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Contents

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MORE ABOUT PETROVIĆ'S INEQUALITY ON COORDINATES VIA m-CONVEX FUNCTIONS AND RELATED RESULTS

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ABSTRACT. In this paper the authors extend Petrović's inequality for coordinated m-convex functions in the plane and also find Lagrange type and Cauchy type mean value theorems for Petrović's inequality for m-convex functions and coordinated m-convex functions. The authors consider functional due to Petrović's inequality in plane and discuss its properties for certain class of coordinated log-m-convex functions.

1. INTRODUCTION

A function $f : [a, b] \to \mathbb{R}$ is said to be convex if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds, for all $x, y \in [a, b]$ and $t \in [0,1]$.

In [6], Dragomir gave the definition of convex functions on coordinates as follows.

Definition 1.1. Let $\Delta = [a, b] \times [c, d] \subseteq \mathbb{R}^2$ and $f : \Delta \to \mathbb{R}$ be a mapping. Define partial mappings

$$f_y : [a, b] \to \mathbb{R} \text{ by } f_y(u) = f(u, y) \tag{1.1}$$

and

$$f_x : [c, d] \to \mathbb{R} \text{ by } f_x(v) = f(x, v). \tag{1.2}$$

Then f is said to be convex on coordinates (or coordinated convex) in Δ if f_y and f_x are convex on $[a, b]$ and $[c, d]$ respectively for all $y \in [c, d]$ and $x \in [a, b]$. A mapping f is said to be strictly convex on coordinates (or strictly coordinated convex) in Δ

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if f_y and f_x are strictly convex on $[a, b]$ and $[c, d]$, respectively, for all $y \in [c, d]$ and $x \in [a, b]$.

In [22], G. Toader gave the definition of m-convexity as follows.

Definition 1.2. The function $f : [0, b] \to \mathbb{R}$, $b > 0$, is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leqslant tf(x) + m(1-t)f(y),$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Remark 1.1. One can note that the notion of m-convexity reduces to convexity if we take $m = 1$. We get starshaped functions from m-convex functions if we take $m = 0$.

Definition 1.3. A function $f : [a, b] \to \mathbb{R}_+$ is called log-convex if

$$f(tx + (1-t)y) \leqslant f^t(x) + f^{(1-t)}(y)$$

holds, for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Log-convex functions have excellent closure properties. The sum and product of two log-convex functions is convex. If f is convex function and g is log-convex function then the functional composition $g \circ f$ is also log-convex.

In [1], Almori and Darus gave the definition of log-convex on coordinates as follows.

Definition 1.4. Let $\Delta = [a, b] \times [c, d]$ and let a function $f : \Delta \to \mathbb{R}_+$ is called log-convex on coordinates in Δ if partial mappings defined in (1.1) and (1.2) are log-convex on $[a, b]$ and $[c, d]$, respectively, for all $y \in [c, d]$ and $x \in [a, b]$.

In [8], Farid et al. gave the definition of coordinated m-convex functions as follows.

Definition 1.5. Let $\Delta = [0, b] \times [0, d] \subset [0, \infty)^2$, then a function $f : \Delta \to \mathbb{R}$ will be called m-convex on coordinates if the partial mappings

$$f_y : [0, b] \to \mathbb{R} \text{ defined by } f_y(u) = f(u, y)$$

and

$$f_x : [0, d] \to \mathbb{R} \text{ defined by } f_x(v) = f(x, v)$$

are m-convex on $[0, b]$ and $[0, d]$, respectively, for all $y \in [0, d]$ and $x \in [0, b]$.

In [17] (see also [15, p. 154]), M. Petrović proved the following result, which is known as Petrović's inequality in the literature.

Theorem 1.1. *Suppose that $(x_1, \ldots, x_n)$ and $(p_1, \ldots, p_n)$ be two non-negative n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_i$ for $i = 1, \ldots, n$ and $\sum_{k=1}^{n} p_k x_k \in [0, a]$. If f is a convex function on $[0, a)$, then the inequality*

$$\sum_{k=1}^{n} p_k f(x_k) \leq f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0) \tag{1.3}$$

is valid.

Remark 1.2. Take $p_k = 1$, $k = 1, \dots, n$ the above inequality becomes

$$\sum_{k=1}^{n} f(x_k) \leq f\left(\sum_{k=1}^{n} x_k\right) + (n-1)f(0).$$

In [2], M. Bakula et al. gave the Petrović's inequality for m-convex function which is stated in the following theorem.

Theorem 1.2. *Let $(x_1, \dots, x_n)$ be non-negative n-tuples and $(p_1, \dots, p_n)$ be positive n-tuples such that*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_j \textit{ for each } j = 1, \dots, n.$$

If $f : [0, \infty) \to \mathbb{R}$ be an m-convex function on $[0, \infty)$ with $m \in (0, 1]$, then

$$\sum_{k=1}^{n} p_k f(x_k) \leqslant \min\left\{ mf\left(\tfrac{\tilde{x}_n}{m}\right) + (P_n - 1)f(0), f(\tilde{x}_n) + m(P_n - 1)f(0) \right\}. \tag{1.4}$$

Remark 1.3. If we take $m = 1$ in Theorem 1.2, we get famous Petrović's inequality stated in Theorem 1.1.

In [19], Rehman et al. gave the Petrović's inequality for coordinated convex functions, which is stated in the following theorem.

Theorem 1.3. *Let $(x_1, \dots, x_n) \in [0, a)^n$, $(y_1, \dots, y_n) \in [0, b)^n$ and $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \in [0, a)$, $\sum_{j=1}^{n} q_j y_j \in [0, b)$, $\sum_{k=1}^{n} p_k \geq 1$,*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \textit{ for each } i = 1, \dots, n,$$

and

$$Q_n := \sum_{j=1}^{n} q_j, \quad 0 \neq \tilde{y}_n = \sum_{j=1}^{n} q_j y_j \geq y_i \textit{ for each } i = 1, \dots, n.$$

If $f : \Delta \to \mathbb{R}$ be a coordinated convex, then

$$\begin{aligned} \sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq & f(\tilde{x}_n, \tilde{y}_n) + (Q_n - 1) f(\tilde{x}_n, 0) \\ & + (P_n - 1)\left(f(0, \tilde{y}_n) + (Q_n - 1) f(0, 0)\right). \end{aligned} \tag{1.5}$$

By considering non-negative difference of (1.5), the authors in [19] defined the following functional

$$\begin{aligned} \Upsilon(f) = & f(\tilde{x}_n, \tilde{y}_n) + (Q_n - 1) f(\tilde{x}_n, 0) + (P_n - 1)\left[f(0, \tilde{y}_n) + (Q_n - 1) f(0, 0)\right] \\ & - \sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j f(x_k, y_j). \end{aligned} \tag{1.6}$$

By considering non-negative difference of (1.3), the authors in [4] defined the following functional

$$\mathcal{P}(f) = f\left(\sum_{k=1}^{n} p_k x_k\right) - \left(\sum_{k=1}^{n} p_k f(x_k)\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0). \tag{1.7}$$

One of the generalizations of convex functions is m-convex functions and it is considered in literature by many researchers and mathematicians, for example, see [7, 10–12, 24] and references there in. In [17] (also see [15, p. 154]), M. Petrović gave the inequality for convex functions known as Petrović's inequality. Many authors worked on this inequality by giving results related to it, for example see [13, 15, 17] and it has been generalized for m-convex functions by M. Bakula et al. in [2]. In [19], Petrović's inequality was generalized on coordinate by using the definition of convex functions on coordinates given by Dragomir in [6].

In this paper the authors extend Petrović's inequality for coordinated m-convex functions in the plane and also find Lagrange type and Cauchy type mean value theorems for Petrović's inequality for m-convex functions and coordinated m-convex functions. The authors consider functional due to Petrović's inequality in plane and discuss its properties for certain class of coordinated log-m-convex functions.

2. Main Result

The following theorem consist the result for Petrović's inequality on coordinated m-convex functions.

Theorem 2.1. *Let $(x_1, \dots, x_n)$, $(y_1, \dots, y_n)$ be non-negative n-tuples and $(p_1, \dots, p_n)$, $(q_1, \dots, q_n)$ be positive n-tuples such that $\sum\limits_{k=1}^{n} p_k \geq 1$,*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \text{ for each } i = 1, \dots, n$$

and

$$Q_n := \sum_{j=1}^{n} q_j, \quad 0 \neq \tilde{y}_n = \sum_{j=1}^{n} q_j y_j \geq y_i \text{ for each } i = 1, \dots, n.$$

If $f : [0, \infty)^2 \to \mathbb{R}$ be an m-convex function on coordinates with $m \in (0, 1]$, then

$$\begin{aligned} \sum_{k=1}^{n} \sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq & \min \left\{ m \min\left\{G_{m,1}(\tilde{x}_n/m), G_{1,m}(\tilde{x}_n/m)\right\} + (P_n - 1) \right. \\ & \times \min\left\{G_{m,1}(0), G_{1,m}(0)\right\}, \min\left\{G_{m,1}(\tilde{x}_n), G_{1,m}(\tilde{x}_n)\right\} \\ & \left. + m(P_n - 1) \min\left\{G_{m,1}(0), G_{1,m}(0)\right\} \right\}, \end{aligned} \tag{2.1}$$

where

$$G_{m,\widetilde{m}}(t) = m f\left(t, \frac{\tilde{y}_n}{m}\right) + \widetilde{m}(Q_n - 1) f(t, 0). \tag{2.2}$$

Proof. Let $f_x : [0, \infty) \to \mathbb{R}$ and $f_y : [0, \infty) \to \mathbb{R}$ be mappings such that $f_x(v) = f(x, v)$ and $f_y(u) = f(u, y)$. Since f is coordinated m-convex on $[0, \infty)^2$, therefore f_y is m-convex on $[0, \infty)$, so by Theorem 1.2, one has

$$\sum_{k=1}^{n} p_k f_y(x_k) \leq \min \left\{ m f_y\left(\tilde{x}_n/m\right) + (P_n - 1) f_y\left(0\right), f_y\left(\tilde{x}_n\right) + m(P_n - 1) f_y\left(0\right) \right\}.$$

This is equivalent to

$$\begin{aligned}\sum_{k=1}^{n} p_k f(x_k, y) \leq \min \{ & m f\left(\tilde{x}_n/m, y\right) + (P_n - 1) f\left(0, y\right), \\ & f\left(\tilde{x}_n, y\right) + m(P_n - 1) f\left(0, y\right) \}.\end{aligned}$$

By setting $y = y_j$, we have

$$\begin{aligned}\sum_{k=1}^{n} p_k f(x_k, y_j) \leq \min \{ & m f\left(\tilde{x}_n/m, y_j\right) + (P_n - 1) f\left(0, y_j\right), \\ & f\left(\tilde{x}_n, y_j\right) + m(P_n - 1) f\left(0, y_j\right) \},\end{aligned}$$

this gives

$$\begin{aligned}\sum_{k=1}^{n} \sum_{j=1}^{n} p_k q_j f(x_k, y_j) \leq \min \Bigg\{ & m \sum_{j=1}^{n} q_j f\left(\tilde{x}_n/m, y_j\right) + (P_n - 1) \sum_{j=1}^{n} q_j f\left(0, y_j\right), \\ & \sum_{j=1}^{n} q_j f\left(\tilde{x}_n, y_j\right) + m(P_n - 1) \sum_{j=1}^{n} q_j f\left(0, y_j\right) \Bigg\}.\end{aligned} \tag{2.3}$$

Now again by Theorem 1.2, one has

$$\begin{aligned}\sum_{j=1}^{n} q_j f\left(\tilde{x}_n/m, y_j\right) \leq \min \{ & m f\left(\tilde{x}_n/m, \tilde{y}_n/m\right) + (Q_n - 1) f\left(\tilde{x}_n/m, 0\right), \\ & f\left(\tilde{x}_n/m, \tilde{y}_n\right) + m(Q_n - 1) f\left(\tilde{x}_n/m, 0\right) \},\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^{n} q_j f\left(0, y_j\right) \leq \min \{ & m f\left(0, \tilde{y}_n/m\right) + (Q_n - 1) f\left(0, 0\right), \\ & f\left(0, \tilde{y}_n\right) + m(Q_n - 1) f\left(0, 0\right) \}\end{aligned}$$

and

$$\begin{aligned}\sum_{j=1}^{n} q_j f\left(\tilde{x}_n, y_j\right) \leq \min \{ & m f\left(\tilde{x}_n, \tilde{y}_n/m\right) + (Q_n - 1) f\left(\tilde{x}_n, 0\right), \\ & f\left(\tilde{x}_n, \tilde{y}_n\right) + m(Q_n - 1) f\left(\tilde{x}_n, 0\right) \}.\end{aligned}$$

Putting these values in inequality (2.3), and using the notation in (2.2), one has the required result. □

Remark 2.1. If we take $m = 1$ in Theorem 2.1, we get Theorem 1.3.

In the following corollary, we gave new Petrović's type inequality for m-convex functions.

Corollary 2.1. *Let $(x_1,\dots,x_n)$, $(y_1,\dots,y_n)$ be non-negative n-tuples and $(p_1,\dots,p_n)$, $(q_1,\dots,q_n)$ be positive n-tuples such that $\sum\limits_{k=1}^{n} p_k \geq 1$ and*

$$P_n := \sum_{k=1}^{n} p_k, \quad 0 \neq \tilde{x}_n = \sum_{k=1}^{n} p_k x_k \geq x_i \text{ for each } i = 1,\dots,n.$$

If $f : [0,\infty)^2 \to \mathbb{R}$ be an m-convex function on coordinates with $m \in (0,1]$, then one has

$$\begin{aligned}(2.4)\qquad \sum_{k=1}^{n} np_k f(x_k) \leq \min\{&m\min\{(m+n-1)f(\tilde{x}_n/m),(mn-m+1)f(\tilde{x}_n/m)\}\\ &+(P_n-1)\min\{(m+n-1)f(0),(mn-m+1)f(0)\},\\ &\min\{(m+n-1)f(\tilde{x}_n),(mn-m+1)f(\tilde{x}_n)\}\\ &+m(P_n-1)\min\{(m+n-1),(mn-m+1)f(0)\}\}.\end{aligned}$$

Proof. If we put $y_j = 0$ and $q_j = 1$, $j = 1,\dots,n$ with $f(x,0) \mapsto f(x)$ in inequality (2.1), we get the required result. □

Remark 2.2. If we take $m = 1$ in inequality (2.4), we get the inequality (1.3).

Let $f : [0,b] \to \mathbb{R}$ be a function. Then we define

$$(2.5)\qquad P_{a,m,f}(x) := \frac{f(x) - mf(a)}{x - ma},$$

for all $x \in [0,b]\backslash\{ma\}$, for fixed $a \in [0,b]$. Also define

$$(2.6)\qquad r_m(x_1,x_2,x_3;f) := \frac{P_{x_1,m}(x_3) - P_{x_1,m}(x_2)}{x_3 - x_2},$$

where $x_1,x_2,x_3 \in [0,b]$, $(x_2 - mx_1)(x_3 - mx_1) > 0$, $x_2 \neq x_3$.

In [11] (see also [7, p. 294]), V. G. Mihesan considered the functions defined in (2.5), (2.6) and proved the following result.

Remark 2.3. If we take $m = 1$ in (2.5) and (2.6), we get divided differences of first and second order respectively.

By considering non-negative difference of (1.4), we defined following functional

$$(2.7)\qquad \mathcal{P}_m(f) = \min\left\{mf\left(\frac{\tilde{x}_n}{m}\right) + (P_n-1)f(0), f(\tilde{x}_n) + m(P_n-1)f(0)\right\} - \sum_{k=1}^{n} p_k f(x_k).$$

Also by considering non-negative difference of (2.1), we defined following functional

$$\begin{aligned}(2.8)\qquad \Upsilon_m(f) = \min\{&m\min\{G_{m,1}(\tilde{x_n}/m),G_{1,m}(\tilde{x_n}/m)\}\\ &+(P_n-1)\min\{G_{m,1}(0),G_{1,m}(0)\},\min\{G_{m,1}(\tilde{x_n}),G_{1,m}(\tilde{x_n})\}\\ &+m(P_n-1)\min\{G_{m,1}(0),G_{1,m}(0)\}\} - \sum_{k=1}^{n}\sum_{j=1}^{n} p_k q_j f(x_k,y_j).\end{aligned}$$

If we take $m = 1$ in the above (2.8), we get $\Upsilon_1(f) = \Upsilon(f)$.

Remark 2.4. Under the suppositions of Theorem 2.1, if f is coordinated m-convex in Δ^2, then $\Upsilon_m(f) \geq 0$.

Here we state an important lemma that is very helpful in proving mean value theorems related to the non-negative functional of Petrović's inequality for m-convex functions.

Lemma 2.1. *Let* $f : [0, b] \to \mathbb{R}$ *be a function such that*

$$m_1 \leqslant \frac{(x - ma)f'(x) - f(x) + mf(a)}{x^2 - 2max + ma^2} \leqslant M_1,$$

for all $x \in [0, b]\backslash\{ma\}$, $a \in (0, b)$ *and* $m \in (0, 1)$.

Consider the functions $\psi_1, \psi_2 : [0, b] \to \mathbb{R}$ *defined as*

$$\psi_1(x) = M_1x^2 - f(x)$$

and

$$\psi_2(x) = f(x) - m_1x^2,$$

then ψ_1 *and* ψ_2 *are* m*-convex in* $[0, b]$.

Proof. Suppose

$$\begin{aligned} P_{a,m,\psi_1}(x) =& \frac{\psi_1(x) - m\psi_1(a)}{x - ma} \\ =& \frac{M_1x^2 - f(x) - mf(a) + mM_1a^2}{x - ma} \\ =& \frac{M_1(x^2 - ma^2)}{x - ma} - \frac{f(x) - mf(a)}{x - ma}. \end{aligned}$$

So we have

$$P'_{a,m,\psi_1}(x) = M_1\frac{x^2 - 2max + ma^2}{(x - ma)^2} - \frac{(x - ma)f'(x) - f(x) + mf(a)}{(x - ma)^2}.$$

Since

$$x^2 - 2max + ma^2 = (x - ma)^2 + m(1 - m)a^2 > 0,$$

by given condition, we have

$$M_1(x^2 - 2max + ma^2) \geq (x - ma)f'(x) - f(x) + mf(a).$$

This leads to

$$M_1\frac{x^2 - 2max + ma^2}{(x - ma)^2} \geq \frac{(x - ma)f'(x) - f(x) + mf(a)}{(x - ma)^2},$$

$$M_1\frac{x^2 - 2max + ma^2}{(x - ma)^2} - \frac{(x - ma)f'(x) - f(x) + mf(a)}{(x - ma)^2} \geq 0.$$

This implies

$$P'_{a,m,\psi_1}(x) \geq 0, \qquad \text{for all } x \in [0, ma) \cup (ma, b].$$

Similarly, one can show that

$$P'_{a,m,\psi_2}(x) \geq 0, \qquad \text{for all } x \in [0, ma) \cup (ma, b].$$

This gives P_{a,m,ψ_1} and P_{a,m,ψ_2} are increasing on $x \in [0, ma) \cup (ma, b]$ for all $a \in [0, b]$. Hence by Lemma 2.1, $\psi_1(x)$ and $\psi_2(x)$ are m-convex in $[0, b]$. □

Here we give mean value theorems related to functional defined for Petrović's inequality for m-convex functions.

Theorem 2.2. *Let $(x_1, \dots, x_n) \in [0, b]$, $(q_1, \dots, q_n)$ and $(p_1, \dots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \dots, n$. Also, let $\phi(x) = x^2$.*

If $f \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\mathcal{P}_m(f) = \frac{(\xi - ma)f'(\xi) - f(\xi) + mf(a)}{\xi^2 - 2ma\xi + ma^2}\mathcal{P}_m(\phi), \tag{2.9}$$

provided that $\mathcal{P}_m(\phi)$ is non zero and $a \in (0, b)$.

Proof. As $f \in C^1([0, b])$, so there exists real numbers m_1 and M_1 such that

$$m_1 \leqslant \frac{(x - ma)f'(x) - f(x) + mf(a)}{x^2 - 2max + ma^2} \leqslant M_1,$$

for each $x \in [0, b]$, $a \in (0, b)$ and $m \in (0, 1)$.

Now let us consider the functions ψ_1 and ψ_2 defined in Lemma 2.1. As ψ_1 is m-convex in $[0, b]$,

$$\mathcal{P}_m(\psi_1) \geq 0,$$

that is

$$\mathcal{P}_m(M_1 x^2 - f(x)) \geq 0,$$

which gives

$$M_1 \mathcal{P}_m(\phi) \geq \mathcal{P}_m(f). \tag{2.10}$$

Similarly ψ_2 is m-convex in $[0, b]$, therefore one has

$$m_1 \mathcal{P}_m(\phi) \leqslant \mathcal{P}_m(f). \tag{2.11}$$

By assumption $\mathcal{P}_m(\phi)$ is non zero, combining inequalities (2.10) and (2.11), one has

$$m_1 \leqslant \frac{\mathcal{P}_m(f)}{\mathcal{P}_m(\phi)} \leqslant M_1.$$

Hence, there exists $\xi \in (0, b)$ such that

$$\frac{\mathcal{P}_m(f)}{\mathcal{P}_m(\phi)} = \frac{(\xi - ma)f'(\xi) - f(\xi) + mf(a)}{\xi^2 - 2ma\xi + ma^2}.$$

Hence, we get the required result. □

Corollary 2.2. *Let $(x_1, \ldots, x_n) \in [0, b]$, $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \ldots, n$. Also let $\phi(x) = x^2$.*

If $f \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\mathcal{P}(f) = \frac{(\xi - a)f'(\xi) - f(\xi) + f(a)}{(\xi - a)^2}\mathcal{P}(\phi),$$

provided that $\mathcal{P}(\phi)$ is non zero and $a \in (0, b)$.

Proof. If we put $m = 1$ in (2.9), we get the required result. □

Corollary 2.3. *Let $(x_1, \ldots, x_n) \in [0, b]$, $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \ldots, n$ and $a \in (0, b)$. Also let $\phi(x) = x^2$.*

If $f \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\mathcal{P}(f) = f''(a)\mathcal{P}(\phi).$$

Proof. If we put $m = 1$ in (2.9), we get

$$\begin{aligned}\frac{\mathcal{P}(f)}{\mathcal{P}(\phi)} &= \frac{(\xi - a)f'(\xi) - f(\xi) + f(a)}{(\xi - a)^2}\\ &= \frac{1}{\xi - a}\left(f'(\xi) - \frac{f(a) - f(\xi)}{a - \xi}\right).\end{aligned}$$

Take limit as $\xi \to a$, we get

$$\begin{aligned}\frac{\mathcal{P}(f)}{\mathcal{P}(\phi)} &= \lim_{\xi \to a} \frac{1}{\xi - a}\left(f'(\xi) - \frac{f(a) - f(\xi)}{a - \xi}\right)\\ &= \lim_{\xi \to a} \frac{1}{\xi - a}\left(f'(\xi) - f'(a)\right).\end{aligned}$$

Again taking limit as $\xi \to a$, we get

$$\frac{\mathcal{P}(f)}{\mathcal{P}(\phi)} = f''(a).$$

Hence, we get the required result. □

Theorem 2.3. *Let $(x_1, \ldots, x_n) \in [0, b]$, $(q_1, \ldots, q_n)$ and $(p_1, \ldots, p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1, 2, \ldots, n$. Also, let $\phi(x) = x^2$.*

If $f_1, f_2 \in C^1([0, b])$, then there exists $\xi \in (0, b)$ such that

$$\frac{\mathcal{P}_m(f_1)}{\mathcal{P}_m(f_2)} = \frac{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)}{(\xi - ma)f_2'(\xi) - f_2(\xi) + mf_2(a)},$$

provided that the denominators are non-zero and $a \in (0, b)$.

Proof. Suppose a function $k \in C^1([0, b])$ be defined as

$$k = c_1 f_1 - c_2 f_2,$$

where c_1 and c_2 are defined as

$$c_1 = \mathcal{P}_m(f_2),$$
$$c_2 = \mathcal{P}_m(f_1).$$

Then using Theorem 2.2 with $f = k$, one has

$$(\xi - ma)((c_1f_1 - c_2f_2)(\xi))' - (c_1f_1 - c_2f_2)(\xi) + m(c_1f_1 - c_2f_2)(a) = 0,$$

that is

$$(\xi - ma)(c_1f_1'(\xi) - c_2f_2'(\xi)) - c_1f_1(\xi) + c_2f_2(\xi) + mc_1f_1(a) - mc_2f_2(a) = 0,$$

which gives

$$(\xi - ma)c_1f_1'(\xi) - (\xi - ma)c_2f_2'(\xi) - c_1f_1(\xi) + c_2f_2(\xi) + mc_1f_1(a) - mc_2f_2(a) = 0,$$

which implies

$$c_1\left\{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)\right\} - c_2\left\{(\xi - ma)f_2'(\xi) + f_2(\xi) - mf_2(a)\right\} = 0,$$
$$c_1\left\{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)\right\} = c_2\left\{(\xi - ma)f_2'(\xi) - f_2(\xi) + mf_2(a)\right\}$$

and

$$\frac{c_2}{c_1} = \frac{(\xi - ma)f_1'(\xi) - f_1(\xi) + mf_1(a)}{(\xi - ma)f_2'(\xi) - f_2(\xi) + mf_2(a)}.$$

After putting the values of c_1 and c_2, we get the required result. □

Here we state an important lemma that is very helpful in proving mean value theorems related to the non-negative functional of Petrović's inequality for coordinated m-convex functions.

Lemma 2.2. *Let $\Delta = [0, b] \times [0, d]$, $m \in (0, 1)$. Also let $f : \Delta \to \mathbb{R}$ be a function such that*

$$m_1 \leqslant \frac{(x - ma)\frac{\partial}{\partial x}f(x, y) - f(x, y) + mf(a, y)}{(x^2 - 2max + ma^2)y^2} \leqslant M_1$$

and

$$m_2 \leqslant \frac{(y - mc)\frac{\partial}{\partial y}f(x, y) - f(x, y) + mf(x, c)}{(y^2 - 2mcy + mc^2)x^2} \leqslant M_2,$$

for all $x \in [0, b]\backslash\{ma\}$, $a \in (0, b)$ and $y \in [0, d]\backslash\{mc\}$, $c \in (0, d)$.

Consider the functions $\alpha_y : [0, b] \to \mathbb{R}$, and $\alpha_x : [0, d] \to \mathbb{R}$, defined as

$$\alpha(x, y) = \max\{M_1, M_2\}x^2y^2 - f(x, y)$$

and

$$\beta(x, y) = f(x, y) - \min\{m_1, m_2\}x^2y^2.$$

Then α and β are coordinated m-convex in Δ.

Proof. Consider the partial mappings $\alpha_y : [0,b] \to \mathbb{R}$ and $\alpha_x : [0,d] \to \mathbb{R}$ defined by $\alpha_y(x) := \alpha(x,y)$ for all $x \in (0,b]$ and $\alpha_x(y) := \alpha(x,y)$ for all $y \in (0,d]$.

$$\begin{aligned} P_{a,m,\alpha_y}(x) &= \frac{\alpha_y(x) - m\alpha_y(a)}{x - ma} \\ &= \frac{\alpha(x,y) - m\alpha(a,y)}{x - ma} \\ &= \frac{M_1 x^2 y^2 - f(x,y) - mM_1 a^2 y^2 + mf(a,y)}{x - ma} \\ &= M_1 \frac{(x^2 - ma^2)y^2}{x - ma} - \frac{f(x,y) - mf(a,y)}{x - ma}. \end{aligned}$$

So we have

$$\begin{aligned} P'_{a,m,\alpha_y}(x) &= M_1 \frac{\partial}{\partial x}\left(\frac{(x^2 - ma^2)y^2}{x - ma}\right) - \frac{\partial}{\partial x}\left(\frac{f(x,y) - mf(a,y)}{x - ma}\right) \\ &= M_1 y^2 \frac{(x^2 - 2max + ma^2)}{(x - ma)^2} - \frac{(x - ma)\frac{\partial}{\partial x} f(x,y) - f(x,y) + mf(a,y)}{(x - ma)^2}. \end{aligned}$$

Since

$$M_1 \geq \frac{(x - ma)\frac{\partial}{\partial x} f(x,y) - f(x,y) + mf(a,y)}{(x^2 - 2max + ma^2)y^2},$$

by given conditions, we have

$$(x^2 - 2max + ma^2)y^2 > 0.$$

This implies

$$\begin{aligned} M_1 y^2 \frac{(x^2 - 2max + ma^2)}{(x - ma)^2} &\geq \frac{(x - ma)\frac{\partial}{\partial x} f(x,y) - f(x,y) + mf(a,y)}{(x - ma)^2} \\ M_1 y^2 \frac{(x^2 - 2max + ma^2)}{(x - ma)^2} &- \frac{(x - ma)\frac{\partial}{\partial x} f(x,y) - f(x,y) + mf(a,y)}{(x - ma)^2} \geq 0. \end{aligned}$$

This implies

$$P'_{a,m,\alpha_y}(x) \geq 0 \text{ for all } x \in [0, ma) \cup (ma, b].$$

Similarly, one can show that

$$P'_{a,m,\alpha_x}(y) \geq 0 \text{ for all } x \in [0, mc) \cup (mc, d].$$

This ensures that P_{a,m,α_y} is increasing on $[0,ma) \cup (ma,b]$ for all $a \in [0,b]$ and P_{a,m,α_x} is increasing on $[0,mc) \cup (mc,d]$ for all $c \in [0,d]$. Hence, by Lemma 2.1, α is m-convex in Δ.

Similarly, one can show that β is m-convex in Δ. $\square$

Here we give mean value theorems related to the functional defined by Petrović's inequality for coordinated m-convex functions.

Theorem 2.4. *Let $\Delta = [0,b] \times [0,d]$, $(x_1,\ldots,x_n) \in [0,b]$, $(y_1,\ldots,y_n) \in [0,d]$ be non-negative n-tuples and $(q_1,\ldots,q_n)$, $(p_1,\ldots,p_n)$ be positive n-tuples such that $\sum_{k=1}^{n} p_k x_k \geq x_j$ for each $j = 1,2,\ldots,n$. Also, let $\varphi(x,y) = x^2y^2$.*

If $f \in C^1(\Delta)$, then there exists (ξ_1,η_1) and (ξ_2,η_2) in the interior of Δ, such that

$$\Upsilon_m(f) = \frac{(\xi_1 - ma)\frac{\partial}{\partial x}f(\xi_1,\eta_1) - f(\xi_1,\eta_1) + mf(a,\eta_1)}{(\xi_1^2 - 2ma\xi_1 + ma^2)\eta_1^2}\Upsilon_m(\varphi) \tag{2.12}$$

and

$$\Upsilon_m(f) = \frac{(\xi_2 - ma)\frac{\partial}{\partial y}f(\xi_2,\eta_2) - f(\xi_2,\eta_2) + mf(a,\eta_2)}{(\xi_2^2 - 2ma\xi_2 + ma^2)\eta_2^2}\Upsilon_m(\varphi), \tag{2.13}$$

and provided that $\Upsilon_m(\varphi)$ is non-zero and $a \in (0,b)$.

Proof. As f has continuous first order partial derivative in Δ, so there exists real numbers m_1, m_2, M_1 and M_2 such that

$$m_1 \leqslant \frac{(x - ma)\frac{\partial}{\partial x}f(x,y) - f(x,y) + mf(a,y)}{(x^2 - 2max + ma^2)y^2} \leqslant M_1$$

and

$$m_2 \leq \frac{(y - ma)\frac{\partial}{\partial y}f(x,y) - f(x,y) + mf(x,a)}{(y^2 - 2may + ma^2)x^2} \leq M_2,$$

for all $x \in (0,b]$, $y \in (0,d]$, $a \in (0,b)$ and $m \in (0,1)$.

Now let us consider the functions α and β defined in Lemma 2.2.

As α is m-convex in Δ, then

$$\Upsilon_m(\alpha) \geq 0,$$

that is

$$\Upsilon_m(M_1x^2y^2 - f(x,y)) \geq 0,$$

which gives

$$M_1\Upsilon_m(\varphi) \geq \Upsilon_m(f). \tag{2.14}$$

Similarly β is m-convex in Δ, therefore one has

$$m_1\Upsilon_m(\varphi) \leqslant \Upsilon_m(f). \tag{2.15}$$

By the assumption $\Upsilon_m(\varphi)$ is non-zero. Combining inequalities (2.14) and (2.15), one has

$$m_1 \leqslant \frac{\Upsilon_m(f)}{\Upsilon_m(\varphi)} \leqslant M_1.$$

Hence there exists (ξ_1,η_1) in the interior of Δ, such that

$$\Upsilon_m(f) = \frac{(\xi_1 - ma)\frac{\partial}{\partial x}f(\xi_1,\eta_1) - f(\xi_1,\eta_1) + mf(a,\eta_1)}{(\xi_1^2 - 2ma\xi_1 + ma^2)\eta_1^2}\Upsilon_m(\varphi).$$

Similarly, one can show that

$$\Upsilon_m(f) = \frac{(\xi_2 - ma)\frac{\partial}{\partial y}f(\xi_2,\eta_2) - f(\xi_2,\eta_2) + mf(a,\eta_2)}{(\xi_2^2 - 2ma\xi_2 + ma^2)\eta_2^2}\Upsilon_m(\varphi),$$

which is the required result. □

Corollary 2.4. *Let* $\Delta = [0,b] \times [0,d]$, $(x_1,\dots,x_n) \in [0,b]$, $(y_1,\dots,y_n) \in [0,d]$ *be non-negative* n*-tuples and* $(q_1,\dots,q_n)$, $(p_1,\dots,p_n)$ *be positive* n*-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_j$ *for each* $j = 1,2,\dots,n$. *Also, let* $\varphi(x,y) = x^2y^2$.

If $f \in C^1(\Delta)$, *then there exists* (ξ_1,η_1) *and* (ξ_2,η_2) *in the interior of* Δ, *such that*

$$\Upsilon(f) = \frac{(\xi_1 - a)\frac{\partial}{\partial x} f(\xi_1,\eta_1) - f(\xi_1,\eta_1) + f(a,\eta_1)}{(\xi_1 - a)^2\eta_1^2}\Upsilon(\varphi)$$

and

$$\Upsilon(f) = \frac{(\xi_2 - a)\frac{\partial}{\partial y} f(\xi_2,\eta_2) - f(\xi_2,\eta_2) + f(a,\eta_2)}{(\xi_2 - a)^2\eta_2^2}\Upsilon(\varphi),$$

provided that $\Upsilon(\varphi)$ *is non-zero and* $a \in (0,b)$.

Proof. If we put m=1 in (2.12) and (2.13), we get the required result. □

Theorem 2.5. *Let* $\Delta = [0,b] \times [0,d]$, $(x_1,\dots,x_n) \in [0,b]$, $(y_1,\dots,y_n) \in [0,d]$ *be non-negative* n*-tuples and* $(q_1,\dots,q_n)$, $(p_1,\dots,p_n)$ *be positive* n*-tuples such that* $\sum_{k=1}^{n} p_k x_k \geq x_j$ *for each* $j = 1,2,\dots,n$. *Also, let* $\varphi(x,y) = x^2y^2$.

If $f_1, f_2 \in C^1(\Delta)$, *then there exists* (ξ_1,η_1) *and* (ξ_2,η_2) *in the interior of* Δ, *such that*

$$\frac{\Upsilon_m(f_1)}{\Upsilon_m(f_2)} = \frac{(\xi_1 - ma)\frac{\partial}{\partial x} f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + mf_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial x} f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + mf_2(a,\eta_2)}$$

and

$$\frac{\Upsilon_m(f_1)}{\Upsilon_m(f_2)} = \frac{(\xi_1 - ma)\frac{\partial}{\partial y} f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + mf_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial y} f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + mf_2(a,\eta_2)},$$

provided that the denominators are non-zero and $a \in (0,b)$.

Proof. Suppose

$$k = c_1 f_1 - c_2 f_2,$$

where c_1 and c_2 are defined by

$$\begin{aligned} c_1 =& \Upsilon_m(f_2), \\ c_2 =& \Upsilon_m(f_1). \end{aligned}$$

Then using Theorem 2.4 with $f = k$, we get

$$
\begin{aligned}
&(\xi - ma)\frac{\partial}{\partial x}(c_1 f_1 - c_2 f_2)(\xi,\eta) - (c_1 f_1 - c_2 f_2)(\xi,\eta) + m(c_1 f_1 - c_2 f_2)(a,\eta) = 0,\\
&(\xi - ma)c_1\frac{\partial}{\partial x} f_1(\xi,\eta) - (\xi - ma)c_2\frac{\partial}{\partial x} f_2(\xi,\eta) - c_1 f_1(\xi,\eta) + c_2 f_2(\xi,\eta)\\
&+ mc_1 f_1(a,\eta) - mc_2 f_2(a,\eta) = 0,\\
&c_1\left\{(\xi - ma)\frac{\partial}{\partial x} f_1(\xi,\eta) - f_1(\xi,\eta) + m f_1(a,\eta)\right\} - c_2\left\{(\xi - ma)\frac{\partial}{\partial x} f_2(\xi,\eta)\right.\\
&\left. + f_2(\xi,\eta) - m f_2(a,\eta)\right\} = 0,\\
&c_1\left\{(\xi - ma)\frac{\partial}{\partial x} f_1(\xi,\eta) - f_1(\xi,\eta) + m f_1(a,\eta)\right\} = c_2\left\{(\xi - ma)\frac{\partial}{\partial x} f_2(\xi,\eta)\right.\\
&\left. - f_2(\xi,\eta) + m f_2(a,\eta)\right\},
\end{aligned}
$$

and

$$\frac{c_2}{c_1} = \frac{(\xi_1 - ma)\frac{\partial}{\partial x} f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + m f_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial x} f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + m f_2(a,\eta_2)}.$$

Similarly, one can show that

$$\frac{c_2}{c_1} = \frac{(\xi_1 - ma)\frac{\partial}{\partial y} f_1(\xi_1,\eta_1) - f_1(\xi_1,\eta_1) + m f_1(a,\eta_1)}{(\xi_2 - ma)\frac{\partial}{\partial y} f_2(\xi_2,\eta_2) - f_2(\xi_2,\eta_2) + m f_2(a,\eta_2)}.$$

After putting the values of c_1 and c_2, we get the required result. □

3. Log Convexity

Here we have defined some families of parametric functions which we use in sequal. Let $I = [0,a)$, $J = [0,b) \subseteq \mathbb{R}$ be intervals and $f_t : I \times J \to \mathbb{R}$ represents some parametric mapping for $t \in (c,d) \subseteq \mathbb{R}$. We define functions

$$f_{t,y} : I \to \mathbb{R} \text{ by } f_{t,y}(u) = f_t(u,y)$$

and

$$f_{t,x} : J \to \mathbb{R} \text{ by } f_{t,x}(v) = f_t(x,v),$$

where $x \in I$ and $y \in J$. Suppose $\mathcal{H}_1$ denotes the class of functions $f_t : I \times J \to \mathbb{R}$ for $t \in (c,d)$ such that the functions

$$t \mapsto r_m(u_0,u_1,u_2,f_{t,y}), \quad \text{for all } u_0,u_1,u_2 \in I$$

and

$$t \mapsto r_m(v_0,v_1,v_2,f_{t,x}), \quad \text{for all } v_0,v_1,v_2 \in J$$

are log-convex functions in Jensen sense on (c,d).

The following lemma is given in [16].

Lemma 3.1. *Let $I \subseteq \mathbb{R}$ be an interval. A function $f : I \to (0, \infty)$ is log-convex in J-sense on I, that is, for each $r, t \in I$*

$$f(r)f(t) \geq f^2\left(\frac{t+r}{2}\right)$$

if and only if the relation

$$m^2 f(t) + 2mnf\left(\frac{t+r}{2}\right) + n^2 f(r) \geq 0$$

holds, for each $m, n \in \mathbb{R}$ and $r, t \in I$.

Our next result comprises properties of functional defined in Theorem 2.1.

Theorem 3.1. *Let $f_t \in \mathcal{H}_1$ and Υ_m be the functional defined in (2.8). Then the function $t \mapsto \Upsilon_m(f_t)$ is log-convex in Jensen sense for each $t \in (c, d)$.*

Proof. Let

$$h(u,v) = m^2 f_t(u,v) + 2mnf_{\frac{t+r}{2}}(u,v) + n^2 f_r(u,v),$$

where $m, n \in \mathbb{R}$ and $t, r \in (c, d)$. Also we can consider that

$$h_y(u) = m^2 f_{t,y}(u) + 2mnf_{\frac{t+r}{2},y}(u) + n^2 f_{r,y}(u)$$

and

$$h_x(v) = m^2 f_{t,x}(v) + 2mnf_{\frac{t+r}{2},x}(v) + n^2 f_{r,x}(v),$$

which gives

$$\begin{aligned} r_m(u_0, u_1, u_2, h_y) =& m^2 r_m(u_0, u_1, u_2, f_{t,y}) + 2mnr_m(u_0, u_1, u_2, f_{\frac{t+r}{2},y}) \\ &+ n^2 r_m(u_0, u_1, u_2, f_{r,y}). \end{aligned}$$

As $r_m[u_0, u_1, u_2, f_{t,y}]$ is log-convex in Jensen sense so by using Lemma 3.1, the right hand side of the above expression is non negative so h_y is m-convex, similarly h_x is also m-convex, so h is m-convex on coordinates, which implies $r_m(h) \geq 0$ and

$$m^2 r_m(f_t) + 2mnr_m(f_{\frac{t+r}{2}}) + n^2 r_m(f_r) \geq 0.$$

Hence, $t \mapsto \Upsilon_m(f_t)$ is log-convex in Jensen sense. □

Theorem 3.2. *Assume that f_t is of class $\mathcal{H}_1$ and Υ_m be the functional defined in (2.8). If the function $\Upsilon_m(f_t)$ is continuous for each $t \in (c, d)$, then $\Upsilon_m(f_t)$ is log-convex for each $t \in (c, d)$.*

Proof. If a function is continuous and log-convex in Jensen sense, then it is log-convex (see [3, p. 48]). It is given that $\Upsilon_m(f_t)$ is continuous for each $t \in (c, d)$, hence $\Upsilon_m(f_t)$ is log-convex for each $t \in (c, d)$. □

Lemma 3.2. *If f is a convex function for all x_1, x_2, x_3 of an open interval I for which $x_1 < x_2 < x_3$, then*

$$(x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0.$$

Theorem 3.3. *Let $f_t \in \mathcal{H}_1$ and Υ_m be the functional defined in* (2.8)*. If $\Upsilon_m(f_t)$ is positive, then for some $r < s < t$, where $r, s, t \in (c, d)$, one has*

$$[\Upsilon_m(f_s)]^{t-r} \leq [\Upsilon_m(f_r)]^{t-s} [\Upsilon_m(f_t)]^{s-r}.$$

Proof. Consider the functional $\Upsilon_m(f_t)$. Also let $r < s < t$, where $r, s, t \in (c, d)$, since $\Upsilon_m(f_t)$ is log-convex, that is, $\log \Upsilon_m(f_t)$ is convex. By taking $f = \log \Upsilon_m$ in Lemma 3.2, we have

$$(t - s) \log \Upsilon_m(f_r) + (r - t) \log \Upsilon_m(f_s) + (s - r) \log \Upsilon_m(f_t) \geq 0,$$

which can be written as

$$[\Upsilon_m(f_s)]^{t-r} \leq [\Upsilon_m(f_r)]^{t-s} [\Upsilon_m(f_t)]^{s-r}.$$

□

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CERTAIN GENERATING MATRIX FUNCTIONS OF LEGENDRE MATRIX POLYNOMIALS USING LIE ALGEBRAIC METHOD

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Abstract. The main aim of this present paper is to investigate a new of interesting generating matrix relation for Legendre matrix polynomials with the help of a Lie group-theoretic method. Certain properties are well known but some of them are believed to be novel families of matrix differential recurrence relations and generating matrix functions for these matrix polynomials. Special cases of new results are also given here as applications.

1. Introduction, Motivation and Preliminaries

Special matrix functions are attaining significant results from both the practical and theoretical examples in different fields of Physics, Mathematics and Lie theory. Theories in connection with the unification of generating matrix relations for various special matrix functions are of greater importance in the study of special matrix functions by Lie group theory. The above idea was originally generated by Weisner group-theoretic method and [22–24] also applied this technique to obtain the generating relation. However, the study of special functions from Lie group-theoretic method approach has been obtained generating relations in the books of McBride [12] and Miller [13]. In [17, 18, 21], the author has earlier introduced and studied the Legendre matrix polynomials. In [3, 9–11, 14–16, 19, 20], certain properties of some special matrix functions via Lie algebra have been proposed as finite series solutions of second-order differential matrix equation.

Motivated by their work, in the present paper, our aim is to establish some results for Legendre matrix polynomials. Here, we give the families of generating matrix

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functions for Legendre matrix polynomials and the differential recurrence matrix relations for these matrix polynomials are also obtained in section 2. In section 3, we study of linear differential operators for Legendre matrix polynomials which generate Lie algebra to apply Weisner's method to obtain some generating matrix relations and apply these linear operators to determine a local representation which makes a one to one correspondence between these Lie algebra with the help of Weisner's method.

