Kragujevac Journal of Mathematics
Volume 44(2) (2020), Pages 313–325.

BASIC INEQUALITIES FOR (m, M)-Ψ-CONVEX FUNCTIONS WHEN $\Psi = -\ln$

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Abstract. In this paper we establish some basic inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

1. Introduction

Assume that the function $\Psi : I \subseteq \mathbb{R} \to \mathbb{R}$ (I is an interval) is convex on I and $m \in \mathbb{R}$. We shall say that the function $\Phi : I \to \mathbb{R}$ is m-Ψ-*lower convex* if $\Phi - m\Psi$ is a convex function on I. We may introduce the class of functions (see [1])

$$\mathcal{L}(I, m, \Psi) := \{\Phi : I \to \mathbb{R} \mid \Phi - m\Psi \text{ is convex on } I\}. \tag{1.1}$$

Similarly, for $M \in \mathbb{R}$ and Ψ as above, we can introduce the class of M-Ψ-*upper convex* functions by (see [1])

$$\mathcal{U}(I, M, \Psi) := \{\Phi : I \to \mathbb{R} \mid M\Psi - \Phi \text{ is convex on } I\}. \tag{1.2}$$

The intersection of these two classes will be called the class of (m, M)-Ψ-*convex functions* and will be denoted by (see [1])

$$\mathcal{B}(I, m, M, \Psi) := \mathcal{L}(I, m, \Psi) \cap \mathcal{U}(I, M, \Psi). \tag{1.3}$$

Remark 1.1. If $\Phi \in \mathcal{B}(I, m, M, \Psi)$, then $\Phi - m\Psi$ and $M\Psi - \Phi$ are convex and then $(\Phi - m\Psi) + (M\Psi - \Phi)$ is also convex which shows that $(M - m)\Psi$ is convex, implying that $M \geq m$ (as Ψ is assumed not to be the trivial convex function $\Psi(t) = 0$, $t \in I$).

Key words and phrases. Convex functions, special convexity, weighted arithmetic and geometric means, logarithmic function.
2010 *Mathematics Subject Classification.* Primary: 26D15. Secondary: 26D10.
DOI 10.46793/KgJMat2002.313D
Received: January 15, 2018.
Accepted: April 24, 2018.

The above concepts may be introduced in the general case of a convex subset in a real linear space, but we do not consider this extension here.

In [7], S. S. Dragomir and N. M. Ionescu introduced the concept of *g-convex dominated functions,* for a function $f : I \to \mathbb{R}$. We recall this, by saying, for a given convex function $g : I \to \mathbb{R}$, the function $f : I \to \mathbb{R}$ is *g-convex dominated* iff $g + f$ and $g - f$ are convex functions on I. In [7], the authors pointed out a number of inequalities for convex dominated functions related to Jensen's, Fuchs', Pečarić's, Barlow-Proschan and Vasić-Mijalković results, etc.

We observe that the concept of g-convex dominated functions can be obtained as a particular case from (m, M)-Ψ-convex functions by choosing $m = -1$, $M = 1$ and $\Psi = g$.

The following lemma holds (see [1]).

Lemma 1.1. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be differentiable functions on $\mathring{I}$, the interior of I and Ψ is a convex function on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$m\left(\Psi(t) - \Psi(s) - \Psi'(s)(t-s)\right) \le \Phi(t) - \Phi(s) - \Phi'(s)(t-s), \tag{1.4}$$

for all $t, s \in \mathring{I}$.

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$\Phi(t) - \Phi(s) - \Phi'(s)(t-s) \le M\left(\Psi(t) - \Psi(s) - \Psi'(s)(t-s)\right), \tag{1.5}$$

for all $t, s \in \mathring{I}$.

(iii) *For $M, m \in \mathbb{R}$ with $M \ge m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both* (1.4) *and* (1.5) *hold.*

Another elementary fact for twice differentiable functions also holds (see [1]).

