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CONVERGENCE OF DOUBLE COSINE SERIES

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Abstract. In this paper we consider double cosine series whose coefficients form a null sequence of bounded variation of order $(p,0)$, $(0,p)$ and (p,p) with the weight $(jk)^{p-1}$ for some $p>1$. We study pointwise convergence, uniform convergence and convergence in L^r-norm of the series under consideration. In a certain sense our results extend the results of Young [7], Kolmogorov [3] and Móricz [4,5].

1. Introduction

Consider the double cosine series

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\lambda_j\lambda_k a_{jk}\cos jx\cos ky, \tag{1.1}$$

on positive quadrant $T=[0,\pi]\times[0,\pi]$ of the two dimensional torus where $\lambda_0=\frac{1}{2}$ and $\lambda_j=1$ for $j=1,2,3,\dots$.

The rectangular partial sums $S_{mn}(x,y)$ and the *Cesàro* means $\sigma_{mn}(x,y)$ of the series (1.1) are defined as

$$\begin{aligned}S_{mn}(x,y)=&\sum_{j=0}^{m}\sum_{k=0}^{n}\lambda_j\lambda_k a_{jk}\cos jx\cos ky,\\ \sigma_{mn}(x,y)=&\frac{1}{(m+1)(n+1)}\sum_{j=0}^{m}\sum_{k=0}^{n}S_{jk}(x,y),\quad m,n>0,\end{aligned}$$

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and for $\lambda > 1$, the truncated Cesáro means are defined by

$$V_{mn}^{\lambda}(x,y) = \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{j=m+1}^{[\lambda m]} \sum_{k=n+1}^{[\lambda n]} S_{jk}(x,y).$$

Now assuming the coefficients $\{a_{jk} : j,k \geq 0\}$ in (1.1) be a double sequence of real numbers which satisfy the following conditions for some positive integer p:

$$|a_{jk}|(jk)^{p-1} \to 0 \text{ as } \max\{j,k\} \to \infty, \tag{1.2}$$

$$\lim_{k\to\infty} \sum_{j=0}^{\infty} |\triangle_{p0} a_{jk}|(jk)^{p-1} = 0, \tag{1.3}$$

$$\lim_{j\to\infty} \sum_{k=0}^{\infty} |\triangle_{0p} a_{jk}|(jk)^{p-1} = 0, \tag{1.4}$$

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{pp} a_{jk}|(jk)^{p-1} < \infty. \tag{1.5}$$

The finite order differences $\triangle_{pq} a_{jk}$ are defined by

$$\begin{aligned} \triangle_{00} a_{jk} &= a_{jk}, \\ \triangle_{pq} a_{jk} &= \triangle_{p-1,q} a_{jk} - \triangle_{p-1,q} a_{j+1,k}, \quad p \geq 1, q \geq 0, \\ \triangle_{pq} a_{jk} &= \triangle_{p,q-1} a_{jk} - \triangle_{p,q-1} a_{j,k+1}, \quad p \geq 0, q \geq 1. \end{aligned}$$

Also a double induction argument gives

$$\triangle_{pq} a_{jk} = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{s+t} \binom{p}{s} \binom{q}{t} a_{j+s,\, k+t}.$$

We can call the above mentioned conditions (1.2)-(1.5) as conditions of bounded variation of order $(p,0), (0,p)$ and (p,p) respectively with the weight $(jk)^{p-1}$. Obviously these conditions generalise the concept of monotone sequences. Also any sequence satisfying (1.5) with $p = 2$ is called a quasi-convex sequence [3, 5]. Clearly the conditions (1.3) and (1.4) can be derived from (1.2) and (1.5) for $p = 1$ and moreover for $p = 1$, the conditions (1.2) and (1.5) reduce to $|a_{jk}| \to 0$ as $\max\{j,k\} \to \infty$ and

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |\triangle_{11} a_{jk}| < \infty.$$

Generally the pointwise convergence of the series (1.1) is defined in Pringsheim's sense ([8], Vol. 2, Ch. 17) which means that the rectangular partial sums of the type

$$S_{mn}(x,y) = \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \cos jx \cos ky, \quad m,n \geq 0,$$

are formed and then by taking both m, n tend to ∞ (independently of one another) the limit $f(x,y)$ (provided it exists) is assigned to the series (1.1) as its sum.

Also let $\|f\|_r$ denotes the $L^r(T^2)$-norm, i.e,

$$\|f\|_r = \left(\int_0^\pi \int_0^\pi |f(x,y)|^r \, dxdy \right)^{1/r}, \quad 1 \leq r < \infty$$

and $\|f\|$ denotes $L^1(T^2)$-norm, i.e,

$$\|f\| = \int_0^\pi \int_0^\pi |f(x,y)| \, dxdy.$$

In this paper, we will investigate the validity of the following statements:

(a) $S_{mn}(x,y)$ converges pointwise to $f(x,y)$ for every $(x,y) \in T^2$;
(b) $S_{mn}(x,y)$ converges uniformly to $f(x,y)$ on T^2;
(c) $\|S_{mn}(x,y) - f(x,y)\|_r = o(1)$ as $\min\{m,n\} \to 0$.

