$ for all $x \in U$ which forces that $\mathcal{N}$ is a commutative ring by Lemma 2.2.
(ii)$\Rightarrow$(iii) Using the same tips that have used in the proof of (i)$\Rightarrow$(iii), we get the desired result. ☐

For $G = \mathcal{F}$, we have the following result.

**Corollary 3.9.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplier $\mathcal{F}$, then the following assertions are equivalent:*

(i) $\mathcal{F}(x)F(y) - yx \in Z(\mathcal{N})$ *for all* $x, y \in U$;

(ii) $\mathcal{F}(x)F(y) + yx \in Z(\mathcal{N})$ *for all* $x, y \in U$;

(iii) $\mathcal{N}$ *is a commutative ring.*

**Theorem 3.5.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits left multipliers $\mathcal{F}$ and $G$ such that $\mathcal{F}(x)G(y) - xy \in Z(\mathcal{N})$ for all $x, y \in U$ or $\mathcal{F}(x)G(y) + xy \in Z(\mathcal{N})$ for all $x, y \in U$, then $\mathcal{N}$ is a commutative ring or $\mathcal{F}$ is a right multiplier.*

*Proof.* If $\mathcal{F} = 0$ or $G = 0$, then $-xy \in Z(\mathcal{N})$ for all $x, y \in U$. Using the same proof that we have used in the beginning of Theorem 3.3, we obtain the required result. Now suppose that

$$\mathcal{F}(x)G(y) - xy \in Z(\mathcal{N}), \quad \text{for all } x, y \in U. \tag{3.19}$$

Putting $yz$ in place of $y$ in (3.19), we get

$$(\mathcal{F}(x)G(y) - xy)z \in Z(\mathcal{N}), \quad \text{for all } x, y, z \in U.$$

Using Lemma 2.1(i) and (3.19), the above expression implies that

$$\mathcal{F}(x)G(y) = xy \text{ or } \ z \in Z(\mathcal{N}), \quad \text{for all } x, y, z \in U$$

By Lemma 2.2, we obtain

$$\mathcal{F}(x)G(y) = xy, \quad \text{for all } x, y \in U \text{ or } \mathcal{N} \text{ is a commutative ring.} \tag{3.20}$$

Assume that $\mathcal{F}(x)G(y) = xy$ for all $x, y \in U$. Taking $xu$ instead of $x$, we get $\mathcal{F}(x)uG(y) = xuy = x\mathcal{F}(u)G(y)$ for all $x, y, u \in U$, so $(\mathcal{F}(x)u - x\mathcal{F}(u)G(y) = 0$ for all $x, y, u \in U$. Replacing $u$ by $ur$, where $r \in \mathcal{N}$, we arrive at $(\mathcal{F}(x)u - x\mathcal{F}(u))\mathcal{N}G(y) = \{0\}$ for all $x, y, u \in U$. By 3-primeness of $\mathcal{N}$, we get either $G(y) = 0$ or $\mathcal{F}(x)u = x\mathcal{F}(u)$ for all $x, y, u \in U$. If $G(y) = 0$ for all $y \in U$, taking $ry$ instead of $y$, we find that $G(r)U = \{0\}$ for all $r \in \mathcal{N}$ and using Lemma 2.1(ii), we get $G = 0$; a contradiction. If $\mathcal{F}(x)u = x\mathcal{F}(u)$ for all $x, u \in U$. Replacing $rx$ in place of $x$, we get $(\mathcal{F}(r)x - r\mathcal{F}(x))U = \{0\}$ for all $x \in U$, $r \in \mathcal{N}$. By Lemma 2.1(ii), we obtain $\mathcal{F}(r)x = r\mathcal{F}(x)$ for all $x \in U$, $r \in \mathcal{N}$. Taking $tx$ instead of $x$, where $t \in \mathcal{N}$ and using Lemma 2.1 (ii) again, we conclude that $\mathcal{F}(r)t = r\mathcal{F}(t)$ for all $r, t \in \mathcal{N}$ which forces that $\mathcal{F}$ is a right multiplier.

If $\mathcal{F}(x)G(y) + xy \in Z(\mathcal{N})$ for all $x, y \in U$, using the same techniques as have used in (i)⇒(iii), we get the required result. □

In particular, when $G = \mathcal{F}$, then we have the following.

**Corollary 3.10.** *Let $\mathcal{N}$ be a 3-prime near-ring, and $U$ be a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplier $\mathcal{F}$ such that $\mathcal{F}(x)\mathcal{F}(y) - xy \in Z(\mathcal{N})$ for all $x, y \in U$ or $\mathcal{F}(x)\mathcal{F}(y) + xy \in Z(\mathcal{N})$ for all $x, y \in U$, then $\mathcal{N}$ is a commutative ring or $\mathcal{F}$ is a right multiplier.*

The following example demonstrates that our results are not true for arbitrary near-rings.

*Example* 3.1. Suppose that $S$ is any right near-ring. Let

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in S \right\} \text{ and } U = \left\{ \begin{pmatrix} 0 & 0 & u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| \, u \in S \right\}.$$

Define maps $\mathcal{F}, G : \mathcal{N} \to \mathcal{N}$ such that

$$\mathcal{F}\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } G\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

Then, it is easy to see that $\mathcal{N}$ is a right near-ring and $\mathcal{F}$, $G$ are left multipliers on $\mathcal{N}$ satisfying the following properties:

(i) $\mathcal{F}(x)G(y) \pm [x, y] \in Z(\mathcal{N})$;
(ii) $\mathcal{F}(x)G(y) \pm x \circ y \in Z(\mathcal{N})$;
(iii) $\mathcal{F}(x)G(y) \pm xy \in Z(\mathcal{N})$;
(iv) $\mathcal{F}(x)G(y) \pm yx \in Z(\mathcal{N})$;
(v) $\mathcal{F}([x, y]) \pm G(x \circ y) \in Z(\mathcal{N})$;

for all $x, y \in U$. However, $\mathcal{N}$ is not commutative.

The following example shows that the condition "$\mathcal{F}([x, y]) \pm G(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in U$" is crucial in Theorem 3.1.

*Example* 3.2. Let $\mathcal{N} = M_2(\mathbb{Z})$ be the $2 \times 2$ matrix ring over $\mathbb{Z}$ and $\mathcal{F}, G : \mathcal{N} \to \mathcal{N}$ such that

$$\mathcal{F}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \quad \text{for all } a, b, c, d \in \mathbb{Z} \text{ and } G = \mathcal{F}$$

It is easy to verify that $\mathcal{N}$ is a non-commutative prime ring which is 2-torsion free and $\mathcal{F}, G$ are left multipliers of $\mathcal{N}$. Moreover, for $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, we have $\mathcal{F}([A, B]) \pm G(A \circ B) \notin Z(\mathcal{N})$.

The following example demonstrate that the existence of "2-torsion free" in the hypotheses of Theorem 3.1 is essential.

*Example* 3.3. Let $\mathcal{N} = M_2(\mathbb{Z}_2)$ be the $2 \times 2$ matrix ring over the field $\mathbb{Z}_2$ and $\mathcal{F} = Id_{\mathcal{N}}$. It is easy to see that $\mathcal{N}$ is a non-commutative prime ring which is not 2-torsion free. Moreover, $\mathcal{N}$ satisfies the condition $\mathcal{F}([x, y]) \pm G(x \circ y) \in Z(\mathcal{N})$ for all $x, y \in \mathcal{N}$.

## References


[1] M. Ashraf and N. Rehman, *On derivations and commutativity in prime rings*, East-West J. Math. **3**(1) (2001), 87–91.
[2] M. Ashraf, A. Ali and S. Ali, *Some commutativity theorems for rings with generalized derivations*, Southeast Asian Bull. Math. **31** (2007), 415–421.
[3] H. E. Bell, *On derivations in near-rings II*, in: *Nearrings, Nearfields and K-Loops*, (Hamburg, 1995), Math. Appl. **426**, Kluwer Academic Publishers, Dordrecht, 1997.
[4] H. E. Bell, A. Boua and L. Oukhtite, *Semigroup ideals and commutativity in* 3*-prime near rings*, Comm. Algebra **43** (2015), 1757–1770.
[5] A. Boua, L. Oukhtite and H. E. Bell, *Differential identities on semigroup ideals of right near-rings*, Asian-Eur. J. Math. **6**(4) (2013), Paper ID 1350050, 8 pages.
[6] A. Boua and L. Taoufiq, *Some algebraic results involving derivations in* 3*-prime near-rings*, Indian J. Math. **59**(2) (2017), 147–160.

[7] M. Brešar, *On the distance of composition of two derivations to the generalized derivations*, Glasg. Math. J. **33** (1991), 89–93.
[8] M. N. Daif and H. E. Bell, *Remarks on derivations on semiprime rings*, Int. J. Math. Math. Sci. **15** (1) (1992), 205–206.
[9] B. Dhara and S. Ali, *On multiplicative (generalized)-derivations in prime and semiprime rings*, Aequationes Math. **86**(1-2) (2013), 65–79.
[10] A. Ali, B. Dhara, S. Khan and F. Ali, *Multiplicative (generalized)-derivations and left ideals in semiprime rings*, Hacet. J. Math. Stat. **44**(6) (2015), 1293–1306
[11] Ö. Gölbasi, *Notes on prime near-rings with generalized derivation*, Southeast Asian Bull. Math. **30** (2006), 49–54.
[12] M. Hongan, *A note on semiprime rings with derivation*, Int. J. Math. Math. Sci. **20**(2) (1997), 413–415.
[13] M. A. Quadri, M. S. Khan and N. Rehman, *Generalized derivations and commutativity of prime rings*, Indian J. Pure Appl. Math. **34**(9) (2003), 1393–1396.
[14] N. Rehman, *On commutativity of rings with generalized derivations*, Math. J. Okayama Univ. **44** (2002), 43–49.
[15] X. K. Wang, *Derivations in prime near-rings*, Proc. Amer. Math. Soc. **121** (1994), 361–366

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# SOFT INTERIOR-HYPERIDEALS IN LEFT REGULAR LA-SEMIHYPERGROUPS

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Abstract. This paper is a contribution to the study of the effective content of LA-hyperstructure. In this paper, we introduce the notion of soft interior-hyperideals. Further, we give several basic properties of these notions and provide different important characterizations in terms of soft interior hyperideals.

## 1. Introduction

Marty [23] introduced the notion of algebraic hyperstructures as natural generalization of classical algebraic structures. The difference between classical algebraic structures and algebraic hyperstructures is that, in algebraic structures the composition of two elements is an element while in algebraic hyperstructure the composition of two elements is a non-empty set. Koskas introduced the notion of semihypergroups. Hasankhani [15] defined ideals in right (left) semihypergroups and discussed some hyper versions of Green's relations.

The concept of LA-semigroup was given by Kazim and Naseeruddin [17]. Faisal et al. [34] characterized left regular LA-semigroup in terms of fuzzy interior ideals. Hila and Dine [16] defined LA-semihypergroups. They studied several properties of hyperideals of LA-semihypergroup. Yaqoob et al. [30] gave some characterizations of LA-semihypergroups using left, right and interior-hyperideals. Yousafzai and Corsini [33] extended the theory of an LA-semigroup in terms of their one sided ideals. They characterized the class of an intra-regular LA-semihypergroup using one-sided hyperideals. Amjad [2] defined generalized hyperideals in locally associative left almost semihypergroups and in [3] they studied pure LA-semihypergroups. Yaqoob and

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Gulistan [31] studied hyperideals and $M$-hypersystem in partially ordered left almost semihypergroups. Recently, many authors [5, 6, 13, 14, 18–20, 28, 29, 32] have worked on LA-semihypergroups.

Our world is surrounded by uncertainties and ambiguities. We pass through many uncertainties in our daily life. Therefore, it is necessary to prepare a model so that we deal such uncertainties and ambiguities. Initially, probability theory was the only mathematical concept for dealing some unplanned activities. To handle some special kind of activity known as fuzziness, Zadeh [35] introduced the notion of fuzzy set as an extension of classical set theory. But there was a difficulty for membership function. How to set the membership function in each particular case. We cannot impose only one way to set the membership function. The reason for these difficulties is, possibly, the inadequacy of the parametrization tool of the theory. To remove this difficulty, Molodtsov [24] introduced a mathematical tool for dealing with hesitant, fuzzy, unpredictable and unsure articles known as soft set. A soft set is a collection of approximate descriptions of an object. Each approximate description has two parts: a predicate and an approximate value set. Further, Maji et al. [22] defined many applications in soft sets. After the beginning of soft set theory, many authors gave a new view to classical mathematics. Cagman and Aktas proposed the concept of soft algebraic structure. They introduced soft group theory [1] and gave the definition of soft group which is analogous to the rough group definition. They correlate soft sets with rough sets and fuzzy sets. After that many authors [8, 12, 27] have worked on soft algebraic structures. Cagman et al. [7] gave a new approach to soft group definition called soft intersection group. This approach is depends on the insertion and intersection of sets. Anvariyeh et al. [4] initiated soft semihypergroups by using the soft set theory. Sezgin [26] studied soft set theory in LA-semigroup with the concept of soft intersection LA-semigroups and soft intersection LA-ideals. Naz and Shabir [25] investigated the basic terms and properties of soft sets. They relate soft sets with the concept of semihypergroups. Farooq et al. [11] characterized regular and left regular ordered semihypergroups using intersection soft generalized bi-hyperideals. Khan et al. [21] introduced the notion of soft intersection (S.I.) hyperideals in LA-semihypergroups and gave some characterizations.

In this paper, we introduce soft interior-hyperideals through new approach called soft intersection (briefly, S.I.) and establish some of their elementary properties. We also define the concept of soft semiprime and study some results on them. We characterize left regular LA-semihypergroups in terms of soft interior-hyperideals and prove that in a left regular LA-semihypergroup, soft interior-hyperideals and soft bi-hyperideals coincide.

## 2. Preliminaries

Throughout this paper we represent:

$\mathbf{H}$ : LA-semihypergroup,

$\mathcal{U}$ : an initial universe,

$E$ : a set of parameters,
$\mathbf{H}(\mathcal{U})$ : set of all soft sets of $\mathbf{H}$ over $\mathcal{U}$,
$P(\mathcal{U})$ : the powerset of $\mathcal{U}$.

**Definition 2.1** ([9, 10])**.** Let $\mathbf{H}$ be a non-empty set and let $\wp^*(\mathbf{H})$ be the set of all non-empty subsets of $\mathbf{H}$. A hyperoperation on $\mathbf{H}$ is a map $\underline{\circ} : \mathbf{H} \times \mathbf{H} \to \wp^*(\mathbf{H})$ and $(\mathbf{H}, \underline{\circ})$ is called a hypergroupoid.

**Definition 2.2** ([9, 10])**.** A hypergroupoid $(\mathbf{H}, \underline{\circ})$ is called a semihypergroup if for all $x$, $y$, $z$ of $\mathbf{H}$ we have $(x \underline{\circ} y) \underline{\circ} z = x \underline{\circ} (y \underline{\circ} z)$, which means that

$$\bigcup_{u \in x \,\underline{\circ}\, y} u \underline{\circ} z = \bigcup_{v \in y \,\underline{\circ}\, z} x \underline{\circ} v.$$

If $x \in \mathbf{H}$ and $A$, $B$ are non-empty subsets of $\mathbf{H}$, then we denote

$$A \underline{\circ} B = \bigcup_{a \in A, b \in B} a \underline{\circ} b,\ x \ o\ A = \{x\} \underline{\circ} A \text{ and } A \underline{\circ} x = A \underline{\circ} \{x\}.$$

**Definition 2.3** ([16])**.** Let $\mathbf{H}$ be non-empty set. A hypergroupoid $\mathbf{H}$ is called an LA-semihypergroup if for every $x$, $y$, $z \in \mathbf{H}$, we have

$$(x \underline{\circ} y) \underline{\circ} z = (z \circ y) \underline{\circ} x.$$

The law is called left invertive law. Every LA-semihypergroup satisfies the following law:

$$(x \underline{\circ} y) \underline{\circ} (z \underline{\circ} w) = (x \underline{\circ} z) \underline{\circ} (y \underline{\circ} w),$$

for all $w, x, y, z \in H$. This law is known as medial law.

**Definition 2.4** ([30])**.** Let $\mathbf{H}$ be an LA-semihypergroup, then an element $e \in \mathbf{H}$ is called left identity (resp., pure left identity) if for all $a \in \mathbf{H}$, $a \in e \underline{\circ} a$ (resp., $a = e \underline{\circ} a$).

An LA- semihypergroup $(\mathbf{H}, \underline{\circ})$ with pure left identity satisfy the following law for all $w, x, y, z \in \mathbf{H}$:

$$(x \underline{\circ} y) \underline{\circ} (z \underline{\circ} w) = (w \underline{\circ} z) \underline{\circ} (y \underline{\circ} x),$$

called a paramedial law, and

$$x \underline{\circ} (y \underline{\circ} z) = y \underline{\circ} (x \underline{\circ} z).$$

**Definition 2.5** ([16])**.** A non-empty subset $T$ of an LA-semihypergroup $\mathbf{H}$ is called sub-LA-semihypergroup of $\mathbf{H}$ if $t_1 \underline{\circ} t_2 \subseteq T$ for every $t_1, t_2 \in T$.

**Definition 2.6** ([30])**.** A sub-LA-semihypergroup $I$ is said to be an interior-hyperideal of $\mathbf{H}$ if $(\mathbf{H} \underline{\circ} I) \underline{\circ} \mathbf{H} \subseteq I$.

**Definition 2.7** ([30])**.** Let $\mathbf{H}$ be an LA-semihypergroup, then a non-empty subset $A$ of $\mathbf{H}$ is called semiprime if for any $a \in \mathbf{H}$ such that $a \underline{\circ} a \subseteq A$ implies $a \in A$.

## 3. Soft Set

**Definition 3.1** ([8, 24])**.** A soft set $\mathcal{F}_A$ over $\mathcal{U}$ is a set defined by $\mathcal{F}_A : E \to P(\mathcal{U})$ such that $\mathcal{F}_A(x) = \emptyset$ if $x \notin A$.

Here $\mathcal{F}_A$ is also called an approximate function. A soft set over $\mathcal{U}$ can be represented by the set of ordered pairs

$$\mathcal{F}_A = \{(x, \mathcal{F}_A(x)) : x \in E, \mathcal{F}_A(x) \in P(\mathcal{U})\}.$$

It is clear that a soft set is a parameterized family of subsets of the set $\mathcal{U}$.

**Definition 3.2** ([8])**.** Let $\mathcal{F}_A, \mathcal{F}_B \in \mathbf{H}(\mathcal{U})$. Then, $\mathcal{F}_A$ is called a soft subset of $\mathcal{F}_B$ and denoted by $\mathcal{F}_A \sqsubseteq \mathcal{F}_B$, if $\mathcal{F}_A(x) \subseteq \mathcal{F}_B(x)$ for all $x \in E$.

**Definition 3.3** ([8])**.** Let $\mathcal{F}_A, \mathcal{F}_B \in \mathbf{H}(\mathcal{U})$. Then, union of $\mathcal{F}_A$ and $\mathcal{F}_B$ denoted by $\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_B$, is defined as $\mathcal{F}_A \widetilde{\bigcup} \mathcal{F}_B = \mathcal{F}_{A \widetilde{\cup} B}$, where $\mathcal{F}_{A \widetilde{\cup} B}(x) = \mathcal{F}_A(x) \bigcup \mathcal{F}_B(x)$ for all $x \in E$.

**Definition 3.4** ([8])**.** Let $\mathcal{F}_A, \mathcal{F}_B \in \mathbf{H}(\mathcal{U})$. Then, intersection of $\mathcal{F}_A$ and $\mathcal{F}_B$ denoted by $\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_B$, is defined as $\mathcal{F}_A \widetilde{\bigcap} \mathcal{F}_B = \mathcal{F}_{A \widetilde{\cap} B}$, where $\mathcal{F}_{A \widetilde{\cap} B}(x) = \mathcal{F}_A(x) \bigcap \mathcal{F}_B(x)$ for all $x \in E$.

**Definition 3.5** ([21])**.** Let $Y$ be a subset of $\mathbf{H}$. We denote the soft characteristic function of $Y$ by $\mathcal{H}_Y$ and is defined as:

$$\mathcal{H}_Y(y) = \begin{cases} \mathcal{U}, & \text{if } y \in Y, \\ \emptyset, & \text{if } y \notin Y. \end{cases}$$

In this paper, we denote an LA-semihypergroup $\mathbf{H}$ as a set of parameters.

Let $\mathbf{H}$ be an LA-semihypergroup. For $x \in \mathbf{H}$, we define $\mathbb{H}_x = \{(y, z) \in \mathbf{H} \times \mathbf{H} : x \in y \,\underline{o}\, z\}$.

**Definition 3.6** ([21])**.** Let $\mathcal{F}_{\mathbf{H}}$ and $\mathcal{G}_{\mathbf{H}}$ be two soft sets of an LA-semihypergroup $\mathbf{H}$ over $\mathcal{U}$. Then, the soft product $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}$ is a soft set of $\mathbf{H}$ over $\mathcal{U}$, defined by

$$(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}})(x) = \begin{cases} \bigcup\limits_{(y,z) \in \mathbb{H}_x} \{\mathcal{F}_{\mathbf{H}}(y) \cap \mathcal{G}_{\mathbf{H}}(z)\}, & \text{if } \mathbb{H}_x \neq \emptyset, \\ \emptyset, & \text{if } \mathbb{H}_x = \emptyset, \end{cases}$$

for all $x \in \mathbf{H}$.

**Theorem 3.1** ([21])**.** *Let $X$ and $Y$ be non-empty subsets of an LA-semihypergroup $\mathbf{H}$. Then*

(1) *If $X \subseteq Y$, then $\mathcal{H}_X \sqsubseteq \mathcal{H}_Y$;*
(2) $\mathcal{H}_X \widetilde{\bigcap} \mathcal{H}_Y = \mathcal{H}_{X \cap Y}$, $\mathcal{H}_X \widetilde{\bigcup} \mathcal{H}_Y = \mathcal{H}_{X \cup Y}$;
(3) $\mathcal{H}_X \hat{\diamond} \mathcal{H}_Y = \mathcal{H}_{X \,\underline{o}\, Y}$.

**Definition 3.7** ([21])**.** A non-null soft set $\mathcal{F}_{\mathbf{H}}$ is said to be an S.I. sub-LA-semihypergroup of $\mathbf{H}$ over $\mathcal{U}$ if

$$\bigcap_{\vartheta \in x \,\underline{o}\, y} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(x) \cap \mathcal{F}_{\mathbf{H}}(y), \quad \text{for all } x, y \in \mathbf{H}.$$

**Definition 3.8.** An S.I. sub-LA-semihypergroup $\mathcal{F}_{\mathbf{H}}$ is said to be an S.I. bi-hyperideal of $\mathbf{H}$ over $\mathcal{U}$ if

$$\bigcap_{\vartheta \in (x \underset{\sim}{\circ} y) \underset{\sim}{\circ} z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(x) \cap \mathcal{F}_{\mathbf{H}}(z), \quad \text{for all } x, y, z \in \mathbf{H}.$$

**Theorem 3.2** ([21])**.** *A non-null soft set $\mathcal{F}_{\mathbf{H}}$ is an S.I. sub-LA-semihypergroup of $\mathbf{H}$ over $\mathcal{U}$ if and only if*

$$\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}.$$

**Corollary 3.1** ([21])**.** *In an LA-semihypergroup $\mathbf{H}$ with left identity, $\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}} = \mathcal{H}_{\mathbf{H}}$.*

**Theorem 3.3** ([21])**.** *Let $\mathbf{H}$ be an LA-semihypergroup and $\mathbf{H}(\mathcal{U})$ be the set of all soft sets of $\mathbf{H}$ over $\mathcal{U}$. Then $(\mathbf{H}(\mathcal{U}), \hat{\diamond})$ is an LA-semigroup.*

**Theorem 3.4** ([21])**.** *If $\mathbf{H}$ is an LA-semihypergroup. Then medial law holds in $\mathbf{H}(\mathcal{U})$.*

**Theorem 3.5** ([21])**.** *Let $\mathbf{H}$ be an LA-semihypergroup with left identity and $\mathcal{F}_{\mathbf{H}}, \mathcal{G}_{\mathbf{H}}, \mathcal{K}_{\mathbf{H}}, \mathcal{L}_{\mathbf{H}} \in \mathbf{H}(\mathcal{U})$. Then following holds:*

(i) $\mathcal{F}_{\mathbf{H}} \hat{\diamond} (\mathcal{G}_{\mathbf{H}} \hat{\diamond} \mathcal{K}_{\mathbf{H}}) = \mathcal{G}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{K}_{\mathbf{H}})$;

(ii) $(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}) \hat{\diamond} (\mathcal{K}_{\mathbf{H}} \hat{\diamond} \mathcal{L}_{\mathbf{H}}) = (\mathcal{L}_{\mathbf{H}} \hat{\diamond} \mathcal{K}_{\mathbf{H}}) \hat{\diamond} (\mathcal{G}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})$.

## 4. Soft Interior-Hyperideals in LA-Semihypergroups

In this section, we define soft interior-hyperideals in LA-semihypergroups and establish some of their elementary properties.

**Definition 4.1.** An S.I. sub-LA-semihypergroup $\mathcal{F}_{\mathbf{H}}$ is said to be an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$ if

$$\bigcap_{\vartheta \in (x \underset{\sim}{\circ} y) \underset{\sim}{\circ} z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y), \quad \text{for all } x, y, z \in \mathbf{H}.$$

*Example* 4.1. An insurance company offers on some insurances to its agents defined in a set $\mathbf{H} = \{$Health Insurance (Hlth. Ins.), Home Insurance (Hme. Ins.), Property Insurance (Prop. Ins.), Vehicle Insurance (V.I.), Computer Insurance (C.I.)$\}$ with the composition Table 1.

Table 1.

| $\underset{\sim}{\circ}$ | Health Ins. | Home Ins. | Prop. Ins. | V. I. | C. I. |
|---|---|---|---|---|---|
| Health Ins. | Health Ins. | Home Ins. | Prop. Ins. | V. I. | C. I. |
| Home Ins. | Prop. Ins. | V. I. | V. I. | {V. I., C. I.} | C.I. |
| Prop. Ins. | Health Ins. | V. I. | V. I. | {V. I., C. I.} | C.I. |
| V. I. | V. I. | {V. I., C. I.} | {V. I., C. I.} | {V. I., C. I.} | C. I. |
| C. I. | C. I. | C. I. | C. I. | C. I. | C. I. |

Let $A =$ 'Husband' and $B =$ 'Wife'. Then the hyperoperation defined in the above composition table as: $(x \circ y) =$ if the agent does $x$ insurance of $A$ and $y$ insurance of $B$, then he will get $X$ insurances free of cost, where $x, y \in \mathbf{H}$ and $X \subseteq \mathbf{H}$. Therefore, $(\mathbf{H}, \circ)$ will be an LA-semihypergroup.

Now, let $\mathcal{U} = \{A_1, A_2, A_3, A_4, A_5\}$ be the set of agents who does insurances to husbands and their wives. Define a soft set $\mathcal{F}_{\mathbf{H}} : \mathbf{H} \to P(\mathcal{U})$ by

$\mathcal{F}_{\mathbf{H}}$(Health Ins.) $= \{A_1, A_2\}$, means the agents who got a health insurance free,
$\mathcal{F}_{\mathbf{H}}$(Home Ins.) $= \{A_1, A_2\}$, means the agents who got a home insurance free,
$\mathcal{F}_{\mathbf{H}}$(Prop. Ins.) $= \{A_1, A_2, A_3\}$, means the agents who got a property insurance free,
$\mathcal{F}_{\mathbf{H}}$(V. I.) $= \{A_1, A_2, A_3, A_4\}$, means the agents who got a vehicle insurance free and
$\mathcal{F}_{\mathbf{H}}$(C. I.) $= \{A_1, A_2, A_3, A_4, A_5\}$, means the agents who got a computer insurance free.

Then, we can verify that $\bigcap_{\vartheta \in (x \circ y) \circ z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y)$ for all $x, y, z \in \mathbf{H}$. Therefore, $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$.

**Theorem 4.1.** *If $\mathcal{F}_{\mathbf{H}}$ and $\mathcal{G}_{\mathbf{H}}$ are two S.I. interior-hyperideals of $\mathbf{H}$ over $\mathcal{U}$. Then $\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}$ is also an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$.*

*Proof.* Assume that $\mathcal{F}_{\mathbf{H}}$ and $\mathcal{G}_{\mathbf{H}}$ are two S.I. interior-hyperideals of $\mathbf{H}$ over $\mathcal{U}$. Then, we have

$$\begin{aligned}(\mathcal{F}_{\mathbf{H}} \widetilde{\bigcap} \mathcal{G}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \widetilde{\bigcap} \mathcal{G}_{\mathbf{H}}) &\sqsubseteq \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \\ &\sqsubseteq \mathcal{F}_{\mathbf{H}}.\end{aligned}$$

In a similar way, $(\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}) \sqsubseteq \mathcal{G}_{\mathbf{H}}$. It implies $(\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}) \sqsubseteq (\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}})$. Also, we have

$$\begin{aligned}(\mathcal{H}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \widetilde{\bigcap} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} \mathcal{H}_{\mathbf{H}} &\sqsubseteq (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \\ &\sqsubseteq \mathcal{F}_{\mathbf{H}}.\end{aligned}$$

In a similar way, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{G}_{\mathbf{H}}$. Therefore, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}$. Hence, $\mathcal{F}_{\mathbf{H}} \tilde{\cap} \mathcal{G}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. □

**Theorem 4.2.** *Let $X$ be any non-empty subset of an LA-semihypergroup $\mathbf{H}$. Then $X$ is an interior-hyperideal of $\mathbf{H}$ if and only if $\mathcal{H}_X$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$.*

*Proof.* Proof is easy, hence omitted. □

**Theorem 4.3.** *An S.I. sub-LA-semihypergroup $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$ if and only if*

$$(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}.$$

*Proof.* Assume that $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, then

$$\bigcap_{\vartheta \in (x \circ y) \circ z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y), \quad \text{for all } x, y, z \in \mathbf{H}.$$

Now, if $\mathbb{H}_x = \emptyset$, then $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})(x) = \emptyset$. Thus, it would yield $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})(x) \subseteq \mathcal{F}_{\mathbf{H}}(x)$. Therefore, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}$.

If $\mathbb{H}_x \neq \emptyset$, then there exists $u, v, p, q \in \mathbf{H}$ such that $x \in u \circ v$ and $u \in p \circ q$. So, $(u, v) \in \mathbb{H}_x$ and $(p, q) \in \mathbb{H}_u$. Thus, we have

$$
\begin{aligned}
((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}})(x) &= \bigcup_{(u,v)\in\mathbb{H}_x} \Big[\Big(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}\Big)(u) \bigcap \mathcal{H}_{\mathbf{H}}(v)\Big] \\
&= \bigcup_{x\in u \circ v} \Big[ \bigcup_{(p,q)\in\mathbb{H}_u} \Big(\mathcal{H}_{\mathbf{H}}(p) \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big) \bigcap \mathcal{H}_{\mathbf{H}}(v)\Big] \\
&= \bigcup_{x\in u \circ v} \Big[ \bigcup_{u\in p \circ q} \Big(\mathcal{H}_{\mathbf{H}}(p) \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big) \bigcap \mathcal{U}\Big] \\
&= \bigcup_{x\in u \circ v} \Big[ \bigcup_{u\in p \circ q} \Big(\mathcal{H}_{\mathbf{H}}(p) \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big)\Big] \\
&= \bigcup_{x\in u \circ v} \Big[ \bigcup_{u\in p \circ q} \Big(\mathcal{U} \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big)\Big] \\
&= \bigcup_{x\in u \circ v} \Big[ \bigcup_{u\in p \circ q} \Big(\mathcal{F}_{S}(q)\Big)\Big] \\
&= \bigcup_{x\in (p \circ q) \circ v} \Big(\mathcal{F}_{\mathbf{H}}(q)\Big) \\
&\subseteq \bigcup_{x\in (p \circ q) \circ v} \Big\{ \bigcap_{\vartheta\in (r \circ q) \circ t} \mathcal{F}_{\mathbf{H}}(\vartheta)\Big\}, \\
&\quad \text{(as } \mathcal{F}_{\mathbf{H}} \text{ is an S.I. interior hyperideal)} \\
&\subseteq \bigcup_{x\in (p \circ q) \circ v} \Big\{ \bigcap_{x\in (r \circ q) \circ t} \mathcal{F}_{\mathbf{H}}(x)\Big\} \\
&\subseteq \bigcup_{x\in (p \circ q) \circ v} \Big\{ \bigcap_{x\in (p \circ q) \circ v} \mathcal{F}_{\mathbf{H}}(x)\Big\} \\
&= \mathcal{F}_{\mathbf{H}}(x).
\end{aligned}
$$

Hence, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}$.

Conversely, suppose that $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}$. Now to show $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, we have

$$
\begin{aligned}
\bigcap_{\vartheta\in (x \circ y) \circ z} \mathcal{F}_{\mathbf{H}}(\vartheta) &\supseteq \bigcap_{\vartheta\in (x \circ y) \circ z} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})(\vartheta) \\
&= \bigcap_{\vartheta\in (x \circ y) \circ z} ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}})(\vartheta) \\
&= \bigcap_{\vartheta\in (x \circ y) \circ z} \Big\{ \bigcup_{(u,v)\in\mathbb{H}_\vartheta} \Big[\Big(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}\Big)(u) \bigcap \mathcal{H}_{\mathbf{H}}(v)\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&= \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, v} \left[ \bigcup_{(p,q)\in\mathbb{H}_u} \Big(\mathcal{H}_{\mathbf{H}}(p) \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big) \bigcap \mathcal{H}_{\mathbf{H}}(v) \right] \right\} \\
&= \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, v} \left[ \bigcup_{u\in p \,\underline{\text{O}}\, q} \Big(\mathcal{H}_{\mathbf{H}}(p) \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big) \bigcap \mathcal{U} \right] \right\} \\
&= \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, v} \left[ \bigcup_{u\in p \,\underline{\text{O}}\, q} \Big(\mathcal{H}_{\mathbf{H}}(p) \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big) \right] \right\} \\
&= \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, v} \left[ \bigcup_{u\in p \,\underline{\text{O}}\, q} \Big(\mathcal{U} \bigcap \mathcal{F}_{\mathbf{H}}(q)\Big) \right] \right\} \\
&= \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, v} \left[ \bigcup_{u\in p \,\underline{\text{O}}\, q} \Big(\mathcal{F}_{\mathbf{H}}(q)\Big) \right] \right\} \\
&\supseteq \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, v} \left[ \bigcup_{u\in x \,\underline{\text{O}}\, y} \Big(\mathcal{F}_{\mathbf{H}}(y)\Big) \right] \right\} \\
&\supseteq \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in u \,\underline{\text{O}}\, z} \left[ \bigcup_{u\in x \,\underline{\text{O}}\, y} \Big(\mathcal{F}_{\mathbf{H}}(y)\Big) \right] \right\} \\
&= \bigcap_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} \left\{ \bigcup_{\vartheta\in(x \,\underline{\text{O}}\, y)\,\underline{\text{O}}\, z} (\mathcal{F}_{\mathbf{H}}(y)) \right\} \\
&= \mathcal{F}_{\mathbf{H}}(y).
\end{aligned}
$$

It follows that $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. □

**Theorem 4.4.** *If $\mathcal{F}_{\mathbf{H}}$ and $\mathcal{G}_{\mathbf{H}}$ are S.I. interior-hyperideals of $\mathbf{H}$ over $\mathcal{U}$ with left identity. Then the S.I. product $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$.*

*Proof.* Let $\mathcal{F}_{\mathbf{H}}$ and $\mathcal{G}_{\mathbf{H}}$ be S.I. interior-hyperideals of $\mathbf{H}$ over $\mathcal{U}$ with left identity. Then, we have

$$(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}) = (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{G}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}) \sqsubseteq \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}.$$

It implies $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}$ is an S.I. sub-LA-semihypergroup of $\mathbf{H}$ over $\mathcal{U}$. Also, we have

$$
\begin{aligned}
(\mathcal{H}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} \mathcal{H}_{\mathbf{H}} &= ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \\
&= ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \\
&= ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}})) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \\
&= ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \\
&\sqsubseteq \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}.
\end{aligned}
$$

This shows that $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{G}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. □

**Definition 4.2.** A soft set $\mathcal{F}_{\mathbf{H}}$ of an LA-semihypergroup $\mathbf{H}$ over $\mathcal{U}$ is said to be idempotent if $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$.

*Example* 4.2. Consider an LA-semihypergroup given in the Example 4.1. Now, let $\mathcal{U} = \{A_1, A_2, A_3, A_4, A_5\}$ be the set of agents who does insurances to husbands and their wives. Define a soft set $\mathcal{F}_{\mathbf{H}} : \mathbf{H} \to P(\mathcal{U})$ by

$\mathcal{F}_{\mathbf{H}}(\text{Health Ins.}) = \emptyset$, means the agents who got a health insurance free,
$\mathcal{F}_{\mathbf{H}}(\text{Home Ins.}) = \emptyset$, means the agents who got a home insurance free,
$\mathcal{F}_{\mathbf{H}}(\text{Prop. Ins.}) = \emptyset$, means the agents who got a property insurance free,
$\mathcal{F}_{\mathbf{H}}(\text{V.I.}) = \{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3\}$, means the agents who got a vehicle insurance free and
$\mathcal{F}_{\mathbf{H}}(\text{C.I.}) = \{\mathrm{A}_1, \mathrm{A}_2, \mathrm{A}_3\}$, means the agents who got a computer insurance free.

Then, we can easily verify that $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$. Hence, $\mathcal{F}_{\mathbf{H}}$ is idempotent.

**Proposition 4.1.** *Every idempotent S.I. bi-hyperideal of* $\mathbf{H}$ *over* $\mathcal{U}$ *with left identity is an S.I. interior-hyperideal of* $\mathbf{H}$ *over* $\mathcal{U}$.

*Proof.* Assume that $\mathcal{F}_{\mathbf{H}}$ is an idempotent S.I. bi-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, then $(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}$. Thus, we have

$$\begin{aligned}
(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} =&(\mathcal{H}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})\\
=&(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} ((\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}})\\
=&\mathcal{H}_{\mathbf{H}} \hat{\diamond} ((\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}})\\
=&(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})\\
=&((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}}\\
=&((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})) \hat{\diamond} \mathcal{F}_{\mathbf{H}}\\
=&((\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})) \hat{\diamond} \mathcal{F}_{\mathbf{H}}\\
=&(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}}\\
\sqsubseteq&\mathcal{F}_{\mathbf{H}}.
\end{aligned}$$

Hence, $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. □

## 5. Characterization of Left Regular LA-Semihypergroups

In this section, we characterize left regular LA-semihypergroup using soft interior-hyperideals.

**Definition 5.1.** An element $l_r$ of an LA-semihypergroup $\mathbf{H}$ is called a left regular element if there exists an element $x \in \mathbf{H}$ such that $l_r \in x \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)$. If every element of $\mathbf{H}$ is left regular, then $\mathbf{H}$ is called a left regular LA-semihypergroup.

**Lemma 5.1.** *Let* $\mathbf{H}$ *be a left regular LA-semihypergroup with left identity. Then for any S.I. interior-hyperideal* $\mathcal{F}_{\mathbf{H}}$ *of* $\mathbf{H}$ *over* $\mathcal{U}$*, we have* $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$.