Lemma 1.2. *Let $\Psi, \Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be twice differentiable on $\mathring{I}$ and Ψ is convex on $\mathring{I}$.*

(i) *For $m \in \mathbb{R}$, the function $\Phi \in \mathcal{L}\left(\mathring{I}, m, \Psi\right)$ if and only if*

$$m\Psi''(t) \le \Phi''(t), \quad \text{for all } t \in \mathring{I}. \tag{1.6}$$

(ii) *For $M \in \mathbb{R}$, the function $\Phi \in \mathcal{U}\left(\mathring{I}, M, \Psi\right)$ if and only if*

$$\Phi''(t) \le M\Psi''(t), \quad \text{for all } t \in \mathring{I}. \tag{1.7}$$

(iii) *For M, $m \in \mathbb{R}$ with $M \ge m$, the function $\Phi \in \mathcal{B}\left(\mathring{I}, m, M, \Psi\right)$ if and only if both* (1.6) *and* (1.7) *hold.*

For various inequalities concerning these classes of function, see the survey paper [3].

In what follows, we consider the class of functions $\mathcal{B}(I, m, M, -\ln)$ for $M, m \in \mathbb{R}$, with $M \ge m$ that is obtained for $\Psi : I \subseteq (0, \infty) \to \mathbb{R}$, $\Psi(t) = -\ln t$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a differentiable function on $\mathring{I}$ then by Lemma 1.1 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$m\left(\ln s - \ln t - \frac{1}{s}(s-t)\right) \le \Phi(t) - \Phi(s) - \Phi'(s)(t-s) \le M\left(\ln s - \ln t - \frac{1}{s}(s-t)\right), \tag{1.8}$$

for any $s, t \in \mathring{I}$.

If $\Phi : I \subseteq (0, \infty) \to \mathbb{R}$ is a twice differentiable function on $\mathring{I}$ then by Lemma 1.2 we have $\Phi \in \mathcal{B}(I, m, M, -\ln)$ if and only if

$$m \le t^2 \Phi''(t) \le M, \tag{1.9}$$

which is a convenient condition to verify in applications.

In this paper we establish some basic inequalities for (m, M)-Ψ-convex functions when $\Psi = -\ln$. Applications for power functions and weighted arithmetic mean and geometric mean are also provided.

For recent results concerning inequalities for weighted arithmetic mean and geometric mean (see [4, 5] and [8–15]).

2. Some Inequalities From Definition of Convexity

We define the weighted arithmetic and geometric means

$$A_\nu(a, b) := (1-\nu) a + \nu b \text{ and } G_\nu(a, b) := a^{1-\nu} b^\nu,$$

where $\nu \in [0, 1]$ and $a, b > 0$. If $\nu = \frac{1}{2}$, then we write for brevity $A(a, b)$ and $G(a, b)$, respectively.

The following double inequality holds, see also [6].

Theorem 2.1. *Let M, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$. Then for any a, $b > 0$ and $\nu \in [0, 1]$ we have*

$$\ln\left(\frac{A_\nu(a, b)}{G_\nu(a, b)}\right)^m \le (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) \le \ln\left(\frac{A_\nu(a, b)}{G_\nu(a, b)}\right)^M. \tag{2.1}$$

Proof. Since $\Phi \in \mathcal{B}((0, \infty), m, M, -\ln)$, then $\Phi_m := \Phi + m\ln$ is convex and by the definition of convexity, we have

$$\begin{aligned}
&\Phi((1-\nu)a + \nu b) + m \ln A_\nu(a, b) \\
&\le (1-\nu)(\Phi(a) + m\ln a) + \nu(\Phi(b) + m \ln b) \\
&= (1-\nu)\Phi(a) + \nu\Phi(b) + (1-\nu) m \ln a + \nu m \ln b \\
&= (1-\nu)\Phi(a) + \nu\Phi(b) + m \ln G_\nu(a, b),
\end{aligned}$$

that is equivalent to

$$m\ln\frac{A_\nu(a,b)}{G_\nu(a,b)}\le(1-\nu)\Phi(a)+\nu\Phi(b)-\Phi((1-\nu)a+\nu b)$$

and the first inequality in (2.1) is proved.