Such type of problems have been studied by Young [7] and Kolmogorov [3] for one-dimensional case (single trigonometric series especially cosine series) and by Móricz [4, 5] and K. Kaur, Bhatia and Ram [2] for double trigonometric series. In [5], Móricz studied both double cosine series and double sine series as far as their integrability and convergence in L^1-norm is concerned where as in [4] he studied double trigonometric series of the form

$$\sum_{-\infty}^{\infty} \sum_{-\infty}^{\infty} c_{jk} e^{i(jx+ky)},$$

under coefficients of bounded variation. All of them discussed the case for $p = 1$ or $p = 2$ only. Our aim in this paper is to extend the above results from $p = 1$ to general cases for double cosine series.

In the results, C_p and C_{pr} denote constants which may not be the same at each occurrence. Also we write $\lambda_n = [\lambda n]$ where n is a positive integer, $\lambda > 1$ is a real number and $[\cdot]$ means greatest integral part.

The first main result reads as follows.

Theorem 1.1. *Assume that conditions* (1.2)–(1.5) *are satisfied for some* $p \geq 1$*, then*

(i) $S_{mn}(x,y)$ *converges pointwise to* $f(x,y)$ *for every* $(x,y) \in T^2$ *such that* $x, y > 0$*;*
(ii) $\|S_{mn}(x,y) - f(x,y)\|_r = o(1)$ *as* $\min\{m,n\} \to \infty$*,* $1 \leq r < \infty$*.*

The above theorem has been proved by Móricz [4, 5] for $p = 1$ and $p = 2$ using suitable estimates for Dirichlet's kernel $D_j(x)$ and Fejér kernel $K_j(x)$. In the case of a single series for $p = 2$, the results regarding convergence have been proved by Kolmogorov [3].

Obviously, condition (1.5) implies any of the following conditions:

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{j=0}^{\infty} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| (jk)^{p-1} = 0, \tag{1.6}$$

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m\to\infty}}\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{\infty}\frac{\lambda_m-j+1}{\lambda_m-m}|\triangle_{pp}a_{jk}|(jk)^{p-1}=0. \tag{1.7}$$

We introduce the following three sums for $m, n \geq 0$ and $\lambda > 1$:

$$\sum\nolimits_{10}^{\lambda}(m,n,x,y)=\sum_{j=m+1}^{\lambda_m}\sum_{k=0}^{n}\frac{\lambda_m-j+1}{\lambda_m-m}\ a_{jk}\cos jx\cos ky,$$

$$\sum\nolimits_{01}^{\lambda}(m,n,x,y)=\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\ a_{jk}\cos jx\cos ky,$$

$$\sum\nolimits_{11}^{\lambda}(m,n,x,y)=\sum_{j=m+1}^{\lambda_m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_m-j+1}{\lambda_m-m}\ \frac{\lambda_n-k+1}{\lambda_n-n}a_{jk}\cos jx\cos ky$$

and we have

$$\sum\nolimits_{11}^{\lambda}(m,n;x,y)=\frac{1}{(\lambda_m-m)}\sum_{u=m+1}^{\lambda_m}\Big(\sum\nolimits_{01}^{\lambda}(u,n;x,y)-\sum\nolimits_{01}^{\lambda}(m,n;x,y)\Big),$$

$$\sum\nolimits_{11}^{\lambda}(m,n;x,y)=\frac{1}{(\lambda_n-n)}\sum_{v=n+1}^{\lambda_n}\Big(\sum\nolimits_{10}^{\lambda}(m,v;x,y)-\sum\nolimits_{10}^{\lambda}(m,n;x,y)\Big).$$

This implies

$$\sum\nolimits_{11}^{\lambda}(m,n;x,y)\leq\left\{\begin{array}{l} 2\sup\limits_{m\leq u\leq\lambda_m}\Big(|\sum_{01}^{\lambda}(u,n;x,y)|\Big)\\ 2\sup\limits_{n\leq v\leq\lambda_n}\Big(|\sum_{10}^{\lambda}(m,v;x,y)|\Big)\end{array}\right\}. \tag{1.8}$$

The second result of this paper is the following theorem.

Theorem 1.2. (i) *Let $E\subset T^2$. Assume that the following conditions are satisfied:*

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|\sum\nolimits_{10}^{\lambda}(m,n;x,y)|\Big)=0, \tag{1.9}$$

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\Big(\sup_{(x,y)\in E}|\sum\nolimits_{01}^{\lambda}(m,n;x,y)|\Big)=0. \tag{1.10}$$

If $V_{mn}^{\lambda}(x,y)$ converges uniformly to $f(x,y)$ on $E\subset T^2$ as $\min\{m,n\}\to\infty$ (that is, in the unrestricted sense), then so does S_{mn}.