Here, the concepts associated with the functional matrix calculus are reviewed. Throughout this article, for a matrix $A \in \mathbb{C}^{N\times N}$, its spectrum $\sigma(A)$ denotes the set of all the eigenvalues of A. We denote by I and $\mathbf{O}$ the identity and null matrix in $\mathbb{C}^{N\times N}$, respectively.

Definition 1.1 ([7]). For a matrix $A \in \mathbb{C}^{N\times N}$ such that $\sigma(A)$ does not contain 0 or a negative integer ($\sigma(A) \cap \mathbb{Z}^{-} = \emptyset$ where $\emptyset$ is an empty set), the matrix form of shifted factorial is defined as

$$(A)_n = \begin{cases} A(A+I)\cdots(A+(n-1)I) = \Gamma(A+nI)\Gamma^{-1}(A), & n \in \mathbb{N}, \\ I, & n = 0, \end{cases} \tag{1.1}$$

where $\Gamma(A)$ is an invertible matrix in $\mathbb{C}^{N\times N}$ and $\Gamma^{-1}(A)$ is inverse Gamma matrix function (see [8]).

For A is an arbitrary matrix in $\mathbb{C}^{N\times N}$ and using (1.1), we have the relations (see Defez and Jódar [4])

$$(A)_{n+k} = (A)_n(A+nI)_k = (A)_k(A+kI)_n,$$

$$(-nI)_k = \begin{cases} \dfrac{(-1)^k\, n!}{(n-k)!} I, & 0 \le k \le n, \\ \mathbf{0}, & k > n, \end{cases}$$

$$(A)_{n-k} = \begin{cases} (-1)^k (A)_n[(I-A-nI)_k]^{-1}, & 0 \le k \le n, \\ \mathbf{0}, & k > n. \end{cases} \tag{1.2}$$

If $\mathrm{Re}(\mu) \in \sigma(A)$ is not an integer and using (1.1), we have the relation

$$\Gamma(I-A-nI)\Gamma^{-1}(I-A) = (-1)^n[(A)_n]^{-1}, \tag{1.3}$$

where $\Gamma(I-A)$ is an invertible matrix.

Lemma 1.1. *If $A(k,n)$ is a matrix in $\mathbb{C}^{N\times N}$ for $k, n \in \mathbb{N}_0$, the relation is satisfied (see Defez and Jódar [4])*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k). \tag{1.4}$$

Definition 1.2 (Jódar and Cortés [7]). For any matrices A, B, and C in $\mathbb{C}^{N\times N}$ such that C is an invertible matrix and for $|z| < 1$, the hypergeometric matrix function is defined as follows

$${}_2F_1(A,B;C;z) = \sum_{k=0}^{\infty} \frac{z^k}{k!}(A)_k(B)_k[(C)_k]^{-1}. \tag{1.5}$$

For $A \in \mathbb{C}^{N\times N}$, we have the relation (see Defez and Jódar [5])

$$D^k\Big[t^{A+mI}\Big] = (A+I)_m (A+I)_{m-k} t^{A+(m-k)I}, \quad k = 0, 1, 2, \dots. \tag{1.6}$$

Theorem 1.1. *For $|z| < 1$ if A, B and C are matrices in $\mathbb{C}^{N\times N}$ where the matrix C satisfies the condition $C + nI$ is an invertible matrix for all integers $n \geq 0$ and C, $C - A$ and $C - B$ are positive stable matrices with all matrices are commutative, then the relation*

$$ {}_2F_1\Big(A, B; C; z\Big) = (1-z)^{C-A-B}\ {}_2F_1\Big(C-A, C-B; C; z\Big). \tag{1.7}$$

Corollary 1.1 ([1,2,6])**.** *Jacobi matrix polynomials have the matrix recurrence relation:*

$$\begin{aligned}&(x-1)\Bigg[(A+B+nI)\frac{d}{dx}P_n^{(A,B)}(x) + (A+nI)\frac{d}{dx}P_{n-1}^{(A,B)}(x)\Bigg]\\ =&(A+B+nI)\Bigg[nP_n^{(A,B)}(x) - (A+nI)P_{n-1}^{(A,B)}(x)\Bigg],\end{aligned} \tag{1.8}$$

where A and B are commutative matrices in $\mathbb{C}^{N\times N}$ such that

$$\operatorname{Re}(z) > -1, \quad \text{for all } z \in \sigma(A) \quad \text{and} \quad \operatorname{Re}(w) > -1, \quad \text{for all } w \in \sigma(B).$$

Definition 1.3 ([18])**.** Let A be a matrix in $\mathbb{C}^{N\times N}$ such that

$$0 < \operatorname{Re}(\lambda) < 1, \quad \text{for all } \lambda \in \sigma(A). \tag{1.9}$$

Legendre matrix polynomials $P_n(x, A)$ is defined by

$$\begin{aligned}P_n(x, A) =& \sum_{k=0}^{n} \frac{(-1)^k (n+k)!}{k!(n-k)!}\left(\frac{1-x}{2}\right)^k \Gamma^{-1}(A+kI)\Gamma(A), \quad n \geq 0\\ =&\ {}_2F_1\left(-nI, (n+1)I; A; \frac{1}{2}(1-x)\right), \quad \left|\frac{1-x}{2}\right| < 1,\end{aligned}$$

such that $A + kI$ is an invertible matrix for all integers $k \geq 0$.

Theorem 1.2 ([18])**.** *For $n \geq 0$, the Legendre matrix polynomials $P_n(x, A)$ satisfy the second order differential matrix equation as*

$$\begin{aligned}&(1-x^2)D^2P_n(x, A) + 2((1-x)I - A)DP_n(x, A) + n(n+1)P_n(x, A) = \mathbf{0},\\ &\left|\frac{x-1}{2}\right| < 1, \quad D = \frac{d}{dx}.\end{aligned} \tag{1.10}$$

Theorem 1.3 ([18])**.** *For the Legendre matrix polynomials $P_n(x, A)$, we have the pure matrix recurrence relation*

$$(A+nI)P_{n+1}(x, A) = (2n+1)xP_n(x, A) + (A-(n+1)I)P_{n-1}(x, A), \quad n \geq 1. \tag{1.11}$$

2. Some New Results for Legendre Matrix Polynomials

Here, we derive families of new results for Legendre matrix polynomials with A a matrix in $\mathbb{C}^{N\times N}$ satisfying the condition (1.9). We have the following main results.

Theorem 2.1. *The generating matrix functions for the Legendre matrix polynomials are*

$$\sum_{n=0}^{\infty} t^n P_n(x,A) = (1-t)^{-1}\ {}_2F_1\left(\frac{1}{2}I, I; A; \frac{2(x-1)t}{(1-t)^2}\right), \tag{2.1}$$

for $\left|\frac{2(x-1)t}{(1-t)^2}\right| < 1$, $|t|<1$, *and*

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} P_n(x,A) = e^t\ {}_1F_1\left((n+1)I; A; \frac{(1-x)t}{2}\right), \quad \left|\frac{(1-x)t}{2}\right| < 1. \tag{2.2}$$

Proof. From the definition of hypergeometric matrix function and multiplying $(1-t)^{-1}$, we have

$$\begin{aligned}
&(1-t)^{-1}\ {}_2F_1\left(\frac{1}{2}I, I; A; \frac{2(x-1)t}{(1-t)^2}\right)\\
=&\sum_{k=0}^{\infty} \frac{2^k}{k!}(1-t)^{-(1+2k)I} t^k (I)_k \left(\frac{1}{2}I\right)_k [(A)_k]^{-1}(x-1)^k\\
=&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{2^k t^{n+k}}{n!k!}((1+2k)I)_n (I)_k \left(\frac{1}{2}I\right)_k [(A)_k]^{-1}(x-1)^k.
\end{aligned}$$

From (1.2), we can write that

$$(I)_{2k} = 2^{2k}(I)_k \left(\frac{1}{2}I\right)_k,$$

which implies

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{2^k t^{n+k}}{n!k!}((1+2k)I)_n 2^{-2k}(I)_{2k}[(A)_k]^{-1}(x-1)^k.$$

Using (1.2), we get

$$(I)_{n+2k} = (I)_{2k}((1+2k)I)_n,$$

which implies

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^{n+k}}{n!k!}(I)_{n+2k}[(A)_k]^{-1}\left(\frac{x-1}{2}\right)^k.$$

Using Lemma 1.1 and replacing n by $n-k$, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{t^n}{(n-k)!k!}(I)_{n+k}[(A)_k]^{-1}\left(\frac{x-1}{2}\right)^k\\
=&\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{n!t^n}{(n-k)!k!}((n+1)I)_k[(A)_k]^{-1}\left(\frac{x-1}{2}\right)^k.
\end{aligned}$$

By using (1.2) in the above equation, we obtain (2.1).

From the definition of hypergeometric matrix function and multiplying e^t, we have

$$e^t\ {}_1F_1\Big((n+1)I; A; \frac{(1-x)t}{2}\Big) = e^t \sum_{k=0}^{\infty} \frac{1}{k!}((n+1)I)_k[(A)_k]^{-1}\Big(\frac{(1-x)t}{2}\Big)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{t^{n+k}}{n!k!}((n+1)I)_k[(A)_k]^{-1}\Big(\frac{1-x}{2}\Big)^k.$$

Using Lemma 1.1 and replacing n by $n-k$ with the help of these Eqs. (1.1), (1.2) and (1.3), we find that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{t^n}{(n-k)!k!}((n-k+1)I)_k[(A)_k]^{-1}\Big(\frac{1-x}{2}\Big)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{t^n}{(n-k)!k!}(-1)^k((n+1)I)_k[(A)_k]^{-1}\Big(\frac{1-x}{2}\Big)^k$$

$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{1}{n!k!}((1+n)I)_k(-nI)_k[(A)_k]^{-1}\Big(\frac{1-x}{2}\Big)^k t^n = \sum_{n=0}^{\infty}\frac{t^n}{n!}P_n(x,A),$$

which completes of the proof (2.2). □

Precisely the same manner as described Theorem 2.1 and using (1.2), (1.3) and (1.4), we can prove the following results.

Theorem 2.2. *For Legendre matrix polynomials, the following generating matrix functions are*

$$\sum_{n=0}^{\infty} t^n P_n(x,A) = (1-t)^{-1}\ {}_2F_1\Big(-nI, I; A; \frac{(1-x)}{2(1-t)}\Big)$$

and

$$\sum_{n=0}^{\infty} t^n P_n(x,A) = (1-t)^{-1}\ {}_2F_1\Big(I, (n+1)I; A; \frac{(x-1)t}{2(1-t)}\Big).$$

Lemma 2.1. *The following equalities for the hypergeometric matrix function satisfy as follows*

$$\frac{d^n}{dz^n}\Big[z^{C-I}\ {}_2F_1\big(A,B;C;z\big)\Big] = (C-nI)_n z^{C-(n+1)I} \times {}_2F_1\big(A,B;C-nI;z\big), \tag{2.3}$$

where C and $C-nI$ are invertible matrices.

$$\frac{d^n}{dz^n}\Big[{}_2F_1\big(A,B;C;z\big)\Big] = (A)_n(B)_n[(C)_n]^{-1} \times {}_2F_1\big(A+nI, B+nI; C+nI; z\big), \tag{2.4}$$

where C and $C+nI$ are invertible matrices, and

$$\frac{d^n}{dz^n}\Big[(1-z)^{A+B-C}\ {}_2F_1\Big(A,B;C;z\Big)\Big] =(C-A)_n(C-B)_n[(C)_n]^{-1} \times (1-z)^{A+B-C-nI}\ {}_2F_1\Big(A,B;C+nI;z\Big), \tag{2.5}$$

where C and $C+nI$ are invertible matrices.

Proof. To prove (2.3), from (1.1) and (1.6), we get

$$\begin{aligned}\frac{d^n}{dz^n}\Big[z^{C+(k-1)I}\Big] =&(C+(k-1)I)(C+(k-2)I)\cdots(C+(k-n)I)z^{C+(k-n+1)I}\\ =&(C)_k(C-nI)_n[(C-nI)_k]^{-1}z^{C+(k-n-1)I}.\end{aligned}$$

Substituting the above expression into the series expression of hypergeometric matrix function, we obtain (2.3).

From (1.5), we get

$$\begin{aligned}\frac{d}{dz}\ {}_2F_1\Big(A,B;C;z\Big) =&\sum_{k=0}^{\infty}\frac{z^k}{k!}(A)_{k+1}(B)_{k+1}[(C)_{k+1}]^{-1}\\ =&AB^{-1}C^{-1}\sum_{k=0}^{\infty}\frac{z^k}{k!}(A+I)_k(B+I)_k[(C+I)_k]^{-1}\\ =&ABC^{-1}\ {}_2F_1\Big(A+I;B+I,C+I;z\Big).\end{aligned} \tag{2.6}$$

By iteration (2.6), for n, one gets (2.4).

In (1.7), we can write

$${}_2F_1\Big(C-A,C-B;C;z\Big) = (1-z)^{A+B-C}\ {}_2F_1\Big(A,B;C;z\Big).$$

Differentiating with respect to z of n times with the help of this eq. (2.3), we have

$$\begin{aligned}&\frac{d^n}{dz^n}\Big[(1-z)^{A+B-C}\ {}_2F_1\Big(A,B;C;z\Big)\Big]\\ =&\frac{d^n}{dz^n}\Big[\ {}_2F_1\Big(C-A,C-B;C;z\Big)\Big]\\ =&(C-A)_n(C-B)_n[(C)_n]^{-1}\ {}_2F_1\Big(C-A+nI,C-B+nI;C+nI;z\Big)\\ =&(C-A)_n(C-B)_n[(C)_n]^{-1}(1-z)^{A+B-C-nI}\ {}_2F_1\Big(A,B;C+nI;z\Big),\end{aligned}$$

and using (1.7), we have the desired recurrence relation. □

Theorem 2.3. *The following differential recurrence matrix relations for Legendre matrix polynomials hold true:*

$$\frac{d^r}{dx^r}\Big[(1-x)^{A-I}\ P_n(x,A)\Big] =(-1)^r(A-rI)_r(1-x)^{A-(r+1)I}P_n(x,A+rI), \tag{2.7}$$

where $A+rI$ is a matrix $\mathbb{C}^{N\times N}$ satisfying the condition (1.9),

$$\frac{d^r}{dx^r}\Big[P_n(x,A)\Big] =(-1)^r2^{-r}(-nI)_r((n+1)I)_r[(A)_r]^{-1}$$

$$\times\ {}_2F_1\left((r-n)I, (n+r+1)I : A+rI; \frac{1-x}{2}\right), \tag{2.8}$$

where $A+rI$ is an invertible matrix $\mathbb{C}^{N\times N}$, and

$$\begin{aligned}\frac{d^r}{dx^r}\Big[(1+x)^{I-A}\ P_n(x,A)\Big] =&(-1)^r(A+nI)_r(A-(n+1)I)_r\\ &\times[(A)_r]^{-1}(1+x)^{I-A-rI}\ P_n(x,A+rI),\end{aligned}$$

where $A+rI$ is a matrix $\mathbb{C}^{N\times N}$ satisfying the condition (1.9).

Proof. To prove (2.7), taking $A\to -nI$, $B\to (n+1)I$, $C\to A$ and $z\to \frac{1-x}{2}$ in equation (2.3), we complete the proof.

Taking $z\to\frac{1-x}{2}$, $A=-nI$, $B=(n+1)I$ and $C=A$ in equation (2.4), which completes of the proof (2.8).

Taking $z\to\frac{1-x}{2}$, $A\to -nI$, $B\to (n+1)I$ and $C\to A$ in equation (2.5), theorem can be proved. □

Therefore, in (1.8) we interchange A and B and replace x by $-x$ with the help $P_n^{(B,A)}(-x)=(-1)^n P_n^{(A,B)}(x)$ to obtain in the following result.

Corollary 2.1. *Jacobi matrix polynomials have the matrix relation:*

$$\begin{aligned}&(x+1)\Big[(A+B+nI)DP_n^{(A,B)}(x)-(B+nI)DP_{n-1}^{(A,B)}(x)\Big]\\ =&(A+B+nI)\Big[nP_n^{(A,B)}(x)+(B+nI)P_{n-1}^{(A,B)}(x)\Big],\quad n\geq 1, D=\frac{d}{dx}.\end{aligned}$$

The relations presented in the following theorem are also interesting.

Theorem 2.4. *Legendre matrix polynomials $P_n(x,A)$ satisfy the following differential recurrence matrix relations:*

$$(x-1)\Big(DP_n(x,A)+DP_{n-1}(x,A)\Big)=n\Big(P_n(x,A)-P_{n-1}(x,A)\Big),\quad n\geq 1, \tag{2.9}$$

$$\begin{aligned}&(x+1)\Big((A+(n-1)I)DP_n(x,A)+(A-(n+1)I)DP_{n-1}(x,A)\Big)\\ =&n\Big((A+(n-1)I)P_n(x,A)-(A-(n+1)I)P_{n-1}(x,A)\Big),\quad n\geq 1\end{aligned} \tag{2.10}$$

and

$$(x^2-1)DP_n(x,A)=\Big((1+nx)I-A\Big)P_n(x,A)+(A-(n+1)I)P_{n-1}(x,A),\quad n\geq 1. \tag{2.11}$$

Proof. To prove 2.9. In the generating matrix relation (2.1). If we put that

$$\begin{aligned}\Phi(x,t,A)=&\sum_{n=0}^{\infty}t^nP_n(x,A)=(1-t)^{-1}\ {}_2F_1\left(\frac{1}{2}I,I;A;\frac{2(x-1)t}{(1-t)^2}\right)\\ =&(1-t)^{-1}\Psi(x,t,A),\end{aligned} \tag{2.12}$$

where $\Psi(x,t,A) = {}_2F_1\left(\frac{1}{2}I, I; A; \frac{2(x-1)t}{(1-t)^2}\right)$.
Differentiating (2.12) with respect to x and t, we obtain

$$\frac{\partial}{\partial x}\Phi(x,t,A) = 2t(1-t)^{-3}\Psi'(x,t,A)$$

and

$$\frac{\partial}{\partial t}\Phi(x,t,A) = (1-t)^{-2}\,\Psi(x,t,A) + 2(x-1)(1+t)(1-t)^{-4}\Psi'(x,t,A).$$

Therefore $\Phi(x,t,A)$ satisfies the partial differential matrix equation

$$(x-1)(1+t)\frac{\partial}{\partial x}\Phi(x,t,A) - t(1-t)\frac{\partial}{\partial t}\Phi(x,t,A) = -t\Phi(x,t,A). \tag{2.13}$$

Equation (2.13) can be put that

$$\begin{aligned}(x-1)\frac{\partial}{\partial x}\Phi(x,t,A) - t\frac{\partial}{\partial t}\Phi(x,t,A) =& -t\Phi(x,t,A) - t^2\frac{\partial}{\partial t}\Phi(x,t,A)\\ &-(x-1)t\frac{\partial}{\partial x}\Phi(x,t,A).\end{aligned} \tag{2.14}$$

Since

$$\Phi(x,t,A) = \sum_{n=0}^{\infty} t^n P_n(x,A),$$

if we differentiate (2.12) with respect to x and t, we get

$$(1-x)\frac{\partial}{\partial x}\Phi(x,t,A) = \sum_{n=0}^{\infty} t^n (1-x)\frac{d}{dx}P_n(x,A)$$

and

$$\frac{\partial}{\partial t}\Phi(x,t,A) = \sum_{n=0}^{\infty} n t^{n-1} P_n(x,A),$$

(2.14) yields that

$$\begin{aligned}&\sum_{n=0}^{\infty} t^n\left((x-1)\frac{d}{dx}P_n(x,A) - nP_n(x,A)\right)\\
&= -\sum_{n=0}^{\infty} t^{n+1}P_n(x,A) - \sum_{n=0}^{\infty} n t^{n+1}P_n(x,A) - \sum_{n=0}^{\infty} t^{n+1}(x-1)\frac{d}{dx}P_n(x,A)\\
&= -\sum_{n=1}^{\infty} t^n P_{n-1}(x,A) - \sum_{n=0}^{\infty}(n-1)t^n P_{n-1}(x,A) - \sum_{n=1}^{\infty} t^n(1-x)\frac{d}{dx}P_{n-1}(x,A)\\
&= \sum_{n=0}^{\infty}(-1-n+1)t^n P_{n-1}(x,A) - \sum_{n=1}^{\infty} t^n(1-x)\frac{d}{dx}P_{n-1}(x,A)\\
&= -\sum_{n=0}^{\infty} n t^n P_{n-1}(x,A) - \sum_{n=1}^{\infty} t^n(1-x)\frac{d}{dx}P_{n-1}(x,A).\end{aligned}$$

Comparing the coefficients of t^n, which leads to (2.9).

If we choose $A = A - I$, $B = I - A$ in Corollary 2.1, we see that the matrix polynomials $P_n^{(A-I,I-A)}(x)$ is $P_n^{(A-I,I-A)}(x) = \frac{(A)_n}{n!} P_n(x, A)$ which leads to the result (2.10).

Let us eliminate $\frac{d}{dx} P_{n-1}(x, A)$ from multiply (2.9) by $(x + 1)(A - (n + 1)I)$ and multiply (2.10) by $(x - 1)$ which gives the result (2.11).

Eliminating $P_{n-1}(x, A)$ from (1.11) and (2.11), one can obtain in the following result. □

Theorem 2.5. *The differential recurrence matrix relation for Legendre matrix polynomials holds*

$$\begin{aligned}(x^2 - 1)DP_n(x, A) =&((1 - (n + 1)x)I - A)P_n(x, A)\\ &- (A + nI)P_{n+1}(x, A).\end{aligned}$$

3. Group-Theoretic Method for Legendre Matrix Polynomials

In order to use Weisner's method. Replacing D by $\frac{\partial}{\partial x}$, n by $y\frac{\partial}{\partial y}$ and $P_n(x, A)$ by $P_n(x, y, A) = y^n P_n(x, A)$ in (1.10) is constructed the partial differential matrix equation

$$\begin{aligned}&(1 - x^2)\frac{\partial^2}{\partial x^2} P_n(x, y, A) + 2\Big[(1 - x)I - A\Big]\frac{\partial}{\partial x} P_n(x, y, A)\\ &+ y^2 \frac{\partial^2}{\partial y^2} P_n(x, y, A) + 2y\frac{\partial}{\partial y} P_n(x, y, A) = \mathbf{0}. \qquad (3.1)\end{aligned}$$

Thus, $P_n(x, y, A) = y^n P_n(x, A)$ is a solution of the partial differential matrix equation (3.1). Linear differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are defined as follows

$$\mathbb{A} = y\frac{\partial}{\partial y} I, \qquad (3.2)$$

$$\mathbb{B} = \frac{1 - x^2}{y}\frac{\partial}{\partial x} I + x\frac{\partial}{\partial y} I + \frac{1}{y}(I - A), \quad y \neq 0, \qquad (3.3)$$

and

$$\mathbb{C} = (1 - x^2)y\frac{\partial}{\partial x} I - xy^2\frac{\partial}{\partial y} I + ((1 - x)I - A)y. \qquad (3.4)$$

Then

$$\mathbb{A}\Big[P_n(x, A)y^n\Big] = nP_n(x, A)y^n, \qquad (3.5)$$

$$\mathbb{B}\Big[P_n(x, A)y^n\Big] = - (A - (n + 1)I)P_{n-1}(x, A)y^{n-1} \qquad (3.6)$$

and

$$\mathbb{C}\Big[P_n(x, A)y^n\Big] = (A + nI)P_{n+1}(x, A)y^{n+1}. \qquad (3.7)$$

From (3.2), (3.3) and (3.4), the following theorem can be stated.

Theorem 3.1. *Linear partial differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ as defined in (3.2), (3.3) and (3.4) have the following properties*

$$\text{(i) } [\mathbb{A},\mathbb{B}] = -\mathbb{B}; \quad \text{(ii) } [\mathbb{A},\mathbb{C}] = \mathbb{C}; \quad \text{(iii) } [\mathbb{B},\mathbb{C}] = -2\mathbb{A} - \mathbb{I}, \tag{3.8}$$

where $\mathbb{I}$ is the identity operator, and the notation $[\mathbb{A},\mathbb{B}] = AB - BA$.

Proof. Now, we proceed to calculate $[\mathbb{A},\mathbb{B}]$. So that, we consider the action of $\mathbb{A}\mathbb{B}$ on the Legendre matrix polynomials $P_n(x,y,A)$

$$\mathbb{A}\mathbb{B}P_n(x,y,A) = y\frac{\partial}{\partial y}\left[\frac{I-x^2}{y}\frac{\partial}{\partial x}I + x\frac{\partial}{\partial y}I + \frac{1}{y}(I-A)\right]P_n(x,y,A).$$

Hence, on simplification, we have

$$\begin{aligned}\mathbb{A}\mathbb{B}P_n(x,y,A) =&(1-x^2)\frac{\partial^2}{\partial y\partial x}P_n(x,y,A) - \frac{1-x^2}{y}\frac{\partial}{\partial x}P_n(x,y,A)\\ &+ xy\frac{\partial^2}{\partial y^2}P_n(x,y,A) + (I-A)\frac{\partial}{\partial y}P_n(x,y,A) - \frac{1}{y}(I-A)P_n(x,y,A).\end{aligned} \tag{3.9}$$

In the similar fashion we can operate $\mathbb{B}\mathbb{A}$ on the $P_n(x,y,A)$ and simplified as

$$\begin{aligned}\mathbb{B}\mathbb{A}P_n(x,y,A) =&(1-x^2)\frac{\partial^2}{\partial x\partial y}P_n(x,y,A) + x\frac{\partial}{\partial y}P_n(x,y,A)\\ &+ xy\frac{\partial^2}{\partial y^2}P_n(x,y,A) + (I-A)\frac{\partial}{\partial y}P_n(x,y,A).\end{aligned} \tag{3.10}$$

Subtracting (3.10) from (3.9) and for $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$, we get

$$\begin{aligned}[\mathbb{A},\mathbb{B}]P_n(x,y,A) =&\Big(\mathbb{A}\mathbb{B} - \mathbb{B}\mathbb{A}\Big)P_n(x,y,A) = -\frac{1-x^2}{y}\frac{\partial}{\partial x}P_n(x,y,A)\\ &- x\frac{\partial}{\partial y}P_n(x,y,A) - \frac{1}{y}(I-A)P_n(x,y,A).\end{aligned}$$

Further simplifying, we get

$$[\mathbb{A},\mathbb{B}]P_n(x,y,A) = -\mathbb{B}P_n(x,y,A).$$

Hence, we have $[\mathbb{A},\mathbb{B}] = -\mathbb{B}$. Similarly, we can calculate each of the results $[\mathbb{A},\mathbb{C}]$ and $[\mathbb{B},\mathbb{C}]$. Thus, the required results are established. □

Now, if we operate the partial differential operator $(1-x^2)\mathbb{L}$ on $P_n(x,y,A)$, we give

$$\begin{aligned}(1-x^2)\mathbb{L}P_n(x,y,A) =&(1-x^2)^2\frac{\partial^2}{\partial x^2}P_n(x,y,A) + (1-x^2)y^2\frac{\partial^2}{\partial y^2}P_n(x,y,A)\\ &+ 2[(1-x)I - A](1-x^2)\frac{\partial}{\partial x}P_n(x,y,A)\\ &+ 2y(1-x^2)\frac{\partial}{\partial y}P_n(x,y,A)\end{aligned}$$

and

$$\begin{aligned}\mathbb{C}\mathbb{B}P_n(x,y,A) =&(1-x^2)^2\frac{\partial^2}{\partial x^2}P_n(x,y,A)+2((1-x)I-A)(1-x^2)\frac{\partial}{\partial x}P_n(x,y,A)\\&+y(1-2x^2)\frac{\partial}{\partial y}P_n(x,y,A)-x^2y^2\frac{\partial^2}{\partial y^2}P_n(x,y,A)\\&+(I-A)^2P_n(x,y,A).\end{aligned}$$

But

$$\mathbb{A}^2P_n(x,y,A)=y\frac{\partial}{\partial y}\left(y\frac{\partial}{\partial y}\right)P_n(x,y,A)=y^2\frac{\partial^2}{\partial y^2}U_n(x,y,A)+y\frac{\partial}{\partial y}P_n(x,y,A).$$

Hence, we get

$$\begin{aligned}(1-x^2)\mathbb{L}P_n(x,y,A)-\mathbb{C}\mathbb{B}P_n(x,y,A) =&y^2\frac{\partial^2}{\partial y^2}P_n(x,y,A)+y\frac{\partial}{\partial y}P_n(x,y,A)\\&-(I-A)^2P_n(x,y,A),\end{aligned}$$

which can be express as:

$$(1-x^2)\mathbb{L}P_n(x,y,A)=\Big[\mathbb{C}\mathbb{B}+\mathbb{A}^2-(I-A)^2\mathbb{I}\Big]P_n(x,y,A).$$

Since, $P_n(x,y,A)$ is the Legendre matrix polynomials, we conclude that

$$(1-x^2)\mathbb{L}=\mathbb{C}\mathbb{B}+\mathbb{A}^2-(I-A)^2\mathbb{I}.$$

Now, we show that

$$\begin{aligned}&\Big[(1-x^2)\mathbb{L},\mathbb{A}\Big]P_n(x,y,A)\\&=\Big((1-x^2)\mathbb{L}\mathbb{A}-\mathbb{A}(1-x^2)\mathbb{L}\Big)P_n(x,y,A)\\&=\Big(\Big(\mathbb{C}\mathbb{B}+\mathbb{A}^2-(I-A)^2\mathbb{I}\Big)\mathbb{A}-\mathbb{A}\Big(\mathbb{C}\mathbb{B}+\mathbb{A}^2-(I-A)^2\mathbb{I}\Big)\Big)P_n(x,y,A)\\&=\Big(\mathbb{C}\mathbb{B}\mathbb{A}-\mathbb{A}\mathbb{C}\mathbb{B}\Big)P_n(x,y,A).\end{aligned}\tag{3.11}$$

Also, with the aid of (3.8), we have

$$\mathbb{C}\mathbb{B}\mathbb{A}-\mathbb{A}\mathbb{C}\mathbb{B}=\mathbb{C}\mathbb{B}\mathbb{A}-(\mathbb{C}+\mathbb{C}\mathbb{A})\mathbb{B}=\mathbb{C}\mathbb{B}-\mathbb{C}\mathbb{B}=\mathbf{0}.$$

So that from (3.11), we get

$$\Big[\Big(1-x^2\Big)\mathbb{L},\mathbb{A}\Big]=\mathbf{0}.$$

Hence, we proved that $(1-x^2)\mathbb{L}$ commute with $\mathbb{A}$. In a similar manner, we can calculate of the operator $(1-x^2)\mathbb{L}$ commute with each of the differential operators $\mathbb{B}$ and $\mathbb{C}$. Thus, we can give in the following.

Theorem 3.2. *The operator $(1-x^2)\mathbb{L}$ commute with each of the linear differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ defined in (3.5), (3.6) and (3.7) as follows*

$$\text{(i) } [(1-x^2)\mathbb{L},\mathbb{A}]=\mathbf{0},\quad \text{(ii) } [(1-x^2)\mathbb{L},\mathbb{B}]=\mathbf{0},\quad \text{(iii) } [(1-x^2)\mathbb{L},\mathbb{C}]=\mathbf{0}.$$

The extended forms of the transformation groups generated by the differential operators $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{C}$ are given by

$$\begin{aligned} e^{a\mathbb{A}}f(x,y,A) =& f\Big(x, e^a y, A\Big), \\ e^{b\mathbb{B}}f(x,y,A) =& f\left(\frac{xy+b}{\sqrt{y^2+2bxy+b^2}}, \sqrt{y^2+2bxy+b^2}, A\right), \end{aligned}$$

for $|y^2+2bxy|<b^2$, $|\frac{y}{2bx}|<1$ and

$$e^{c\mathbb{C}}f(x,y,A) = (c^2y^2+2cxy+1)^{-\frac{1}{2}} f\left(\frac{x+cy}{\sqrt{c^2y^2+2cxy+1}}, \frac{y}{\sqrt{c^2y^2+2cxy+1}}, A\right),$$

where $|c^2y^2+2cxy|<1$ and $\left|\frac{cy}{2x}\right|<1$, $f(x,y,A)$ is an arbitrary matrix function, and a, b and c are arbitrary constants.

From the above equations, we get

$$\begin{aligned} e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}f(x,y,A) =& f\left(\frac{y(x+cy)+b(c^2y^2+2cxy+1)}{\sqrt{c^2y^2+2cxy+1}\sqrt{b^2(c^2y^2+2cxy+1)+2by(x+cy)+y^2}},\right. \\ & \left. e^a\frac{\sqrt{b^2(c^2y^2+2cxy+1)+2by(x+cy)+y^2}}{(c^2y^2+2cxy+1)^{\frac{3}{2}}}, A\right). \end{aligned} \tag{3.12}$$

3.1. **Generating matrix functions.** From (3.5), $P_n(x,y,A)=P_n(x,A)y^n$ is a solution of the system

$$\mathbb{L}P_n(x,y,A)=\mathbf{0} \quad \text{and} \quad (\mathbb{A}-n\mathbb{I})P_n(x,y,A)=\mathbf{0}.$$

From (3.12), we get

$$e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}(1-x^2)\mathbb{L}[P_n(x,A)y^n] = (1-x^2)\mathbb{L}e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}[P_n(x,A)y^n].$$

Therefore, the transform $e^{c\mathbb{C}}e^{b\mathbb{B}}e^{a\mathbb{A}}[P_n(x,A)y^n]$ is annulled by $(1-x^2)\mathbb{L}$.

If we choose $a=0$ and $P_n(x,y,A)=P_n(x,A)y^n$ in (3.12), we get

$$\begin{aligned} & e^{c\mathbb{C}}e^{b\mathbb{B}}[P_n(x,A)y^n] \\ =& \Big(b^2(c^2y^2+2cxy+1)+2by(x-cy)+y^2\Big)^{\frac{1}{2}n}(c^2y^2+2cxy+1)^{-\left(\frac{1}{2}+\frac{3}{2}n\right)} \\ & \times P_n\left(\frac{y(x+cy)+b(c^2y^2+2cxy+1)}{\sqrt{c^2y^2+2cxy+1}\sqrt{b^2(c^2y^2+2cxy+1)+2by(x+cy)+y^2}}, A\right). \end{aligned}$$

On the other hand we get

$$\begin{aligned} e^{c\mathbb{C}}e^{b\mathbb{B}}[P_n(x,A)y^n] =& \sum_{m=0}^{\infty}\frac{c^m}{m!}\sum_{k=0}^{\infty}\frac{b^k}{k!}(A+(n-k)I)_m((n+1)I-A)_k \\ & \times y^{n-k+m}P_{n-k+m}(x,A). \end{aligned}$$

Equating the results (3.6) and (3.7), we get

$$\Big(b^2(c^2y^2+2cxy+1)+2by(x+cy)+y^2\Big)^{\frac{1}{2}n}\Big(c^2y^2+2cxy+1\Big)^{-\left(\frac{1}{2}+\frac{3}{2}nI\right)}$$

$$\times P_n\Bigg(\frac{y(x+cy)+b(c^2y^2+2cxy+1)}{\sqrt{c^2y^2+2cxy+1}\sqrt{b^2(c^2y^2+2cxy+1)+2by(x+cy)+y^2}}, A\Bigg)$$

$$=\sum_{m=0}^{\infty}\sum_{k=0}^{n}\frac{c^m b^k}{m!k!}(A+(n-k)I)_m((n+1)I-A)_k y^{n-k+m}P_{n-k+m}(x,A). \tag{3.13}$$

Here, we derive of some interesting results as the particular case of generating matrix relations (2.11), we need to consider three cases.

Case 1: $b=-1$, $c=0$.

If we substitute $b=-1$ and $c=0$ in (3.13), then it will gives us

$$e^{-\mathbb{B}}f(x,y,A)=f\Big(\frac{xy-1}{\sqrt{y^2-2xy+1}}, \sqrt{y^2-2xy+1}, A\Big).$$

Hence, if we take $f(x,y,A)=P(x,y,A)=P_n(x,A)y^n$, we find

$$^{-\mathbb{B}}\Big[P_n(x,A)y^n\Big]=\Big(1-2xy+y^2\Big)^{\frac{1}{2}n}P_n\Bigg(\frac{xy-1}{\sqrt{1-2xy+y^2}}, A\Bigg), \tag{3.14}$$

since

$$\begin{aligned}\mathbb{B}\Big[P_n(x,A)y^n\Big]=&\frac{1-x^2}{y}\frac{\partial}{\partial x}\Big(P_n(x,A)y^n\Big)+x\frac{\partial}{\partial y}\Big(P_n(x,A)y^n\Big)\\=&((n+1)I-A)P_{n-1}(x,A)y^{n-1}.\end{aligned}$$

On another hand, we can expand left-hand side of (3.14) in a series form and then repeated application of (3.6) on the same side of (3.14), we have

$$e^{-\mathbb{B}}\Big[P_n(x,A)y^n\Big]=\sum_{k=0}^{n}\frac{1}{k!}(A-(n+1)I)_k P_{n-k}(x,A)y^{n-k}. \tag{3.15}$$

Equating expressions (3.14) and (3.15), we get

$$\sum_{k=0}^{n}\frac{1}{k!}(A-(n+1)I)_k P_{n-k}(x,A)y^{n-k}=\Big(1-2xy+y^2\Big)^{\frac{1}{2}n}P_n\Bigg(\frac{xy-1}{\sqrt{1-2xy+y^2}}, A\Bigg).$$

Replacing y^{-1} by t, we obtain of a generating matrix relation

$$\begin{aligned}&\sum_{k=0}^{n}\frac{1}{k!}(A-(n+1)I)_k P_{n-k}(x,A)t^k\\=&\Big(1-2xt+t^2\Big)^{\frac{1}{2}n}P_n\Bigg(\frac{x-t}{\sqrt{1-2xt+t^2}}, A\Bigg).\end{aligned}$$

Case 2. If we choose $b=0$ and $c=1$ in (3.13) we have

$$e^{\mathbb{C}}f(x,y,A)=\Big(y^2+2xy+1\Big)^{-\frac{1}{2}}f\Bigg(\frac{x+y}{\sqrt{y^2+2xy+1}}, \frac{y}{\sqrt{y^2+2xy+1}}, A\Bigg).$$

Hence, if we take $f(x,y,A)=P(x,y,A)=P_n(x,A)y^n$, we find

$$e^{\mathbb{C}}P(x,y,A)=\Big(y^2+2cxy+1\Big)^{-\frac{1}{2}}P\Bigg(\frac{x+y}{\sqrt{y^2+2xy+1}}, \frac{y}{\sqrt{y^2+2xy+1}}, A\Bigg)$$

and

$$e^{\mathbb{C}}P(x,y,A)=y^n\Big(y^2+2cxy+1\Big)^{-\frac{n+1}{2}}P\left(\frac{x+y}{\sqrt{y^2+2xy+1}},A\right). \tag{3.16}$$

Since we have

$$\begin{aligned}\mathbb{C}\Big[P_n(x,A)y^n\Big]=&(1-x^2)y\frac{\partial}{\partial x}\Big(P_n(x,A)y^n\Big)-xy^2\frac{\partial}{\partial y}\Big(P_n(x,A)y^n\Big)\\&-xy\Big(P_n(x,A)y^n\Big)=(A+nI)P_{n+1}(x,A)y^{n+1}.\end{aligned}$$

On other hand, we can expand left hand side of (3.16) in a series form and then repeated application of (3.7) on the same side of (3.16), we have

$$e^{\mathbb{C}}\Big[P_n(x,A)y^n\Big]=\sum_{k=0}^{n}\frac{1}{k!}(A+nI)_kP_{n+k}(x,A)y^{n+k}. \tag{3.17}$$

Equating expressions (3.16) and (3.17) we get

$$\sum_{k=0}^{n}\frac{1}{k!}(A+nI)_kP_{n+k}(x,A)y^k=\Big(1+2xy+y^2\Big)^{-\frac{n+1}{2}}P_n\left(\frac{x+y}{\sqrt{1+2xy+y^2}},A\right),$$

and replacing y by $-t$, we get of a generating matrix relation

$$\begin{aligned}&\sum_{k=0}^{n}\frac{(-1)^k}{k!}(A+nI)_kP_{n+k}(x,A)t^k\\&=\Big(1-2xt+t^2\Big)^{-\frac{n+1}{2}}P_n\left(\frac{x-t}{\sqrt{1-2xt+t^2}},A\right).\end{aligned}$$

Case 3: $b=1$, $c=-1$.

Let us take $b=1$ and $c=-1$, so that (3.17) becomes

$$\begin{aligned}e^{-\mathbb{C}}e^{\mathbb{B}}[P_n(x,A)y^n]&=\sum_{r,k=0}^{\infty}\frac{(-1)^r}{r!k!}\mathbb{C}^r\mathbb{B}^k[P_n(x,A)y^n]\\&=\sum_{r,k=0}^{\infty}\frac{(-1)^r}{r!k!}((A+nI))_r\mathbb{B}^k[P_{n+r}(x,A)y^{n+r}]\\&=\sum_{k=0}^{\infty}\sum_{r}^{n}\frac{(-1)^r}{r!k!}\Gamma(A+nI)\Gamma^{-1}(A+(n-r)I)\mathbb{B}^k[P_n(x,A)y^n]\end{aligned}$$

and

$$\begin{aligned}e^{-\mathbb{C}}e^{\mathbb{B}}[P_n(x,A)y^n=&\sum_{k=0}^{\infty}\sum_{r}^{n}\frac{(-1)^r}{r!k!}\Gamma(A+nI)\Gamma^{-1}(A+(n-r)I)\\&\times(A-(n+1)I)_kP_{n-k}(x,A)y^{n-k}.\end{aligned} \tag{3.18}$$

Using (3.13) and (3.18), we get

$$e^{-\mathbb{C}}e^{\mathbb{B}}[P_n(x,A)y^n]=\Big(y^2-2xy+1\Big)^{-(\frac{n+1}{2})}P_n\left(\frac{1-xy}{\sqrt{y^2-2xy+1}},A\right)$$

$$=\sum_{k=0}^{\infty}\sum_{r}^{n}\frac{(-1)^r}{r!k!}\Gamma(A+nI)\Gamma^{-1}(A+(n-r)I)(A-(n+1)I)_k P_{n-k}(x,A)y^{n-k} \tag{3.19}$$

and

$$\Big(y^2-2xy+1\Big)^{-(\frac{n+1}{2})}P_n\left(\frac{1-xy}{\sqrt{y^2-2xy+1}},A\right)$$
$$=\sum_{k=0}^{\infty}\sum_{r}^{n}\frac{(-1)^r}{r!k!}\Gamma(A+nI)\Gamma^{-1}(A+(n-r)I)(A-(n+1)I)_k P_{n-k}(x,A)y^{n-k}.$$

Replacing y^{-1} by t in (3.19) we get

$$t^{2n+1}\Big(t^2-2xt+1\Big)^{-(\frac{n+1}{2})}P_n\left(\frac{t-x}{\sqrt{t^2-2tx+1}},A\right)$$
$$=\sum_{k=0}^{\infty}\sum_{r}^{n}\frac{(-1)^r}{r!k!}\Gamma(A+nI)\Gamma^{-1}(A+(n-r)I)(A-(n+1)I)_k P_{n-k}(x,A)t^{k}.$$

4. Conclusion

A novel approach has been obtained in this paper for studying many interesting results of Legendre matrix polynomials viz certain generating matrix relations, matrix recurrence relation, matrix differential recurrence relation and matrix differential equation. Lie algebra method developed in this work can also be used to study some other Legendre matrix polynomials which play as applications and a vital role in Mathematical Physics in the future. However, the merging of these matrix polynomials with a Lie algebraic techniques is also stimulating for further research work.

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ON THE HERMITE-HADAMARD TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATOR

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ABSTRACT. In this paper, using a general class of fractional integral operators, we establish new fractional integral inequalities of Hermite-Hadamard type. The main results are used to derive Hermite-Hadamard type inequalities involving the familiar Riemann-Liouville fractional integral operators.

1. INTRODUCTION

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality [5]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite-Hadamard inequalities.

In [2], Dragomir and Agarwal proved the following results connected with the right part of (1.1).

Lemma 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $f' \in L[a, b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \tag{1.2}$$

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Theorem 1.1. *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}\left(|f'(a)|+|f'(b)|\right). \tag{1.3}$$

Meanwhile, in [8], Sarikaya et al. gave the following interesting Riemann-Liouville integral inequalities of Hermite-Hadamard type.

Theorem 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1([a, b])$. If f is a convex function on $[a, b]$, then the following inequalities for fractional integrals hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{1.4}$$

with $\alpha > 0$.