*Proof.* Assume that $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, then by the Theorem 4.3, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}}$. Now, it is only remains to prove that $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} \sqsupseteq$

$\mathcal{F}_{\mathbf{H}}$. By assumption, $\mathbf{H}$ is left regular, thus for any $l_r \in \mathbf{H}$, there exists $x \in \mathbf{H}$ such that $l_r \in x \,\underline{o}\, (l_r \,\underline{o}\, l_r)$. Let $e \in \mathbf{H}$ be the left identity, then we have

$$\begin{aligned}
l_r \in & x \,\underline{o}\, (l_r \,\underline{o}\, l_r) \\
\subseteq & (e \,\underline{o}\, x) \,\underline{o}\, (l_r \,\underline{o}\, l_r) \\
= & (l_r \,\underline{o}\, l_r) \,\underline{o}\, (x \,\underline{o}\, e) \\
= & ((x \,\underline{o}\, e) \,\underline{o}\, l_r) \,\underline{o}\, l_r \\
\subseteq & ((x \,\underline{o}\, e) \,\underline{o}\, (x \,\underline{o}\, (l_r \,\underline{o}\, l_r))) \,\underline{o}\, l_r \\
= & (x \,\underline{o}\, ((x \,\underline{o}\, e) \,\underline{o}\, (l_r \,\underline{o}\, l_r))) \,\underline{o}\, l_r \\
= & (x \,\underline{o}\, ((l_r \,\underline{o}\, l_r) \,\underline{o}\, (e \,\underline{o}\, x))) \,\underline{o}\, l_r.
\end{aligned}$$

It implies there exists $a \in e \,\underline{o}\, x$ such that $l_r \in (x \,\underline{o}\, ((l_r \,\underline{o}\, l_r) \,\underline{o}\, a)) \,\underline{o}\, l_r$, there exists $b \in ((l_r \,\underline{o}\, l_r) \,\underline{o}\, a)$ such that $l_r \in (x \,\underline{o}\, b) \,\underline{o}\, l_r$ and there exists $c \in x \,\underline{o}\, b$ such that $l_r \in c \,\underline{o}\, l_r$. So, $(c, l_r) \in \mathbb{H}_{l_r}$ and $(x, b) \in \mathbb{H}_c$. Thus, we have

$$\begin{aligned}
((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}})(l_r) = & \bigcup_{(u,v) \in \mathbb{H}_{l_r}} \left[ \Big( \mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \Big)(u) \cap \mathcal{H}_{\mathbf{H}}(v) \right] \\
\supseteq & \Big( \mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \Big)(c) \cap \mathcal{H}_{\mathbf{H}}(l_r) \\
= & \Big( \mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \Big)(c) \cap \mathcal{U} \\
= & \bigcup_{(p,q) \in \mathbb{H}_c} \Big( \mathcal{H}_{\mathbf{H}}(p) \cap \mathcal{F}_{\mathbf{H}}(q) \Big) \\
\supseteq & \mathcal{H}_{\mathbf{H}}(x) \cap \mathcal{F}_{\mathbf{H}}(b) \\
= & \mathcal{F}_{\mathbf{H}}(b).
\end{aligned} \tag{5.1}$$

As $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, we have $\bigcap_{\vartheta \in (x \,\underline{o}\, y) \,\underline{o}\, z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y)$ for all $x, y, z \in \mathbf{H}$. Since $b \in (l_r \,\underline{o}\, l_r) \,\underline{o}\, a$, it would imply that $\mathcal{F}_{\mathbf{H}}(b) \supseteq \mathcal{F}_{\mathbf{H}}(l_r)$. Therefore, from equation (5.1), we have

$$\begin{aligned}
((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}})(l_r) \supseteq & \mathcal{F}_{\mathbf{H}}(b) \\
= & \mathcal{F}_{\mathbf{H}}(l_r).
\end{aligned}$$

Hence, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$. □

**Lemma 5.2.** *If $\mathbf{H}$ is a left regular LA-semihypergroup with left identity. Then for every S.I. interior-hyperideal $\mathcal{F}_{\mathbf{H}}$ of $\mathbf{H}$ over $\mathcal{U}$, we have*

$$\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}.$$

*Proof.* Let $\mathcal{F}_{\mathbf{H}}$ be an S.I. interior-hyperideal of a left regular LA-semihypergroup $\mathbf{H}$ over $\mathcal{U}$ with left identity. By Lemma 5.1, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$. Thus, we have

$$\begin{aligned}
\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = & (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}} = (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \\
= & (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) = ((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}.
\end{aligned}$$

Also, we have

$$\begin{aligned}\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}} =& \mathcal{F}_{\mathbf{H}} \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) = \mathcal{H}_{\mathbf{H}} \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) = (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})\\ =&(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) = (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}})\\ =&((\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}\end{aligned}$$

and

$$\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{F}_{\mathbf{H}} = (\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} \mathcal{H}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}.$$

Hence, $\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{H}_{\mathbf{H}}$. □

**Definition 5.2.** A soft set $\mathcal{F}_{\mathbf{H}}$ is said to be soft semiprime if for all $l_r \in \mathbf{H}$, $\mathcal{F}_{\mathbf{H}}(l_r) \supseteq \bigcap_{\vartheta \in l_r \,\underline{o}\, l_r} \mathcal{F}_{\mathbf{H}}(\vartheta)$.

*Example* 5.1. Consider an LA-semihypergroup given in the Example 4.1. Now, let $\mathcal{U} = \{A_1, A_2, A_3, A_4, A_5\}$ be the set of agents who does insurances to husbands and their wives. Define a soft set $\mathcal{F}_{\mathbf{H}} : \mathbf{H} \to P(\mathcal{U})$ by

$\mathcal{F}_{\mathbf{H}}$(Health Ins.) $= \{A_1, A_2, A_3\}$, means the agents who got a health insurance free,
$\mathcal{F}_{\mathbf{H}}$(Home Ins.) $= \{A_1, A_2, A_3, A_4\}$, means the agents who got a home insurance free,
$\mathcal{F}_{\mathbf{H}}$(Prop. Ins.) $= \{A_1, A_2, A_3\}$, means the agents who got a property insurance free,
$\mathcal{F}_{\mathbf{H}}$(V.I.) $= \{A_1, A_2\}$, means the agents who got a vehicle insurance free and
$\mathcal{F}_{\mathbf{H}}$(C.I.) $= \{A_1, A_2, A_3, A_4, A_5\}$, means the agents who got a computer insurance free.

Then, we can easily verify that for all $l_r \in \mathbf{H}$, $\mathcal{F}_{\mathbf{H}}(l_r) \supseteq \bigcap_{\vartheta \in l_r \,\underline{o}\, l_r} \mathcal{F}_{\mathbf{H}}(\vartheta)$. Hence, $\mathcal{F}_{\mathbf{H}}$ is soft semiprime.

**Lemma 5.3.** *Let* $\mathbf{H}$ *be an LA-semihypergroup. Then* $A$ *is semiprime if and only if* $\mathcal{H}_A$ *is soft semiprime.*

*Proof.* Proof is easy, hence omitted. □

**Lemma 5.4.** *Let* $\mathbf{H}$ *be an LA-semihypergroup with left identity. Then for any* $l_r \in \mathbf{H}$, $(l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}$ *is an interior-hyperideal of* $\mathbf{H}$.

*Proof.* Firstly, we will show that $(l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}$ is a sub-LA-semihypergroup of $\mathbf{H}$, for some $l_r \in \mathbf{H}$. So, we have

$$\begin{aligned}((l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}) \,\underline{o}\, ((l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}) =& (((l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}) \,\underline{o}\, \mathbf{H}) \,\underline{o}\, (l_r \,\underline{o}\, l_r)\\ =&((\mathbf{H} \,\underline{o}\, \mathbf{H}) \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \,\underline{o}\, (l_r \,\underline{o}\, l_r)\\ \subseteq&(\mathbf{H} \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \,\underline{o}\, (l_r \,\underline{o}\, l_r)\\ \subseteq&(\mathbf{H} \,\underline{o}\, (\mathbf{H} \,\underline{o}\, \mathbf{H})) \,\underline{o}\, (l_r \,\underline{o}\, l_r)\\ \subseteq&(\mathbf{H} \,\underline{o}\, \mathbf{H}) \,\underline{o}\, (l_r \,\underline{o}\, l_r)\\ \subseteq&(l_r \,\underline{o}\, l_r) \,\underline{o}\, (\mathbf{H} \,\underline{o}\, \mathbf{H})\\ \subseteq&(l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}.\end{aligned}$$

Also,

$$(\mathbf{H} \,\underline{o}\, ((l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H})) \,\underline{o}\, \mathbf{H} = (\mathbf{H} \,\underline{o}\, ((\mathbf{H} \,\underline{o}\, l_r) \,\underline{o}\, l_r)) \,\underline{o}\, \mathbf{H}$$

$$
\begin{aligned}
&=((\mathbf{H} \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} l_r)) \underset{\sim}{\circ} \mathbf{H}\\
&=((\mathbf{H} \underset{\sim}{\circ} \mathbf{H}) \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)) \underset{\sim}{\circ} \mathbf{H}\\
&=((l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} \mathbf{H})) \underset{\sim}{\circ} \mathbf{H}\\
&\subseteq((l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} \mathbf{H}) \underset{\sim}{\circ} \mathbf{H}\\
&=(\mathbf{H} \underset{\sim}{\circ} \mathbf{H}) \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)\\
&=(l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} \mathbf{H})\\
&\subseteq(l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} \mathbf{H}.
\end{aligned}
$$

Hence, $(l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} \mathbf{H}$ is an interior-hyperideal of $\mathbf{H}$. □

**Theorem 5.1.** *Let $\mathbf{H}$ be an LA-semihypergroup with left identity, then the following statements are equivalent.*

(1) *$\mathbf{H}$ is left regular.*
(2) *$\mathcal{M} \subseteq \mathcal{M}^2$ and $\mathcal{M}$ is semiprime, where $\mathcal{M}$ is an interior-hyperideal of $\mathbf{H}$.*
(3) *$\mathcal{F}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}$ and $\mathcal{F}_{\mathbf{H}}$ is soft semiprime, where $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior hypeerideal of $\mathbf{H}$ over $\mathcal{U}$.*

*Proof.* (1)⇒(3) Let $\mathbf{H}$ be a left regular LA-semihypergroup with left identity, thus for any $l_r \in \mathbf{H}$, there exists $x \in \mathbf{H}$ such that $l_r \in x \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)$. Now we have

$$
\begin{aligned}
l_r \in& x \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)\\
=& l_r \underset{\sim}{\circ} (x \underset{\sim}{\circ} l_r)\\
\subseteq& l_r \underset{\sim}{\circ} (x \underset{\sim}{\circ} (x \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)))\\
\subseteq& l_r \underset{\sim}{\circ} ((e \underset{\sim}{\circ} x) \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} (x \underset{\sim}{\circ} l_r)))\\
=& l_r \underset{\sim}{\circ} (((x \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (x \underset{\sim}{\circ} e)).
\end{aligned}
$$

Then, there exists $b \in x \underset{\sim}{\circ} l_r$ and $c \in x \underset{\sim}{\circ} e$ such that $l_r \in l_r \underset{\sim}{\circ}((b \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} c)$. Again, there exists $d \in ((b \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} c)$ such that $l_r \in l_r \underset{\sim}{\circ} d$. So, $(l_r, d) \in \mathbb{H}_{l_r}$. Thus, we have

$$
\begin{aligned}
(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})(l_r) =& \bigcup_{(y,z)\in\mathbb{H}_{l_r}} \left\{\mathcal{F}_{\mathbf{H}}(y) \ \cap \ \mathcal{F}_{\mathbf{H}}(z)\right\}\\
\supseteq& \mathcal{F}_{\mathbf{H}}(l_r) \ \cap \ \mathcal{F}_{\mathbf{H}}(d). \qquad (5.2)
\end{aligned}
$$

As $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, we have $\bigcap_{\vartheta\in(x \underset{\sim}{\circ} y) \underset{\sim}{\circ} z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y)$ for all $x, y, z \in \mathbf{H}$. Since $d \in ((b \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} c)$ it would imply that $\mathcal{F}_{\mathbf{H}}(d) \supseteq \mathcal{F}_{\mathbf{H}}(l_r)$. Therefore from equation (5.2), we have

$$
\begin{aligned}
(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})(l_r) \supseteq& \mathcal{F}_{\mathbf{H}}(l_r) \ \cap \ \mathcal{F}_{\mathbf{H}}(d)\\
\supseteq& \mathcal{F}_{\mathbf{H}}(l_r) \ \cap \ \mathcal{F}_{\mathbf{H}}(l_r)\\
=& \mathcal{F}_{\mathbf{H}}(l_r).
\end{aligned}
$$

Hence, $\mathcal{F}_{\mathbf{H}} \sqsubseteq \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}$.

Now, it remains to show that $\mathcal{F}_{\mathbf{H}}$ is soft semiprime. Let $l_r \in \mathbf{H}$, then there exists $x \in \mathbf{H}$ such that $l_r \in x \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)$. Therefore, we have

$$\begin{aligned} l_r \in & x \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r) \\ \subseteq & (e \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r) \\ = & (l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, (x \,\underline{\circ}\, e) \\ \subseteq & (l_r \,\underline{\circ}\, (x \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r))) \,\underline{\circ}\, (x \,\underline{\circ}\, e) \\ = & (x \,\underline{\circ}\, (l_r \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r))) \,\underline{\circ}\, (x \,\underline{\circ}\, e) \\ \subseteq & (x \,\underline{\circ}\, ((e \,\underline{\circ}\, l_r) \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r))) \,\underline{\circ}\, (x \,\underline{\circ}\, e) \\ = & (x \,\underline{\circ}\, ((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, (l_r \,\underline{\circ}\, e))) \,\underline{\circ}\, (x \,\underline{\circ}\, e) \\ = & ((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, (x \,\underline{\circ}\, (l_r \,\underline{\circ}\, e))) \,\underline{\circ}\, (x \,\underline{\circ}\, e) \\ = & (((l_r \,\underline{\circ}\, e) \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)) \,\underline{\circ}\, (x \,\underline{\circ}\, e). \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{F}_{\mathbf{H}}(l_r) \supseteq & \bigcap_{\vartheta \in (((l_r \,\underline{\circ}\, e) \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)) \,\underline{\circ}\, (x \,\underline{\circ}\, e)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\ \supseteq & \bigcap_{\vartheta \in (l_r \,\underline{\circ}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta). \end{aligned}$$

It implies $\mathcal{F}_{\mathbf{H}}$ is soft semiprime.
(3)⇒(2) Assume that $\mathcal{M}$ is an interior-hyperideal of $\mathbf{H}$, then by Theorem 4.2 $\mathcal{H}_{\mathcal{M}}$ will be an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. Let $m \in \mathcal{M}$, then we have $\mathcal{H}_{\mathcal{M}}(m) = \mathcal{U}$. Now

$$\begin{aligned} \mathcal{U} = & \mathcal{H}_{\mathcal{M}}(m) \\ \subseteq & (\mathcal{H}_{\mathcal{M}} \hat{\diamond} \mathcal{H}_{\mathcal{M}})(m) \\ = & \mathcal{H}_{(\mathcal{M} \,\underline{\circ}\, \mathcal{M})}(m). \end{aligned}$$

It would yield $m \in \mathcal{M} \,\underline{\circ}\, \mathcal{M}$. Therefore, $\mathcal{M} \subseteq \mathcal{M} \,\underline{\circ}\, \mathcal{M}$. Now, let $m \,\underline{\circ}\, m \subseteq \mathcal{M}$ for some $m \in \mathcal{M}$, then $\bigcap_{\vartheta \in (m \,\underline{\circ}\, m)} \mathcal{H}_{\mathcal{M}}(\vartheta) = \mathcal{U}$. As $\mathcal{H}_{\mathcal{M}}$ is soft semiprime, thus we have

$$\begin{aligned} \mathcal{H}_{\mathcal{M}}(m) \supseteq & \bigcap_{\vartheta \in (m \,\underline{\circ}\, m)} \mathcal{H}_{\mathcal{M}}(\vartheta) \\ = & \mathcal{U}. \end{aligned}$$

It follows that $m \in \mathcal{M}$. Hence, $\mathcal{M}$ is semiprime.
(2)⇒(1) By Lemma 5.4, $(l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}$ is an interior-hyperideal of $\mathbf{H}$. Now, $l_r \,\underline{\circ}\, l_r \subseteq (l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}$ for some $l_r \in \mathbf{H}$, then by assumption $(l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}$ will be semiprime. Thus, it would imply that $l_r \in (l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}$. Therefore, we have

$$\begin{aligned} l_r \in & (l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H} \\ \subseteq & ((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}) \,\underline{\circ}\, ((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}) \\ = & (\mathbf{H} \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)) \,\underline{\circ}\, (\mathbf{H} \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)) \end{aligned}$$

$$
\begin{aligned}
&\subseteq (\mathbf{H} \underset{\sim}{\circ} \mathbf{H}) \underset{\sim}{\circ} ((e \underset{\sim}{\circ} \mathbf{H}) \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r))\\
&\subseteq \mathbf{H} \underset{\sim}{\circ} ((l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} e))\\
&\subseteq (l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} \mathbf{H}))\\
&\subseteq (l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (\mathbf{H} \underset{\sim}{\circ} \mathbf{H})\\
&= (\mathbf{H} \underset{\sim}{\circ} \mathbf{H}) \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)\\
&\subseteq \mathbf{H} \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r).
\end{aligned}
$$

Hence, $\mathbf{H}$ is left regular. □

**Theorem 5.2.** *If $\mathbf{H}$ is a left regular LA-semihypergroup with left identity, then every S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$ is idempotent.*

*Proof.* Let $\mathbf{H}$ be an LA-semihypergroup with left identity and let $l_r \in \mathbf{H}$. As $\mathbf{H}$ is left regular, thus for any $l_r \in \mathbf{H}$, there exists $x \in \mathbf{H}$ such that

$$
\begin{aligned}
l_r \in\; & x \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)\\
=\; & l_r \underset{\sim}{\circ} (x \underset{\sim}{\circ} l_r)\\
\subseteq\; & (x \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)) \underset{\sim}{\circ} (x \underset{\sim}{\circ} l_r)\\
=\; & (l_r \underset{\sim}{\circ} x) \underset{\sim}{\circ} ((l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} x)\\
=\; & (((l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} x) \underset{\sim}{\circ} x) \underset{\sim}{\circ} l_r\\
=\; & ((x \underset{\sim}{\circ} x) \underset{\sim}{\circ} (l_r \underset{\sim}{\circ} l_r)) \underset{\sim}{\circ} l_r\\
=\; & ((l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (x \underset{\sim}{\circ} x)) \underset{\sim}{\circ} l_r.
\end{aligned}
$$

Then, there exists $b \in (l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (x \underset{\sim}{\circ} x)$ such that $l_r \in b \underset{\sim}{\circ} l_r$. Therefore $(b, l_r) \in \mathbb{H}_{l_r}$. Suppose $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, then by Lemma 5.2, $\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$. Thus, $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \sqsubseteq \mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$. Now, it remain to prove that $\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \sqsupseteq \mathcal{F}_{\mathbf{H}}$. For this, we have

$$
\begin{aligned}
(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})(l_r) = & \bigcup_{(y,z)\in\mathbb{H}_{l_r}} \Big\{\mathcal{F}_{\mathbf{H}}(y) \;\cap\; \mathcal{F}_{\mathbf{H}}(z)\Big\}\\
\supseteq & \mathcal{F}_{\mathbf{H}}(b) \;\cap\; \mathcal{F}_{\mathbf{H}}(l_r). \qquad (5.3)
\end{aligned}
$$

As $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, we have $\bigcap_{\vartheta \in (x \underset{\sim}{\circ} y) \underset{\sim}{\circ} z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y)$. Since $b \in (l_r \underset{\sim}{\circ} l_r) \underset{\sim}{\circ} (x \underset{\sim}{\circ} x)$, it would imply that $\mathcal{F}_{\mathbf{H}}(b) \supseteq \mathcal{F}_{\mathbf{H}}(l_r)$. Hence, from (5.3), we have

$$
\begin{aligned}
(\mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}})(l_r) \supseteq & \mathcal{F}_{\mathbf{H}}(b) \;\cap\; \mathcal{F}_{\mathbf{H}}(l_r)\\
\supseteq & \mathcal{F}_{\mathbf{H}}(l_r) \;\cap\; \mathcal{F}_{\mathbf{H}}(l_r)\\
= & \mathcal{F}_{\mathbf{H}}(l_r).
\end{aligned}
$$

This shows that every S.I. interior-hyperideal of $\mathbf{H}$ is idempotent. □

**Theorem 5.3.** *If $\mathbf{H}$ is an LA-semihypergroup with left identity, then the following statements are equivalent.*

(1) $\mathbf{H}$ *is left regular.*
(2) *Every S.I. interior-hyperideal of* $\mathbf{H}$ *over* $\mathcal{U}$ *is soft semiprime and idempotent.*

*Proof.* (1)⇒(2) Let $\mathbf{H}$ be a left regular LA-semihypergroup with left identity and let $\mathcal{F}_{\mathbf{H}}$ be an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, then by Theorem 5.2, $\mathcal{F}_{\mathbf{H}}$ will be idempotent. Thus, it is only remains to show that $\mathcal{F}_{\mathbf{H}}$ is soft semiprime. As $\mathbf{H}$ is left regular, thus for any $l_r \in \mathbf{H}$ there exists $x \in \mathbf{H}$ such that

$$\begin{aligned} l_r \in & x \mathbin{\underline{\circ}} (l_r \mathbin{\underline{\circ}} l_r) \\ \subseteq & (e \mathbin{\underline{\circ}} x) \mathbin{\underline{\circ}} (l_r \mathbin{\underline{\circ}} l_r) \\ = & (l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} (x \mathbin{\underline{\circ}} e) \\ \subseteq & (l_r \mathbin{\underline{\circ}} (e \mathbin{\underline{\circ}} l_r)) \mathbin{\underline{\circ}} (x \mathbin{\underline{\circ}} e) \\ = & (e \mathbin{\underline{\circ}} (l_r \mathbin{\underline{\circ}} l_r)) \mathbin{\underline{\circ}} (x \mathbin{\underline{\circ}} e). \end{aligned}$$

Then, we have

$$\begin{aligned} \mathcal{F}_{\mathbf{H}}(l_r) \supseteq & \bigcap_{\vartheta \in ((e \mathbin{\underline{\circ}} (l_r \mathbin{\underline{\circ}} l_r)) \mathbin{\underline{\circ}} (x \mathbin{\underline{\circ}} e))} \mathcal{F}_{\mathbf{H}}(\vartheta) \\ \supseteq & \bigcap_{\vartheta \in l_r \mathbin{\underline{\circ}} l_r} \mathcal{F}_{\mathbf{H}}(\vartheta). \text{ As } \mathcal{F}_{\mathbf{H}} \text{ is an S.I. interior hyperideal of } \mathbf{H}. \end{aligned}$$

Hence, $\mathcal{F}_{\mathbf{H}}$ is soft semiprime.
(2)⇒(1) Suppose that every S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$ with left identity is idempotent and soft semiprime. By Lemma 5.4, $(l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} \mathbf{H}$ is an interior-hyperideal of $\mathbf{H}$. Therefore, by Theorem 4.2, characteristic soft function $\mathcal{H}_{(l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} \mathbf{H}}$ will be S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. By assumption, $\mathcal{H}_{(l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} \mathbf{H}}$ is soft semiprime. So, by Lemma 5.3, $(l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} \mathbf{H}$ will be semiprime. Thus, for any $l_r \in \mathbf{H}$, we have

$$\begin{aligned} (l_r \mathbin{\underline{\circ}} l_r) \subseteq & (e \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} l_r \\ = & (l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} e \\ \subseteq & (l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} H. \end{aligned}$$

This yield $l_r \in (l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} \mathbf{H}$. Therefore, we have

$$\begin{aligned} l_r \in & (l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} \mathbf{H} \\ = & (l_r \mathbin{\underline{\circ}} l_r) \mathbin{\underline{\circ}} (\mathbf{H} \mathbin{\underline{\circ}} \mathbf{H}) \\ = & (\mathbf{H} \mathbin{\underline{\circ}} \mathbf{H}) \mathbin{\underline{\circ}} (l_r \mathbin{\underline{\circ}} l_r) \\ \subseteq & \mathbf{H} \mathbin{\underline{\circ}} (l_r \mathbin{\underline{\circ}} l_r). \end{aligned}$$

Hence, $\mathbf{H}$ is left regular. □

**Theorem 5.4.** *Let* $\mathbf{H}$ *be a left regular LA-semihypergroup with left identity, then* $(\mathcal{H}_{\mathbf{H}} \mathbin{\hat{\diamond}} \mathcal{F}_{\mathbf{H}}) \mathbin{\hat{\diamond}} (\mathcal{H}_{\mathbf{H}} \mathbin{\hat{\diamond}} \mathcal{F}_{\mathbf{H}}) = \mathcal{F}_{\mathbf{H}}$*, for every S.I. interior-hyperideal* $\mathcal{F}_{\mathbf{H}}$ *of* $\mathbf{H}$ *over* $\mathcal{U}$.

*Proof.* Assume that $\mathbf{H}$ is a left regular LA-semihypergroup with left identity. Let $\mathcal{F}_{\mathbf{H}}$ be any S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, then by Theorem 5.3, $\mathcal{F}_{\mathbf{H}}$ will be soft semiprime and idempotent. Also, by Lemma 5.2, $\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} = \mathcal{F}_{\mathbf{H}}$. Thus, we have

$$\begin{aligned}(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) =& \mathcal{F}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}} \\ =& \mathcal{F}_{\mathbf{H}}.\end{aligned}$$

Hence, $(\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) \hat{\diamond} (\mathcal{H}_{\mathbf{H}} \hat{\diamond} \mathcal{F}_{\mathbf{H}}) = \mathcal{F}_{\mathbf{H}}$. □

**Theorem 5.5.** *Let* $\mathbf{H}$ *be an LA-semihypergroup with left identity, then the following statements are equivalent.*

(1) $\mathbf{H}$ *is left regular.*
(2) *Every S.I. interior-hyperideal of* $\mathbf{H}$ *over* $\mathcal{U}$ *is soft semiprime.*
(3) $\mathcal{F}_{\mathbf{H}}(h) = \bigcap_{\vartheta \in l_r \,\underline{o}\, l_r} \mathcal{F}_{\mathbf{H}}(\vartheta)$, *for every S.I. interior-hyperideal* $\mathcal{F}_{\mathbf{H}}$ *of* $\mathbf{H}$ *over* $\mathcal{U}$, *for all* $l_r \in \mathbf{H}$.

*Proof.* (1)⇒(2) Suppose that $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. As $\mathbf{H}$ is left regular, thus for any $l_r \in \mathbf{H}$, there exists $x \in \mathbf{H}$ such that $l_r \in x \,\underline{o}\, (l_r \,\underline{o}\, l_r)$. Now, we have

$$\begin{aligned}l_r \in& x \,\underline{o}\, (l_r \,\underline{o}\, l_r) \\ \subseteq& x \,\underline{o}\, ((x \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \,\underline{o}\, l_r) \\ =& (x \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \,\underline{o}\, (x \,\underline{o}\, l_r).\end{aligned}$$

As $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$, we have $\bigcap_{\vartheta \in (x \,\underline{o}\, y) \,\underline{o}\, z} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(y)$ for all $x, y, z \in \mathbf{H}$. Since $l_r \in (x \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \,\underline{o}\, (x \,\underline{o}\, l_r)$, it would imply that $\mathcal{F}_{\mathbf{H}}(l_r) \supseteq \bigcap_{\vartheta \in (x \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \,\underline{o}\, (x \,\underline{o}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \bigcap_{\vartheta \in (l_r \,\underline{o}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta)$. Therefore, $\mathcal{F}_{\mathbf{H}}$ is soft semiprime.

(2)⇒(3) Here, we only need to show that $\bigcap_{\vartheta \in l_r \,\underline{o}\, l_r} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(l_r)$. For this, we have

$$\begin{aligned}l_r \,\underline{o}\, l_r \subseteq& l_r \,\underline{o}\, (x \,\underline{o}\, (l_r \,\underline{o}\, l_r)) \\ =& l_r \,\underline{o}\, (l_r \,\underline{o}\, (x \,\underline{o}\, l_r)) \\ \subseteq& (e \,\underline{o}\, l_r) \,\underline{o}\, (l_r \,\underline{o}\, (x \,\underline{o}\, l_r)) \\ =& ((x \,\underline{o}\, l_r) \,\underline{o}\, l_r) \,\underline{o}\, (l_r \,\underline{o}\, e).\end{aligned}$$

Then, we have

$$\begin{aligned}\bigcap_{\vartheta \in (l_r \,\underline{o}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq& \bigcap_{\vartheta \in ((x \,\underline{o}\, l_r) \,\underline{o}\, l_r) \,\underline{o}\, (l_r \,\underline{o}\, e)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\ \supseteq& \mathcal{F}_{\mathbf{H}}(l_r) \\ & \text{(as } \mathcal{F}_{\mathbf{H}} \text{ is an S.I. interior hyperideal).}\end{aligned}$$

It follows that $\mathcal{F}_{\mathbf{H}}(l_r) = \bigcap_{\vartheta \in l_r \,\underline{o}\, l_r} \mathcal{F}_{\mathbf{H}}(\vartheta)$.

(3)⇒(1) By Lemma 5.4, $(l_r \,\underline{o}\, l_r) \,\underline{o}\, \mathbf{H}$ is an interior-hyperideal of $\mathbf{H}$. Now

$$(l_r \,\underline{o}\, l_r) \subseteq (e \,\underline{o}\, l_r) \,\underline{o}\, l_r$$

$$\begin{aligned}&=(l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, e\\&\subseteq(l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}.\end{aligned}$$

Then, by Theorem 4.2, $\mathcal{H}_{((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H})}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. Now, $(l_r \,\underline{\circ}\, l_r) \subseteq (l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}$, it would imply $\bigcap_{\vartheta\in l_r \,\underline{\circ}\, l_r} \mathcal{H}_{((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H})}(\vartheta) = \mathcal{U}$. By assumption, $\mathcal{H}_{((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H})}(l_r) = \bigcap_{\vartheta\in l_r \,\underline{\circ}\, l_r} \mathcal{H}_{((l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H})}(\vartheta) = \mathcal{U}$. This yield $l_r \in (l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}$. Therefore,

$$\begin{aligned}l_r \in&(l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, \mathbf{H}\\=&(l_r \,\underline{\circ}\, l_r) \,\underline{\circ}\, (\mathbf{H} \,\underline{\circ}\, \mathbf{H})\\=&(\mathbf{H} \,\underline{\circ}\, \mathbf{H}) \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)\\\subseteq&\mathbf{H} \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r).\end{aligned}$$

Hence, $\mathbf{H}$ is left regular. ☐

**Theorem 5.6.** *Let* $\mathbf{H}$ *be a left regular LA-semihypergroup with left identity, then the following statements are equivalent:*

(1) $\mathcal{F}_{\mathbf{H}}$ *is an S.I. interior-hyperideal of* $\mathbf{H}$ *over* $\mathcal{U}$*;*
(2) $\mathcal{F}_{\mathbf{H}}$ *is an S.I. bi-hyperideal of* $\mathbf{H}$ *over* $\mathcal{U}$*.*

*Proof.* (1)⇒(2) Let $\mathbf{H}$ be a left regular LA-semihypergroup with left identity, thus for $a, b \in \mathbf{H}$, there exists $a', b' \in \mathbf{H}$ such that $a \in a' \,\underline{\circ}\, (a \,\underline{\circ}\, a)$ and $b \in b' \,\underline{\circ}\, (b \,\underline{\circ}\, b)$. Suppose that $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. Then, we have

$$\begin{aligned}\bigcap_{\vartheta\in((a \,\underline{\circ}\, l_r) \,\underline{\circ}\, b)} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq& \bigcap_{\vartheta\in((a' \,\underline{\circ}\, (a \,\underline{\circ}\, a)) \,\underline{\circ}\, l_r) \,\underline{\circ}\, b} \mathcal{F}_{\mathbf{H}}(\vartheta)\\=& \bigcap_{\vartheta\in((a \,\underline{\circ}\, (a' \,\underline{\circ}\, a)) \,\underline{\circ}\, l_r) \,\underline{\circ}\, b} \mathcal{F}_{\mathbf{H}}(\vartheta)\\=& \bigcap_{\vartheta\in((l_r \,\underline{\circ}\, (a' \,\underline{\circ}\, a)) \,\underline{\circ}\, a) \,\underline{\circ}\, b} \mathcal{F}_{\mathbf{H}}(\vartheta)\\\supseteq&\mathcal{F}_{\mathbf{H}}(a).\end{aligned}$$

Also, we have

$$\begin{aligned}\bigcap_{\vartheta\in((a \,\underline{\circ}\, l_r) \,\underline{\circ}\, b)} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq& \bigcap_{\vartheta\in((a \,\underline{\circ}\, l_r) \,\underline{\circ}\, (b' \,\underline{\circ}\, (b \,\underline{\circ}\, b)))} \mathcal{F}_{\mathbf{H}}(\vartheta)\\=& \bigcap_{\vartheta\in((a \,\underline{\circ}\, l_r) \,\underline{\circ}\, (b \,\underline{\circ}\, (b' \,\underline{\circ}\, b)))} \mathcal{F}_{\mathbf{H}}(\vartheta)\\=& \bigcap_{\vartheta\in((a \,\underline{\circ}\, b) \,\underline{\circ}\, (l_r \,\underline{\circ}\, (b' \,\underline{\circ}\, b)))} \mathcal{F}_{\mathbf{H}}(\vartheta)\\\supseteq&\mathcal{F}_{\mathbf{H}}(b).\end{aligned}$$

This show that $\bigcap_{\vartheta\in((a \,\underline{\circ}\, l_r) \,\underline{\circ}\, b)} \mathcal{F}_{\mathbf{H}}(\vartheta) \supseteq \mathcal{F}_{\mathbf{H}}(a) \cap \mathcal{F}_{\mathbf{H}}(b)$. Hence, $\mathcal{F}_{\mathbf{H}}$ is an S.I. bi-hyperideal of $\mathbf{H}$ over $\mathcal{U}$.

(2)⇒(1) Suppose that $\mathbf{H}$ is a left regular LA-semihypergroup with left identity $'e'$ and $\mathcal{F}_{\mathbf{H}}$ an S.I. bi-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. Let $l_r \in \mathbf{H}$, then there exists $l_r^{'} \in \mathbf{H}$ such that $l_r \in l_r^{'} \underline{\circ} (l_r \underline{\circ} l_r)$. Then for any $x, y \in \mathbf{H}$, we have

$$\begin{aligned}
\bigcap_{\vartheta \in ((x \,\underline{\circ}\, l_r) \,\underline{\circ}\, y)} \mathcal{F}_{\mathbf{H}}(\vartheta) &\supseteq \bigcap_{\vartheta \in ((x \,\underline{\circ}\, l_r) \,\underline{\circ}\, (e \,\underline{\circ}\, y))} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in ((y \,\underline{\circ}\, e) \,\underline{\circ}\, (l_r \,\underline{\circ}\, x))} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in (l_r \,\underline{\circ}\, ((y \,\underline{\circ}\, e) \,\underline{\circ}\, x)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&\supseteq \bigcap_{\vartheta \in ((l_r^{'} \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)) \,\underline{\circ}\, ((y \,\underline{\circ}\, e) \,\underline{\circ}\, x))} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in ((l_r \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, l_r)) \,\underline{\circ}\, ((y \,\underline{\circ}\, e) \,\underline{\circ}\, x)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&\supseteq \bigcap_{\vartheta \in ((l_r \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)))) \,\underline{\circ}\, ((y \,\underline{\circ}\, e) \,\underline{\circ}\, x))} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in ((((y \,\underline{\circ}\, e) \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r \,\underline{\circ}\, l_r)))) \,\underline{\circ}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in ((((y \,\underline{\circ}\, e) \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, l_r)))) \,\underline{\circ}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in ((((y \,\underline{\circ}\, e) \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, l_r)))) \,\underline{\circ}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&= \bigcap_{\vartheta \in ((l_r \,\underline{\circ}\, (((y \,\underline{\circ}\, e) \,\underline{\circ}\, x) \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, (l_r^{'} \,\underline{\circ}\, l_r)))) \,\underline{\circ}\, l_r)} \mathcal{F}_{\mathbf{H}}(\vartheta) \\
&\supseteq \mathcal{F}_{\mathbf{H}}(l_r) \cap \mathcal{F}_{\mathbf{H}}(l_r) \\
&= \mathcal{F}_{\mathbf{H}}(l_r).
\end{aligned}$$

Therefore, $\mathcal{F}_{\mathbf{H}}$ is an S.I. interior-hyperideal of $\mathbf{H}$ over $\mathcal{U}$. □

**Conclusion.** In this paper, we have introduced soft interior-hyperideals in LA-semihypergroups and characterized left regular LA-semihypergroups in terms of soft interior-hyperideals. Based on the results of this paper, some further work can be done on the properties of soft interior-hyperideals in other structures.

## References


[1] H. Aktas and N. Cagman, *Soft sets and soft groups*, Inform. Sci. **177** (2007), 2726–2735.

[2] V. Amjad, K. Hila and F. Yousafzai, *Generalized hyperideals in locally associative left almost semihypergroups*, New York J. Math. **20** (2014), 1063–1076.

[3] V. Amjad and F. Yousafzai, *On pure LA-semihypergroups*, Konuralp J. Math. **2** (2014), 53–63.

[4] S. M. Anvariyeh, S. Mirvakili, O. Kazanci and B. Davvaz, *Algebraic hyperstructures of soft sets associated to semihypergroups*, Southeast Asian Bull. Math. **35** (2011), 911–925.

[5] M. Aslam, T. Aroob and N. Yaqoob, *On cubic G-hyperideals in left almost G-semihypergroups*, Ann. Fuzzy Math. Inform. **5** (2013), 169–182.

[6] M. Azhar, M. Gulistan, N. Yaqoob and S. Kadry, *On fuzzy ordered LA-semihypergroups*, International Journal of Analysis and Applications **16** (2018), 276–289.

[7] N. Cagman, F. Citak and H. Aktas, *Soft int-group and its applications to group theory*, Neural Computing and Applications **21** (2012), 151–158.

[8] N. Cagman and S. Enginoglu, *Soft set theory and uni-int decision making*, European J. Oper. Res. **207** (2010), 848–855.

[9] P. Corsini, *Prolegomena of Hypergroup Theory*, Second Edition, Aviani Editore, Aviani, 1993.

[10] B. Davvaz and V. L. Fotea, *Hyperring Theory and Applications*, International Academic Press, Gulf Breeze, Fla, USA, 2007.

[11] M. Farooq, A. Khan, and B. Davvaz, *Characterizations of ordered semihypergroups by the properties of their intersectional-soft generalized bi-hyperideals*, Soft Computing **9**(22) (2018), 3001–3010.

[12] F. Feng, Y. B. Jun and X. Zhao, *Soft semirings*, Comput. Math. Appl. **56** (2008), 2621–2628.

[13] M. Gulistan, M. Khan, N. Yaqoob and M. Shahzad, *Structural properties of cubic sets in regular LA-semihypergroups*, Fuzzy Inf. Eng. **9** (2017), 93–116.

[14] M. Gulistan, N. Yaqoob and M. Shahzad, *A note on Hv-LA-semigroups*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **77** (2015), 93–106.

[15] A. Hasankhani, *Ideals in a semihypergroup and greens relations*, Ratio Mathematica **13** (1999), 29–36.

[16] K. Hila and J. Dine, *On hyperideals in left almost semihypergroups*, ISRN Algebra **2011** (2011), Article ID 953124, 8 pages.

[17] M. A. Kazim and M. Naseeruddin, *On almost semigroups*, Aligarh Bull. Math. **2** (1972), 1–7.

[18] A. Khan, M. Farooq, M. Izhar and B. Davvaz, *Fuzzy hyperideals of left almost semihypergroups*, International Journal of Analysis and Applications **15** (2016), 155–171.

[19] M. Khan, N. Y. M. Gulistan and F. Hussain, *General cubic hyperideals of LA-semihypergroups*, Afr. Mat. **27** (2016), 731–751.

[20] M. Khan, N. Y. M. Gulistan and F. Hussain, *General cubic hyperideals of LA-semihypergroups*, Afr. Mat. **27** (2016), 731–751.