Similarly, by the convexity of $\Phi_M := -M\ln -\Phi$ we get the second part of (2.1). □

For m, M with $M>m>0$ we define

$$M_p:=\begin{cases}M^p, & \text{if } p>1,\\ m^p, & \text{if } p<0,\end{cases}\quad\text{and } m_p:=\begin{cases}m^p, & \text{if } p>1,\\ M^p, & \text{if } p<0.\end{cases}\tag{2.2}$$

Consider the function $\Phi(t)=t^p$, $p\in(-\infty,0)\cup(1,\infty)$. This is a convex function and $\Phi''(t)=p(p-1)t^{p-2}$, $t>0$. Consider $\kappa(t):=t^2\Phi''(t)=p(p-1)t^p$. We observe that

$$\max_{t\in[m,M]}\kappa(t)=p(p-1)M_p \text{ and } \min_{t\in[m,M]}\kappa(t)=p(p-1)m_p.$$

Corollary 2.1. *Let m, M with $M>m>0$ and $p\in(-\infty,0)\cup(1,\infty)$. Then for any a, $b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)m_p}\le(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\le\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)M_p},\tag{2.3}$$

where M_p and m_p are defined by (2.2).

By taking the exponential in (2.3) we get the equivalent inequality

$$\exp\left(\frac{(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p}{p(p-1)M_p}\right)\le\frac{A_\nu(a,b)}{G_\nu(a,b)}\le\exp\left(\frac{(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p}{p(p-1)m_p}\right),\tag{2.4}$$

for any $p\in(-\infty,0)\cup(1,\infty)$, $\nu\in[0,1]$ and any a, $b\in[m,M]$.

If we take $p=2$ in (2.4) and perform the calculations, then we get

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{M^2}\right)\le\frac{A_\nu(a,b)}{G_\nu(a,b)}\le\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{m^2}\right),\tag{2.5}$$

for any $a,b\in[m,M]$.

If a, $b>0$ then by taking $M=\max\{a,b\}$ and $m=\min\{a,b\}$ in (2.5) we have

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max^2\{a,b\}}\right)\le\frac{A_\nu(a,b)}{G_\nu(a,b)}\le\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min^2\{a,b\}}\right).\tag{2.6}$$

Since
$$\frac{(b-a)^2}{\max^2\{a,b\}}=\left(\frac{b-a}{\max\{a,b\}}\right)^2=\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2$$
and
$$\frac{(b-a)^2}{\min^2\{a,b\}}=\left(\frac{b-a}{\min\{a,b\}}\right)^2=\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2$$
for any $a,\ b>0$, then (2.6) can be written as

$$\begin{aligned}\exp&\left(\frac{1}{2}(1-\nu)\nu\left(1-\frac{\min\{a,b\}}{\max\{a,b\}}\right)^2\right)\\ \le&\frac{A_\nu(a,b)}{G_\nu(a,b)}\\ \le&\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right).\end{aligned}\tag{2.7}$$

This inequality was obtained in a different way in [5].

If we take $p=-1$ in (2.4) and perform the calculations, then we get

$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{m(b-a)^2}{abA_\nu(a,b)}\right)\le\frac{A_\nu(a,b)}{G_\nu(a,b)}\le\exp\left(\frac{1}{2}(1-\nu)\nu\frac{M(b-a)^2}{abA_\nu(a,b)}\right),\tag{2.8}$$

for any $a,\ b\in[m,M]$ and $\nu\in[0,1]$.

If $a,\ b>0$ then by taking $M=\max\{a,b\}$ and $m=\min\{a,b\}$ in (2.8) and since $ab=\max\{a,b\}\min\{a,b\}$ we have

$$\begin{aligned}\exp&\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right)\\ \le&\frac{A_\nu(a,b)}{G_\nu(a,b)}\\ \le&\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right),\end{aligned}\tag{2.9}$$

for any $\nu\in[0,1]$.