Lemma 1.2. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $f' \in L[a, b]$, then the following equality for fractional integrals holds:*

$$\begin{aligned}\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\\ =\frac{b-a}{2}\int_0^1\left[(1-t)^\alpha-t^\alpha\right]f'(ta+(1-t)b)\,dt.\end{aligned} \tag{1.5}$$

Theorem 1.3. *Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$. If $|f'|$ is convex on $[a, b]$, then the following inequality for fractional integrals holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a+}^\alpha f(b)+J_{b-}^\alpha f(a)\right]\right|\\ \leq\frac{b-a}{2(\alpha+1)}\left(1-\frac{1}{2^\alpha}\right)\left[|f'(a)|+|f'(b)|\right].\end{aligned} \tag{1.6}$$

For some recent results connected with fractional integral inequalities see ([8]-[11])

In [7], Raina defined the following results connected with the general class of fractional integral operators

$$\mathcal{F}_{\rho,\lambda}^{\sigma}(x)=\mathcal{F}_{\rho,\lambda}^{\sigma(0),\sigma(1),\ldots}(x)=\sum_{k=0}^{\infty}\frac{\sigma(k)}{\Gamma(\rho k+\lambda)}x^k, \quad \rho,\lambda>0, |x|<\mathcal{R}, \tag{1.7}$$

where the coefficents $\sigma(k)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, is a bounded sequence of positive real numbers and $\mathcal{R}$ is the real number. With the help of (1.7), Raina and Agarwal et al. defined the following left-sided and right-sided fractional integral operators, respectively, as follows:

$$\mathcal{J}_{\rho,\lambda,a+;\omega}^{\sigma}\varphi(x)=\int_a^x (x-t)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[\omega(x-t)^\rho\right]\varphi(t)dt, \quad x>a, \tag{1.8}$$

$$\mathcal{J}_{\rho,\lambda,b-;\omega}^{\sigma}\varphi(x)=\int_x^b (t-x)^{\lambda-1}\mathcal{F}_{\rho,\lambda}^{\sigma}\left[\omega(t-x)^\rho\right]\varphi(t)dt, \quad x<b, \tag{1.9}$$

where $\lambda, \rho > 0$, $\omega \in \mathbb{R}$, and $\varphi(t)$ is such that the integrals on the right side exists.

It is easy to verify that $\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(x)$ and $\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(x)$ are bounded integral operators on $L(a,b)$, if

$$\mathfrak{M} := \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right] < \infty. \tag{1.10}$$

In fact, for $\varphi \in L(a,b)$, we have

$$\left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi(x)\right\|_1 \leq \mathfrak{M}(b-a)^{\lambda}\|\varphi\|_1 \tag{1.11}$$

and

$$\left\|\mathfrak{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi(x)\right\|_1 \leq \mathfrak{M}(b-a)^{\lambda}\|\varphi\|_1, \tag{1.12}$$

where

$$\|\varphi\|_p := \left(\int_a^b |\varphi(t)|^p\,dt\right)^{\frac{1}{p}}.$$

The importance of these operators stems indeed from their generality. Many useful fractional integral operators can be obtained by specializing the coefficient $\sigma(k)$. Here, we just point out that the classical Riemann-Liouville fractional integrals I^{α}_{a+} and I^{α}_{b-} of order α defined by (see, [3,4,6])

$$\left(I^{\alpha}_{a^+}\varphi\right)(x) := \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\varphi(t)dt, \quad x > a, \alpha > 0 \tag{1.13}$$

and

$$\left(I^{\alpha}_{b^-}\varphi\right)(x) := \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\varphi(t)dt, \quad x < b, \alpha > 0, \tag{1.14}$$

follow easily by setting

$$\lambda = \alpha,\ \sigma(0) = 1 \text{ and } w = 0 \tag{1.15}$$

in (1.8) and (1.9), and the boundedness of (1.13) and (1.14) on $L(a,b)$ is also inherited from (1.11) and (1.12), (see [1]).

In this paper, using a general class of fractional integral operators, we establish new fractional integral inequalities of Hermite-Hadamard type. The main results are used to derive Hermite-Hadamard type inequalities involving the familiar Riemann-Liouville fractional integral operators.

2. Main Results

In this section, using fractional integral operators, we start with stating and proving the fractional integral counterparts of Lemma 1.1, Theorem 1.1 and Theorem 1.2. Then some other refinements will folllow. We begin by the following theorem.

Theorem 2.1. *Let $\varphi : [a,b] \to \mathbb{R}$ be a convex function on $[a,b]$, with $a < b$, then the following inequalities for fractional integral operators hold:*

$$\varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(a)\right] \leq \frac{\varphi(a)+\varphi(b)}{2}, \tag{2.1}$$

with $\lambda > 0$.

Proof. For $t \in [0,1]$, let $x = ta + (1-t)b$, $y = (1-t)a + tb$. The convexity of φ yields

$$\varphi\left(\frac{a+b}{2}\right) = \varphi\left(\frac{x+y}{2}\right) \leq \frac{\varphi(x)+\varphi(y)}{2}, \tag{2.2}$$

i.e.,

$$2\varphi\left(\frac{a+b}{2}\right) \leq \varphi(ta+(1-t)b) + \varphi((1-t)a+tb). \tag{2.3}$$

Multiplying both sides of (2.3) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned}
&2\varphi\left(\frac{a+b}{2}\right)\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]dt \\
\leq &\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]\varphi(ta+(1-t)b)\,dt \\
&+\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]\varphi((1-t)a+tb)\,dt.
\end{aligned}$$

Calculating the following integrals by using (1.7), we have

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]dt = \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right],$$

$$\begin{aligned}
&\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]\varphi(ta+(1-t)b)\,dt \\
=&\frac{1}{(b-a)^{\lambda}}\int_a^b (b-x)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-x)^{\rho}\right]\varphi(x)\,dx
\end{aligned}$$

and

$$\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega(b-a)^{\rho}t^{\rho}\right]\varphi((1-t)a+tb)\,dt$$

$$= \frac{1}{(b-a)^{\lambda}} \int_a^b (x-a)^{\lambda-1} \mathcal{F}^{\sigma}_{\rho,\lambda} \left[\omega (x-a)^{\rho}\right] \varphi(x)\, dx.$$

As consequence, we obtain

$$2\mathcal{F}^{\sigma}_{\rho,\lambda+1} \left[\omega (b-a)^{\rho}\right] \varphi\left(\frac{a+b}{2}\right) \leq \frac{1}{(b-a)^{\lambda}} \left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(a)\right] \tag{2.4}$$

and the first inequality is proved.

Now, we prove the other inequality in (2.1), Since φ is convex, for every $t \in [0,1]$, we have

$$\varphi(ta + (1-t)b) + \varphi((1-t)a + tb) \leq \varphi(a) + \varphi(b). \tag{2.5}$$

Then multiplying both hand sides of (2.5) by $t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (b-a)^{\rho} t^{\rho}\right]$ and integrating the resulting inequality with respect to t over $[0,1]$, we obtain

$$\begin{aligned} &\int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (b-a)^{\rho} t^{\rho}\right] \varphi(ta + (1-t)b)\, dt \\ &\quad + \int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (b-a)^{\rho} t^{\rho}\right] \varphi((1-t)a + tb)\, dt \\ &\leq \left[\varphi(a) + \varphi(b)\right] \int_0^1 t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega (b-a)^{\rho} t^{\rho}\right] dt. \end{aligned}$$

Using the similar arguments as above we can show that

$$\frac{1}{(b-a)^{\lambda}} \left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(a)\right] \quad \leq \quad \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega (b-a)^{\rho}\right] \left[\varphi(a) + \varphi(b)\right]$$

and the second inequality is proved. □

Remark 2.1. If in Theorem 2.1 we set $\lambda = \alpha$, $\sigma(0) = 1$, $w = 0$, then the inequalities (2.1) become the inequalities (1.4) of Theorem 1.2.

Remark 2.2. If in Theorem 2.1 we set $\lambda = 1$, $\sigma(0) = 1$, $w = 0$, then the inequalities (2.1) become the inequalities (1.1).

Before starting and proving our next result, we need the following lemma.

Lemma 2.1. *Let $\varphi : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$ and $\lambda > 0$. If $\varphi' \in L[a,b]$, then the following equality for fractional integrals holds:*

(2.6)

$$\frac{\varphi(a) + \varphi(b)}{2} - \frac{1}{2(b-a)^{\lambda} \mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega (b-a)^{\rho}\right]} \left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b) + \left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(a)\right]$$

$$=\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}\right]}\left[\int_{0}^{1}\left(1-t\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}\left(1-t\right)^{\rho}\right]\varphi'\left(ta+(1-t)b\right)dt\right.$$
$$\left.-\int_{0}^{1}t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}t^{\rho}\right]\varphi'\left(ta+(1-t)b\right)dt\right].$$

Proof. Here, we apply integration by parts in integrals of right hand side of (2.6), then we have

$$\begin{aligned}
&\int_{0}^{1}\left(1-t\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}\left(1-t\right)^{\rho}\right]\varphi'\left(ta+(1-t)b\right)dt \\
&\quad-\int_{0}^{1}t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}t^{\rho}\right]\varphi'\left(ta+(1-t)b\right)dt \\
&=\left(1-t\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}\left(1-t\right)^{\rho}\right]\left.\frac{\varphi\left(ta+(1-t)b\right)}{a-b}\right|_{0}^{1} \\
&\quad-\frac{1}{b-a}\int_{0}^{1}\left(1-t\right)^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(b-a\right)^{\rho}\left(1-t\right)^{\rho}\right]\varphi\left(ta+(1-t)b\right)dt \\
&\quad+t^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}t^{\rho}\right]\left.\frac{\varphi\left(ta+(1-t)b\right)}{b-a}\right|_{0}^{1} \\
&\quad-\frac{1}{b-a}\int_{0}^{1}t^{\lambda-1}\mathcal{F}^{\sigma}_{\rho,\lambda}\left[\omega\left(b-a\right)^{\rho}t^{\rho}\right]\varphi\left(ta+(1-t)b\right)dt.
\end{aligned}\tag{2.7}$$

Now we use the substitution rule last integrals in (2.7), after by using definition of left and right-sided fractional integral operator, we obtain proof of this lemma. □

Remark 2.3. If in Lemma 2.1 we set $\lambda=\alpha$, $\sigma(0)=1$, and $w=0$, then the inequalities (2.6) become the equality (1.5) of Lemma 1.2.

Remark 2.4. If in Lemma 2.1 we set $\lambda=1$, $\sigma(0)=1$, and $w=0$, then the inequalities (2.6) become the equality (1.2) of Lemma 1.1.

We have the following results.

Theorem 2.2. *Let $\varphi:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$ and $\lambda>0$. If $|\varphi'|$ is convex on $[a,b]$, then the following inequality for fractional integrals holds:*

(2.8)
$$\left|\frac{\varphi(a)+\varphi(b)}{2}-\frac{1}{2\left(b-a\right)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\left(b-a\right)^{\rho}\right]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(a)\right]\right|$$

$$\leq (b-a)\,\frac{\mathcal{F}^{\sigma'}_{\rho,\lambda+2}\left[\omega\,(b-a)^{\rho}\right]}{\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\,(b-a)^{\rho}\right]}\frac{\left|\varphi'\,(a)\right|+\left|\varphi'\,(b)\right|}{2},$$

where

$$\sigma'\,(k) := \sigma\,(k)\left(1-\frac{1}{2^{\rho k+\lambda}}\right).$$

Proof. Using Lemma 2.1 and the convexity of $|\varphi'|$, we find that

$$\left|\frac{\varphi\,(a)+\varphi\,(b)}{2}-\frac{1}{2\,(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\,(b-a)^{\rho}\right]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;\omega}\varphi\right)(a)\right]\right|$$

$$\leq\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\,(b-a)^{\rho}\right]}\left[\sum_{k=0}^{\infty}\frac{\sigma\,(k)\,\omega^{k}\,(b-a)^{\rho k}}{\Gamma\,(\rho k+\lambda+1)}\right.$$

$$\left.\times\left(\int_0^1\left|(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right|\left[t\left|\varphi'\,(a)\right|+(1-t)\left|\varphi'\,(b)\right|\right]dt\right)\right]$$

$$=\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\,(b-a)^{\rho}\right]}\left[\sum_{k=0}^{\infty}\frac{\sigma\,(k)\,\omega^{k}\,(b-a)^{\rho k}}{\Gamma\,(\rho k+\lambda+1)}\right.$$

$$\times\left\{\int_0^{\frac{1}{2}}\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]\left[t\left|\varphi'\,(a)\right|+(1-t)\left|\varphi'\,(b)\right|\right]dt\right.$$

$$\left.\left.+\int_{\frac{1}{2}}^{1}\left[t^{\rho k+\lambda}-(1-t)^{\rho k+\lambda}\right]\left[t\left|\varphi'\,(a)\right|+(1-t)\left|\varphi'\,(b)\right|\right]dt\right\}\right]$$

$$=\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega\,(b-a)^{\rho}\right]}\left(\mathcal{F}^{\sigma'}_{\rho,\lambda+2}\left[\omega\,(b-a)^{\rho}\right]\right)\left(\left|\varphi'\,(a)\right|+\left|\varphi'\,(b)\right|\right).$$

This completes the proof. ☐

Remark 2.5. If in Theorem 2.2 we set $\lambda=\alpha$, $\sigma\,(0)=1$, and $w=0$, then the inequality (2.8) become the inequalities (1.6) of Theorem 1.3.

Remark 2.6. If in Theorem 2.2 we set $\lambda=1$, $\sigma\,(0)=1$, and $w=0$, then, the inequality (2.8) become the inequalities (1.3) of Theorem 1.1.

Theorem 2.3. *Let $\varphi:[a,b]\to\mathbb{R}$ be a differentiable mapping on (a,b) with $a<b$. If $|\varphi'|^{q}$ is convex on $[a,b]$ for some $q>1$, then the following inequality for fractional integrals holds:*

$$\left|\frac{\varphi\,(a)+\varphi\,(b)}{2}-\frac{1}{2\,(b-a)^{\lambda}\,\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\,(b-a)^{\rho}\right]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;w}\varphi\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b^-;w}\varphi\right)(a)\right]\right|$$

$$\leq\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w\,(b-a)^{\rho}\right]}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w\,(b-a)^{\rho}\right]$$

$$\times\left[\left(\frac{\left|\varphi'(a)\right|^{q}+3\left|\varphi'(b)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|\varphi'(a)\right|^{q}+\left|\varphi'(b)\right|^{q}}{8}\right)^{\frac{1}{q}}\right],$$

where

$$\sigma_1(k):=\sigma(k)\left(\frac{1}{(\rho k+\lambda)p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{(\rho k+\lambda)p}}\right)^{\frac{1}{p}},$$

with $\frac{1}{p}+\frac{1}{q}=1,\ \lambda>0.$

Proof. Using Lemma 2.1 and the convexity of $|\varphi'|^q$, and Hölder's inequality, we obtain

$$\begin{aligned}
&\left|\frac{\varphi(a)+\varphi(b)}{2}-\frac{1}{2(b-a)^{\lambda}\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\left(\mathcal{J}^{\sigma}_{\rho,\lambda,a+;\omega}\varphi\right)(b)+\left(\mathcal{J}^{\sigma}_{\rho,\lambda,b-;\omega}\varphi\right)(a)\right]\right|\\
\leq&\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)\,\omega^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\right.\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left[(1-t)^{\rho k+\lambda}-t^{\rho k+\lambda}\right]^{p}dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[t\left|\varphi'(a)\right|^{q}+(1-t)\left|\varphi'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left[t^{\rho k+\lambda}-(1-t)^{\rho k+\lambda}\right]^{p}dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[t\left|\varphi'(a)\right|^{q}+(1-t)\left|\varphi'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right\}\right]\\
\leq&\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[\omega(b-a)^{\rho}\right]}\left[\sum_{k=0}^{\infty}\frac{\sigma(k)\,\omega^{k}(b-a)^{\rho k}}{\Gamma(\rho k+\lambda+1)}\right.\\
&\times\left\{\left(\int_0^{\frac{1}{2}}\left[(1-t)^{(\rho k+\lambda)p}-t^{(\rho k+\lambda)p}\right]dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left[t\left|\varphi'(a)\right|^{q}+(1-t)\left|\varphi'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right.\\
&\left.\left.+\left(\int_{\frac{1}{2}}^{1}\left[t^{(\rho k+\lambda)p}-(1-t)^{(\rho k+\lambda)p}\right]dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^{1}\left[t\left|\varphi'(a)\right|^{q}+(1-t)\left|\varphi'(b)\right|^{q}\right]dt\right)^{\frac{1}{q}}\right\}\right]\\
=&\frac{(b-a)}{2\mathcal{F}^{\sigma}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]}\mathcal{F}^{\sigma_1}_{\rho,\lambda+1}\left[w(b-a)^{\rho}\right]\\
&\times\left[\left(\frac{\left|\varphi'(a)\right|^{q}+3\left|\varphi'(b)\right|^{q}}{8}\right)^{\frac{1}{q}}+\left(\frac{3\left|\varphi'(a)\right|^{q}+\left|\varphi'(b)\right|^{q}}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Here, we use $(A-B)^p\leq A^p-B^p$ for any $A>B\geq 0$ and $p\geq 1$.

This completes the proof. □

Corollary 2.1. *Under the assumption of Theorem* 2.3 *with* $\lambda = \alpha$, $\sigma(0) = 1$ *and* $w = 0$, *we have*

$$\begin{aligned}
&\left|\frac{\varphi(a)+\varphi(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a+}\varphi(b) + J^{\alpha}_{b-}\varphi(a)\right]\right| \\
\leq &\frac{(b-a)}{2}\left(\frac{1}{\alpha p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{\alpha p}}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{|\varphi'(a)|^q + 3|\varphi'(b)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|\varphi'(a)|^q + |\varphi'(b)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Corollary 2.2. *If we take* $\alpha = 1$ *in Corollary* 2.1, *we have*

$$\begin{aligned}
&\left|\frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{(b-a)}\int_a^b \varphi(t)\,dt\right| \\
\leq &\left(\frac{b-a}{2}\right)\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(1-\frac{1}{2^{p}}\right)^{\frac{1}{p}} \\
&\times\left[\left(\frac{|\varphi'(a)|^q + 3|\varphi'(b)|^q}{8}\right)^{\frac{1}{q}} + \left(\frac{3|\varphi'(a)|^q + |\varphi'(b)|^q}{8}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

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AN APPROXIMATE APPROACH FOR SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BASED ON TAYLOR EXPANSION

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Abstract. The main purpose of this work is to present an efficient approximate approach for solving linear systems of fractional integro-differential equations based on a new application of Taylor expansion. Using the mth-order Taylor polynomial for unknown functions and employing integration method the given system of fractional integro-differential equations will be converted into a system of linear equations with respect to unknown functions and their derivatives. The solutions of this system yield the approximate solutions of fractional integro-differential equations system. The Riemann-Liouville fractional derivative is applied in calculations. An error analysis is discussed as well. The accuracy and the efficiency of the suggested method is illustrated by considering five numerical examples.

1. Introduction

During the past decades, fractional calculus and fractional differential equations have found various applications in sciences and engineering, such as electrical networks, rheology, acoustics, electroanalytical chemistry, neuron modeling, viscoelasticity, material sciences, fluid flow, diffusive transport akin to diffusion, probability, electromagnetic theory, and so on (see [7, 13, 18, 24, 26]).

Since most of FDEs do not have exact solutions, approximate and numerical techniques have received considerable attention to solve fractional differential equations.

Key words and phrases. Fractional differential equation (FDE), systems of fractional integro-differential equations (SFIDE), Riemann-Liouville fractional derivative, Taylor expansion, error analysis.

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So far, several analytical and numerical methods have been proposed to solve fractional differential equations which the interested reader can refer to [1–5, 10–12, 16, 19–23, 25, 27–30, 34] and the references therein.

In this paper, we investigate the approximate solutions of linear fractional integro-differential equations systems based on a new application of Taylor expansion (see [6, 8–10, 14, 15, 17, 31–33]). By expanding unknown functions as an mth-order Taylor polynomial and employing integration method, we can convert the given system of fractional integro-differential equations into a system of linear equations with respect to unknown functions and their derivatives. Approximate solutions can be obtained by solving the resulting system of equations according to a standard rule. The results of the obtained approximations of the suggested method are then compared with the referenced methods for several examples. In the present investigation, the main property of this approximate method besides simplicity and reliability is that an mth-order approximation is equal to exact solution if the exact solution is a polynomial of degree at most m. The present work may be viewed as an extension of the results obtained in [10].

The remainder of this paper is organized as follows. In Section 2, some definitions of fractional calculus are recalled. In Section 3, we describe the proposed method. In Section 4, we give an error analysis. In Section 5, we investigate some examples, which demonstrate the effectiveness of our approach. In Section 6, our findings are concluded.

2. Preliminaries and Basic Definitions

Let's describe some basic concepts, and properties of the fractional calculus, which will be used later.

Definition 2.1. A real function $\phi(x)$, $x > 0$, is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $p\ (> \mu)$, such that $\phi(x) = x^p \phi_1(x)$, where $\phi_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $\phi^{(n)} \in C_\mu,\ n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $\phi \in C_\mu,\ \mu \geq -1$, is considered as follows

$$\begin{aligned} J^\alpha \phi(x) =& \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} \phi(t) dt, \quad \alpha > 0, x > 0, \\ J^0 \phi(x) =& \phi(x). \end{aligned}$$

Definition 2.3. The Caputo fractional derivative of $\phi(x)$ is considered as follows

$$D_*^\alpha \phi(x) = J^{n-\alpha} \left(\frac{d^n}{dx^n} \phi(x) \right) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1} \phi^{(n)}(t) dt,$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $x > 0$, $\phi \in C_{-1}^n$.

Definition 2.4. The Riemann-Liouville fractional derivative of $\phi(x)$ is considered as follows

$$D^{\alpha}\phi(x) = \frac{d^n}{dx^n}\left(J^{n-\alpha}\phi(x)\right),$$

for $n-1<\alpha\leq n$, $n\in\mathbb{N}$, $x>0$, $\phi\in C_{-1}^{n}$.

3. Description of the Method

Consider the following system of linear fractional integro-differential equations

$$\begin{aligned}D^{\alpha_i}\psi_i(x) + \lambda_1\int_0^1\sum_{j=1}^{\nu}K_{1_{ij}}(x,t)\psi_j(t)dt + \lambda_2\int_0^x\sum_{j=1}^{\nu}K_{2_{ij}}(x,t)\psi_j(t)dt\\ =f_i(x),\quad i=1,\dots,\nu,\end{aligned}\tag{3.1}$$

with initial conditions

$$\psi_i^{(\kappa)}(0)=0,\quad \kappa=0,1,\dots,n-1, n-1<\alpha_i\leq n, n\in\mathbb{N},\tag{3.2}$$

where $D^{\alpha_i}\psi_i(x)$ indicates Riemann-Liouville fractional derivative of order α_i, and λ_1, λ_2 are constants, $K_{1_{ij}}(x,t)$, $K_{2_{ij}}(x,t)$, $f_i(x)$ are given known functions which satisfy certain conditions so that system (3.2) has a unique solution, and $\psi_i(x)$ are unknown functions.

According to definition (2.4), system of fractional integro-differential equation (3.1) can be rewritten as

$$\frac{d^n}{dx^n}\left(J^{n-\alpha_i}\psi_i(x)\right) + \lambda_1\int_0^1\sum_{j=1}^{\nu}K_{1_{ij}}(x,t)\psi_j(t)dt + \lambda_2\int_0^x\sum_{j=1}^{\nu}K_{2_{ij}}(x,t)\psi_j(t)dt = f_i(x),$$

or equivalently by using definition (2.2), we have

$$\begin{aligned}\frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha_i)}\int_0^x(x-t)^{n-\alpha_i-1}\psi_i(t)dt\right) + \lambda_1\int_0^1\sum_{j=1}^{\nu}K_{1_{ij}}(x,t)\psi_j(t)dt\\ + \lambda_2\int_0^x\sum_{j=1}^{\nu}K_{2_{ij}}(x,t)\psi_j(t)dt = f_i(x).\end{aligned}\tag{3.3}$$

In the following, by integrating both hand side of (3.3), n times with respect to x from 0 to s and with the help of changing the order of the integrations, we obtain

$$\begin{aligned}\frac{1}{\Gamma(n-\alpha_i)}\int_0^x(x-t)^{n-\alpha_i-1}\psi_i(t)dt + \lambda_1\sum_{j=1}^{\nu}\int_0^1\int_0^x\frac{(x-s)^{l-1}}{(l-1)!}K_{1_{ij}}(s,t)\psi_j(t)dsdt\\ + \lambda_2\sum_{j=1}^{\nu}\int_0^x\int_t^x\frac{(x-s)^{l-1}}{(l-1)!}K_{2_{ij}}(s,t)\psi_j(t)dsdt = F_i(x),\quad l=1,\dots,n,\end{aligned}\tag{3.4}$$

where

$$F_i(x) = \int_0^x\frac{(x-t)^{l-1}}{(l-1)!}f_i(t)dt,\quad i=1,\dots,\nu,$$

in which the variable s has been replaced by x, for simplicity. Hence we transformed the system of fractional integro-differential equations (3.1) into a system of mixed

Volterra-Fredholm integral equations. To approximately solve the resulting system, we reduce Eq. (3.4) into a system of linear equations with respect to unknown functions and their derivatives. Toward this goal, the method assumes that the desired solutions $\psi_j(t)$ to be $m+1$ times continuously differentiable on the interval I, in other words $\psi_j \in C^{m+1}(I)$. Therefore, for $\psi_j \in C^{m+1}(I)$, $\psi_j(t)$ can be expressed in terms of the mth-order Taylor series at an arbitrary point $x \in I$ as

$$\psi_j(t) = \psi_j(x) + \psi_j'(x)(t-x) + \cdots + \frac{1}{m!}\psi_j^{(m)}(x)(t-x)^m + E_{j,m}(t,x),$$

where $E_{j,m}(t,x)$ indicates the Lagrange error bound

$$E_{j,m}(t,x) = \frac{\psi_j^{(m+1)}(\xi_j)}{(m+1)!}(t-x)^{m+1},$$

for some point ξ_j between x and t. Generally, the Lagrange error bound $E_{j,m}(t,x)$ becomes sufficiently small as m gets great enough. Especially, if the solutions $\psi_j(t)$ are polynomials of degree up to m, then the last Lagrange error bound becomes zero, namely, the obtained approximate solutions of system (3.1) yield the true solutions. With due attention to aforementioned assumption, by omitting the last Lagrange error bound, we consider the truncated Taylor expansion $\psi_j(t)$ as

$$\psi_j(t) \approx \sum_{k=0}^{m} \psi_j^{(k)}(x)\frac{(t-x)^k}{k!}. \tag{3.5}$$

Inserting the approximate relation (3.5), for unknown functions $\psi_j(t)$, into (3.4) we obtain

$$\begin{aligned}
&\sum_{k=0}^{m} \frac{(-1)^k}{k!}\psi_j^{(k)}(x)\int_0^x \frac{(x-t)^{k+n-\alpha_i-1}}{\Gamma(n-\alpha_i)}dt \\
&+ \lambda_1 \sum_{j=1}^{\nu}\sum_{k=0}^{m} \frac{\psi_j^{(k)}(x)}{k!}\int_0^1\int_0^x \frac{(x-s)^{l-1}}{(l-1)!}(t-x)^k K_{1_{ij}}(s,t)dsdt \\
&+ \lambda_2 \sum_{j=1}^{\nu}\sum_{k=0}^{m} \frac{\psi_j^{(k)}(x)}{k!}\int_0^x\int_t^x \frac{(x-s)^{l-1}}{(l-1)!}(t-x)^k K_{2_{ij}}(s,t)dsdt \\
=&F_i(x), \quad i=1,\dots,\nu.
\end{aligned} \tag{3.6}$$

In fact, system (3.1) was converted into a linear system of ordinary differential equations with respect to $\psi_j(x)$ and its derivatives up to order m. In other word, we have obtained ν linear equations in (3.6) with respect to $\nu \times (m+1)$ unknown functions $\psi_j^{(k)}$, for $k=0,\dots,m,\ j=1,\dots,\nu$. In the following, we want to determine $\psi_j^{(k)}$ by solving a system of linear equations. In order to achieve this goal, other $\nu \times m$ independent linear equations with respect to $\psi_j(x),\dots,\psi_j^{(m)}(x)$ are needed, which can be achieved by integrating both sides of Eq.(3.4) m times with respect to x. Thus, we have

$$\int_0^x \frac{(x-t)^{\gamma+n-\alpha_i-1}}{\Gamma(\gamma+n-\alpha_i)}\psi_i(t)dt + \lambda_1 \sum_{j=1}^{\nu} \int_0^1 \int_0^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!} K_{1_{ij}}(s,t)\psi_j(t)dsdt$$
$$+ \lambda_2 \sum_{j=1}^{\nu} \int_0^x \int_t^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!} K_{2_{ij}}(s,t)\psi_j(t)dsdt = F_i^{(\gamma)}(x), \quad \gamma = 1,\ldots,m, \tag{3.7}$$

where

$$F_i^{(\gamma)}(x) = \int_0^x \frac{(x-t)^{\gamma-1}}{(\gamma-1)!} F_i(t)dt, \quad i = 1,\ldots,\nu, \gamma = 1,\ldots,m.$$

We apply the Taylor expansion again and substituting (3.5) for $\psi_j(t)$ into E(3.7) leads to

$$\sum_{k=0}^{m} \frac{(-1)^k}{k!}\psi_j^{(k)}(x) \int_0^x \frac{(x-t)^{k+\gamma+n-\alpha_i-1}}{\Gamma(\gamma+n-\alpha_i)}dt$$
$$+ \lambda_1 \sum_{j=1}^{\nu} \sum_{k=0}^{m} \frac{\psi_j^{(k)}(x)}{k!} \int_0^1 \int_0^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!}(t-x)^k K_{1_{ij}}(s,t)dsdt +$$
$$\lambda_2 \sum_{j=1}^{\nu} \sum_{k=0}^{m} \frac{\psi_j^{(k)}(x)}{k!} \int_0^x \int_t^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!}(t-x)^k K_{2_{ij}}(s,t)dsdt$$
$$= F_i^{(\gamma)}(x), \quad \gamma = 1,\ldots,m. \tag{3.8}$$

In this way, (3.4) and (3.8) construct a system of linear equations with resect to unknown functions $\psi_j(x)$ and its derivatives up to order m. The obtained system is indicated as follows

$$\mathbf{Q}(x)\Psi(x) = F(x),$$

where

$$\mathbf{Q}(x) = \begin{bmatrix}
q_{10}^{10}(x) & \cdots & q_{\nu 0}^{10}(x) & \cdots & q_{1k}^{10}(x) & \cdots & q_{\nu k}^{10}(x) & \cdots & q_{1m}^{10}(x) & \cdots & q_{\nu m}^{10}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{\nu 0}(x) & \cdots & q_{\nu 0}^{\nu 0}(x) & \cdots & q_{1k}^{\nu 0}(x) & \cdots & q_{\nu k}^{\nu 0}(x) & \cdots & q_{1m}^{\nu 0}(x) & \cdots & q_{\nu m}^{\nu 0}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{1\gamma}(x) & \cdots & q_{\nu 0}^{1\gamma}(x) & \cdots & q_{1k}^{1\gamma}(x) & \cdots & q_{\nu k}^{1\gamma}(x) & \cdots & q_{1m}^{1\gamma}(x) & \cdots & q_{\nu m}^{1\gamma}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{\nu\gamma}(x) & \cdots & q_{\nu 0}^{\nu\gamma}(x) & \cdots & q_{1k}^{\nu\gamma}(x) & \cdots & q_{\nu k}^{\nu\gamma}(x) & \cdots & q_{1m}^{\nu\gamma}(x) & \cdots & q_{\nu m}^{\nu\gamma}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{1m}(x) & \cdots & q_{\nu 0}^{1m}(x) & \cdots & q_{1k}^{1m}(x) & \cdots & q_{\nu k}^{1m}(x) & \cdots & q_{1m}^{1m}(x) & \cdots & q_{\nu m}^{1m}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{\nu m}(x) & \cdots & q_{\nu 0}^{\nu m}(x) & \cdots & q_{1k}^{\nu m}(x) & \cdots & q_{\nu k}^{\nu m}(x) & \cdots & q_{1m}^{\nu m}(x) & \cdots & q_{\nu m}^{\nu m}(x)
\end{bmatrix}, \tag{3.9}$$

$$\mathbf{F}(x) = \left[\ F_1(x),\ldots,F_\nu(x),\ldots,F_1^{(\gamma)}(x),\ldots,F_\nu^{(\gamma)}(x),\ldots,F_1^{(m)}(x),\ldots,F_\nu^{(m)}(x)\ \right]^T,$$

$$\mathbf{\Psi}(x) = \left[\ \psi_1(x), \ldots, \psi_\nu(x), \ldots, \psi_1^{(k)}(x), \ldots, \psi_\nu^{(k)}(x), \ldots, \psi_1^{(m)}(x), \ldots, \psi_\nu^{(m)}(x) \ \right]^T.$$

In coefficient matrix (3.9), the first ν rows refer to coefficients of $\psi_j^{(k)}(x)$ in (3.4) for $k = 0, \ldots, m$, $j = 1, \ldots, \nu$ and the other rows refer to coefficients of $\psi_j^{(k)}(x)$ in (3.8) for $\gamma = 1, \ldots, m$. Application of a standard rule to the resulting new system yields an mth-order approximate solution of (3.1) as $\psi_{im}(x)$. It is to be noted that not only $\psi_j(x)$ but also $\psi_j^{(k)}(x)$, for $k = 1, \ldots, m$, are determined by solving the resulting new system but in point of fact, it is $\psi_j(x)$ that we need.

4. Error Analysis

In this section, we expand the error analysis proposed in [9] for derived mth-order approximate solution of fractional integro-differential equations system (3.1). We assume that the exact solutions $\psi_j(t)$ are infinitely differentiable on the interval I; so $\psi_j(t)$ can be expressed as an uniformly convergent Taylor series in I as follows

$$\psi_j(t) = \sum_{k=0}^{\infty} \psi_j^{(k)}(x) \frac{(t-x)^k}{k!}.$$

Using the proposed method in the previous section, system of fractional integro-differential equations (3.1) can be converted into an equivalent system of linear equations with respect to unknown functions $\psi_i^{(k)}(x)$, $k = 0, 1, \ldots$ as

$$\mathbf{Q\Psi} = \mathbf{F},$$

where

$$\mathbf{Q} = \lim_{\nu \longrightarrow \infty} \mathbf{Q}_{\nu\nu}^{\nu\nu}, \qquad \mathbf{\Psi} = \lim_{\nu \longrightarrow \infty} \mathbf{\Psi}_\nu, \qquad \mathbf{F} = \lim_{\nu \longrightarrow \infty} \mathbf{F}_\nu,$$

in which $\mathbf{Q}_{\nu\nu}^{\nu\nu}$, $\mathbf{\Psi}_\nu$, and $\mathbf{F}_\nu$, as shown in the previous section, are defined as follows

$$\mathbf{Q}_{\nu\nu}^{\nu\nu} = \left[q_{ij}^{pq}(x) \right]_{\nu(m+1)\times\nu(m+1)}, \ \mathbf{\Psi}_\nu = \left[\psi_i^{(k)}(x) \right]_{\nu(m+1)\times 1}, \ \mathbf{F}_\nu = \left[f_i^{(l)}(x) \right]_{\nu(m+1)\times 1}.$$

Hence, under the solvability conditions for the above system and letting $\mathbf{B} = \mathbf{Q}^{-1}$, the unique solution is represented as

$$\mathbf{\Psi} = \mathbf{BF}. \tag{4.1}$$

We rewrite relation (4.1) in an alternative matrix form as

$$\begin{bmatrix} \mathbf{\Psi}_\nu \\ \mathbf{\Psi}_\infty \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\nu\nu}^{\nu\nu} & \mathbf{B}_{\nu\infty}^{\nu\infty} \\ \mathbf{B}_{\infty\nu}^{\infty\nu} & \mathbf{B}_{\infty\infty}^{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{F}_\nu \\ \mathbf{F}_\infty \end{bmatrix}. \tag{4.2}$$

Accordingly, we can find out that the vector $\mathbf{\Psi}_\nu$ consists of the first $\nu(m+1)$ elements of the exact solution vector $\mathbf{\Psi}$ must satisfy the following relation

$$\mathbf{\Psi}_\nu = \mathbf{B}_{\nu\nu}^{\nu\nu}\mathbf{F}_\nu + \mathbf{B}_{\nu\infty}^{\nu\infty}\mathbf{F}_\infty. \tag{4.3}$$

According to the proposed process, the unique solution of SFIDE (3.1) can be denoted as

$$\widetilde{\mathbf{\Psi}}_\nu = \mathbf{Q}_{\nu\nu}^{\nu\nu^{-1}}\mathbf{F}_\nu, \tag{4.4}$$

where $\mathbf{\Psi}_\nu$ is replaced by $\widetilde{\mathbf{\Psi}}_\nu$ as its approximate solution.

Subtracting (4.4) from (4.3) leads to

$$\mathbf{\Psi}_\nu - \widetilde{\mathbf{\Psi}}_\nu = \mathbf{D}^{\nu\nu}_{\nu\nu}\mathbf{F}_\nu + \mathbf{B}^{\nu\infty}_{\nu\infty}\mathbf{F}_\infty, \tag{4.5}$$

where

$$\mathbf{D}^{\nu\nu}_{\nu\nu} = \mathbf{B}^{\nu\nu}_{\nu\nu} - \mathbf{Q}^{\nu\nu^{-1}}_{\nu\nu}.$$

In the following, we expand the right-hand side of (4.5) and the first ν elements of the vector at the left-hand side of (4.5) can be expressed as

$$\psi^\nu(x) - \widetilde{\psi}^\nu(x) = \sum_{j=0}^{m}\sum_{i=1}^{\nu} d^{p0}_{ij}(x) f_i^{(j)}(x) + \sum_{j=m+1}^{\infty}\sum_{i=1}^{\nu} b^{p0}_{ij}(x) f_i^{(j)}(x), \quad p = 1, \dots, \nu,$$

where

$$\psi^\nu(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_\nu(x) \end{bmatrix}, \qquad \widetilde{\psi}^\nu(x) = \begin{bmatrix} \widetilde{\psi}_1(x) \\ \widetilde{\psi}_2(x) \\ \vdots \\ \widetilde{\psi}_\nu(x) \end{bmatrix},$$

and $d^{p0}_{ij}(x)$, $b^{p0}_{ij}(x)$ are the elements of $\mathbf{D}^{\nu\nu}_{\nu\nu}$ and $\mathbf{B}^{\nu\infty}_{\nu\infty}$, respectively. Thus, according to the Cauchy-Schwarz inequality we have

$$\begin{aligned} \left|\psi^\nu(x) - \widetilde{\psi}^\nu(x)\right| \le & \left(\sum_{j=0}^{m}\sum_{i=1}^{\nu}\left|d^{p0}_{ij}(x)\right|^2\right)^{\frac{1}{2}} \left(\sum_{j=0}^{m}\sum_{i=1}^{\nu}\left|f_i^{(j)}(x)\right|^2\right)^{\frac{1}{2}} \\ & + \left(\sum_{j=m+1}^{\infty}\sum_{i=1}^{\nu}\left|b^{p0}_{ij}(x)\right|^2\right)^{\frac{1}{2}} \left(\sum_{j=m+1}^{\infty}\sum_{i=1}^{\nu}\left|f_i^{(j)}(x)\right|^2\right)^{\frac{1}{2}}. \end{aligned}$$

It is to be noted that as $\lim\limits_{\nu\longrightarrow\infty} \mathbf{D}^{\nu\nu}_{\nu\nu} = 0$ and $\lim\limits_{\nu\longrightarrow\infty} \mathbf{B}^{\nu\infty}_{\nu\infty} = 0$, we have

$$\lim_{\nu\longrightarrow\infty} |\psi^\nu(x) - \widetilde{\psi}^\nu(x)| = 0.$$

5. Illustrative Examples

In this section, the efficiency and the accuracy of the proposed approach is illustrated by considering some numerical problems. The obtained numerical results are compared with some existing approaches and it was found that the proposed approximate approach produces acceptable results and even more accurate results in comparison with some existing methods. All computations are performed using Mathematica 8.

Example 5.1. Consider the following system of fractional integro-differential equations (see [5, 29]):

$$\begin{cases} D^{\frac{1}{2}}\psi_1(x) - \int_0^1 (\psi_1(t) + \psi_2(t))\, dt = \dfrac{2\sqrt{x}}{\sqrt{\pi}} - \dfrac{5}{6}, \\ D^{\frac{3}{2}}\psi_1(x) - \int_0^1 (\psi_1(t) + \psi_2(t))\, dt = \dfrac{4\sqrt{x}}{\sqrt{\pi}} - \dfrac{x}{6}, \end{cases} \tag{5.1}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x$ and $\psi_2(x) = x^2$.

Using the present method, the first-order and the second-order approximate solutions at equidistant points are computed. The obtained results and the results given in [5, 29] are listed in Tables 1 and 2. From Tables 1 and 2, we observe that the second-order approximate solution yields the exact solution as expected, since the exact solution is a polynomial function of degree 2.

TABLE 1. Absolute errors of Example 5.1 for $\psi_1(x)$.

x	Method in [29]	Method in [5]	Suggested method	
			$m=1$	$m=2$
0.1	8.75559×10^{-2}	2.78470×10^{-3}	1.73688×10^{-1}	0
0.2	1.23823×10^{-1}	3.93816×10^{-3}	5.59324×10^{-1}	0
0.3	1.51651×10^{-1}	4.82324×10^{-3}	1.98751	0
0.4	1.75112×10^{-1}	5.56940×10^{-3}	4.08095	0
0.5	1.95781×10^{-1}	6.22678×10^{-3}	1.10827	0
0.6	2.14467×10^{-1}	6.82110×10^{-3}	5.81370×10^{-1}	0
0.7	2.31651×10^{-1}	7.36763×10^{-3}	3.21226×10^{-1}	0
0.8	2.47646×10^{-1}	7.87633×10^{-3}	1.50704×10^{-1}	0
0.9	2.62668×10^{-1}	8.35411×10^{-3}	2.74544×10^{-2}	0
1.0	2.76876×10^{-1}	8.80600×10^{-3}	6.20423×10^{-2}	0

TABLE 2. Absolute errors of Example 5.1 for $\psi_2(x)$.

x	Method in [5]	Method in [30]	Suggested method	
			$m=1$	$m=2$
0.1	1.93140×10^{-4}	1.29824×10^{-4}	3.56504×10^{-5}	0
0.2	1.09257×10^{-3}	3.77788×10^{-4}	3.25545×10^{-3}	0
0.3	3.01076×10^{-3}	7.13496×10^{-4}	3.28085×10^{-2}	0
0.4	6.18049×10^{-3}	1.12845×10^{-3}	1.35422×10^{-1}	0
0.5	1.07969×10^{-2}	1.61892×10^{-3}	6.60271×10^{-2}	0
0.6	1.70314×10^{-2}	2.18315×10^{-3}	6.09208×10^{-2}	0
0.7	2.50391×10^{-2}	2.82043×10^{-3}	6.26674×10^{-2}	0
0.8	3.49621×10^{-2}	3.53063×10^{-3}	6.66494×10^{-2}	0
0.9	4.69331×10^{-2}	4.31399×10^{-3}	7.19976×10^{-2}	0
1.0	6.10763×10^{-2}	5.17100×10^{-3}	7.88615×10^{-2}	0

It is important to note that after converting system (5.1) into a system of linear equations, the Mathematica command 'LinearSolve' is used for the new system.

Example 5.2. Consider the following system of fractional integro-differential equations (see [29]):

$$\begin{cases} D^{\frac{1}{2}}\psi_1(x) - \int_0^1 x\psi_2(t)dt = \dfrac{2\sqrt{x}}{\sqrt{\pi}} - \dfrac{x}{2}, \\ D^{\frac{1}{2}}\psi_2(x) - \int_0^1 x\psi_1(t)dt = \dfrac{2\sqrt{x}}{\sqrt{\pi}} - \dfrac{1}{3}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x$ and $\psi_2(x) = x$.

We employ the approach described in Section 3 to evaluate the approximate solutions. For this case, we can find that $\psi_m(x)$ yields the exact solution only by setting $m = 1$. Moreover, we present the results given in [29] in Table 3.

TABLE 3. Absolute errors of Example 5.2 in [29] for $(\psi_1(x), \psi_2(x))$.

x	Methode in [29]
0.1	$(5.02704 \times 10^{-5},\ 5.02704 \times 10^{-4})$
0.2	$(1.42186 \times 10^{-4},\ 7.10931 \times 10^{-4})$
0.3	$(2.61213 \times 10^{-4},\ 8.70709 \times 10^{-4})$
0.4	$(4.02163 \times 10^{-4},\ 1.00541 \times 10^{-3})$
0.5	$(5.62040 \times 10^{-4},\ 1.12408 \times 10^{-3})$
0.6	$(7.38821 \times 10^{-4},\ 1.23137 \times 10^{-3})$
0.7	$(9.31021 \times 10^{-4},\ 1.33003 \times 10^{-3})$
0.8	$(1.13749 \times 10^{-3},\ 1.42186 \times 10^{-3})$
0.9	$(1.35730 \times 10^{-3},\ 1.50811 \times 10^{-3})$
1.0	$(1.58969 \times 10^{-3},\ 1.58969 \times 10^{-3})$

Example 5.3. Consider the following system of fractional integro-differential equations (see [16, 30]):

$$\begin{cases} D^{\frac{3}{4}}\psi_1(x) - \displaystyle\int_0^1 (x+t)\,[\psi_1(t) + \psi_2(t)]\,dt = -\frac{1}{20} - \frac{x}{12} + \frac{4x^{\frac{1}{4}}}{\Gamma(\frac{1}{4})} - \frac{128x^{\frac{9}{4}}}{15\Gamma(\frac{1}{4})}, \\[2ex] D^{\frac{3}{4}}\psi_2(x) - \displaystyle\int_0^1 \sqrt{x}t^2\,[\psi_1(t) - \psi_2(t)]\,dt = -\frac{2\sqrt{x}}{15} - \frac{4x^{\frac{1}{4}}}{\Gamma(\frac{1}{4})} + \frac{32x^{\frac{5}{4}}}{5\Gamma(\frac{1}{4})}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x - x^3$ and $\psi_2(x) = x^2 - x$.