[21] S. A. Khan, M. Y. Abbasi and A. F. Talee, *A new approach to soft hyperideals in LA-semihypergroups*, in: N. Yadav, A. Yadav, J. C. Bansal, K. Deep and J. H. Kim (Eds.), *Harmony Search and Nature Inspired Optimization Algorithm, Theory and Applications*, ICHSA-2018, Advances in Intelligent System and Computing **29** (2019), 293–303, DOI 10.1007/978-981-13-0761-4-29.

[22] P. K. Maji, R. Biswas and A. R. Roy, *Soft set theory*, Comput. Math. Appl. **45** (2003), 555–562.

[23] F. Marty, *Sur une generalization de la notion de group*, in: *Proceedings of 8th Congress Math. Scandinaves*, Stockholm, Sweden, 1934, 45–49.

[24] D. Molodtsov, *Soft set theory-first results*, Comput. Math. Appl. **37** (1999), 19–31.

[25] S. Naz and M. Shabir, *On soft semihypergroups*, Journal of Intelligent and Fuzzy Systems **26** (2014), 2203–2213.

[26] A. Sezgin, *A new approach to LA-semigroup theory via the soft sets*, Journal of Intelligent and Fuzzy Systems **26** (2014), 2483–2495.

[27] A. Sezgin and A. O. Atagun, *On operations of soft sets*, Comput. Math. Appl. **61** (2011), 1457–1467.

[28] N. Yaqoob, M. Akram and M. Aslam, *Intuitionistic fuzzy soft groups induced by $(t, s)$-norm*, Indian Journal of Science and Technology **6** (2013), 4282–4289.

[29] N. Yaqoob, M. Aslam and Faisal, *On soft G-hyperideals over left almost G-semihypergroups*, Journal of Advanced Research in Dynamical and Control Systems **4** (2012), 1–12.

[30] N. Yaqoob, P. Corsini and F. Yousafzai, *On intra-regular left almost semihypergroups with pure left identity*, J. Math. **2013** (2013), Article ID 510790, 10 pages.

[31] N. Yaqoob and M. Gulistan, *Partially ordered left almost semihypergroups*, J. Egyptian Math. Soc. **23** (2015), 231–235.

[32] F. Yousafzai, K. Hila, P. Corsini and A. Zeb, *Existence of non-associative algebraic hyperstructures and related problems*, Afr. Mat. **26** (2015), 981–995.

[33] F. Yousafzai and C. Piersiulio, *Some characterization problems in LA-semihypergroups*, Journal of Algebra, Number Theory: Advances and Applications **10** (2013), 41–55.

[34] F. Yousafzai, N. Yaqoob and A. Ghareeb, *Left regular AG-groupoids in terms of fuzzy interior ideals*, Afr. Mat. **24** (2013), 577–587.

[35] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

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# SOME RESULTS ON SUPER EDGE-MAGIC DEFICIENCY OF GRAPHS

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Abstract. An edge-magic total labeling of a graph $G$ is a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, |V(G)| + |E(G)|\}$, where there exists a constant $k$ such that $f(u) + f(uv) + f(v) = k$, for every edge $uv \in E(G)$. Moreover, if the vertices are labeled with the numbers $1, 2, \dots, |V(G)|$ such a labeling is called a super edge-magic total labeling. The super edge-magic deficiency of a graph $G$, denoted by $\mu_s(G)$, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic total labeling or is defined to be $\infty$ if there exists no such $n$.

In this paper we study the super edge-magic deficiencies of two types of snake graph and a prism graph $D_n$ for $n \equiv 0 \pmod 4$. We also give an exact value of super edge-magic deficiency for a ladder $P_n \times K_2$ with 1 pendant edge attached at each vertex of the ladder, for $n$ odd, and an exact value of super edge-magic deficiency for a square of a path $P_n$ for $n \geq 3$.

## 1. Introduction

In this paper, we consider only finite, simple and undirected graphs. We denote the vertex set and edge set of a graph $G$ by $V(G)$ and $E(G)$, respectively. Let $|V(G)| = p$ and $|E(G)| = q$.

An *edge-magic total labeling* of a graph $G$ is a bijection $f : V(G) \cup E(G) \to \{1, 2, \dots, p + q\}$, where there exists a constant $k$ such that

$$f(u) + f(uv) + f(v) = k,$$

for every edge $uv \in E(G)$. The constant $k$ is called a *magic constant*. An edge-magic total labeling $f$ is called *super edge-magic total* if the vertices are labeled with the

*Key words and phrases.* Super edge-magic total labeling, super edge-magic deficiency, block graph, snake graph, prism, corona of graphs, square of graph.

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smallest possible labels, i.e., with the numbers $1, 2, \dots, p$. A graph that admits a (super) edge-magic total labeling is called (*super*) *edge-magic total*.

The concept of edge-magic total labeling was given by Kotzig and Rosa [8]. Super edge-magic total labelings were originally defined by Enomoto et al. in [3]. However Acharya and Hegde had introduced in [1] the concept of strongly indexable graphs that is equivalent to the one of super edge-magic total labeling.

Kotzig and Rosa [8] proved that for any graph $G$ there exists an edge-magic graph $H$ such that $H \cong G \cup nK_1$ for some nonnegative integer $n$. This fact leads to the concept of *edge-magic deficiency* of a graph $G$, which is the minimum nonnegative integer $n$ such that $G \cup nK_1$ is edge-magic total and it is denoted by $\mu(G)$. In particular,

$$\mu(G) = \min\{n \geq 0 : G \cup nK_1 \text{ is edge-magic total}\}.$$

In the same paper, Kotzig and Rosa gave an upper bound for the edge-magic deficiency of a graph $G$ with $n$ vertices,

$$\mu(G) \leq F_{n+2} - 2 - n - \frac{n(n-1)}{2},$$

where $F_n$ is the $n$th Fibonacci number.

Motivated by Kotzig and Rosa's concept of edge-magic deficiency, Figueroa-Centeno, Ichishima and Muntaner-Batle [5] defined a similar concept for the super edge-magic total labelings. The *super edge-magic deficiency* of a graph $G$, denoted by $\mu_s(G)$, is the minimum nonnegative integer $n$ such that $G \cup nK_1$ has a super edge-magic total labeling, or is defined to be $\infty$ if there exists no such $n$. More precisely, if

$$M(G) = \{n \geq 0 : G \cup nK_1 \text{ is a super edge-magic total graph}\},$$

then

$$\mu_s(G) = \begin{cases} \min M(G), & \text{if } M(G) \neq \emptyset, \\ \infty, & \text{if } M(G) = \emptyset. \end{cases}$$

It is easy to see that for every graph $G$ it holds

$$\mu(G) \leq \mu_s(G).$$

In [5, 7] Figueroa-Centeno, Ichishima and Muntaner-Batle found the exact values of the super edge-magic deficiencies of several classes of graphs, such as cycles, complete graphs, 2-regular graphs and complete bipartite graphs $K_{2,m}$. They also proved that all forests have finite deficiency. In particular, they proved that

$$\mu_s(nK_2) = \begin{cases} 0, & \text{if } n \text{ is odd}, \\ 1, & \text{if } n \text{ is even}. \end{cases}$$

In [10] Ngurah, Simanjuntak and Baskoro gave some upper bounds for the super edge-magic deficiency of fans, double fans and wheels. In [6] Figueroa-Centeno, Ichishima and Muntaner-Batle proved

$$\mu_s(P_m \cup K_{1,n}) = \begin{cases} 1, & \text{if } m = 2 \text{ and } n \text{ is odd or } m = 3 \text{ and } n \not\equiv 0 \pmod 3, \\ 0, & \text{otherwise.} \end{cases}$$

In the same paper, they showed that

$$\mu_s(K_{1,m} \cup K_{1,n}) = \begin{cases} 0, & \text{if } m \text{ is a multiple of } n+1 \text{ or } n \text{ is a multiple of } m+1, \\ 1, & \text{otherwise.} \end{cases}$$

They also conjectured that every forest with two components has super edge-magic deficiency less than or equal to 1. Baig, Baskoro and Semaničová–Feňovčíková [2] have determined the super edge magic deficiency of a star forest. Santhosh and Singh [11] studied the corona product of $K_2$ and $C_n$ and they showed that $\mu_s(K_2 \odot C_n) \leq \frac{n-3}{2}$, for $n$ odd at least 3.

In this paper we study the super edge-magic deficiencies for several classes of graphs. We give upper bounds for the super edge-magic deficiencies of two types of snake graph and for prism graph $D_n$ for $n \equiv 0 \pmod 4$. We also give an exact value of super edge-magic deficiency for a ladder $P_n \times K_2$ with 1 pendant edge attached at each vertex of the ladder, for $n$ odd, and an exact value of super edge-magic deficiency for a square of a path $P_n$ for every positive integer $n$, $n \geq 3$.

To prove the results presented in this paper, we frequently use the following lemma.

**Lemma 1.1.** [4] *A graph $G$ with $p$ vertices and $q$ edges is super edge-magic total if and only if there exists a bijective function $f : V(G) \to \{1, 2, \ldots, p\}$ such that the set $\{f(u) + f(v) : uv \in E(G)\}$ consists of $q$ consecutive integers. In such a case, $f$ extends to a super edge-magic total labeling of $G$.*

## 2. Upper Bounds

In graph theory a *block graph* is a graph in which every bi-connected component (block) is a clique. Block graphs are sometimes erroneously said to be "Husimi trees", but that name more properly refers to cactus graphs, graphs in which every nontrivial bi-connected component is a cycle. In graph theory block graphs may be described as the intersection graphs of the blocks of arbitrary undirected graphs.

Let $G$ be a graph and $u$ and $v$ are two fixed vertices in $G$. The $G^n$*-snake* is a graph obtained from $n$ copies of $G$ by identifying the vertex corresponding to the vertex $v$ in the $i$th copy of $G$ with the vertex corresponding to the vertex $u$ in the $(i+1)$th copy of $G$, for $i = 1, 2, \ldots, n-1$. The wheel $W_k$, $k \geq 3$ is a graph obtained by joining every vertex of a cycle $C_k$ with a new vertex.

In the following theorem we will deal with the super edge-magic deficiency of $W_4^n$-snake. Let us denote the vertex set and the edge set of $W_4^n$-snake such that

$$\begin{aligned} V(W_4^n\text{-snake}) = &\{x_i : i = 1, 2, \ldots, 2n\} \cup \{y_i : i = 1, 2, \ldots, n\} \\ &\cup \{z_i : i = 1, 2, \ldots, n+1\}, \end{aligned}$$

$$\begin{aligned}
E(W_4^n\text{-snake}) =& \{x_i x_{n+i} : i = 1, 2, \dots, n\} \cup \{z_i z_{i+1} : i = 1, 2, \dots, n\} \\
& \cup \{x_i z_i, x_{n+i} z_{i+1} : i = 1, 2, \dots, n\} \\
& \cup \{y_i x_i, y_i x_{n+i} : i = 1, 2, \dots, n\} \\
& \cup \{y_i z_i, y_i z_{i+1} : i = 1, 2, \dots, n\}.
\end{aligned}$$

**Theorem 2.1.** *The graph $W_4^n$-snake has super edge-magic deficiency at most* 1.

*Proof.* Let us denote the vertices and edges of $G \cong W_4^n \cup K_1$ such that $V(G) = V(W_4^n\text{-snake}) \cup \{v\}$ and $E(G) = E(W_4^n\text{-snake})$. The graph $G$ has $4n+2$ vertices and $8n$ edges.

We define the vertex labeling $f$ of $G$ in the following way

$$\begin{aligned}
f(x_i) &= 4i - 3, \quad \text{if } i = 1, 2, \dots, n, \\
f(x_{n+i}) &= 4i - 1, \quad \text{if } i = 1, 2, \dots, n, \\
f(y_i) &= 4i, \quad \text{if } i = 1, 2, \dots, n, \\
f(z_i) &= 4i - 2, \quad \text{if } i = 1, 2, \dots, n+1, \\
f(v) &= 4n + 1.
\end{aligned}$$

It is easy to see that the vertices of $G$ are labeled with the numbers $1, 2, 3, \dots, 4n+2$ as the sets of vertex labels are

$$\begin{aligned}
\{f(x_i) : i = 1, 2, 3, \dots, n\} &= \{1, 5, 9, \dots, 4n - 3\}, \\
\{f(z_i) : i = 1, 2, 3, \dots, n, n+1\} &= \{2, 6, 10, \dots, 4n - 2, 4n + 2\}, \\
\{f(x_i) : i = n+1, n+2, n+3, \dots, 2n\} &= \{3, 7, 11, \dots, 4n - 1\}, \\
\{f(y_i) : i = 1, 2, 3, \dots, n\} &= \{4, 8, 12, \dots, 4n\}, \\
f(v) &= 4n + 1.
\end{aligned}$$

Next we will count the edge sums of the edges in the blocks. For $i = 1, 2, \dots, n$ it holds

$$\begin{aligned}
f(x_i) + f(z_i) &= (4i - 3) + (4i - 2) = 8i - 5, \\
f(x_i) + f(x_{i+n}) &= (4i - 3) + (4i - 1) = 8i - 4, \\
f(x_i) + f(y_i) &= (4i - 3) + 4i = 8i - 3, \\
f(y_i) + f(z_i) &= 4i + (4i - 2) = 8i - 2, \\
f(x_{i+n}) + f(y_i) &= (4i - 1) + 4i = 8i - 1, \\
f(z_i) + f(z_{i+1}) &= (4i - 2) + (4(i+1) - 2) = 8i, \\
f(x_{i+n}) + f(z_{i+1}) &= (4i - 1) + (4(i+1) - 2) = 8i + 1, \\
f(y_i) + f(z_{i+1}) &= 4i + (4(i+1) - 2) = 8i + 2.
\end{aligned}$$

It means that the edge sums are consecutive integers $3, 4, \dots, 8n+2$. According to Lemma 1.1 the labeling $f$ can be extended to a super edge-magic total labeling of $G$ with magic constant $12n + 5$. □

A graph is called a *cactus graph* if every block is either a cycle or a complete graph $K_2$. Next we will deal with a special type of a cactus graph called an alternate quadrilateral snake. An *alternate quadrilateral snake* $A(C_4^n)$ is obtained from a path $x_1x_2\dots x_n$ by joining the vertices $x_i$, $x_{i+1}$, for every odd $i$, to new vertices $y_i$, $y_{i+1}$, respectively and then joining $y_i$ and $y_{i+1}$. That is every alternate edge of the path is replaced by a cycle $C_4$. More precisely, the vertex set and the edge set of $A(C_4^n)$ are the following

$$V(A(C_4^n)) = \{x_i, y_i : i = 1, 2, \dots, n\}$$

and

$$\begin{aligned} E(A(C_4^n)) =& \{x_ix_{i+1} : i = 1, 2, \dots, n-1\} \cup \{x_iy_i : i = 1, 2, \dots, n\} \\ &\cup \{y_iy_{i+1} : i = 1, 3, \dots, n-1\}. \end{aligned}$$

**Theorem 2.2.** *For every even positive integer $n$, $n \geq 4$, for super edge-magic deficiency of the alternate quadrilateral snake $A(C_4^n)$ we have*

$$\mu_s(A(C_4^n)) \leq \frac{n}{2}.$$

*Proof.* Let $n$ be an even positive integer. Let us denote the vertex set and the edge set of $G \cong A(C_4^n) \cup \frac{n}{2}K_1$ as follows $V(G) = V(A(C_4^n)) \cup \{v_i : i = 1, 2, \dots, \frac{n}{2}\}$ and $E(G) = E(A(C_4^n))$.

We define the vertex labeling of the graph $G$ in the following way

$$f(x_i) = \begin{cases} i, & \text{if } i = 1, 3, \dots, n-1, \\ n + \dfrac{3i}{2}, & \text{if } i = 2, 4, \dots, n, \end{cases}$$

$$f(y_i) = \begin{cases} n + \dfrac{3i-1}{2}, & \text{if } i = 1, 3, \dots, n-1, \\ i, & \text{if } i = 2, 4, \dots, n. \end{cases}$$

The remaining $\frac{n}{2}$ numbers $n+2, n+5, \dots, \frac{5n}{2}-1$ are used to label the isolated vertices $v_1, v_2, \dots, v_{\frac{n}{2}}$ of the graph $G$ arbitrary.

It is easy to see that $f$ is a bijection from the vertex set of $G$ onto the set of integers $1, 2, \dots, \frac{5n}{2}$.

For the edge sums we have the following. The edge sum of the edges $x_iy_i$, $y_iy_{i+1}$, $x_ix_{i+1}$ and $y_{i+1}x_{i+1}$, for $i = 1, 3, \dots, n-1$, are

$$\begin{aligned} f(x_i) + f(y_i) =& i + \left(n + \frac{3i-1}{2}\right) = n + \frac{5i-1}{2}, \\ f(y_i) + f(y_{i+1}) =& \left(n + \frac{3i-1}{2}\right) + (i+1) = n + \frac{5i-1}{2} + 1, \\ f(x_i) + f(x_{i+1}) =& i + \left(n + \frac{3(i+1)}{2}\right) = n + \frac{5i-1}{2} + 2, \\ f(x_{i+1}) + f(y_{i+1}) =& \left(n + \frac{3(i+1)}{2}\right) + (i+1) = n + \frac{5i-1}{2} + 3. \end{aligned}$$

The edge sum of the edge $x_{i+1}x_{i+2}$, for $i = 1, 3, \dots, n-3$, is

$$f(x_{i+1}) + f(x_{i+2}) = \left(n + \frac{3(i+1)}{2}\right) + (i+2) = n + \frac{5i-1}{2} + 4.$$

Moreover, for $i = 1, 3, \dots, n-3$, we have

$$f(x_{i+2}) + f(y_{i+2}) = (i+2) + \left(n + \frac{3(i+2)-1}{2}\right) = n + \frac{5i-1}{2} + 5.$$

Hence the edge sums are consecutive integers $n+2, n+3, \dots, \frac{7n}{2}$. Thus, according to Lemma 1.1, the labeling $f$ can be extended to the super edge-magic total labeling of $G$ with the magic constant $6n+1$. ☐

The graph $A(C_4^2)$ is isomorphic to the cycle $C_4$. Figueroa-Centeno, Ichishima and Muntaner-Batle [5] proved that $\mu_s(C_4) = 1$.

A *prism graph* $D_n$, sometimes also called a circular ladder graph, is a graph corresponding to the skeleton of an $n$-prism. Prism graphs are both planar and polyhedral. An $n$-prism graph consist of $2n$ vertices and $3n$ edges, which is equivalent to generalized Petersen graph $P_{n,1}$. The $n$-prism is isomorphic to circulant graph $Ci_{2n}(2, n)$ for odd $n$, and can be showed by rotating the inner cycle by 180°, and its radius is equal to that of the outer cycle in the top embedding above. In addition, for odd $n, D_n$ is isomorphic to $Ci_{2n}(4, n), Ci_{2n}(6, n), \dots, Ci_{2n}(n-1, n)$. The prism $D_n$ is isomorphic to the Cartesian product $C_n \times K_2$, where $C_n$ is the cycle on $n$ vertices and $K_2$ is the complete graph of order 2. The prism graph $D_n$ is equivalent to the Cayley graph of the dihedral group $D_n$, with respect to the generating set $\{x, x^{-1}, y\}$.

We denote the vertices and edges of $D_n$ such that

$$V(D_n) = \{x_i, y_i : i = 1, 2, \dots, n\}$$

and

$$E(D_n) = \{x_i x_{i+1}, y_i y_{i+1} : i = 1, 2, \dots, n-1\} \cup \{x_1 x_n, y_1 y_n\} \cup \{x_i y_i : i = 1, 2, \dots, n\}.$$

The cardinality of the vertex set and the edge set of $D_n$ is $2n$ and $3n$, respectively.

In [4] Figueroa-Centeno, Ichishima and Muntaner-Batle proved that for $n$ odd the graph $D_n$ is super edge-magic total. Ngurah and Baskoro [9] showed that for $n$ even the prism $D_n$ is not edge-magic total. In the following theorem we are dealing with the case when $n$ is divisible by 4.

**Theorem 2.3.** *Let $n$ be a positive integer, $n \equiv 0 \pmod 4$. The super edge-magic deficiency of $D_n$ is*

$$\mu_s(D_n) \le \frac{3n}{2} - 1.$$

*Proof.* Let $n$ be a positive integer, $n \equiv 0 \pmod 4$. Let us denote the isolated vertices of $G \cong D_n \cup (\frac{3n}{2} - 1)K_1$ by the symbols $v_1, v_2, \dots, v_{\frac{3n}{2}-1}$.

We define the vertex labeling $f$ of $G$ in the following way.

$$f(x_i) = \begin{cases} \dfrac{i+1}{2}, & \text{if } i = 1,3,\ldots,n-1, \\ \dfrac{9n}{4} - 1 + \dfrac{i}{2}, & \text{if } i = 2,4,\ldots,\dfrac{n}{2}, \\ \dfrac{5n}{4} + \dfrac{i}{2}, & \text{if } i = \dfrac{n}{2}+2, \dfrac{n}{2}+4,\ldots,n, \end{cases}$$

$$f(y_i) = \begin{cases} \dfrac{11n}{4}, & \text{if } i = 1, \\ n + \dfrac{i}{2}, & \text{if } i = 2,4,\ldots,n, \\ \dfrac{13n}{4} + \dfrac{i-3}{2} & \text{if } i = 3,5,\ldots,\dfrac{n}{2}+1, \\ \dfrac{9n}{4} + \dfrac{i-1}{2} & \text{if } i = \dfrac{n}{2}+3, \dfrac{n}{2}+5,\ldots n-1, \end{cases}$$

and the vertices $v_i$, $i = 1,2,\ldots,\frac{3n}{2}-1$ are labeled arbitrary with $\frac{3n}{2}-1$ unused numbers from the set $\{1,2,\ldots,\frac{7n}{2}-1\}$. It is not difficult to check that the vertices $v_i$, $i = 1,2,\ldots,\frac{3n}{2}-1$ are labeled with the numbers $\frac{n}{2}+1, \frac{n}{2}+2,\ldots,\frac{3n}{2}, \frac{7n}{4}+1, \frac{7n}{4}+2,\ldots,\frac{9n}{4}-1, \frac{5n}{2}, \frac{11n}{4}+1, \frac{11n}{4}+2,\ldots,\frac{13n}{4}-1$.

Next we prove that the edge sums are consecutive integers. Indeed, we have

$$f(x_1) + f(x_n) = \frac{1+1}{2} + \left(\frac{5n}{4} + \frac{n}{2}\right) = \frac{7n}{4} + 1,$$

$$f\left(x_{\frac{n}{2}+1}\right) + f\left(x_{\frac{n}{2}+2}\right) = \frac{\left(\frac{n}{2}+1\right)+1}{2} + \left(\frac{5n}{4} + \frac{\frac{n}{2}+2}{2}\right) = \frac{7n}{4} + 2,$$

$$f\left(x_{\frac{n}{2}+2}\right) + f\left(x_{\frac{n}{2}+3}\right) = \left(\frac{5n}{4} + \frac{\frac{n}{2}+2}{2}\right) + \frac{\left(\frac{n}{2}+1\right)+3}{2} = \frac{7n}{4} + 3,$$

$$\vdots$$

$$f(x_{n-1}) + f(x_n) = \frac{(n-1)+1}{2} + \left(\frac{5n}{4} + \frac{n}{2}\right) = \frac{9n}{4},$$

$$f(x_1) + f(x_2) = \frac{1+1}{2} + \left(\frac{9n}{4} - 1 + \frac{2}{2}\right) = \frac{9n}{4} + 1,$$

$$f(x_2) + f(x_3) = \left(\frac{9n}{4} - 1 + \frac{2}{2}\right) + \frac{3+1}{2} = \frac{9n}{4} + 2,$$

$$\vdots$$

$$f\left(x_{\frac{n}{2}}\right) + f\left(x_{\frac{n}{2}+1}\right) = \left(\frac{9n}{4} - 1 + \frac{\frac{n}{2}}{2}\right) + \frac{\left(\frac{n}{2}+1\right)+1}{2} = \frac{11n}{4},$$

$$\vdots$$

$$
\begin{aligned}
f(x_1)+f(y_1) &= \frac{1+1}{2}+\frac{11n}{4}=\frac{11n}{4}+1,\\
f\left(x_{\frac{n}{2}+2}\right)+f\left(y_{\frac{n}{2}+2}\right) &= \left(\frac{5n}{4}+\frac{\frac{n}{2}+2}{2}\right)+\left(n+\frac{\frac{n}{2}+2}{2}\right)=\frac{11n}{4}+2,\\
f\left(x_{\frac{n}{2}+3}\right)+f\left(y_{\frac{n}{2}+3}\right) &= \frac{\left(\frac{n}{2}+1\right)+3}{2}+\left(\frac{9n}{4}+\frac{\left(\frac{n}{2}+3\right)-1}{2}\right)=\frac{11n}{4}+3,\\
&\ \vdots\\
f(x_n)+f(y_n) &= \left(\frac{5n}{4}+\frac{n}{2}\right)+\left(n+\frac{n}{2}\right)=\frac{13n}{4},\\
f(x_2)+f(y_2) &= \left(\frac{9n}{4}-1+\frac{2}{2}\right)+\left(n+\frac{2}{2}\right)=\frac{13n}{4}+1,\\
f(x_3)+f(y_3) &= \frac{3+1}{2}+\left(\frac{13n}{4}+\frac{3-3}{2}\right)=\frac{13n}{4}+2,\\
&\ \vdots\\
f\left(x_{\frac{n}{2}+1}\right)+f\left(y_{\frac{n}{2}+1}\right) &= \frac{\left(\frac{n}{2}+1\right)+1}{2}+\left(\frac{13n}{4}+\frac{\left(\frac{n}{2}+1\right)-3}{2}\right)=\frac{15n}{4},\\
f(y_1)+f(y_2) &= \frac{11n}{4}+\left(n+\frac{2}{2}\right)=\frac{15n}{4}+1,\\
f\left(y_{\frac{n}{2}+2}\right)+f\left(y_{\frac{n}{2}+3}\right) &= \left(n+\frac{\frac{n}{2}+2}{2}\right)+\left(\frac{9n}{4}+\frac{\left(\frac{n}{2}+3\right)-1}{2}\right)=\frac{15n}{4}+2,\\
f\left(y_{\frac{n}{2}+3}\right)+f\left(y_{\frac{n}{2}+4}\right) &= \left(\frac{9n}{4}+\frac{\left(\frac{n}{2}+3\right)-1}{2}\right)+\left(n+\frac{\frac{n}{2}+4}{2}\right)=\frac{15n}{4}+3,\\
&\ \vdots\\
f(y_1)+f(y_n) &= \frac{11n}{4}+\left(n+\frac{n}{2}\right)=\frac{17n}{4},\\
f(y_2)+f(y_3) &= \left(n+\frac{2}{2}\right)+\left(\frac{13n}{4}+\frac{3-3}{2}\right)=\frac{17n}{4}+1,\\
f(y_3)+f(y_4) &= \left(\frac{13n}{4}+\frac{3-3}{2}\right)+\left(n+\frac{4}{2}\right)=\frac{17n}{4}+2,\\
&\ \vdots\\
f\left(y_{\frac{n}{2}+1}\right)+f\left(y_{\frac{n}{2}+2}\right) &= \left(\frac{13n}{4}+\frac{\frac{n}{2}+1-3}{2}\right)+\left(n+\frac{\frac{n}{2}+2}{2}\right)=\frac{19n}{4}.
\end{aligned}
$$

Hence the edge sums are the numbers $\frac{7n}{4}+1, \frac{7n}{4}+2, \dots, \frac{19n}{4}$. According to Lemma 1.1 the labeling $f$ can be extended to the super edge-magic total labeling of $G$ with the magic constant $\frac{33n}{4}$. $\square$

## 3. Exact Values

If $G$ has order $p$, the *corona* of $G$ with $H$, denoted by $G \odot H$, is the graph obtained by taking one copy of $G$ and $p$ copies of $H$ and joining the $i$th vertex of $G$ with an edge to every vertex in the $i$th copy of $H$.

Let us consider the Cartesian product $P_n \times K_2$, where $P_n$ is the path on $n$ vertices and $K_2$ is the complete graph of order 2. This graph is also called a ladder. In this section we deal with the super edge-magic deficiency of a ladder $P_n \times K_2$ with 1 pendant edge attached at each vertex of $P_n \times K_2$, i.e., the corona $(P_n \times K_2) \odot K_1$.

**Theorem 3.1.** *For every odd positive integer $n$ the graph $(P_n \times K_2) \odot K_1$ is super edge-magic total, i.e.,*

$$\mu_s((P_n \times K_2) \odot K_1) = 0.$$

*Proof.* Let $n$ be a positive odd integer. We denote the vertex set and the edge set of $G \cong (P_n \times K_2) \odot K_1$ as follows

$$V(G) = \{x_i, s_i, b_i, d_i : i = 1, 2, \dots, n\},$$
$$E(G) = \{x_i s_i, x_i b_i, s_i d_i : i = 1, 2, \dots, n\} \cup \{x_i x_{i+1}, s_i s_{i+1} : i = 1, 2, \dots, n-1\}.$$

The graph $G$ is of order $4n$ and of size $5n-2$.

For $n \geq 5$ we define the vertex labeling $f$ of $G$ such that

$$f(x_i) = \begin{cases} \dfrac{4n+1+i}{2}, & \text{if } i = 1, 3, \dots, n, \\ \dfrac{5n+1+i}{2}, & \text{if } i = 2, 4, \dots, n-1, \end{cases}$$

$$f(s_i) = \begin{cases} \dfrac{3n+i}{2}, & \text{if } i = 1, 3, \dots, n, \\ \dfrac{2n+i}{2}, & \text{if } i = 2, 4, \dots, n-1, \end{cases}$$

$$f(b_i) = \begin{cases} \dfrac{n-1}{2}, & \text{if } i = 1, \\ \dfrac{6n+i}{2}, & \text{if } i = 2, 4, \dots, n-1, \\ \dfrac{7n+i}{2}, & \text{if } i = 3, 5, \dots, n-2, \\ n, & \text{if } i = n, \end{cases}$$

$$f(d_i) = \begin{cases} n-1, & \text{if } i = 1, \\ \dfrac{7n+1}{2}, & \text{if } i = 2, \\ 4n, & \text{if } i = 3, \\ \dfrac{n-3+i}{2}, & \text{if } i = 4, 6, \dots, n-1, \\ \dfrac{i-3}{2}, & \text{if } i = 5, 7, \dots, n. \end{cases}$$

It is easy to see that the vertices of $G$ are labeled with the numbers $1, 2, \dots, 4n$ as the sets of vertex labels are the following ones.

$$\begin{aligned} \{f(s_i) : i = 1, 2, \dots, n\} =& \{n+1, n+2, \dots, 2n\}, \\ \{f(x_i) : i = 1, 2, \dots, n\} =& \{2n+1, 2n+2, \dots, 3n\}, \\ \{f(b_i) : i = 1, 2, \dots, n\} =& \Big\{\tfrac{n-1}{2}, n, 3n+1, 3n+2, \dots, \tfrac{7n-1}{2}, \tfrac{7n+3}{2}, \tfrac{7n+5}{2}, \dots, 4n-1\Big\}, \\ \{f(d_i) : i = 1, 2, \dots, n\} =& \Big\{1, 2, \dots, \tfrac{n-3}{2}, \tfrac{n+1}{2}, \tfrac{n+3}{2}, \dots, n-2, n-1, \tfrac{7n+1}{2}, 4n\Big\}. \end{aligned}$$

Thus $f$ is a bijection.

The edge sums under the labeling $f$ are consecutive integers from the set $\{\frac{3n+5}{2}, \frac{3n+7}{2}+1, \dots, \frac{13n-1}{2}\}$ since we have

$$\begin{aligned} f(s_4 d_4) &= \frac{2n+4}{2} + \frac{n-3+4}{2} = \frac{3n+5}{2}, \\ f(s_5 d_5) &= \frac{3n+5}{2} + \frac{5-3}{2} = \frac{3n+7}{2}, \\ &\vdots \\ f(s_n d_n) &= \frac{3n+n}{2} + \frac{n-3}{2} = \frac{5n-3}{2}, \\ f(s_1 d_1) &= \frac{3n+1}{2} + (n-1) = \frac{5n-1}{2}, \\ f(x_1 b_1) &= \frac{4n+1+1}{2} + \frac{n-1}{2} = \frac{5n+1}{2}, \\ f(s_1 s_2) &= \frac{3n+1}{2} + \frac{2n+2}{2} = \frac{5n+3}{2}, \\ f(s_2 s_3) &= \frac{2n+2}{2} + \frac{3n+3}{2} = \frac{5n+5}{2}, \\ &\vdots \\ f(s_{n-1} s_n) &= \frac{2n+(n-1)}{2} + \frac{3n+n}{2} = \frac{7n-1}{2}, \\ f(x_n b_n) &= \frac{4n+1+n}{2} + n = \frac{7n+1}{2}, \end{aligned}$$

$$\begin{aligned}
f(s_1x_1) &= \frac{3n+1}{2} + \frac{4n+1+1}{2} = \frac{7n+3}{2},\\
f(s_2x_2) &= \frac{2n+2}{2} + \frac{5n+1+2}{2} = \frac{7n+5}{2},\\
&\vdots\\
f(s_nx_n) &= \frac{3n+n}{2} + \frac{4n+1+n}{2} = \frac{9n+1}{2},\\
f(s_2d_2) &= \frac{2n+2}{2} + \frac{7n+1}{2} = \frac{9n+3}{2},\\
f(x_1x_2) &= \frac{4n+1+1}{2} + \frac{5n+1+2}{2} = \frac{9n+5}{2},\\
f(x_2x_3) &= \frac{5n+1+2}{2} + \frac{4n+1+3}{2} = \frac{9n+7}{2},\\
&\vdots\\
f(x_{n-1}x_n) &= \frac{5n+1+(n-1)}{2} + \frac{4n+1+n}{2} = \frac{11n+1}{2},\\
f(s_3d_3) &= \frac{3n+3}{2} + 4n = \frac{11n+3}{2},\\
f(x_2b_2) &= \frac{5n+1+2}{2} + \frac{6n+2}{2} = \frac{11n+5}{2},\\
f(x_3b_3) &= \frac{4n+1+3}{2} + \frac{7n+3}{2} = \frac{11n+7}{2},\\
&\vdots\\
f(x_{n-1}b_{n-1}) &= \frac{5n+1+(n-1)}{2} + \frac{6n+(n-1)}{2} = \frac{13n-1}{2}.
\end{aligned}$$

According to Lemma 1.1 the labeling $f$ can be extended to the super edge-magic total labeling of $G \cong (P_n \times K_2) \odot K_1$, for $n \geq 5$ with the magic constant $\frac{21n+1}{2}$.

On Figures 1 and 2 are illustrated super edge-magic total labelings of $(P_1 \times K_2) \odot K_1 \cong P_4$ and $(P_3 \times K_2) \odot K_1$, respectively.

This concludes the proof.

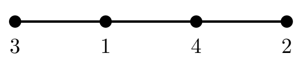


FIGURE 1. A super edge-magic total labeling of $(P_1 \times K_2) \odot K_1 \cong P_4$.

□

## 4. CONCLUSION

In this paper we have dealt with the problem of finding super edge-magic deficiency of graphs. We were trying to find the exact values of super edge-magic deficiencies

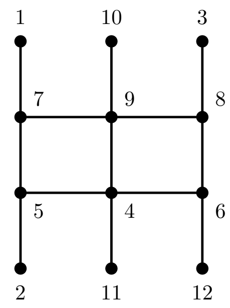


FIGURE 2. A super edge-magic total labeling of $(P_3 \times K_2) \odot K_1$.

of some graphs or to find the upper bound of this parameter for several classes of graphs.

In Theorem 2.3 we described the upper bound of the super edge-magic deficiency of prism $D_n$ for $n \equiv 0 \pmod 4$. As it is known, see [4], that for $n$ odd the prism $D_n$ is super edge-magic. To conclude the problem of finding the super edge-magic deficiency of prism $D_n$ also for $n$ even, for further investigation we state the following open problem.

**Open Problem.** Find the super edge-magic deficiency of prism $D_n$, for $n \equiv 2 \pmod 4$.

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## References

[1] B. D. Acharya and S. M. Hegde, *Strongly indexable graphs*, Discrete Math. **93** (1991), 123–129.
[2] A. Q. Baig, E. T. Baskoro and A. Semaničová–Feňovčíková, *On the super edge-magic deficiency of a star forest*, Ars Combin. **115** (2014), 3–12.
[3] H. Enomoto, A. S. Lladó, T. Nakamigawa and G. Ringel, *Super edge-magic graphs*, SUT J. Math. **34** (1998), 105–109.
[4] R. M. Figueroa, R. Ichishima and F. A. Muntaner-Batle, *The place of super edge-magic labeling among other classes of labeling*, Discrete Math. **231** (2001), 153–168.
[5] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, *On the super edge magic deficiency of graphs*, Electron. Notes Discrete Math. **11** (2002), 299–314.
[6] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, *Some new results on the super edge-magic deficiency of graphs*, J. Combin. Math. Combin. Comput. **55** (2005), 17–31.
[7] R. M. Figueroa-Centeno, R. Ichishima and F. A. Muntaner-Batle, *On the super edge-magic deficiency of graphs*, Ars Combin. **78** (2006), 33–45.
[8] A. Kotzig and A. Rosa, *Magic valuaton of finite graphs*, Canad. Math. Bull. **13**(4) (1970), 451–461.

[9] A. A. G. Ngurah and E. T. Baskoro, *On magic and antimagic total labelings of generalized Petersen graph*, Util. Math. **63** (2003), 97–107.

[10] A. A. G. Ngurah, R. Simanjuntak and E. T. Baskoro, *On the super edge-magic deficiencies of graphs,* Australas. J. Combin. **40** (2008), 3–14.

[11] G. Santhosh and G. Singh, *A note on the super edge-magic deficiency of graphs*, Far East J. Appl. Math. **24** (2006), 233–242.

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# LAPLACE TRANSFORM AND HOMOTOPY PERTURBATION METHODS FOR SOLVING THE PSEUDOHYPERBOLIC INTEGRODIFFERENTIAL PROBLEMS WITH PURELY INTEGRAL CONDITIONS

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Abstract. In this paper we defined and investigated the various properties of a class of pseudohyperbolic equation defined on purely integral (nonlocal) conditions. We proved the uniqueness and the existence of the solution using energy inequality (A priori estimates). We found a semi analytical solution using the Laplace transform and Stehfest algorithm method. Next, we used another method called the Homotopy perturbation. Finally, we give some examples for illustration.

## 1. Introduction

Some problems of modern physics and technology can be effectively described in terms of nonlocal (integral) conditions. These nonlocal conditions raise mainly when the data on the boundary cannot be measured directly. The presence of integral terms in the boundary conditions can, greatly, complicate the application of standard numerical techniques such as finite difference procedures, finite elements methods, spectral techniques,..., endless. So far, not much seems to be done for obtaining an explicit solution of heat and wave equations. However, the solvability of these equations has been theoretically studied in terms of existence and uniqueness of a solution. There are two tools used in this paper. The first one is **Laplace transformation** and the use of its inversed transformation to obtain the numerical solution. The Laplace transformation method has been used to approximate the solution of different classes of linear partial differential equations. **Suying** et al [25], established

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a numerical method based on the Laplace transformation for solving initial problem nonlinear dynamic differential equations. The main difficulty in using the Laplace transformation method consists in finding its inverse, because the inverse of transformation is very complex in few situations. To overcome this difficulty, there are many numerical techniques available to invert Laplace transformation. In this work, we focus exclusively on the **STEHFEST** algorithm [26] in order to efficiently and accurately invert the Laplace transformation (which cannot be done analytically). The application of Lapalce transformation on the equations reduces the problem to a second order inhomogeneous ordinary differential equations with nonlocal conditions. The reduced problem can be solved by the method of variation of parameters. After discretization, the numerical method for inverting the Laplace transformation is used to get an approximation of solution. The second tool is the homotopy perturbation method (**HPM**), which has been developed by scientists and engineers, in nonlinear problems. This method reduces difficulties of the problem under study into a simple problem which is easy to solve. Most perturbation methods assume the existence of a small parameter, but most nonlinear problems have no small parameter at all. Recently, the application of homotopy perturbation theory has appeared in the work of many scientists [2, 3, 7–10, 12–18, 23], which has become a powerful mathematical tool. Recently, **S. Abbasbandy** [2] applied this method on functional integral equations. The aim of this paper is to establish the existence, uniqueness and the continuous dependence of the data with the solution of second order pseudohyperbolic integrodifferential equations with nonlocal conditions [5, 6]. The proofs are based on a priori estimates and a combination of **Laplace transformation** technique with **Stehfest algorithm**. Furthermore, some examples are given to compare between the approximate and the exact solutions. Later, another semi-analytical technique called **Homotopy perturbation method** is used. Some testing examples are given to show the efficiency of this method.