Since
$$\frac{1}{\max\{a,b\}}\le\frac{1}{A_\nu(a,b)}\le\frac{1}{\min\{a,b\}},$$
hence,
$$\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\min\{a,b\}}{\max\{a,b\}}-1\right)^2\right)\le\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\max\{a,b\}A_\nu(a,b)}\right)$$
and
$$\exp\left(\frac{1}{2}(1-\nu)\nu\frac{(b-a)^2}{\min\{a,b\}A_\nu(a,b)}\right)\le\exp\left(\frac{1}{2}(1-\nu)\nu\left(\frac{\max\{a,b\}}{\min\{a,b\}}-1\right)^2\right),$$

showing that the double inequality (2.9) is better than (2.7).

3. Some Perturbed Inequalities

Recall the following result obtained by Dragomir in 2006 [2] that provides a refinement and a reverse for the weighted Jensen's discrete inequality:

$$\begin{aligned} &n \min_{j\in\{1,2,\dots,n\}} \{p_j\} \left(\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^{n} x_j \right)\right) \\ \leq &\frac{1}{P_n}\sum_{j=1}^{n} p_j f(x_j) - f\left(\frac{1}{P_n}\sum_{j=1}^{n} p_j x_j\right) \\ \leq &n \max_{j\in\{1,2,\dots,n\}} \{p_j\} \left(\frac{1}{n} \sum_{j=1}^{n} f(x_j) - f\left(\frac{1}{n}\sum_{j=1}^{n} x_j \right)\right), \end{aligned} \tag{3.1}$$

where $f : C \to \mathbb{R}$ is a convex function defined on convex subset C of the linear space X, $\{x_j\}_{j\in\{1,2,\dots,n\}}$ are vectors in C and $\{p_j\}_{j\in\{1,2,\dots,n\}}$ are nonnegative numbers with $P_n = \sum_{j=1}^{n} p_j > 0$.

For $n = 2$, we deduce from (3.1) that

$$\begin{aligned} &2r\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right) \\ \leq &\nu f(x) + (1-\nu) f(y) - f(\nu x + (1-\nu) y) \\ \leq &2R\left(\frac{f(x)+f(y)}{2} - f\left(\frac{x+y}{2}\right)\right), \end{aligned} \tag{3.2}$$

for any $x, y \in C$ and $\nu \in [0,1]$, where $r := \min\{\nu, 1-\nu\}$ and $R := \max\{\nu, 1-\nu\}$.

Theorem 3.1. *Let M, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. Then for any a, $b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned} &\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right)^m \\ &\leq (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) - 2r\left(\frac{\Phi(a)+\Phi(b)}{2} - \Phi\left(\frac{a+b}{2}\right)\right) \\ &\leq \ln\left(\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^M \end{aligned} \tag{3.3}$$

and

$$\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^m \tag{3.4}$$

$$\leq 2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-\left(\nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)\right)$$
$$\leq \ln\left(\frac{G_{\nu}\left(a,b\right)}{A_{\nu}\left(a,b\right)}\left(\frac{A\left(a,b\right)}{G\left(a,b\right)}\right)^{2R}\right)^{M},$$

where $r := \min\left\{\nu, 1-\nu\right\}$ *and* $R := \max\left\{\nu, 1-\nu\right\}.$

Proof. Since $\Phi \in \mathcal{B}\left((0,\infty), m, M, -\ln\right)$, then $f_m := \Phi + m\ln$ is convex and by (3.2) we have

$$2r\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-2rm\ln\frac{A\left(a,b\right)}{G\left(a,b\right)} \tag{3.5}$$
$$\leq \nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)-m\ln\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}$$
$$\leq 2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-2Rm\ln\frac{A\left(a,b\right)}{G\left(a,b\right)},$$

for any $a, b > 0$ and $\nu \in [0,1]$.