We apply the approach described in Section 3 to determine the approximate solutions. For this case, we can find that $\psi_m(x)$ yields the exact solution only by setting $m = 3$. We present our results when $m = 1, 2, 3$, and the results given in [30] in Tables 4 and 5.

Example 5.4. Consider the following system of fractional integro-differential equations (see [16, 30])

$$\begin{cases} D^{\frac{4}{5}}\psi_1(x) - \displaystyle\int_0^1 2xt\,[\psi_1(t) - \psi_2(t)]\,dt = \frac{83}{80}x - \frac{25x^{\frac{6}{5}}}{3\Gamma(\frac{1}{5})} + \frac{125x^{\frac{11}{5}}}{11\Gamma(\frac{1}{5})}, \\ D^{\frac{4}{5}}\psi_2(x) - \displaystyle\int_0^1 (x+t)\,[\psi_1(t) + \psi_2(t)]\,dt = -\frac{67}{160} - \frac{13}{24}x + \frac{125x^{\frac{6}{5}}}{8\Gamma(\frac{1}{5})}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x^3 - x^2$ and $\psi_2(x) = \frac{15}{8}x^2$.

TABLE 4. Absolute errors of Example 5.3 for $\psi_1(x)$

x	Method in [30]	Suggested method		
		$m=1$	$m=2$	$m=3$
0.1	1.86460×10^{-3}	2.33950×10^{-2}	4.37610×10^{-3}	0
0.2	3.38103×10^{-3}	6.86709×10^{-2}	1.69027×10^{-3}	0
0.3	4.91496×10^{-3}	1.21870×10^{-1}	1.70008×10^{-3}	0
0.4	6.51082×10^{-3}	1.73108×10^{-1}	3.93799×10^{-3}	0
0.5	8.18437×10^{-3}	2.11497×10^{-1}	4.52983×10^{-3}	0
0.6	9.94249×10^{-3}	2.25976×10^{-1}	3.55933×10^{-3}	0
0.7	1.17883×10^{-2}	2.06732×10^{-1}	1.36667×10^{-3}	0
0.8	1.37235×10^{-2}	1.47035×10^{-1}	1.59402×10^{-3}	0
0.9	1.57484×10^{-2}	4.52912×10^{-2}	4.82795×10^{-3}	0
1.0	1.78631×10^{-2}	9.29796×10^{-2}	7.84433×10^{-3}	0

TABLE 5. Absolute errors of Example 5.3 for $\psi_2(x)$

x	Method in [30]	Suggested method		
		$m=1$	$m=2$	$m=3$
0.1	1.99879×10^{-4}	1.46339×10^{-2}	3.62132×10^{-3}	0
0.2	4.75397×10^{-4}	3.25600×10^{-2}	1.64100×10^{-2}	0
0.3	7.89170×10^{-4}	4.88261×10^{-2}	2.95774×10^{-2}	0
0.4	1.13069×10^{-3}	6.04406×10^{-2}	3.63960×10^{-2}	0
0.5	1.49445×10^{-3}	6.45455×10^{-2}	3.60909×10^{-2}	0
0.6	1.87697×10^{-3}	5.84157×10^{-2}	3.03835×10^{-2}	0
0.7	2.27584×10^{-2}	3.97032×10^{-2}	2.15300×10^{-2}	0
0.8	2.68925×10^{-2}	6.79901×10^{-3}	1.17235×10^{-2}	0
0.9	3.11582×10^{-2}	4.07493×10^{-2}	2.96446×10^{-3}	0
1.0	3.55442×10^{-2}	1.01834×10^{-1}	2.95048×10^{-3}	0

Applying the approach described in this paper, we determine the approximate solutions. For this case, we can find that $\psi_m(x)$ yields the exact solution only by setting $m = 3$. We present our numerical results obtained by proposed Taylor expansion method for $m = 1, 2, 3$ and the results obtained in [30] in Tables 6 and 7.

Example 5.5. Consider the following system of fractional integro-differential equations

$$\begin{cases} D^{\frac{3}{4}}\psi_1(x) - \displaystyle\int_0^x \frac{\psi_1(t)+\psi_2(t)}{\sqrt{x-t}}dt = -\frac{16x^{\frac{5}{2}}}{15} - \frac{32x^{\frac{7}{2}}}{35} + \frac{32x^{\frac{5}{4}}}{5\Gamma(\frac{1}{4})}, \\ \\ D^{\frac{1}{2}}\psi_2(x) - \displaystyle\int_0^x \frac{\psi_1(t)+\psi_2(t)}{(x-t)^{\frac{2}{3}}}dt = -\frac{27x^{\frac{7}{3}}}{14} + \frac{16x^{\frac{5}{2}}}{5\sqrt{\pi}} - \frac{243x^{\frac{10}{3}}}{140}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x^2$ and $\psi_2(x) = x^3$.

Based on the proposed method in Section 3, we obtain the approximate results by setting $m = 1, 2, 3$ and we observe that the third-order approximate solution yields the

TABLE 6. Absolute errors of Example 5.4 for $\psi_1(x)$.

x	Method in [30]		Suggested method	
		$m=1$	$m=2$	$m=3$
0.1	1.96792×10^{-4}	1.66987×10^{-2}	4.37610×10^{-3}	0
0.2	6.85268×10^{-4}	4.54650×10^{-2}	1.69027×10^{-3}	0
0.3	1.42175×10^{-3}	7.48952×10^{-2}	1.70008×10^{-3}	0
0.4	2.38624×10^{-3}	9.69101×10^{-2}	3.93799×10^{-3}	0
0.5	3.56576×10^{-3}	1.05439×10^{-1}	4.52983×10^{-3}	0
0.6	4.95084×10^{-3}	9.62850×10^{-2}	3.55933×10^{-3}	0
0.7	6.53406×10^{-3}	6.71607×10^{-2}	1.36667×10^{-3}	0
0.8	8.30938×10^{-3}	1.77783×10^{-2}	1.59402×10^{-3}	0
0.9	1.02717×10^{-2}	5.00357×10^{-2}	4.82795×10^{-3}	0
1.0	1.24167×10^{-2}	1.32209×10^{-1}	7.84433×10^{-3}	0

TABLE 7. Absolute errors of Example 5.4 for $\psi_2(x)$.

x	Method in [30]		Suggested method	
		$m=1$	$m=2$	$m=3$
0.1	8.20450×10^{-4}	1.35222×10^{-1}	4.98795×10^{-2}	0
0.2	1.58553×10^{-3}	1.88478×10^{-1}	8.22827×10^{-2}	0
0.3	2.41026×10^{-3}	2.17328×10^{-1}	9.64328×10^{-2}	0
0.4	3.30743×10^{-3}	2.25836×10^{-1}	9.56954×10^{-2}	0
0.5	4.28071×10^{-3}	2.16061×10^{-1}	8.41589×10^{-2}	0
0.6	5.33111×10^{-3}	1.89798×10^{-1}	6.57542×10^{-2}	0
0.7	6.45864×10^{-3}	1.49181×10^{-1}	4.42508×10^{-2}	0
0.8	7.66286×10^{-3}	9.71051×10^{-2}	2.32810×10^{-2}	0
0.9	8.94313×10^{-3}	3.76493×10^{-2}	6.32948×10^{-3}	0
1.0	1.02987×10^{-2}	2.34213×10^{-2}	3.30327×10^{-3}	0

exact solution as expected. In the following, our results for $m = 1, 2, 3$ at equidistant points in $[0, 1]$ are tabulated in Tables 8 and 9.

TABLE 8. Absolute errors of Example 5.5 for $\psi_1(x)$.

x	$m=1$	$m=2$	$m=3$
0.1	4.39572×10^{-4}	5.63735×10^{-8}	0
0.2	2.02649×10^{-3}	1.49505×10^{-6}	0
0.3	6.38129×10^{-3}	1.61418×10^{-5}	0
0.4	1.85611×10^{-2}	1.16368×10^{-4}	0
0.5	4.69815×10^{-2}	6.32737×10^{-4}	0
0.6	9.46103×10^{-2}	2.86770×10^{-3}	0
0.7	1.53109×10^{-1}	1.22967×10^{-2}	0
0.8	2.14122×10^{-1}	7.00457×10^{-2}	0
0.9	2.76101×10^{-1}	2.65058×10^{-1}	0
1.0	3.40830×10^{-1}	1.19614×10^{-1}	0

TABLE 9. Absolute errors of Example 5.5 for $\psi_2(x)$.

x	$m=1$	$m=2$	$m=3$
0.1	1.17689×10^{-4}	2.02948×10^{-5}	0
0.2	1.61962×10^{-3}	1.53357×10^{-4}	0
0.3	9.65962×10^{-3}	4.69785×10^{-4}	0
0.4	3.89089×10^{-2}	8.58738×10^{-4}	0
0.5	1.13095×10^{-1}	4.68815×10^{-4}	0
0.6	2.40454×10^{-1}	4.31091×10^{-3}	0
0.7	3.98040×10^{-1}	2.88928×10^{-2}	0
0.8	5.63382×10^{-1}	1.89377×10^{-1}	0
0.9	7.33038×10^{-1}	7.60116×10^{-1}	0
1.0	9.12716×10^{-1}	3.53191×10^{-1}	0

6. Conclusion

In this paper, we have proposed an approximate method for solving systems of fractional integro-differential equations. In the proposed technique, the SFIDE to be solved, has been converted into integral equations. Then Taylor expansion for unknown functions and integration method have employed to convert the resulting integral equations into a system of linear equations with respect to unknown functions and their derivatives. By applying a standard method the resulting system has been solved. In particular for such cases when the exact solutions are polynomial functions of degree up to m, the derived mth-order approximations are exact.

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ON CAPUTO FRACTIONAL DERIVATIVES VIA CONVEXITY

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Abstract. In this paper some estimations of Caputo fractional derivatives via convexity have been presented. By using convexity of any positive integer order differentiable function some novel results are given.

1. Introduction

Caputo fractional derivatives are defined as follows (see [1]).

Definition 1.1. Let $\alpha > 0$ and $\alpha \notin \{1, 2, 3, \dots\}$, $n = [\alpha] + 1$, $f \in AC^n[a, b]$, the space of functions having nth derivatives absolutely continuous. The left-sided and right-sided Caputo fractional derivatives of order α are defined as follows:

$$\left({}^C D^{\alpha}_{a+} f\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \quad x > a$$

and

$$\left({}^C D^{\alpha}_{b-} f\right)(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n+1}} dt, \quad x < b.$$

If $\alpha = n \in \{1, 2, 3, \dots\}$ and usual derivative $f^{(n)}(x)$ of order n exists, then Caputo fractional derivative $\left({}^C D^{n}_{a+} f\right)(x)$ coincides with $f^{(n)}(x)$ whereas $\left({}^C D^{n}_{b-} f\right)(x)$ coincides with $f^{(n)}(x)$ with exactness to a constant multiplier $(-1)^n$. In particular we have

$$\left({}^C D^{0}_{a+} f\right)(x) = \left({}^C D^{0}_{b-} f\right)(x) = f(x),$$

where $n = 1$ and $\alpha = 0$.

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Since the inequalities always have been proved worthy in establishing the mathematical models and their solutions in almost all branches of applied sciences (see [2,3]). Especially the convexity takes very important role in the optimization theory. The aim of this paper is to introduce some fractional inequalities for the Caputo fractional derivatives via the convexity property of the functions which have derivatives of any integer order.

2. Main Results

First we give the following estimate of the sum of left and right handed Caputo fractional derivatives.

Theorem 2.1. *Let $f : I \to \mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $f^{(n)}$ is a positive convex function, then for $a, b \in I$, $a < b$ and $\alpha, \beta \geq 1$, the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned} \Gamma(n-\alpha+1)\left({}^{C}D_{a+}^{\alpha-1}f\right)(x) + \Gamma(n-\beta+1)\left({}^{C}D_{b-}^{\beta-1}f\right)(x) \\ \leq \frac{(x-a)^{n-\alpha+1}f^{(n)}(a) + (b-x)^{n-\beta+1}f^{(n)}(b)}{2} \\ + f^{(n)}(x)\left[\frac{(x-a)^{n-\alpha+1} + (b-x)^{n-\beta+1}}{2}\right]. \end{aligned} \tag{2.1}$$

Proof. Let us consider the function f on the interval $[a, x]$, $x \in [a, b]$. For $t \in [a, x]$, the following inequality holds

$$(x-t)^{n-\alpha} \leq (x-a)^{n-\alpha}. \tag{2.2}$$

Since $f^{(n)}$ is convex therefore for $t \in [a, x]$ we have

$$f^{(n)}(t) \leq \frac{x-t}{x-a}f^{(n)}(a) + \frac{t-a}{x-a}f^{(n)}(x). \tag{2.3}$$

Multiplying inequalities (2.3) and (2.2), then integrating with respect to t over $[a, x]$ we have

$$\int_a^x (x-t)^{n-\alpha}f^{(n)}(t)dt \leq \frac{(x-a)^{n-\alpha}}{x-a}\left[f^{(n)}(a)\int_a^x (x-t)dt + f^{(n)}(x)\int_a^x (t-a)dt\right],$$

$$\Gamma(n-\alpha+1)\left({}^{C}D_{a+}^{\alpha-1}f\right)(x) \leq \frac{(x-a)^{n-\alpha+1}}{2}[f^{(n)}(a) + f^{(n)}(x)]. \tag{2.4}$$

Now we consider function f on the interval $[x, b]$, $x \in [a, b]$. For $t \in [x, b]$ the following inequality holds

$$(t-x)^{n-\beta} \leq (b-x)^{n-\beta}. \tag{2.5}$$

Since $f^{(n)}$ is convex on $[a, b]$, therefore, for $t \in [x, b]$ we have

$$f^{(n)}(t) \leq \frac{t-x}{b-x}f^{(n)}(b) + \frac{b-t}{b-x}f^{(n)}(x). \tag{2.6}$$

Multiplying inequalities (2.5) and (2.6), then integrating with respect to t over $[x,b]$ we have

$$\int_x^b (t-x)^{n-\beta} f^{(n)}(t)dt \leq \frac{(b-x)^{n-\beta}}{b-x}\left[f^{(n)}(b)\int_x^b (t-x)dt + f^{(n)}(x)\int_x^b (b-t)dt\right],$$

$$\Gamma(n-\beta+1)\left({}^C D_{b-}^{\beta-1} f\right)(x) \leq \frac{(b-x)^{n-\beta+1}}{2}[f^{(n)}(b)+f^{(n)}(x)]. \tag{2.7}$$

Adding (2.4) and (2.7) we get the required inequality in (2.1). □

It is nice to see that the following implication holds.

Corollary 2.1. *By setting $\alpha=\beta$ in (2.1) we get the following fractional integral inequality*

$$\begin{aligned}&\Gamma(n-\alpha+1)\left(\left({}^C D_{a+}^{\alpha-1} f\right)(x) + \left({}^C D_{b-}^{\alpha-1} f\right)(x)\right)\\ \leq&\frac{(x-a)^{n-\alpha+1} f^{(n)}(a) + (b-x)^{n-\alpha+1} f^{(n)}(b)}{2} + f^{(n)}(x)\left[\frac{(x-a)^{n-\alpha+1}+(b-x)^{n-\alpha+1}}{2}\right].\end{aligned}$$

Now we give the next result stated in the following theorem.

Theorem 2.2. *Let $f: I \to \mathbb{R}$ be a real valued n-time differentiable function, where n is a positive integer. If $|f^{(n+1)}|$ is convex function, then for $a,b \in I$ $a<b$ and $\alpha,\beta>0$, the following inequality for Caputo fractional derivatives holds*

$$\begin{aligned}&\Big|\Gamma(n-\alpha+1)\left({}^C D_{a+}^{\alpha} f\right)(x) + \Gamma(n-\beta+1)\left({}^C D_{b-}^{\beta} f\right)(x)\\ &-\left((x-a)^{n-\alpha} f^{(n)}(a) + (b-x)^{n-\beta} f^{(n)}(b)\right)\Big|\\ \leq&\frac{(x-a)^{\alpha+1}|f^{(n+1)}(a)| + (b-x)^{\beta+1}|f^{(n+1)}(b)|}{2}\\ &+\frac{|f^{(n+1)}(x)|\left((x-a)^{\alpha+1}+(b-x)^{\beta+1}\right)}{2}.\end{aligned} \tag{2.8}$$

Proof. Since $|f^{(n+1)}|$ is convex, therefore, for $t \in [a,x]$ we have

$$|f^{(n+1)}(t)| \leq \frac{x-t}{x-a}|f^{(n+1)}(a)| + \frac{t-a}{x-a}|f^{(n+1)}(x)|$$

from which we can write

$$\begin{aligned}&-\left(\frac{x-t}{x-a}|f^{(n+1)}(a)| + \frac{t-a}{x-a}|f^{(n+1)}(x)|\right) \leq f^{(n+1)}(t)\\ &\qquad\leq \frac{x-t}{x-a}|f^{(n+1)}(a)| + \frac{t-a}{x-a}|f^{(n+1)}(x)|.\end{aligned} \tag{2.9}$$

We consider the second inequality of inequality (2.9)

$$f^{(n+1)}(t) \leq \frac{x-t}{x-a}|f^{(n+1)}(a)| + \frac{t-a}{x-a}|f^{(n+1)}(x)|.$$

Now for $\alpha > 0$ we have

$$(x-t)^{n-\alpha} \leq (x-a)^{n-\alpha}, t \in [a,x]. \tag{2.10}$$

The product of last two inequalities give

$$(x-t)^{n-\alpha} f^{(n+1)}(t) \leq (x-a)^{n-\alpha-1} \left((x-t)|f^{(n+1)}(a)| + (t-a)|f^{(n+1)}(x)| \right).$$

Integrating with respect to t over $[a,x]$ we have

$$\begin{aligned} &\int_a^x (x-t)^{n-\alpha} f^{(n+1)}(t)dt \\ \leq &(x-a)^{n-\alpha-1} \left(|f^{(n+1)}(a)| \int_a^x (x-t)dt + |f^{(n+1)}(x)| \int_a^x (t-a)dt \right) \\ = &(x-a)^{n-\alpha+1} \left(\frac{|f^{(n+1)}(a)| + |f^{(n+1)}(x)|}{2} \right) \end{aligned} \tag{2.11}$$

and

$$\begin{aligned} \int_a^x (x-t)^{n-\alpha} f^{(n+1)}(t)dt &= f^{(n)}(t)(x-t)^{n-\alpha}|_a^x + (n-\alpha) \int_a^x (x-t)^{n-\alpha-1} f^{(n)}(t)dt \\ &= -f^{(n)}(a)(x-a)^{n-\alpha} + \Gamma(n-\alpha+1) \left({}^C D_{a+}^{\alpha} f \right)(x). \end{aligned}$$

Therefore, (2.11) takes the form

$$\begin{aligned} &\Gamma(n-\alpha+1) \left({}^C D_{a+}^{\alpha} f \right)(x) - f^{(n)}(a)(x-a)^{n-\alpha} \\ \leq &(x-a)^{n-\alpha+1} \left(\frac{|f^{(n+1)}(a)| + |f^{(n+1)}(x)|}{2} \right). \end{aligned} \tag{2.12}$$

If one consider from (2.9) the first inequality and proceed as we did for the second inequality, then following inequality can be obtained

$$\begin{aligned} &f^{(n)}(a)(x-a)^{n-\alpha} - \Gamma(n-\alpha+1) \left({}^C D_{a+}^{\alpha} f \right)(x) \\ \leq &(x-a)^{n-\alpha+1} \left(\frac{|f^{(n+1)}(a)| + |f^{(n+1)}(x)|}{2} \right). \end{aligned} \tag{2.13}$$

From (2.12) and (2.13) we get

$$\begin{aligned} &\left| \Gamma(n-\alpha+1) \left({}^C D_{a+}^{\alpha} f \right)(x) - f^{(n)}(a)(x-a)^{n-\alpha} \right| \\ \leq &(x-a)^{n-\alpha+1} \left(\frac{|f^{(n+1)}(a)| + |f^{(n+1)}(x)|}{2} \right). \end{aligned} \tag{2.14}$$

On the other hand for $t \in [x,b]$ using convexity of $|f^{(n+1)}|$ we have

$$|f^{(n+1)}(t)| \leq \frac{t-x}{b-x}|f^{(n+1)}(b)| + \frac{b-t}{b-x}|f^{(n+1)}(x)|. \tag{2.15}$$

Also for $t \in [x,b]$ and $\beta > 0$ we have

$$(t-x)^{n-\beta} \leq (b-x)^{n-\beta}. \tag{2.16}$$

By adopting the same treatment as we have done for (2.9) and (2.10) one can obtain from (2.15) and (2.16) the following inequality

$$\begin{aligned}
(2.17)\qquad &\left|\Gamma(n-\beta+1)\left({}^C D^{\beta}_{b-} f\right)(x) - f^{(n)}(b)(b-x)^{n-\beta}\right| \\
&\leq (b-x)^{n-\beta+1}\left(\frac{|f^{(n+1)}(b)| + |f^{(n+1)}(x)|}{2}\right).
\end{aligned}$$

By combining the inequalities (2.14) and (2.17) via triangular inequality we get the required inequality. □

It is interesting to see the following inequalities as a special case.

Corollary 2.2. *By setting $\alpha = \beta$ in (2.8) we get the following fractional integral inequality*

$$\begin{aligned}
&\Big|\Gamma(n-\alpha+1)[\left({}^C D^{\alpha}_{a+} f\right)(x) + \left({}^C D^{\alpha}_{b-} f\right)(x)] \\
&\quad - \left((x-a)^{n-\alpha} f^{(n)}(a) + (b-x)^{n-\alpha} f^{(n)}(b)\right)\Big| \\
&\leq \frac{(x-a)^{n-\alpha+1}|f^{(n+1)}(a)| + (b-x)^{n-\alpha+1}|f^{(n+1)}(b)|}{2} \\
&\quad + \frac{|f^{(n+1)}(x)|\left((x-a)^{n-\alpha+1} + (b-x)^{n-\alpha+1}\right)}{2}.
\end{aligned}$$

Before going to the next theorem we observe the following result.

Lemma 2.1. *Let $f : [a,b] \to \mathbb{R}$, be a convex function. If f is symmetric about $\frac{a+b}{2}$, then the following inequality holds*

$$(2.18)\qquad f\left(\frac{a+b}{2}\right) \leq f(x), \quad x \in [a,b].$$

Proof. We have

$$\frac{a+b}{2} = \frac{1}{2}\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}x\right) + \frac{1}{2}\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right).$$

Since f is convex, therefore we have

$$\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left(f\left(\frac{x-a}{b-a}b + \frac{b-x}{b-a}x\right) + f\left(\frac{x-a}{b-a}a + \frac{b-x}{b-a}b\right)\right) \\
&= \frac{1}{2}\left(f(x) + f(a+b-x)\right).
\end{aligned}$$

Also f is symmetric about $\frac{a+b}{2}$, therefore, we have $f(a+b-x) = f(x)$ and inequality in (2.18) holds. □

Theorem 2.3. *Let $f : I \to \mathbb{R}$ be a real valued n-time differentiable function where n is a positive integer. If $f^{(n)}$ is a positive convex and symmetric about $\frac{a+b}{2}$, then for*

$a, b \in I$, $a < b$ and $\alpha, \beta \geq 1$, the following inequality for Caputo fractional derivatives holds

$$\begin{aligned}
(2.19) \qquad & \frac{1}{2}\left(\frac{1}{n-\alpha+1}+\frac{1}{n-\beta+1}\right) f^{(n)}\left(\frac{a+b}{2}\right) \\
& \leq \frac{\Gamma(n-\beta+1)\left({}^{C}D_{b-}^{\beta-1} f\right)(a)}{2(b-a)^{n-\beta+1}}+\frac{\Gamma(n-\alpha+1)\left({}^{C}D_{a+}^{\alpha-1} f\right)(b)}{2(b-a)^{n-\alpha+1}} \\
& \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.
\end{aligned}$$

Proof. For $x \in [a, b]$ we have

$$(x-a)^{n-\beta} \leq (b-a)^{n-\beta}. \tag{2.20}$$

Also f is convex function we have

$$f^{(n)}(x) \leq \frac{x-a}{b-a} f^{(n)}(b)+\frac{b-x}{b-a} f^{(n)}(a). \tag{2.21}$$

Multiplying (2.20) and (2.21) and then integrating with respect to x over $[a, b]$ we have

$$\int_{a}^{b}(x-a)^{n-\beta} f^{(n)}(x) dx \leq \frac{(b-a)^{n-\beta}}{b-a}\left(\int_{a}^{b}\left(f^{(n)}(b)(x-a)+f^{(n)}(a)(b-x)\right) dx\right).$$

From which we have

$$\frac{\Gamma(n-\beta+1)\left({}^{C}D_{b-}^{\beta-1} f\right)(a)}{(b-a)^{n-\beta+1}} \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \tag{2.22}$$

On the other hand for $x \in [a, b]$ we have

$$(b-x)^{n-\alpha} \leq (b-a)^{n-\alpha}. \tag{2.23}$$

Multiplying (2.21) and (2.23) and then integrating with respect to x over $[a, b]$ we get

$$\int_{a}^{b}(b-x)^{n-\alpha} f^{(n)}(x) dx \leq (b-a)^{n-\alpha+1} \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.$$

From which we have

$$\frac{\Gamma(n-\alpha+1)\left({}^{C}D_{a+}^{\alpha-1} f\right)(b)}{(b-a)^{n-\alpha+1}} \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \tag{2.24}$$

Adding (2.22) and (2.24) we get the second inequality

$$\frac{\Gamma(n-\beta+1)\left({}^{C}D_{b-}^{\beta-1} f\right)(a)}{2(b-a)^{n-\beta+1}}+\frac{\Gamma(n-\alpha+1)\left({}^{C}D_{a+}^{\alpha-1} f\right)(b)}{2(b-a)^{n-\alpha+1}} \leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}.$$

Since $f^{(n)}$ is convex and symmetric about $\frac{a+b}{2}$ using Lemma 2.1 we have

$$f^{(n)}\left(\frac{a+b}{2}\right) \leq f^{(n)}(x), \quad x \in [a, b]. \tag{2.25}$$

Multiplying with $(x-a)^{n-\beta}$ on both sides and then integrating over $[a,b]$ we have

$$f^{(n)}\left(\frac{a+b}{2}\right)\int_a^b (x-a)^{n-\beta}dx \leq \int_a^b (x-a)^{n-\beta}f^{(n)}(x)dx.$$

By definition of Caputo fractional derivatives one can has

$$f^{(n)}\left(\frac{a+b}{2}\right)\frac{1}{2(n-\beta+1)} \leq \frac{\Gamma(n-\beta+1)\left({}^{C}D_{b-}^{\beta-1}f\right)(a)}{2(b-a)^{n-\beta+1}}. \tag{2.26}$$

Multiplying (2.25) with $(b-x)^{n-\alpha}$, then integrating over $[a,b]$ one can get

$$f^{(n)}\left(\frac{a+b}{2}\right)\frac{1}{2(n-\alpha+1)} \leq \frac{\Gamma(n-\alpha+1)\left({}^{C}D_{a+}^{\alpha-1}f\right)(b)}{2(b-a)^{n-\alpha+1}}. \tag{2.27}$$

Adding (2.26) and (2.27) we get the first inequality. □

Corollary 2.3. *If we put $\alpha=\beta$ in* (2.19)*, then we get*

$$\begin{aligned} f^{(n)}\left(\frac{a+b}{2}\right)\frac{1}{\alpha+1} &\leq \frac{\Gamma(n-\alpha+1)}{2(b-a)^{\alpha+1}}\left(\left({}^{C}D_{b-}^{\alpha+1}f\right)(a)+\left({}^{C}D_{a+}^{\alpha+1}f\right)(b)\right) \\ &\leq \frac{f^{(n)}(a)+f^{(n)}(b)}{2}. \end{aligned}$$

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BEST PROXIMITY POINT RESULTS VIA SIMULATION FUNCTIONS IN METRIC-LIKE SPACES

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ABSTRACT. In this paper, we discuss the existence of best proximity points of certain mappings via simulation functions in the frame of complete metric-like spaces. Some consequences and examples are given of the obtained results.

1. INTRODUCTION

Khojasteh et al. introduced in [13] the notion of simulation function in order to unify several fixed point results obtained by various authors. These functions were later utilized by Karapinar and Khojasteh in [9] to solve some problems concerning best proximity points.

On the other hand, spaces more general than metric and fixed point and related problems in them have been lately a wide field of interest of huge number of mathematicians. Among them, metric-like spaces, introduced by Amini-Harandi in [2], took a prominent place.

In this paper, we are going to extend these investigations to best proximity points of mappings acting in complete metric-like spaces, using conditions involving simulation functions. The results will be illustrated by several examples, showing the strength of these results compared with others existing in the literature.

2. PRELIMINARIES

Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^+$, $\mathbb{R}_0^+$ will denote the set of real numbers, the set of positive real numbers and the set of nonnegative real numbers, respectively. Also, $\mathbb{N}_0$ and $\mathbb{N}$ will denote the set of nonnegative, resp. positive integers.

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We shall first recall some basic definitions and some results from [1, 5, 13].

Definition 2.1 ([13])**.** A simulation function is a mapping $\zeta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;
(ζ_2) $\zeta(t,s) < s - t$ for all $t, s > 0$;
(ζ_3) if $\{t_n\}$ and $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = l \in (0,\infty)$, then $\limsup_{n\to\infty} \zeta(t_n, s_n) < 0$.

Note that, according to the axiom (ζ_2), each simulation function ζ satisfies $\zeta(t,t) < 0$ for all $t > 0$. The family of all simulation functions will be denoted by $\mathcal{Z}$.

Example 2.1 (See, e.g., [1,5,7,13]). For $i = 1, 2, \dots, 6$, define mappings $\zeta_i : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$, as follows.

(i) $\zeta_1(t,s) = \phi_1(s) - \phi_2(t)$ for all $t, s \in \mathbb{R}_0^+$, where $\phi_1, \phi_2 : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ are continuous functions, with $\phi_i(t) = 0$ if and only if $t = 0$ and $\phi_1(t) < t \le \phi_2(t)$ for all $t > 0$.
(ii) $\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)} t$ for all $t, s \in \mathbb{R}_0^+$, where $f, g : {\mathbb{R}_0^+}^2 \to \mathbb{R}_0^+$ are two functions, continuous with respect to each variable and such that $f(t,s) > g(t,s)$ for all $t, s > 0$.
(iii) $\zeta_3(t,s) = s - \phi(s) - t$ for all $t, s \in \mathbb{R}_0^+$, where $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous functions, with $\phi(t) = 0$ if and only if $t = 0$.
(iv) If $\varphi : \mathbb{R}_0^+ \to [0,1)$ is a function such that $\limsup_{t\to r^+} \varphi(t) < 1$ for all $r > 0$, let
$$\zeta_4(t,s) = s\varphi(s) - t, \qquad \text{for all } t, s \in \mathbb{R}_0^+.$$
(v) If $\eta : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is an upper semi-continuous function such that $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$, let
$$\zeta_5(t,s) = \eta(s) - t, \qquad \text{for all } t, s \in \mathbb{R}_0^+.$$
(vi) If $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a function such that $\int_0^\epsilon \phi(u)\,du > \epsilon$ for each $\epsilon > 0$, let
$$\zeta_6(t,s) = s - \int_0^t \phi(u)\,du, \qquad \text{for all } t, s \in \mathbb{R}_0^+.$$

It is clear that each function ζ_i, $i = 1, 2, \dots, 6$, is a simulation function.

Definition 2.2 ([2])**.** Let X be a nonempty set, and a mapping $\sigma : X \times X \to \mathbb{R}_0^+$ is such that, for all $x, y, z \in X$,

(σ_1) $\sigma(x,y) = 0$ implies $x = y$;
(σ_2) $\sigma(x,y) = \sigma(y,x)$;
(σ_3) $\sigma(x,y) \le \sigma(x,z) + \sigma(z,y)$.

Then (X,σ) is said to be a metric-like space.

As is well known, each partial metric space is an example of a metric-like space. The converse is not true. The following example illustrates this statement.

Example 2.2. Take $X = \{1, 2, 3\}$ and consider the metric-like $\sigma : X \times X \to \mathbb{R}_0^+$ given by

$$\sigma(1,1) = 0, \qquad \sigma(2,2) = 1, \qquad \sigma(3,3) = \frac{2}{3},$$
$$\sigma(2,1) = \sigma(1,2) = \frac{9}{10}, \quad \sigma(1,3) = \sigma(3,1) = \frac{7}{10}, \quad \sigma(2,3) = \sigma(3,2) = \frac{4}{5}.$$

Since $\sigma(2,2) \neq 0$, σ is not a metric and since $\sigma(2,2) > \sigma(2,1)$, σ is not a partial metric.

Every metric-like σ on X generates a topology τ_σ whose base is the family of all open σ-balls

$$\{\, B_\sigma(x,\delta) : x \in X, \delta > 0 \,\},$$

where $B_\sigma(x,\delta) = \{\, y \in X : |\sigma(x,y) - \sigma(x,x)| < \delta \,\}$, for all $x \in X$ and $\delta > 0$.

Definition 2.3 ([2])**.** Let (X, σ) be a metric-like space, let $\{x_n\}$ be a sequence in X and $x \in X$. Then

(i) $\{x_n\}$ is said to converge to x, w.r.t. τ_σ, if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$;
(ii) $\{x_n\}$ is called a Cauchy sequence in (X, σ) if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists (and is finite);
(iii) (X, σ) is called complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_σ to a point $x \in X$ such that

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x);$$

(iv) a function $f : X \to X$ is continuous if for any sequence $\{x_n\}$ in X such that $\sigma(x_n, x) \to \sigma(x, x)$ as $n \to \infty$, we have $\sigma(fx_n, fx) \to \sigma(fx, fx)$ as $n \to \infty$.

Note that the limit of a sequence in a metric-like space might not be unique.

Lemma 2.1 ([11])**.** *Let (X, σ) be a metric-like space. Let $\{x_n\}$ be a sequence in X such that $x_n \to x$ where $x \in X$ and $\sigma(x, x) = 0$. Then for all $y \in X$, we have*

$$\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y).$$

Ψ will denote the family of non-decreasing functions $\psi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfying the following conditions:

(i) $\psi(t) < t$, for any $t \in \mathbb{R}^+$;
(ii) ψ is continuous at 0.

Let (X, σ) be a metric-like space, and U and V be two non-empty subsets of X. Recall the following standard notation:

$$\sigma(U, V) := \inf\{\sigma(u, v) : u \in U, v \in V\},$$
$$U_0 := \{u \in U : \sigma(u, v) = \sigma(U, V) \text{ for some } v \in V\},$$
$$V_0 := \{v \in V : \sigma(u, v) = \sigma(U, V) \text{ for some } u \in U\}.$$

Consider now a non-self mapping $T : U \to V$ and the equation $Tu = u$ ($u \in U$). As is well known, a solution of this equation, if it exists, is called a fixed point of T. If such solution does not exist, an approximate solution $u^* \in U$ have the least possible error when $\sigma(u^*, Tu^*) = \sigma(U, V)$. In this case, u^* is called a best proximity point of the mapping $T : U \to V$.

Finally, recall the following useful notions.

Definition 2.4 ([6])**.** Let U and V be nonempty subsets of a metric-like space (X, σ), and $\alpha : U \times U \to \mathbb{R}_0^+$ be a function. We say that the mapping T is α-proximal admissible if

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) \;\Rightarrow\; \alpha(u, v) \geq 1,$$

for all $x, y, u, v \in X$.

If $\sigma(U, V) = 0$, then T reduces from α-proximal admissible to α-admissible.

Definition 2.5 ([8, 10])**.** Let $T : X \to X$ be a mapping and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. We say that the mapping T is triangular weakly-α-admissible if

$$\alpha(x, y) \geq 1 \text{ and } \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1.$$

3. Main Results

Definition 3.1. Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\psi \in \Psi$, $\alpha : X \times X \to \mathbb{R}_0^+$ and $\zeta \in \mathcal{Z}$. We say that $T : U \to V$ is an α-ψ-ζ-contraction if T is α-proximal admissible and

(3.1)

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) \;\Rightarrow\; \zeta(\alpha(x, y)\sigma(u, v), \psi(\sigma(x, y))) \geq 0,$$

for all $x, y, u, v \in U$.

Definition 3.2. Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\alpha : X \times X \to \mathbb{R}_0^+$ and $\zeta \in \mathcal{Z}$. We say that $T : U \to V$ is an α-ζ-contraction if T is α-proximal admissible and

(3.2)

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) \;\Rightarrow\; \zeta(\alpha(x, y)\sigma(u, v), \sigma(x, y)) \geq 0,$$

for all $x, y, u, v \in U$.

Notice that Definition 3.2 is not a special case of Definition 3.1 since the function $\psi(t) = t$ does not belong to Ψ.

The following lemma provides a standard step in proving that the given sequence is Cauchy in a certain space.

Lemma 3.1 (See, e.g., [14])**.** *Let (X, σ) be a metric-like space and let $\{x_n\}$ be a sequence in X such that $\sigma(x_{n+1}, x_n)$ is non-increasing and that $\lim_{n\to\infty} \sigma(x_{n+1}, x_n) = 0$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and two sequences $\{m_k\}$*

and $\{n_k\}$ *of positive integers such that the following four sequences tend to* ϵ *when* $k \to \infty$*:*

$$\sigma(x_{m_k}, x_{n_k}),\ \sigma(x_{m_k+1}, x_{n_k+1}),\ \sigma(x_{m_k-1}, x_{n_k}),\ \sigma(x_{m_k}, x_{n_k-1}).$$

Now we present the main results of this article.

Theorem 3.1. *Let* (X, σ) *be a metric-like space,* U *and* V *be two non-empty subsets of* X*,* $\alpha : X \times X \to \mathbb{R}_0^+$*,* $\psi \in \Psi$ *and* $\zeta \in \mathcal{Z}$ *is non-decreasing with respect to its second argument. Suppose that* $T : U \to V$ *is an* α*-*ψ*-*ζ*-contraction and*

(1) T *is triangular weakly-*α*-admissible;*
(2) U *is closed with respect to the topology* τ_σ*;*
(3) $T(U_0) \subset V_0$*;*
(4) *there exist* $x_0, x_1 \in U$ *such that* $\sigma(x_1, Tx_0) = \sigma(U, V)$ *and* $\alpha(x_0, x_1) \geq 1$*;*
(5) T *is continuous.*

Then, T *has a best proximity point, that is, there exists* $z \in U$ *such that* $\sigma(z, Tz) = \sigma(U, V)$*.*

Proof. Take $x_0, x_1 \in U$ given as in (4). Taking (3) into account, we conclude that $Tx_1 \in V_0$ which implies that there exists $x_2 \in U$ such that $\sigma(x_2, Tx_1) = \sigma(U, V)$. Since $\alpha(x_0, x_1) \geq 1$ and T is α-proximal admissible, we conclude that $\alpha(x_1, x_2) \geq 1$. Recursively, a sequence $\{x_n\} \subset U$ can be chosen satisfying

$$\sigma(x_{n+1}, Tx_n) = \sigma(U, V) \text{ and } \alpha(x_n, x_{n+1}) \geq 1, \quad \text{for all } n \in \mathbb{N}_0. \tag{3.3}$$

If $x_k = x_{k+1}$ for some $k \in \mathbb{N}_0$, then $\sigma(x_k, Tx_k) = \sigma(x_{k+1}, Tx_k) = \sigma(U, V)$, meaning that x_k is the required best proximal point. Hence, we will further assume that

$$x_n \neq x_{n+1}, \quad \text{for all } n \in \mathbb{N}_0. \tag{3.4}$$

Using relations (3.3) and (3.4), we get that $\sigma(x_n, Tx_{n-1}) = \sigma(x_{n+1}, Tx_n) = \sigma(U, V)$, for all $n \in \mathbb{N}$. Furthermore, by (3.1)

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \psi(\sigma(x_{n-1}, x_n))) \geq 0, \quad \text{for all } n \in \mathbb{N}, \tag{3.5}$$

since $T : U \to V$ is an α-ψ-ζ-contraction. Regarding (3.4) and (ζ_2), the inequality (3.5) implies that

$$\sigma(x_n, x_{n+1}) \leq \alpha(x, y)\sigma(x_n, x_{n+1}) \leq \psi(\sigma(x_{n-1}, x_n)) < \sigma(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\{\sigma(x_n, x_{n+1})\}$ is a non-increasing sequence bounded from below and there exists $L \in \mathbb{R}_0^+$ such that $\sigma(x_n, x_{n+1}) \to L$ as $n \to \infty$. We shall prove that $L = 0$. Suppose, on the contrary, that $L > 0$. Taking the upper limit in (3.5) as $n \to \infty$, regarding (ζ_3), property (i) of $\psi \in \Psi$ and that ζ is non-decreasing with respect to the second argument, we deduce

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \psi(\sigma(x_n, x_{n-1}))) \\ &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \sigma(x_n, x_{n-1})) < 0, \end{aligned}$$

which is a contradiction. We conclude that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$.

We shall now prove that the sequence $\{x_n\}$ is Cauchy. Suppose that it is not. Then, there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, so that $n_k > m_k > k$ and

$$\sigma(x_{m_k}, x_{n_k}) \geq \epsilon \text{ and } \sigma(x_{m_k}, x_{n_k-1}) < \epsilon. \tag{3.6}$$

By Lemma 2.1, we have

$$\lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) = \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k-1}) = \epsilon.$$

Since T is triangular weakly-α-admissible, from (3.3), we get that

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } n, m \in \mathbb{N}_0 \text{ with } n > m.$$

Hence,

$$\alpha(x_{m_k}, x_{n_k}) \geq 1 \text{ and } \sigma(x_{m_k}, Tx_{m_k-1}) = \sigma(x_{n_k}, Tx_{n_k-1}) = \sigma(U, V), \quad \text{for all } k \in \mathbb{N}. \tag{3.7}$$

Since T is an α-ψ-ζ-contraction, the obtained relations (3.7) yield the following inequality:

$$0 \leq \zeta(\alpha(x_n, x_{n-1})\sigma(x_{m_k}, x_{n_k}), \psi(\sigma(x_{m_k}, x_{n_k}))), \quad \text{for all } k \in \mathbb{N}.$$

Letting $k \to \infty$, using (3.6) and (ζ_3), and regarding properties of $\psi \in \Psi$ and that ζ is non-decreasing with respect to the second argument, we obtain

$$\begin{aligned} 0 &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_{m_k}, x_{n_k}), \psi(\sigma(x_{m_k-1}, Tx_{n_k-1}))) \\ &\leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_{m_k}, x_{n_k}), \sigma(x_{m_k-1}, Tx_{n_k-1})) < 0, \end{aligned}$$

which is a contradiction. Thus, we conclude that the sequence $\{x_n\}$ is Cauchy in U.

Since U is a closed subset of a complete metric-like space (X, σ), there exists $z \in U$ such that

$$\lim_{n\to\infty} \sigma(x_n, z) = 0. \tag{3.8}$$

Since T is continuous, we deduce that

$$\lim_{n\to\infty} \sigma(Tx_n, Tz) = 0. \tag{3.9}$$

From (3.3), using the triangle inequality together with (3.8) and (3.9), we find that

$$\sigma(U, V) = \lim_{n\to\infty} \sigma(x_{n+1}, Tx_n) = \sigma(z, Tz).$$

Thus, $z \in U$ is a best proximity point of the mapping T. □

The continuity hypothesis in Theorem 3.1 can be omitted if we assume the following additional condition on U:

(P) if a sequence $\{u_n\}$ in U converges to $u \in U$ and is such that $\alpha(u_n, u_{n+1}) \geq 1$ for $n \geq 1$, then there is a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ with $\alpha(u_{n(k)}, u) \geq 1$ for all k.

Theorem 3.2. *Let all the conditions of Theorem* 3.1 *hold, except that the condition* (5) *is replaced by*

(5') (P) *holds.*

Then T has a best proximity point.