## 2. Statement of the Problem

In the rectangular domain $Q = \Omega \times I = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, we consider a pseudohyperbolic integrodifferential equation:

$$\mathcal{L}v = \frac{\partial^2 v}{\partial t^2} - \alpha\frac{\partial^2 v}{\partial x^2} - \beta\frac{\partial^3 v}{\partial t \partial x^2} + \gamma v - \int_0^t a(t-s)v(x,s)ds = g(x,t), \tag{2.1}$$

subject to the initial conditions

$$\ell v = v(x,0) = \Phi(x), \quad 0 < x < 1, \tag{2.2}$$

$$qv = v_t(x,0) = \Psi(x), \quad 0 < x < 1, \tag{2.3}$$

and the purely integral conditions

$$\int_0^1 v(x,t)dx = n(t), \quad 0 < t \leq T, \tag{2.4}$$

$$\int_0^1 xv(x,t)dx = m(t), \quad 0 < t \leq T, \tag{2.5}$$

where $g, \Phi, \Psi, a, n,$ and $m$ are known functions, $\alpha, \beta, \gamma$ and $T$ are known positive constants.

## 3. Reformulation of the Problem

Since the integral boundary conditions are inhomogeneous, it is convenient to convert the problem (2.1)–(2.5) to an equivalent problem with homogeneous integral conditions. For this, we introduce a new function $u(x,t)$ as follow:

$$v(x,t) = u(x,t) + U(x,t), \tag{3.1}$$

where

$$U(x,t) = (-6x+4)n(t) + (12x-6)m(t). \tag{3.2}$$

Problem (2.1)–(2.5) with inhomogeneous integral conditions (2.4)–(2.5) can be equivalently reduced to the problem of finding a function $u$ satisfying:

$$\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial^3 u}{\partial t \partial x^2} + \gamma u - \int_0^t a(t-s)u(x,s)ds = f(x,t), \tag{3.3}$$

with the initial conditions

$$\ell u = u(x,0) = \varphi(x), \quad 0 < x < 1, \tag{3.4}$$

$$qu = u_t(x,0) = \psi(x), \quad 0 < x < 1, \tag{3.5}$$

and the purely nonlocal conditions

$$\int_0^1 u(x,t)\, dx = 0, \quad 0 < t \leq T, \tag{3.6}$$

$$\int_0^1 xu(x,t)\, dx = 0, \quad 0 < t \leq T, \tag{3.7}$$

where

$$f(x,t) = g(x,t) - \mathcal{L}U(x,t) \tag{3.8}$$

and

$$\varphi(x) = \Phi(x) - \ell U(x,t), \tag{3.9}$$

$$\psi(x) = \Psi(x) - qU(x,t). \tag{3.10}$$

Hence, instead of looking for $v$, we simply look for $u$. The solution of problem (2.1)–(2.5) will be obtained by the relations (3.1)–(3.2).

## 4. A Priori Estimates and its Consequences

The solution of the problem (3.3)–(3.7) can be considered as a solution of the problem in the operational form:

$$Lu = \mathscr{F},$$

where $L = (\mathcal{L}, \ell, q)$ is considered from $B$ to $F$, where $B$ is the Banach space of the functions $u \in L^2(Q)$, whose norm is:

$$\|u\|_B = \left( \sup_{0\leqslant\tau\leqslant T} \int_0^1 \left( \left\| \Im_x \frac{\partial u}{\partial t}(x,\tau) \right\|^2 + \|\Im_x u(x,\tau)\|^2 \right) dx \right)^{\frac{1}{2}},$$

which is finite, and $F$ is the Hilbert space consisting of all the elements $\mathscr{F} = (f, \varphi, \psi)$ whose norm is:

$$\|\mathscr{F}\|_F = \left( \int_{Q_\tau} \|f\|^2 \, dxdt + \int_0^1 \left( \|\psi(x)\|^2 + \|\varphi(x)\|^2 \right) dx \right)^{\frac{1}{2}},$$

which is finite.

The domain $D(L)$ of the operator $L$ is the set of all the functions $u$ such that $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial t^2}, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial t \partial x^2} \in L^2(Q)$ and $u$ satisfies (3.6) as well as (3.7).

First an a priori estimates is established. Then, the uniqueness and continuous dependence of the solution with respect to the data are immediately conveyed.

**Theorem 4.1.** *If $u(x,t)$ is a solution of problem* (3.3)–(3.7) *and $|a(t)| \leq a_1$, $0 \leq t \leq T$ and $\beta$ satisfying the condition $\beta \geqslant \frac{T^4 a_1 \varepsilon_1^2 + a_1}{8\varepsilon_1} + \frac{1}{8\varepsilon_2}$ , $f \in C\left(\overline{Q}\right)$, then*

$$\|u\|_B \leq C \|\mathscr{F}\|_F,$$

*where $C$ is a positive constant independent of $u$, $u \in D(L)$ and*

$$C = \left( \frac{\max\{\frac{1}{2}, \varepsilon_2, \frac{1}{2}\gamma + \alpha\}}{\min\{1, \gamma + 2\alpha\}} \right)^{\frac{1}{2}}.$$

*Proof.* We put $\Im_x u = \int_0^x u(\xi,t)\, d\xi$ and $\Im_x^2 u = \int_0^x \int_0^\eta u(\xi,t)\, d\xi d\eta$. Multiplying the equation (3.3) by the integro differential operator $Mu = -\Im_x^2 \frac{\partial u}{\partial t}$ and integrating on the subdomain $Q_\tau = (0,1) \times (0,\tau)$, where $0 \leq \tau \leq T$, we obtain:

(4.1)
$$\begin{aligned} &- \int_{Q_\tau} \frac{\partial^2 u}{\partial t^2} \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt + \alpha \int_{Q_\tau} \frac{\partial^2 u}{\partial x^2} \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt + \beta \int_{Q_\tau} \frac{\partial^3 u}{\partial t \partial x^2} \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt \\ &+ \beta \int_{Q_\tau} \frac{\partial^3 u}{\partial t \partial x^2} \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt - \gamma \int_{Q_\tau} u \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt \\ = &- \int_{Q_\tau} \left( \int_0^t a(t-s) u(x,s) ds \right) \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt - \int_{Q_\tau} f \cdot \Im_x^2 \left( \frac{\partial u}{\partial t} \right) dxdt. \end{aligned}$$

The integration by parts of each term on the left-hand side of the equation (4.1) gives:

$$- \int_{Q_\tau} \frac{\partial^2 u}{\partial t^2} \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt = \frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial u}{\partial t}(x,\tau) \right\|^2 dx - \frac{1}{2} \int_0^1 \|\Im_x \psi(x)\|^2 dx, \tag{4.2}$$

$$\alpha \int_{Q_\tau} \frac{\partial^2 u}{\partial x^2} \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt = \frac{1}{2}\alpha \int_0^1 \|u(x,\tau)\|^2 dx - \frac{1}{2}\alpha \int_0^1 \|\varphi(x)\|^2 dx, \tag{4.3}$$

$$\beta \int_{Q_\tau} \frac{\partial^3 u}{\partial t \partial x^2} \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt = \beta \int_{Q_\tau} \left\| \frac{\partial u}{\partial t} \right\|^2 dxdt, \tag{4.4}$$

$$-\gamma \int_{Q_\tau} u \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt = \frac{1}{2}\gamma \int_0^1 \|\Im_x u(x,\tau)\|^2 dx - \frac{1}{2}\gamma \int_0^1 \|\Im_x \varphi(x)\|^2 dx. \tag{4.5}$$

The substitution of $(4.2), (4.3), (4.4)$ and $(4.5)$ into $(4.1)$ gives:

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left\| \Im_x \frac{\partial u}{\partial t}(x,\tau) \right\|^2 dx + \frac{1}{2}\gamma \int_0^1 \|\Im_x u(x,\tau)\|^2 dx \\ &+ \frac{1}{2}\alpha \int_0^1 \|u(x,\tau)\|^2 dx + \beta \int_{Q_\tau} \left\| \frac{\partial u}{\partial t} \right\|^2 dxdt \\ =&\frac{1}{2}\int_0^1 \|\Im_x \psi(x)\|^2 dx + \frac{1}{2}\gamma \int_0^1 \|\Im_x \varphi(x)\|^2 dx + \frac{1}{2}\alpha \int_0^1 \|\varphi(x)\|^2 dx \\ &- \int_{Q_\tau} f \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt - \int_{Q_\tau} (\int_0^t a(t-s)u(x,s)ds) \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt. \end{aligned} \tag{4.6}$$

Using Poincare's inequality type

$$\begin{aligned} \int_0^1 \left\| \Im_x^2 u(x,\tau) \right\|^2 dx \leq& \frac{1}{2}\int_0^1 \Im_x \|u(x,\tau)\|^2 dx, \\ \int_0^1 \|\Im_x u(x,\tau)\|^2 dx \leq& \frac{1}{2}\int_0^1 \|u(x,\tau)\|^2 dx, \\ \int_{Q_\tau} \left\| \int_0^t u(x,s)ds \right\|^2 dxdt \leq& \frac{T^2}{2}\int_{Q_\tau} \|u\|^2 dxdt, \end{aligned}$$

we obtain:

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left\| \Im_x \frac{\partial u}{\partial t}(x,\tau) \right\|^2 dx + \left(\frac{1}{2}\gamma + \alpha\right) \int_0^1 \Im_x \|u(x,\tau)\|^2 dx + \beta \int_{Q_\tau} \left\| \frac{\partial u}{\partial t} \right\|^2 dxdt \\ \leq& \frac{1}{4}\int_0^1 \|\psi(x)\|^2 dx + \left(\frac{1}{4}\gamma + \frac{1}{2}\alpha\right) \int_0^1 \|\varphi(x)\|^2 dx - \int_{Q_\tau} f \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt \\ &- \int_{Q_\tau} \left(\int_0^t a(t-s)u(x,s)ds\right) \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt. \end{aligned} \tag{4.7}$$

Using the Cauchy inequality with $\varepsilon$, the right-hand side of (4.1) is bounded

$$\begin{aligned} &\int_{Q_\tau} f \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt - \int_{Q_\tau} \left(\int_0^t a(t-s)u(x,s)ds\right) \cdot \Im_x^2 \frac{\partial u}{\partial t} dxdt \\ \leqslant& \frac{\varepsilon_2}{2}\int_{Q_\tau} \|f\|^2 dxdt + \frac{1}{8\varepsilon_2}\int_{Q_\tau} \left\| \frac{\partial u}{\partial t} \right\|^2 dxdt + \frac{a_1 \varepsilon_1}{2}\int_{Q_\tau} \left\| \int_0^t u(x,s)ds \right\|^2 dxdt \\ &+ \frac{a_1}{2\varepsilon_1}\int_{Q_\tau} \left\| \Im_x^2 \frac{\partial u}{\partial t} \right\|^2 dxdt, \end{aligned}$$

we get

$$\int_{Q_\tau} f\cdot\Im_x^2\frac{\partial u}{\partial t}dxdt-\int_{Q_\tau}\left(\int_0^t a(t-s)u(x,s)ds\right)\cdot\Im_x^2\frac{\partial u}{\partial t}dxdt$$

$$\leq\frac{\varepsilon_2}{2}\int_{Q_\tau}\|f\|^2dxdt+\left(\frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1}+\frac{1}{8\varepsilon_2}\right)\int_{Q_\tau}\left\|\frac{\partial u}{\partial t}\right\|^2dxdt, \tag{4.8}$$

using (4.8) into (4.7) we obtain:

$$\begin{aligned}&\int_0^1\left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2dx+(\gamma+2\alpha)\int_0^1\|\Im_x u(x,\tau)\|^2dx\\&+2\left(\beta-\frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1}-\frac{1}{8\varepsilon_2}\right)\int_{Q_\tau}\left\|\frac{\partial u}{\partial t}\right\|^2dxdt\\&\leq\varepsilon_2\int_{Q_\tau}\|f\|^2dxdt+\frac12\int_0^1\|\psi(x)\|^2dx+\left(\frac12\gamma+\alpha\right)\int_0^1\|\varphi(x)\|^2dx,\end{aligned}$$

if $\beta$ satisfies the condition $\beta\geqslant\frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1}+\frac{1}{8\varepsilon_2}$ we obtain:

$$\begin{aligned}&\int_0^1\left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2dx+(\gamma+2\alpha)\int_0^1\|\Im_x u(x,\tau)\|^2dx\\&\leq\varepsilon_2\int_{Q_\tau}\|f\|^2dxdt+\frac12\int_0^1\|\psi(x)\|^2dx+\left(\frac12\gamma+\alpha\right)\int_0^1\|\varphi(x)\|^2dx.\end{aligned}\tag{4.9}$$

Since the right-hand side of (4.9) is independent of $\tau$, we take the supremum with respect to $\tau$ from 0 to $T$ in the left-hand side we obtain:

$$\begin{aligned}&\sup_{0\leqslant\tau\leqslant T}\left\{\int_0^1\left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2dx+\int_0^1\|\Im_x u(x,\tau)\|^2dx\right\}\\&\leqslant C\left(\int_{Q_\tau}\|f\|^2dxdt+\int_0^1\|\psi(x)\|dx+\int_0^1\|\varphi(x)\|^2dx\right).\end{aligned}$$

We thus obtain inequality

$$\|u\|_B\leq C\|\mathscr{F}\|_F,$$

with

$$C=\left(\frac{\max\{\frac12,\varepsilon_2,\frac12\gamma+\alpha\}}{\min\{1,\gamma+2\alpha\}}\right)^{\frac12}.$$

□

**Corollary 4.1.** *If problem* (3.3)–(3.7) *has a solution, then this solution is unique and depends continuously on* $(f,\varphi,\psi)$.

## 5. The Existence of Solution

**Theorem 5.1.** *If* $\beta$ *satisfies the condition* $\beta\geq\frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1}+\frac{1}{8\varepsilon_2}$, *then the problem* (3.3)–(3.7) *admits a unique strong solution* $u=\overline{L}^{-1}(f,\varphi,\psi)=\overline{L^{-1}}(f,\varphi,\psi)$.

*Proof.* To prove that the problem (3.3)–(3.7) admits a strong solution for any arbitrary $(f, \varphi, \psi) \in F$, it is sufficient to prove that $R(L)$ is dense in $F$, first of all, for the case where $L$ is reduced to $L_0 = (\mathcal{L}, \ell, q)$ where its domain is $D(L_0) = \{u/u \in D(L) : \ell u = 0 \text{ and } qu = 0\}$. For this purpose, we demonstrate the following proposition. □

**Proposition 5.1.** *Under the conditions of Theorem* 5.1*, for* $\omega \in L^2(Q)$ *and for all* $u \in D(L_0)$*, we have*

$$\int_Q \mathcal{L}u \cdot \omega \, dxdt = 0, \tag{5.1}$$

*then* $\omega$ *renders null almost everywhere in* $Q$.

*Proof.* The equality (5.1) can be written as follows

$$\begin{aligned}\int_{Q_\tau} \frac{\partial^2 u}{\partial t^2} \cdot \omega dxdt =& \alpha \int_{Q_\tau} \frac{\partial^2 u}{\partial x^2} \cdot \omega dxdt + \beta \int_{Q_\tau} \frac{\partial^3 u}{\partial t \partial x^2} \cdot \omega dxdt - \gamma \int_{Q_\tau} u \cdot \omega dxdt \\ &+ \int_{Q_\tau} \left( \int_0^t a(t-s) u(x,s) ds \right) \cdot \omega dxdt.\end{aligned} \tag{5.2}$$

From the equality (5.2), we give the function $\omega$ in terms of $u$ as follows:

$$\omega = -\Im_x^2 \frac{\partial u}{\partial t}. \tag{5.3}$$

By substituting $\omega$ in (5.2) by its representation (5.3), we obtain:

$$\begin{aligned}&\int_{Q_\tau} \frac{\partial^2 u}{\partial t^2} \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt = \alpha \int_{Q_\tau} \frac{\partial^2 u}{\partial x^2} \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt \\ &+ \beta \int_{Q_\tau} \frac{\partial^3 u}{\partial t \partial x^2} \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt - \gamma \int_{Q_\tau} u \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt \\ &+ \int_{Q_\tau} \left( \int_0^t a(t-s) u(x,s) ds \right) \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt,\end{aligned} \tag{5.4}$$

Integrating by parts and taking into account conditions (3.6) and (3.7), we obtain:

$$\begin{aligned}\int_{Q_\tau} \frac{\partial^2 u}{\partial t^2} \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt =& \frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial u}{\partial t}(x,\tau) \right\|^2 dx - \frac{1}{2} \int_0^1 \left\| \Im_x q u \right\|^2 dx \\ =& \frac{1}{2} \int_0^1 \left\| \Im_x \frac{\partial u}{\partial t}(x,\tau) \right\|^2 dx,\end{aligned} \tag{5.5}$$

$$\begin{aligned}\alpha \int_{Q_\tau} \frac{\partial^2 u}{\partial x^2} \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt =& -\frac{1}{2} \alpha \int_0^1 \left\| u(x,\tau) \right\|^2 dx + \frac{1}{2} \alpha \int_0^1 \left\| \ell u \right\|^2 dx \\ =& -\frac{1}{2} \alpha \int_0^1 \left\| u(x,\tau) \right\|^2 dx,\end{aligned} \tag{5.6}$$

$$\beta \int_{Q_\tau} \frac{\partial^3 u}{\partial t \partial x^2} \cdot \left( -\Im_x^2 \frac{\partial u}{\partial t} \right) dxdt = -\beta \int_{Q_\tau} \left\| \frac{\partial u}{\partial t} \right\|^2 dxdt, \tag{5.7}$$

$$-\gamma\int_{Q_\tau} u\cdot\left(-\Im_x^2\frac{\partial u}{\partial t}\right)dxdt = -\frac{1}{2}\gamma\int_0^1 \|\Im_x u(x,\tau)\|^2\,dx + \frac{1}{2}\gamma\int_0^1 \|\Im_x \ell u\|^2\,dx$$

$$= -\frac{1}{2}\gamma\int_0^1 \|\Im_x u(x,\tau)\|^2\,dx. \tag{5.8}$$

By substituting (5.5), (5.6), (5.7) and (5.8) into (5.4) we obtain:

$$\begin{aligned}\frac{1}{2}\int_0^1 \left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2 dx + \beta\int_{Q_\tau}\left\|\frac{\partial u}{\partial t}\right\|^2 dxdt \\ +\frac{1}{2}\gamma\int_0^1 \|\Im_x u(x,\tau)\|^2\,dx + \frac{1}{2}\alpha\int_0^1\|u(x,\tau)\|^2\,dx \\ \le \int_{Q_\tau}\left(\int_0^t a(t-s)u(x,s)ds\right)\cdot\left(-\Im_x^2\frac{\partial u}{\partial t}\right)dxdt.\end{aligned}\tag{5.9}$$

By the use of Cauchy inequality with $\varepsilon$, the right-hand side of (5.9) is bounded

$$\int_{Q_\tau}\left(\int_0^t a(t-s)u(x,s)ds\right)\cdot\left(-\Im_x^2\frac{\partial u}{\partial t}\right)dxdt \le \frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1}\int_{Q_\tau}\left\|\frac{\partial u}{\partial t}\right\|^2 dxdt. \tag{5.10}$$

By using (5.10) into (5.9) we obtain:

$$\begin{aligned}\frac{1}{2}\int_0^1 \left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2 dx + \left(\beta - \frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1}\right)\int_{Q_\tau}\left\|\frac{\partial u}{\partial t}\right\|^2 dxdt \\ +\frac{1}{2}\gamma\int_0^1 \|\Im_x u(x,\tau)\|^2\,dx + \frac{1}{2}\alpha\int_0^1\|u(x,\tau)\|^2\,dx \le 0,\end{aligned}$$

if $\beta$ satisfies the condition $\beta \ge \frac{T^4a_1\varepsilon_1^2+a_1}{8\varepsilon_1} + \frac{1}{8\varepsilon_2}$ we obtain:

$$\frac{1}{2}\int_0^1 \left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2 dx + \frac{1}{2}\gamma\int_0^1 \|\Im_x u(x,\tau)\|^2\,dx + \frac{1}{2}\alpha\int_0^1\|u(x,\tau)\|^2\,dx \le 0. \tag{5.11}$$

Since the right-hand side of (5.11) is independent from $\tau$. We take the supremum, with respect to $\tau$ from 0 to $T$, in the left-hand side we obtain:

$$\sup_{0\leqslant\tau\leqslant T}\left\{\frac{1}{2}\int_0^1 \left\|\Im_x\frac{\partial u}{\partial t}(x,\tau)\right\|^2 dx + \frac{1}{2}\gamma\int_0^1 \|\Im_x u(x,\tau)\|^2\,dx + \frac{1}{2}\alpha\int_0^1\|u(x,\tau)\|^2\,dx\right\} \le 0.$$

We get $u = 0$. Now, we put $u = 0$ in (5.3), which gives $\omega \equiv 0$ in $L^2(Q)$. □

## 6. Laplace Transform Method and Stehfest Algorithm

6.1. **Laplace transform method.** Laplace transform is an efficient method for solving many differential equations and partial differential equations. The main difficulty with Laplace transform method, is inverting the domain of Laplace solution into the real domain (see [4, 20–22]). In this section we shall apply the Laplace transform technique to find solutions of partial differential equations.

Supposing that $v(x,t)$ is defined and is of exponential order for $t \ge 0$, i.e., it exists $A$, $\gamma > 0$ and $t_0 > 0$ such that $|v(x,t)| \le A\exp(\gamma t)$ for $t \ge t_0$. Then the Laplace transform $V(x,s)$, exists and is given by

$$V(x,s) = \mathscr{L}\{v(x,t) : t \to s\} = \int_0^\infty v(x,t)\exp(-st)\,dt,$$

where $s$ is a positive real parameter. Taking the Laplace transform on both sides of (2.1), we get

$$-(\alpha+s\beta)\frac{\partial^2 V}{\partial x^2}(x,s)+(s^2+\gamma-A(s))V(x,s) = G(x,s)+\psi(x)+s\varphi(x)-\beta\varphi''(x), \tag{6.1}$$

where $G(x,s) = \mathscr{L}\{g(x,t)\,; t \to s\}$ and $A(s) = \mathscr{L}\{a(t)\,; t \to s\}$.

Similarly, we have

$$\int_0^1 V(x,s)\,dx = N(s), \tag{6.2}$$

$$\int_0^1 xV(x,s)\,dx = M(s), \tag{6.3}$$

where

$$\begin{aligned} N(s) &= \mathscr{L}\{n(t) : t \to s\}, \\ M(s) &= \mathscr{L}\{m(t) : t \to s\}. \end{aligned}$$

Thus, the considered equation is reduced into a boundary-value problem governed by a second-order inhomogeneous ordinary differential equation.

Now, we distinguish the following cases.

**Case 1.** If $A(s) = s^2 + \gamma$, we obtain a general solution of (6.1) as follows

$$\begin{aligned} V(x,s) =& \frac{-1}{(\alpha+s\beta)}\int_0^x (x-\tau)\left[G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)\right]d\tau \\ &+ C_1(s)x + C_2(s), \end{aligned} \tag{6.4}$$

where $C_1$ and $C_2$ are arbitrary functions of $s$. By substituting (6.4) into (6.2) and (6.3), we get

$$\begin{aligned} C_1 =& \frac{1}{(\alpha+s\beta)}\int_0^1 \left(G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)\right)\left(2\tau^3-3\tau^2+1\right)d\tau \\ &+ 12M(s) - 6N(s), \end{aligned} \tag{6.5}$$

$$\begin{aligned} C_2 =& \frac{-2}{(\alpha+s\beta)}\int_0^1 \left(G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)\right)\left(-\tau^3+2\tau^2-\tau\right)d\tau \\ &- 6M(s) + 4N(s). \end{aligned} \tag{6.6}$$

In general it is impossible to evaluate the integrals in (6.4)–(6.6) exactly. So one may have to resort to numerical integration in order to compute them, for example, the Gauss's formula (25.4.30) given in Abramowitz and Stegun [1] may be employed to calculate these integrals numerically, we have the following approximations for the integrals:

(6.7)
$$\int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))\left(2\tau^3-3\tau^2+1\right)d\tau$$
$$\simeq\frac{1}{2}\sum_{i=1}^{N} w_i\left[G\left(\frac{1}{2}[x_i+1];s\right)+\psi\left(\frac{1}{2}[x_i+1]\right)+s\varphi\left(\frac{1}{2}[x_i+1]\right)-\beta\varphi''\left(\frac{1}{2}[x_i+1]\right)\right]$$
$$\times\frac{1}{4}(x_i-1)^2(x_i+2),$$

(6.8)
$$\int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))\left(-\tau^3+2\tau^2-\tau\right)d\tau$$
$$\simeq\frac{1}{2}\sum_{i=1}^{N} w_i\left[G\left(\frac{1}{2}[x_i+1];s\right)+\psi\left(\frac{1}{2}[x_i+1]\right)+s\varphi\left(\frac{1}{2}[x_i+1]\right)-\beta\varphi''\left(\frac{1}{2}[x_i+1]\right)\right]$$
$$\times\frac{1}{8}\left[-x_i^3+x_i^2+x_i-5\right],$$

$$\int_0^x (x-\tau)\left[G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)\right]d\tau$$

(6.9)
$$\simeq\frac{x}{2}\sum_{i=1}^{N} w_i\left[G\left(\frac{x}{2}[x_i+1];s\right)+\psi\left(\frac{x}{2}[x_i+1]\right)+s\varphi\left(\frac{x}{2}[x_i+1]\right)-\beta\varphi''\left(\frac{x}{2}[x_i+1]\right)\right].$$

**Case 2.** If $A(s)<s^2+\gamma$, we obtain a general solution of (6.1) as follows
$$V(x,s)=\frac{-1}{R(\alpha+s\beta)}\int_0^x (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))\sinh R(x-\tau)d\tau$$
$$+C_1(s)e^{Rx}+C_2(s)e^{-Rx}, \tag{6.10}$$
where
$$R=\sqrt{\frac{s^2+\gamma-A(s)}{\alpha+s\beta}}$$
and $C_1$, $C_2$ are arbitrary functions of $s$. By substituting (6.10) into (6.2) and (6.3), we get
$$(e^R-1)C_1+(1-e^{-R})C_2$$
$$=\frac{1}{R(\alpha+s\beta)}\int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))(\cosh R(1-\tau)-1)d\tau+RN(s)$$
$$\times\left[(R-1)e^R+1\right]C_1+\left[-(R+1)e^R+1\right]C_2$$
$$=\frac{1}{R(\alpha+s\beta)}\int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))$$
$$\times\left[R(\cosh R(1-\tau)-\tau)-\sinh R(1-\tau)\right]d\tau+R^2M(s),$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix},$$

and

$$\begin{aligned}
a_{11}(s) =& (e^R - 1),\\
a_{12}(s) =& (1 - e^{-R}),\\
a_{21}(s) =& (R-1)e^R + 1,\\
a_{22}(s) =& -(R+1)e^R + 1,\\
b_1(s) =& \frac{1}{R(\alpha + s\beta)} \int_0^1 (G(\tau, s) + \psi(\tau) + s\varphi(\tau) - \beta\varphi''(\tau))(\cosh R(1-\tau) - 1)d\tau\\
&+ RN(s),\\
b_2(s) =& \frac{1}{R(\alpha + s\beta)} \int_0^1 (G(\tau, s) + \psi(\tau) + s\varphi(\tau) - \beta\varphi''(\tau))\\
&\times \left[R(\cosh R(1-\tau) - \tau) - \sinh R(1-\tau)\right] d\tau + R^2 M(s).
\end{aligned}$$

We have the following approximations for the integrals:

(6.11)

$$\begin{aligned}
&\int_0^1 (G(\tau, s) + \psi(\tau) + s\varphi(\tau) - \beta\varphi''(\tau)) (\cosh R(1-\tau) - 1)d\tau\\
\simeq& \frac{1}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}[x_i+1]; s\right) + \psi\left(\frac{1}{2}[x_i+1]\right) + s\varphi\left(\frac{1}{2}[x_i+1]\right) - \beta\varphi''\left(\frac{1}{2}[x_i+1]\right)\right]\\
&\times \left(\cosh R\left(1 - \frac{1}{2}[x_i+1]\right) - 1\right),
\end{aligned}$$

(6.12)

$$\begin{aligned}
&\int_0^1 (G(\tau, s) + \psi(\tau) + s\varphi(\tau) - \beta\varphi''(\tau)) \left[R(\cosh R(1-\tau) - \tau) - \sinh R(1-\tau)\right] d\tau\\
\simeq& \frac{1}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}[x_i+1]; s\right) + \psi\left(\frac{1}{2}[x_i+1]\right) + s\varphi\left(\frac{1}{2}[x_i+1]\right) - \beta\varphi''\left(\frac{1}{2}[x_i+1]\right)\right]\\
&\times \left[R(\cosh R(1 - \frac{1}{2}[x_i+1]) - \frac{1}{2}[x_i+1]) - \sinh R(1 - \frac{1}{2}[x_i+1])\right],
\end{aligned}$$

(6.13)

$$\begin{aligned}
&\int_0^x \left[G(\tau, s) + \psi(\tau) + s\varphi(\tau) - \beta\varphi''(\tau)\right] \sinh R(x-\tau)d\tau\\
\simeq& \frac{x}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}[x_i+1]; s\right) + \psi\left(\frac{x}{2}[x_i+1]\right) + s\varphi\left(\frac{x}{2}[x_i+1]\right) - \beta\varphi''\left(\frac{x}{2}[x_i+1]\right)\right]\\
&+ \sinh R\left(x - \frac{x}{2}[x_i+1]\right).
\end{aligned}$$

**Case 3.** If $A(s) > s^2 + \gamma$, we obtain a general solution of (6.1) as follows

$$\begin{aligned} V(x,s) =& \frac{-1}{R(\alpha+s\beta)} \int_0^x \left[G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)\right] \sin R(x-\tau) d\tau \\ &+ C_1(s)\cos Rx + C_2(s)\sin Rx, \end{aligned} \tag{6.14}$$

where

$$R = \sqrt{\frac{s^2+\gamma-A(s)}{\alpha+s\beta}}$$

and $C_1$, $C_2$ are arbitrary functions of $s$. By substituting (6.14) into (6.2) and (6.3), we get

$$\begin{aligned} &\sin RC_1 + (1-\cos R)C_2 \\ =& \frac{1}{R(\alpha+s\beta)} \int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))(-\cos R(1-\tau)+1)d\tau + RN(s) \\ &\times (R\sin R+\cos R-1)C_1 + (-R\cos R+\sin R)\, C_2 \\ =& \frac{1}{R(\alpha+s\beta)} \\ &\times \int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))(R\cos R(1-\tau)+R\tau+\sin R(1-\tau))d\tau \\ &+ R^2M(s), \end{aligned}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix},$$

and

$$\begin{aligned} a_{11}(s) =& \sin R, \\ a_{12}(s) =& (1-\cos R), \\ a_{21}(s) =& R\sin R+\cos R-1), \\ a_{22}(s) =& (-R\cos R+\sin R), \\ b_1(s) =& \frac{1}{R(\alpha+s\beta)} \int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))(-\cos R(1-\tau)+1)d\tau \\ &+ RN(s), \\ b_2(s) =& \frac{1}{R(\alpha+s\beta)} \int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)) \\ &\times (R\cos R(1-\tau)+R\tau+\sin R(1-\tau))d\tau + R^2M(s). \end{aligned}$$

We have the following approximations for the integrals:

(6.15)

$$\int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))\,(-\cos R(1-\tau)+1)d\tau$$

$$\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+\psi\left(\frac{1}{2}\left[x_i+1\right]\right)+s\varphi\left(\frac{1}{2}\left[x_i+1\right]\right)-\beta\varphi''\left(\frac{1}{2}\left[x_i+1\right]\right)\right]$$
$$\times\left(-\cos R\left(1-\frac{1}{2}\left[x_i+1\right]\right)+1\right),$$

(6.16)

$$\int_0^1 (G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau))(R\cos R(1-\tau)+R\tau+\sin R(1-\tau))d\tau$$
$$\simeq \frac{1}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{1}{2}\left[x_i+1\right];s\right)+\psi\left(\frac{1}{2}\left[x_i+1\right]\right)+s\varphi\left(\frac{1}{2}\left[x_i+1\right]\right)-\beta\varphi''\left(\frac{1}{2}\left[x_i+1\right]\right)\right]$$
$$\times\left[R\cos R\left(1-\frac{1}{2}\left[x_i+1\right]\right)+\frac{R}{2}\left[x_i+1\right]+\sin R\left(1-\frac{1}{2}\left[x_i+1\right]\right)\right],$$

(6.17)

$$\int_0^x \left[G(\tau,s)+\psi(\tau)+s\varphi(\tau)-\beta\varphi''(\tau)\right]\sin R(x-\tau)d\tau$$
$$\simeq \frac{x}{2}\sum_{i=1}^{N} w_i \left[G\left(\frac{x}{2}\left[x_i+1\right];s\right)+\psi\left(\frac{x}{2}\left[x_i+1\right]\right)+s\varphi\left(\frac{x}{2}\left[x_i+1\right]\right)-\beta\varphi''\left(\frac{x}{2}\left[x_i+1\right]\right)\right]$$
$$\times\sin R\left(x-\frac{x}{2}\left[x_i+1\right]\right),$$

where $x_i$ and $w_i$ the abscissa and weights, are defined as

$$x_i : i^{\text{th}} \text{ zero of } P_n\left(x\right), \quad w_i = \frac{2}{(1-x_i^2)}\left(P_n'\left(x\right)\right)^2.$$

Their tabulated values can be found in [1] for different values of $N$.

## 7. The Stehfest Algorithm (Numerical Inversion of Laplace Transform)

Sometimes, an analytical inversion of a domain of Laplace solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by H. Hassanzadeh et al [19]. In this work we use the Stehfest's algorithm [26] that is easy to implement. This numerical technique was first introduced by Graver [11] and its algorithm then offered by [26]. Stehfest's algorithm approximates the time domain solution as follows

$$v\left(x,t\right)\approx\frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n V\left(x;\frac{n\ln 2}{t}\right), \tag{7.1}$$

where $m$ is a positive integer,

$$\beta_n=(-1)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min\{n,m\}}\frac{k^m\left(2k\right)!}{(m-k)!k!\left(k-1\right)!\left(n-k\right)!\left(2k-n\right)!}, \tag{7.2}$$

and $[q]$ denotes the integer part of the real number $q$.

7.1. **Numerical examples.** In this section, we report some results of numerical computations using Laplace transformation method proposed in the previous section. These techniques are applied to solve the problem defined by (2.1)–(2.5) for particular functions $g$, $\Phi$, $\Psi$, $n$, $m$ and positive constants $\alpha$, $\beta$ and $\gamma$. The method of solution is easily implemented on the computer, used Matlab 7.9.3 program. The numerical results are obtained by $N = 8$ in (6.7)–(6.9), (6.11)–(6.13), (6.15)–(6.17) and $m = 5$ in (7.1)–(7.2). Then, we compared the exact solution with numerical solution. An excellent agreement was found between the two of them.

*Example* 7.1. We take

$$\begin{aligned} g(x,t) &= -\, e^{-x}\sinh(t), a(t) = 0, \quad 0 < x < 1,\ 0 < t \leq T \text{ and } \alpha = 1, \beta = 1, \gamma = 0, \\ \Phi(x) &= \exp(-x), \quad 0 < x < 1, \\ \Psi(x) &= 0, \quad 0 < x < 1, \\ n(t) &= \left(1 - e^{-1}\right)\cosh(t), \quad 0 < t \leq T, \\ m(t) &= \left(1 - 2e^{-1}\right)\cosh(t), \quad 0 < t \leq T. \end{aligned}$$

In this case, the exact solution given by

$$v(x,t) = e^{-x}\cosh(t), \quad 0 < x < 1,\ 0 < t \leq T.$$

For $t = 0.10$, $x \in [0.10, 0.90]$, we calculate $v$ numerically using the proposed method of solution and compare it with the exact solution in Table 1.

TABLE 1.

| $x$ | 0.10 | 0.30 | 0.50 | 0.70 | 0.90 |
|---|---|---|---|---|---|
| $v$ exact | 0.9093654 | 0.7445254 | 0.6095658 | 0.490703 | 0.4086042 |
| $v$ numerical | 0.9093851 | 0.7443921 | 0.6097452 | 0.500183 | 0.4080919 |
| Relative error | 0.000217 | −0.0001790 | 0.0002943 | 0.0022295 | −0.0012538 |

*Example* 7.2. We take

$$\begin{aligned} g(x,t) &= (3e^{-t} - 1)\cos(2\pi x), a(t) = 1, \quad 0 < x < 1,\ 0 < t \leq T \text{ and } \alpha = \beta = \gamma = 1, \\ \Phi(x) &= \cos(2\pi x), \quad 0 < x < 1, \\ \Psi(x) &= -\cos(2\pi x), \quad 0 < x < 1, \\ n(t) &= 0, \quad 0 < t \leq T, \\ m(t) &= 0, \quad 0 < t \leq T. \end{aligned}$$

In this case, the exact solution given by

$$v(x,t) = e^{-t}\cos(2\pi x), \quad 0 < x < 1,\ 0 < t \leq T.$$

For $t = 0.1$, $x \in [0.1, 0.9]$, we calculate $v$ numerically using the proposed method of solution and compare it with the exact solution in Table 2.

TABLE 2.

| $x$ | 0.10 | 0.30 | 0.50 | 0.70 | 0.90 |
|---|---|---|---|---|---|
| $v$ exact | 0.7320288 | −0.2796101 | −0.9048374 | −0.2796101 | 0.7320288 |
| $v$ numerical | 0.7324162 | −0.2795921 | −0.9047562 | −0.2795421 | 0.7321329 |
| Relative error | 0.0005292 | −0.0000644 | −0.0000897 | −0.0002432 | 0.0001422 |

*Example* 7.3. We take

$$\begin{aligned} g(x,t) &= -e^{x}\sinh t, a(t)=0, \quad 0<x<1,\ 0<t\le T \text{ and } \alpha=1,\beta=1,\gamma=0,\\ \Phi(x) &= \exp(x), \quad 0<x<1,\\ \Psi(x) &= 0, \quad 0<x<1,\\ n(t) &= (e-1)\cosh(t), \quad 0<t\le T\ ,\\ m(t) &= \cosh(t), \quad 0<t\le T. \end{aligned}$$

In this case, the exact solution given by

$$v(x,t) = e^{x}\cosh(t), \quad 0<x<1,\ 0<t\le T.$$

For $t = 0.1$, $x \in [0.1, 0.9]$, we calculate $v$ numerically using the proposed method of solution and compare it with the exact solution in Table 3.