Since $\Phi \in \mathcal{B}\left((0,\infty), m, M, -\ln\right)$, then also $f_M := -\Phi - M\ln$ is convex and by (3.2) we have

$$2r\left(\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi(a)+\Phi(b)}{2}\right)+2rM\ln\frac{A\left(a,b\right)}{G\left(a,b\right)} \tag{3.6}$$
$$\leq \Phi\left(\nu a+\left(1-\nu\right)b\right)-\nu\Phi\left(a\right)-\left(1-\nu\right)\Phi\left(b\right)+M\ln\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}$$
$$\leq 2R\left(\Phi\left(\frac{a+b}{2}\right)-\frac{\Phi(a)+\Phi(b)}{2}\right)+2RM\ln\frac{A\left(a,b\right)}{G\left(a,b\right)},$$

for any $a, b > 0$ and $\nu \in [0,1]$.

From the first inequality in (3.5) we have

$$\ln\left(\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}\left(\frac{G\left(a,b\right)}{A\left(a,b\right)}\right)^{2r}\right)^{m}$$
$$\leq \nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)-2r\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right),$$

while from the first inequality in (3.6) we also have

$$\nu\Phi\left(a\right)+\left(1-\nu\right)\Phi\left(b\right)-\Phi\left(\nu a+\left(1-\nu\right)b\right)-2r\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)$$
$$\leq \ln\left(\left(\frac{G\left(a,b\right)}{A\left(a,b\right)}\right)^{2r}\frac{A_{\nu}\left(a,b\right)}{G_{\nu}\left(a,b\right)}\right)^{M},$$

for any $a, b > 0$ and $\nu \in [0,1]$.

These prove the desired result (3.3).

From the second inequality in (3.5) we have

$$\begin{aligned}
&\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^{m}\\
\leq &2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-\left(\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)\right),
\end{aligned}$$

while from the second inequality in (3.6) we also have

$$\begin{aligned}
&2R\left(\frac{\Phi(a)+\Phi(b)}{2}-\Phi\left(\frac{a+b}{2}\right)\right)-\left(\nu\Phi(a)+(1-\nu)\Phi(b)-\Phi(\nu a+(1-\nu)b)\right)\\
\leq &\ln\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^{M},
\end{aligned}$$

for any $a,b>0$ and $\nu\in[0,1]$.

These prove the desired result (3.4). □

Corollary 3.1. *Let m,M with $M>m>0$ and $p\in(-\infty,0)\cup(1,\infty)$. Then for any $a,b\in[m,M]$ and $\nu\in[0,1]$ we have*

$$\begin{aligned}
&\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right)^{p(p-1)m_p}\\
&\leq(1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p-2r\left(\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right)\\
&\leq\ln\left(\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{p(p-1)M_p}
\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}
&\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^{p(p-1)m_p}\\
&\leq 2R\left(\frac{a^p+b^p}{2}-\left(\frac{a+b}{2}\right)^p\right)-\left((1-\nu)a^p+\nu b^p-((1-\nu)a+\nu b)^p\right)\\
&\leq\ln\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^{p(p-1)M_p},
\end{aligned}\tag{3.8}$$

where $r:=\min\{\nu,1-\nu\}$ and $R:=\max\{\nu,1-\nu\}$ and M_p and m_p are defined by (2.2).

Observe, by simple calculation, we have that

$$(1-\nu)a^2+\nu b^2-((1-\nu)a+\nu b)^2-2r\left(\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\right) \tag{3.9}$$
$$=(1-\nu)\nu(b-a)^2-\frac{r}{2}(b-a)^2=r\left(R-\frac{1}{2}\right)(b-a)^2$$

and

$$2R\left(\frac{a^2+b^2}{2}-\left(\frac{a+b}{2}\right)^2\right)-\left((1-\nu)a^2+\nu b^2-((1-\nu)a+\nu b)^2\right) \tag{3.10}$$
$$=\frac{R}{2}(b-a)^2-(1-\nu)\nu(b-a)^2=R\left(\frac{1}{2}-r\right)(b-a)^2,$$

for any $a,b\in[m,M]$ and $\nu\in[0,1]$.