Proof. As in the proof of Theorem 3.1 we conclude that there exists a sequence $\{x_n\}$ in U_0 which converges to $z \in U_0$. Using (3), we note that $Tz \in V_0$ and hence

$$\sigma(u_1, Tz) = \sigma(U, V), \quad \text{for some } u_1 \in U_0.$$

Notice that from (P), we have $\alpha(x_{n_k}, z) \geq 1$ for all $k \in \mathbb{N}$. Since T is α-proximal admissible and

$$\sigma(u_1, Tz) = \sigma(x_{n_k+1}, Tx_{n_k}) = \sigma(U, V), \tag{3.10}$$

we obtain that $\alpha(x_{n_k+1}, u_1) \geq 1$ for all $k \in \mathbb{N}$ and

$$\zeta(\alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}), \psi(\sigma(z, x_{n_k}))) \geq 0.$$

Then, (ζ_2) implies that

$$\sigma(u_1, x_{n_k+1}) \leq \alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}) \leq \psi(\sigma(z, x_{n_k})) < \sigma(z, x_{n_k})$$

and so $\lim_{k\to\infty} \sigma(u_1, x_{n_k+1}) \to 0$. Thus, $u_1 = z$ and by (3.10) we have $\sigma(z, Tz) = \sigma(U, V)$. □

Theorem 3.3. *Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\zeta \in \mathcal{Z}$ and $\alpha : X \times X \to \mathbb{R}_0^+$. Suppose that $T : U \to V$ is an α-ζ-contraction and that conditions* (1)-(4) *of Theorem* 3.1 *are satisfied, as well as*

$(5'')$ *T is continuous or* (P) *holds.*

Then, T has a best proximity point.

Proof. By following the lines in the proof of Theorem 3.1, we easily construct a sequence $\{x_n\}$ in U which converges to some $z \in U$, moreover

$$\lim_{n\to\infty} \sigma(x_n, z) = 0. \tag{3.11}$$

Suppose first that T is continuous. Then

$$\lim_{n\to\infty} \sigma(Tx_n, Tz) = 0. \tag{3.12}$$

From (3.3), the triangle inequality together with (3.11) and (3.12) imply

$$\sigma(U, V) = \lim_{n\to\infty} \sigma(x_{n+1}, Tx_n) = \sigma(z, Tz).$$

In other words, $z \in U$ is a best proximity of the mapping T.

Suppose now that (P) holds. Regarding (3), we note that $Tz \in V_0$ and hence

$$\sigma(u_1, Tz) = \sigma(U, V), \quad \text{for some } u_1 \in U_0.$$

Notice that from (P), we have $\alpha(x_{n_k}, z) \geq 1$ for all $k \in \mathbb{N}$. Since T is α-proximal admissible, and

$$\sigma(u_1, Tz) = \sigma(x_{n_k+1}, Tx_{n_k}) = \sigma(U, V),$$

we get that $\alpha(x_{n_k+1}, u_1) \geq 1$ for all $k \in \mathbb{N}$ and

$$\zeta(\alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}), \sigma(z, x_{n_k})) \geq 0. \tag{3.13}$$

Then, (ζ_2) implies that $\sigma(u_1, x_{n_k+1}) \leq \alpha(x_{n_k+1}, u_1)\sigma(u_1, x_{n_k+1}) \leq \sigma(z, x_{n_k})$ and so

$$\lim_{k\to\infty} \sigma(u_1, x_{n_k+1}) \to 0.$$

Thus, $u_1 = z$ and by (3.13) we have $\sigma(z, Tz) = \sigma(U, V)$ and the proof is completed. □

Notice that Theorem 3.3 cannot be obtained by combining Theorems 3.1 and 3.2, since the function $\psi(t) = t$ does not belong to Ψ. Furthermore, in Theorems 3.1 and 3.2, we have an additional condition that ζ is non-decreasing in its second argument.

Definition 3.3. Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X, $\alpha : X \times X \to \mathbb{R}_0^+$ and $\zeta \in \mathcal{Z}$. We say that $T : U \to V$ is a generalized α-ζ-contraction if T is α-proximal admissible and

(3.14)

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) \ \Rightarrow\ \zeta(\alpha(x, y)\sigma(u, v), r(x, y)) \geq 0,$$

for all $x, y, u, v \in U$ with $x \neq y$, where

$$r(x, y) = \max\left\{\sigma(x, y), \frac{\sigma(x, u)\sigma(y, v)}{\sigma(x, y)}\right\}.$$

Theorem 3.4. *Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X and $\alpha : X \times X \to \mathbb{R}_0^+$, $\zeta \in \mathcal{Z}$. Suppose that $T : U \to V$ is a generalized α-ζ-contraction and conditions* (1)-(5) *of Theorem* 3.1 *are satisfied. Then T has a best proximity point.*

Proof. As in the proof of Theorem 3.1, we can construct a sequence $\{x_n\}$ in X satisfying conditions (3.3) and (3.4). Combining these relations with (3.14), we get that $\sigma(x_n, Tx_{n-1}) = \sigma(x_{n+1}, Tx_n) = \sigma(U, V)$ for all $n \in \mathbb{N}$ and

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), r(x_{n-1}, x_n)) \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

Here,

$$\begin{aligned} r(x_{n-1}, x_n) &= \max\left\{\frac{\sigma(x_{n-1}, x_n)\sigma(x_n, x_{n+1})}{\sigma(x_{n-1}, x_n)}, \sigma(x_{n-1}, x_n)\right\} \\ &= \max\left\{\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\right\}. \end{aligned}$$

Suppose that for some $n \in \mathbb{N}$

$$\max\left\{\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)\right\} = \sigma(x_n, x_{n+1}).$$

Since $\sigma(x_n, x_{n+1}) > 0$, using the property (2) of the simulation function, we obtain

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \sigma(x_n, x_{n+1})) < 0,$$

which is a contradiction. It follows that $r(x_{n-1}, x_n) = \sigma(x_{n-1}, x_n)$ for all $n \in \mathbb{N}$, implying that

$$\zeta(\alpha(x_{n-1}, x_n)\sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n)) \geq 0, \quad \text{for all } n \in \mathbb{N}. \tag{3.15}$$

Using (ζ_2), the inequality (3.15) yields that

$$\sigma(x_n, x_{n+1}) \leq \sigma(x_{n-1}, x_n), \quad \text{for all } n \in \mathbb{N}.$$

Hence, $\{\sigma(x_n, x_{n+1})\}$ is a non-increasing sequence, bounded from below, converging to some $L \geq 0$. Suppose that $L > 0$. Taking the upper limit as $n \to \infty$ in (3.15), using (ζ_3), we get

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_n, x_{n-1})\sigma(x_n, x_{n+1}), \psi(\sigma(x_{n-1}, x_n))) < 0,$$

which is a contradiction. Hence, we conclude that $\lim_{n\to\infty} \sigma(x_n, x_{n+1}) = 0$.

In order to prove that $\{x_n\}$ is a Cauchy sequence, suppose the contrary. Then, as in the proof of Theorem 3.1, there exist $\epsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{m_k}\}$ of $\{x_n\}$, so that for $n_k > m_k > k$ we have

$$\sigma(x_{m_k}, x_{n_k}) \geq \epsilon \text{ and } \sigma(x_{m_k}, x_{n_k-1}) < \epsilon.$$

Also, in the same way, the following inequalities hold:

$$\begin{gathered}\lim_{k\to\infty} \sigma(x_{m_k}, x_{n_k}) = \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k-1}) = \epsilon,\\ \lim_{k\to\infty} \sigma(x_{m_k-1}, x_{n_k}) = \lim_{k\to\infty} \sigma(x_{n_k-1}, x_{m_k}) = \epsilon.\end{gathered} \tag{3.16}$$

Since T is triangular weakly-α-admissible, we derive that

$$\alpha(x_n, x_m) \geq 1, \quad \text{for all } n, m \in \mathbb{N}_0 \text{ with } n > m.$$

Thus, we have

$$\alpha(x_{m_k}, x_{n_k}) \geq 1 \text{ and } \sigma(x_{m_k}, Tx_{m_k-1}) = \sigma(x_{n_k}, Tx_{n_k-1}) = \sigma(U, V), \tag{3.17}$$

for all $k \in \mathbb{N}$. Since T is a generalized α-ζ-contraction, the obtained relations (3.17) imply

$$0 \leq \zeta(\alpha(x_{m_k-1}, x_{n_k-1})\sigma(x_{m_k}, x_{n_k}), r(x_{m_k-1}, x_{n_k-1})), \quad \text{for all } k \in \mathbb{N}.$$

Since

$$r(x_{m_k-1}, x_{n_k-1}) = \max\left\{\frac{\sigma(x_{m_k-1}, x_{m_k})\sigma(x_{n_k-1}, x_{n_k})}{\sigma(x_{m_k-1}, x_{n_k-1})}, \sigma(x_{m_k-1}, x_{n_k-1})\right\}, \tag{3.18}$$

taking limits of both sides of (3.18), we conclude that $\lim_{k\to\infty} r(x_{m_k-1}, x_{n_k-1}) = \epsilon$. Letting $k \to \infty$ and keeping (3.16) and (ζ_3) in mind, we get

$$0 \leq \limsup_{n\to\infty} \zeta(\alpha(x_{m_k-1}, x_{n_k-1})\sigma(x_{m_k}, x_{n_k}), r(x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Thus, we conclude that the sequence $\{x_n\}$ is Cauchy in U.

The final step of the proof is the same as for Theorem 3.1. □

4. Corollaries and Examples

Using Example 2.1, it is possible to get a number of consequences of our main results by choosing the simulation function ζ and $\alpha(x, y)$ in a proper way. We skip making such a list of corollaries since they seem clear. We just state the following one as a sample

Corollary 4.1. *Let (X, σ) be a metric-like space, U and V be two non-empty subsets of X and $\alpha : X \times X \to \mathbb{R}_0^+$, $\psi \in \Psi$. Suppose that $T : U \to V$ is a given α-proximal admissible mapping such that*

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) \;\Rightarrow\; \alpha(x, y)\sigma(u, v) \leq \psi(\sigma(x, y))),$$

for all $x, y, u, v \in U$. Suppose also

(a) *T is triangular weakly-α-admissible;*
(b) *U is closed with respect to the topology induced by τ_σ;*
(c) *$T(U_0) \subset V_0$;*
(d) *there exist $x_0, x_1 \in U$ such that $\sigma(x_1, Tx_0) = \sigma(U, V)$ and $\alpha(x_0, x_1) \geq 1$;*
(e) *T is continuous or* (P) *holds.*

Then, T has a best proximity point.

In particular, if the given space (X, σ) is also endowed with a partial order $\preceq$, by taking

$$\alpha(x, y) \geq 1 \Leftrightarrow x \succeq y,$$

one can get standard variations of the given results in a partially ordered space.

The following illustrative examples show how our results can be used for certain mappings acting in metric-like spaces.

Example 4.1. Consider $X = \{a, b, c, d\}$ equipped with $\sigma : X \times X \to \mathbb{R}_0^+$ defined by

$$\begin{gathered}\sigma(a,a)=\frac{1}{2},\quad \sigma(b,b)=0,\quad \sigma(c,c)=2,\quad \sigma(d,d)=\frac{1}{3},\quad \sigma(a,b)=3,\\ \sigma(a,c)=\frac{5}{2},\quad \sigma(a,d)=\frac{3}{2}\quad \sigma(b,c)=2,\quad \sigma(b,d)=\frac{3}{2},\quad \sigma(c,d)=\frac{5}{2},\end{gathered}$$

and $\sigma(x, y) = \sigma(y, x)$ for $x, y \in X$. It is clear that (X, σ) is a complete metric-like space. Take $U = \{b, c\}$ and $V = \{c, d\}$. Consider the mapping $T : U \to V$ defined by $Tb = d$, and $Tc = c$. Remark that $\sigma(U, V) = \sigma(b, d) = \frac{3}{2}$. Also, $U_0 = \{b\}$ and $V_0 = \{d\}$. Note that $T(U_0) \subseteq V_0$. Take $\psi(t) = \frac{5}{6}t$, and $\zeta(t, s) = \frac{3}{4}s - t$ for all $t, s \geq 0$. Define $\alpha : X \times X \to \mathbb{R}_0^+$ by

$$\alpha(x, y) = \begin{cases} 1, & x, y \in U, \\ 0, & \text{otherwise.} \end{cases}$$

Let $x, y, u, v \in U$ be such that

$$\alpha(x, y) \geq 1 \text{ and } \sigma(u, Tx) = \sigma(v, Ty) = \sigma(U, V) = \frac{3}{2}.$$

Then, necessarily, we have $x = y = u = v = b$. So, $\alpha(u, v) \geq 1$, that is, T is α-proximal admissible.

We need to prove that T is an α-ψ-ζ contraction. By the previous conclusion, the only case to be checked is when $x = y = u = v = b$. Then we have

$$\zeta(\alpha(b, b)\sigma(b, b), \psi(\sigma(b, b))) = \zeta(1 \cdot 0, \psi(0)) = 0.$$

Thus, all the conditions of Theorem 3.1 are satisfied. So T has a best proximity point (which is $z = b$). On the other hand, e.g., Corollary 2.2 (with $k = 2$) of [4] is not applicable for the standard metric.

Example 4.2. Consider the set $X = \{a, b, c, d\}$ equipped with the following complete metric-like σ:

$$\sigma(a, a) = \sigma(b, b) = \frac{1}{4}, \quad \sigma(c, c) = \sigma(d, d) = 2,$$
$$\sigma(a, b) = \sigma(c, d) = \frac{1}{2}, \quad \sigma(a, c) = \sigma(b, d) = 1, \quad \sigma(a, d) = \sigma(b, c) = \frac{3}{2},$$

and $\sigma(x, y) = \sigma(y, x)$ for all $x, y \in X$. Let $U = \{a, b\}$ and $V = \{c, d\}$; then $\sigma(U, V) = 1$, $U_0 = U$ and $V_0 = V$. Consider, further, the mappings $T : U \to V$ given by $Ta = c$, $Tb = c$, $\alpha : X \times X \to [0, +\infty)$ given by

$$\alpha(x, y) = \begin{cases} 1, & \text{if } x, y \in U, \\ 0, & \text{otherwise,} \end{cases}$$

and $\zeta \in \mathcal{Z}$ given by $\zeta(t, s) = s - \frac{2+t}{1+t}t$. Let us check that the mapping T is a generalized α-ζ-contraction. Let $x, y, u, v \in U$ be such that $x \neq y$, $\alpha(x, y) \geq 1$, $\sigma(u, Tx) = \sigma(v, Ty) = 1$. Then it must be $u = v = a$ and either $x = a$, $y = b$ or $x = b$, $y = a$. In both cases, it is $\alpha(u, v) \geq 1$. In order to check condition (3.14), it is enough to consider the case $x = a$, $y = b$, $u = v = a$ (the other is treated symmetrically). Then,

$$\begin{aligned} \zeta(\alpha(x, y)\sigma(u, v), r(x, y)) &= \zeta\left(1 \cdot \frac{1}{4}, \max\left\{\frac{1}{2}, \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{2}}\right\}\right) = \zeta\left(\frac{1}{4}, \frac{1}{2}\right) \\ &= \frac{1}{2} - \frac{2 + \frac{1}{4}}{1 + \frac{1}{4}} \cdot \frac{1}{4} = \frac{1}{20} > 0, \end{aligned}$$

and the condition is satisfied. All other conditions of Theorem 3.4 are fulfilled, hence, we conclude that the mapping T has a best proximity point (which is $z = a$).

5. Application to Best Proximity Results on a Metric-like Space with a Graph

Throughout this section, (X, σ) will denote a metric-like space and $G = (V(G), E(G))$ will be a directed graph such that its set of vertices $V(G) = X$ and the set of edges $E(G)$ contains all loops, i.e., $\Delta := \{(x; x) : x \in X\} \subseteq E(G)$. We need in the sequel the following hypothesis:

(P_G) if a sequence $\{u_n\}$ in X converges to $u \in A$ such that $(u_n, u_{n+1}) \in E(G)$, then there is a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ with $(u_{n(k)}, u) \in E(G)$ for all k.

Definition 5.1. Let U and V be two non-empty subsets of X and $\alpha : X \times X \to \mathbb{R}_0^+$. We say that $T : U \to V$ is a G-proximal mapping if

$$\left.\begin{array}{c}(x,y) \in E(G), \ \ \alpha(x,y) \geq 1, \\ \sigma(u,Tx) = \sigma(v,Ty) = \sigma(U,V)\end{array}\right\} \Rightarrow (u,v) \in E(G), \tag{5.1}$$

for all $x, y, u, v \in U$.

Definition 5.2 ([8,10])**.** Let U and V be two non-empty subsets of X, let $T : U \to V$ be a mapping and $\alpha : X \times X \to \mathbb{R}_0^+$ be a function. We say that T is triangular weakly-G-admissible if

$$\alpha(x,y) \in E(G) \text{ and } \alpha(y,z) \in E(G) \ \Rightarrow \ \alpha(x,z) \in E(G).$$

Corollary 5.1. *Let U and V be two non-empty subsets of X and $\psi \in \Psi$. Suppose that $T : U \to V$ is a mapping such that*

$$\sigma(Tx,Ty) \leq \psi(\sigma(x,y)),$$

for all $x, y \in U$ such that $(x,y) \in E(G)$. Suppose also:

(a) *T is triangular weakly-G-admissible;*
(b) *$T(U_0) \subset V_0$;*
(c) *there exist $x_0, x_1 \in U$ such that $\sigma(x_1, Tx_0) = \sigma(U,V)$ and $(x_0,x_1) \in E(G)$;*
(d) *T is continuous or (R_G) holds.*

Then, T has a best proximity point.

Proof. It suffices to consider $\alpha : X \times X \to \mathbb{R}_0^+$ such that

$$\alpha(x,y) = \begin{cases} 1, & \text{if } (x,y) \in E(G), \\ 0, & \text{if not.} \end{cases}$$

All the hypotheses of Corollary 4.1 are satisfied. □

In this way, we can derive all results and consequences of the paper [15], extending them to partially ordered metric-like spaces. Similarly, we can extend the frame of several other existing results from, e.g., [3, 10, 12, 16].

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NUMERICAL RADIUS INEQUALITIES IN 2-INNER PRODUCT SPACES

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Abstract. In this paper, we have obtained the analogue results on numerical radius inequalities from the classical inner product spaces to 2-inner product spaces. We have established several related reverse inequalities and some well known results in 2-inner product spaces.

1. Introduction and Preliminaries

Let $\mathscr{X}$ be a linear space of dimension greater than 1 over the field $K = \mathbb{R}$ of real numbers or the field $K = \mathbb{C}$ of complex numbers. Suppose that $(\cdot, \cdot|\cdot)$ is a K-valued function defined on $\mathscr{X} \times \mathscr{X} \times \mathscr{X}$ satisfying the following conditions:

(I1) $(x, x|z) \geq 0$, and $(x, x|z) = 0$ if and only if x and z are linearly dependent;
(I2) $(x, x|z) = (z, z|x)$;
(I3) $(y, x|z) = \overline{(x, y|z)}$;
(I4) $(\alpha x, y|z) = \alpha\, (x, y|z)$ for any scalar $\alpha \in K$;
(I5) $(x + x', y|z) = (x, y|z) + (x', y|z)$.

$(\cdot, \cdot|\cdot)$ is called a 2-inner product on $\mathscr{X}$ and $(\mathscr{X}, (\cdot, \cdot|\cdot))$ is called a 2-inner product space (or 2-pre-Hilbert sapce). Some basic properties of 2-inner product $(\cdot, \cdot|\cdot)$ can be immediately obtained as follows (see [3]):

(P1) $(0, y|z) = (x, 0|z) = (x, y|0) = 0$;
(P2) $(x, \alpha y|z) = \overline{\alpha}\, (x, y|z)$;
(P3) $(x, y|\alpha z) = |\alpha|^2\, (x, y|z)$, for all $x, y, z \in \mathscr{X}$ and $\alpha \in K$.

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Using the above properties, one has proved that Cauchy-Schwartz inequality (see [5])

$$|(x,y|z)|^2 \le (x,x|z)\,(y,y|z)\,.$$

It should be noticed that, the most standard example for a linear 2-inner product $(\cdot,\cdot|\cdot)$ is defined on $\mathscr{X}$ by

$$(x,y|z) := \det\begin{pmatrix} \langle x,y\rangle & \langle x,z\rangle \\ \langle z,y\rangle & \langle z,z\rangle \end{pmatrix}, \tag{1.1}$$

for all $x,y,z \in \mathscr{X}$. In [2], it is shown that, in any given 2-inner product space $(\mathscr{X},(\cdot,\cdot|\cdot))$, we can define a function

$$\|x,z\| = \sqrt{(x,x|z)}, \tag{1.2}$$

for all $x,z \in \mathscr{X}$. It is not hard to see that this function satisfies the following conditions (see [6]):

(N1) $\|x,y\| = 0$ if and only if x and y are linearly dependent;
(N2) $\|x,y\| = \|y,x\|$;
(N3) $\|\alpha x,y\| = |\alpha|\,\|x,y\|$ for any real number α;
(N4) $\|x,y+z\| \le \|x,y\| + \|x,z\|$.

Any function $\|\cdot,\cdot\|$ defined on $\mathscr{X}\times\mathscr{X}$ and satisfying the above conditions is called a 2-norm induced from a 2-inner product on $\mathscr{X}$ and $(\mathscr{X},\|\cdot,\cdot\|)$ is called linear 2-normed space.

Some of the basic properties of 2-norms are that they are non-negative and $\|x,y+\alpha x\| = \|x,y\|$, for all $x,y \in \mathscr{X}$ and all $\alpha \in \mathbb{R}$. Whenever a 2-inner product space $(\mathscr{X},(\cdot,\cdot|\cdot))$ is given, we consider it as a linear 2-normed space $(\mathscr{X},\|\cdot,\cdot\|)$ with the 2-norm defined by (1.2).

An operator $A \in \mathcal{B}(\mathscr{X})$ is said to be bounded if there exists a real number $M > 0$ such that

$$\|Ax,y\| \le M\,\|x,y\|\,,$$

for every $x,y \in \mathscr{X}$. The norm of the b-operator is defined by [9]:

$$\|A\|_b = \sup\{\|Ax,b\| : \|x,b\| = 1\}\,, \tag{1.3}$$

where b is fixed element in $\mathscr{X}$. We can easily verify that the left-hand side of (1.3), is equivalent with $\sup\{|(Ax,x|b)| : \|x,b\| \le 1\}$.

Harikrishnan et al. in [8] proved the Riesz theorem in 2-inner product spaces. As a consequence of their work, we have

$$(Ax,y|b) = (x,A^*y|b)\,,$$

for each $x,y \in \mathscr{X}$ and fixed element $b \in \mathscr{X}$.

Recently, M. E. Omidvar et al. [10] established various reverses of the Cauchy-Schwarz and triangle inequalities in 2-inner product spaces.

In this paper, we introduce the concepts of b-numerical radius in 2-inner product spaces. Some fundamental inequalities related to the b-numerical radius of bounded linear operators in 2-inner product spaces are established.

2. Main Results

We first review some basic facts about numerical range and numerical radius in Hilbert space $\mathscr{H}$, then try to define them in a 2-inner product space. Let $(\mathscr{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $\mathcal{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. An operator $A \in \mathcal{B}(\mathscr{H})$ is called positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathscr{H}$. We write $A \geq 0$ if A is positive. The numerical radius is defined by

$$\omega(A) = \sup\{|\lambda| : \lambda \in W(A)\},$$

where $W(A)$ is the numerical range of A given by

$$W(A) = \{\langle Ax, x\rangle : x \in \mathscr{H}, \|x\| = 1\}.$$

The following properties of $W(A)$ are immediate:
(a) $W(\alpha I + \beta A) = \alpha + \beta W(A)$ for $\alpha, \beta \in \mathbb{C}$;
(b) $W(A^*) = \left\{\overline{\lambda} : \lambda \in W(A)\right\}$, where A^* is the adjoint operator of A;
(c) $W(U^*AU) = W(A)$ for any unitary operator U.

The most important classical fact about the geometry of the numerical range is that it is convex and its closure contains the spectrum of the operator. The usual operator norm of A, is defined by

$$\|A\| = \sup_{\|x\|=1} \|Ax\|, \quad \text{for all } x \in \mathscr{H},$$

where $\|x\| = \langle x, x\rangle^{\frac{1}{2}}$. It is well known that $\omega(\cdot)$ defines a norm on $\mathcal{B}(\mathscr{H})$ and that for every $A \in \mathcal{B}(\mathscr{H})$, we have

$$\frac{1}{2}\|A\| \leq \omega(A) \leq \|A\|. \tag{2.1}$$

Thus, the usual operator norm and the numerical radius norm are equivalent. See [7] for a discussion and further references.

Now we are in a position to state the main result of this section. The b-numerical range of $A \in \mathcal{B}(\mathscr{X})$, denoted by $W_b(A)$, is the subset of the complex numbers given by

$$W_b(A) = \{(Ax, x|b) : \|x, b\| \leq 1\}.$$

The b-numerical radius of $A \in \mathcal{B}(\mathscr{X})$, denoted by $\omega_b(A)$, is defined by

$$\omega_b(A) = \sup\{|(Ax, x|b)| : \|x, b\| \leq 1\}.$$

It is easy to see that, for any $(x, b) \in \mathscr{X} \times \langle b\rangle$, we have

$$|(Ax, x|b)| \leq \omega_b(A)\|x, b\|^2.$$

The b-numerical radius $\omega_b(A)$ of an operator A on $\mathscr{X}$ is a norm on $\mathcal{B}(\mathscr{X})$, this norm is equivalent to the b-operator norm. In order to get our main result, we need the following lemmas:

Lemma 2.1 ([1]). *Let $A \in \mathcal{B}(\mathscr{X})$, then*

$$\begin{aligned} 4(Ax, y|z) = &(A(x+y), x+y|z) - (A(x-y), x-y|z) \\ &+ i(A(x+iy), x+iy|z) - i(A(x-iy), x-iy|z), \end{aligned}$$

for any $x, y, z \in \mathscr{X}$.

Lemma 2.2 ([4]). *For every $x, y \in \mathscr{X}$, we have*

$$\|x+y, b\|^2 + \|x-y, b\|^2 = 2\left(\|x, b\|^2 + \|y, b\|^2\right).$$

We shall, however, present another result, which is a possible generalization of (2.1).

Proposition 2.1. *For each $A \in \mathcal{B}(\mathscr{X})$, we get*

$$\frac{1}{2}\|A\|_b \le \omega_b(A) \le \|A\|_b.$$

Proof. If $\lambda = (Ax, x|b)$ with $\|x, b\| \le 1$, by Schwartz inequality we obtain

$$|\lambda| \le |(Ax, x|b)| \le \|Ax, b\| \, \|x, b\| \le \|A\|_b.$$

On the other hand, by Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned} 4|(Ax, y|b)| &\le \omega_b(A)\left[\|x+y, b\|^2 + \|x-y, b\|^2 + \|x+iy, b\|^2 + \|x-iy, b\|^2\right] \\ &= 2\omega_b(A)\left[\|x, b\|^2 + \|y, b\|^2 + \|x, b\|^2 + \|iy, b\|^2\right] \\ &\le 8\omega_b(A). \end{aligned}$$

By taking supremum over $\|x, b\| = \|y, b\| = 1$, we deduce the desired result. □

Theorem 2.1. *Let $A, B \in \mathcal{B}(\mathscr{X})$ and $AB = BA$, then*

$$\omega_b(AB) \le 2\omega_b(A)\,\omega_b(B).$$

Proof. We may assume $\omega_b(A) = \omega_b(B) = 1$ and show that $\omega_b(AB) \le 2$. By the triangle inequality, the power inequality theorem, and the subadditivity of $\omega(\cdot)$, we have

$$\begin{aligned} \omega_b(AB) &\equiv \omega_b\left(\frac{1}{4}\left[(A+B)^2 - (A-B)^2\right]\right) \\ &\le \frac{1}{4}\omega_b\left[(A+B)^2 - (A-B)^2\right] \\ &\le \frac{1}{4}\left[(\omega_b(A+B))^2 + (\omega_2(A-B))^2\right] \\ &\le \frac{1}{4}\left[(\omega_b(A) + \omega_b(B))^2 + (\omega_b(A) + \omega_b(B))^2\right] \\ &= 2, \end{aligned}$$

as desired. □

The following simple result provides a connection between the numerical radius and b-numerical radius as follows:

Theorem 2.2. *Let $A \in \mathcal{B}(\mathscr{X})$, then*

$$\omega(A) \leq \omega_b(A) + \|A\|_b', \tag{2.2}$$

where

$$\|A\|_b' = \sup\{|(Ax, x|b)| : \|x, b\| \leq 1\},$$

and $b \in \mathscr{X}$ is a fixed element.

Proof. We observe that

$$\begin{aligned} |(Ax, x|b)| &= \left|(Ax, x)\|b\|^2 - (Ax, b)(b, x)\right| \quad \text{(by (1.1))} \\ &\geq |(Ax, x)|\|b\|^2 - |(Ax, b)||(b, x)|. \end{aligned}$$

By taking supremum over $\|x, b\| \leq 1$ we deduce the desired result (2.2). □

The following inequalities may be stated as well.

Theorem 2.3. *Let $A \in \mathcal{B}(\mathscr{X})$ be a bounded linear operator on the linear 2-normed space $\mathscr{X}$. If $\lambda \in \mathbb{C}\setminus\{0\}$ and $\alpha > 0$ are such that*

$$\|A - \lambda I\|_b \leq \alpha, \tag{2.3}$$

where I is the identity operator on $\mathscr{X}$, then

$$\|A\|_b - \omega_b(A) \leq \frac{1}{2}\frac{\alpha^2}{|\lambda|}. \tag{2.4}$$

Proof. For $(x, b) \in \mathscr{X}, \langle b\rangle$ with $\|x, b\| = 1$, we have from (2.3) that

$$\|(A - \lambda)x, b\| \leq \|A - \lambda I\|_b \leq \alpha,$$

giving

$$\begin{aligned} \|Ax, b\|^2 + |\lambda|^2 &\leq 2\operatorname{Re}\left[\overline{\lambda}(Ax, x|b)\right] + \alpha^2 \\ &\leq 2|\lambda|(Ax, x|b) + \alpha^2. \end{aligned} \tag{2.5}$$

Taking supremum over $(x, b) \in \mathscr{X}, \langle b\rangle$, with $\|x, b\| = 1$ we get the following inequality

$$\|A\|_b^2 + |\lambda|^2 \leq 2\omega_b(A)|\lambda| + \alpha^2. \tag{2.6}$$

Since

$$2\|A\|_b|\lambda| \leq \|A\|_b^2 + |\lambda|^2, \tag{2.7}$$

hence by (2.6) and (2.7) we deduce the desired inequality (2.4). □

Corollary 2.1. *In particular, if $\|A - \lambda I\|_b \leq \alpha$ and $|\lambda| = \omega_b(A)$, $\lambda \in \mathbb{C}$, then*

$$\|A\|_b - \omega_b^2(A) \leq \alpha^2.$$

Proposition 2.2. *Let $A \in \mathcal{B}(\mathscr{X})$ be a non zero bounded linear operator on the linear 2-normed space $\mathscr{X}$ and $\lambda \in \mathbb{C}\backslash\{0\}$ and $\alpha > 0$ with $|\lambda| > \alpha$. If*

$$\|A - \lambda I\|_b \le \alpha,$$

then

$$\sqrt{1 - \frac{\alpha^2}{|\lambda|^2}} \le \frac{\omega_b(A)}{\|A\|_b}. \tag{2.8}$$

Proof. From (2.6) of Theorem 2.3, we have

$$\|A\|_b^2 + |\lambda|^2 - \alpha^2 \le 2|\lambda|\,\omega_b(A),$$

which implies, on dividing with $\sqrt{|\lambda|^2 - \alpha^2} > 0$ that

$$\frac{\|A\|_b^2}{\sqrt{|\lambda|^2 - \alpha^2}} + \sqrt{|\lambda|^2 - \alpha^2} \le \frac{2|\lambda|\,\omega_b(A)}{\sqrt{|\lambda|^2 - \alpha^2}}. \tag{2.9}$$

Whence

$$2\|A\|_b \le \frac{\|A\|_b^2}{\sqrt{|\lambda|^2 - \alpha^2}} + \sqrt{|\lambda|^2 - \alpha^2},$$

and by (2.9) we deduce

$$\|A\|_b \le \frac{\omega_b(A)\,|\lambda|}{\sqrt{|\lambda|^2 - \alpha^2}},$$

which is equivalent to (2.8). □

Corollary 2.2. *Squaring* (2.8)*, we get the inequality*

$$\|A\|_b^2 - \omega_b^2(A) \le \frac{\alpha^2}{|\lambda|^2}\|A\|_b^2.$$

Corollary 2.3. *Let $A \in \mathcal{B}(\mathscr{X})$ be a bounded linear operator on the linear 2-normed space and $\lambda \in \mathbb{C}\backslash\{0\}$ and $\alpha > 0$ with $|\lambda| > \alpha$ then $-\frac{\sqrt{3}}{2} \le \frac{\alpha}{|\lambda|} \le \frac{\sqrt{3}}{2}$.*

Proof. From Proposition 2.1, we infer that $\frac{1}{2} \le \frac{\omega_b(A)}{\|A\|_b}$.

By (2.8) we have $\sqrt{1 - \frac{\alpha^2}{|\lambda|^2}} \le \frac{\omega_b(A)}{\|A\|_b}$. Combining the above two inequalities one can obtain $\sqrt{1 - \frac{\alpha^2}{|\lambda|^2}} \ge \frac{1}{2}$ implies $\left(\frac{\alpha}{|\lambda|}\right)^2 \le \frac{3}{4}$, which implies $-\frac{\sqrt{3}}{2} \le \frac{\alpha}{|\lambda|} \le \frac{\sqrt{3}}{2}$. □

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PARACONTACT METRIC $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-HARMONIC MANIFOLDS

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Abstract. We give classifications of paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ manifolds M^{2n+1} with harmonic curvature for $n > 1$ and $n = 1$.

1. Introduction

Paracontact metric structures were introduced in [5], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ have been studied by many authors in the recent years, particularly since the appearance of [10]. An important class among paracontact metric manifolds is that of the $(\tilde{\kappa}, \tilde{\mu})$-spaces, which satisfy the nullity condition (see [4])

$$\tilde{R}(X,Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y), \tag{1.1}$$

for all X, Y vector fields on M, where $\tilde{\kappa}$ and $\tilde{\mu}$ are constants and $\tilde{h} = \frac{1}{2}\mathcal{L}_\xi \tilde{\varphi}$.

This class includes the para-Sasakian manifolds (see [5, 10]), the paracontact metric manifolds satisfying $\tilde{R}(X,Y)\xi = 0$, for all X, Y (see [11]), etc.

In [4], the authors showed that while the values of $\tilde{\kappa}$ and $\tilde{\mu}$ change the form of (1.1) remains unchanged under $\mathcal{D}$-homothetic deformations. There are differences between a contact metric (κ, μ)-space $(M^{2n+1}, \varphi, \xi, \eta, g)$ and a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-space $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$. Namely, unlike in the contact Riemannian case, a paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} = -1$ in general is not para-Sasakian. In fact, there are paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds such that $\tilde{h}^2 = 0$ (which is equivalent to take $\tilde{\kappa} = -1$) but with $\tilde{h} \neq 0$. For 5-dimensional, Cappelletti Montano and Di Terlizzi gave the first example of paracontact metric $(-1, 2)$-space $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ with $\tilde{h}^2 = 0$ but

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$\tilde{h} \neq 0$ in [3] and then Cappelletti Montano et al. gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary n in [4]. Later, for 3-dimensional, the first numerical example was given in [6]. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ, μ)-spaces the constant κ can not be greater than 1, paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-space has no restriction for the constants $\tilde{\kappa}$ and $\tilde{\mu}$.

Contact metric R-harmonic manifolds were studied in [1], [9]. But no effort has been made for paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds. Hence, in this paper, we give some characterizations for paracontact $(\tilde{\kappa}, \tilde{\mu})$ R-harmonic manifolds, i.e, for paracontact metric manifolds whose characteristic vector ξ belongs to the $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution and whose curvature tensor $\tilde{R}$ satisfies the condition $(div\tilde{R})(X, Y, Z) = 0$.

The outline of the article goes as follows. In Section 2, we recall basic facts which we will need throughout the paper. In Section 3, we deal with some results related with paracontact metric manifolds with characteristic vector field ξ belongs to the $(\tilde{\kappa}, \tilde{\mu})$-nullity distribution. Section 4 is devoted to paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ R-Harmonic manifolds. For such manifolds, our first result is that a paracontact metric R-harmonic manifold $M^{2n+1}$where $n > 1$, for which the characteristic vector field ξ belongs to the $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution is either locally product of a flat $(n + 1)$-dimensional manifold and n-dimensional of negative constant curvature equal to -4, or Ricci operator of the manifold has the form $\tilde{Q} = (n^2+n+2)I+(3n+1)\tilde{h}-(3n^2+7n+2)\eta\otimes\xi$ with $\tilde{\kappa} \leq -5$, or the manifold is an Einstein manifold. Our second result is that a paracontact metric R-harmonic manifold M^3, for which the characteristic vector field ξ belongs to the $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution is either flat, or Ricci operator of the manifold has the form $\tilde{Q} = 4I + 4\tilde{h} - 12\eta \otimes \xi$ with $\tilde{\kappa} = -4$.

2. Preliminaries

In this section we collect the formulas and results we need on paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [5], [10] and references therein for more information about paracontact metric geometry.

An $(2n + 1)$-dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1, 1)$-tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\tilde{\varphi}^2 = I - \eta \otimes \xi$;

(ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e., the ± 1-eigendistributions, $\mathcal{D}^{\pm} = \mathcal{D}_{\tilde{\varphi}}(\pm 1)$ of $\tilde{\varphi}$ have equal dimension n.

From the definition it follows that $\tilde{\varphi}\xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the endomorphism $\tilde{\varphi}$ has rank $2n$. We denote by $[\tilde{\varphi}, \tilde{\varphi}]$ the Nijenhius torsion

$$[\tilde{\varphi}, \tilde{\varphi}](X, Y) = \tilde{\varphi}^2[X, Y] + [\tilde{\varphi}X, \tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X, Y] - \tilde{\varphi}[X, \tilde{\varphi}Y].$$

When the tensor field $N_{\tilde\varphi} = [\tilde\varphi,\tilde\varphi] - 2d\eta\otimes\xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde g$ such that

$$\tilde g(\tilde\varphi X,\tilde\varphi Y) = -\tilde g(X,Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M,\tilde\varphi,\xi,\eta,\tilde g)$ is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1,n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1,\dots,X_n,Y_1,\dots,Y_n,\xi\}$, such that $\tilde g(X_i,X_j) = \delta_{ij}$, $\tilde g(Y_i,Y_j) = -\delta_{ij}$, $\tilde g(X_i,Y_j) = 0$, $\tilde g(\xi,X_i) = \tilde g(\xi,Y_j) = 0$, and $Y_i = \tilde\varphi X_i$, for any $i,j\in\{1,\dots,n\}$. Such basis is called a $\tilde\varphi$-basis.

We can now define the *fundamental form* of the almost paracontact metric manifold by $F(X,Y) = \tilde g(X,\tilde\varphi Y)$. If $d\eta(X,Y) = \tilde g(X,\tilde\varphi Y)$, then $(M,\tilde\varphi,\xi,\eta,\tilde g)$ is said to be *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde h = \frac{1}{2}\mathcal{L}_\xi\tilde\varphi$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [10] that $\tilde h$ anti-commutes with $\tilde\varphi$ and satisfies $\tilde h\xi = 0$, tr$\tilde h$ =tr$\tilde h\tilde\varphi = 0$ and

$$\tilde\nabla\xi = -\tilde\varphi + \tilde\varphi\tilde h, \tag{2.1}$$

where $\tilde\nabla$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M,\tilde g)$. Let $\tilde R$ be Riemannian curvature operator

$$\tilde R(X,Y)Z = (\tilde\nabla^2_{X,Y}Z) - (\tilde\nabla^2_{Y,X}Z) = [\tilde\nabla_X,\tilde\nabla_Y]Z - \tilde\nabla_{[X,Y]}Z.$$

Moreover $\tilde h = 0$ if and only if ξ is Killing vector field. In this case $(M,\tilde\varphi,\xi,\eta,\tilde g)$ is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$\tilde R(X,Y)\xi = -(\eta(Y)X - \eta(X)Y).$$

3. Paracontact Metric $(\tilde\kappa,\tilde\mu)$-Manifolds

In this section we recall several notions and results which will be needed throughout the paper.

Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a paracontact manifold. The $(\tilde\kappa,\tilde\mu)$-nullity distribution of a $(M,\tilde\varphi,\xi,\eta,\tilde g)$ for the pair $(\tilde\kappa,\tilde\mu)$ is a distribution

$$\begin{aligned} N(\tilde\kappa,\tilde\mu) : p \to N_p(\tilde\kappa,\tilde\mu) = \{Z\in T_pM \mid \tilde R(X,Y)Z &= \tilde\kappa(\tilde g(Y,Z)X - \tilde g(X,Z)Y)\\ &+ \tilde\mu(\tilde g(Y,Z)\tilde hX - \tilde g(X,Z)\tilde hY)\}, \end{aligned}$$

for some real constants $\tilde\kappa$ and $\tilde\mu$. If the characteristic vector field ξ belongs to the $(\tilde\kappa,\tilde\mu)$-nullity distribution we have (1.1). [4] is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (the condition (1.1), for some real numbers $\tilde\kappa$ and $\tilde\mu$).

Lemma 3.1 ([4]). *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-manifold of dimension $2n+1$. Then the following identity holds:*

$$
\begin{aligned}
(\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X = & -(1+\tilde{\kappa})(2\tilde{g}(X, \tilde{\varphi}Y)\xi + \eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) \\
& + (1-\tilde{\mu})(\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X),
\end{aligned} \tag{3.1}
$$

for any vector fields X, Y on M.

Lemma 3.2 ([4]). *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} \neq -1$. Then the operator $\tilde{h}$ in the case $\tilde{\kappa} > -1$ and the operator $\tilde{\varphi}\tilde{h}$ in the case $\tilde{\kappa} < -1$ are diagonalizable and admit three eigenvalues: 0, associated to the eigenvector ξ, $\tilde{\lambda}$ and -$\tilde{\lambda}$, of multiplicity n, where $\tilde{\lambda} := \sqrt{|1+\tilde{\kappa}|}$. The corresponding eigendistributions $\mathcal{D}_{\tilde{h}}(0) = \mathbb{R}\xi$, $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ and $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(0) = \mathbb{R}\xi$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ are mutually orthogonal and one has $\tilde{\varphi}\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_{\tilde{h}}(\mp\tilde{\lambda})$ and $\tilde{\varphi}\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) = \mathcal{D}_{\tilde{\varphi}\tilde{h}}(\mp\tilde{\lambda})$. Furthermore,*

$$
\mathcal{D}_{\tilde{h}}(\pm\tilde{\lambda}) = \left\{ X \pm \frac{1}{\sqrt{1+\tilde{\kappa}}}\tilde{h}X \mid X \in \Gamma(\mathcal{D}^{\mp}) \right\},
$$

in the case $\tilde{\kappa} > -1$, and

$$
\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\pm\tilde{\lambda}) = \left\{ X \pm \frac{1}{\sqrt{-1-\tilde{\kappa}}}\tilde{\varphi}\tilde{h}X \mid X \in \Gamma(\mathcal{D}^{\mp}) \right\},
$$

in the case $\tilde{\kappa} < -1$, where $\mathcal{D}^+$ and $\mathcal{D}^-$ denote the eigendistributions of $\tilde{\varphi}$ corresponding to the eigenvalues 1 and -1, respectively. Finally any two among the four distributions $\mathcal{D}^+$, $\mathcal{D}^-$, $\mathcal{D}_{\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{h}}(-\tilde{\lambda})$ in the case $\tilde{\kappa} > -1$ or $\mathcal{D}^+$, $\mathcal{D}^-$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(\tilde{\lambda})$, $\mathcal{D}_{\tilde{\varphi}\tilde{h}}(-\tilde{\lambda})$ in the case $\tilde{\kappa} < -1$ are mutually transversal.

Theorem 3.1 ([4]). *Any positive or negative definite paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} < -1$ carries a canonical contact Riemannian structure (ϕ, ξ, η, g) given by*

$$
\phi := \pm\frac{1}{\sqrt{-1-\tilde{\kappa}}}\tilde{h}, \qquad g := -d\eta(\cdot, \phi\cdot) + \eta \otimes \eta,
$$

where the sign $\pm$ depends on the positive or negative definiteness of the paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold. Moreover, (ϕ, ξ, η, g) is a contact metric (κ, μ)-structure, where

$$
\kappa = \tilde{\kappa} + 2 - \left(1 - \frac{\tilde{\mu}}{2}\right)^2, \quad \mu = 2.
$$

Lemma 3.3 ([4]). *In any $(2n+1)$-dimensional paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ such that $\tilde{\kappa} \neq -1$, the Ricci operator $\tilde{Q}$ is given by*

$$
\tilde{Q} = (2(1-n) + n\tilde{\mu})I + (2(n-1) + \tilde{\mu})\tilde{h} + (2(n-1) + n(2\tilde{\kappa} - \tilde{\mu}))\eta \otimes \xi. \tag{3.2}
$$

Lemma 3.4. *Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} \neq -1$. Then the following identity holds:*

$$
\begin{aligned}
(\tilde{\nabla}_X \tilde{S})(Y, Z) = & \, [2(n-1) + \tilde{\mu}] \, (\tilde{g}((\tilde{\nabla}_X \tilde{h})Y, Z) \\
& + [2(n-1) + n(2\tilde{\kappa} - \tilde{\mu})] \, (\tilde{g}(\tilde{\nabla}_X \xi, Y)\eta(Z) + \tilde{g}(Z, \tilde{\nabla}_X \xi)\eta(Y)),
\end{aligned} \tag{3.3}
$$

for any vector fields X, Y, Z on M.