TABLE 3.

| $x$ | 0.10 | 0.30 | 0.50 | 0.70 | 0.90 |
|---|---|---|---|---|---|
| $v$ exact | 1.1107014 | 1.3566137 | 1.6569717 | 2.0238299 | 2.4719114 |
| $v$ numerical | 1.1106841 | 1.3565926 | 1.6569459 | 2.0237984 | 2.4718729 |
| Relative error | 0.0000155 | 0.0000155 | 0.0000155 | 0.0000155 | 0.0000155 |

*Example* 7.4. We take

$$\begin{aligned} g(x,t) &= 0, a(t)=0, \quad 0<x<1,\ 0<t\le T \text{ and } \alpha=\frac{1}{2},\beta=\frac{1}{2},\gamma=0,\\ \Phi(x) &= \exp(x), \quad 0<x<1,\\ \Psi(x) &= \exp(x), \quad 0<x<1,\\ n(t) &= (e-1)\exp(t), \quad 0<t\le T,\\ m(t) &= \exp(t), \quad 0<t\le T. \end{aligned}$$

In this case, the exact solution given by

$$v(x,t) = e^{x+t}, \quad 0<x<1,\ 0<t\le T.$$

For $t = 0.1$, $x \in [0.1, 0.9]$, we calculate $v$ numerically using the proposed method of solution and compare it with the exact solution in Table 4.

Table 4.

| $x$ | 0.10 | 0.30 | 0.50 | 0.70 | 0.90 |
|---|---|---|---|---|---|
| $v$ exact | 1.221403 | 1.491825 | 1.822119 | 2.225541 | 2.718281 |
| $v$ numerical | 1.221407 | 1.491830 | 1.822125 | 2.22555 | 2.718291 |
| Relative error | 0.0000031 | 0.0000033 | 0.0000032 | 0.000004 | 0.0000036 |

## 8. Homotopy Perturbation Method with Laplace Transform (LT- HPM)

8.1. **Basic idea of homotopy perturbation method.** The homotopy perturbation method was proposed first by He in 1998 [12] and was developed and improved by He [13–17]. To illustrate the basic ideas of this method, we consider the following non-linear functional equation:

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{8.1}$$

with the following boundary condition:

$$B\left(u; \frac{\partial u}{\partial \eta}\right) = 0, \quad r \in \Gamma, \tag{8.2}$$

where $A$ is a general functional operator, $B$ a boundary operator, $f(r)$ is a known analytical function and $\Gamma$ is the boundary of the domain. The operator $A$ can be decomposed into two operators $L$ and $N$, where $L$ is linear, and $N$ is nonlinear operator. Equation (8.1) can be, therefore, written as follows:

$$L(u) + N(u) - f(r) = 0. \tag{8.3}$$

Using the homotopy technique, we construct an homotopy:

$$v(r;p) : \Omega \times [0;1] \to R,$$

which satisfies:

$$H(v;p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0,1], r \in \Omega, \tag{8.4}$$

or

$$H(v;p) = L(v) - L(u_0) + p\,[L(u_0) + N(v) - f(r)] = 0, \tag{8.5}$$

where $p \in [0;1]$ is an embedding parameter, $u_0$ is an initial approximation for the solution of equation (8.1), which satisfies the boundary conditions. Obviously, from equations. (8.4) and (8.5) we will have:

$$H(v;0) = L(v) - L(u_0) = 0, \tag{8.6}$$

$$H(v;1) = A(v) - f(r) = 0. \tag{8.7}$$

The changing values of $p$ from zero to unity are just that of $v(r;p)$ from $u_0(r)$ to $u(r)$.In topology, this is called deformation, and $L(v) - L(u_0)$, $A(v) - f(r)$ are called homotopic. In 1998 J. H. He, used the imbedding parameter $p$ as a “small parameter”, and assume that the solution of equations (8.4) and (8.5) can be written as a power series in $p$:

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{8.8}$$

We take $p \to 1$, results in the approximation to the solution of equation (8.1),

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots . \tag{8.9}$$

The combination of the perturbation method and the homotopy method is called the homotopy perturbation method (HPM), which has eliminated limitations of the traditional perturbation techniques. The series (8.9) is convergent for more cases. Some criteria are suggested for convergence of the Series (8.9), in [12].

8.2. **Laplace transform HPM.** Taking the Laplace transform of (2.1) we obtain a new partial differential equation (6.1).

According to HPM, for solving equation (6.1) we construct an homotopy by Madani et al [18], as the following form:

(8.10)

$$(s^2 + \gamma - A(s))V(x,s) = p\left[-(\alpha + s\beta)\frac{\partial^2 V}{\partial x^2}(x,s)\right] + G(x,s) + \psi(x) + s\varphi(x) - \beta\varphi''(x).$$

Now, let us present the solution of equation (8.10) as the following form:

$$V(x,s) = \sum_{j=0}^{\infty} p^j V_j(x,s), \tag{8.11}$$

where $v_j(x;s)$, $j = 0, 1, 2, \ldots$ are functions which should be determined. By substituting (8.11) into (8.10), we get

$$\begin{aligned}(s^2 + \gamma - A(s))\sum_{j=0}^{\infty} p^j V_j(x,s) =& p\left[-(\alpha + s\beta)\frac{\partial^2}{\partial x^2}\left(\sum_{j=0}^{\infty} p^j V_j(x,s)\right)\right] \\ &+ G(x,s) + \psi(x) + s\varphi(x) - \beta\varphi''(x).\end{aligned} \tag{8.12}$$

Equating the coefficients of $p$ with the same powers in (8.12) leads to

$$\begin{aligned}
p^0 &: (s^2 + \gamma - A(s))V_0(x,s) = G(x,s) + \psi(x) + s\varphi(x) - \beta\varphi''(x),\\
p^1 &: (s^2 + \gamma - A(s))V_1(x,s) = (\alpha + s\beta)\frac{\partial^2}{\partial x^2}V_0(x,s),\\
p^2 &: (s^2 + \gamma - A(s))V_2(x,s) = (\alpha + s\beta)\frac{\partial^2}{\partial x^2}V_1(x,s),\\
&\vdots\\
p^{n+1} &: (s^2 + \gamma - A(s))V_{n+1}(x,s) = (\alpha + s\beta)\frac{\partial^2}{\partial x^2}V_n(x,s).
\end{aligned}$$

From where

$$\begin{cases} V_{n+1}(x,s) = \dfrac{\alpha+s\beta}{s^2+\gamma-A(s)}\dfrac{\partial^2}{\partial x^2}V_n(x,s), \\ V_0(x,s) = \dfrac{G(x,s)+\psi(x)+s\varphi(x)-\beta\varphi''(x)}{s^2+\gamma-A(s)}, \end{cases}$$

we obtain:

$$V_n(x,s) = \left(\frac{\alpha+s\beta}{s^2+\gamma-A(s)}\right)^n \frac{G_x^{(2n)}(x,s)+\psi_x^{(2n)}(x)+s\varphi_x^{(2n)}(x)-\beta\varphi_x^{(2n+2)}(x)}{s^2+\gamma-A(s)},$$

when $p \to 1$, (8.11) becomes the approximate solution of equation (6.1), i.e.,

$$V(x;s) \cong H_n(x;s) = \sum_{j=0}^{n} V_j(x,s). \tag{8.13}$$

The physical solution $v(x;t)$ can be recovered approximately from $H_n(x;s)$ according to the Stehfest's algorithm [26].

Taking the inverse $\mathscr{L}^{-1}$ from both sides of (8.13) we get the approximate solution of (2.1)–(2.5):

$$v(x;t) \cong \mathscr{L}^{-1}(H_n(x;s)) = \mathscr{L}^{-1}\left(\sum_{j=0}^{n} V_j(x,s)\right).$$

*Example* 8.1. We take

$$\begin{aligned} g(x,t) &= -e^x \sinh t,\ a(t)=0, \quad 0<x<1,\ 0<t\le T \text{ and } \alpha=1,\beta=1,\gamma=0,\\ \Phi(x) &= \exp(x), \quad 0<x<1,\\ \Psi(x) &= 0, \quad 0<x<1,\\ n(t) &= (e-1)\cosh(t), \quad 0<t\le T,\\ m(t) &= \cosh(t), \quad 0<t\le T. \end{aligned}$$

We obtain

$$\begin{aligned} V_n(x,s) &= \left(\frac{s+1}{s^2}\right)^n \frac{s^2-s-1}{s^2(s^2-1)}e^x,\\ V(x;s) \cong H_n(x;s) &= \sum_{j=0}^{n} V_j(x,s) = e^x\left[1-\left(\frac{s+1}{s^2}\right)^{n+1}\right]\frac{s}{(s^2-1)},\\ v(x;t) \cong e^x\mathscr{L}^{-1}&\left\{\left[1-\left(\frac{s+1}{s^2}\right)^{n+1}\right]\frac{s}{(s^2-1)}\right\}, \end{aligned}$$

when $n=1,\dots,4$ (see Table 5).

The closed form of the series $1+t+\frac{1}{2!}t^2+\frac{1}{3!}t^3+\frac{1}{4!}t^4+\cdots$ is $e^t$ which gives an exact solution of the problem $v(x;t)=e^x\cosh t$.

*Example* 8.2. We take

$$g(x,t) = 0,\ a(t)=0, \quad 0<x<1,\ 0<t\le T \text{ and } \alpha=\frac{1}{2},\ \beta=\frac{1}{2},\ \gamma=0,$$

TABLE 5.

| $n$ | $v(x;t)$ |
|---|---|
| 1 | $\left(\cosh t + 2\left(-e^t + 1 + t + \frac{1}{4}t^2\right)\right)e^x$ |
| 2 | $\left(\cosh t + 4\left(-e^t + 1 + t + \frac{1}{2!}t^2 + \frac{1}{8}t^3 + \frac{1}{96}t^4\right)\right)e^x$ |
| 3 | $\left(\cosh t + 8\left(-e^t + 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{7}{192}t^4 + \frac{1}{240}t^5 + \frac{1}{5760}t^6\right)\right)e^x$ |
| 4 | $\left(\cosh t + 16\left(-e^t + 1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \frac{1}{128}t^5 + \frac{11}{11520}t^6 + \frac{1}{16128}t^7 + \frac{1}{645120}t^8\right)\right)e^x$ |

$$
\begin{aligned}
\Phi(x) &= \exp(x), \quad 0 < x < 1,\\
\Psi(x) &= \exp(x), \quad 0 < x < 1,\\
n(t) &= (e-1)\,e^t, \quad 0 < t \leq T,\\
m(t) &= e^t, \quad 0 < t \leq T.
\end{aligned}
$$

We obtain

$$
\begin{aligned}
V_n(x,s) &= \left(\frac{s+1}{2s^2}\right)^n \frac{2s+1}{2s^2}e^x,\\
V(x;s) \cong H_n(x;s) &= \sum_{j=0}^{n} V_j(x,s) = e^x\left(1 - \left(\frac{s+1}{2s^2}\right)^{n+1}\right)\frac{2s+1}{(2s^2-s-1)},\\
v(x;t) &\cong e^x \mathscr{L}^{-1}\left\{\left(1 - \left(\frac{s+1}{2s^2}\right)^{n+1}\right)\frac{2s+1}{(2s^2-s-1)}\right\},
\end{aligned}
$$

when $n = 1, \ldots, 4$ (see Table 6).

TABLE 6.

| $n$ | $v(x;t)$ |
|---|---|
| 1 | $\left(\frac{1}{24}t^3 + \frac{3}{8}t^2 + t + 1\right)e^x$ |
| 2 | $\left(\frac{1}{960}t^5 + \frac{1}{48}t^4 + \frac{7}{48}t^3 + \frac{1}{2!}t^2 + t + 1\right)e^x$ |
| 3 | $\left(\frac{1}{80\,640}t^7 + \frac{1}{2304}t^6 + \frac{11}{1920}t^5 + \frac{5}{128}t^4 + \frac{1}{3!}t^3 + \frac{1}{2!}t^2 + t + 1\right)e^x$ |
| 4 | $\left(\frac{1}{11\,612\,160}t^9 + \frac{1}{215\,040}t^8 + \frac{1}{10\,080}t^7 + \frac{13}{11\,520}t^6 + \frac{31}{3840}t^5 + \frac{1}{4!}t^4 + \frac{1}{3!}t^3 + \frac{1}{2!}t^2 + t + 1\right)e^x.$ |

The closed form of the series $1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \frac{1}{4!}t^4 + \cdots$ is $e^t$ which gives an exact solution of the problem $v(x;t) = e^{x+t}$.

*Example* 8.3. We take

$$
\begin{aligned}
g(x,t) &= \left(t^2 + 2t + 2\right)x^2 - 4e^t,\ a(t) = t^2, \quad 0 < x < 1,\ 0 < t \leq T\\
&\quad \text{and } \alpha = \beta = \gamma = 1,\\
\Phi(x) &= x^2, \quad 0 < x < 1,\\
\Psi(x) &= x^2, \quad 0 < x < 1,
\end{aligned}
$$

$$\begin{aligned} n(t) =& \frac{1}{3}e^t, \quad 0 < t \leq T, \\ m(t) =& \frac{1}{4}e^t, \quad 0 < t \leq T. \end{aligned}$$

We obtain

$$\begin{aligned} V_0(x,s) =& -\frac{1}{(s-1)(s^5+s^3-2)}\left(-s^5x^2+2s^4-s^3x^2+2s^3+2x^2\right), \\ V_1(x,s) =& 2s^3\frac{s+1}{(s^5+s^3-2)^2}\left(s^4+s^3+2s^2+2s+2\right), \\ V_n(x,s) =& 0, \quad n \geq 2, \\ V(x;s) =& \sum_{i=0}^{i=1} V_i(x,s) = \frac{x^2}{s-1}, \\ v(x;t) =& \mathscr{L}^{-1}\left\{\frac{x^2}{s-1}\right\} = x^2e^t, \end{aligned}$$

which gives an exact solution of the problem $v(x;t) = x^2e^t$.

*Example* 8.4. We take

$$\begin{aligned} g(x,t) =& -\frac{1}{6}x^2\left(t^3x^2+3t^2x^2+72t+144\right), \ a(t) = t, \quad 0 < x < 1, \ 0 < t \leq T \\ & \text{and } \alpha = \beta = 1, \ \gamma = 0, \\ \Phi(x) =& x^4, \quad 0 < x < 1, \\ \Psi(x) =& x^4, \quad 0 < x < 1, \\ n(t) =& \frac{1}{5}t + \frac{1}{5}, \quad 0 < t \leq T, \\ m(t) =& \frac{1}{6}t + \frac{1}{6}, \quad 0 < t \leq T, \\ V_0(x,s) =& \frac{-s^4x^4+12s^3x^2+12s^2x^2+x^4}{-s^5+s^4-s^3+s^2}, \\ V_1(x,s) =& -\frac{12}{(s^4-1)^2}(s+1)^3\left(-s^3x^2+s^2x^2+2s^2-sx^2+x^2\right), \\ V_2(x,s) =& \frac{s^2}{(s^4-1)^3}(s+1)^2\left(24s^5+24s^4-24s-24\right), \\ V_n(x,s) =& 0, \quad \text{for all } n \geq 3, \\ V(x,s) =& \sum_{i=0}^{i=2} V_i = \frac{1}{s^2}x^4(s+1), \\ v(x;t) =& \mathscr{L}^{-1}\left\{\frac{1}{s^2}x^4(s+1)\right\} = x^4(t+1), \end{aligned}$$

which gives an exact solution of the problem $v(x;t) = x^4(t+1)$.

## References


[1] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, Dover, New York, 1972.

[2] S. Abbasbandy, *Numerical solutions of the integral equations: homotopy perturbation method and Adomian's decomposition method*, Appl. Math. Comput. **173**(2-3) (2006), 493–500.

[3] G. A. Afrouzi, D. D. Ganji, H. Hosseinzadeh and R. A. Talarposhti, *Fourth order Volterra integro-differential equations using modied homotopy-perturbation method*, Turkish Journal of Mathematics and Computer Science **3**(2) (2011), 179–191.

[4] A. Bouziani and A. Merad, *The Laplace transform method for one-dimensional hyperbolic equationwith purely integral conditions*, Rom. J. Math. Comput. Sci. **3**(2) (2013), 191–204.

[5] A. Bouziani and R. Mechri, *The Rothe method to a parabolic integro-differential equation with a nonclassical boundary conditions*, Int. J. Stoch. Anal. (2010), Article ID 519684, 16 pages, DOI: 10.1155/519684/(2010)

[6] A. Bouziani and M. S. Temsi, *On a pseudohyperbolic equation with nonlocal boundary condition*, Kobe J. Math. **21** (2004), 15–31.

[7] D. D. Ganji, G. A. Afrouzi, H. Hosseinzadeh and R. A. Talarposhti, *Application of Hmotopy perturbation method to the second kind of nonlinear integral equations*, Phys. Lett. A **371**(1-2) (2007), 20–25.

[8] D. D. Ganji and A. Sadighi, *Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations*, Int. J. Nonlinear Sci. Numer. Simul. **7**(4) (2006), 411–418.

[9] D. D. Ganji and A. Rajabi, *Assessment of homotopy-perturbation and perturbation methods in heat radiation equations*, International Communications in Heat and Mass Transfer **33** (2006), 391–400.

[10] D. D. Ganji and M. Rafei, *Solitary wave solutions for a generalized Hirota-Satsuma coupled KdV equation by homotopy perturbation method*, Phys. Lett. A **356** (2006), 131–137.

[11] D. P. Graver, *Observing stochastic processes and aproximate transform inversion*, Oper. Res. **14** (1966), 444–459.

[12] J. H. He, *Homotopy perturbation technique*, Comput. Methods Appl. Mech. Engrg. **178** (1999), 257–262.

[13] J. H. He, *New interpretation of homotopy perturbation method*, Internat. J. Modern Phys. B **20** (2006), 2561–2568.

[14] J. H. He, *A coupling method of homotopy technique and perturbation technique for nonlinear problems*, International Journal of Non-Linear Mechanics **35**(1) (2000), 37–43.

[15] J. H. He, *Limit cycle and bifurcation of nonlinear problems*, Chaos Solitons Fractals **26** (2005), 827–833.

[16] J. H. He, *Homotopy perturbation method for bifurcation of nonlinear problems*, Int. J. Nonlinear Sci. Numer. Simul. **6**(2) (2005), 207–208.

[17] J. H. He, *Application of homotopy perturbation method to nonlinear wave equations*, Chaos Solitons Fractals **26** (2005), 695–700.

[18] M. Madani, M. Fathizadeh, Y. Khan and A. Yildrim, *On coupling the homotpy perturbation method ans Laplace transformation*, Math. Comput. Modelling **53**(2011) (1970), 1937–1945.

[19] H. Hassanzadeh and M. Pooladi-Darvish, *Comparision of different numerical Laplace inversion methods for engineering applications*, Appl. Math. Comput. **189** (2007), 1966–1981.
[20] A. Merad and A. Bouziani, *Laplace transform technique for pseudoparabolic equation with nonlocal conditiond*, Transylvanian Journal of Mathematics and Mechanics **5**(1) (2013), 59–64.
[21] A. Merad and A. Bouziani, *A method of solution of integro-differential parabolic equation with purely integral conditions*, in: *Advances in Applied Mathematics and Approximation Theory*, Springer Proceeding in Mathematics and Statistics **41**, New York, 2013.
[22] A. Merad and A. Bouziani, *Solvability the telegraph equation with purely integral conditions*, TWMS J. Appl. Eng. Math. **3**(2) (2013), 117–125.
[23] M. El-Shahed, *Application of He's homotopy perturbation method to Volterra integrodifferential equation*, Int. J. Nonlinear Sci. Numer. Simul. **6**(2) (2005), 163–168.
[24] A. D. Shruti, *Numerical solution for nonlocal Sobolev-type differential equations*, Electron. J. Differential Equations **19** (2010), 75–83.
[25] Z. Suying, Z. Minzhen, D. Zichen and L. Wencheng, *Solution of nonlinear dynamic differential equations based on numerical Laplace transform inversion*, Appl. Math. Comput. **189** (2007), 79–86.
[26] H. Stehfest, *Numerical inversion of the Laplace transform*, Communications of the ACM **13** (1970), 47–49.

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# SOME MATRIX AND COMPACT OPERATORS OF THE ABSOLUTE FIBONACCI SERIES SPACES

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ABSTRACT. In the present paper, we introduce the absolute Fibonacci space $|F_u|_k$, give some inclusion relations and investigate topological and algebraic structure such as $BK$-space, $\alpha$-, $\beta$-, $\gamma$- duals and Schauder basis. Further, we characterize certain matrix and compact operators on these spaces, also determine their norms and Hausdroff meausures of noncompactness.

## 1. INTRODUCTION

Let $\omega$ be the set of all sequences of complex numbers. We write $c$, $\ell_\infty$, $c_s, b_s$ and $\ell_k$, $k \geq 1$, for the sequence space of all convergent, bounded sequences; for the spaces of all convergent, bounded, $k$-absolutely convergent series, respectively. Let $X$ and $Y$ be two subspaces of $\omega$ and $A = (a_{nv})$ be an arbitrary infinite matrix of complex numbers. If the series

$$A_n(x) = \sum_{v=0}^{\infty} a_{nv} x_v,$$

converges for all $n \in \mathbb{N} = \{0, 1, 2, \ldots\}$, then, by $A(x) = (A_n(x))$, we denote the $A$-transform of the sequence $x = (x_v)$. Also, we say that $A$ defines a matrix transformation from $X$ into $Y$, and denote it by $A \in (X, Y)$ or $A : X \to Y$ if $Ax = (A_n(x)) \in Y$ for every $x \in X$. The $\alpha$-, $\beta$-, $\gamma$- duals of $X$ and the domain of the matrix $A$ in $X$ are defined by

$$X^\alpha = \{\epsilon \in \omega : (\epsilon_n x_n) \in \ell \text{ for all } x \in X\},$$
$$X^\beta = \{\epsilon \in \omega : (\epsilon_n x_n) \in c_s \text{ for all } x \in X\},$$
$$X^\gamma = \{\epsilon \in \omega : (\epsilon_n x_n) \in b_s \text{ for all } x \in X\}$$

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and

$$X_A = \{x = (x_n) \in \omega : A(x) \in X\}, \tag{1.1}$$

respectively. Further, $X$ is said to be a $BK$-space if it is a complete normed space with continuous coordinates $p_n : X \to \mathbb{C}$ defined by $p_n(x) = x_n$ for all $n \in \mathbb{N}$. If there exists unique sequence of coefficients $(x_k)$ such that, for each $x \in X$,

$$\left\| x - \sum_{k=0}^{m} x_k b_k \right\| \to 0, \quad m \to \infty,$$

then, the sequence $(b_k)$ is called the Schauder basis (or briefly basis) for a normed sequence space $X$, and in this case we write $x = \sum_{k=0}^{\infty} x_k b_k$. For instance, the sequence $(e^{(j)})$ is the Schauder basis of the space $\ell_k$, where $e^{(j)}$ is the sequence whose only non-zero term is 1 in $j$th place for each $j \in \mathbb{N}$.

Now take $\sum x_v$ as an infinite series with $n$th partial sum $s_n$ and let $(u_n)$ be a sequence of positive terms. Then, the series $\sum x_v$ is said to be summable $|A, u_n|_k$, $k \geq 1$, if (see [32])

$$\sum_{n=0}^{\infty} u_n^{k-1} |\Delta A_n(s)|^k < \infty,$$

where $\Delta A_n(s) = A_n(s) - A_{n-1}(s)$, $A_{-1}(s) = 0$.

Note that this method includes some well known methods. For example, if $A$ is the matrix of weighted mean $\left(\bar{N}, p_n\right)$ (resp. $u_n = P_n/p_n$), then it reduces to the summability $\left|\bar{N}, p_n, u_n\right|_k$ [36] (the summability $\left|\bar{N}, p_n\right|_k$ [10]). Also if we take $A$ as the matrix of Cesàro mean of order $\alpha > -1$ and $u_n = n$, then we get summability $|C, \alpha|_k$ in Flett's notation [11].

A large literature has recently grown up, concerned with producing sequence spaces by means of matrix domain of a special limitation method and studying their algebraic, topological structure and matrix transformations (see [1–7, 15–18, 25]). Also, some series spaces have been derived and studied by absolute summability methods from a different point of view (see [9–14, 23–26, 28–34, 36]). The aim of this paper is to define the space $|F_u|_k$ combining absolute summability and Fibonacci matrix given by Kara [15], investigate some inclusion relation, construct their $\alpha$-, $\beta$-, $\gamma$- duals, basis and characterize some matrix operators related to that space, and also determine their norms and Hausdroff measures of noncompactness.

Firstly, we mention some properties of Fibonacci numbers as follows: the sequence $(f_n)$ of Fibonacci numbers is given by the relations

$$f_0 = f_1 = 1 \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n \geq 2$$

that is, each term is equal to the sum of the previous two terms. The sequences of Fibonacci numbers have been important for artist, architects, physicists and mathematicians since the old. The ratio of Fibonacci numbers converges to the golden ratio which is one of the most interesting irrationals having an important role in number theory, algorithms, network theory, etc. Also, Fibonacci numbers have the following properties [19]:

$$\sum_n \frac{1}{f_n} \text{ converges,}$$
$$f_{n-1}^2 + f_n f_{n-1} - f_n^2 = (-1)^{n+1}, \quad n \geq 1,$$
$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = 1.61803398875....$$

Fibonacci matrix $F = (\hat{f}_{nv})$ has recently been defined by Kara [15] as follows:

$$\hat{f}_{nv} = \begin{cases} \dfrac{-f_{n+1}}{f_n}, & v = n-1, \\ \dfrac{f_n}{f_{n+1}}, & v = n, \\ 0, & v > n \text{ or } 0 \leq v < n-1, \end{cases}$$

where $f_n$ be the $n$th Fibonacci number for every $n \in \mathbb{N}$. Note that if we take the Fibonacci matrix instead of $A$, then $|A, u_n|_k$ summability reduces to the absolute Fibonacci summability. On the other hand, since $(s_n)$ is a sequence of partial sum of the series $\sum x_v$, we get

$$A_n(s) = \sum_{v=0}^{n} \hat{f}_{nv} s_v = \sum_{j=0}^{n} x_j \sum_{v=j}^{n} \hat{f}_{nv} = x_n \hat{f}_{nn} + \sum_{j=0}^{n-1} (\hat{f}_{nn} + \hat{f}_{n,n-1}) x_j$$

and so,

$$\begin{aligned} \Delta A_n(s) =& x_n \frac{f_n}{f_{n+1}} + x_{n-1}\left(\frac{(-1)^n}{f_n f_{n+1}} - \frac{f_{n+1}}{f_n}\right) + \sum_{j=0}^{n-2} (-1)^n \frac{f_{n-1}+f_{n+1}}{f_{n-1} f_n f_{n+1}} x_j \\ =& \sum_{j=0}^{n} \sigma_{nj} x_j, \end{aligned}$$

where

$$\sigma_{nj} = \begin{cases} \dfrac{f_n}{f_{n+1}}, & j = n, \\ \dfrac{(-1)^n}{f_n f_{n+1}} - \dfrac{f_{n+1}}{f_n}, & j = n-1, \\ (-1)^n \dfrac{f_{n-1}+f_{n+1}}{f_{n-1} f_n f_{n+1}}, & 0 \leq j \leq n-2, \\ 0, & j > n. \end{cases}$$

Now, we introduce the absolute Fibonacci space as follows:

$$|F_u|_k = \left\{ x \in \omega : \sum_{n=0}^{\infty} u_n^{k-1} \left| \sum_{j=0}^{n} \sigma_{nj} x_j \right|^k < \infty \right\}.$$

Also, it may be written that

$$(E^{(k)} \circ T)_n(x) = u_n^{1/k^*} (T_n(x) - T_{n-1}(x)), \tag{1.2}$$

where

$$t_{nv}=\begin{cases}\dfrac{f_n}{f_{n+1}}, & v=n,\\ \dfrac{f_n^2-f_{n+1}^2}{f_nf_{n+1}}, & 0\le v\le n-1,\\ 0, & v>n,\end{cases}$$

$$e_{nv}^{(k)}=\begin{cases}u_n^{1/k^*}, & v=n,\\ -u_n^{1/k^*}, & v=n-1,\\ 0, & v\neq n,n-1,\end{cases}$$

and $k^*$ is the conjugate of $k$, i.e., $1/k+1/k^*=1$ for $k>1$, and $1/k^*=0$ for $k=1$. With these matrices $T=(t_{nv})$ and $E^{(k)}=(e_{nv}^{(k)})$, according to the notation (1.1), it is obvious that $|F_u|_k=(\ell_k)_{E^{(k)}\circ T}$. Further, since every triangle matrix has a unique inverse which also is a triangle [37], $T$ and $E^{(k)}$ have a unique inverse $\tilde{T}=(\tilde{t}_{nv})$ and $\tilde{E}^{(k)}=(\tilde{e}_{nv})$ given by

$$\tilde{t}_{nv}=\begin{cases}\dfrac{f_{n+1}}{f_n}, & v=n,\\ \dfrac{f_{n+1}^2-f_n^2}{f_vf_{v+1}}, & 0\le v\le n-1,\\ 0, & v>n,\end{cases}\tag{1.3}$$

$$\tilde{e}_{nv}^{(k)}=\begin{cases}u_v^{-1/k^*}, & 0\le v\le n,\\ 0, & v>n,\end{cases}\tag{1.4}$$

respectively.

Before the main theorems, we point out some well known lemmas which are needed in the proofs of theorems.

**Lemma 1.1** ([35]). *Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if*

$$\|A\|_{(\ell_k,\ell)}=\sup_{N\in\mathfrak{F}}\left\{\sum_{v=0}^{\infty}\left|\sum_{n=0}^{\infty}a_{nv}\right|^{k^*}\right\}^{1/k^*},$$

*where $\mathfrak{F}$ denotes the collection of all finite subsets of $\mathbb{N}$.*

Lemma 1.1 exposes a rather difficult condition to apply in applications. So the following lemma is more useful in many cases, which gives equivalent norm.

**Lemma 1.2** ([29]). *Let $1<k<\infty$. Then, $A\in(\ell_k,\ell)$ if and only if*

$$\|A\|'_{(\ell_k,\ell)}=\left\{\sum_{v=0}^{\infty}\left(\sum_{n=0}^{\infty}|a_{nv}|\right)^{k^*}\right\}^{1/k^*}<\infty.$$

*Moreover, since*

$$\|A\|_{(\ell_k,\ell)}\le\|A\|'_{(\ell_k,\ell)}\le 4\,\|A\|_{(\ell_k,\ell)},$$

*there exists $1\le\xi\le 4$ such that $\|A\|'_{(\ell_k,\ell)}=\xi\,\|A\|_{(\ell_k,\ell)}$.*

**Lemma 1.3** ([20]). *Let $1 \leq k < \infty$. Then, $A \in (\ell, \ell_k)$ if and only if*

$$\|A\|_{(\ell,\ell_k)} = \sup_v \left\{ \sum_{n=0}^{\infty} |a_{nv}|^k \right\}^{\frac{1}{k}}.$$

**Lemma 1.4** ([35]).

(a) $A \in (\ell, c) \Leftrightarrow (i)\ \lim_n a_{nv}$ *exists for* $v \geq 0$, $(ii)\ \sup_{n,v} |a_{nv}| < \infty$;
(b) $A \in (\ell, \ell_\infty) \Leftrightarrow (ii)$ *holds;*
(c) *If* $1 < k < \infty$, *then,* $A \in (\ell_k, c) \Leftrightarrow (i)$ *holds,* $(iii)\ \sup_n \sum_{v=0}^{\infty} |a_{nv}|^{k^*} < \infty$;
(d) *If* $1 < k < \infty$, *then,* $A \in (\ell_k, \ell_\infty) \Leftrightarrow (iii)$ *holds.*

## 2. The Hausdorff Measure of Noncompactness

If $S$ and $H$ are subsets of a metric space $(X, d)$ and, for every $h \in H$, there exists an $s \in S$ such that $d(h, s) < \varepsilon$ then, $S$ is called an $\varepsilon$-net of $H$; if $S$ is finite, then the $\varepsilon$-net $S$ of $H$ is called a finite $\varepsilon$-net of $H$. Let $X$ and $Y$ be Banach spaces. A linear operator $L : X \to Y$ is called compact if its domain is all of $X$ and, for every bounded sequence $(x_n)$ in $X$, the sequence $(L(x_n))$ has a convergent subsequence in $Y$. We denote the class of such operators by $\mathcal{C}(X, Y)$. If $Q$ is a bounded subset of the metric space $X$, then the Hausdorff measure of noncompactness of $Q$ is defined by

$$\chi(Q) = inf \{\varepsilon > 0 : Q \text{ has a finite } \varepsilon - \text{net in } X\},$$

and $\chi$ is called the Hausdorff measure of noncompactness.

The following lemma is very important to calculate the Hausdorff measure of noncompactness of a bounded subset of the space $\ell_k$.

**Lemma 2.1** ([27]). *Let $Q$ be a bounded subset of the normed space $X$ where $X = \ell_k$ for $1 \leq k < \infty$ or $X = c_0$. If $P_n : X \to X$ is the operator defined by $P_n(x) = (x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ for all $x \in X$, then*

$$\chi(Q) = \lim_{r\to\infty} \left( \sup_{x\in Q} \|(I - P_r)(x)\| \right).$$

Let $X$ and $Y$ be Banach space and $\chi_1$ and $\chi_2$ be Hausdorff measures on $X$ and $Y$, the linear operator $L : X \to Y$ is said to be $(\chi_1, \chi_2)$- bounded if $L(Q)$ is a bounded subset of $Y$ and there exists a positive constant $M$ such that $\chi_2(L(Q)) \leq M\chi_1(L(Q))$ for every bounded subset $Q$ of $X$. If an operator $L$ is $(\chi_1, \chi_2)$- bounded, then the number

$$\|L\|_{(\chi_1,\chi_2)} = \inf \{M > 0 : \chi_2(L(Q)) \leq M\chi_1(L(Q)) \text{ for all bounded set } Q \subset X\}$$

is called the $(\chi_1, \chi_2)$-measure noncompactness of $L$. In particular, if $\chi_1 = \chi_2 = \chi$ then we write $\|L\|_{(\chi,\chi)} = \|L\|_\chi$.

**Lemma 2.2** ([22]). *Let $X$ and $Y$ be Banach spaces and $L \in \mathcal{B}(X,Y)$. Also $S_x = \{x \in X : \|x\| \leq 1\}$ be the unit sphere in $X$. Then,*

$$\|L\|_\chi = \chi\left(L\left(S_x\right)\right)$$

*and*

$$L \in \mathcal{C}(X,Y) \Leftrightarrow \|L\|_\chi = 0.$$

**Lemma 2.3** ([21]). *Let $X$ be a normed sequence space, $T = (t_{nv})$ be an infinite triangle matrix, $\chi_T$ and $\chi$ denote the Hausdroff measures of noncompactness on $M_{X_T}$ and $M_X$, the collections of all bounded sets in $X_T$ and $X$, respectively. Then, $\chi_T(Q) = \chi(T(Q))$ for all $Q \in M_{X_T}$.*

## 3. Absolute Fibonacci Space $|F_u|_k$

In this section, we investigate some inclusion relations, topological and algebraic structures of the space $|F_u|_k$. Also we characterize some classes of compact matrix operators on that space and compute their norms and Hausdroff measure of noncompactness.

Firstly, since $|F_u|_k$ is generated from $\ell_k$, to explain a relation between the spaces $\ell_k$ and $|F_u|_k$, we begin with the following theorem.

**Theorem 3.1.** *Let $u = (u_n) \in \ell_\infty$ and $1 \leq k < \infty$. Then, $\ell_k \subset |F_u|_k$.*

*Proof.* To prove the inclusion $\ell_k \subset |F_u|_k$, it is sufficient to show that

$$\|x\|_{|F_u|_k} \leq O(1)\,\|x\|_{\ell_k},$$

for all $x \in \ell_k$. The proof is clear for the case $k = 1$, and so it is omitted. Let $k > 1$. Then, since the series $\sum_n \frac{1}{f_n}$ is convergent and $(\frac{1}{f_n})$ is decreasing sequence, it follows from Abel's Theorem, $\frac{n}{f_n} \to 0$ as $n \to \infty$, $\sum_{v=0}^{n} |\sigma_{nv}| = O(1)$ and $\sum_{n=v}^{\infty} |\sigma_{nv}| = O(1)$. Now applying Hölder's inequality, we get

$$\begin{aligned}
\|x\|_{|F_u|_k} &= \left\{\sum_{n=0}^{\infty} u_n^{k-1} \left|\sum_{v=0}^{n} \sigma_{nv} x_v\right|^k\right\}^{1/k} \\
&\leq \left\{\sum_{n=0}^{\infty} u_n^{k-1} \sum_{v=0}^{n} |\sigma_{nv}|\,|x_v|^k \left(\sum_{v=0}^{n} |\sigma_{nv}|\right)^{k/k^*}\right\}^{1/k} \\
&= O(1)\left\{\sum_{v=0}^{\infty} |x_v|^k \sum_{n=v}^{\infty} |\sigma_{nv}|\right\}^{1/k} \\
&= O(1)\left\{\sum_{v=0}^{\infty} |x_v|^k\right\}^{1/k} = O(1)\,\|x\|_{\ell_k},
\end{aligned}$$

which completes the proof. ☐

**Theorem 3.2.** *Let $1 \leq k \leq q < \infty$. If there is a constant $M > 0$ such that $u_n \leq M$ for all $n \in \mathbb{N}$, then $|F_u|_k \subset |F_u|_q$.*

*Proof.* Take $x \in |F_u|_k$. Since $\ell_k \subset \ell_q$, then $\left(u_n^{\frac{1}{k^*}} \sum_{j=0}^{n} \sigma_{nj} x_j\right) \in \ell_q$ and also, since $u_n \leq M$ for all $n \in \mathbb{N}$,

$$M^{\frac{q}{k^*}-\frac{q}{q^*}} \left| u_n^{\frac{1}{q^*}} \sum_{j=0}^{n} \sigma_{nj} x_j \right|^q \leq \left| u_n^{1/k^*} \sum_{j=0}^{n} \sigma_{nj} x_j \right|^q ,$$

where $k^*$ and $q^*$ are the conjugate of exponent of $k$ and $q$, respectively. So this gives that $x \in |F_u|_q$, which completes the proof. □

**Theorem 3.3.** *Let $1 \leq k < \infty$. Then, $|F_u|_k$ is $BK$-space with respect to the norm*

$$\|x\|_{|F_u|_k} = \left\| E^{(k)} \circ T(x) \right\|_{\ell_k} .$$

*Also, the sequence $b^{(j)} = (b_n^{(j)})$ is a Schauder basis for the space $|F_u|_k$, where*

$$b_n^{(j)} = \begin{cases} u_j^{-1/k^*} \dfrac{f_{n+1}}{f_n} + u_j^{-1/k^*} \sum\limits_{r=j}^{n-1} \dfrac{f_{n+1}^2 - f_n^2}{f_r f_{r+1}}, & 0 \leq j \leq n-1 \\ u_n^{-1/k^*} \dfrac{f_{n+1}}{f_n}, & j = n \\ 0, & j > n. \end{cases}$$

*Proof.* We note that $\ell_k$ is a $BK$-space for $1 \leq k < \infty$. Further, since $E^{(k)} \circ T$ is a triangle matrix, it follows from Theorem 4.3.2 of [37], $|F_u|_k = (\ell_k)_{E^{(k)} \circ T}$ is a $BK$-space. Since the sequence $(e^{(j)})$ is the Schauder basis of the space $\ell_k$, it can be written from Theorem 2.3 in [14] that $b^{(j)} = (\widetilde{T}_n(\widetilde{E}^{(k)}(e^{(j)})))$ is a Schauder basis of the space $|F_u|_k$. □

**Theorem 3.4.** *Let $1 \leq k < \infty$. Then, the space $|F_u|_k$ is isomorphic to the space $\ell_k$ that is, $|F_u|_k \cong \ell_k$.*

*Proof.* To prove the theorem, we should show that there exists a linear bijection between the spaces $|F_u|_k$ and $\ell_k$ where $1 \leq k < \infty$. Let consider the transformations $T : |F_u|_k \to (\ell_k)_{E^{(k)}}, E^{(k)} : (\ell_k)_{E^{(k)}} \to \ell_k$ given in (1.3) and (1.4). Since the matrices corresponding these transformations are triangles, it can be easily seen that $T$ and $E^{(k)}$ are linear bijections. So, the composite function $E^{(k)} \circ T$ is a linear bijective operator. Furthermore,

$$\|x\|_{|F_u|_k} = \left\| E^{(k)} \circ T(x) \right\|_{\ell_k} ,$$

i.e., it preserves the norm. So the proof is completed. □

In the following theorems, for the simplicity of presentation we take

$$\xi_{vr} = \left( \frac{f_{v+1}}{f_v} + \left( f_{v+1}^2 - f_v^2 \right) \sum_{j=r}^{v-1} \frac{1}{f_j f_{j+1}} \right)$$

and define

$$D_1 = \left\{ \epsilon \in \omega : \sum_{v=r+1}^{\infty} \xi_{vr}\epsilon_v \text{ exists for all } r \right\},$$

$$D_2 = \left\{ \epsilon \in \omega : \sup_m \left( \frac{1}{u_m} \left| \epsilon_m \frac{f_{m+1}}{f_m} \right|^{k^*} + \sum_{r=0}^{m-1} \frac{1}{u_r} \left| \epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr}\epsilon_v \right|^{k^*} \right) < \infty \right\},$$

$$D_3 = \left\{ \epsilon \in \omega : \sup_{m,r} \left( \left| \epsilon_m \frac{f_{m+1}}{f_m} \right| + \left| \epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr}\epsilon_v \right| \right) < \infty \right\},$$

$$D_4 = \left\{ \epsilon \in \omega : \sum_{r=0}^{\infty} \frac{1}{u_r} \left( \sum_{v=r+1}^{\infty} |\xi_{vr}\epsilon_v| + \left| \epsilon_r \frac{f_{r+1}}{f_r} \right| \right)^{k^*} < \infty \right\},$$

$$D_5 = \left\{ \epsilon \in \omega : \sup_r \left( \sum_{v=r+1}^{\infty} |\xi_{vr}\epsilon_v| + \left| \epsilon_r \frac{f_{r+1}}{f_r} \right| \right) < \infty \right\}.$$

**Theorem 3.5.** *Let $1 < k < \infty$ and $u = (u_n)$ be a sequence of positive numbers. Then,*
(i) $\{|F_u|\}^{\alpha} = D_5$, $\{|F_u|_k\}^{\alpha} = D_4$;
(ii) $\{|F_u|\}^{\beta} = D_1 \cap D_3$, $\{|F_u|_k\}^{\beta} = D_1 \cap D_2$;
(iii) $\{|F_u|\}^{\gamma} = D_3$, $\{|F_u|_k\}^{\gamma} = D_2$.