If we write the inequalities (3.7) and (3.8) for $p=2$, then we get

$$\ln\left(\frac{A_\nu(a,b)}{G_\nu(a,b)}\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\right)^{2m^2}\le r\left(R-\frac{1}{2}\right)(b-a)^2 \tag{3.11}$$
$$\le\ln\left(\left(\frac{G(a,b)}{A(a,b)}\right)^{2r}\frac{A_\nu(a,b)}{G_\nu(a,b)}\right)^{2M^2}$$

and

$$\ln\left(\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\frac{G_\nu(a,b)}{A_\nu(a,b)}\right)^{2m^2}\le R\left(\frac{1}{2}-r\right)(b-a)^2 \tag{3.12}$$
$$\le\ln\left(\frac{G_\nu(a,b)}{A_\nu(a,b)}\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\right)^{2M^2},$$

for any $a,b\in[m,M]$ and $\nu\in[0,1]$.

From the first inequality in (3.11) we have

$$\frac{A_\nu(a,b)}{G_\nu(a,b)}\le\left(\frac{A(a,b)}{G(a,b)}\right)^{2r}\exp\left(\frac{1}{2m^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right), \tag{3.13}$$

while from the second inequality in (3.11) we have

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{2r}\exp\left(\frac{1}{2M^2}r\left(R-\frac{1}{2}\right)(b-a)^2\right)\le\frac{A_\nu(a,b)}{G_\nu(a,b)}. \tag{3.14}$$

From the first inequality in (3.12) we have

$$\left(\frac{A(a,b)}{G(a,b)}\right)^{2R}\exp\left(-\frac{1}{2m^2}R\left(\frac{1}{2}-r\right)(b-a)^2\right)\le\frac{A_\nu(a,b)}{G_\nu(a,b)}, \tag{3.15}$$

while from the second inequality in (3.12) we have

$$\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)} \le \left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2M^2} R\left(\frac{1}{2}-r\right)(b-a)^2\right). \tag{3.16}$$

In conclusion, from (3.13)-(3.16) we have the following result:

$$\begin{aligned}
&\max\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r} \exp\left(\frac{1}{2M^2} r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right.\\
&\left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2m^2} R\left(\frac{1}{2}-r\right)(b-a)^2\right)\right\}\\
\le&\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\\
\le&\min\left\{\left(\frac{A(a,b)}{G(a,b)}\right)^{2r} \exp\left(\frac{1}{2m^2} r\left(R-\frac{1}{2}\right)(b-a)^2\right),\right.\\
&\left.\left(\frac{A(a,b)}{G(a,b)}\right)^{2R} \exp\left(-\frac{1}{2M^2} R\left(\frac{1}{2}-r\right)(b-a)^2\right)\right\},
\end{aligned} \tag{3.17}$$

for any $a,b \in [m,M]$ and $\nu \in [0,1]$.

We need the following lemma (see [4]).

Lemma 3.1. *If the function $f : I \subset \mathbb{R} \to \mathbb{R}$ is a differentiable convex function on $\mathring{I}$, then for any a, $b \in \mathring{I}$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
0 &\le (1-\nu) f(a) + \nu f(b) - f((1-\nu)a + \nu b)\\
&\le \nu(1-\nu)(b-a)(f'(b) - f'(a)).
\end{aligned} \tag{3.18}$$

We have the following theorem.

Theorem 3.2. *Let M, $m \in \mathbb{R}$ with $M > m$ and $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$. Then for any $a,b > 0$ and $\nu \in [0,1]$ we have*

$$\begin{aligned}
&m\left(\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right)\\
\le&\nu(1-\nu)(b-a)(\Phi'(b) - \Phi'(a))\\
&-((1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b))\\
\le&M\left(\nu(1-\nu)\frac{(b-a)^2}{ab} - \ln\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}\right).
\end{aligned} \tag{3.19}$$