Proof. Differentiating $\tilde{S}$ covariantly with respect to X, we have

$$(\tilde{\nabla}_X\tilde{S})(Y,Z)=\tilde{\nabla}_X\tilde{S}(Y,Z)-\tilde{S}(\tilde{\nabla}_XY,Z)-\tilde{S}(Y,\tilde{\nabla}_XZ). \tag{3.4}$$

By means of $\tilde{S}(Y,Z)=\tilde{g}(\tilde{Q}Y,Z)$ and (3.2), we find

$$\begin{aligned}\tilde{\nabla}_X\tilde{S}(Y,Z)=&(2(1-n)+n\tilde{\mu})(\tilde{g}(\tilde{\nabla}_XY,Z)+\tilde{g}(Y,\tilde{\nabla}_XZ))\\&+(2(n-1)+\tilde{\mu})(\tilde{g}(\tilde{\nabla}_X\tilde{h}Y,Z)+\tilde{g}(\tilde{h}Y,\tilde{\nabla}_XZ))\\&+(2(n-1)+n(2\tilde{\kappa}-\tilde{\mu}))(\tilde{g}(\tilde{\nabla}_XY,\xi)+\tilde{g}(Y,\tilde{\nabla}_X\xi))\eta(Z)\\&+(2(n-1)+n(2\tilde{\kappa}-\tilde{\mu}))(\tilde{g}(\tilde{\nabla}_XZ,\xi)+\tilde{g}(Z,\tilde{\nabla}_X\xi))\eta(Y).\end{aligned} \tag{3.5}$$

Taking into account again (3.2), we get

$$\begin{aligned}-\tilde{S}(\tilde{\nabla}_XY,Z)=&-(2(1-n)+n\tilde{\mu})\tilde{g}(\tilde{\nabla}_XY,Z)\\&-(2(n-1)+\tilde{\mu})(\tilde{g}(\tilde{h}\tilde{\nabla}_XY,Z)\\&-(2(n-1)+n(2\tilde{\kappa}-\tilde{\mu}))\eta(\tilde{\nabla}_XY)\eta(Z)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}-\tilde{S}(Y,\tilde{\nabla}_XZ)=&-(2(1-n)+n\tilde{\mu})\tilde{g}(\tilde{\nabla}_XZ,Y)\\&-(2(n-1)+\tilde{\mu})\tilde{g}(\tilde{h}Y,\tilde{\nabla}_XZ)\\&-(2(n-1)+n(2\tilde{\kappa}-\tilde{\mu}))\eta(\tilde{\nabla}_XZ)\eta(Y).\end{aligned} \tag{3.7}$$

Using (3.5)-(3.7) in (3.4), we obtain the requested equation. □

4. Paracontact Metric $(\tilde{\kappa},\tilde{\mu})$ $\tilde{R}$-Harmonic Manifolds

In this section, we will investigate harmonicity of the curvature tensor of a pseudo-Riemannian manifold. It is well known that, if the divergence of the curvature tensor of a pseudo-Riemannian manifold is equal to zero, then this curvature tensor is called harmonic.

Proposition 4.1. *Let $\tilde{R}$ be a curvature tensor field which satisfies the second Bianchi identity. If $\tilde{S}$ is the associated Ricci tensor field, then*

$$(\mathrm{div}\tilde{R})(X,Y,Z)=(\tilde{\nabla}_X\tilde{S})(Y,Z)-(\tilde{\nabla}_Y\tilde{S})(X,Z).$$

Definition 4.1 ([7])**.** A curvature tensor field $\tilde{R}$ is *harmonic if*

$$(\mathrm{div}\tilde{R})(X,Y,Z)=0.$$

A pseudo-Riemannian manifold M is said to be *$\tilde{R}$-harmonic* if its curvature tensor field $\tilde{R}$ is harmonic. Following [8], a pseudo- Riemannian manifold has harmonic curvature tensor if and only if the Ricci operator Q, which is given by $\tilde{S}(X,Y)=\tilde{g}(\tilde{Q}X,Y)$ where S is the Ricci tensor, satisfies

$$(\tilde{\nabla}_X\tilde{Q})Y-(\tilde{\nabla}_Y\tilde{Q})X=0, \tag{4.1}$$

for any vector fields X, Y on M .

Theorem 4.1 ([11]). *Let M^{2n+1} be a paracontact metric manifold and suppose that $\tilde{R}(X,Y)\xi = 0$ for all vector fields X and Y. Then locally M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4.*

Theorem 4.2. *Let M^{2n+1} be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-harmonic manifold where $n > 1$. If $\tilde{\kappa} \neq -1$, then M is either*

i) *locally product of a flat $(n+1)$-dimensional manifold and n-dimensional of negative constant curvature equal to -4, or*
ii) *the Ricci operator of the manifold has the form*
$$\tilde{Q} = (n^2 + n + 2)I + (3n+1)\tilde{h} - (3n^2 + 7n + 2)\eta \otimes \xi,$$
with $\tilde{\kappa} \leq -5$, or
iii) *M is an Einstein manifold.*

Proof. Using (3.3) and (4.1), we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{Q})Y - (\tilde{\nabla}_Y \tilde{Q})X = &[2(n-1) + \tilde{\mu}]\,((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X) \\ &+ [2(n-1) + n(2\tilde{\kappa} - \tilde{\mu})]\,(\tilde{g}(\tilde{\nabla}_X \xi, Y)\xi + \eta(Y)\tilde{\nabla}_X \xi \\ &- \tilde{g}(\tilde{\nabla}_Y \xi, X)\xi - \eta(X)\tilde{\nabla}_Y \xi).\end{aligned} \tag{4.2}$$

With the help of (3.1) and *$\tilde{R}$-harmonic manifold definition*, (4.2) returns to

$$\begin{aligned}(\tilde{\nabla}_X \tilde{Q})Y - (\tilde{\nabla}_Y \tilde{Q})X = &[2(n-1) + \tilde{\mu}]\,[-(1+\tilde{\kappa})(2\tilde{g}(X, \tilde{\varphi}Y)\xi + \eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) \\ &+ (1-\tilde{\mu})(\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X)] \\ &+ [2(n-1) + n(2\tilde{\kappa} - \tilde{\mu})]\,[\tilde{g}(\tilde{\nabla}_X \xi, Y)\xi + \eta(Y)\tilde{\nabla}_X \xi \\ &- \tilde{g}(\tilde{\nabla}_Y \xi, X)\xi - \eta(X)\tilde{\nabla}_Y \xi] \\ =&0\end{aligned} \tag{4.3}$$

If we take the inner product of (4.3) with ξ and use (2.1), one can easily show that

$$0 = 2\tilde{g}(X, \tilde{\varphi}Y)\,[\tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1)]\,.$$

Taking into account that $\tilde{g}(X, \tilde{\varphi}Y) = d\eta(X,Y) \neq 0$, we can conclude that

$$\tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1) = 0. \tag{4.4}$$

Replacing X by ξ in (4.3), by direct computations we get

$$[\tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1)]\,\tilde{\varphi}Y + [-2n\tilde{\kappa} + \tilde{\mu}(3-n-\tilde{\mu})]\,\tilde{\varphi}\tilde{h}Y = 0.$$

In virtue of (4.4), we have

$$[-2n\tilde{\kappa} + \tilde{\mu}(3-n-\tilde{\mu})]\,\tilde{\varphi}\tilde{h}Y = 0. \tag{4.5}$$

From the last equation, precisely following cases occurs

$$\begin{aligned} \tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1) =& 0 \quad \text{and} \quad -2n\tilde{\kappa} + \tilde{\mu}(3-n-\tilde{\mu}) = 0, \\ \tilde{\varphi}\tilde{h}Y =& 0. \end{aligned} \tag{4.6}$$

We now check, case by case, whether (4.5) give rise to a local classifcation.

First of all, solving the system of (4.6), we have following possibilities:

(i) $\tilde{\kappa} = \tilde{\mu} = 0$;
(ii) $\tilde{\kappa} = -(n+3) = -\tilde{\mu}$;
(iii) $\tilde{\kappa} = \frac{(1-n)(1+n)}{n}$, $\tilde{\mu} = 2 - 2n$.

If the first (i) equality holds, then using Theorem 4.1, we conclude that M is locally product of a flat $(n+1)$-dimensional manifold and n-dimensional of negative constant curvature equal to -4. If the second (ii) equality holds, then we can deduce that the Ricci operator of the manifold has the form $\tilde{Q} = (n^2+n+2)I + (3n+1)\tilde{h} - (3n^2 + 7n+2)\eta \otimes \xi$ with $\tilde{\kappa} \leq -5$.

If the third (iii) equality holds, using (3.2), we obtain M is an Einstein manifold.

Secondly, suppose $\tilde{\varphi}\tilde{h}Y = 0$. By (2.1), we have $\tilde{\nabla}_Y \xi = -\tilde{\varphi}Y$ which means that M is K-paracontact and hence $\tilde{h} = 0$. Using the fact that $h^2 = (1+\tilde{\kappa})\tilde{\varphi}^2$, we obtain $\tilde{\kappa} = -1$. But this contradicts with the chosen of $\tilde{\kappa}$. So, we omit this case. □

Using the same method for the proof, we can give following result.

Theorem 4.3. *Let M^3 be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-harmonic manifold. If $\tilde{\kappa} \neq -1$, then M is either*

i) *flat, or*
ii) *the Ricci operator of the manifold has the form $\tilde{Q} = 4I + 4\tilde{h} - 12\eta \otimes \xi$ with $\tilde{\kappa} = -4$.*

Remark 4.1. Using Theorem 3.1 and Theorem 4.2, we can say that if M^{2n+1} be a paracontact metric $\tilde{R}$*-harmonic manifold with* ξ belonging to $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution ,then M^{2n+1} carries a canonical contact metric (κ, μ)-structure where either $\kappa = 1$, $\mu = 2$ or $\kappa = \frac{-n^2-6n-5}{4}$, $\mu = 2$ or $\kappa = \frac{1-n^2+2n-n^3}{n}$, $\mu = 2$.

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ON THE NORMALIZED LAPLACIAN SPECTRUM OF SOME GRAPHS

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Abstract. In this paper we determine the normalized Laplacian spectrum of duplication vertex join of two graphs, duplication graph, splitting graph and double graph of a regular graph. Here we investigate some graph invariants like the normalized Laplacian energy, Kemeny's constant and number of spanning tree of these graphs.

1. Introduction

All graphs explained in this paper are undirected, without parallel edges and loops. Let $G = G(V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The adjacency matrix, $A(G) = (a_{ij})_{n\times n}$, is an $n \times n$ symmetric matrix with rows and columns are indexed by vertices of G where $a_{ij} = 1$ if the vertices v_i and v_j are adjacent in G, 0 elsewhere. The characteristic polynomial of A is of the form $f_G(A : x) = det(xI_n - A)$ where I_n is the identity matrix of order n. The roots of $f_G(A : x) = 0$ constitute the eigenvalues of G. We denote these as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and form the A - spectrum of G.

Let d_i be the degree of the vertex v_i in G and $D(G) = \operatorname{diag}(d_1, d_2, \dots, d_n)$ be the diagonal degree matrix of G. The matrix $D^{-1/2}$ is a diagonal matrix with diagonal entries $\frac{1}{\sqrt{d_i}}$ for all i. *Chung* in [5] introduced a new matrix called, *normalized Laplacian matrix* of a graph G. It is defined to be the matrix $\tilde{L}(G) = D^{-1/2}LD^{-1/2}$, whose

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$(i,j)^{th}$ - entry is given by,

$$\tilde{L}_{ij} = \begin{cases} 1, & \text{if } v_i = v_j \text{ and } d_i \neq 0, \\ \dfrac{-1}{\sqrt{d_i d_j}}, & \text{if } v_i \text{ and } v_j \text{ are adjacent}, \\ 0 & \text{otherwise}. \end{cases}$$

The roots of the characteristic equation of $\tilde{L}$ are known as the normalized Laplacian eigenvalues of G. Since $\tilde{L}(G)$ is symmetric and positive semi definite matrix, its eigenvalues are all real and non negative of the form $0 = \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$. These eigenvalues together their multiplicities is called normalized Laplacian spectrum or $\tilde{L}$-spectrum of G and is denoted by $\tilde{\text{L}}\text{Spec}(G)$.

The mathematicians like *Chen* and *Zhang* express the resistance distance in terms of normalized Laplacian eigenvalues and vectors of the graph G [4]. Also they propose degree-Kirchhoff index is closely related to spectrum of the normalized Laplacian. The concept of limit point for the normalized Laplacian eigenvalues are used by *Kirkkland* in [9]. In [1] *Banergee* and *Jost* investigated, how the normalized spectrum is affected by some operations like mofit doubling, graph splitting or joining. *Renny* and *Susha* defined some new join and corona based on duplication graph of an arbitrary graph (see [13,14]).

Motivated by these, in this paper we are interested in finding the normalized Laplacian spectrum of duplication, splitting and double graph of a regular graph G. Also we define and determine the normalized Laplacian spectrum of Duplication vertex join of two regular graphs G_1 and G_2.

The arrangement of the paper in section wise as follows. Section 2 describes the necessary preliminaries. In Section 3, we determine the normalized Laplacian spectrum of duplication vertex join of two graphs, duplication, splitting, double graph of a regular graph. Then in the last section we discuss some applications such as normalized Laplacian energy, the Kemeny's constant and number of spanning tree of these graphs.

2. Preliminaries

Definition 2.1 ([8,11,12]). Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $U(G) = \{u_1, u_2, \ldots, u_n\}$ be the vertex set of another copy of G. The *double graph*, $D_2(G)$, is the graph obtained by joining u_i to every vertices in $\mathcal{N}(v_i)$, the neighbourhood set of v_i of G, for each i. If we remove the edges of the copy of G in vertex set $U(G)$ in the double graph we get the *splitting graph*, $\text{splt}(G)$, of G. Removing the edges of two copies of G in the double graph, then it is called the *duplication graph*, $\mathcal{DG}$, of G.

Lemma 2.1 ([6]). *Let* $M = \begin{bmatrix} M_1 & M_2 \\ M_2 & M_1 \end{bmatrix}$ *be a symmetric block matrix of order* 2×2. *Then the eigenvalues of* M *are those of* $M_1 + M_2$ *together with* $M_1 - M_2$.

Proposition 2.1 ([6]). *Let P_0, P_1, P_2 and P_3 be matrices of order $n_1 \times n_1, n_1 \times n_2, n_2 \times n_1, n_2 \times n_2$ respectively. Then*

$$\det \begin{bmatrix} P_0 & P_1 \\ P_2 & P_3 \end{bmatrix} = \begin{cases} \det(P_0)\det(P_3 - P_2P_0^{-1}P_1), & \text{if } P_0 \text{ is invertible} \\ \det(P_3)\det(P_0 - P_1P_3^{-1}P_2), & \text{if } P_3 \text{ is invertible.} \end{cases}$$

Remark 2.1. Let G be a r-regular graph with adjacency matrix A. Then normalized Laplacian matrix is $I - \frac{A}{r}$ [5].

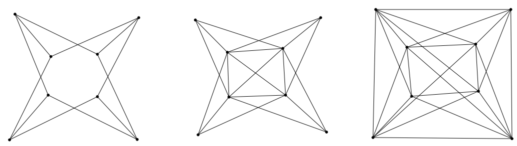

FIGURE 1. Duplication, splitting and double graph of K_4

3. Normalized Laplacian Spectrum of Some Graphs

In this section we determine the normalized Laplacian spectrum of duplication vertex join of two graphs, duplication, double and splitting graph of a regular graph.

3.1. **Normalized Laplacian spectrum of duplication vertex join.**

Definition 3.1. For $i = 1, 2$, let G_i be graphs on n_i vertices. Let $\mathcal{DG}_1$ be the duplication graph of G_1. The duplication vertex join of G_1 and G_2 is denoted by $G_1 \overline{\nabla} G_2$ and is the graph obtained from $\mathcal{DG}_1$ and G_2, by joining every vertex of G_1 to all the vertices of G_2.

Example 3.1. The following, Figure 2 illustrate the Definition 3.1.

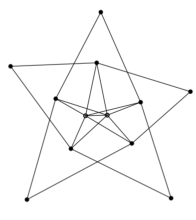

FIGURE 2. Duplication vertex join of C_5 and K_2.

Let G_i, $i = 1, 2$, be r_i-regular graphs on n_i vertices and m_i edges. Then $G_1 \overline{\nabla} G_2$ has $2n_1 + n_2$ vertices and $2m_1 + m_2 + n_1n_2$ edges.

Theorem 3.1. *For $i = 1, 2$, let G_i be r_i-regular graphs on n_i vertices with spectrum $\lambda_{i1}(G) \geq \lambda_{i2}(G) \geq \cdots \geq \lambda_{in_i}(G)$. Then the normalized Laplacian spectrum of $G_1 \overline{\nabla} G_2$ is* $0, 1 - \frac{\lambda_{2k}}{n_1+r_2}, 1 \pm \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}$, $i = 2, 3, \dots, n_1$, $k = 2, 3, \dots, n_2$. *Together with the roots of the equation*

$$x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2}x + \frac{2n_1n_2 + 2n_1r_1 + n_2r_2}{(n_1 + r_2)(n_2 + r_1)} = 0.$$

Proof. Let G_i, $i = 1, 2$, be r_i-regular graphs on n_i vertices. Let $V(G_1) = \{v_1, v_2, \dots, v_{n_1}\}$ be the vertex set of G_1 and $U(G_1) = \{x_1, x_2, \dots, x_{n_1}\}$ is the additional vertices corresponding to each vertex of G_1. Let $V(G_2) = \{u_1, u_2, \dots, u_{n_2}\}$ be the vertex set of G_2.

Under this vertex partitioning the adjacency matrix of $G_1 \overline{\nabla} G_2$ is,

$$A = \begin{bmatrix} 0_{n_1} & A_1 & J_{n_1 \times n_2} \\ A_1 & 0_{n_1} & 0_{n_1 \times n_2} \\ J_{n_2 \times n_1} & 0_{n_2 \times n_1} & A_2 \end{bmatrix},$$

where A_1 and A_2 are the adjacency matrix of G_1 and G_2 respectively. J denote matrix with all entries equal to 1 and 0 is the zero matrix of appropriate order. The degree of the vertices of $G_1 \overline{\nabla} G_2$ are $d_{G_1 \overline{\nabla} G_2}(v_i) = n_2 + r_1$, $d_{G_1 \overline{\nabla} G_2}(x_i) = r_1$, $i = 1, 2, \dots, n_1$ and $d_{G_1 \overline{\nabla} G_2}(u_j) = n_1 + r_2$, $j = 1, 2, \dots, n_2$.

The diagonal degree matrix of $G_1 \overline{\nabla} G_2$ is

$$D = \begin{bmatrix} (r_1 + n_2)I_{n_1} & 0 & 0 \\ 0 & r_1 I_{n_1} & 0 \\ 0 & 0 & (n_1 + r_2)I_{n_2} \end{bmatrix}.$$

Hence, the Laplace adjacency matrix of $G_1 \overline{\nabla} G_2$ is

$$L = \begin{bmatrix} (r_1 + n_2)I & -A_1 & -J_{n_1 \times n_2} \\ -A_1 & r_1 I & 0_{n_1 \times n_2} \\ -J_{n_2 \times n_1} & 0_{n_2 \times n_1} & n_1 I_{n_2} + L_2 \end{bmatrix},$$

where L_2 is the Laplacian matrix of G_2. Also,

$$D^{-1/2} = \begin{bmatrix} \frac{I_{n_1}}{\sqrt{r_1 + n_2}} & 0 & 0 \\ 0 & \frac{I_{n_1}}{\sqrt{r_1}} & 0 \\ 0 & 0 & \frac{I_{n_2}}{\sqrt{n_1 + r_2}} \end{bmatrix}.$$

By simple calculation we get

$$D^{-1/2}LD^{-1/2} = \tilde{L} = \begin{bmatrix} I_{n_1} & \dfrac{-A_1}{\sqrt{r_1(n_2+r_1)}} & \dfrac{-J_{n_1\times n_2}}{\sqrt{(n_1+r_2)(n_2+r_1)}} \\ \dfrac{-A_1}{\sqrt{r_1(n_2+r_1)}} & I_{n_1} & 0 \\ \dfrac{-J_{n_2\times n_1}}{\sqrt{(n_1+r_2)(n_2+r_1)}} & 0 & I_{n_2} - \dfrac{A_2}{n_1+r_2} \end{bmatrix}.$$

Since G_i is r_i-regular, it has an eigenvector $\mathbf{j}_{n_i}$, a vector with all entries equal to 1, corresponding to the eigenvalue r_i. All other eigenvectors are orthogonal to $\mathbf{j}_{n_i}$. Let λ_{2i} be an eigenvalue of G_2 with eigenvector Z such that $\mathbf{j}_{n_2}^T Z = 0$ Then $(0, 0, Z)^T$ is an eigenvector of $\tilde{L}$ corresponding to the eigenvalue $1 - \frac{\lambda_{2i}}{n_1+r_2}$.

This is because,

$$\tilde{L}\begin{pmatrix} 0 \\ 0 \\ Z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ Z - \frac{A_2 Z}{n_1+r_2} \end{pmatrix} = \left(1 - \frac{\lambda_{2i}}{n_1+r_2}\right)\begin{pmatrix} 0 \\ 0 \\ Z \end{pmatrix}.$$

Therefore, $1 - \frac{\lambda_{2i}}{n_1+r_2}$ for $i = 2, 3, \ldots, n_2$, is an eigenvalue corresponding to the eigenvector $(0, 0, Z)^T$.

Let X be an eigenvector corresponding to the eigenvalue λ_{1i} of G_1. Then $(X, X, 0)^T$ is an eigenvector corresponding to the eigenvalue $1 - \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}$. For,

$$\tilde{L}\begin{pmatrix} X \\ X \\ 0 \end{pmatrix} = \begin{pmatrix} X - \dfrac{A_1 X}{\sqrt{(r_1(n_2+r_1)}} \\ \dfrac{-A_1 X}{\sqrt{(r_1(n_2+r_1)}} + X \\ 0 \end{pmatrix} = \left(1 - \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}\right)\begin{pmatrix} X \\ X \\ 0 \end{pmatrix}.$$

Therefore, $1 - \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}$ for $i = 2, 3, \ldots, n_1$, is an eigenvalue corresponding to the eigenvector $(X, X, 0)^T$. Similarly we can prove $(-X, X, 0)^T$ is an eigenvector corresponding to the eigenvalue $1 + \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}$ for $i = 2, 3, \ldots, n_1$.

Thus we obtain $n_2 - 1 + 2(n_1 - 1) = 2n_1 + n_2 - 3$ eigenvalues of $\tilde{L}$ all orthogonal to $(\mathbf{j}, 0, 0)^T, (0, \mathbf{j}, 0)^T$ and $(0, 0, \mathbf{j})^T$.

The remaining three vectors of $\tilde{L}$ are of the form $\tau = (\alpha\mathbf{j}, \beta\mathbf{j}, \gamma\mathbf{j})^T$ for $(\alpha, \beta, \gamma) \neq (0, 0, 0)$. Let v be an eigenvalue of $\tilde{L}$ with eigenvector τ. Then from $\tilde{L}\tau = v\tau$ we get,

$$\alpha - \frac{r_1}{\sqrt{r_1(n_2+r_1)}}\beta - \frac{n_2}{\sqrt{(n_1+r_2)(n_2+r_1)}}\gamma = v\alpha, \tag{3.1}$$

$$-\frac{r_1}{\sqrt{r_1(n_2+r_1)}}\alpha + \beta + 0\gamma = v\beta, \tag{3.2}$$

$$-\frac{n_1}{\sqrt{(n_1+r_2)(n_2+r_1)}}\alpha + 0\beta + (1 - \frac{r_2}{n_1+r_2}\gamma = v\gamma. \tag{3.3}$$

By solving above three equations we get the cubic equation as,

$$x^3 - \frac{3n_1+2r_2}{n_1+r_2}x^2 + \frac{2n_1n_2+2n_1r_1+n_2r_2}{(n_1+r_2)(n_2+r_1)}x = 0. \tag{3.4}$$

Now the theorem follows. □

Corollary 3.1. *If $G_2 \cong \overline{K}_{n_2}$ (totally disconnected graph with n_2 vertices), then the normalized Laplacian of $G_1 \overline{\nabla} G_2$ consists of $0, 2, \alpha_i$ and β_i together with 1, repeats n_2 times, where $\alpha_i = 1 - \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}$, $\beta_i = 1 + \frac{\lambda_{1i}}{\sqrt{r_1(n_2+r_1)}}$, $i = 2, 3, \dots, n_1$.*

Proof. If G_2 is totally disconnected or $\overline{K}_{n_2}$ then $r_2 = 0$. The cubic equation (3.4) reduces to

$$x^3 - 3x^2 + 2x = 0.$$

On solving we get the solution as $x = 0, 1, 2$. The remaining eigenvalues are obtained from Theorem 3.1. Hence the corollary is proved. □

3.2. Normalized Laplacian spectrum of duplication, splitting and double graph.

Theorem 3.2. *Let G be a r-regular graph on n vertices with adjacency spectrum $\{r = \lambda_1, \lambda_2, \dots, \lambda_n\}$. Then the normalized Laplacian spectrum of the duplication graph, $\mathcal{DG}$, consists of $1 \pm \frac{\lambda_i}{r}$ for $i = 1, 2, \dots, n$.*

Proof. Let A be the adjacency matrix of G. The Laplacian and normalized Laplacian matrix of $\mathcal{DG}$ are

$$L = \begin{bmatrix} rI_n & -A \\ -A & rI_n \end{bmatrix} \quad \text{and} \quad \tilde{L} = \begin{bmatrix} I_n & \dfrac{-A}{r} \\ \dfrac{-A}{r} & I_n \end{bmatrix}.$$

Since G is r-regular with n vertices, the duplication graph $\mathcal{DG}$ is also an r-regular graph on $2n$ vertices with eigenvalues $\pm\lambda_i$, $i = 1, 2, \dots, n$. By Remark 2.1, the normalized Laplacian eigenvalues of $\mathcal{DG}$ are $1 \pm \frac{\lambda_i}{r}$, $i = 1, 2, \dots, n$. □

Theorem 3.3. *Let G be an r-regular graph on n vertices with adjacency spectrum $\{r = \lambda_1, \lambda_2, \dots, \lambda_n\}$. Then the normalized Laplacian spectrum of the splitting graph, $\operatorname{splt}(G)$, consists of $1 - \frac{\lambda_i}{r}$, $1 + \frac{\lambda_i}{2r}$ for $i = 1, 2, \dots, n$.*

Proof. Let A and D be respectively the adjacency matrix and diagonal degree matrix of G. The Laplacian matrix of splt(G) is $L = \begin{bmatrix} 2rI_n - A & -A \\ -A & rI_n \end{bmatrix}$.

Also $D = \begin{bmatrix} 2rI_n & 0 \\ 0 & rI_n \end{bmatrix}$ and $D^{-1/2} = \begin{bmatrix} \frac{I_n}{\sqrt{2r}} & 0 \\ 0 & \frac{I_n}{\sqrt{r}} \end{bmatrix}$.

The normalized Laplacian matrix is

$$\tilde{L} = D^{-1/2}LD^{-1/2} = \begin{bmatrix} I_n - \dfrac{A}{2r} & \dfrac{-A}{r\sqrt{2}} \\ \dfrac{-A}{r\sqrt{2}} & I_n \end{bmatrix}.$$

The characteristic polynomial of $\tilde{L}$ is

$$\det(xI - \tilde{L}) = \det \begin{bmatrix} (x-1)I_n + \dfrac{A}{2r} & \dfrac{A}{r\sqrt{2}} \\ \dfrac{A}{r\sqrt{2}} & (x-1)I_n \end{bmatrix}.$$

Using Proposition 2.1 and the result [6] that, if λ_i is an eigenvalue of A then $P(\lambda_i)$ is an eigenvalue of $P(A)$, for any polynomial $P(x)$. We arrive at

$$\begin{aligned} f_G(\tilde{L} : x) &= (x-1)^n \det\left((x-1)I_n + \frac{A}{2r} - \frac{A^2}{2r^2(x-1)}\right) \\ &= \det\left((x-1)^2 I_n + (x-1)\frac{A}{2r} - \frac{A^2}{2r^2}\right) \\ &= \prod_{i=1}^{n}\left((x-1)^2 + (x-1)\frac{\lambda_i}{2r} - \frac{\lambda_i^2}{2r^2}\right) \\ &= \prod_{i=1}^{n}\left(x^2 - (\frac{4r-\lambda_i}{2r})x + \frac{2r^2 - r\lambda_i - \lambda_i^2}{2r^2}\right) \\ &= \prod_{i=1}^{n}\left(x - 1 - \frac{\lambda_i}{2r}\right)\left(x - 1 + \frac{\lambda_i}{r}\right). \end{aligned}$$

Thus we obtain the normalized Laplacian spectrum. □

Theorem 3.4. *Let G be an r-regular graph on n vertices with adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Then the normalized Laplacian spectrum of the double graph, $D_2(G)$, consists of 1, repeats n times and $1 - \frac{\lambda_i}{r}$ for $i = 1, 2, \ldots, n$.*

Proof. Let A be the adjacency matrix of G. The Laplacian and normalized Laplacian matrix of $D_2(G)$ are

$$L = \begin{bmatrix} 2rI_n - A & -A \\ -A & 2rI_n - A \end{bmatrix} \quad \text{and} \quad \tilde{L} = \begin{bmatrix} I_n - \dfrac{A}{2r} & \dfrac{-A}{2r} \\ \dfrac{-A}{2r} & I_n - \dfrac{A}{2r} \end{bmatrix}.$$

As like the proof of the Theorem 3.2 and using Remark (2.1), we get the normalized Laplacian eigenvalues of $D_2(G)$. □

4. Applications

In this section we discuss some applications of normalized Laplacian spectrum. Here we determine the normalized Laplacian energy, Kemeny's constant and number of spanning tree of the different graphs under consideration.

4.1. **Normalized Laplacian energy.** In [10], I. Gutman defined the graph energy, $E(G)$, as the sum of the absolute value of its eigenvalues. Let G be a graph on n vertices with adjacency spectrum $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ then energy

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

Let G be a graph on n vertices and normalized Laplacian spectrum $0 = \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$. The normalized Laplacian energy is denoted by $\tilde{L}E(G)$ and is defined in [3] as

$$\tilde{L}E(G) = \sum_{i=1}^{n} |\sigma_i - 1|. \tag{4.1}$$

Theorem 4.1. *Let G_1 be an r_1 regular graph on n_1 vertices and $G_2 \cong \overline{K}_{n_2}$, totally disconnected graph. Then,*

$$\tilde{L}E(G_1 \nabla G_2) = 2 + \frac{2(E(G_1) - r_1)}{\sqrt{r_1(n_2 + r_1)}}.$$

Proof. We have $\lambda_1 = r_1$ and $E(G) = \sum_{i=1}^{n_1} |\lambda_{1i}| = r_1 + \sum_{i=2}^{n_1} |\lambda_{1i}|$. By Corollary 3.1 and (4.1) we get,

$$\begin{aligned} \tilde{L}E(G_1 \nabla G_2) &= n_2 \times 0 + 2 + \sum_{i=2}^{n_1} \frac{|\lambda_{1i}|}{\sqrt{r_1(n_2 + r_1)}} + \sum_{i=2}^{n_1} \frac{|-\lambda_{1i}|}{\sqrt{r_1(n_2 + r_1)}} \\ &= 2 + \frac{2}{\sqrt{r_1(n_2 + r_1)}} \sum_{i=2}^{n_1} |\lambda_{1i}| \\ &= 2 + \frac{2(E(G_1) - r_1)}{\sqrt{r_1(n_2 + r_1)}}. \end{aligned}$$ □

Theorem 4.2. *Let G be a r-regular graph with n vertices. Then*

(a) $\tilde{L}E(\mathcal{D}\mathcal{G}) = \frac{2}{r}E(G)$;
(b) $\tilde{L}E(D_2G) = \frac{1}{r}E(G)$;
(c) $\tilde{L}E(splt(G)) = \frac{3}{2r}E(G)$.

Proof. The proof follows from Theorem 3.2, Theorem 3.4 and Theorem 3.3. □

4.2. **Kemeny's constant.** *Kemeny's constant* $K(G)$, of a graph G is defined as the expected number of steps required for the transition from a starting vertex v_i called origin to a destination vertex, which is chosen randomly according to a stationary distribution of unbiased random walks on G [2, 7]. Also $K(G)$ is a constant and is independent of the choice of the origin v_i. Let G be a graph on n vertices and normalized Laplacian spectrum $0 = \sigma_1 \leq \sigma_2 \leq \cdots \leq \sigma_n$ then Kemeny's constant is the sum of all reciprocal normalized Laplacian eigenvalues except $1/\sigma_1$. Thus we can write,

$$K(G) = \sum_{i=2}^{n} \frac{1}{\sigma_i}. \tag{4.2}$$

Theorem 4.3. *For $i = 1, 2$, let G_i be r_i-regular graph on n_i vertices with adjacency spectrum $\{r_i = \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}\}$. Then the Kemeny's constant of $G_1 \overline{\nabla} G_2$ is*

$$K(G_1 \overline{\nabla} G_2) = \frac{(3n_1 + 2r_2)(n_2 + r_1)}{2n_1n_2 + 2n_1r_1 + n_2r_2} + \sum_{i=2}^{n_2} \frac{n_1 + r_2}{n_1 + r_2 - \lambda_{2i}} + \sum_{j=2}^{n_1} \frac{2r_1(n_2 + r_1)}{n_2r_1 + r_1^2 - \lambda_{1j}^2}.$$

Proof. Since for $i = 1, 2$, G_i is r_i-regular graph on n_i vertices and let η_1 and η_2 be the roots of the quadratic equation $x^2 - \frac{3n_1+2r_2}{n_1+r_2}x + \frac{2n_1n_2+2n_1r_1+n_2r_2}{(n_1+r_2)(n_2+r_1)} = 0$. Then

$$\begin{aligned}
\frac{1}{\eta_1} + \frac{1}{\eta_2} &= \frac{\eta_1 + \eta_2}{\eta_1\eta_2} \\
&= \frac{(3n_1 + 2r_2)(n_2 + r_1)}{2n_1n_2 + 2n_1r_1 + n_2r_2}, \\
K(G_1 \overline{\nabla} G_2) &= \sum_{i=2}^{n_2} \frac{n_1 + r_2}{n_1 + r_2 - \lambda_{2i}} + \frac{1}{\eta_1} + \frac{1}{\eta_2} \\
&\quad + \sum_{j=2}^{n_1} \left[\frac{\sqrt{r_1(n_2 + r_1)}}{\sqrt{r_1(n_2 + r_1)} + \lambda_{1j}} + \frac{\sqrt{r_1(n_2 + r_1)}}{\sqrt{r_1(n_2 + r_1)} - \lambda_{1j}} \right] \\
&= \frac{(3n_1 + 2r_2)(n_2 + r_1)}{2n_1n_2 + 2n_1r_1 + n_2r_2} + \sum_{i=2}^{n_2} \frac{n_1 + r_2}{n_1 + r_2 - \lambda_{2i}} \\
&\quad + \sqrt{r_1(n_2 + r_1)} \sum_{j=2}^{n_1} \frac{2\sqrt{r_1(n_2 + r_1)}}{r_1(n_2 + r_1) - \lambda_{1j}^2}.
\end{aligned}$$

On simplification we get the required result. ∎

Theorem 4.4. *Let G be an r-regular graph on n vertices with adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let $K(G)$ be the Kemeny's constant of G, then*

(1) $K(\mathcal{DG}) = K(G) + r \sum_{i=1}^{n} \frac{1}{r+\lambda_i}$;
(2) $K(splt(G)) = K(G) + 2r \sum_{i=1}^{n} \frac{1}{2r+\lambda_i}$;
(3) $K(D_2(G)) = K(G) + n$.

Proof. (1) Since G is r-regular with adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$, the normalized Laplacian spectrum consists of $1 - \frac{\lambda_i}{r}$, for $i = 1, 2, \ldots, n$. Therefore, $K(G) = \sum_{i=2}^{n}(1 - \frac{\lambda_i}{r})^{-1}$.

By Theorem 3.2 and (4.2) we get the Kemney's constant as

$$\begin{aligned} K(\mathcal{D}\mathcal{G}) &= \sum_{i=2}^{n}\left(1 - \frac{\lambda_i}{r}\right)^{-1} + \sum_{i=1}^{n}\left(1 + \frac{\lambda_i}{r}\right)^{-1} \\ &= K(G) + r\sum_{i=1}^{n}\frac{1}{r + \lambda_i}. \end{aligned}$$

The other results obtained from Theorem 3.4, Theorem 3.3 and (4.2). □

4.3. **Number of spanning tree.** Let $t(G)$ denote the number of spanning tree of the graph G, the total number of distinct spanning subgraphs of G that are trees. If G is a connected graph with n vertices and the normalized Laplacian spectrum $0 = \sigma_1(G) \leq \sigma_2(G) \cdots \leq \sigma_n(G)$ then the number of spanning tree (see [5])

$$t(G) = \frac{\prod_{i=1}^{n} d_i \ \prod_{i=2}^{n} \sigma_i}{\sum_{i=1}^{n} d_i}. \tag{4.3}$$

Theorem 4.5. *For $i = 1, 2$ let G_i be r_i-regular graph on n_i vertices with adjacency spectrum $\{r_i = \lambda_{i1}, \lambda_{i2}, \ldots, \lambda_{in_i}\}$. Then the number of spanning tree of $G_1 \overline{\nabla} G_2$ is*

$$t(G_1 \overline{\nabla} G_2) = r_1 \prod_{i=2}^{n_2}(n_1 + r_2 - \lambda_{2i}) \prod_{i=2}^{n_1}(n_2 r_1 + r_1^2 - \lambda_{1i}^2).$$

Proof. Since for $i = 1, 2$, G_i is a r_i-regular graph with n_i vertices, there are n_1 vertices of degree $n_2 + r_1$, another n_1 vertices are of degree r_1 and n_2 vertices are of degree $n_1 + r_2$.

Let η_1 and η_2 be the roots of the quadratic equation

$$x^2 - \frac{3n_1 + 2r_2}{n_1 + r_2}x + \frac{2n_1 n_2 + 2n_1 r_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)} = 0,$$

then we have

$$\begin{aligned} \eta_1\eta_2 &= \frac{2n_1 n_2 + 2n_1 r_1 + n_2 r_2}{(n_1 + r_2)(n_2 + r_1)}, \\ \sum d_i &= n_1(n_2 + r_1) + n_1 r_1 + n_2(n_1 + r_2) \\ &= 2n_1 n_2 + 2n_1 r_1 + n_2 r_2, \\ \prod d_i &= (n_2 + r_1)^{n_1} r_1^{n_1}(n_1 + r_2)^{n_2}. \end{aligned}$$

Hence, from (4.3), we get,

$$\begin{aligned} t(G_1 \overline{\nabla} G_2) &= \frac{(n_2 + r_1)^{n_1} r_1^{n_1}(n_1 + r_2)^{n_2}}{2n_1 n_2 + 2n_1 r_1 + n_2 r_2}\eta_1\eta_2 \prod_{i=2}^{n_2}\frac{n_1 + r_2 - \lambda_{2i}}{n_1 + r_2} \prod_{j=2}^{n_1}\frac{r_1(n_2 + r_1) - \lambda_{1j}^2}{r_1(n_2 + r_1)} \\ &= r_1 \prod_{i=2}^{n_2}(n_1 + r_2 - \lambda_{2i}) \prod_{i=2}^{n_1}(n_2 r_1 + r_1^2 - \lambda_{1i}^2). \end{aligned}$$

□

Theorem 4.6. *Let G be a r-regular graph on n vertices with adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$. Let $t(G)$ be the number of spanning tree of G then,*

(1) $t(\mathcal{DG}) = \frac{t(G)}{2} \prod_{i=1}^{n} (r + \lambda_i)$;
(2) $t(splt(G)) = \frac{t(G)}{3} \prod_{i=1}^{n} (2r + \lambda_i)$;
(3) $t(D_2(G)) = 2^{2n-2} r^n \, t(G)$.

Proof. (1) Since G is r-regular with adjacency spectrum $\{r = \lambda_1, \lambda_2, \ldots, \lambda_n\}$, the normalized Laplacian spectrum of $t(\mathcal{DG})$ consists of $1-\frac{\lambda_i}{r}$, for $i = 1, 2, \ldots, n$. Therefore $t(G) = \frac{1}{n} \prod_{i=2}^{n} (r - \lambda_i)$. Also $\prod_{i=2}^{n} d_i = r^{2n} and \sum_{i=1}^{n} d_i = 2nr$.

By Theorem 3.2 and (4.3) we get the

$$\begin{aligned} t(\mathcal{DG}) &= \frac{r^{2n} \ \prod_{i=2}^{n} \frac{r-\lambda_i}{r} \ \prod_{i=1}^{n} \frac{r+\lambda_i}{r}}{2nr} \\ &= \frac{t(G)}{2} \prod_{i=1}^{n} (r + \lambda_i). \end{aligned}$$

The other results follows from Theorem 3.4, Theorem 3.3 and (4.3). ☐

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CONVERGENCE OF DOUBLE COSINE SERIES

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Abstract. In this paper we consider double cosine series whose coefficients form a null sequence of bounded variation of order $(p,0)$, $(0,p)$ and (p,p) with the weight $(jk)^{p-1}$ for some $p>1$. We study pointwise convergence, uniform convergence and convergence in L^r-norm of the series under consideration. In a certain sense our results extend the results of Young [7], Kolmogorov [3] and Móricz [4, 5].

1. Introduction

Consider the double cosine series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k a_{jk}\cos jx\cos ky, \tag{1.1}$$

on positive quadrant $T=[0,\pi]\times[0,\pi]$ of the two dimensional torus where $\lambda_0=\frac{1}{2}$ and $\lambda_j=1$ for $j=1,2,3,\dots$.

The rectangular partial sums $S_{mn}(x,y)$ and the *Cesàro* means $\sigma_{mn}(x,y)$ of the series (1.1) are defined as

$$\begin{aligned} S_{mn}(x,y)=&\sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k a_{jk}\cos jx\cos ky,\\ \sigma_{mn}(x,y)=&\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}S_{jk}(x,y),\quad m,n>0, \end{aligned}$$

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and for $\lambda > 1$, the truncated Cesáro means are defined by

$$V_{mn}^{\lambda}(x,y) = \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} S_{jk}(x,y).$$

Now assuming the coefficients $\{a_{jk} : j,k \geq 0\}$ in (1.1) be a double sequence of real numbers which satisfy the following conditions for some positive integer p:

$$|a_{jk}|(jk)^{p-1} \to 0 \text{ as } \max\{j,k\} \to \infty, \tag{1.2}$$

$$\lim_{k\to\infty} \sum_{j=0}^{\infty} |\triangle_{p0} a_{jk}|(jk)^{p-1} = 0, \tag{1.3}$$

$$\lim_{j\to\infty} \sum_{k=0}^{\infty} |\triangle_{0p} a_{jk}|(jk)^{p-1} = 0, \tag{1.4}$$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{pp} a_{jk}|(jk)^{p-1} < \infty. \tag{1.5}$$

The finite order differences $\triangle_{pq} a_{jk}$ are defined by

$$\begin{aligned}
\triangle_{00} a_{jk} &= a_{jk}, \\
\triangle_{pq} a_{jk} &= \triangle_{p-1,q} a_{jk} - \triangle_{p-1,q} a_{j+1,k}, \quad p \geq 1, q \geq 0, \\
\triangle_{pq} a_{jk} &= \triangle_{p,q-1} a_{jk} - \triangle_{p,q-1} a_{j,k+1}, \quad p \geq 0, q \geq 1.
\end{aligned}$$

Also a double induction argument gives

$$\triangle_{pq} a_{jk} = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{s+t} \binom{p}{s} \binom{q}{t} a_{j+s,\, k+t}.$$

We can call the above mentioned conditions (1.2)-(1.5) as conditions of bounded variation of order $(p,0), (0,p)$ and (p,p) respectively with the weight $(jk)^{p-1}$. Obviously these conditions generalise the concept of monotone sequences. Also any sequence satisfying (1.5) with $p = 2$ is called a quasi-convex sequence [3,5]. Clearly the conditions (1.3) and (1.4) can be derived from (1.2) and (1.5) for $p = 1$ and moreover for $p = 1$, the conditions (1.2) and (1.5) reduce to $|a_{jk}| \to 0$ as $\max\{j,k\} \to \infty$ and

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{11} a_{jk}| < \infty.$$

Generally the pointwise convergence of the series (1.1) is defined in Pringsheim's sense ([8], Vol. 2, Ch. 17) which means that the rectangular partial sums of the type

$$S_{mn}(x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \cos jx \cos ky, \quad m,n \geq 0,$$

are formed and then by taking both m,n tend to ∞ (independently of one another) the limit $f(x,y)$ (provided it exists) is assigned to the series (1.1) as its sum.