*Proof.* (ii) Let's recall that $\epsilon \in \{|F_u|_k\}^{\beta}$ if and only if $\epsilon x = (\epsilon_n x_n) \in c_s$ for all $x \in |F_u|_k$. By (1.3) and (1.4), it can be seen immediately that

$$\begin{aligned}
\sum_{v=0}^{m} \epsilon_v x_v =& \epsilon_0 x_0 + \sum_{v=1}^{m} \epsilon_v \left( \frac{f_{v+1}}{f_v} y_v + \left( f_{v+1}^2 - f_v^2 \right) \sum_{j=0}^{v-1} \frac{y_j}{f_j f_{j+1}} \right) \\
=& \sum_{r=0}^{m} u_r^{-1/k^*} \sum_{v=r}^{m} \epsilon_v \frac{f_{v+1}}{f_v} z_r + \sum_{r=0}^{m-1} \left( \sum_{v=r+1}^{m} \epsilon_v (f_{v+1}^2 - f_v^2) \sum_{j=r}^{v-1} \frac{1}{f_j f_{j+1}} \right) u_r^{-1/k^*} z_r \\
=& u_m^{-1/k^*} \epsilon_m \frac{f_{m+1}}{f_m} z_m + \sum_{r=0}^{m-1} u_r^{-1/k^*} \left( \epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \epsilon_v \xi_{vr} \right) z_r \\
=& \sum_{r=0}^{m} h_{mr} z_r \quad (y = T(x), z = E^{(k)}(y))
\end{aligned}$$

where $H = (h_{mr})$ is defined by

$$h_{mr} = \begin{cases} u_r^{-1/k^*} \left( \epsilon_r \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \epsilon_v \xi_{vr} \right), & 0 \le r \le m-1, \\ u_m^{-1/k^*} \frac{f_{m+1}}{f_m} \epsilon_m, & r = m \\ 0, & r > m. \end{cases}$$

Therefore, $\epsilon \in \{|F_u|_k\}^{\beta}$ if and only if $H \in (\ell_k, c)$. Applying Lemma 1.4 to the matrix $H$, we get $\{|F_u|_k\}^{\beta} = D_1 \cap D_2$, which completes the proof.

The proofs of other parts can similarly be proved, so we omit. □

**Theorem 3.6.** *Let $1 \leq k < \infty$, $A = (a_{nv})$ be an infinite matrix of complex numbers for each $n, v \in \mathbb{N}$ and define the matrix $B^{(n)} = \left(b^{(n)}_{mr}\right)$ by*

$$b^{(n)}_{mr} = \begin{cases} a_{nr}\dfrac{f_{r+1}}{f_r} + \sum\limits_{v=r+1}^{m} a_{nv}\xi_{vr}, & 0 \leq r \leq m-1, \\ \dfrac{f_{m+1}}{f_m} a_{nm}, & r = m, \\ 0, & r > m. \end{cases}$$

*Further, let $\bar{B} = (\bar{b}_{nv})$ be a matrix given by $\bar{b}_{nv} = \lim\limits_{m} b^{(n)}_{mv}$ and $\widetilde{B} = E^{(k)} \circ T \circ \bar{B}$. Then, $A \in (|F_u|, |F_u|_k)$ if and only if*

$$\sum_{v=r+1}^{\infty} \xi_{vr} a_{nv} \text{ exists for all } r \tag{3.1}$$

$$\sup_{m,r} \left\{ \left| a_{nm}\frac{f_{m+1}}{f_m} \right| + \left| a_{nr}\frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr} a_{nv} \right| \right\} < \infty, \tag{3.2}$$

$$\sup_{r} \sum_{n=0}^{\infty} \frac{1}{u_r} \left| \widetilde{b}_{nr} \right|^k < \infty, \tag{3.3}$$

*If $A \in (|F_u|, |F_u|_k)$, then $A$ is a bounded linear operator,*

$$\|A\|_{\left(|F_u|,|F_u|_k\right)} = \left\| \widetilde{B} \right\|_{(l,l_k)}$$

*and*

$$\|A\|_{\chi} = \lim_{v \to \infty} \left\{ \sup_{r} \sum_{n=v+1}^{\infty} \frac{1}{u_r} \left| \widetilde{b}_{nr} \right|^k \right\}^{\frac{1}{k}}.$$

*Proof.* $A \in (|F_u|, |F_u|_k)$ if and only if $(a_{nv})_{v=0}^{\infty} \in \{|F_u|\}^{\beta}$ and $A(x) \in |F_u|_k$ for all $x \in |F_u|$. Now, it can be easily seen from Theorem 3.5, $(a_{nv})_{v=0}^{\infty} \in \{|F_u|\}^{\beta}$ if and only if (3.1) and (3.2) hold. On the other hand, if a matrix $R = (r_{nv}) \in (\ell, c)$, then the series $R_n(x) = \sum_{v=0}^{\infty} r_{nv} x_v$ converges uniformly in $n$, because, the remaining term of the series tends to zero uniformly in $n$, since

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \sup_{v} |r_{nv}| \sum_{v=m}^{\infty} |x_v| \to 0, \quad m \to \infty.$$

So we obtain

$$\lim_{n} R_n(x) = \sum_{v=0}^{\infty} \lim_{n} r_{nv} x_v. \tag{3.4}$$

Using (1.3), (1.4) and (3.4) it can be written that

$$A_n(x) = \lim_{m} \sum_{k=0}^{m} a_{nk} x_k = \lim_{m} \sum_{r=0}^{m} b^{(n)}_{mr} z_r = \sum_{r=0}^{\infty} \bar{b}_{nr} z_r.$$

Besides, according to Theorem 3.4, since $|F_u|_k \cong \ell_k$ for $1 \leq k < \infty$, it follows that $A(x) \in |F_u|_k$ for all $x \in |F_u|$ if and only if $\bar{B} \in (\ell, |F_u|_k)$, or equivalently, since $|F_u|_k = (\ell_k)_{E^{(k)} \circ T}$, $\widetilde{B} \in (\ell, \ell_k)$. Also, it is clear that the terms of matrix $\widetilde{B}$ can be expressed as

$$\begin{aligned}\hat{b}_{nr} &= \sum_{v=0}^{n} t_{nv}\bar{b}_{vr} = \frac{f_n}{f_{n+1}}\bar{b}_{nr} + \sum_{v=0}^{n-1}\frac{f_n^2 - f_{n+1}^2}{f_n f_{n+1}}\bar{b}_{vr},\\ \widetilde{b}_{nr} &= u_r^{1/k^*}\left(\hat{b}_{nr} - \hat{b}_{n-1,r}\right), \quad n \geq 1 \text{ and } \widetilde{b}_{0r} = \bar{b}_{0r}.\end{aligned}$$

Hence, applying Lemma 1.3 to the matrix $\widetilde{B}$, we have (3.3), which completes the first part of the proof.

Also, if $A \in (|F_u|, |F_u|_k)$, then, since the spaces $|F_u|_k$ and $|F_u|$ are $BK$-spaces, it is a bounded operator. In order to determine the operator norm of $A$, consider the isomorphisms $T : |F_u|_k \to (\ell_k)_{E^{(k)}}, E^{(k)} : (\ell_k)_{E^{(k)}} \to \ell_k$ defined as in Theorem 3.4. Then, it is easy to see that $A = \widetilde{T} \circ \widetilde{E}^{(k)} \circ \widetilde{B} \circ E^{(1)} \circ T$ and so,

$$\begin{aligned}\|A\|_{\left(|F_u|,|F_u|_k\right)} &= \sup_{x\neq 0}\frac{\|A(x)\|_{|F_u|_k}}{\|x\|_{|F_u|}} = \sup_{x\neq 0}\frac{\left\|\widetilde{T} \circ \widetilde{E}^{(k)} \circ \widetilde{B} \circ E^{(1)} \circ T(x)\right\|_{|F_u|_k}}{\|x\|_{|F_u|}}\\ &= \sup_{z\neq 0}\frac{\left\|\widetilde{B}(z)\right\|_{\ell_k}}{\|z\|_{\ell}} = \left\|\widetilde{B}\right\|_{(\ell,\ell_k)} (z = E^{(1)} \circ T(x)).\end{aligned}$$

Finally, assume that $Q$ is a unique ball in $|F_u|$. Since $E^{(k)} \circ T \circ AQ = \widetilde{B} \circ E^{(1)} \circ TQ$, we get that

$$\begin{aligned}\|A\|_{\chi} = \chi(AQ) &= \chi\left(E^{(k)} \circ T \circ AQ\right) = \chi\left(\widetilde{B} \circ E^{(1)} \circ TQ\right)\\ &= \lim_{v\to\infty}\left(\sup_{z\in E^{(1)}(T(Q))}\left\|(I - P_v)\left(\widetilde{B}(z)\right)\right\|\right)\\ &= \lim_{v\to\infty}\left\{\sup_r\left(\sum_{n=v+1}^{\infty}\frac{1}{u_r}\left|\widetilde{b}_{nr}\right|\right)^k\right\}^{\frac{1}{k}}.\end{aligned}$$

This completes the proof. □

By Theorem 3.6 and Lemma 2.2, the compact operators in this class are characterized as follows.

**Corollary 3.1.** *Under the hypothesis of Theorem* 3.6

$$A \in (|F_u|, |F_u|_k) \text{ is compact if and only if } \lim_{v\to\infty}\left\{\sup_r\left(\sum_{n=v+1}^{\infty}\frac{1}{u_r}\left|\widetilde{b}_{nr}\right|\right)^k\right\}^{\frac{1}{k}} = 0.$$

**Theorem 3.7.** *Let* $1 < k < \infty$, $A = (a_{nv})$ *be an infinite matrix of complex numbers for all* $n, v \in \mathbb{N}$ *and* $B^{(n)} = \left(b_{mv}^{(n)}\right)$ *be as in Theorem* 3.6. *Besides, define* $\bar{H} = (h_{nv})$

*by* $\bar{h}_{nv} = \lim_m u_v^{-1/k^*} b_{mv}^{(n)}$ *and* $\widetilde{H} = E^{(1)} \circ T \circ \bar{H}$. *Then,* $A \in (|F_u|_k, |F_u|)$ *if and only if*

$$\sum_{v=r+1}^{\infty} \xi_{vr} a_{nv} \text{ exist for all } r, \tag{3.5}$$

$$\sup_m \left\{ \frac{1}{u_m} \left| a_{nm} \frac{f_{m+1}}{f_m} \right|^{k^*} + \sum_{r=0}^{m-1} \frac{1}{u_r} \left| a_{nr} \frac{f_{r+1}}{f_r} + \sum_{v=r+1}^{m} \xi_{vr} a_{nv} \right|^{k^*} \right\} < \infty, \tag{3.6}$$

$$\sum_{r=0}^{\infty} \left( \sum_{n=0}^{\infty} \left| \widetilde{h}_{nr} \right| \right)^{k^*} < \infty. \tag{3.7}$$

*Moreover, if* $A \in (|F_u|_k, |F_u|)$, *then* $A$ *is a bounded linear operator,*

$$\|A\|_{\left(|F_u|_k, |F_u|\right)} = \left\| \widetilde{H} \right\|_{(\ell_k, \ell)}$$

*and*

$$\|A\|_{\chi} = \frac{1}{\xi} \lim_{v \to \infty} \left\{ \sum_{r=0}^{\infty} \left( \sum_{n=v+1}^{\infty} \left| \widetilde{h}_{nr} \right| \right)^{k^*} \right\}^{\frac{1}{k^*}},$$

*where* $1 \leq \xi \leq 4$.

*Proof.* $A \in (|F_u|_k, |F_u|)$ if and only if $A_n = (a_{nv})_{v=0}^{\infty} \in \{|F_u|_k\}^{\beta}$ and $A(x) \in |F_u|$ where $x \in |F_u|_k$. By Theorem 3.5, it can be easily seen that $A_n \in \{|F_u|_k\}^{\beta}$ if and only if (3.5) and (3.6) hold. Also, if any matrix $R = (r_{nv}) \in (\ell_k, c)$, then the series $R_n(x) = \sum_{v=0}^{\infty} r_{nv} x_v$ converges uniformly in $n$. Because, the remaining term of the series tends to zero uniformly in $n$, since

$$\left| \sum_{v=m}^{\infty} r_{nv} x_v \right| \leq \left( \sum_{v=m}^{\infty} |r_{nv}|^{k^*} \right)^{\frac{1}{k^*}} \left( \sum_{v=m}^{\infty} |x_v|^k \right)^{\frac{1}{k}} \to 0, \quad m \to \infty$$

and so, it can be written that

$$\lim_n R_n(x) = \sum_{v=0}^{\infty} \lim_n r_{nv} x_v. \tag{3.8}$$

Then, using (3.8), with a few calculations, we get

$$A_n(x) = \lim_m \sum_{k=0}^{m} a_{nk} x_k = \lim_m \sum_{r=0}^{m} u_r^{-1/k^*} b_{mr}^{(n)} z_r = \sum_{r=0}^{\infty} \bar{h}_{nr} z_r.$$

Since $|F_u|_k \cong \ell_k$ for $1 \leq k < \infty$, by the Theorem 3.4, then, $A(x) \in |F_u|$ for every $x \in |F_u|_k$ if and only if $\bar{H}(z) \in |F_u|$, i.e., $\widetilde{H}(z) = E^{(1)} \circ T \circ \bar{H}(z) \in \ell$ for every $z \in \ell_k$, where $z = E^{(k)} \circ T(x)$. This means that $\widetilde{H} \in (\ell_k, \ell)$. Thus applying Lemma 1.2 to the matrix $\widetilde{H}$, we get (3.7). This completes the proof of first part.

Since $|F_u|_k$ is $BK$-spaces for every $k \geq 1$, $A$ is a bounded operator by Theorem 4.2.8 of [37].

Additionally, as Theorem 3.4, it can be written that $A=\widetilde{T}\circ\widetilde{E}^{(1)}\circ\widetilde{H}\circ E^{(k)}\circ T$ and so,

$$\begin{aligned}\|A\|_{\left(|F_u|_k,|F_u|\right)} &= \sup_{x\neq 0}\frac{\|A(x)\|_{|F_u|}}{\|x\|_{|F_u|_k}}=\sup_{x\neq 0}\frac{\left\|\widetilde{H}\circ E^{(k)}\circ T(x)\right\|_{\ell}}{\|E^{(k)}\circ T(x)\|_{\ell_k}}\\ &= \sup_{z\neq 0}\frac{\left\|\widetilde{H}(z)\right\|_{\ell}}{\|z\|_{\ell_k}}=\left\|\widetilde{H}\right\|_{(\ell_k,\ell)}.\end{aligned}$$

Finally, let $Q=S_{|F_u|}$. Since $E^{(1)}\circ T\circ AQ=\widetilde{H}\circ E^{(k)}\circ TQ$, it follows by Lemma 2.1, Lemma 2.3 and Lemma 1.2 that

$$\begin{aligned}\|A\|_{\chi} &= \chi\left(AQ\right)=\chi\left(E^{(1)}\circ T\circ AQ\right)=\chi\left(\widetilde{H}\circ E^{(k)}\circ TQ\right)\\ &= \lim_{v\to\infty}\left(\sup_{z\in E^{(k)}(T(Q))}\left\|(I-P_v)(\widetilde{H}(z))\right\|_{\ell_k}\right)\\ &= \frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^{k^*}\right\}^{\frac{1}{k^*}},\end{aligned}$$

which completes the proof. □

Also, the compact operators can immediately be characterized by Lemma 2.2 and Theorem 3.7 as follows.

**Corollary 3.2.** *Under the conditions of Theorem* 3.7

$$A\in\mathcal{C}\left(|F_u|_k,|F_u|\right)\Leftrightarrow\frac{1}{\xi}\lim_{v\to\infty}\left\{\sum_{r=0}^{\infty}\left(\sum_{n=v+1}^{\infty}\left|\widetilde{h}_{nr}\right|\right)^{k^*}\right\}^{\frac{1}{k^*}}=0.$$

## References

[1] B. Altay and F. Başar, *The fine spectrum and the matrix domain of the difference operator* $\Delta$ *on the sequence space* $\ell_p$ $(0<p<1)$, Commun. Math. Anal. **2**(2) (2007), 1–11.
[2] B. Altay and F. Başar, *Some paranormed Riesz sequence spaces of non-absolute type*, Southeast Asian Bull. Math. **30**(5) (2006), 591–608.
[3] B. Altay, F. Başar and M. Mursaleen, *On the Euler sequence spaces which include the spaces* $l_p$ *and* $l_\infty$ *I*, Inform. Sci. **176**(10) (2005), 1450–1462.
[4] F. Başar and N. L. Braha, *Euler-Cesàro difference spaces of bounded, convergent and null sequences*, Tamkang J. Math. **47**(4) (2016), 405–420.
[5] F. Başar, *Summability Theory and Its Applications*, Bentham Science Publishers, e-books, Monographs, Istanbul, 2012.
[6] F. Başar and B. Altay, *On the space of sequences of p-bounded variation and related matrix map- pings*, (English, Ukrainian summary) Ukrain. Mat. Zh. **55**(1) (2003), 108–118, reprinted in Ukrainian Math. J. **55**(1) (2003), 136–147.
[7] M. Başarır, F. Başar and E. E. Kara, *On the spaces of Fibonacci difference absolutely p-summable, null and convergent sequences*, Sarajevo J. Math. **12**(25) (2016), 167–182.
[8] J. Boos and P. Cass, *Classical and Modern Methods in Summability*, Oxford University Press, New York, 2000.

[9] H. Bor, *On summability factors of infinite series*, Tamkang J. Math. **16**(1) (1985), 13–20.

[10] H. Bor and B. Thorpe, *On some absolute summability methods*, Analysis **7**(2) (1987), 145–152.

[11] T. M. FLett, *On an extension of absolute summability and some theorems of Littlewood and Paley*, Proc. Lond. Math. Soc. **7** (1957), 113–141.

[12] G. C. Hazar and M. A. Sarıgöl, *Compact and matrix operators on the space* $|C,-1|_k$, J. Comput. Anal. Appl. **25** (2018), 1014–1024.

[13] G. C. Hazar and F. Gökçe, *Characterizations of matrix transformatios on the series spaces derived by absolute factorable summability*, Developments in Science and Engineering (2016), 411–426.

[14] A. M. Jarrah and E. Malkowsky, *Ordinary absolute and strong summability and matrix transformations*, Filomat **17** (2003), 59–78.

[15] E. E. Kara, *Some topological and geometrical properties of new Banach sequence spaces*, J. Inequal. Appl. **2013**(1) (38) (2013), 16 pages.

[16] E. E. Kara and M. Ilkhan, *On some Banach sequence spaces derived by a new band matrix*, Br. J. Math. Comput. Sci. **9**(2) (2015), 141–159.

[17] E. E. Kara and M. Ilkhan, *Some properties of generalized Fibonacci sequence spaces*, Linear Multilinear Algebra **64**(11) (2016), 2208–2223.

[18] V. Karakaya, A. K. Noman and H. Polat, *On paranormed* $\lambda$*-sequence spaces of non-absolute type*, Math. Comput. Modelling **54**(5) (2011), 1473–1480.

[19] T. Koshy, *Fibonacci and Lucas Numbers with Applications*, John Wiley and Sons, New York, 2011.

[20] I. J. Maddox, *Elements of Functinal Analysis*, Cambridge University Press, London, New York, 1970.

[21] E. Malkowsky and V. Rakocević, *On matrix domains of triangles*, Appl. Math. Comput. **189**(2) (2007), 1146–1163.

[22] E. Malkowsky and V. Rakocević, *An introduction into the theory of sequence space and measures of noncompactness*, Zb. Rad. (Beogr.) **9**(17) (2000), 143–234.

[23] M. R. Mehdi, *Summability factors for generalized absolute summability I*, Proc. Lond. Math. Soc. **3**(1) (1960), 180–200.

[24] R. N. Mohapatra and M. A. Sarıgöl, *On matrix operators on the series spaces* $\left|\bar{N}_p^{\theta}\right|_k$, Ukrain Mat. Zh. **69**(11) (2017), 1524–1533.

[25] M. Mursaleen, F. Başar and B. Altay, *On the Euler sequence spaces which include the spaces* $l_p$ *and* $l_\infty$ *II*, Nonlinear Anal. **65**(3) (2006), 707–717.

[26] C. Orhan and M. A. Sarıgöl, *On absolute weighted mean summability*, Rocky Mountain J. Math. **23**(3) (1993), 1091–1097.

[27] V. Rakocević, *Measures of noncompactness and some applications*, Filomat **12**(2) (1998), 87–120.

[28] M. A. Sarıgöl, *Spaces of series summable by absolute Cesàro and matrix operators*, Communications in Mathematics and Applications **7**(1) (2016), 11–22.

[29] M. A. Sarıgöl, *Extension of Mazhar's theorem on summability factors*, Kuwait J. Sci. **42**(3) (2015), 28–35.

[30] M. A. Sarıgöl, *Matrix transformations on fields of absolute weighted mean summability*, Studia Sci. Math. Hungar. **48**(3) (2011), 331–341.

[31] M. A. Sarıgöl, *Necessary and sufficient conditions for the equivalence of the summability methods* $\left|\bar{N},p_n\right|_k$ *and* $|C,1|_k$, Indian J. Pure Appl. Math. **22**(6) (1991), 483–489.

[32] M. A. Sarıgöl, *On the local properties of factored Fourier series*, Appl. Math. Comput. **216**(11) (2010), 3386–3390.

[33] M. A. Sarıgöl, *On two absolute Riesz summability factors of infinite series*, Proc. Amer. Math. Soc. **118**(2) (1993), 485–488.

[34] M. A. Sarıgöl, *On absolute weighted mean summability methods*, Proc. Amer. Math. Soc. **115**(1) (1992), 157–160.

[35] M. Stieglitz and H. Tietz, *Matrix transformationen von Folgenraumen. Eine Ergebnisübersicht*, Math. Z. **154**(1) (1977), 1–16.

[36] W. T. Sulaiman, *On summability factors of infinite series*, Proc. Amer. Math. Soc. **115** (1992), 313–317.

[37] A. Wilansky, *Summability Through Functional Analysis*, Mathematics Studies **85**, North Holland, Amsterdam, 1984.

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# A NOTE ON ALMOST ANTI-PERIODIC FUNCTIONS IN BANACH SPACES

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Abstract. The main aim of this note is to introduce the notion of an almost anti-periodic function in Banach space. We prove some characterizations for this class of functions, investigating also its relationship with the classes of anti-periodic functions and almost periodic functions in Banach spaces.

## 1. Introduction and Preliminaries

As mentioned in the abstract, the main aim of this note is to introduce the notion of an almost anti-periodic function in Banach space as well as to prove some characterizations for this class of functions. Any anti-periodic function is almost anti-periodic, and any almost anti-periodic function is almost periodic. Unfortunately, almost anti-periodic functions do not have a linear vector structure with the usually considered operations of pointwise addition of functions and multiplication with scalars. The main result of paper is Theorem 2.3, in which we completely profile the closure of linear span of almost anti-periodic functions in the space of almost periodic functions. We also prove some other statements regarding almost anti-periodic functions, and introduce the concepts of Stepanov almost anti-periodic functions, asymptotically almost anti-periodic functions and Stepanov asymptotically almost anti-periodic functions. We investigate the almost anti-periodic properties of convolution products, providing also a few elementary examples and applications.

Let $(X, \| \cdot \|)$ be a complex Banach space. By $C_b([0,\infty) : X)$ we denote the space consisting of all bounded continuous functions from $[0,\infty)$ into $X$, the symbol $C_0([0,\infty) : X)$ denotes the closed subspace of $C_b([0,\infty) : X)$ consisting of functions

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vanishing at infinity. By $BUC([0,\infty) : X)$ we denote the space consisted of all bounded uniformly continuous functions from $[0,\infty)$ to $X$. This space becomes one of Banach's endowed with the sup-norm.

The concept of almost periodicity was introduced by Danish mathematician H. Bohr around 1924-1926 and later generalized by many other authors (cf. [6–9] and [16] for more details on the subject). Let $I = \mathbb{R}$ or $I = [0,\infty)$, and let $f : I \to X$ be continuous. Given $\epsilon > 0$, we call $\tau > 0$ an $\epsilon$-period for $f(\cdot)$ if and only if

$$\|f(t+\tau) - f(t)\| \leq \epsilon, \quad t \in I.$$

The set constituted of all $\epsilon$-periods for $f(\cdot)$ is denoted by $\vartheta(f,\epsilon)$. It is said that $f(\cdot)$ is almost periodic, a.p. for short, if and only if for each $\epsilon > 0$ the set $\vartheta(f,\epsilon)$ is relatively dense in $I$, which means that there exists $l > 0$ such that any subinterval of $I$ of length $l$ meets $\vartheta(f,\epsilon)$.

The space consisted of all almost periodic functions from the interval $I$ into $X$ will be denoted by $AP(I : X)$. Equipped with the sup-norm, $AP(I : X)$ becomes a Banach space.

For the sequel, we need some preliminary results appearing already in the pioneering paper [2] by H. Bart and S. Goldberg, who introduced the notion of an almost periodic strongly continuous semigroup there (see [1] for more details on the subject). The translation semigroup $(W(t))_{t\geq 0}$ on $AP([0,\infty) : X)$, given by $[W(t)f](s) := f(t+s)$, $t \geq 0$, $s \geq 0$, $f \in AP([0,\infty) : X)$ is consisted solely of surjective isometries $W(t)$ $(t \geq 0)$ and can be extended to a $C_0$-group $(W(t))_{t\in\mathbb{R}}$ of isometries on $AP([0,\infty) : X)$, where $W(-t) := W(t)^{-1}$ for $t > 0$. Furthermore, the mapping $E : AP([0,\infty) : X) \to AP(\mathbb{R} : X)$, defined by

$$[Ef](t) := [W(t)f](0), \quad t \in \mathbb{R},\ f \in AP([0,\infty) : X),$$

is a linear surjective isometry and $Ef$ is the unique continuous almost periodic extension of a function $f(\cdot)$ from $AP([0,\infty) : X)$ to the whole real line. We have that $[E(Bf)] = B(Ef)$ for all $B \in L(X)$ and $f \in AP([0,\infty) : X)$.

The most intriguing properties of almost periodic vector-valued functions are collected in the following two theorems (in the case that $I = \mathbb{R}$, these assertions are well-known in the existing literature; in the case that $I = [0,\infty)$, then these assertions can be deduced by using their validity in the case $I = \mathbb{R}$ and the properties of extension mapping $E(\cdot)$; see [14] for more details).

**Theorem 1.1.** *Let $f \in AP(I : X)$. Then the following holds:*

(i) $f \in BUC(I : X)$;
(ii) *if $g \in AP(I : X)$, $h \in AP(I : \mathbb{C})$, $\alpha,\ \beta \in \mathbb{C}$, then $\alpha f + \beta g$ and $hf \in AP(I : X)$;*
(iii) *Bohr's transform of $f(\cdot)$,*

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_0^t e^{-irs} f(s)\, ds,$$

*exists for all* $r \in \mathbb{R}$ *and*

$$P_r(f) := \lim_{t\to\infty} \frac{1}{t} \int_{\alpha}^{t+\alpha} e^{-irs} f(s)\, ds,$$

*for all* $\alpha \in I,\ r \in \mathbb{R}$*;*

(iv) *if* $P_r(f) = 0$ *for all* $r \in \mathbb{R}$, *then* $f(t) = 0$ *for all* $t \in I$*;*

(v) $\sigma(f) := \{r \in \mathbb{R} : P_r(f) \neq 0\}$ *is at most countable;*

(vi) *if* $c_0 \nsubseteq X$, *which means that* $X$ *does not contain an isomorphic copy of* $c_0$, $I = \mathbb{R}$ *and* $g(t) = \int_0^t f(s)\, ds$ $(t \in \mathbb{R})$ *is bounded, then* $g \in AP(\mathbb{R} : X)$*;*

(vii) *if* $(g_n)_{n\in\mathbb{N}}$ *is a sequence in* $AP(I : X)$ *and* $(g_n)_{n\in\mathbb{N}}$ *converges uniformly to* $g$, *then* $g \in AP(I : X)$*;*

(viii) *if* $I = \mathbb{R}$ *and* $f' \in BUC(\mathbb{R} : X)$, *then* $f' \in AP(\mathbb{R} : X)$*;*

(ix) (*spectral synthesis*) $f \in \overline{\mathrm{Span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}}$*;*

(x) $R(f)$ *is relatively compact in* $X$*;*

(xi) *we have*

$$\|f\|_\infty = \sup_{t\geq t_0} \|f(t)\|, \quad t_0 \in I.$$

**Theorem 1.2** (Bochner's criterion)**.** *Let* $f \in BUC(\mathbb{R} : X)$. *Then* $f(\cdot)$ *is almost periodic if and only if for any sequence* $(b_n)$ *of numbers from* $\mathbb{R}$ *there exists a subsequence* $(a_n)$ *of* $(b_n)$ *such that* $(f(\cdot + a_n))$ *converges in* $BUC(\mathbb{R} : X)$.

Theorem 1.2 has served S. Bochner to introduce the notion of an almost automorphic function, which slightly generalize the notion of an almost periodic function [4]. For more details about almost periodic and almost automorphic solutions of abstract Volterra integro-differential equations, we refer the reader to the monographs by T. Diagana [6], G. M. N'Guérékata [9], M. Kostić [14] and M. Levitan, V. V. Zhikov [16].

By either $AP(\Lambda : X)$ or $AP_\Lambda(I : X)$, where $\Lambda$ is a non-empty subset of $I$, we denote the vector subspace of $AP(I : X)$ consisting of all functions $f \in AP(I : X)$ for which the inclusion $\sigma(f) \subseteq \Lambda$ holds good. It can be easily seen that $AP(\Lambda : X)$ is a closed subspace of $AP(I : X)$ and therefore Banach space itself.

## 2. Almost Anti-Periodic Functions

Assume that $I = \mathbb{R}$ or $I = [0, \infty)$, as well as that $f : I \to X$ is continuous. Given $\epsilon > 0$, we call $\tau > 0$ an $\epsilon$-antiperiod for $f(\cdot)$ if and only if

$$\|f(t + \tau) + f(t)\| \leq \epsilon, \quad t \in I. \tag{2.1}$$

In what follows, by $\vartheta_{ap}(f, \epsilon)$ we denote the set of all $\epsilon$-antiperiods for $f(\cdot)$.

We introduce the notion of an almost anti-periodic function as follows.

**Definition 2.1.** It is said that $f(\cdot)$ is almost anti-periodic if and only if for each $\epsilon > 0$ the set $\vartheta_{ap}(f, \epsilon)$ is relatively dense in $I$.

Suppose that $\tau > 0$ is an $\epsilon$-antiperiod for $f(\cdot)$. Applying (2.1) twice, we get that

$$\begin{aligned}\|f(t+2\tau)-f(t)\| &= \|[f(t+2\tau)+f(t+\tau)]-[f(t+\tau)+f(t)]\| \\ &\leq \|f(t+2\tau)+f(t+\tau)\|+\|f(t+\tau)+f(t)\| \leq 2\epsilon, \quad t\in I.\end{aligned}$$

Taking this inequality in account, we obtain almost immediately from elementary definitions that $f(\cdot)$ needs to be almost periodic. Further on, assume that $f : I \to X$ is anti-periodic, i.e, there exists $\omega > 0$ such that $f(t+\omega) = -f(t)$, $t \in I$. Then we obtain inductively that $f(t+(2k+1)\omega) = -f(t)$, $k \in \mathbb{Z}$, $t \in I$. Since the set $\{(2k+1)\omega : k \in \mathbb{Z}\}$ is relatively dense in $I$, the above implies that $f(\cdot)$ is almost anti-periodic. Therefore, we have proved the following theorem.

**Theorem 2.1.** (i) *Assume $f : I \to X$ is almost anti-periodic. Then $f : I \to X$ is almost periodic.*
(ii) *Assume $f : I \to X$ is anti-periodic. Then $f : I \to X$ is almost anti-periodic.*

It is well known that any anti-periodic function $f : I \to X$ is periodic since, with the notation used above, we have that $f(t+2k\omega) = f(t)$, $k \in \mathbb{Z}\setminus\{0\}$, $t \in I$. But, the constant non-zero function is a simple example of a periodic function (therefore, almost periodic function) that is neither anti-periodic nor almost anti-periodic.

*Example* 2.1. (i) Consider the function $f(t) := \sin(\pi t)+\sin(\pi t\sqrt{2})$, $t \in \mathbb{R}$. This is an example of an almost anti-periodic function that is not a periodic function. This can be verified as it has been done by A. S. Besicovitch [3, Introduction, p. ix].
(ii) The function $g(t) := f(t)+5$, $t \in \mathbb{R}$, where $f(\cdot)$ is defined as above, is almost periodic, not almost anti-periodic and not periodic.

We continue by noting the following simple facts. Let $f : I \to X$ be continuous, and let $\epsilon' > \epsilon > 0$. Then the following holds true.

(i) $\vartheta_{ap}(f,\epsilon) \subseteq \vartheta_{ap}(f,\epsilon')$.
(ii) If $I = \mathbb{R}$ and (2.1) holds with some $\tau > 0$, then (2.1) holds with $-\tau$.
(iii) If $I = \mathbb{R}$ and $\tau_1, \tau_2 \in \vartheta_{ap}(f,\epsilon)$, then $\tau_1 \pm \tau_2 \in \vartheta(f,\epsilon)$.

Furthermore, the argumentation contained in the proofs of structural results of [3, pp. 3-4] shows that the following holds.

**Theorem 2.2.** *Let $f : I \to X$ be almost anti-periodic. Then we have the following.*
(i) *$cf(\cdot)$ is almost anti-periodic for any $c \in \mathbb{C}$.*
(ii) *If $X = \mathbb{C}$ and $\inf_{x\in\mathbb{R}} |f(x)| = m > 0$, then $1/f(\cdot)$ is almost anti-periodic.*
(iii) *If $(g_n : I \to X)_{n\in\mathbb{N}}$ is a sequence of almost anti-periodic functions and $(g_n)_{n\in\mathbb{N}}$ converges uniformly to a function $g : I \to X$, then $g(\cdot)$ is almost anti-periodic.*

Concerning products and sums of almost anti-periodic functions, the situation is much more complicated than for the usually examined class of almost periodic functions.

*Example* 2.2. (i) The product of two scalar almost anti-periodic functions need not be almost anti-periodic. To see this, consider the functions $f_1(t) = f_2(t) = \cos t$, $t \in \mathbb{R}$, which are clearly (almost) anti-periodic. Then $f_1(t) \cdot f_2(t) = \cos^2 t$, $t \in \mathbb{R}$, $\cos^2(t+\tau) + \cos^2 t \geq \cos^2 t$, $\tau$, $t \in \mathbb{R}$, and therefore $\vartheta_{ap}(f_1 \cdot f_2, \epsilon) = \emptyset$ for any $\epsilon \in (0,1)$.

(ii) The sum of two scalar almost anti-periodic functions need not be almost anti-periodic, so that the almost anti-periodic functions do not form a vector space. To see this, consider the functions $f_1(t) = 2^{-1} \cos 4t$ and $f_2(t) = 2\cos 2t$, $t \in \mathbb{R}$, which are clearly (almost) anti-periodic. Then

$$f_1(t) + f_2(t) = 4\cos^4 t - \frac{3}{2}, \quad t \in \mathbb{R}.$$

Asssume that $f_1 + f_2$ is almost anti-periodic. Then the above identity implies that the function $t \mapsto 8\cos^4 t - 3$, $t \in \mathbb{R}$, is almost anti-periodic, as well. This, in particular, yields that for any $\epsilon \in (0,1)$ we can find $\tau \in \mathbb{R}$ such that

$$\Big| 8\cos^4(t+\tau) + 8\cos^4 t - 6 \Big| \leq \epsilon, \quad t \in \mathbb{R}.$$

Plugging $t = \pi$, we get that $8\cos^4 \tau + 2 \leq \epsilon$, which is a contradiction. Finally, we would like to point out that there exists a large number of much simpler examples which can be used for verification of the statement clarified in this part; for example, the interested reader can easily check that the function $t \mapsto \cos t + \cos 2t$, $t \in \mathbb{R}$, is not almost anti-periodic.

Assume that $f : I \to X$ is almost anti-periodic. Then it can be easily seen that $f(\cdot + a)$ and $f(b\,\cdot)$ are likewise almost anti-periodic, where $a \in I$ and $b \in I \setminus \{0\}$.

Denote now by $ANP_0(I : X)$ the linear span of almost anti-periodic functions $I \mapsto X$. By Theorem 2.1(i), $ANP_0(I : X)$ is a linear subspace of $AP(I : X)$. Let $ANP(I : X)$ be the linear closure of $ANP_0(I : X)$ in $AP(I : X)$. Then, clearly, $ANP(I : X)$ is a Banach space. Furthermore, we have the following result.