Proof. Since $\Phi \in \mathcal{B}((0,\infty), m, M, -\ln)$, then $f_m := \Phi + m\ln$ is convex and by (3.18) we have

$$0 \le (1-\nu)\Phi(a) + \nu\Phi(b) - \Phi((1-\nu)a + \nu b) - m\ln\frac{A_{\nu}(a,b)}{G_{\nu}(a,b)}$$

$$\begin{aligned}
&\leq \nu (1-\nu) (b-a) \left(\Phi' (b) - \Phi' (a) + \frac{m}{b} - \frac{m}{a} \right) \\
&= \nu (1-\nu) (b-a) (\Phi' (b) - \Phi' (a)) - \frac{m}{ab} \nu (1-\nu) (b-a)^2 ,
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&m \left(\nu (1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \right) \\
&\leq \nu (1-\nu) (b-a) (\Phi' (b) - \Phi' (a)) - ((1-\nu) \Phi (a) + \nu \Phi (b) - \Phi ((1-\nu) a + \nu b)) ,
\end{aligned}$$

for any a, $b \in [m, M]$ and $\nu \in [0, 1]$ and the first inequality in (3.19) is proved.

Since $\Phi \in \mathcal{B} ((0, \infty), m, M, -\ln)$, then also $f_M := -\Phi - M \ln$ is convex and by (3.18) we have

$$\begin{aligned}
0 &\leq - (1-\nu) \Phi (a) - \nu \Phi (b) + f ((1-\nu) a + \nu b) + M \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \\
&\leq - \nu (1-\nu) (b-a) (\Phi' (b) - \Phi' (a)) + M \nu (1-\nu) \frac{(b-a)^2}{ab},
\end{aligned}$$

that is equivalent to

$$\begin{aligned}
&\nu (1-\nu) (b-a) (\Phi' (b) - \Phi' (a)) - (1-\nu) \Phi (a) - \nu \Phi (b) + f ((1-\nu) a + \nu b) \\
&\leq M \left(\nu (1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \right),
\end{aligned}$$

for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ and the second inequality in (3.19) is proved. □

Corollary 3.2. *Let m, M with $M > m > 0$ and $p \in (-\infty, 0) \cup (1, \infty)$. Then for any $a, b \in [m, M]$ and $\nu \in [0, 1]$ we have*

$$\begin{aligned}
&p (p-1) m_p \left(\nu (1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \right) \\
&\leq p \nu (1-\nu) (b-a) \left(b^{p-1} - a^{p-1} \right) - ((1-\nu) a^p + \nu b^p - ((1-\nu) a + \nu b)^p) \\
&\leq p (p-1) M_p \left(\nu (1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \right),
\end{aligned} \tag{3.20}$$

where M_p and m_p are defined by (2.2).

The case $p = 2$ is of interest. Observe that

$$\begin{aligned}
&2\nu (1-\nu) (b-a)^2 - \left((1-\nu) a^2 + \nu b^2 - ((1-\nu) a + \nu b)^2 \right) \\
&= 2\nu (1-\nu) (b-a)^2 - \nu (1-\nu) (b-a)^2 = \nu (1-\nu) (b-a)^2
\end{aligned}$$

and by (3.20) we have

$$2m^2 \left(\nu (1-\nu) \frac{(b-a)^2}{ab} - \ln \frac{A_\nu (a,b)}{G_\nu (a,b)} \right) \leq \nu (1-\nu) (b-a)^2$$

$$\leq 2M^2\left(\nu\left(1-\nu\right)\frac{\left(b-a\right)^2}{ab}-\ln\frac{A_\nu\left(a,b\right)}{G_\nu\left(a,b\right)}\right),$$

which is equivalent to

(3.21)
$$\exp\left(\nu\left(1-\nu\right)\left(b-a\right)^2\left(\frac{1}{ab}-\frac{1}{2m^2}\right)\right)\leq\frac{A_\nu\left(a,b\right)}{G_\nu\left(a,b\right)}$$
$$\leq\exp\left(\nu\left(1-\nu\right)\left(b-a\right)^2\left(\frac{1}{ab}-\frac{1}{2M^2}\right)\right),$$

for any a, $b\in[m,M]$ and $\nu\in[0,1]$.

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