Also let $\|f\|_r$ denotes the $L^r(T^2)$-norm, i.e,

$$\|f\|_r = \left(\int_0^\pi \int_0^\pi |f(x,y)|^r \, dxdy \right)^{1/r}, \quad 1 \le r < \infty$$

and $\|f\|$ denotes $L^1(T^2)$-norm, i.e,

$$\|f\| = \int_0^\pi \int_0^\pi |f(x,y)| \, dxdy.$$

In this paper, we will investigate the validity of the following statements:

(a) $S_{mn}(x,y)$ converges pointwise to $f(x,y)$ for every $(x,y) \in T^2$;
(b) $S_{mn}(x,y)$ converges uniformly to $f(x,y)$ on T^2;
(c) $\|S_{mn}(x,y) - f(x,y)\|_r = o(1)$ as $\min\{m,n\} \to 0$.

Such type of problems have been studied by Young [7] and Kolmogorov [3] for one-dimensional case (single trigonometric series especially cosine series) and by Móricz [4, 5] and K. Kaur, Bhatia and Ram [2] for double trigonometric series. In [5], Móricz studied both double cosine series and double sine series as far as their integrability and convergence in L^1-norm is concerned where as in [4] he studied double trigonometric series of the form

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_{jk} e^{i(jx+ky)},$$

under coefficients of bounded variation. All of them discussed the case for $p = 1$ or $p = 2$ only. Our aim in this paper is to extend the above results from $p = 1$ to general cases for double cosine series.

In the results, C_p and C_{pr} denote constants which may not be the same at each occurrence. Also we write $\lambda_n = [\lambda n]$ where n is a positive integer, $\lambda > 1$ is a real number and $[\cdot]$ means greatest integral part.

The first main result reads as follows.

Theorem 1.1. *Assume that conditions* (1.2)–(1.5) *are satisfied for some* $p \ge 1$, *then*

(i) $S_{mn}(x,y)$ *converges pointwise to* $f(x,y)$ *for every* $(x,y) \in T^2$ *such that* $x, y > 0$;
(ii) $\|S_{mn}(x,y) - f(x,y)\|_r = o(1)$ *as* $\min\{m,n\} \to \infty$, $1 \le r < \infty$.

The above theorem has been proved by Móricz [4, 5] for $p = 1$ and $p = 2$ using suitable estimates for Dirichlet's kernel $D_j(x)$ and Fejér kernel $K_j(x)$. In the case of a single series for $p = 2$, the results regarding convergence have been proved by Kolmogorov [3].

Obviously, condition (1.5) implies any of the following conditions:

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{j=0}^{\infty} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| (jk)^{p-1} = 0, \tag{1.6}$$

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m\to\infty}}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{\infty}\frac{\lambda_m-j+1}{\lambda_m-m}|\triangle_{pp}a_{jk}|(jk)^{p-1}=0. \tag{1.7}$$

We introduce the following three sums for $m,n\geq 0$ and $\lambda>1$:

$$\sum\nolimits_{10}^{\lambda}(m,n,x,y)=\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\,a_{jk}\cos jx\cos ky,$$

$$\sum\nolimits_{01}^{\lambda}(m,n,x,y)=\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\,a_{jk}\cos jx\cos ky,$$

$$\sum\nolimits_{11}^{\lambda}(m,n,x,y)=\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_m-j+1}{\lambda_m-m}\,\frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}\cos jx\cos ky$$

and we have

$$\sum\nolimits_{11}^{\lambda}(m,n;x,y)=\frac{1}{(\lambda_m-m)}\sum_{u=m+1}^{\lambda_m}\Big(\sum\nolimits_{01}^{\lambda}(u,n;x,y)-\sum\nolimits_{01}^{\lambda}(m,n;x,y)\Big),$$

$$\sum\nolimits_{11}^{\lambda}(m,n;x,y)=\frac{1}{(\lambda_n-n)}\sum_{v=n+1}^{\lambda_n}\Big(\sum\nolimits_{10}^{\lambda}(m,v;x,y)-\sum\nolimits_{10}^{\lambda}(m,n;x,y)\Big).$$

This implies

$$\sum\nolimits_{11}^{\lambda}(m,n;x,y)\leq\left\{\begin{array}{c}2\sup\limits_{m\leq u\leq\lambda_m}\Big(|\sum\nolimits_{01}^{\lambda}(u,n;x,y)|\Big)\\ 2\sup\limits_{n\leq v\leq\lambda_n}\Big(|\sum\nolimits_{10}^{\lambda}(m,v;x,y)|\Big)\end{array}\right\}. \tag{1.8}$$

The second result of this paper is the following theorem.

Theorem 1.2. (i) *Let $E\subset T^2$. Assume that the following conditions are satisfied:*

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|\sum\nolimits_{10}^{\lambda}(m,n;x,y)|\Big)=0, \tag{1.9}$$

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|\sum\nolimits_{01}^{\lambda}(m,n;x,y)|\Big)=0. \tag{1.10}$$

If $V_{mn}^{\lambda}(x,y)$ converges uniformly to $f(x,y)$ on $E\subset T^2$ as $\min\{m,n\}\to\infty$ (that is, in the unrestricted sense), then so does S_{mn}.

(ii) *Assume that the following conditions are satisfied for some $r\geq 1$:*

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\Big(\|\sum\nolimits_{10}^{\lambda}(m,n;x,y)\|_r\Big)=0,$$

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\Big(\|\sum\nolimits_{01}^{\lambda}(m,n;x,y)\|_r\Big)=0. \tag{1.11}$$

If $\|V_{mn}^{\lambda}-f\|_r\to 0$ unrestictedly then $\|S_{mn}-f\|_r\to 0$ as $\min\{m,n\}\to\infty$.

We will also prove the following theorem.

Theorem 1.3. *Assume that the conditions* (1.2)–(1.4) *and* (1.6)–(1.7) *are satisfied for some* $p \geq 1$*, then*

(i) *if* $V^{\lambda}_{mn}(x,y)$ *converges uniformly to* $f(x,y)$ *as* $\min\{m,n\} \to \infty$*, then so does* S_{mn}*;*

(ii) *if* $\|V^{\lambda}_{mn} - f\|_r \longrightarrow 0$ *unrestictedly for some* r *with* $1 \leq r < \infty$*, then* $\|S_{mn} - f\|_r \longrightarrow 0$ *as* $\min\{m,n\} \to \infty$*.*

2. Notation and Formulas

We define for every $\alpha = 0, 1, 2, \ldots$ the sequence $S^{\alpha}_0, S^{\alpha}_1, S^{\alpha}_2, \ldots$ by the conditions

$$S^0_n = S_n, \quad S^{\alpha}_n = \sum_{u=0}^{n} S^{\alpha-1}_u, \quad \alpha \geq 1$$

and

$$A^0_n = 1, \quad A^{\alpha}_n = \sum_{u=0}^{n} A^{\alpha-1}_u, \quad \alpha \geq 1,$$

denotes binomial coefficients. Also

$$A^{\alpha}_n = \binom{n+\alpha}{n} \simeq \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \quad \alpha \neq -1, -2, -3, \ldots.$$

The Cesàro means T^{α}_n of order α of $\sum a_n$ will be defined by $T^{\alpha}_n = \frac{S^{\alpha}_n}{A^{\alpha}_n}$ and also it is known [8] that $\int_0^{\pi} |T^{\alpha}_n(x)| dx$, $\alpha > 0$, is bounded for all n.

3. Lemmas

We require the following lemmas for the proof of our results.

Lemma 3.1. *For* $m, n \geq 0$ *and* $p > 1$*, the following representation holds:*

$$\begin{aligned} S_{mn}(x,y) =& \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \cos jx \cos ky \\ =& \sum_{j=0}^{m} \sum_{k=0}^{n} \triangle_{pp} a_{jk} S^{p-1}_j(x) S^{p-1}_k(y) + \sum_{j=0}^{m} \sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S^{p-1}_j(x) S^t_n(y) \\ &+ \sum_{k=0}^{n} \sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} S^s_m(x) S^{p-1}_k(y) + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S^s_m(x) S^t_n(y). \end{aligned}$$

Lemma 3.2 ([1])**.** *For* $m, n \geq 0$ *and* $\lambda > 1$*, the following representation holds:*

$$\begin{aligned} S_{mn} - \sigma_{mn} =& \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{\lambda_m, \lambda_n} - \sigma_{\lambda_m, n} - \sigma_{m, \lambda_n} + \sigma_{mn}) \\ &+ \frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m, n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) \\ &- \sum\nolimits_{11}^{\lambda}(m,n,x,y) - \sum\nolimits_{10}^{\lambda}(m,n,x,y) - \sum\nolimits_{01}^{\lambda}(m,n,x,y). \end{aligned}$$

Lemma 3.3. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$V_{mn}^{\lambda} - S_{mn} = \sum\nolimits_{11}^{\lambda}(m,n,x,y) + \sum\nolimits_{10}^{\lambda}(m,n,x,y) + \sum\nolimits_{01}^{\lambda}(m,n,x,y).$$

Proof. We have

$$V_{mn}^{\lambda}(x,y) = \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} S_{jk}(x,y).$$

Performing double summation by parts, we have

$$\begin{aligned}
V_{mn}^{\lambda} =& \frac{\lambda_m + 1}{\lambda_m - m} \, \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{\lambda_m, \lambda_n} - \frac{\lambda_m + 1}{\lambda_m - m} \, \frac{n+1}{\lambda_n - n} \sigma_{\lambda_m, n} \\
& - \frac{m+1}{\lambda_m - m} \, \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{m, \lambda_n} + \frac{m+1}{\lambda_m - m} \, \frac{n+1}{\lambda_n - n} \sigma_{mn} \\
=& \frac{\lambda_m + 1}{\lambda_m - m} \, \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{\lambda_m, \lambda_n} - \sigma_{\lambda_m, n} - \sigma_{m, \lambda_n} + \sigma_{mn}) \\
& + \frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m, n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \sigma_{mn}.
\end{aligned}$$

The use of Lemma 3.2, gives

$$\begin{aligned}
V_{mn}^{\lambda} - S_{mn} =& \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \, \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \cos jx \cos ky \\
& + \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} \, a_{jk} \cos jx \cos ky \\
& + \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \, a_{jk} \cos jx \cos ky.
\end{aligned}$$

□

Lemma 3.4. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$\begin{aligned}
\sum\nolimits_{10}^{\lambda}(m,n;x,y) =& \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} \, a_{jk} \cos jx \cos ky \\
=& \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} \, \triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y) \\
& + \sum_{j=m+1}^{\lambda_m} \sum_{t=0}^{p-1} \frac{\lambda_m - j + 1}{\lambda_m - m} \, \triangle_{pt} a_{j,n+1} S_j^{p-1}(x) S_n^{t}(y) \\
& + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \sum_{k=0}^{n} \triangle_{sp} a_{j+1,k} S_j^{s}(x) S_k^{p-1}(y) \\
& + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \triangle_{st} a_{j+1,n+1} S_j^{s}(x) S_n^{t}(y)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s=0}^{p-1}\sum_{k=0}^{n} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) \\
&-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_m^s(x) S_n^t(y).
\end{aligned}
$$

Proof. We have by summation by parts,

$$
\begin{aligned}
&\textstyle\sum_{10}^{\lambda}(m,n;x,y) \\
&=\sum_{k=0}^{n} \cos ky \left(\sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, a_{jk} \cos jx \right) \\
&=\sum_{k=0}^{n} \cos ky \left(\sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, \triangle_{p0} a_{jk} S_j^{p-1}(x) \right. \\
&\quad \left. + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \triangle_{s0} a_{j+1,k} S_j^s(x) - \sum_{s=0}^{p-1} \triangle_{s0} a_{m+1,k} S_m^s(x) \right) \\
&=\sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, S_j^{p-1}(x) \left(\sum_{k=0}^{n} \triangle_{p0} a_{jk} \cos ky \right) \\
&\quad + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{s0} a_{j+1,k} \cos ky \right) S_j^s(x) \\
&\quad - \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{s0} a_{m+1,k} \cos ky \right) S_m^s(x) \\
&=\sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, S_j^{p-1}(x) \left(\sum_{k=0}^{n} \triangle_{pp} a_{jk} S_k^{p-1}(y) + \sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S_n^t(y) \right) \\
&\quad + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{sp} a_{j+1,k} S_k^{p-1}(y) + \sum_{t=0}^{p-1} \triangle_{st} a_{j+1,n+1} S_n^t(y) \right) S_j^s(x) \\
&\quad - \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{sp} a_{m+1,k} S_k^{p-1}(y) + \sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_n^t(y) \right) S_m^s(x).
\end{aligned}
$$

Similarly we can have representation for $\sum_{01}^{\lambda}(m,n;x,y)$. □

4. Proof of Theorems

Proof of Theorem 1.1. For $m, n \geq 0$ and $p > 1$, we have from Lemma 3.1,

$$
\begin{aligned}
S_{mn}(x,y) =& \sum_{j=0}^{m}\sum_{k=0}^{n} \triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y) + \sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S_j^{p-1}(x) S_n^t(y) \\
&+ \sum_{k=0}^{n}\sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) + \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_m^s(x) S_n^t(y)
\end{aligned}
$$

$$=\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4.$$

Using the results as given in [6] that $S_j^p(x) = O\left(\frac{1}{x^p}\right)$, for all $p \geq 2$, $0 < x \leq \pi$, etc, we have for $0 < x, y \leq \pi$,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y)| < \infty \quad \text{(by (1.2))}$$

and also by (1.3)–(1.5), we have

$$\begin{aligned}\sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} &\leq \sum_{t=0}^{p-1}\sum_{v=0}^{t} \binom{t}{v} \left(\sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}|\right) \\ &\leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| \\ &\leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty.\end{aligned}$$

Thus,

$$\sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S_j^{p-1}(x) S_n^t(y) \to 0 \text{ as } \min\{m,n\} \to \infty.$$

And similarly

$$\begin{aligned}\sum_{s=0}^{p-1}\sum_{k=0}^{n} \triangle_{sp} a_{m+1,k} &\leq \sum_{s=0}^{p-1}\sum_{u=0}^{s} \binom{s}{u} (\sum_{k=0}^{n} |\triangle_{0p} a_{m+u+1,k}|) \\ &\leq \sup_{m<j\leq m+p} \sum_{k=0}^{n} |\triangle_{0p} a_{jk}| \\ &\leq \sup_{m<j\leq m+p} \sum_{k=0}^{n} |\triangle_{0p} a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty.\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}\sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) \to 0,$$

as $\min\{m,n\} \to \infty$. Also

$$\begin{aligned}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} &\leq \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t} \binom{s}{u}\binom{t}{v} |\triangle_{00} a_{m+u+1,n+v+1}| \\ &\leq \sup_{j>m,k>n} |a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty.\end{aligned}$$

So,

$$\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_m^s(x) S_n^t(y) \to 0 \text{ as } \min\{m,n\} \to \infty.$$

Consequently, series (1.1) converges to the function $f(x,y)$ where

$$f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\quad\text{and}\quad\lim_{m,n\to\infty}S_{mn}(x,y)=f(x,y).$$

Now we will calculate $\|\sum_1\|$, $\|\sum_2\|$, $\|\sum_3\|$ and $\|\sum_4\|$ in the following way:

$$\begin{aligned}
\left\|\sum\nolimits_1\right\| =&\left\|\sum_{j=0}^{m}\sum_{k=0}^{n}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right\|\\
\le&\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|\int_0^{\pi}\int_0^{\pi}|S_j^{p-1}(x)S_k^{p-1}(y)|dxdy\\
\le&\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|A_j^{p-1}A_k^{p-1}\int_0^{\pi}\int_0^{\pi}|T_j^{p-1}(x)T_k^{p-1}(y)|dxdy\\
\le&C_p\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1},\\
\left\|\sum\nolimits_2\right\| =&\left\|\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}S_j^{p-1}(x)S_n^{t}(y)\right\|\\
\le&\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\left(\sum_{j=0}^{m}|\triangle_{p0}a_{j,n+v+1}|\right)A_j^{p-1}A_n^{t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|T_j^{p-1}(x)T_n^{t}(y)|dxdy\\
\le&C_p\sup_{n<k\le n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\;j^{p-1}\left(\sum_{t=0}^{p-1}n^t\right)\\
\le&C_p\sup_{n<k\le n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\;j^{p-1}k^{p-1},\\
\left\|\sum\nolimits_3\right\| =&\left\|\sum_{s=0}^{p-1}\sum_{k=0}^{n}\triangle_{sp}a_{m+1,k}S_m^{s}(x)S_k^{p-1}(y)\right\|\\
\le&\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\left(\sum_{k=0}^{n}|\triangle_{0p}a_{m+u+1,k}|\right)A_m^{s}A_k^{p-1}\int_0^{\pi}\int_0^{\pi}|T_m^{s}(x)T_k^{p-1}(y)|dxdy\\
\le&C_p\sup_{m<j\le m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\;k^{p-1}\left(\sum_{s=0}^{p-1}m^s\right)\\
\le&C_p\sup_{m<j\le m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\;j^{p-1}k^{p-1},\\
\left\|\sum\nolimits_4\right\| =&\left\|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}S_m^{s}(x)S_n^{t}(y)\right\|
\end{aligned}$$

$$\leq \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\triangle_{00}a_{m+u+1,n+v+1}|A_m^s A_n^t \int_0^\pi\int_0^\pi |T_m^s(x)T_n^t(y)|dxdy$$
$$\leq C_p \sup_{j>m,k>n}|a_{jk}|\, j^{p-1}k^{p-1}.$$

Now let R_{mn} consists of all (j,k) with $j>m$ or $k>n$, that is,

$$\sum\sum\nolimits_{(j,k)\in R_{mn}} = \sum_{j=m+1}^{\infty}\sum_{k=0}^{n} + \sum_{j=0}^{\infty}\sum_{k=n+1}^{\infty} + \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}.$$

Then

$$\begin{aligned}
\|f-S_{mn}\|_r = &\left(\int_0^\pi\int_0^\pi |f(x,y)-S_{mn}(x,y)|^r\,dxdy\right)^{1/r}, \quad 1\le r<\infty,\\
\leq &\left\|\sum_{(j,k)}\sum_{\in R_{mn}} \triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right\|_r\\
&+\left\|\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}S_j^{p-1}(x)S_n^t(y)\right\|_r\\
&+\left\|\sum_{k=0}^{n}\sum_{s=0}^{p-1}\triangle_{sp}a_{m+1,k}S_m^s(x)S_k^{p-1}(y)\right\|_r\\
&+\left\|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}S_m^s(x)S_n^t(y)\right\|_r\\
\leq &C_{pr}\Bigg\{\left(\sum_{(j,k)\in R_{mn}}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1}\right)\\
&+\left(\sup_{n<k\le n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\, j^{p-1}k^{p-1}\right)\\
&+\left(\sup_{m<j\le m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\, j^{p-1}k^{p-1}\right)\\
&+\left(\sup_{j>m,k>n}|a_{jk}|\, j^{p-1}k^{p-1}\right)\Bigg\} \quad \text{(as discussed above)}\\
\to &0 \quad \text{as } \min\{m,n\}\to\infty \quad \text{(by (1.2)-(1.5))},
\end{aligned}$$

which proves (ii) part.

Proof of Theorem 1.2. Using the relation (1.8), we find that (1.9) or (1.10) implies

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}|\textstyle\sum_{11}^{\lambda}(m,n;x,y)|\right)=0. \tag{4.1}$$

Assume that $V_{mn}^{\lambda}(x,y)$ converges uniformly on E to $f(x,y)$. Then by Lemma 3.3, we get

$$\begin{aligned}\overline{\lim_{m,n\to\infty}}\left(\left|\sup_{(x,y)\in E}\Big(S_{mn}(x,y)-V_{mn}^{\lambda}(x,y)\Big)\right|\right)\leq{}&\overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits_{10}^{\lambda}(m,n;x,y)\right|\right)\\&+\overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits_{01}^{\lambda}(m,n;x,y)\right|\right)\\&+\overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits_{11}^{\lambda}(m,n;x,y)\right|\right).\end{aligned}$$

After taking $\lambda\downarrow 1$ the result follows from (1.9), (1.10) and (4.1).

For (ii) part of theorem, we have

$$\begin{aligned}\left\|\sum\nolimits_{11}^{\lambda}(m,n;x,y)\right\|_r&=\frac{1}{\lambda_m-m}\sum_{u=m+1}^{\lambda_m}\left(\left\|\sum\nolimits_{01}^{\lambda}(u,n;x,y)\right\|_r+\left\|\sum\nolimits_{01}^{\lambda}(m,n;x,y)\right\|_r\right)\\&\leq 2\left(\sup_{m\leq u\leq\lambda_m}\left(\left\|\sum\nolimits_{01}^{\lambda}(u,n;x,y)\right\|_r\right)\right).\end{aligned}$$

Thus (1.11) implies

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\left\|\sum\nolimits_{11}^{\lambda}(m,n;x,y)\right\|_r=0.$$

Thus, the result of Theorem 1.2 (ii) follows.

Proof of Theorem 1.3. Using the Lemma 3.4, we can write the expression for $\sum_{01}^{\lambda}(m,n;x,y)$ as

$$\begin{aligned}\sum\nolimits_{01}^{\lambda}(m,n;x,y)={}&\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\;a_{jk}\cos jx\cos ky\\={}&\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\;\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\\&+\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\frac{\lambda_n-k+1}{\lambda_n-n}\;\triangle_{sp}a_{m+1,k}S_m^{s}(x)S_k^{p-1}(y)\\&+\frac{1}{\lambda_n-n}\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,k+1}S_j^{p-1}(x)S_k^{t}(y)\\&+\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,k+1}S_m^{s}(x)S_k^{t}(y)\\&-\sum_{t=0}^{p-1}\sum_{j=0}^{m}\triangle_{pt}a_{j,n+1}S_j^{p-1}(x)S_n^{t}(y)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}S_m^s(x)S_n^t(y)\\
=&I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

Now by using (1.2)–(1.4) and (1.6) along with estimates of $S_j^{p-1}(x)$ etc., as mentioned in [6], we have the following estimates in brief:

$$
\begin{aligned}
|I_1| =&\left|\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right|\\
\le&\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\left|\triangle_{pp}a_{jk}\right|\\
\to&0 \quad \text{as } \min\{m,n\}\to\infty.
\end{aligned}
$$

Consequently, $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|I_1|\Big)\to 0$ as $\min\{m,n\}\to\infty$. Also,

$$
\begin{aligned}
|I_2| =&\left|\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{sp}a_{m+1,k}S_m^s(x)S_k^{p-1}(y)\right|\\
\le&\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\sum_{k=n+1}^{\lambda_n}|\triangle_{0p}a_{m+u+1,k}|\\
\le&\sup_{m<j\le m+p}\sum_{k=n+1}^{\lambda_n}|\triangle_{0p}a_{jk}| \to 0 \quad \text{as } \min\{m,n\}\to\infty.
\end{aligned}
$$

So, $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\left(\sup\limits_{(x,y)\in E}|I_2|\right)\to 0$ as $\min\{m,n\}\to\infty$. Also,

$$
\begin{aligned}
|I_3| \le&\sup_{n<k\le\lambda_n}\sum_{t=0}^{p-1}\sum_{j=0}^{m}|\triangle_{pt}a_{j,k+1}|\\
\le&\sup_{n<k\le\lambda_n}\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m}|\triangle_{pt}a_{j,k+v+1}|\\
\le&\sup_{n<k\le\lambda_n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\to 0 \text{ as } \min\{m,n\}\to\infty,
\end{aligned}
$$

which implies $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\left(\sup\limits_{(x,y)\in E}|I_3|\right)\to 0$ as $\min\{m,n\}\to\infty$. Now,

$$
\begin{aligned}
|I_4| \le&\sup_{n<k\le\lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}|\triangle_{st}a_{m+1,k+1}|\\
\le&\sup_{j>m,k>n}|a_{jk}|\to 0 \text{ as } \min\{m,n\}\to\infty.
\end{aligned}
$$

Thus $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\left(\sup\limits_{(x,y)\in E}|I_4|\right)\to 0$ as $\min\{m,n\}\to\infty$. Also,

$$|I_5|\leq\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m}|\triangle_{p0}a_{j,n+v+1}|\leq\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\to\ \ 0 \text{ as } \min\{m,n\}\to\infty,$$

which implies $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\left(\sup\limits_{(x,y)\in E}|I_5|\right)\to 0$ as $\min\{m,n\}\to\infty$. Also,

$$\begin{aligned}|I_6|&\leq\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\triangle_{00}a_{m+u+1,n+v+1}|\\&\leq\sup_{j>m,k>n}|a_{jk}|\to 0 \text{ as } \min\{m,n\}\to\infty,\end{aligned}$$

and

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}|I_6|\right)\to 0 \text{ as } \min\{m,n\}\to\infty.$$

Thus, combining all these, we have

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits_{01}^{\lambda}(m,n;x,y)\right|\right)=0.$$

Similarly (1.2)–(1.4) and (1.7) results in

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits_{10}^{\lambda}(m,n;x,y)\right|\right)=0.$$

Thus, first part of theorem follows from Theorem 1.2.

Proof of (ii). We have

$$\|S_{mn}-f\|_r\leq\|S_{mn}-V_{mn}^{\lambda}\|_r+\|V_{mn}^{\lambda}-f\|_r.$$

By assumption $\|V_{mn}^{\lambda}-f\|_r\to 0$, so it is sufficient to show that

$$\|S_{mn}-V_{mn}^{\lambda}\|_r\to 0 \text{ as } \min\{m,n\}\to\infty.$$

By Lemma 3.3, we have

$$\begin{aligned}\|S_{mn}-V_{mn}^{\lambda}\|_r\leq&\left\|\sum\nolimits_{10}^{\lambda}(m,n;x,y)\right\|_r+\left\|\sum\nolimits_{01}^{\lambda}(m,n;x,y)\right\|_r\\&+\left\|\sum\nolimits_{11}^{\lambda}(m,n;x,y)\right\|_r.\end{aligned}$$

Now in order to estimate $\|\sum_{01}^{\lambda}(m,n;x,y)\|_r$, we first find $\|I_1\|$, $\|I_2\|$, $\|I_3\|$, $\|I_4\|$, $\|I_5\|$ and $\|I_6\|$, so we have

$$\begin{aligned}\|I_1\|&=\left\|\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right\|\\&\leq\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{pp}a_{jk}A_j^{p-1}A_k^{p-1}\int_0^{\pi}\int_0^{\pi}|T_j^{p-1}(x)T_k^{p-1}(y)|dxdy\end{aligned}$$

$$
\begin{aligned}
&\leq C_p \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| j^{p-1} k^{p-1}, \\
\|I_2\| &= \left\| \sum_{k=n+1}^{\lambda_n} \sum_{s=0}^{p-1} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) \right\| \\
&\leq C_p \sum_{s=0}^{p-1} \sum_{u=0}^{s} \binom{s}{u} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{m+u+1,k}| k^{p-1} m^s \\
&\leq C_p \sup_{m<j\leq m+p} \left(\sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| k^{p-1} \right) \left(\sum_{s=0}^{p-1} m^s \right) \\
&\leq C_p \sup_{m<j\leq m+p} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| j^{p-1} k^{p-1}, \\
\|I_3\| &\leq C_p \sup_{n<k\leq \lambda_n} \sum_{t=0}^{p-1} \sum_{j=0}^{m} |\triangle_{pt} a_{j,k+1}| j^{p-1} k^t \\
&\leq C_p \sup_{n<k\leq \lambda_n} \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{j=0}^{m} |\triangle_{pt} a_{j,k+v+1}| j^{p-1} k^t \\
&\leq C_p \sup_{n<k\leq \lambda_n + p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1}, \\
\|I_4\| &\leq C_p \sup_{n<k\leq \lambda_n} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} |\triangle_{st} a_{m+1,k+1}| m^s k^t \\
&\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}, \\
\|I_5\| &\leq C_p \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}| j^{p-1} n^t \\
&\leq C_p \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1}, \\
\|I_6\| &\leq C_p \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{s} \sum_{v=0}^{t} \binom{s}{u} \binom{t}{v} |\triangle_{00} a_{m+u+1,n+v+1}| m^s n^t \\
&\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}.
\end{aligned}
$$

Thus, we can estimate

$$
\begin{aligned}
\left\| \sum\nolimits_{01}^{\lambda}(m,n;x,y) \right\|_r \leq & C_{pr} \sum_{k=n+1}^{\lambda_n} \sum_{j=0}^{m} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| j^{p-1} k^{p-1} \\
&+ C_{pr} \left(\sup_{m<j\leq m+p} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| j^{p-1} k^{p-1} \right)
\end{aligned}
$$

$$+ C_{pr}\left(\sup_{n<k\le\lambda_n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|j^{p-1}k^{p-1}\right)$$
$$+ C_{pr}\left(\sup_{j>m,k>n}|a_{jk}|j^{p-1}k^{p-1}\right)$$
$$+ C_{pr}\left(\sup_{n<k\le n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|j^{p-1}k^{p-1}\right)$$
$$+ C_{pr}\left(\sup_{j>m,k>n}|a_{jk}|j^{p-1}k^{p-1}\right).$$

By (1.2)–(1.4) and (1.6), we conclude that

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\left(\|{\textstyle\sum}_{01}^{\lambda}(m,n;x,y)\|_r\right)=0.$$

Similarly, by conditions (1.2)–(1.4) and (1.7), we get

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\left(\|{\textstyle\sum}_{10}^{\lambda}(m,n;x,y)\|_r\right)=0.$$

Also, by (1.8), we have

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\left(\|{\textstyle\sum}_{11}^{\lambda}(m,n;x,y)\|_r\right)=0.$$

Thus, $\|S_{mn}-V_{mn}^{\lambda}\|_r \to 0$ as $\min\{m,n\}\to\infty$.

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NECESSARY AND SUFFICIENT CONDITION FOR OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF SECOND-ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, necessary and sufficient conditions are obtained for oscillatory and asymptotic behaviour of solutions of second-order neutral delay differential equations of the form

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}[x(t)+p(t)x(\tau(t))]\right]+q(t)G\left(x(\sigma(t))\right)=0, \quad \text{for } t\geq t_0,$$

under the assumption $\int^{\infty}\frac{1}{r(\eta)}d\eta=\infty$ for various ranges of the bounded neutral coefficient p. Our main tools are Lebesgue's dominated convergence theorem and Banach's contraction mapping principle. Further, an illustrative example showing the applicability of the new results is included.

1. Introduction

Consider a class of nonlinear neutral delay differential equations of the form:

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\Big[x(t)+p(t)x(\tau(t))\Big]\right]+q(t)G\Big(x(\sigma(t))\Big)=0, \tag{1.1}$$

where

($A1$) $r,q,\tau,\sigma\in C(\mathbb{R}_+,\mathbb{R}_+)$, $p\in C(\mathbb{R}_+,\mathbb{R})$ such that $\tau(t)\leq t$, $\sigma(t)\leq t$ for $t\geq t_0$, $\tau(t)\to\infty$, $\sigma(t)\to\infty$ as $t\to\infty$, with invertible τ when necessary;

($A2$) $G\in C(\mathbb{R},\mathbb{R})$ is nondecreasing with satisfying the property $uG(u)>0$ for $u\neq 0$

and

($A3$) $R(t)=\int_0^t\frac{d\eta}{r(\eta)}\to+\infty$ as $t\to\infty$.

Key words and phrases. Oscillation, nonoscillation, neutral, delay, nonlinear, Lebesgue's dominated convergence theorem, Banach's contraction mapping principle.

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Baculikova *et al.* [3] have studied the linear counterpart of (1.1),

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}[x(t)+p(t)x(\tau(t))]\right]+q(t)x(\sigma(t))=0, \tag{1.2}$$

when $0 \le p(t) \le p_0 < \infty$ and $(A3)$ holds. The authors have obtained sufficient conditions for oscillation of solutions of (1.2) through some comparison results, where the comparison results are unpredictable. In [6], Džurina have studied (1.2) when $0 \le p(t) \le p_0 < \infty$ and $(A3)$ holds true. He has established sufficient condition for oscillation of solutions of (1.2) by comparison techniques. In [16], under various ranges of p, Santra studied oscillatory behaviour of the solutions of the following neutral differential equations

$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+q(t)G\left(x(t-\sigma)\right)=0$$

and

$$\frac{d}{dt}[x(t)+p(t)x(t-\tau)]+q(t)G\left(x(t-\sigma)\right)=f(t). \tag{1.3}$$

Also, sufficient conditions are obtained for existence of bounded positive solutions of (1.3). Tripathy *et al.* [18] have studied and obtained the sufficient conditions for oscillation, nonoscillation and asymptotic behavior of solutions of (1.1) provided G could be linear or nonlinear. The motivation of the present work come from the above studies. Hence, in this work, an attempt is made to study the more general form of (1.2) without making any comparison. It seems that this method is the next alternative to the works [3, 6] when p is bounded.

The neutral differential equations find numerous applications in natural sciences and technology. For instance, they are frequently used for the study of distributed networks containing lossless transmission lines (see, for e.g., [8]). In this paper, we restrict our attention to study (1.1), which includes a class of nonlinear functional differential equations of neutral type. In this direction we refer the reader to some of the works (see [1, 4, 5, 10, 13, 19, 20]) and the references cited therein.

By a solution to equation (1.1), we mean a function $x \in \mathrm{C}([T_x,\infty),\mathbb{R})$, where $T_x \ge t_0$, such that $rz' \in \mathrm{C}^1([T_x,\infty),\mathbb{R})$, where

$$z(t) := x(t)+p(t)x(\tau(t)), \quad \text{for } t \ge T_x, \tag{1.4}$$

and satisfies (1.1) on the interval $[T_x,\infty)$. A solution x of (1.1) is said to be proper if x is not identically zero eventually, i.e., $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. We assume that (1.1) possesses such solutions. A solution of (1.1) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise, it is said to be nonoscillatory. (1.1) itself is said to be oscillatory if all of its solutions are oscillatory.

Remark 1.1. When the domain is not specified explicitly, all functional inequalities considered in this paper are assumed to hold eventually, i.e., they are satisfied for all t large enough.

2. Main Results

In this section, necessary and sufficient conditions are obtained for oscillatory and asymptotic behaviour of solutions of second order nonlinear neutral differential equations of the form (1.1).

Lemma 2.1. *Assume that* $(A1)$-$(A3)$ *hold. If* x *is an eventually positive solution of* (1.1) *such that the companion function* z *defined by* (1.4) *is also eventually positive, then* z *satisfies*

$$z'(t) > 0 \quad \text{and} \quad (rz')'(t) < 0, \quad \text{for all large } t. \tag{2.1}$$

Proof. Suppose that $x(t) > 0$ and $z(t) > 0$ for $t \geq t_1$, where $t \geq t_0$. By $(A1)$, we may assume without loss of generality that $x\big(\sigma(t)\big) > 0$ for $t \geq t_1$. From (1.1) and $(A2)$, it follows that

$$(rz')'(t) = -q(t)G\left(x(\sigma(t))\right) < 0, \quad \text{for } t \geq t_1. \tag{2.2}$$

Consequently, rz' is nonincreasing on $[t_1, \infty)$ and thus either $z'(t) < 0$ or $z'(t) > 0$ for $t \geq t_2$, where $t_2 \geq t_1$. If $z'(t) < 0$, then there exists $\varepsilon > 0$ such that $r(t)z'(t) \leq -\varepsilon$ for $t \geq t_2$, which yields upon integration over $[t_2, t) \subset [t_2, \infty)$ after dividing through by r that

$$z(t) \leq z(t_2) - \varepsilon \int_{t_2}^{t} \frac{1}{r(\eta)} d\eta, \quad \text{for } t \geq t_2. \tag{2.3}$$

In view of $(A3)$, letting $t \to \infty$ in (2.3) yields $z(t) \to -\infty$, which is a contradiction. Therefore, $z'(t) > 0$ for $t \geq t_2$. This completes the proof. ☐

Remark 2.1. It follows from Lemma 2.1 that $\lim_{t\to\infty} z(t) > 0$, i.e., there exists $\varepsilon > 0$ such that $z(t) \geq \varepsilon$ for all large t.

Lemma 2.2. *Assume that* $(A1)$-$(A3)$ *hold. If* x *is an eventually positive solution of* (1.1) *such that the companion function* z *defined by* (1.4) *is bounded, then* z *satisfies* (2.1) *for all large* t.

Theorem 2.1. *Assume that* $(A1)$-$(A3)$ *hold and* $-1 < -a \leq p(t) \leq 0$, $a \geq 0$ *for* $t \in \mathbb{R}_+$. *Furthermore, assume that*

$(A4)$ *G is strictly sublinear, that is,* $\frac{G(u)}{u^\beta} \geq \frac{G(v)}{v^\beta}$, $0 < u \leq v$, $\beta < 1$,

holds. Then every unbounded solution of (1.1) *oscillates if and only if*

$(A5)$ $\int_T^\infty q(\eta)G\Big(\varepsilon R(\sigma(\eta))\Big)d\eta = +\infty$, $T > 0$ *for every* $\varepsilon > 0$.

Proof. Suppose the contrary that x is a nonoscillatory solution of (1.1). Then, there exists $t_1 \geq t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \geq t_1$. Assume that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $z(t) < 0$ for $t \geq t_2$. As x is unbounded, there exists $T \geq t_2$ such that $x(T) = \max\{x(\eta) : \ t_2 \leq \eta \leq T\}$. Then, from (1.4), we have $x(T) \leq z(T) + x(\tau(T)) < x(T)$, which is a contradiction.

Case 2. Let $z(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2.1) holds for $t \geq t_3$. Note that $\lim_{t\to\infty} r(t)z'(t)$ exists. Upon using $z(t) \leq x(t)$ in (2.2) and then integrating the final inequality from t to $+\infty$, we obtain

$$\int_t^\infty q(\eta)G\Big(z(\sigma(\eta))\Big)d\eta \leq r(t)z'(t),$$

that is,

$$z'(t) \geq \frac{1}{r(t)} \int_t^\infty q(\eta)G\Big(z(\sigma(\eta))\Big)d\eta, \tag{2.4}$$

for $t \geq t_3$. Let $t_4 > t_3$ be a point such that

$$R(t) - R(t_4) \geq \frac{1}{2}R(t), \quad t \geq t_4.$$

Then integrating (2.4) from t_4 to $t(> t_4)$, we get

$$\begin{aligned} z(t) - z(t_4) &\geq \int_{t_4}^t \frac{1}{r(\eta)} \int_\eta^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta d\eta \\ &\geq \int_{t_4}^t \frac{1}{r(\eta)} \int_t^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta d\eta, \end{aligned}$$

that is,

$$\begin{aligned} z(t) \geq& \Big(R(t) - R(t_4)\Big) \int_t^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta \\ \geq& \frac{1}{2}R(t) \int_t^\infty q(\zeta)G\Big(z(\sigma(\zeta))\Big)d\zeta, \quad t \geq t_4. \end{aligned} \tag{2.5}$$

Using the fact that $r(t)z'(t)$ is nonincreasing on $[t_4, \infty)$, we can find a constant $\varepsilon > 0$ and $t_5 > t_4$ such that $r(t)z'(t) \leq \varepsilon$ for $t \geq t_5$ and hence $z(t) \leq \varepsilon R(t)$, $t \geq t_5$. On the otherhand, $(A3)$ implies that

$$\begin{aligned} G\Big(z(\sigma(\zeta))\Big) &= \frac{G\Big(z(\sigma(\zeta))\Big)}{z^\beta\Big(\sigma(\zeta)\Big)} z^\beta\Big(\sigma(\zeta)\Big) \\ &\geq \frac{G\Big(\varepsilon R(\sigma(\zeta))\Big)}{\varepsilon^\beta R^\beta\Big(\sigma(\zeta)\Big)} z^\beta\Big(\sigma(\zeta)\Big). \end{aligned}$$

Consequently, (2.5) becomes

$$z(t) \geq \frac{R(t)}{2} \int_t^\infty \frac{q(\zeta)G\Big(\varepsilon R(\sigma(\zeta))\Big)z^\beta(\sigma(\zeta))}{\varepsilon^\beta R^\beta(\sigma(\zeta))} d\zeta,$$

for $t \geq t_5$. If we define

$$w(t) = \frac{1}{2}\int_t^\infty \frac{q(\zeta)G\Big(\varepsilon R(\sigma(\zeta))\Big)z^\beta(\sigma(\zeta))}{\varepsilon^\beta R^\beta(\sigma(\zeta))}d\zeta,$$

then $z(t) \geq R(t)w(t)$ for $t \geq t_5$. Now,

$$\begin{aligned} w'(t) \leq & -\frac{1}{2}\frac{q(t)G\Big(\varepsilon R(\sigma(t))\Big)z^\beta(\sigma(t))}{\varepsilon^\beta R^\beta(\sigma(t))} \\ \leq & -\frac{1}{2}\frac{q(t)G\Big(\varepsilon R(\sigma(t))\Big)}{\varepsilon^\beta}w^\beta(\sigma(t)) \leq 0, \quad t \geq t_5, \end{aligned}$$

implies that $w(t)$ is nonincreasing on $[t_5, \infty)$ and $\lim_{t\to\infty} w(t)$ exists. It is easy to verify that

$$\begin{aligned} \Big[w^{1-\beta}(t)\Big]' \leq & -\frac{(1-\beta)}{2}w^{-\beta}(t)\frac{q(t)G\Big(\varepsilon R(\sigma(t))\Big)}{\varepsilon^\beta}w^\beta\Big(\sigma(t)\Big) \\ \leq & -\frac{(1-\beta)}{2}w^{-\beta}(t)\frac{q(t)G\Big(\varepsilon R(\sigma(t))\Big)}{\varepsilon^\beta}w^\beta(t) \\ \leq & -\frac{(1-\beta)}{2\varepsilon^\beta}q(t)G\Big(\varepsilon R(\sigma(t))\Big), \end{aligned} \tag{2.6}$$

for $t \geq t_5$. Integrating (2.6) from t_5 to $t(> t_5)$, we obtain

$$\frac{(1-\beta)}{2\varepsilon^\beta}\int_{t^5}^t q(\eta)G\Big(\varepsilon R(\sigma(\eta))\Big)d\eta \quad \leq \quad -\Big[w^{1-\beta}(\eta)\Big]_{t_5}^t < w^{1-\beta}(t_5) < \infty,$$

a contradiction to $(A5)$.

If $x(t) < 0$ for $t \geq t_1$, then we set $y(t) := -x(t)$ for $t \geq t_1$ in (1.1). Using $(A2)$, we find

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}[y(t) + p(t)y(\tau(t))]\right] + q(t)H\left(y(\sigma(t))\right) = 0, \quad \text{for } t \geq t_1,$$

where $H(u) := -G(-u)$ for $u \in \mathbb{R}$. Clearly, H also satisfies $(A2)$. Then, proceeding as above, we find the same contradiction.

Next, we suppose that $(A5)$ does not hold. For $\varepsilon > 0$, let us assume that

$$\int_T^\infty q(\eta)G\Big(\varepsilon R(\sigma(\eta))\Big)d\eta \leq \frac{\varepsilon}{3}.$$

Consider

$$\begin{aligned} M = \Big\{ & x : x \in C([t_0, \infty), \mathbb{R}), x(t) = 0 \text{ for } t \in [t_0, T] \text{ and} \\ & \frac{\varepsilon}{3}[R(t) - R(T)] \leq x(t) \leq \varepsilon[R(t) - R(T)]\Big\}, \end{aligned}$$

and define

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T), & t \in [t_0, T], \\ -p(t)x\Big(\tau(t)\Big) + \int_T^t \frac{1}{r(\eta)} \left[\frac{\varepsilon}{3} + \int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right] d\eta, & t \geq T. \end{cases}$$

For every $x \in M$,

$$\begin{aligned}(\Phi x)(t) \geq & \int_T^t \frac{1}{r(\eta)} \left[\frac{\varepsilon}{3} + \int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right] d\eta \\ \geq & \frac{\varepsilon}{3} \int_T^t \frac{d\eta}{r(\eta)} = \frac{\varepsilon}{3}[R(t) - R(T)]\end{aligned}$$

and $x(t) \leq \varepsilon R(t)$ implies that

$$\begin{aligned}(\Phi x)(t) \leq & - p(t)x\Big(\tau(t)\Big) + \frac{2\varepsilon}{3} \int_T^t \frac{d\eta}{r(\eta)} \\ \leq & a\varepsilon\Big[R(\tau(t)) - R(T)\Big] + \frac{2\varepsilon}{3}\Big[R(t) - R(T)\Big] \\ \leq & a\varepsilon\Big[R(t) - R(T)\Big] + \frac{2\varepsilon}{3}\Big[R(t) - R(T)\Big] \\ = & \left(a + \frac{2}{3}\right)\varepsilon\Big[R(t) - R(T)\Big] \\ \leq & \varepsilon\Big[R(t) - R(T)\Big]\end{aligned}$$

implies that $(\Phi x)(t) \in M$. Define $u_n : [t_0, +\infty) \to \mathbb{R}$ by the recursive formula

$$u_n(t) = \Big(\Phi u_{n-1}\Big)(t), \quad n \geq 1,$$

with the initial condition

$$u_0(t) = \begin{cases} 0, & t \in [t_0, T], \\ \frac{\varepsilon}{3}[R(t) - R(T)], & t \geq T. \end{cases}$$

Inductively it is easy to verify that

$$\frac{\varepsilon}{3}\Big[R(t) - R(T)\Big] \leq u_{n-1}(t) \leq u_n(t) \leq \varepsilon\Big[R(t) - R(T)\Big],$$

for $t \geq T$. Therefore, for $t \geq t_0$, $\lim_{n\to\infty} u_n(t)$ exists. By the Lebesgue's dominated convergence theorem, $u \in M$ and $(\Phi u)(t) = u(t)$, where $u(t)$ is a solution of (1.1) such that $u(t) > 0$. Hence, $(A5)$ is necessary. This completes the proof of the theorem. ☐

Theorem 2.2. *Assume that* $(A1)$*-*$(A3)$ *hold and* $-1 < -a \leq p(t) \leq 0$*,* $a > 0$ *for* $t \in \mathbb{R}_+$*. Then every unbounded solution of* (1.1) *oscillates if and only if* $(A5)$ *holds for every* $\varepsilon > 0$*.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive unbounded solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x\Big(\tau(t)\Big) > 0$ and $x\Big(\sigma(t)\Big) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. We have the following two possible cases.