**Theorem 2.3.** $ANP(I : X) = AP_{\mathbb{R}\setminus\{0\}}(I : X)$.

*Proof.* Since the mapping $E : AP([0,\infty) : X) \to AP(\mathbb{R} : X)$ is a linear surjective isometry, it suffices to consider the case in which $I = \mathbb{R}$. Assume first that $f \in AP_{\mathbb{R}\setminus\{0\}}(I : X)$. By spectral synthesis (see Theorem 1.1(ix)), we have that

$$f \in \overline{\operatorname{Span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\}},$$

where the closure is taken in the space $AP(\mathbb{R} : X)$. Since $\sigma(f) \subseteq \mathbb{R} \setminus \{0\}$ and the function $t \mapsto e^{i\mu t}$, $t \in \mathbb{R}$, $\mu \in \mathbb{R} \setminus \{0\}$ is anti-periodic, we have that $\operatorname{Span}\{e^{i\mu\cdot}x : \mu \in \sigma(f),\ x \in R(f)\} \subseteq ANP_0(\mathbb{R} : X)$. Hence, $f \in ANP(\mathbb{R} : X)$. The converse statement immediately follows if we prove that, for any fixed function $f \in ANP(\mathbb{R} : X)$, we have that $P_0(f) = 0$, i.e.,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(s)\, ds = 0. \tag{2.2}$$

By almost periodicity of $f(\cdot)$, the limit in (2.2) exists. Hence, it is enough to show that for any given number $\epsilon > 0$ we can find a sequence $(\omega_n)_{n\in\mathbb{N}}$ of positive reals such that $\lim_{n\to\infty}\omega_n = \infty$ and

$$\left\| \frac{1}{2\omega_n}\int_0^{2\omega_n} f(s)\,ds \right\| \leq \frac{\epsilon}{2}, \quad n \in \mathbb{N}. \tag{2.3}$$

By definition of almost anti-periodicity, we have the existence of a number $l > 0$ such that any interval $I_n = [nl, (n+1)l]$ $(n \in \mathbb{N})$ contains a number $\omega_n$ that is anti-period for $f(\cdot)$. The validity of (2.3) is a consequence of the following computation:

$$\begin{aligned}\left\| \int_0^{2\omega_n} f(s)\,ds \right\| &= \left\| \int_0^{\omega_n} f(s)\,ds + \int_{\omega_n}^{2\omega_n} f(s)\,ds \right\| \\ &= \left\| \int_0^{\omega_n} [f(s) + f(s+\omega_n)]\,ds \right\| \\ &\leq \int_0^{\omega_n} \|f(s)+f(s+\omega_n)\|\,ds \leq \epsilon\omega_n, \quad n \in \mathbb{N},\end{aligned}$$

finishing the proof of theorem. □

Let $f \in AP(I : X)$ and $\emptyset \neq \Lambda \subseteq \mathbb{R}$. Since

$$\sigma(f) \subseteq \Lambda \text{ if and only if } P_r(f) = 0,\ r \in \mathbb{R}\backslash\Lambda \text{ if and only if } P_0\Big(e^{-ir\cdot} f(\cdot)\Big) = 0,\ r \in \mathbb{R}\backslash\Lambda,$$

we have the following corollary of Theorem 2.3 (see also [1, Corollary 4.5.9]).

**Corollary 2.1.** *Let $f \in AP(I : X)$ and $\emptyset \neq \Lambda \subseteq \mathbb{R}$. Then $f \in AP_\Lambda(I : X)$ if and only if $e^{-ir\cdot} f(\cdot) \in ANP(I : X)$ for all $r \in \mathbb{R} \setminus \Lambda$.*

Further on, Theorem 2.3 combined with the obvious equality $\sigma(Ef) = \sigma(f)$ immediately implies that the unique ANP extension of a function $f \in ANP([0,\infty) : X)$ to the whole real axis is $Ef(\cdot)$. As the next proposition shows, this also holds for almost anti-periodic functions.

**Proposition 2.1.** *Suppose that $f : [0,\infty) \to X$ is almost anti-periodic. Then $Ef : \mathbb{R} \to X$ is a unique almost anti-periodic extension of $f(\cdot)$ to the whole real axis.*

*Proof.* The uniqueness of an almost anti-periodic extension of $f(\cdot)$ follows from the uniqueness of an almost periodic extension of $f(\cdot)$. It remains to be proved that $Ef : \mathbb{R} \to X$ is almost anti-periodic. To see this, let $\epsilon > 0$ be given. Then there exists $l > 0$ such that any interval $I \subseteq [0,\infty)$ of length $l$ contains a number $\tau \in I$ such that $\|f(s+\tau)+f(s)\| \leq \epsilon$, $s \geq 0$. We only need to prove that any interval $I \subseteq \mathbb{R}$ of length $2l$ contains a number $\tau \in I$ such that

$$\|[Ef](t+\tau) + [Ef](t)\| = \|[W(t+\tau)f + W(t)f](0)\| \leq \epsilon, \quad t \in \mathbb{R}.$$

If $I \subseteq [0,\infty)$, then the situation is completely clear. Suppose now that $I \subseteq (-\infty, 0]$. Then $-I \subseteq [0,\infty)$ and there exists a number $-\tau \in -I$ such that

$$\sup_{s\geq 0} \|f(s-\tau) + f(s)\| \leq \epsilon.$$

Then the conclusion follows from the computation

$$\begin{aligned}\|[W(t+\tau)f+W(t)f](0)\| &\leq \|W(t+\tau)f+W(t)f\|_{L^\infty([0,\infty))}\\ &\leq \|W(t+\tau)\|_{L^\infty([0,\infty))}\|W(-\tau)f+f\|_{L^\infty([0,\infty))}\\ &= \sup_{s\geq 0}\|f(s-\tau)+f(s)\| \leq \epsilon, \quad t\in\mathbb{R}.\end{aligned}$$

Finally, if $I = I_1 \cup I_2$, where $I_1 = [a, 0]$ $(a < 0)$ and $I_2 = [0, b]$ $(b > 0)$, then $|a| \geq l$ or $b \geq l$. In the case that $|a| \geq l$, then the conclusion follows similarly as in the previously considered case. If $b \geq l$, then the conclusion follows from the computation

$$\begin{aligned}\|[W(t+\tau)f+W(t)f](0)\| &\leq \|W(t+\tau)f+W(t)f\|_{L^\infty([0,\infty))}\\ &\leq \|W(t)\|_{L^\infty([0,\infty))}\|W(\tau)f+f\|_{L^\infty([0,\infty))}\\ &= \sup_{s\geq 0}\|f(s+\tau)+f(s)\| \leq \epsilon, \quad t\in\mathbb{R},\end{aligned}$$

where $\tau \in I_2$ is an $\epsilon$-antiperiod of $f(\cdot)$. □

For various generalizations of almost periodic functions, we refer the reader to [14]. In the following definition, we introduce the notion of a Stepanov almost anti-periodic function.

**Definition 2.2.** Let $1 \leq p < \infty$, and let $f \in L^p_{loc}(I : X)$. Then we say that $f(\cdot)$ is Stepanov $p$-almost anti-periodic function, $S^p$-almost anti-periodic shortly, if and only if the function $\hat{f} : I \to L^p([0,1] : X)$, defined by

$$\hat{f}(t)(s) := f(t+s), \quad t \in I,\ s \in [0,1],$$

is almost anti-periodic.

It can be easily seen that any almost anti-periodic function needs to be $S^p$-almost anti-periodic, as well as that any $S^p$-almost anti-periodic function has to be $S^p$-almost periodic, $1 \leq p < \infty$.

## 3. Almost Anti-Periodic Properties of Convolution Products

Since almost anti-periodic functions do not form a vector space, we will focus our attention here to the almost anti-periodic properties of finite and infinite convolution product, which is undoubtedly a safe and sound way for providing certain applications to abstract PDEs.

**Proposition 3.1.** *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that $M := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} < \infty$. If $g : \mathbb{R} \to X$ is $S^p$-almost anti-periodic, then the function $G(\cdot)$, given by*

$$G(t) := \int_{-\infty}^{t} R(t-s)g(s)\,ds, \quad t \in \mathbb{R}, \tag{3.1}$$

*is well-defined and almost anti-periodic.*

*Proof.* It can be easily seen that, for every $t \in \mathbb{R}$, we have $G(t) = \int_0^\infty R(s)g(t-s)\,ds$. Since $g(\cdot)$ is $S^p$-almost periodic, we can apply [15, Proposition 2.11] in order to see that $G(\cdot)$ is well-defined and almost periodic. It remains to be proved that $G(\cdot)$ is almost anti-periodic. Let a number $\epsilon > 0$ be given in advance. Then we can find a finite number $l > 0$ such that any subinterval $I$ of $\mathbb{R}$ of length $l$ contains a number $\tau \in I$ such that $\int_t^{t+1} \|g(s+\tau)+g(s)\|^p\,ds \leq \epsilon^p$, $t \in \mathbb{R}$. Applying Hölder inequality and this estimate, similarly as in the proof of above-mentioned proposition, we get that

$$\begin{aligned}
\|G(t+\tau)+G(t)\| &\leq \int_0^\infty \|R(r)\| \cdot \|g(t+\tau-r)+g(t-r)\|\,dr \\
&= \sum_{k=0}^{\infty} \int_k^{k+1} \|R(r)\| \cdot \|g(t+\tau-r)+g(t-r)\|\,dr \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left( \int_k^{k+1} \|g(t+\tau-r)+g(t-r)\|^p\,dr \right)^{1/p} \\
&= \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]} \left( \int_{t-k-1}^{t-k} \|g(s+\tau)+g(s)\|^p\,ds \right)^{1/p} \\
&\leq \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[k,k+1]}\epsilon = M\epsilon, \quad t \in \mathbb{R},
\end{aligned}$$

which clearly implies that the set of all $\epsilon$-antiperiods of $G(\cdot)$ is relatively dense in $\mathbb{R}$. $\square$

In order to relax our exposition, we shall introduce the notion of an asymptotically ($S^p$-)almost anti-periodic function in the following way (cf. also [12, Lemma 1.1]):

**Definition 3.1.** (i) Let $f \in C_b([0,\infty) : X)$. Then we say that $f(\cdot)$ is asymptotically almost anti-periodic if and only if there are two locally functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:
  (a) $g$ is almost anti-periodic;
  (b) $q$ belongs to the class $C_0([0,\infty) : X)$;
  (c) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

(ii) Let $1 \leq p < \infty$, and let $f \in L^p_{loc}([0,\infty) : X)$. Then we say that $f(\cdot)$ is asymptotically Stepanov $p$-almost anti-periodic, asymptotically $S^p$-almost anti-periodic shortly, if and only if there are two locally $p$-integrable functions $g : \mathbb{R} \to X$ and $q : [0,\infty) \to X$ satisfying the following conditions:
  (a) $g$ is $S^p$-almost anti-periodic;
  (b) $\hat{q}$ belongs to the class $C_0([0,\infty) : L^p([0,1] : X))$;
  (c) $f(t) = g(t) + q(t)$ for all $t \geq 0$.

Keeping in mind Proposition 3.1 and the proof of [15, Propostion 2.13], we can simply clarify the following result.

**Proposition 3.2.** *Suppose that $1 \leq p < \infty$, $1/p + 1/q = 1$ and $(R(t))_{t>0} \subseteq L(X)$ is a strongly continuous operator family satisfying that, for every $s \geq 0$, we have that*

$$m_s := \sum_{k=0}^{\infty} \|R(\cdot)\|_{L^q[s+k,s+k+1]} < \infty.$$

*Suppose, further, that $f : [0, \infty) \to X$ is asymptotically $S^p$-almost anti-periodic as well as that the locally $p$-integrable functions $g : \mathbb{R} \to X$, $q : [0, \infty) \to X$ satisfy the conditions from* Definition 3.1(ii). *Let there exist a finite number $M > 0$ such that the following holds:*

(i) $\lim_{t\to+\infty} \int_t^{t+1} \Big[\int_M^s \|R(r)\|\|q(s-r)\|\, dr\Big]^p\, ds = 0$;

(ii) $\lim_{t\to+\infty} \int_t^{t+1} m_s^p\, ds = 0$.

*Then the function $H(\cdot)$, given by*

$$H(t) := \int_0^t R(t-s)f(s)\, ds, \quad t \geq 0,$$

*is well-defined, bounded and asymptotically $S^p$-almost anti-periodic.*

Before providing some applications, we want to note that our conclusions from [15, Remark 2.14] and [14, Proposition 2.7.5] can be reformulated for asymptotical almost anti-periodicity.

It is clear that we can apply results from this section in the study of existence and uniqueness of almost anti-periodic solutions of fractional Cauchy inclusion

$$D_{t,+}^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), \quad t \in \mathbb{R},$$

where $D_{t,+}^{\gamma}$ denotes the Riemann-Liouville fractional derivative of order $\gamma \in (0,1)$ and $f : \mathbb{R} \to X$ satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator (see [8] for the notion). Furthermore, we can analyze the existence and uniqueness of asymptotically ($S^p$-) almost anti-periodic solutions of fractional Cauchy inclusion

$$(\mathrm{DFP})_{f,\gamma} : \begin{cases} \mathbf{D}_t^{\gamma} u(t) \in \mathcal{A}u(t) + f(t), & t \geq 0, \\ u(0) = x_0, \end{cases}$$

where $\mathbf{D}_t^{\gamma}$ denotes the Caputo fractional derivative of order $\gamma \in (0,1]$, $x_0 \in X$ and $f : [0,\infty) \to X$, satisfies certain properties, and $\mathcal{A}$ is a closed multivalued linear operator (cf. [14] for more details). Arguing so, we can analyze the existence and uniqueness of (asymptotically $S^p$-) almost anti-periodic solutions of the fractional Poisson heat equations

$$\begin{cases} D_{t,+}^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \in \mathbb{R},\ x \in \Omega, \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \end{cases}$$

and

$$\begin{cases} \mathbf{D}_t^{\gamma}[m(x)v(t,x)] = (\Delta - b)v(t,x) + f(t,x), & t \geq 0,\ x \in \Omega, \\ v(t,x) = 0, \quad (t,x) \in [0,\infty) \times \partial\Omega, \\ m(x)v(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

in the space $X := L^p(\Omega)$, where $\Omega$ is a bounded domain in $\mathbb{R}^n$, $b > 0$, $m(x) \geq 0$ a.e. $x \in \Omega$, $m \in L^\infty(\Omega)$, $\gamma \in (0, 1)$ and $1 < p < \infty$, see [8] and [14] for further information in this direction.

For some other references regarding the existence and uniqueness of anti-periodic and Bloch periodic solutions of certain classes of abstract Volterra integro-differential equations, we refer the reader to [5, 7, 10–13, 17, 18].

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## References

[1] W. Arendt, C. J. K. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems*, Monographs in Mathematics **96**, Birkhäuser Verlag, Basel, 2001.

[2] H. Bart and S. Goldberg, *Characterizations of almost periodic strongly continuous groups and semigroups*, Math. Ann. **236** (1978), 105–116.

[3] A. S. Besicovitch, *Almost Periodic Functions*, Dover Publications Inc. New York, 1954.

[4] S. Bochner, *A new approach to almost periodicity*, Proc. Nat. Acad. Sci. USA **48** (1962), 2039–2043.

[5] Y. Q. Chen, *Anti-periodic solutions for semilinear evolution equations*, J. Math. Anal. Appl. **315** (2006), 337–348.

[6] T. Diagana, *Almost Automorphic Type and Almost Periodic Type Functions in Abstract Spaces*, Springer, New York, 2013.

[7] W. Dimbour and V. Valmorin, *Asymptotically antiperiodic solutions for a nonlinear differential equation with piecewise constant argument in a Banach space*, Appl. Math. **7** (2016), 1726–1733.

[8] A. Favini and A. Yagi, *Degenerate Differential Equations in Banach Spaces*, Chapman and Hall/CRC Pure and Applied Mathematics, New York, 1998.

[9] G. M. N'Guérékata, *Almost Automorphic and Almost Periodic Functions in Abstract Spaces*, Kluwer Acad. Publ. Dordrecht, 2001.

[10] G. M. N'Guérékata and V. Valmorin, *Antiperiodic solutions of semilinear integrodifferential equations in Banach spaces*, Appl. Math. Comput. **218** (2012), 1118–1124.

[11] M. F. Hasler and G. M. N'Guérékata, *Bloch-periodic functions and some applications*, Nonlinear Stud. **21** (2014), 21–30.

[12] H. R. Henríquez, *On Stepanov-almost periodic semigroups and cosine functions of operators*, J. Math. Anal. Appl. **146** (1990), 420–433.

[13] G. Mophou, G. M. N'Guérékata and V. Valmorin, *Asymptotic behavior of mild solutions of some fractional functional integro-differential equations*, Afr. Diaspora J. Math. **16** (2013), 70–81.

[14] M. Kostić, *Almost Periodic Functions, Almost Automorphic Functions and Integro-Differential Equations*, Book Manuscript, 2017.

[15] M. Kostić, *Existence of generalized almost periodic and asymptotic almost periodic solutions to abstract Volterra integro-differential equations*, Electron. J. Differential Equations **2017**(239) (2017), 1–30.

[16] M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge University Press, London, 1982.

[17] J. Liu and L. Zhang, *Existence of anti-periodic (differentiable) mild solutions to semilinear differential equations with nondense domain*, Springerplus **5**(704) (2016), DOI 10.1186/s40064-016-2315-1

[18] J. H. Liu, X. Q. Song and L. T. Zhang, *Existence of anti-periodic mild solutions to semilinear nonautonomous evolution equations,* J. Math. Anal. Appl. **425** (2015), 295–306.

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# HYERS-ULAM STABILITY OF A FREE AND FORCED VIBRATIONS

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Abstract. In this paper, we study the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the general differential equation of the Free Damped Vibrations, Undamped Vibrations and Forced Vibrations by using initial conditions.

## 1. Introduction

In every day life we come across numerous things that move. These motions are of two types, viz (i) the motion in wich the body moves about a mean position or a fixed point an (ii) the motion in which the body moves from one place to another place with respect to time. The first type of motion of a body about a mean position is called oscillatory motion. A moving train, flying aeroplane and moving ball etc., corresponds to the second type of motion. An oscillating pendulum, vibrations of a stretched string, movement of water in a cup, vibration of electrons are the examples of oscillatry motions. Vibration and oscillation are part of our every day life. With in a minute of waking up, we may well experience vibrations in a wide variety of forms: the buzzing of the alarm clock; the bounce of a bed; the oscillation of a loud speaker, electric tooth brush or an electric razor, musical instruments, machinery and traffic, noisy and annoying. A common feature of many of these vibrations of oscillations is that the motion is repetitive of periodic. Such a motions are called periodic motions. In nature, all these mechanical vibrations are Simple Harmonic Motion.

The theory of stability is an important branch of the qualitative theory of differential equations. In 1940, Ulam [28] posed a problem concerning the stability of functional

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equation: "Give conditions in order for a linear function near an approximately linear function to exist." A year later, Hyers [11] gave an answer to the problem of Ulam for additive functions defined on Banach spaces. Thereafter, Aoki [3], Bourgin [4] and Rassias [21] improved the result reported in [11]. After that, many mathematicians have extended the Ulam's problem to other functional equations on various spaces in different directions (see [1, 5–8, 12, 15, 24, 25]).

Definitions of both Hyers-Ulam stability and Hyers-Ulam-Rassias stability have applicable significance since it means that if one is studying an Hyers-Ulam stable or Hyers-Ulam-Rassias stable system then one does not have to reach the exact solution. (Which is usually is quite difficult or time consuming). This is quite useful in many applications. For example, numerical analysis, optimization, biology, economics, dynamic programming, wireless sensor networks, physics, chemistry, geometry and etc., where finding the exact solution is quite difficult.

A generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations: The differential equation

$$\phi\left(f, x, x', x'', \ldots, x^{(n)}\right) = 0$$

has the Hyers-Ulam stability if for a given $\epsilon > 0$ and a function $x$ such that $\left|\phi\left(f, x, x', x'', \ldots, x^{(n)}\right)\right| \leq \epsilon$, there exists a solution $x_a$ of the differential equation such that $|x(t) - x_a(t)| \leq K(\epsilon)$ and $\lim_{\epsilon \to 0} K(\epsilon) = 0$.

Oblaza seems to be the first author who has investigated the Hyers-Ulam stability of linear differential equations (see [19, 20]). Thereafter, Alsina and Ger published their paper [2], which handles the Hyers-Ulam stability of the linear differential equation $y'(t) = y(t)$. Those previous results were extended to the Hyers-Ulam stability of linear differential equations of first order and higher orders in [9, 10, 13, 14, 16–18, 22, 23, 26, 27, 29, 30].

These days, the Hyers-Ulam stability of differential equation is investigated and the investigation is ongoing. Motivated and connected by above results, we prove the Hyers-Ulam stability of the general differential equation of Free Damped Vibrations (FDV),

$$x''(t) + 2lx'(t) + k^2x(t) = 0, \tag{1.1}$$

Undamped Vibrations (UV)

$$x''(t) + k^2x(t) = 0 \tag{1.2}$$

and Forced Vibrations (FV)

$$x''(t) + 2lx'(t) + k^2x(t) = L\sin pt, \tag{1.3}$$

by using initial conditions

$$x(a) = x'(a) = 0, \tag{1.4}$$

where $x(t) \in C^2(I)$, $I = [a, b] \subseteq \mathbb{R}$.

## 2. Preliminaries

Simple Harmonic Motion plays an important role and it is the one of the most important example of periodic motion. Clock, car shock absorbers, musical instru ments, bungee jumping, diving board, the process of hearing, earth quake, proof building metronome, a swing, a rocking chair, clock pendulum, heart beat and breathing these are the some common examples of Simple Harmonic Motion in real life, but there are countless more applications.

**Simple Harmonic Motion.** When a body moves such that its acceleration is always directed towards a certain fixed point and varies directly as its distance from that point, the body is said to execute harmonic motion.

For such a motion to take place the force acting on the body should be directed towards the fixed point and should also be proportional to the displacement, i.e., the distance from the fixed point. The function of the force is to bring the body back to its equilibrium position and hence this force is often known as restoring force.

Consider a particle of mass $m$ executing simple harmonic motion. If $x$ be the displacement of the particle from equilibrium position at any instant $t$, the resulting force $F$ acting on the particle is given by $F \propto x$ (or) $F = -sx$, where $s$ is the force constant of proportionality or stiffness. The negative sign is used to indicate that the direction of the force is opposite to the direction of increasing displacement.

If $\dfrac{d^2x}{dt^2}$ is the acceleration of the particle at time $t$, then

$$m\frac{d^2x}{dt^2} = -sx \text{ (or) } \frac{d^2x}{dt^2} + \frac{s}{m}x = 0,$$

substituting $k^2 = \frac{s}{m}$, we get $\frac{d^2x}{dt^2} + k^2x = 0$. This is the general differential equation of motion of a simple harmonic oscillator.

**Free Vibrations.** When the bob of a simple pendulum (in vacuum) is displaced from its mean position and left, it executes simple harmonic motion. The time period of oscillation depends only on the length of the pendulum and the acceleration due to the gravity at the place. The pendulum will continue to oscillate with the same time periodic and amplitude for any length of time. In such cases there is no loss of energy by friction or otherwise. In all similar cases, the vibrations will be undamped free vibrations. The amplitude swing remains constant.

**Damping.** The amplitude of a vibrating string, a sounding tuning fork and an oscillating pendulum goes on gradually decreasing and ultimately these bodies stop vibrating. It is because some energy is inevitably lost due to resistive or viscous forces. For example, in the case of a simple pendulum, energy is lost due to friction at the supports and resistance of air. The resistance offered by a damping force is known as damping. When the damping is small it does not produce any significant change in the undamped motion of the vibrating body. In such a case, the damping force is proportional to the velocity of the vibrating body.

**Damped vibrations.** In actual particle, when the pendulum vibrates in air medium, there are frictional forces and consequently energy is dissipated in each vibration. The amplitude of swing decreases continuously with time and finally the oscillations die out. Such vibrations are called free damped vibrations. The dissipated energy appears as heat either within the system itself or in the surrounding medium. The dissipative force due to friction etc. (resistance in $LCR$ circuit) is proportional to the velocity of the particle at that instant. That is, in an ideal simple harmonic motion, the displacement follows a sine curve for an infinite time. This is because the total energy remains constant. In actual practice, the simple harmonic oscillator always experiences frictional or resistive forces due to which some energy is lost and the oscillations get damped. The amplitude of vibration decreases gradually an ultimately the body comes to rest.

The decay of amplitude with time is called damping. Those simple harmonic vibrations where amplitude decreases with the passage of time are called damped simple harmonic vibrations.

Let $\mu\frac{dx}{dt}$ be the dissipative force due to friction or other phenomenon. Therefore, the differential equation in the case of Free Damped Vibration is,

$$m\frac{d^2x}{dt^2} + Tx + \mu\frac{dx}{dt} = 0 \tag{2.1}$$

or

$$\frac{d^2x}{dt^2} + \left(\frac{\mu}{m}\right)\frac{dx}{dt} + \left(\frac{T}{m}\right)x = 0.$$

The above equation is similar to a general differential equation of the form

$$\frac{d^2x}{dt^2} + 2l\frac{dx}{dt} + k^2x = 0$$

or

$$x''(t) + 2lx'(t) + k^2x(t) = 0, \tag{2.2}$$

where $k^2 = \frac{T}{m}$, $l$ is known as the damping coefficient and $2l$ gives the force due to resistance of the medium per unit mass per unit velocity. Equation (2.2) is known as differential equation of damped simple harmonic motion. Also, it can be written in the form of $x''(t) + a\ x'(t) + b\ x(t) = 0$ whose solution is,

$$x(t) = \begin{cases} e^{\left(\frac{-1}{2}at\right)}\left[c_1e^{\left(\frac{1}{2}\lambda t\right)} + c_2e^{\left(\frac{-1}{2}\lambda t\right)}\right], & \text{if } \lambda^2 = a^2 - 4b > 0, \\ e^{\left(\frac{-1}{2}at\right)}\left[c_1\sin\left(\frac{1}{2}\lambda t\right) + c_2\cos\left(\frac{1}{2}\lambda t\right)\right], & \text{if } \lambda^2 = 4b - a^2 > 0, \\ e^{\left(\frac{-1}{2}at\right)}\left[c_1x + c_2\right], & \text{if } a^2 = 4b. \end{cases}$$

**Undamped Vibrations.** For a simple harmonic vibrating particle, the kinetic energy for displacement $x$ is given by $\frac{1}{2}m\left(\frac{dx}{dt}\right)^2$. At the same instant, the potential energy of the particle is $\dfrac{1}{2}Kx^2$, where $K$ is the restoring force per unit displacement. The differential equation in the case of undamped free vibrations is,

$$m\frac{d^2x}{dt^2} + Tx = 0 \text{ or } \frac{d^2x}{dt^2} + \left(\frac{T}{m}\right)x = 0,$$

it takes the form

$$x''(t) + k^2x(t) = 0,$$

where $k^2 = \left(\frac{T}{m}\right)$. This is only an ideal case. Here it has been assumed that the vibrations are free and undamped.

**Forced Vibrations.** The time period of a body executing simple harmonic motion depends on the dimension of the body and its elastic properties The vibrations of such a body die out with time due to dissipation of energy. If some external periodic force is constantly applied on the body, the body continues to oscillate under the influence of such external force. Such vibrations of the body are called forced vibrations.

Initially, the amplitude of the swing increases, then decreases with time, becomes minimum and again increases. This will be repeated if the external periodic force is constantly applied on the system. In such cases the body will finally be forced to vibrate with the same frequency as that of the applied force. The frequency of the forced vibration is different from the natural frequency of vibration of the body. The amplitude of the forced vibration of the body depends on the difference between the natural frequency and the frequency applied force. The amplitude will be large of difference in frequency is small and vice versa. For forced vibrations, equation (2.1) is modified in the form

$$m\frac{d^2x}{dt^2} + Tx + \mu\frac{dx}{dt} = F\sin pt,$$

$$x''(t) + 2lx'(t) + k^2x(t) = L\sin pt.$$

Here $p$ is angular frequency of the applied periodic force and $L = \left(\frac{F}{m}\right)$.

Now, we give the definition of Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the general differential equation of Free Damped Vibrations (FDV) (1.1), Undamped Vibrations (UV) (1.2) and Forced Vibrations (FV) (1.3) by using initial conditions (1.4).

**Definition 2.1.** We say that the general differential equation of Free Damped Vibrations (1.1) has the Hyers-Ulam stability with initial conditions (1.4), if there exists a positive constant $K$ with the following property: For every $\epsilon > 0$, $x \in C^2(I)$ satisfies the inequality

$$\left|x''(t) + 2lx'(t) + k^2x(t)\right| \leq \epsilon,$$

then there exists some $y \in C^2(I)$ satisfies $y''(t) + 2ly'(t) + k^2y(t) = 0$ with initial conditions $y(a) = y'(a) = 0$ sucht that $|x(t) - y(t)| \leq K\epsilon$. We call such $K$ as a Hyers-Ulam stability constant for the differential equation of Free Damped Vibrations (1.1) with initial conditions (1.4).

**Definition 2.2.** We say that the general differential equation of Free Damped Vibrations (1.1) has the Hyers-Ulam-Rassias stability with $\phi(\cdot)$, where $\phi : \mathbb{R} \to [0, \infty)$ and initial conditions (1.4), if there exists a positive constant $K$ with the following property: If $x \in C^2(I)$ is such that satisfying the inequality

$$\left| x''(t) + 2lx'(t) + k^2x(t) \right| \leq \phi(t),$$

then there exists some $y \in C^2(I)$ satisfying $y''(t) + 2ly'(t) + k^2y(t) = 0$ with initial conditions $y(a) = y'(a) = 0$ such that $|x(t) - y(t)| \leq K\phi(t)$. We call such $K$ as a Hyers-Ulam-Rassias stability constant for the differential equation of Free Damped Vibrations (1.1) with initial conditions (1.4).

**Definition 2.3.** We say that the general differential equation of Undamped Vibrations (1.2) has the Hyers-Ulam stability with initial conditions (1.4), if there exists a positive constant $K$ with the following property: for every $\epsilon > 0$, $x \in C^2(I)$ satisfying the inequality

$$\left| x''(t) + k^2x(t) \right| \leq \epsilon,$$

then there exists some $y \in C^2(I)$ satisfying the differential equation $y''(t) + k^2y(t) = 0$ with initial conditions $y(a) = y'(a) = 0$ sucht that $|x(t) - y(t)| \leq K\epsilon$. We call such $K$ as a Hyers-Ulam stability constant for the differential equation of Undamped Vibrations (1.2) with initial conditions (1.4).

**Definition 2.4.** We say that the general differential equation of Undamped Vibrations (1.2) has the Hyers-Ulam-Rassias stability with $\phi(\cdot)$, where $\phi : \mathbb{R} \to [0, \infty)$ and initial conditions (1.4), if there exists a positive constant $K$ with the following property: If $x \in C^2(I)$ is such that satisfying the inequality

$$\left| x''(t) + k^2x(t) \right| \leq \phi(t),$$

then there exists some $y \in C^2(I)$ satisfying the differential equation $y''(t) + k^2y(t) = 0$ with initial conditions $y(a) = y'(a) = 0$ sucht that $|x(t) - y(t)| \leq K\phi(t)$. We call such $K$ as a Hyers-Ulam-Rassias stability constant for the differential equation of Undamped Vibrations (1.2) with initial conditions (1.4).

**Definition 2.5.** We say that the general differential equation of Forced Vibrations (1.3) has the Hyers-Ulam stability with initial conditions (1.4), if there exists a positive constant $K$ with the following property: For every $\epsilon > 0$, $x \in C^2(I)$ satisfying

$$\left| x''(t) + 2lx'(t) + k^2x(t) - L\sin pt \right| \leq \epsilon,$$

then there exists some $y \in C^2(I)$ satisfies the differential equation

$$y''(t) + 2ly'(t) + k^2y(t) = L\sin pt,$$

with initial conditions $y(a) = y'(a) = 0$ sucht that $|x(t) - y(t)| \le K\epsilon$. We call such $K$ as a Hyers-Ulam stability constant for the differential equation of Forced Vibrations (1.3) with initial conditions (1.4).

**Definition 2.6.** We say that the general differential equation of Forced Vibrations (1.3) has the Hyers-Ulam-Rassias stability with $\phi(\cdot)$, where $\phi : \mathbb{R} \to [0, \infty)$ and initial conditions (1.4), if there exists a positive constant $K$ with the following property: If $x \in C^2(I)$ is such that satisfies

$$\Big| x''(t) + 2lx'(t) + k^2 x(t) - L \sin pt \Big| \le \phi(t),$$

then there exists some $y \in C^2(I)$ satisfying the differential equation

$$y''(t) + 2ly'(t) + k^2 y(t) = L \sin pt,$$

with initial conditions $y(a) = y'(a) = 0$ such that $|x(t) - y(t)| \le K\phi(t)$. We call such $K$ as a Hyers-Ulam-Rassias stability constant for the differential equation of Forced Vibrations (1.3) with initial conditions (1.4).

## 3. Hyers-Ulam Stability

In this section, we prove the Hyers-Ulam stability of a general differential equation of FDV (1.1), UV (1.2) and FV (1.3) by using initial conditions (1.4). Now, we prove the Hyers-Ulam stability of the general differential equation of FDV (1.1) by using the initial conditions (1.4).

**Theorem 3.1.** *If $x : I \to \mathbb{R}$ be a twice continuously differentiable function such that $|x'(t)| \le |x(t)|$ satisfies the differential inequality*

$$\Big| x''(t) + 2lx'(t) + k^2 x(t) \Big| \le \epsilon, \tag{3.1}$$

*with initial conditions* (1.4) *then the differential equation* (1.1) *has the Hyers-Ulam stability.*

*Proof.* For every $\epsilon > 0$, there exists $x \in C^2(I)$ such that $|x'(t)| \le |x(t)|$ satisfies the differential inequality (3.1) with initial conditions (1.4) and define $M = \max_{t \in I} |x(t)|$. Then by the inequality (3.1), we have

$$-\epsilon \le x''(t) + 2lx'(t) + k^2 x(t) \le \epsilon,$$

multiplying the above inequality by $x'(t)$ and then integrate, we obtain that

$$\int_a^t -\epsilon x'(\tau) d\tau \le \int_a^t \Big\{ x''(\tau) + 2lx'(\tau) + k^2 x(\tau) \Big\} x'(\tau) d\tau \le \int_a^t \epsilon x'(\tau) d\tau,$$

$$-\epsilon \int_a^t x'(\tau) d\tau \le \int_a^t x''(\tau) x'(t) d\tau + 2l \int_a^t x'(\tau)^2 d\tau + k^2 \int_a^t x(\tau) x'(\tau) d\tau \le \epsilon \int_a^t x'(\tau) d\tau,$$

$$-2\epsilon x(t) \leq x'(t)^2 + k^2 x(t)^2 + 4l \int\limits_a^t x'(\tau)^2 d\tau \leq 2\epsilon x(t),$$

from which we get that

$$k^2 x(t)^2 \leq 2\epsilon x(t) - 4l \int\limits_a^t x'(\tau)^2 d\tau,$$

$$M^2 \leq \frac{2\epsilon}{k^2} M + \frac{4l}{k^2} M^2 (b-a).$$

Choose $\gamma = \frac{4l(b-a)}{k^2}$, we get

$$M \leq \frac{2\epsilon}{k^2(1-\gamma)},$$

which gives $|x(t)| \leq \frac{2\epsilon}{k^2(1-\gamma)}$. Hence, $|x(t)| \leq K\epsilon$, where $K = \frac{2}{k^2(1-\gamma)}$. It is easy to see that, $y(t_0) \equiv 0$ is a solution of $x''(t) + 2lx'(t) + k^2 x(t) = 0$ with initial condition (1.4) such that $|x(t) - y(t_0)| \leq K\epsilon$. Hence, by the virtue of Definition 2.1, the differential equation of FDV (1.1) has the Hyers-Ulam stability. □

Now, we prove the Hyers-Ulam stability of the general differential equation of UV (1.2) with initial conditions (1.4).

**Theorem 3.2.** *If $x : I \to \mathbb{R}$ be a twice continuously differentiable function satisfies the inequality*

$$\left| x''(t) + k^2 x(t) \right| \leq \epsilon, \tag{3.2}$$

*with initial conditions* (1.4) *then the differential equation* (1.2) *has the Hyers-Ulam stability.*

*Proof.* For every $\epsilon > 0$, there exists $x \in C^2(I)$, satisfies the differential inequality (3.2) with initial conditions (1.4) and define $M = \max_{t \in I} |x(t)|$. Then by the inequality (3.2), we have

$$-\epsilon \leq x''(t) + k^2 x(t) \leq \epsilon,$$

multiplying the above inequality by $x'(t)$ and then integrating $a$ to $t$, we get

$$-\epsilon \int\limits_a^t x'(\tau) d\tau \leq \int\limits_a^t \left\{ x''(\tau) + k^2 x(\tau) \right\} x'(\tau) d\tau \leq \epsilon \int\limits_a^t x'(\tau) d\tau,$$

$$-2\epsilon x(t) \leq x'(t)^2 + k^2 x(t)^2 \leq 2\epsilon x(t),$$

from which we obtain that $k^2 x(t)^2 \leq 2\epsilon x(t)$. Hence, $|x(t)| \leq K\epsilon$, where $K = \frac{2}{k^2}$. It is easy to see that, $y(t_0) \equiv 0$ is a solution of $x''(t) + k^2 x(t) = 0$ with initial condition (1.4) such that

$$|x(t) - y(t_0)| \leq K\epsilon.$$

Hence by the virtue of Definition 2.3, the differential equation of UV (1.2) has the Hyers-Ulam stability. □

Finally, in this section we would like to prove the Hyers-Ulam stability of the general differential equation of FV (1.3) with initial conditions (1.4).

**Theorem 3.3.** *If $x : I \to \mathbb{R}$ be a twice continuously differentiable function such that $|x'(t)| \leq |x(t)|$ satisfies the differential inequality*

$$\Big|x''(t) + 2lx'(t) + k^2x(t) - L\sin pt\Big| \leq \epsilon, \tag{3.3}$$

*with initial conditions* (1.4) *then the differential equation* (1.3) *has the Hyers-Ulam stability.*

*Proof.* For every $\epsilon > 0$, there exists $x \in C^2(I)$ such that $|x'(t)| \leq |x(t)|$ satisfies the differential inequality (3.3) with initial conditions (1.4) and define $M = \max_{t\in I} |x(t)|$. Then by the inequality (3.3), we have

$$-\epsilon \leq x''(t) + 2lx'(t) + k^2x(t) - L\sin pt \leq \epsilon.$$

Multiplying the above inequality by $x'(t)$ and then integrate, we obtain that

$$\int_a^t -\epsilon x'(\tau)d\tau \leq \int_a^t \Big\{x''(\tau) + 2lx'(\tau) + k^2x(\tau) - L\sin p\tau\Big\}\, x'(\tau)d\tau \leq \int_a^t \epsilon x'(\tau)d\tau,$$

$$-\epsilon\int_a^t x'(\tau)d\tau \leq \int_a^t x''(\tau)x'(\tau)d\tau + 2l\int_a^t x'(\tau)^2d\tau + k^2\int_a^t x(\tau)x'(\tau)d\tau,$$

$$-L\int_a^t x'(\tau)\sin p\tau d\tau \leq \epsilon\int_a^t x'(\tau)d\tau,$$

$$\begin{aligned}-2\epsilon\ x(t) &\leq x'(t)^2 + k^2\ x(t)^2 + 4l\int_a^t\ x'(\tau)^2d\tau - 2L\int_a^t x'(\tau)\sin p\tau d\tau\\ &\leq 2\epsilon x(\tau),\end{aligned}$$

from which we get that

$$k^2x(t)^2 \leq 2\epsilon x(t) - 4l\int_a^t x'(\tau)^2d\tau + 2L\int_a^t x'(\tau)\sin p\tau d\tau,$$

$$M^2 \leq \frac{2\epsilon}{k^2}M + \frac{4l}{k^2}M^2(b-a) + \frac{2LM}{k^2}\sin pa.$$

Choose $\gamma = \frac{4l(b-a)}{k^2}$, we get $M \leq \frac{2\epsilon+2L\sin pa}{k^2(1-\gamma)}$, which gives

$$|x(t)| \leq \frac{2\epsilon + 2L\sin pa}{k^2(1-\gamma)}.$$

Hence $|x(t)| \leq K(\epsilon)$, where $K(\epsilon) = \frac{2\epsilon+2L\sin pa}{k^2(1-\gamma)}$ for all $t \in I$. Obviously, $y(t_0) \equiv 0$ is a solution of the differential equation $x''(t) + 2lx'(t) + k^2x(t) - L\sin pt$ with (1.4) such that $|x(t) - y(t_0)| \leq K(\epsilon)$.

Hence by the virtue of Definition 2.5, the differential equation of FV (1.3) has the Hyers-Ulam stability. ☐

## 4. Hyers-Ulam-Rassias Stability

In this section, we prove the Hyers-Ulam-Rassias stability of a general differential equation of FDV (1.1), UV (1.2) and FV (1.3) by using initial conditions (1.4). Now, we prove the Hyers-Ulam-Rassias stability of the general differential equation of FDV (1.1) by using the initial conditions (1.4).

**Theorem 4.1.** *If $x : I \to \mathbb{R}$ be a twice continuously differentiable function such that $|x'(t)| \leq |x(t)|$ with $\phi(t)$, where $\phi : \mathbb{R} \to [0, \infty)$ satisfies the differential inequality*

$$\left| x''(t) + 2lx'(t) + k^2 x(t) \right| \leq \phi(t), \tag{4.1}$$

*with initial conditions* (1.4) *then the differential equation* (1.1) *has the Hyers-Ulam-Rassias stability with respect to $\phi(t)$ such that $\phi(a) = 0$.*

*Proof.* If $x \in C^2(I)$ is such that $|x'(t)| \leq |x(t)|$ with $\phi : \mathbb{R} \to [0, \infty)$ satisfies the differential inequality (4.1) with initial conditions (1.4) and define $M = \max_{t \in I} |x(t)|$. Then by the inequality (4.1), we have

$$-\phi(t) \leq x''(t) + 2lx'(t) + k^2 x(t) \leq \phi(t),$$

multiplying the above inequality by $x'(t)$ and then integrating with the limits $a$ to $t$, we obtain that

$$\int_a^t -\phi(\tau) x'(\tau) d\tau \leq \int_a^t \left\{ x''(\tau) + 2lx'(\tau) + k^2 x(\tau) \right\} x'(\tau) d\tau \leq \int_a^t \phi(\tau) x'(\tau) d\tau,$$

$$-2 \int_a^t x'(\tau)\phi(\tau) d\tau \leq x'(t)^2 + k^2 x(t)^2 + 4l \int_a^t x'(\tau)^2 d\tau \leq 2 \int_a^t x'(\tau)\phi(\tau) d\tau,$$

from which we get that

$$k^2 x(t)^2 \leq 2 \int_a^t x'(\tau)\phi(\tau) d\tau - 4l \int_a^t x'(\tau)^2 d\tau,$$

$$M^2 \leq \frac{2}{k^2} M\phi(t) + \frac{4l}{k^2} M^2 (b - a).$$

Choose $\gamma = \frac{4l(b-a)}{k^2}$, we get

$$M \leq \frac{2\phi(t)}{k^2(1 - \gamma)},$$

which gives $|x(t)| \leq \frac{2\phi(t)}{k^2(1-\gamma)}$. Hence, $|x(t)| \leq K\phi(t)$ for all $t \in I$, where $K = \frac{2}{k^2(1-\gamma)}$. Obviously, $y(t_0) \equiv 0$ is a solution of the differential equation (1.1) with (1.4) such that

$$|x(t) - y(t_0)| \leq K\phi(t).$$

Hence by the virtue of Definition 2.2, FDV (1.1) has the Hyers-Ulam-Rassias stability with initial conditions (1.4). □

Now, we are going to prove the Hyers-Ulam-Rassias stability of the differential equation of UV (1.2) with initial conditions (1.4).

**Theorem 4.2.** *If $x : I \to \mathbb{R}$ be a twice continuously differentiable function such that $|x'(t)| \le |x(t)|$ with $\phi(t)$, where $\phi : \mathbb{R} \to [0, \infty)$ satisfies the differential inequality*

$$\left| x''(t) + k^2 x(t) \right| \le \phi(t), \tag{4.2}$$

*with initial conditions* (1.4) *then the general differential equation of UV* (1.2) *has the Hyers-Ulam-Rassias stability with respect to $\phi(t)$ such that $\phi(a) = 0$.*

*Proof.* If $x \in C^2(I)$ is such that $|x'(t)| \le |x(t)|$ with $\phi : \mathbb{R} \to [0, \infty)$ satisfies the differential inequality (4.2) with initial conditions (1.4) and define $M = \max_{t \in I} |x(t)|$. Then by the inequality (4.2), we have

$$-\phi(t) \le x''(t) + k^2 x(t) \le \phi(t),$$

multiplying the above inequality by $x'(t)$ and then integrating $a$ to $t$, we get

$$-\int_a^t x'(\tau)\phi(\tau)d\tau \le \int_a^t \left\{ x''(\tau) + k^2 x(\tau) \right\} x'(\tau)d\tau \le \int_a^t x'(\tau)\phi(\tau)d\tau,$$

$$-2\int_a^t x'(\tau)\phi(\tau)d\tau \le x'(t)^2 + k^2 x(t)^2 \le 2\int_a^t x'(\tau)\phi(\tau)d\tau,$$

from which we obtain that

$$k^2 x(t)^2 \le 2\int_a^t x'(\tau)\phi(\tau)d\tau,$$

$$M^2 \le \frac{2}{k^2} M\phi(t).$$

Hence, we get that $|x(t)| \le K\phi(t)$, where $K = \frac{2}{k^2}$. It is easy to see that, $y(t_0) \equiv 0$ is a solution of the differential equation (1.2) with (1.4) such that $|x(t) - y(t_0)| \le K\phi(t)$. Hence by the virtue of Definition 2.4, the differential equation (1.2) has the Hyers-Ulam-Rassias stability. □

Finally, we would like to prove the Hyers-Ulam-Rassias stability of the general differential equation of FV (1.3) with initial conditions (1.4).

**Theorem 4.3.** *If $x : I \to \mathbb{R}$ be a twice continuously differentiable function such that $|x'(t)| \le |x(t)|$ with $\phi(t)$, where $\phi : \mathbb{R} \to [0, \infty)$ satisfies the differential inequality*

$$\left| x''(t) + 2lx'(t) + k^2 x(t) - L \sin pt \right| \le \phi(t), \tag{4.3}$$

*with initial conditions* (1.4) *then the differential equation* (1.3) *has the Hyers-Ulam-Rassias stability with respect to $\phi(t)$ such that $\phi(a) = 0$.*

*Proof.* Assume that $x \in C^2(I)$ is such that $|x'(t)| \leq |x(t)|$ with $\phi : \mathbb{R} \to [0, \infty)$ satisfies the differential inequality (4.3) with initial conditions (1.4) and define $M = \max_{t\in I} |x(t)|$. Then by the inequality (4.3), we have

$$-\phi(t) \leq x''(t) + 2lx'(t) + k^2x(t) - L\sin pt \leq \phi(t).$$

Multiplying the above inequality by $x'(t)$ and then integrating $a$ to $t$, we obtain that

$$\begin{aligned}
\int_a^t -x'(\tau)\phi(\tau)d\tau &\leq \int_a^t \Big\{x''(\tau) + 2lx'(\tau) + k^2x(\tau) - L\sin p\tau\Big\}\, x'(\tau)d\tau \\
&\leq \int_a^t x'(\tau)\phi(\tau)d\tau, \\
-\int_a^t x'(t)\phi(\tau)d\tau &\leq \int_a^t x''(\tau)x'(\tau)d\tau + 2l\int_a^t x'(\tau)^2d\tau + k^2\int_a^t x(\tau)x'(\tau)d\tau \\
&\quad - L\int_a^t x'(\tau)\sin p\tau d\tau \leq \int_a^t x'(\tau)\phi(\tau)d\tau, \\
-2\int_a^t x'(\tau)\phi(\tau)d\tau &\leq x'(t)^2 + k^2x(t)^2 + 4l\int_a^t x'(\tau)^2d\tau - 2L\int_a^t x'(\tau)\sin p\tau d\tau \\
&\leq 2\int_a^t x'(\tau)\phi(\tau)d\tau,
\end{aligned}$$

from which we get that

$$\begin{aligned}
k^2x(t)^2 &\leq 2\int_a^t x'(\tau)\phi(\tau)d\tau - 4l\int_a^t x'(\tau)^2\, d\tau + 2L\int_a^t x'(\tau)\sin p\tau d\tau, \\
M^2 &\leq \frac{2}{k^2}M\phi(t) + \frac{4l}{k^2}M^2(b-a) + \frac{2LM}{k^2}\sin pa.
\end{aligned}$$

Choose $\gamma = \frac{4l(b-a)}{k^2}$, we get $M \leq \frac{2\phi(t)+2L\sin pa}{k^2(1-\gamma)}$, which gives

$$|x(t)| \leq \frac{2\phi(t) + 2L\sin pa}{k^2(1-\gamma)}.$$

Hence, $|x(t)| \leq K\phi(t)$, where

$$K\phi(t) = \frac{2\phi(t) + 2L\sin pa}{k^2(1-\gamma)},$$

for all $t \in I$. It is easy to see that, $y(t_0) \equiv 0$ is a solution of the differential equation (1.3) with the initial condition (1.4) such that

$$|x(t) - y(t_0)| \leq K\phi(t).$$

Hence by the virtue of Definition 2.6, the differential equation (1.3) has the Hyers-Ulam-Rassias stability. □

## Conclusion

In this paper, we have proved the Hyers-Ulam stability and Hyers-Ulam-Rassias stability of the general differential equation of the Free Damped Vibrations (FDV), Undamped Vibrations (UV) and Forced Vibrations (FV) by using initial conditions. It will be very useful to the readers to study the stability problem for various physical Phenomenon.

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## References

[1] M. Almahalebi, A. Chahbi and S. Kabbaj, *A fixed point approach to the stability of a bi-cubic functional equations in 2-Banach spaces*, Palest. J. Math. **5**(2) (2016), 220–227.

[2] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. **2** (1998), 373–380.

[3] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan **2** (1950), 64–66.

[4] D. G. Bourgin, *Classes of transformations and bordering transformations*, Bull. Amer. Math. Soc. **57** (1951), 223–237.

[5] N. Brillouet-Belluot, J. Brzdek and K. Cieplinski, *On some recent developements in Ulam's type stability*, Abstr. Appl. Anal. **2012** (2012), Article ID 716936, 41 pages.

[6] M. Burger, N. Ozawa and A. Thom, *On Ulam stability*, Israel J. Math. **193** (2013), 109–129.

[7] L. Cadariu, L. Gavruta and P. Gavruta, *Fixed points and generalized Hyers-Ulam stability*, Abstr. Appl. Anal. **2012** (2012), Article ID 712743, 10 pages.

[8] J. Chung, *Hyers-Ulam stability theorems for Pexiders equations in the space of Schwartz distributions*, Arch. Math. **84** (2005), 527–537.

[9] J. Huang and Y. Li, *Hyers-Ulam stability of linear differential equations*, J. Math. Anal. Appl. **426** (2015), 1192–1200.

[10] J. Huang, S. M. Jung and Y. Li, *On Hyers-Ulam stability of non-linear differential equations*, Bull. Korean Math. Soc. **52**(2) (2015), 685–697.

[11] D. H. Hyers, *On the stability of the linear functional equation*, in: N. Raikhel (Ed.), *Proceeding of the National Academy of Sciences of the United States of America*, **27** (1941), 222–224.

[12] S. M. Jung, *Hyers-Ulam-Rassias Stability of Functional Equation in Nonlinear Analysis*, Springer Optimization and Its Applications **48**, Springer, New York, 2011.

[13] Y. Li and Y. Shen, *Hyers-Ulam stability fo linear differential equations of second order*, Appl. Math. Lett. **23** (2010), 306–309.

[14] M. I. Modebei, O. O. Olaiya and I. Otaide, *Generalized Hyers-Ulam stability of second order linear ordinary differential equation with initial condition*, Advances in Inequalities and Applications **2014**(36) (2014), 1–7.

[15] R. Murali, M. J. Rassias and V. Vithya, *The general solution and stability of nonadecic functional equation in matrix normed spaces*, Malaya Journal of Matematik **5**(2) (2017), 416–427.

[16] R. Murali and A. P. Selvan, *On the generalized Hyers-Ulam stability of linear ordinary differential equations of higher order*, International Journal of Pure and Applied Mathematics **117**(12) (2017), 317–326.

[17] R. Murali and A. P. Selvan, *Hyers-Ulam stability of nth order differential equation*, Contemporary Studies in Discrete Mathematics (CSDM) **2**(1) (2018), 45–50.

[18] R. Murali and A. P. Selvan, *Hyers-Ulam-Rassias stability for the linear ordinary differential equation of third order*, Kragujevac J. Math. **42**(4) (2018), 579–590.

[19] M. Obloza, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Matematyczne **14** (1997), 141–146.

[20] M. Obloza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Matematyczne **13** (1993), 259–270.

[21] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978), 297–300.

[22] K. Ravi, R. Murali and A. P. Selvan, *Ulam stability of a general nth order linear differential equation with constant coefficients*, Asian Journal of Mathematics and Computer Research **11**(1) (2016), 61–68.

[23] K. Ravi, R. Murali and A. P. Selvan, *Hyers-Ulam stability of nth order linear differential equation with initial and boundary condition*, Asian Journal of Mathematics and Computer Research **11**(3) (2016), 201–207.

[24] K. Ravi, J. M. Rassias and B. V. S. Kumar, *Ulam-Hyers stability of undecic functional equation in quasi-beta-normed spaces fixed point method*, Tbilisi Math. J. **9**(2) (2016), 83–103.

[25] K. Ravi, J. M. Rassias, S. Pinelas and S. Suresh, *General solution and stability of quattuordecic functional equation in quasi-beta-normed spaces*, Advances in Pure Mathematics **6** (2016), 921–941.

[26] I. A. Rus, *Ulam stabilities of ordinary differential equations in Banach space*, Carpathian J. Math. **26**(1) (2010), 103–107.

[27] S. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach space valued differential equation* $y'(t) = \lambda y(t)$, Bull. Korean Math. Soc. **39** (2002), 309–315.

[28] S. M. Ulam, *A Collection of Mathematical Problems*, Interscience Publishers, New York, 1960.

[29] T. Z. Xu, *On the stability of Multi-Jensens mappings in* $\beta$*-normed spaces*, Appl. Math. Lett. **25**(11) (2012), 1866–1870.

[30] A. Zada, O. Shah and R. Shah, *Hyers-Ulam stability of non-autonomous systems in terms of boundedness of cauchy problems*, Appl. Math. Comput. **271** (2015), 512–518.

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# BASIC INEQUALITIES FOR $(m, M)$-$\Psi$-CONVEX FUNCTIONS WHEN $\Psi = -\ln$

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Abstract. In this paper we establish some basic inequalities for $(m, M)$-$\Psi$-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

## 1. Introduction

Assume that the function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ ($I$ is an interval) is convex on $I$ and $m \in \mathbb{R}$. We shall say that the function $\Phi : I \to \mathbb{R}$ is $m$-$\Psi$-*lower convex* if $\Phi - m\Psi$ is a convex function on $I$. We may introduce the class of functions (see [1])

$$\mathcal{L}(I, m, \Psi) := \{\Phi : I \to \mathbb{R} \mid \Phi - m\Psi \text{ is convex on } I\}. \tag{1.1}$$

Similarly, for $M \in \mathbb{R}$ and $\Psi$ as above, we can introduce the class of $M$-$\Psi$-*upper convex* functions by (see [1])

$$\mathcal{U}(I, M, \Psi) := \{\Phi : I \to \mathbb{R} \mid M\Psi - \Phi \text{ is convex on } I\}. \tag{1.2}$$

The intersection of these two classes will be called the class of $(m, M)$-$\Psi$-*convex functions* and will be denoted by (see [1])

$$\mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi). \tag{1.3}$$

*Remark* 1.1. If $\Phi \in \mathcal{B}(I, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as $\Psi$ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in I$).

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The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [7], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions,* for a function $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is *g-convex dominated* iff $g + f$ and $g - f$ are convex functions on $I$. In [7], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of $g$-convex dominated functions can be obtained as a particular case from $(m, M)$-$\Psi$-convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds (see [1]).

**Lemma 1.1.** *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$, the interior of $I$ and $\Psi$ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$m\left(\Psi\left(t\right) - \Psi\left(s\right) - \Psi'\left(s\right)\left(t - s\right)\right) \le \Phi\left(t\right) - \Phi\left(s\right) - \Phi'\left(s\right)\left(t - s\right), \tag{1.4}$$

*for all $t, s \in \mathring{I}$.*

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$\Phi\left(t\right) - \Phi\left(s\right) - \Phi'\left(s\right)\left(t - s\right) \le M\left(\Psi\left(t\right) - \Psi\left(s\right) - \Psi'\left(s\right)\left(t - s\right)\right), \tag{1.5}$$

*for all $t, s \in \mathring{I}$.*

(iii) *For $M, m \in \mathbb{R}$ with $M \ge m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both* (1.4) *and* (1.5) *hold.*

Another elementary fact for twice differentiable functions also holds (see [1]).

**Lemma 1.2.** *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and $\Psi$ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$m\Psi''\left(t\right) \le \Phi''\left(t\right), \quad \text{for all } t \in \mathring{I}. \tag{1.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$\Phi''\left(t\right) \le M\Psi''\left(t\right), \quad \text{for all } t \in \mathring{I}. \tag{1.7}$$

(iii) *For $M$, $m \in \mathbb{R}$ with $M \ge m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both* (1.6) *and* (1.7) *hold.*

For various inequalities concerning these classes of function, see the survey paper [3].

In what follows, we consider the class of functions $\mathcal{B}\left(I, m, M, -\ln\right)$ for $M, m \in \mathbb{R}$, with $M \ge m$ that is obtained for $\Psi : I \subseteq (0, \infty) \to \mathbb{R}$, $\Psi\left(t\right) = -\ln t$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a differentiable function on $\mathring{I}$ then by Lemma 1.1 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$\begin{aligned}(1.8) \qquad m\left(\ln s - \ln t - \frac{1}{s}(s-t)\right) &\leq \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \\ &\leq M\left(\ln s - \ln t - \frac{1}{s}(s-t)\right),\end{aligned}$$

for any $s, t \in \mathring{I}$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a twice differentiable function on $\mathring{I}$ then by Lemma 1.2 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$(1.9) \qquad m \leq t^2 \Phi''(t) \leq M,$$

which is a convenient condition to verify in applications.

In this paper we establish some basic inequalities for $(m, M)$-$\Psi$-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

For recent results concerning inequalities for weighted arithmetic mean and geometric mean (see [4, 5] and [8–15]).

## 2. Some Inequalities From Definition of Convexity

We define the weighted arithmetic and geometric means

$$A_\nu(a, b) := (1-\nu)a + \nu b \text{ and } G_\nu(a, b) := a^{1-\nu} b^\nu,$$

where $\nu \in [0, 1]$ and $a$, $b > 0$. If $\nu = \frac{1}{2}$, then we write for brevity $A(a, b)$ and $G(a, b)$, respectively.

The following double inequality holds, see also [6].

**Theorem 2.1.** *Let $M$, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$. Then for any $a$, $b > 0$ and $\nu \in [0, 1]$ we have*

$$\begin{aligned}(2.1) \qquad \ln\left(\frac{A_\nu(a, b)}{G_\nu(a, b)}\right)^m &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \\ &\leq \ln\left(\frac{A_\nu(a, b)}{G_\nu(a, b)}\right)^M.\end{aligned}$$

*Proof.* Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $\Phi_m := \Phi + m\ln$ is convex and by the definition of convexity, we have

$$\begin{aligned}&\Phi((1-\nu)a + \nu b) + m\ln A_\nu(a, b) \\ &\leq (1-\nu)(\Phi(a) + m\ln a) + \nu(\Phi(b) + m\ln b) \\ &= (1-\nu)\Phi(a) + \nu\Phi(b) + (1-\nu)m\ln a + \nu m\ln b \\ &= (1-\nu)\Phi(a) + \nu\Phi(b) + m\ln G_\nu(a, b),\end{aligned}$$

that is equivalent to

$$m \ln \frac{A_\nu(a,b)}{G_\nu(a,b)} \le (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b)$$

and the first inequality in (2.1) is proved.

Similarly, by the convexity of $\Phi_M := -M \ln - \Phi$ we get the second part of (2.1). $\square$

For $m$, $M$ with $M > m > 0$ we define

$$M_p := \begin{cases} M^p, & \text{if } p > 1, \\ \\ m^p, & \text{if } p < 0, \end{cases} \text{ and } m_p := \begin{cases} m^p, & \text{if } p > 1, \\ \\ M^p, & \text{if } p < 0. \end{cases} \tag{2.2}$$

Consider the function $\Phi(t) = t^p$, $p \in (-\infty, 0) \cup (1, \infty)$. This is a convex function and $\Phi''(t) = p(p-1)t^{p-2}$, $t > 0$. Consider $\kappa(t) := t^2\Phi''(t) = p(p-1)t^p$. We observe that

$$\max_{t\in[m,M]} \kappa(t) = p(p-1)M_p \text{ and } \min_{t\in[m,M]} \kappa(t) = p(p-1)m_p.$$

**Corollary 2.1.** *Let $m$, $M$ with $M > m > 0$ and $p \in (-\infty, 0) \cup (1, \infty)$. Then for any $a$, $b \in [m, M]$ and $\nu \in [0, 1]$ we have*

$$\begin{aligned} \ln \left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)m_p} &\le (1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p \\ &\le \ln \left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)M_p}, \end{aligned} \tag{2.3}$$

*where $M_p$ and $m_p$ are defined by (2.2).*

By taking the exponential in (2.3) we get the equivalent inequality

$$\begin{aligned} &\exp\left(\frac{(1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p}{p(p-1)M_p}\right) \\ \le&\frac{A_\nu(a,b)}{G_\nu(a,b)} \\ \le&\exp\left(\frac{(1-\nu)a^p + \nu b^p - ((1-\nu)a + \nu b)^p}{p(p-1)m_p}\right), \end{aligned} \tag{2.4}$$

for any $p \in (-\infty, 0) \cup (1, \infty)$, $\nu \in [0, 1]$ and any $a$, $b \in [m, M]$.

If we take $p = 2$ in (2.4) and perform the calculations, then we get

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{M^2}\right) \le \frac{A_\nu(a,b)}{G_\nu(a,b)} \le \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{m^2}\right), \tag{2.5}$$

for any $a, b \in [m, M]$.

If $a$, $b > 0$ then by taking $M = \max\{a, b\}$ and $m = \min\{a, b\}$ in (2.5) we have

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max^2\{a,b\}}\right) \le \frac{A_\nu(a,b)}{G_\nu(a,b)} \le \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min^2\{a,b\}}\right). \tag{2.6}$$

Since

$$\frac{(b-a)^2}{\max^2\{a,b\}} = \left(\frac{b-a}{\max\{a,b\}}\right)^2 = \left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2$$

and

$$\frac{(b-a)^2}{\min^2\{a,b\}} = \left(\frac{b-a}{\min\{a,b\}}\right)^2 = \left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2$$

for any $a$, $b > 0$, then (2.6) can be written as

$$\begin{aligned} &\exp\left(\frac{1}{2}(1-\nu)\nu\left(1 - \frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right) \\ \le &\frac{A_\nu(a,b)}{G_\nu(a,b)} \\ \le &\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right). \end{aligned} \tag{2.7}$$

This inequality was obtained in a different way in [5].

If we take $p = -1$ in (2.4) and perform the calculations, then we get

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{m(b-a)^2}{abA_\nu(a,b)}\right) \le \frac{A_\nu(a,b)}{G_\nu(a,b)} \le \exp\left(\frac{1}{2}(1-\nu)\nu\frac{M(b-a)^2}{abA_\nu(a,b)}\right), \tag{2.8}$$

for any $a$, $b \in [m, M]$ and $\nu \in [0,1]$.

If $a$, $b > 0$ then by taking $M = \max\{a,b\}$ and $m = \min\{a,b\}$ in (2.8) and since $ab = \max\{a,b\}\min\{a,b\}$ we have

$$\begin{aligned} &\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right) \\ \le &\frac{A_\nu(a,b)}{G_\nu(a,b)} \\ \le &\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right), \end{aligned} \tag{2.9}$$

for any $\nu \in [0,1]$.

Since

$$\frac{1}{\max\{a,b\}} \le \frac{1}{A_\nu(a,b)} \le \frac{1}{\min\{a,b\}},$$

hence,

$$\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\min\{a,b\}}{\max\{a,b\}} - 1\right)^2\right) \le \exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right)$$

and

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right) \le \exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}} - 1\right)^2\right),$$

showing that the double inequality (2.9) is better than (2.7).

## 3. Some Perturbed Inequalities

Recall the following result obtained by Dragomir in 2006 [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n \min_{j\in\{1,2,\ldots,n\}} \{p_j\} \left(\frac{1}{n}\sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right) \\ \leq &\frac{1}{P_n}\sum_{j=1}^{n} p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\ \leq &n \max_{j\in\{1,2,\ldots,n\}} \{p_j\} \left(\frac{1}{n}\sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^{n} x_j\right)\right), \end{aligned} \tag{3.1}$$

where $f : C \to \mathbb{R}$ is a convex function defined on convex subset $C$ of the linear space $X$, $\{x_j\}_{j\in\{1,2,\ldots,n\}}$ are vectors in $C$ and $\{p_j\}_{j\in\{1,2,\ldots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (3.1) that

$$\begin{aligned} &2r\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right) \\ \leq &\nu f(x) + (1-\nu) f(y) - f(\nu x + (1-\nu) y) \\ \leq &2R\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right), \end{aligned} \tag{3.2}$$

for any $x, y \in C$ and $\nu \in [0,1]$, where $r := \min\{\nu, 1-\nu\}$ and $R := \max\{\nu, 1-\nu\}$.

**Theorem 3.1.** *Let $M$, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. Then for any $a$, $b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned} &\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right)^m \\ &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) - 2r\left(\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right) \\ &\leq \ln\left(\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^M \end{aligned} \tag{3.3}$$

*and*

(3.4)

$$\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^m$$

$$\leq 2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-\left(\nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)\right)$$
$$\leq \ln\left(\frac{G_{\nu}\left(a,b\right)}{A_{\nu}\left(a,b\right)}\left(\frac{A\left(a,b\right)}{G\left(a,b\right)}\right)^{2R}\right)^{M},$$

*where* $r:=\min\left\{\nu,1-\nu\right\}$ *and* $R:=\max\left\{\nu,1-\nu\right\}.$

*Proof.* Since $\Phi\in\mathcal{B}\left(\left(0,\infty\right),m,M,-\ln\right)$, then $f_m:=\Phi+m\ln$ is convex and by (3.2) we have

$$\begin{aligned}(3.5)\qquad & 2r\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-2rm\ln\frac{A\left(a,b\right)}{G\left(a,b\right)}\\ &\leq\nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)-m\ln\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}\\ &\leq 2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-2Rm\ln\frac{A\left(a,b\right)}{G\left(a,b\right)},\end{aligned}$$

for any $a,b>0$ and $\nu\in[0,1]$.

Since $\Phi\in\mathcal{B}\left(\left(0,\infty\right),m,M,-\ln\right)$, then also $f_M:=-\Phi-M\ln$ is convex and by (3.2) we have

$$\begin{aligned}(3.6)\qquad & 2r\left(\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi(a)+\Phi(b)}{2}\right)+2rM\ln\frac{A\left(a,b\right)}{G\left(a,b\right)}\\ &\leq\Phi\left(\nu a+\left(1-\nu\right)b\right)-\nu\Phi\left(a\right)-\left(1-\nu\right)\Phi\left(b\right)+M\ln\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}\\ &\leq 2R\left(\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi(a)+\Phi(b)}{2}\right)+2RM\ln\frac{A\left(a,b\right)}{G\left(a,b\right)},\end{aligned}$$

for any $a,b>0$ and $\nu\in[0,1]$.

From the first inequality in (3.5) we have

$$\ln\left(\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}\left(\frac{G\left(a,b\right)}{A\left(a,b\right)}\right)^{2r}\right)^{m}$$
$$\leq\nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)-2r\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right),$$

while from the first inequality in (3.6) we also have

$$\nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)-2r\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)$$
$$\leq\ln\left(\left(\frac{G\left(a,b\right)}{A\left(a,b\right)}\right)^{2r}\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}\right)^{M},$$

for any $a,b>0$ and $\nu\in[0,1]$.

These prove the desired result (3.3).

From the second inequality in (3.5) we have

$$\begin{aligned}
&\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^{m}\\
\leq&2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-\left(\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)\right),
\end{aligned}$$

while from the second inequality in (3.6) we also have

$$\begin{aligned}
&2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-\left(\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)\right)\\
\leq&\ln\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^{M},
\end{aligned}$$

for any $a,b>0$ and $\nu\in[0,1]$.

These prove the desired result (3.4). □

**Corollary 3.1.** *Let $m,M$ with $M>m>0$ and $p\in(-\infty,0)\cup(1,\infty)$. Then for any $a,b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}
&\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right)^{p(p-1)m_p}\\
&\leq(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p-2r\left(\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right)\\
&\leq\ln\left(\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)M_p}
\end{aligned}\tag{3.7}$$

*and*

$$\begin{aligned}
&\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^{p(p-1)m_p}\\
&\leq 2R\left(\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right)-\left((1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\right)\\
&\leq\ln\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^{p(p-1)M_p},
\end{aligned}\tag{3.8}$$

*where $r:=\min\{\nu,1-\nu\}$ and $R:=\max\{\nu,1-\nu\}$ and $M_p$ and $m_p$ are defined by* (2.2).

Observe, by simple calculation, we have that

$$(1-\nu)\,a^2+\nu b^2-((1-\nu)\,a+\nu b)^2-2r\left(\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\right) \tag{3.9}$$
$$=(1-\nu)\,\nu\,(b-a)^2-\frac{r}{2}\,(b-a)^2=r\left(R-\frac{1}{2}\right)(b-a)^2$$

and

$$2R\left(\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\right)-\left((1-\nu)\,a^2+\nu b^2-((1-\nu)\,a+\nu b)^2\right) \tag{3.10}$$
$$=\frac{R}{2}\,(b-a)^2-(1-\nu)\,\nu\,(b-a)^2=R\left(\frac{1}{2}-r\right)(b-a)^2,$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$.

If we write the inequalities (3.7) and (3.8) for $p = 2$, then we get

$$\ln\left(\frac{A_\nu\,(a,b)}{G_\nu\,(a,b)}\left(\frac{G\,(a,b)}{A\,(a,b)}\right)^{2r}\right)^{2m^2} \le r\left(R-\frac{1}{2}\right)(b-a)^2 \tag{3.11}$$
$$\le \ln\left(\left(\frac{G\,(a,b)}{A\,(a,b)}\right)^{2r}\frac{A_\nu\,(a,b)}{G_\nu\,(a,b)}\right)^{2M^2}$$

and

$$\ln\left(\left(\frac{A\,(a,b)}{G\,(a,b)}\right)^{2R}\frac{G_\nu\,(a,b)}{A_\nu\,(a,b)}\right)^{2m^2} \le R\left(\frac{1}{2}-r\right)(b-a)^2 \tag{3.12}$$
$$\le \ln\left(\frac{G_\nu\,(a,b)}{A_\nu\,(a,b)}\left(\frac{A\,(a,b)}{G\,(a,b)}\right)^{2R}\right)^{2M^2},$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$.

From the first inequality in (3.11) we have

$$\frac{A_\nu\,(a,b)}{G_\nu\,(a,b)} \le \left(\frac{A\,(a,b)}{G\,(a,b)}\right)^{2r}\exp\left(\frac{1}{2m^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right), \tag{3.13}$$

while from the second inequality in (3.11) we have

$$\left(\frac{A\,(a,b)}{G\,(a,b)}\right)^{2r}\exp\left(\frac{1}{2M^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right) \le \frac{A_\nu\,(a,b)}{G_\nu\,(a,b)}. \tag{3.14}$$

From the first inequality in (3.12) we have

$$\left(\frac{A\,(a,b)}{G\,(a,b)}\right)^{2R}\exp\left(-\frac{1}{2m^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right) \le \frac{A_\nu\,(a,b)}{G_\nu\,(a,b)}, \tag{3.15}$$

while from the second inequality in (3.12) we have

$$\frac{A_\nu(a,b)}{G_\nu(a,b)} \le \left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2M^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right). \tag{3.16}$$

In conclusion, from (3.13)-(3.16) we have the following result:

$$\begin{aligned}
&\max\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r} \exp\left(\frac{1}{2M^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right. \\
&\qquad \left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2m^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right)\right\} \\
&\le \frac{A_\nu(a,b)}{G_\nu(a,b)} \\
&\le \min\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r} \exp\left(\frac{1}{2m^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right. \\
&\qquad \left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2M^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right)\right\},
\end{aligned} \tag{3.17}$$

for any $a, b \in [m, M]$ and $\nu \in [0,1]$.

We need the following lemma (see [4]).

**Lemma 3.1.** *If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any $a$, $b \in \mathring{I}$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
0 &\le (1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b) \\
&\le \nu(1-\nu)(b-a)(f'(b) - f'(a)).
\end{aligned} \tag{3.18}$$

We have the following theorem.

**Theorem 3.2.** *Let $M$, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. Then for any $a, b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
&m\left(\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right) \\
&\le \nu(1-\nu)(b-a)(\Phi'(b) - \Phi'(a)) \\
&\quad - ((1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b)) \\
&\le M\left(\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\right).
\end{aligned} \tag{3.19}$$

*Proof.* Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then $f_m := \Phi + m\ln$ is convex and by (3.18) we have

$$0 \le (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) - m\ln\frac{A_\nu(a,b)}{G_\nu(a,b)}$$

$$
\begin{aligned}
&\leq \nu (1-\nu)(b-a)\left(\Phi'(b)-\Phi'(a)+\frac{m}{b}-\frac{m}{a}\right)\\
&= \nu (1-\nu)(b-a)(\Phi'(b)-\Phi'(a))-\frac{m}{ab}\nu (1-\nu)(b-a)^2,
\end{aligned}
$$

that is equivalent to

$$
\begin{aligned}
&m\left(\nu (1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu (a,b)}{G_\nu (a,b)}\right)\\
&\leq \nu (1-\nu)(b-a)(\Phi'(b)-\Phi'(a))-((1-\nu)\Phi (a)+\nu\Phi (b)-\Phi ((1-\nu) a+\nu b)),
\end{aligned}
$$

for any $a$, $b\in [m,M]$ and $\nu \in [0,1]$ and the first inequality in (3.19) is proved.

Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then also $f_M := -\Phi - M\ln$ is convex and by (3.18) we have

$$
\begin{aligned}
0 &\leq -(1-\nu)\Phi (a)-\nu\Phi (b)+f((1-\nu) a+\nu b)+M\ln\frac{A_\nu (a,b)}{G_\nu (a,b)}\\
&\leq -\nu (1-\nu)(b-a)(\Phi'(b)-\Phi'(a))+M\nu (1-\nu)\frac{(b-a)^2}{ab},
\end{aligned}
$$

that is equivalent to

$$
\begin{aligned}
&\nu (1-\nu)(b-a)(\Phi'(b)-\Phi'(a))-(1-\nu)\Phi (a)-\nu\Phi (b)+f((1-\nu) a+\nu b)\\
&\leq M\left(\nu (1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu (a,b)}{G_\nu (a,b)}\right),
\end{aligned}
$$

for any $a,b\in [m,M]$ and $\nu \in [0,1]$ and the second inequality in (3.19) is proved. □

**Corollary 3.2.** *Let $m, M$ with $M > m > 0$ and $p \in (-\infty,0)\cup(1,\infty)$. Then for any $a,b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$
\begin{aligned}
&p(p-1)m_p\left(\nu (1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu (a,b)}{G_\nu (a,b)}\right)\\
&\leq p\nu (1-\nu)(b-a)\left(b^{p-1}-a^{p-1}\right)-((1-\nu) a^p+\nu b^p-((1-\nu) a+\nu b)^p)\\
&\leq p(p-1)M_p\left(\nu (1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu (a,b)}{G_\nu (a,b)}\right),
\end{aligned}
\tag{3.20}
$$

*where $M_p$ and $m_p$ are defined by (2.2).*

The case $p = 2$ is of interest. Observe that

$$
\begin{aligned}
&2\nu (1-\nu)(b-a)^2-\left((1-\nu) a^2+\nu b^2-((1-\nu) a+\nu b)^2\right)\\
&=2\nu (1-\nu)(b-a)^2-\nu (1-\nu)(b-a)^2=\nu (1-\nu)(b-a)^2
\end{aligned}
$$

and by (3.20) we have

$$
2m^2\left(\nu (1-\nu)\frac{(b-a)^2}{ab}-\ln\frac{A_\nu (a,b)}{G_\nu (a,b)}\right)\leq \nu (1-\nu)(b-a)^2
$$

$$\leq 2M^2 \left( \nu (1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \right),$$

which is equivalent to

(3.21)

$$\exp \left( \nu (1-\nu) (b-a)^2 \left( \frac{1}{ab} - \frac{1}{2m^2} \right) \right) \leq \frac{A_\nu (a,b)}{G_\nu (a,b)}$$
$$\leq \exp \left( \nu (1-\nu) (b-a)^2 \left( \frac{1}{ab} - \frac{1}{2M^2} \right) \right),$$

for any $a$, $b \in [m, M]$ and $\nu \in [0,1]$.

## References


[1] S. S. Dragomir, *On a reverse of Jessen's inequality for isotonic linear functionals*, Journal of Inequalities in Pure and Applied Mathematics **2**(3) (2001), Article ID 36.

[2] S. S. Dragomir, *Bounds for the normalized Jensen functional,* Bull. Aust. Math. Soc. **74**(3) (2006), 417–478.

[3] S. S. Dragomir, *A survey on Jessen's type inequalities for positive functionals*, in: P. M. Pardalos et al. (Eds.), *Nonlinear Analysis*, Springer Optimization and Its Applications 68, In Honor of Themistocles M. Rassias on the Occasion of his 60th Birthday, Springer Science+Business Media, LLC 2012, DOI 10.1007/978-1-4614-3498-6_12.

[4] S. S. Dragomir, *A note on Young's inequality*, Research Group in Mathematical Inequalities and Applications **18** (2015), Article ID 126, [`http://rgmia.org/papers/v18/v18a126.pdf`].

[5] S. S. Dragomir, *A note on new refinements and reverses of Young's inequality*, Research Group in Mathematical Inequalities and Applications **18** (2015), Article ID 131, [`http://rgmia.org/papers/v18/v18a131.pdf`].

[6] S. S. Dragomir, *Additive inequalities for weighted harmonic and arithmetic operator means*, Research Group in Mathematical Inequalities and Applications **19** (2016), Article ID 6, [`http://rgmia.org/papers/v19/v19a06.pdf`].

[7] S. S. Dragomir and N. M. Ionescu, *On some inequalities for convex-dominated functions*, L'Anal. Num. Théor. L'Approx. **19**(1) (1990), 21–27.

[8] S. Furuichi, *Refined Young inequalities with Specht's ratio*, J. Egyptian Math. Soc. **20** (2012), 46–49.

[9] S. Furuichi, *On refined Young inequalities and reverse inequalities*, J. Math. Inequal. **5** (2011), 21–31.

[10] S. Furuichi and N. Minculete, *Alternative reverse inequalities for Young's inequality*, J. Math. Inequal. **5**(4) (2011), 595–600.

[11] F. Kittaneh and Y. Manasrah, *Improved Young and Heinz inequalities for matrix*, J. Math. Anal. Appl. **361** (2010), 262–269.

[12] F. Kittaneh and Y. Manasrah, *Reverse Young and Heinz inequalities for matrices*, Linear Multilinear Algebra **59** (2011), 1031–1037.

[13] W. Liao, J. Wu and J. Zhao, *New versions of reverse Young and Heinz mean inequalities with the Kantorovich constant*, Taiwanese J. Math. **19**(2) (2015), 467–479.

[14] M. Tominaga, *Specht's ratio in the Young inequality*, Sci. Math. Jpn. **55** (2002), 583–588.

[15] G. Zuo, G. Shi and M. Fujii, *Refined Young inequality with Kantorovich constant*, J. Math. Inequal. **5** (2011), 551–556.

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