Case 1. Let $z(t) < 0$ for $t \geq t_2$. The case is same as in proof of Theorem 2.1.

Case 2. Let $z(t) > 0$ for $t \geq t_2$. By Lemma 2.1, (2.1) holds for $t \geq t_3$. Since $z(t)$ is unbounded and monotonic increasing, then it follows that

$$\lim_{t\to\infty} \frac{z(t)}{R(t)} = \lim_{t\to\infty} \frac{z'(t)}{R'(t)} = \lim_{t\to\infty} r(t)z'(t) = \alpha < \infty.$$

If $\alpha = 0$, then $\lim_{t\to\infty} R(t) = +\infty$ implies that $\lim_{t\to\infty} z(t) < +\infty$, which is absurd (because of unbounded $z(t)$). Hence $\alpha \neq 0$. Therefore, there exists a constant $\varepsilon > 0$ and a $t_2 > t_1$ such that $z(t) \geq \varepsilon R(t)$ for $t \geq t_2$. Consequently, $x(t) \geq z(t) \geq \varepsilon R(t)$ for $t \geq t_2$. Using $x(t) \geq \varepsilon R(t)$ in (2.2) and then integrating from t_2 to $+\infty$, we obtain a contradiction to $(A5)$ for every $\varepsilon > 0$.

The case where x is eventually negative unbounded solution is very similar and we omit it here.

The necessary part is same as in Theorem 2.1. This completes the proof of the theorem. □

Theorem 2.3. *Assume that* $(A1)$*-*$(A4)$ *hold and* $-1 < -a \leq p(t) \leq 0$*, where* $a > 0$*,* $t \in \mathbb{R}_+$*. Then every solution of* (1.1) *oscillates or converges to zero if and only if* $(A5)$ *holds for every* $\varepsilon > 0$*.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and, rz' and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. By Lemma 2.1, we have the following three possible cases.

Case 1. Let $z(t) < 0$, $r(t)z'(t) < 0$ for $t \geq t_2$. Since $z(t) < 0$ implies $z(t)$ is bounded due to Theorem 2.1 and $r(t)z'(t) < 0$ implies that $z(t)$ is unbounded due to Lemma 2.1, a contradiction.

Case 2. Assume that $z(t) < 0$, $r(t)z'(t) > 0$ holds for $t \geq t_2$. Therefore,

$$\begin{aligned} 0 \geq \lim_{t\to\infty} z(t) &= \limsup_{t\to\infty} z(t) \\ &\geq \limsup_{t\to\infty} \Big(x(t) - a x(\tau(t)) \Big) \\ &\geq \limsup_{t\to\infty} x(t) + \liminf_{t\to\infty} \Big(-a x(\tau(t)) \Big) \\ &= (1-a) \limsup_{t\to\infty} x(t), \end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ and hence $\lim_{t\to\infty} x(t) = 0$.

Case 3. Let $z(t) > 0$, $r(t)z'(t) > 0$ for $t \geq t_2$. The case follows from Theorem 2.1.

Hence, $(A5)$ is a sufficient condition. The case where x is negative solution is similar and we omit it here.

The necessary part is same as in the Theorem 2.1. Thus, the proof of the theorem is complete. □

Theorem 2.4. *Assume that* $(A1)$-$(A3)$ *hold and* $-1 < -a \le p(t) \le 0$ *such that* $r(t) \ge r\big(\sigma(t)\big)$ *for* $a > 0$, $t \in \mathbb{R}_+$. *Furthermore, assume that*

$(A6)$ *G is strictly superlinear, that is,* $\frac{G(u)}{u^\beta} \ge \frac{G(v)}{v^\beta}$, $u \ge v > 0$, $\beta > 1$,

holds. Then every solution of (1.1) *either oscillates or converges to zero if and only if*

$(A7)$ $\int_0^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta\right] d\eta = +\infty.$

Proof. For the sufficient part, we use the same type of argument as in the proof of Theorem 2.3 for first two cases of the pair $z(t)$ and $r(t)z'(t)$. Let us consider the **Case 3** for $t \ge t_1$. By Remark 2.1, there exists a constant $\varepsilon > 0$ and $t_2 > t_1$ such that $z\big(\sigma(t)\big) \ge \varepsilon$ for $t \ge t_2$. Consequently,

$$\begin{aligned} G\big(z(\sigma(t))\big) =& \frac{G\big(z(\sigma(t))\big)}{z^\beta\big(\sigma(t)\big)} z^\beta\big(\sigma(t)\big) \\ \ge& \frac{G(\varepsilon)}{\varepsilon^\beta} z^\beta\big(\sigma(t)\big), \end{aligned}$$

for $t \ge t_2$. Therefore, (2.4) becomes

$$\begin{aligned} r(t)z'(t) \ge& \frac{G(\varepsilon)}{\varepsilon^\beta} \int_t^\infty q(\eta) z^\beta\big(\sigma(\eta)\big) d\eta, \\ \ge& \frac{G(\varepsilon)}{\varepsilon^\beta} \left[\int_t^\infty q(\eta)d\eta\right] z^\beta\big(\sigma(t)\big), \end{aligned}$$

that is,

$$r\big(\sigma(t)\big) z'\big(\sigma(t)\big) \ge \frac{G(\varepsilon)}{\varepsilon^\beta} \left[\int_t^\infty q(\eta)d\eta\right] z^\beta\big(\sigma(t)\big),$$

for $t \ge t_2$, implies that

$$\begin{aligned} z'\big(\sigma(t)\big) \ge& \frac{G(\varepsilon)}{\varepsilon^\beta r\big(\sigma(t)\big)} \left[\int_t^\infty q(\eta)d\eta\right] z^\beta\big(\sigma(t)\big) \\ \ge& \frac{G(\varepsilon)}{\varepsilon^\beta} \frac{z^\beta\big(\sigma(t)\big)}{r(t)} \left[\int_t^\infty q(\eta)d\eta\right]. \end{aligned}$$

Integrating the last inequality from t_2 to $+\infty$, we get

$$\frac{G(\varepsilon)}{\varepsilon^\beta} \int_{t_2}^\infty \frac{1}{r(\eta)} \left[\int_\eta^\infty q(\zeta)d\zeta\right] d\eta \le \int_{t_2}^\infty \frac{z'\big(\sigma(\eta)\big)}{z^\beta\big(\sigma(\eta)\big)} d\eta < \infty,$$

which is a contradiction to $(A7)$.

The case where x is eventually negative solution is omitted since it can be dealt similarly.

Next, we show that $(A7)$ is necessary. Assume that $(A7)$ fails to hold and let

$$G(\varepsilon)\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta \le \frac{\varepsilon}{3}, \quad T \ge T^*,$$

where $\varepsilon > 0$ is a constant. Consider

$$M = \left\{x \in C([t_0,\infty),\mathbb{R}) : x(t) = \frac{\varepsilon}{3},\ t \in [t_0,T],\ \frac{\varepsilon}{3} \le x(t) \le \varepsilon,\ \text{for } t \ge T\right\},$$

and define

$$(\Phi x)(t) = \begin{cases} \dfrac{\varepsilon}{3}, & t \in [t_0,T],\\ -p(t)x\big(\tau(t)\big) + \dfrac{\varepsilon}{3} + \displaystyle\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta, & t \ge T, \end{cases}$$

for every $x \in M$, $(\Phi x)(t) \ge \frac{\varepsilon}{3}$ and

$$\begin{aligned}(\Phi x)(t) \le& a\varepsilon + \frac{\varepsilon}{3} + G(\varepsilon)\int_T^t \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta\\ \le& a\varepsilon + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \left(a + \frac{2}{3}\right)\varepsilon\\ \le& \varepsilon,\end{aligned}$$

implies that $\Phi x \in M$. The rest of the proof follows from Theorem 2.1. This completes the proof of the theorem. □

Theorem 2.5. *Assume that* $(A1)$-$(A3)$, $(A6)$ *hold and* $0 \le p(t) \le a < 1$ *such that* $r(t) \ge r\big(\sigma(t)\big)$ *for* $t \in \mathbb{R}_+$. *Furthermore, assume that* G *is Lipschitzian on the interval of the form* $[c,d]$, $0 < c < d < \infty$. *Then every solution of* (1.1) *oscillates if and only if* $(A7)$ *holds.*

Proof. Suppose the contrary that x is a nonoscillatory solution of (1.1). Then, there exists $t_1 \ge t_0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \ge t_1$. Assume that $x(t) > 0$, $x\big(\tau(t)\big) > 0$ and $x\big(\sigma(t)\big) > 0$ for $t \ge t_1$. Clearly, z defined by (2.1) is positive on $[t_1,\infty)$. By Lemma 2.1 and Remark 2.1, there exists $\varepsilon > 0$ such that $z(t) \ge \varepsilon$ for $t \ge t_2$, where $t_2 \ge t_1$. On the other hand, z being increasing implies that

$$\begin{aligned}(1-a)z(t) \le& \big(1-p(t)\big)z(t) \le z(t) - p(t)z(\tau(t))\\ =& x(t) - p(t)p(\tau(t))x(\tau(\tau(t))) \le x(t),\end{aligned}$$

for $t \ge t_3$, where $t_3 \ge t_2$. Consequently, (1.1) becomes

$$\Big(r(t)z'(t)\Big)' + q(t)G\Big((1-a)z(\sigma(t))\Big) \le 0,$$

for $t \geq t_3$. Using $(A6)$ it follows that

$$\begin{aligned} G\Big((1-a)z(\sigma(t))\Big) =& \frac{G\Big((1-a)z(\sigma(t))\Big)}{(1-a)^\beta z^\beta\Big(\sigma(t)\Big)}(1-a)^\beta z^\beta\Big(\sigma(t)\Big) \\ \geq& \frac{G\Big(\varepsilon(1-a)\Big)}{\varepsilon^\beta(1-a)^\beta}(1-a)^\beta z^\beta\Big(\sigma(t)\Big). \end{aligned}$$

The remaining portion of the sufficient part follows from Theorem 2.4.

Conversely, suppose that $(A7)$ fails to hold. Then there exists $T \geq T^*$ such that

$$\int_T^\infty \frac{1}{r(\eta)}\Big[\int_\eta^\infty q(\zeta)d\zeta\Big]d\eta < \frac{1-a}{5K},$$

where $K = \max\{K_1, G(1)\}$ and K_1 is the Lipschitz constant of G on $\Big[\frac{7(1-a)}{10}, 1\Big]$ for $t \geq t_0$. Let $X = BC([t_0,\infty),\mathbb{R})$ be the space of real valued continuous functions on $[t_0,\infty)$. Indeed, X is a Banach space with respect to sup norm defined by

$$\|x\| = \sup\{|x(t)| : t \ \geq t_0\}.$$

Define

$$S = \Big\{u \in X : \frac{7(1-a)}{10} \leq u(t) \leq 1,\ t \geq t_0\Big\}.$$

We notice that S is a closed convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases} (\Phi x)(T), & t \in [t_0, T], \\ -p(t)x\Big(\tau(t)\Big) + \dfrac{9+a}{10} - \displaystyle\int_t^\infty \frac{1}{r(\eta)}\Big[\int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\Big]d\eta, & t \geq T. \end{cases}$$

For every $x \in X$, $(\Phi x)(t) \leq \frac{9+a}{10} \leq 1$ and

$$(\Phi x)(t) \geq -a + \frac{9+a}{10} - \frac{1-a}{5} = \frac{7}{10}(1-a),$$

implies that $\Phi(x) \in S$. Now for $x_1, x_2 \in S$, we have

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| \leq& a|x_1\Big(\tau(t)\Big) - x_2\Big(\tau(t)\Big)| \\ &+ \int_t^\infty \frac{1}{r(\eta)}\Big[\int_\eta^\infty q(\zeta)|G\Big(x_1(\sigma(\zeta))\Big) - G\Big(x_2(\sigma(\zeta))\Big)|d\zeta\Big]d\eta, \end{aligned}$$

that is,

$$\begin{aligned} |(\Phi x_1)(t) - (\Phi x_2)(t)| \leq& a\|x_1 - x_2\| + \|x_1 - x_2\|K_1 \int_t^\infty \frac{1}{r(\eta)}\Big[\int_\eta^\infty q(\zeta)d\zeta\Big]d\eta \\ \leq& \Big(a + \frac{1-a}{5}\Big)\|x_1 - x_2\| \\ =& \frac{1+4a}{5}\|x_1 - x_2\|. \end{aligned}$$

Therefore, $\|\Phi x_1 - \Phi x_2\| \leq \frac{1+4a}{5}\|x_1 - x_2\|$ implies that Φ is a contraction. By using Banach's contraction mapping principle, it follows that Φ has a unique fixed point $x(t)$ in $\left[\frac{7(1-a)}{10}, 1\right]$. Hence, $(A7)$ is the necessary condition for oscillation of (1.1). This completes the proof of the theorem. □

Theorem 2.6. *Assume that* $(A1)$-$(A3)$ *hold and* $0 \leq p(t) \leq a < 1$ *for* $t \in \mathbb{R}_+$. *Furthermore, assume that* G *be Lipschitzian on intervals of the form* $[c, d]$, $0 < c < d < \infty$. *Then every bounded solutions of* (1.1) *oscillates if and only if* $(A7)$ *holds.*

Proof. Proceeding as in proof of the Theorem 2.5 we have obtained $x(t) \geq (1-a)z(t) \geq (1-a)\varepsilon = \varepsilon_1$. Consequently, (1.1) becomes

$$\Big(r(t)z'(t)\Big)' + q(t)G(\varepsilon_1) \leq 0.$$

Twice integration on last inequality yields a contradiction to $(A7)$. The necessary part is same as in the proof of Theorem 2.5. Hence the details are omitted. Thus the proof of theorem is complete. □

Theorem 2.7. *Assume that* $(A1)$-$(A3)$ *hold and* $-\infty < -a_1 \leq p(t) \leq -a_2 < -1$ *such that* $3a_2 > a_1$ *for* $t \in \mathbb{R}_+$ *where* $a_1, a_2 > 0$. *Let* G *be Lipschitzian on intervals of the form* $[c, d]$, $0 < c < d < \infty$. *Then every bounded solution of* (1.1) *oscillates or tends to zero if and only if* $(A7)$ *holds.*

Proof. Without loss of generality, suppose the contrary that x is an eventually positive solution of (1.1). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$. Proceeding as in the proof of Lemma 2.1, we see rz' is nonincreasing and, rz' and z is monotonic on $[t_2, \infty)$, where $t_2 \geq t_1$. Since $x(t)$ is bounded, then by (1.4), $z(t)$ is bounded and hence $\lim_{t\to\infty} z(t)$ exists. It is easy to see that the case $z(t) < 0$, $r(t)z'(t) < 0$ is not possible. Using the proof of Lemma 2.2, we conclude that the case $z(t) > 0$, $r(t)z'(t) < 0$ does not arise. Therefore, we have following two cases.

Case 1. Let $z(t) > 0$, $r(t)z'(t) > 0$ for $[t_3, \infty)$, $t_3 > t_2$. Then we can find a constant $\varepsilon > 0$ and $t_4 > t_3$ such that $z\Big(\sigma(t)\Big) \geq \varepsilon$ for $t \geq t_4$, that is, $x\Big(\sigma(t)\Big) \geq z\Big(\sigma(t)\Big) \geq \varepsilon$ for $t \geq t_4$. Hence, (1.1) becomes

$$\Big(r(t)z'(t)\Big)' + G(\varepsilon)q(t) \leq 0,\ t \geq t_4.$$

Twice integration on last inequality gives a contradiction to $(A7)$.

Case 2. Let $z(t) < 0$, $r(t)z'(t) > 0$ for $[t_3, \infty)$, $t_3 > t_2$. We claim that $\lim_{t\to\infty} z(t) = 0$. If not, there exist $\alpha < 0$ and $t_4 > t_3$ such that $z\Big(\tau^{-1}(\sigma(t))\Big) < \alpha$ for $t \geq t_4$. Hence, $z(t) \geq -a_1 x\Big(\tau(t)\Big)$ implies that $x(t) \geq -a_1^{-1} z\Big(\tau^{-1}(t)\Big)$, that is, $x\Big(\sigma(t)\Big) \geq -a_1^{-1} z\Big(\tau^{-1}(\sigma(t))\Big) \geq -a_1^{-1}\alpha$ for $t \geq t_4$. Consequently, (1.1) reduces to

$$\Big(r(t)z'(t)\Big)' + G\Big(-a_1^{-1}\alpha\Big)q(t) \leq 0,$$

for $t \geq t_4$. Using the same type of argument as in the former case, we get a contradiction to $(A7)$. Thus, our claim holds and hence

$$\begin{aligned}0 = \lim_{t\to\infty} z(t) =& \liminf_{t\to\infty}\Big(x(t)+p(t)x(\tau(t))\Big)\\ \leq& \liminf_{t\to\infty}\Big(x(t)-a_2x(\tau(t))\Big)\\ \leq& \limsup_{t\to\infty} x(t)+\liminf_{t\to\infty}\Big(-a_2x(\tau(t))\Big)\\ =&(1-a_2)\limsup_{t\to\infty} x(t),\end{aligned}$$

implies that $\limsup_{t\to\infty} x(t) = 0$ $[\because 1-a_2<0]$. Therefore, $\lim_{t\to\infty} x(t) = 0$.

The case where x is negative bounded solution is very similar and we omit it here.

For the necessary part, it is possible to find $T \geq T^*$ such that

$$\int_T^\infty \frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)d\zeta\right]d\eta < \frac{a_2-1}{3K},$$

where $K = \max\{K_1, G(1)\}$ and K_1 is the Lipschitz constants of G on $[a,1]$, where $a = \frac{(a_2-1)(3a_2-a_1)}{3a_1a_2}$. Let $X = BC([t_0,\infty),\mathbb{R})$ be the space of real valued continuous functions defined on $[t_0,\infty)$. Indeed, X is a Banach space with the sup norm defined by

$$\|x\| = \sup\{|x(t)| : t \ \geq t_0\}.$$

Define

$$S = \{u \in X : a \leq u(t) \leq 1,\ t \geq t_0\}$$

and we note that S is a closed convex subspace of X. Let $\Phi : S \to S$ be such that

$$(\Phi x)(t) = \begin{cases}(\Phi x)(T), & t\in[t_0,T],\\ -\dfrac{x\Big(\tau^{-1}(t)\Big)}{p\Big(\tau^{-1}(t)\Big)}-\dfrac{a_2-1}{p\Big(\tau^{-1}(t)\Big)} & \\ +\dfrac{1}{p\Big(\tau^{-1}(t)\Big)}\displaystyle\int_T^{\tau^{-1}(t)}\frac{1}{r(\eta)}\left[\int_\eta^\infty q(\zeta)G\Big(x(\sigma(\zeta))\Big)d\zeta\right]d\eta, & t\geq T.\end{cases}$$

For every $x \in S$,

$$(\Phi x)(t) \leq -\frac{x\Big(\tau^{-1}(t)\Big)}{p\Big(\tau^{-1}(t)\Big)}-\frac{a_2-1}{p\Big(\tau^{-1}(t)\Big)} \leq \frac{1}{a_2}+\frac{a_2-1}{a_2} = 1$$

and

$$
\begin{aligned}
(\Phi x)(t) \geq & -\frac{a_2-1}{p\Big(\tau^{-1}(t)\Big)}+\frac{1}{p\Big(\tau^{-1}(t)\Big)} \int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^{\infty} q(\zeta) G\Big(x(\sigma(\zeta))\Big) d\zeta\right] d\eta \\
\geq & -\frac{a_2-1}{a_1}+\frac{G(1)}{p\Big(\tau^{-1}(t)\Big)} \int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^{\infty} q(\zeta) d\zeta\right] d\eta \\
\geq & -\frac{a_2-1}{a_1}-\frac{G(1)}{a_2} \int_T^{\infty} \frac{1}{r(\eta)}\left[\int_\eta^{\infty} q(\zeta) d\zeta\right] d\eta \\
\geq & -\frac{a_2-1}{a_1}-\frac{a_2-1}{3 a_2}=a,
\end{aligned}
$$

implies that $\Phi x \in S$. Now for $x_1, x_2 \in S$, we have

$$
\begin{aligned}
|(\Phi x_1)(t)-(\Phi x_2)(t)| \leq & \frac{1}{\Big|p\Big(\tau^{-1}(t)\Big)\Big|}\left|x_1\Big(\tau^{-1}(t)\Big)-x_2\Big(\tau^{-1}(t)\Big)\right|+\frac{K_1}{\Big|p\Big(\tau^{-1}(t)\Big)\Big|} \\
& \times \int_T^{\tau^{-1}(t)} \frac{1}{r(\eta)}\left[\int_\eta^{\infty}\left|x_1\Big(\sigma(\zeta)\Big)-x_2\Big(\sigma(\zeta)\Big)\right| q(\zeta) d\zeta\right] d\eta \\
\leq & \frac{1}{a_2}\|x_1-x_2\|+\frac{a_2-1}{3 a_2}\|x_1-x_2\| \\
= & \gamma\|x_1-x_2\|,
\end{aligned}
$$

implies that

$$
\|\Phi x_1-\Phi x_2\| \leq \gamma\|x_1-x_2\|,
$$

where $\gamma=\frac{1}{a_2}(1+\frac{a_2-1}{3})<1$. Therefore, Φ is a contraction. Hence by the Banach's contraction mapping principle Φ has a unique fixed point $x \in S$. It is easy to see that $\lim_{t \to \infty} x(t) \neq 0$. This completes the proof of the theorem. □

3. Discussion and Example

It is worth observation that we could succeed partially to establish the oscillation of all solutions of the nonlinear equation (1.1), when $|p(t)|<\infty$. We failed to obtain the necessary and sufficient conditions in the range $1 \leq p(t)<\infty$ and $p(t) \equiv -1$. Therefore, the undertaken problem is incomplete for all range of $p(t)$.

Remark 3.1. In Theorems 2.2, 2.6 and 2.7, G could be linear, sublinear or superlinear.

We conclude this section with the following examples to illustrate our main results:

Example 3.1. Consider the delay differential equations

$$
\frac{d}{dt}\left[t \frac{d}{dt}[x(t)-3 x(e^{-\pi} t)]\right]+\frac{4}{t} x(t)=0, \quad \text{for } t \geq 1, \tag{3.1}
$$

where $r(t):=t$, $p(t):\equiv-3$, $\tau(t):=e^{-\pi} t$, $q(t):=\frac{4}{t^2}$, $\sigma(t):=t$ and $G(u):=u$ for $t \geq 1$ and $u \in \mathbb{R}$. It can be easily shown that Theorem 2.7 applies to (3.1). Thus, every bounded solution oscillates or converges to zero asymptotically. Obviously, $x(t) = \sin(\ln(t^2))$ for $t \geq 1$ is an oscillating solution.

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SOME ESTIMATES FOR HOLOMORPHIC FUNCTIONS AT THE BOUNDARY OF THE UNIT DISC

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ABSTRACT. In this paper, for holomorphic function $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ belong to the class of $\mathcal{N}(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ on the boundary point of the unit disc.

1. INTRODUCTION

Let f be a holomorphic function in the unit disc $E = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc E, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = ze^{i\theta}$, where θ is a real number ([8], p. 329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2, 7]).

The basic tool in proving our results is the following lemma due to Jack.

Lemma 1.1 (Jack's lemma)**.** *Let $f(z)$ be holomorphic function in the unit disc E with $f(0) = 0$. Then if $|f(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in E$, then there exists a real number $k \geq 1$ such that*

$$\frac{z_0 f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + c_2z^2 + c_3z^3 + \cdots ,$$

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that are holomorphic in the unit disc E. Also, $\mathcal{N}(\lambda)$ be the subclass of $\mathcal{A}$ consisting of all functions $f(z)$ which satisfy

$$\left|\frac{zf'(z)}{f(z)}\right|^{\alpha}\left|z\left(\frac{zf'(z)}{f(z)}\right)'\right|^{\beta}<\left(\frac{1}{2}\lambda\right)^{\beta}, \tag{1.1}$$

for some real $\alpha \geq 0$, $\beta > 0$ and $\lambda = \frac{\beta}{\beta+\alpha}$.

Let $f(z) \in \mathcal{N}(\lambda)$ and define $\phi(z)$ in E by

$$\phi(z)=\frac{(h(z))^{\frac{1}{\lambda}}-1}{(h(z))^{\frac{1}{\lambda}}+1}, \tag{1.2}$$

where $h(z) = \frac{zf'(z)}{f(z)}$.

Obviously, $\phi(z)$ is holomorphic function in the unit disc E and $\phi(0) = 0$. We want to prove $|\phi(z)| < 1$ for $|z| < 1$. Differentiating (1.2) and simplifiying, we obtain

$$\left(\frac{zf'(z)}{f(z)}\right)'=\frac{2\lambda\phi'(z)}{(1-\phi(z))^2}\left(\frac{1+\phi(z)}{1-\phi(z)}\right)^{\lambda-1}$$

and, so

$$\begin{aligned}\left|\frac{zf'(z)}{f(z)}\right|^{\alpha}\left|z\left(\frac{zf'(z)}{f(z)}\right)'\right|^{\beta}&=\left|\frac{1+\phi(z)}{1-\phi(z)}\right|^{\alpha\beta+\beta(\lambda-1)}\left|\frac{2\lambda z\phi'(z)}{(1-\phi(z))^2}\right|^{\beta}\\&=\left|\frac{2\lambda z\phi'(z)}{(1-\phi(z))^2}\right|^{\beta}<\left(\frac{\lambda}{2}\right)^{\beta}.\end{aligned}$$

If there exists a point $z_0 \in E$ such that

$$\max_{|z|\leq|z_0|}|\phi(z)|=|\phi(z_0)|=1,$$

then Jack's lemma gives us that $\phi(z_0) = e^{i\theta}$ and $z_0\phi'(z_0) = k\phi(z_0)$, $k \geq 1$.

Thus we have

$$\begin{aligned}\left|\frac{z_0f'(z_0)}{f(z_0)}\right|^{\alpha}\left|z_0\left(\frac{z_0f'(z_0)}{f(z_0)}\right)'\right|^{\beta}&=\left|\frac{2\lambda z_0\phi'(z_0)}{(1-\phi(z_0))^2}\right|^{\beta}=\left|\frac{2\lambda k e^{i\theta}}{(1-e^{i\theta})^2}\right|^{\beta}\\&=\frac{(2\lambda k)^{\beta}}{|1-e^{i\theta}|^{2\beta}}\geq\frac{(2\lambda)^{\beta}}{2^{2\beta}}=\left(\frac{\lambda}{2}\right)^{\beta}.\end{aligned}$$

This contradict (1.1). So, there is no point $z_0 \in E$ such that $\phi(z_0) = 1$. This means that $|\phi(z)| < 1$ for $|z| < 1$. Thus, from the Schwarz lemma, we obtain

$$|c_2|\leq\frac{2\beta}{\beta+\alpha}.$$

Moreover, the equality $|c_2| = \frac{2\beta}{\beta+\alpha}$ occurs for the function

$$f(z)=e^{\int_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda}dt}.$$

That proves the following lemma.

Lemma 1.2. *If $f(z) \in \mathcal{N}(\lambda)$, then we have*

$$|c_2| \leq \frac{2\beta}{\beta+\alpha}. \tag{1.3}$$

The equality in (1.3) *occurs for the function*

$$f(z) = e^{\int\limits_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt}.$$

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [21] and then rediscovered and partially improved by Osserman in [17].

Lemma 1.3. *Let $f(z)$ be a holomorphic function self-mapping of $E = \{z : |z| < 1\}$, that is $|f(z)| < 1$ for all $z \in E$. Assume that there is a $b \in \partial E$ so that f extend continuously to b, $|f(b)| = 1$ and $f'(b)$ exists. Then*

$$|f'(b)| \geq \frac{2}{1+|f'(0)|}. \tag{1.4}$$

The equality in (1.4) *holds if and only if f is of the form*

$$f(z) = -z\frac{a-z}{1-az}, \quad \text{for all } z \in E,$$

for some constant $a \in (-1, 0]$.

Corollary 1.1. *Under the hypotheses lemma, we have*

$$|f'(b)| \geq 1, \tag{1.5}$$

with equality only if f is of the form

$$f(z) = ze^{i\theta},$$

where θ is a real number.

The following Lemma 1.4 and Corollary 1.2, known as the Julia-Wolff lemma, is needed in the sequel [15].

Lemma 1.4 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in E, $f(0) = 0$ and $f(E) \subset E$. If, in addition, the function f has an angular limit $f(b)$ at $b \in \partial E$, $|f(b)| = 1$, then the angular derivative $f'(b)$ exists and $1 \leq |f'(b)| \leq \infty$.*

Corollary 1.2. *The holomorphic function f has a finite angular derivative $f'(b)$ if and only if f' has the finite angular limit $f'(b)$ at $b \in \partial E$.*

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [8, 18]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1, 2, 5–7, 15–17, 19, 20] and references therein).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwar lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, with a zero set $\{z_k\}$ (see [5]).

S. G. Krantz and D. M. Burns [3] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. According to M. Mateljević's studies, some other types of results which are related to the subject can be found in ([13,14] and [12]). In addition, [11] was posed on ResearchGate where is discussed concerning results in more general aspects.

Also, M. Jeong [10] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [9] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, for holomorphic function $f(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ belong to the class of $\mathcal{N}(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ on the boundary point of the unit disc.

Theorem 2.1. *Let $f(z) \in \mathcal{N}(\lambda)$. Assume that, for some $b \in \partial E$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)} = i^\lambda$. Then we have the inequality*

$$\left| \left(\frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{\beta}{\beta + \alpha}. \tag{2.1}$$

The equality in (2.1) *occurs for the function*

$$f(z) = e^{\int\limits_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt},$$

where $\lambda = \frac{\beta}{\beta+\alpha}$.

Proof. Consider the function

$$\phi(z) = \frac{(h(z))^{\frac{1}{\lambda}} - 1}{(h(z))^{\frac{1}{\lambda}} + 1},$$

where $h(z) = \frac{zf'(z)}{f(z)}$ and $\lambda = \frac{\beta}{\beta+\alpha}$. $\phi(z)$ is a holomorphic function in the unit disc E and $\phi(0) = 0$. From the Jack's lemma and since $f(z) \in \mathcal{N}(\lambda)$, we obtain $|\phi(z)| < 1$ for $|z| < 1$. Also, we have $|\phi(b)| = 1$ for $b \in \partial E$.

From (1.5), we obtain

$$1 \leq |\phi'(b)| = \frac{2}{\lambda} \left| \frac{(h(b))^{\frac{1}{\lambda}-1} h'(b)}{\left(1 + (h(b))^{\frac{1}{\lambda}}\right)^2} \right| = \frac{2}{\lambda} \left| \frac{\left(i^\lambda\right)^{\frac{1}{\lambda}-1} h'(b)}{\left(1 + (i^\lambda)^{\frac{1}{\lambda}}\right)^2} \right| = \frac{2}{\lambda} \left| \frac{\left(i^\lambda\right)^{\frac{1}{\lambda}-1} h'(b)}{\left(1 + (i^\lambda)^{\frac{1}{\lambda}}\right)^2} \right|$$

and

$$1 \leq \frac{2}{\lambda} \frac{|h'(b)|}{|1+i|^2} = \frac{|h'(b)|}{\lambda}.$$

So, we take the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp. Let

$$f(z) = e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt}.$$

Then, we have

$$\begin{aligned}
\ln f(z) =& \ln e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt} = \int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt,\\
\frac{f'(z)}{f(z)} =& \frac{1}{z}\left(\frac{1+z}{1-z}\right)^\lambda,\\
h(z) =& z\frac{f'(z)}{f(z)} = \left(\frac{1+z}{1-z}\right)^\lambda
\end{aligned}$$

and

$$h'(z) = \lambda \left(\frac{1+z}{1-z}\right)^{\lambda-1} \frac{2}{(1-z)^2}.$$

Therefore, we obtain

$$h'(i) = \lambda \left(\frac{1+i}{1-i}\right)^{\lambda-1} \frac{2}{(1-i)^2}$$

and

$$|h'(i)| = \lambda = \frac{\beta}{\beta+\alpha}.$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{4\beta^2}{(\beta+\alpha)\left(2\beta+(\beta+\alpha)|c_2|\right)}. \tag{2.2}$$

The inequality (2.2) *is sharp with equality for the function*

$$f(z) = e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt},$$

where $\lambda = \frac{\beta}{\beta+\alpha}$.

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. Using the inequality (1.4) for the function $\phi(z)$, we obtain

$$\frac{2}{1+|\phi'(0)|} \leq |\phi'(b)| = \frac{2}{\lambda}\left|\frac{(h(b))^{\frac{1}{\lambda}-1}h'(b)}{\left(1+(h(b))^{\frac{1}{\lambda}}\right)^2}\right| = \frac{2}{\lambda}\frac{|h'(b)|}{|1+i|^2} = \frac{|h'(b)|}{\lambda}.$$

Since

$$\phi'(z)=\frac{2}{\lambda}\frac{(h(z))^{\frac{1}{\lambda}-1}h'(z)}{\left(1+(h(z))^{\frac{1}{\lambda}}\right)^2}$$

and

$$|\phi'(0)|=\frac{2}{\lambda}\left|\frac{(h(0))^{\frac{1}{\lambda}-1}h'(0)}{\left(1+(h(0))^{\frac{1}{\lambda}}\right)^2}\right|=\frac{2}{\lambda}\frac{|c_2|}{4}=\frac{|c_2|}{2\lambda},$$

we have

$$\frac{2}{1+\frac{|c_2|}{2\lambda}}\leq\frac{|h'(b)|}{\lambda}$$

and

$$|h'(b)|\geq\frac{4\lambda^2}{2\lambda+|c_2|}.$$

So, we obtain the inequality (2.2).

To show that the inequality (2.2) is sharp, take the holomorphic function

$$f(z)=e^{\int\limits_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt}.$$

Then

$$h(z)=z\frac{f'(z)}{f(z)}=\left(\frac{1+z}{1-z}\right)^\lambda$$

and

$$|h'(i)|=\lambda.$$

Since $|c_2|=2\lambda$ is satisfied with equality. That is;

$$\frac{4\lambda^2}{2\lambda+|c_2|}=\frac{4\lambda^2}{2\lambda+2\lambda}=\lambda.$$

□

Theorem 2.3. *Let $f(z)\in\mathcal{N}(\lambda)$. Assume that, for some $b\in\partial E$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)}=i^\lambda$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right|\geq\lambda\left(1+\frac{2\left(2\lambda-|c_2|\right)^2}{4\lambda^2-|c_2|^2+|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\right), \tag{2.3}$$

where $\lambda=\frac{\beta}{\beta+\alpha}$. The inequality (2.3) is sharp with equality for the function

$$f(z)=e^{\int\limits_0^z\frac{1}{t}\left(\frac{1+t}{1-t}\right)^\lambda dt}.$$

Proof. Let $\phi(z)$ be as in the proof of Theorem 2.1. By the maximum principle for each $z\in E$, we have $|\phi(z)|\leq|z|$. So,

$$\psi(z)=\frac{\phi(z)}{z}$$

is a holomorphic function in E and $|\psi(z)|<1$ for $|z|<1$. For any real number $\mu=\frac{1}{\lambda}$ that is not a non-negative integer

$$k^{\mu}=\sum_{n=0}^{\infty}\binom{\mu}{n}(k-1)^{n},$$

where $k=\frac{zf'(z)}{f(z)}=1+c_2z+\left(2c_3-c_2^2\right)z^2+\cdots$.

From equality of $\psi(z)$, we have

$$\psi(z)=\frac{\phi(z)}{z}=\frac{1}{z}\frac{(h(z))^{\frac{1}{\lambda}}-1}{(h(z))^{\frac{1}{\lambda}}+1}=\frac{1}{z}\frac{(k)^{\mu}-1}{(k)^{\mu}+1}.$$

Thus, we take

$$|\psi(0)|=\frac{|c_2|}{2\lambda}\leq 1 \tag{2.4}$$

and

$$|\psi'(0)|=\frac{|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}{4\lambda^2}.$$

Moreover, it can be seen that

$$\frac{b\phi'(b)}{\phi(b)}=|\phi'(b)|\geq\left|(b^p)'\right|=\frac{b\,(b^p)'}{b^p}.$$

The function

$$\Phi(z)=\frac{\psi(z)-\psi(0)}{1-\overline{\psi(0)}\psi(z)}$$

is a holomorphic in the unit disc E, $|\Phi(z)|<1$ for $|z|<1$, $\Phi(0)=0$ and $|\Phi(b)|=1$ for $b\in\partial E$.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|}\leq|\Phi'(b)|&=\frac{1-|\psi(0)|^2}{\left|1-\overline{\psi(0)}\psi(b)\right|^2}|\psi'(b)|\leq\frac{1+|\psi(0)|}{1-|\psi(0)|}|\psi'(b)|\\&=\frac{1+|\psi(0)|}{1-|\psi(0)|}\left\{|\phi'(b)|-1\right\}.\end{aligned}$$

Since

$$\begin{aligned}\Phi'(z)=&\frac{1-|\psi(0)|^2}{\left(1-\overline{\psi(0)}\psi(z)\right)^2}\psi'(z),\\|\Phi'(0)|=&\frac{|\psi'(0)|}{1-|\psi(0)|^2}=\frac{\frac{\left|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2\right|}{4\lambda^2}}{1-\left(\frac{|c_2|}{2\lambda}\right)^2}=\frac{|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}{4\lambda^2-|c_2|^2},\end{aligned}$$

we take

$$\frac{2}{1+\frac{\left|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2\right|}{4\lambda^2-|c_2|^2}} \le \frac{1+\frac{|c_2|}{2\lambda}}{1-\frac{|c_2|}{2\lambda}}\left\{\frac{|h'(b)|}{\lambda}-1\right\}$$
$$=\frac{2\lambda+|c_2|}{2\lambda-|c_2|}\left\{\frac{|h'(b)|}{\lambda}-1\right\}.$$

Therefore, we obtain

$$1+\frac{2\left(4\lambda^2-|c_2|^2\right)}{4\lambda^2-|c_2|^2+|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\frac{2\lambda-|c_2|}{2\lambda+|c_2|}\le\frac{|h'(b)|}{\lambda}$$

and

$$|h'(b)|\ge\lambda\left(1+\frac{2\left(2\lambda-|c_2|\right)^2}{4\lambda^2-|c_2|^2+|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\right).$$

So, we obtain the inequality (2.3).

To show that the inequality (2.3) is sharp, take the holomorphic function

$$f(z)=e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda}dt}.$$

Then

$$h(z)=z\frac{f'(z)}{f(z)}=\left(\frac{1+z}{1-z}\right)^{\lambda}$$

and

$$|h'(i)|=\lambda.$$

Since $|c_2|=2\lambda$, (2.3) is satisfied with equality. □

If $\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}-1$ has no zeros different from $z=0$ in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

Theorem 2.4. *Let $f(z)\in\mathcal{N}(\lambda)$ and $\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}}-1$ has no zeros in E except $z=0$ and $c_2>0$. Assume that, for some $b\in\partial E$, f has angular limit $f(b)$ at b and $\frac{bf'(b)}{f(b)}=i^{\lambda}$. Then we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right|\ge\lambda\left(1-\frac{2\lambda\,|c_2|\ln^2\left(\frac{|c_2|}{2\lambda}\right)}{2\lambda\,|c_2|\ln\left(\frac{|c_2|}{2\lambda}\right)-|4\lambda c_3-c_2^2(2\lambda-1)+(1-\lambda)c_2|}\right), \tag{2.5}$$

where $\lambda=\frac{\beta}{\beta+\alpha}$. In addition, the equality in (2.5) occurs for the function

$$f(z)=e^{\int_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda}dt},$$

where $\lambda=\frac{\beta}{\beta+\alpha}$.

Proof. Let $c_2 > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function $\left(\frac{zf'(z)}{f(z)}\right)^{\frac{1}{\lambda}} - 1$ has no zeros in E except $E - \{0\}$, we denote by $\ln \psi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln \psi(0) = \ln \left(\frac{|c_2|}{2\lambda}\right) < 0.$$

The auxiliary function

$$\Delta(z) = \frac{\ln \psi(z) - \ln \psi(0)}{\ln \psi(z) + \ln \psi(0)}$$

is a holomorphic in the unit disc E, $|\Delta(z)| < 1$, $\Delta(0) = 0$ and $|\Delta(b)| = 1$ for $b \in \partial E$.

From (1.4), we obtain

$$\begin{aligned}\frac{2}{1 + |\Delta'(0)|} \leq |\Delta'(b)| &= \frac{|2 \ln \psi(0)|}{|\ln \psi(b) + \ln \psi(0)|^2} \left|\frac{\psi'(b)}{\psi(b)}\right| \\ &= \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \left\{|\phi'(b)| - 1\right\}.\end{aligned}$$

Since

$$\begin{aligned}|\Delta'(0)| &= \frac{-1}{\ln\left(\frac{|c_2|}{2\lambda}\right)} \frac{\frac{\left|4\lambda c_3 - c_2^2(2\lambda-1) + (1-\lambda)c_2\right|}{4\lambda^2}}{\frac{|c_2|}{2\lambda}} \\ &= \frac{-1}{\ln\left(\frac{|c_2|}{2\lambda}\right)} \frac{|4\lambda c_3 - c_2^2(2\lambda - 1) + (1-\lambda)c_2|}{2\lambda\, |c_2|}\end{aligned}$$

and replacing $\arg^2 \psi(b)$ by zero, then we have

$$\frac{1}{1 - \frac{\left|4\lambda c_3 - c_2^2(2\lambda-1) + (1-\lambda)c_2\right|}{2\lambda |c_2| \ln\left(\frac{|c_2|}{2\lambda}\right)}} \leq \frac{-1}{\ln\left(\frac{|c_2|}{2\lambda}\right)} \left\{\frac{|h'(b)|}{\lambda} - 1\right\}$$

and

$$1 - \frac{2\lambda\, |c_2| \ln^2\left(\frac{|c_2|}{2\lambda}\right)}{2\lambda\, |c_2| \ln\left(\frac{|c_2|}{2\lambda}\right) - |4\lambda c_3 - c_2^2(2\lambda - 1) + (1-\lambda)c_2|} \leq \frac{|h'(b)|}{\lambda}.$$

Thus, we obtain the inequality (2.5) with an obvious equality case. $\square$

The following inequality (2.6) is weaker, but is simpler than (2.5) and does not contain the coeffient c_3.

Theorem 2.5. *Under the hypotheses of Theorem 2.4, we have the inequality*

$$\left|\left(\frac{zf'(z)}{f(z)}\right)'_{z=b}\right| \geq \frac{\beta}{\beta + \alpha}\left[1 - \ln\left((\beta + \alpha)\frac{|c_2|}{2\beta}\right)\right]. \tag{2.6}$$

Moreover, the result is sharp and the extremal function is

$$f(z) = e^{\int\limits_0^z \frac{1}{t}\left(\frac{1+t}{1-t}\right)^{\lambda} dt},$$

where $\lambda = \frac{\beta}{\beta+\alpha}$.

Proof. Let $c_2 > 0$. Using the inequality (1.5) for the function $\Phi(z)$, we obtain

$$1 \leq |\Delta'(b)| = \frac{|2\ln\psi(0)|}{|\ln\psi(b)+\ln\psi(0)|^2}\left|\frac{\psi'(b)}{\psi(b)}\right| = \frac{-2\ln\psi(0)}{\ln^2\psi(0)+\arg^2\psi(b)}\left\{|\phi'(b)|-1\right\}.$$

Replacing $\arg^2\varphi(b)$ by zero, then we have

$$1 \leq \frac{-1}{\ln\left(\frac{|c_2|}{2\lambda}\right)}\left\{\frac{|h'(b)|}{\lambda}-1\right\}$$

and

$$|h'(b)| \geq \lambda\left[1-\ln\left(\frac{|c_2|}{2\lambda}\right)\right].$$

Thus, we obtain the inequality (2.6) with an obvious equality case. □

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