(ii) *Assume that the following conditions are satisfied for some $r\geq 1$:*

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\Big(\|\sum\nolimits_{10}^{\lambda}(m,n;x,y)\|_r\Big)=0,$$

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\Big(\|\sum\nolimits_{01}^{\lambda}(m,n;x,y)\|_r\Big)=0. \tag{1.11}$$

If $\|V_{mn}^{\lambda}-f\|_r\to 0$ unrestictedly then $\|S_{mn}-f\|_r\to 0$ as $\min\{m,n\}\to\infty$.

We will also prove the following theorem.

Theorem 1.3. *Assume that the conditions* (1.2)–(1.4) *and* (1.6)–(1.7) *are satisfied for some* $p \geq 1$, *then*

(i) *if* $V^{\lambda}_{mn}(x, y)$ *converges uniformly to* $f(x, y)$ *as* $\min\{m, n\} \to \infty$, *then so does* S_{mn};

(ii) *if* $\|V^{\lambda}_{mn} - f\|_r \longrightarrow 0$ *unrestictedly for some* r *with* $1 \leq r < \infty$, *then* $\|S_{mn} - f\|_r \longrightarrow 0$ *as* $\min\{m, n\} \to \infty$.

2. Notation and Formulas

We define for every $\alpha = 0, 1, 2, \ldots$ the sequence $S^{\alpha}_0, S^{\alpha}_1, S^{\alpha}_2, \ldots$ by the conditions

$$S^0_n = S_n, \quad S^{\alpha}_n = \sum_{u=0}^{n} S^{\alpha-1}_u, \quad \alpha \geq 1$$

and

$$A^0_n = 1, \quad A^{\alpha}_n = \sum_{u=0}^{n} A^{\alpha-1}_u, \quad \alpha \geq 1,$$

denotes binomial coefficients. Also

$$A^{\alpha}_n = \binom{n+\alpha}{n} \simeq \frac{n^{\alpha}}{\Gamma(\alpha+1)}, \quad \alpha \neq -1, -2, -3, \ldots.$$

The Cesàro means T^{α}_n of order α of $\sum a_n$ will be defined by $T^{\alpha}_n = \frac{S^{\alpha}_n}{A^{\alpha}_n}$ and also it is known [8] that $\int_0^{\pi} |T^{\alpha}_n(x)| dx$, $\alpha > 0$, is bounded for all n.

3. Lemmas

We require the following lemmas for the proof of our results.

Lemma 3.1. *For* $m, n \geq 0$ *and* $p > 1$, *the following representation holds:*

$$\begin{aligned} S_{mn}(x, y) &= \sum_{j=0}^{m} \sum_{k=0}^{n} \lambda_j \lambda_k a_{jk} \cos jx \cos ky \\ &= \sum_{j=0}^{m} \sum_{k=0}^{n} \triangle_{pp} a_{jk} S^{p-1}_j(x) S^{p-1}_k(y) + \sum_{j=0}^{m} \sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S^{p-1}_j(x) S^t_n(y) \\ &\quad + \sum_{k=0}^{n} \sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} S^s_m(x) S^{p-1}_k(y) + \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S^s_m(x) S^t_n(y). \end{aligned}$$

Lemma 3.2 ([1])**.** *For* $m, n \geq 0$ *and* $\lambda > 1$, *the following representation holds:*

$$\begin{aligned} S_{mn} - \sigma_{mn} =& \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{\lambda_m, \lambda_n} - \sigma_{\lambda_m, n} - \sigma_{m, \lambda_n} + \sigma_{mn}) \\ &+ \frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m, n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) \\ &- \sum\nolimits_{11}^{\lambda}(m, n, x, y) - \sum\nolimits_{10}^{\lambda}(m, n, x, y) - \sum\nolimits_{01}^{\lambda}(m, n, x, y). \end{aligned}$$

Lemma 3.3. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$V_{mn}^{\lambda} - S_{mn} = \sum\nolimits_{11}^{\lambda}(m, n, x, y) + \sum\nolimits_{10}^{\lambda}(m, n, x, y) + \sum\nolimits_{01}^{\lambda}(m, n, x, y).$$

Proof. We have

$$V_{mn}^{\lambda}(x, y) = \frac{1}{(\lambda_m - m)(\lambda_n - n)} \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} S_{jk}(x, y).$$

Performing double summation by parts, we have

$$\begin{aligned}
V_{mn}^{\lambda} =& \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{\lambda_m, \lambda_n} - \frac{\lambda_m + 1}{\lambda_m - m} \frac{n + 1}{\lambda_n - n} \sigma_{\lambda_m, n} \\
& - \frac{m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} \sigma_{m, \lambda_n} + \frac{m + 1}{\lambda_m - m} \frac{n + 1}{\lambda_n - n} \sigma_{mn} \\
=& \frac{\lambda_m + 1}{\lambda_m - m} \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{\lambda_m, \lambda_n} - \sigma_{\lambda_m, n} - \sigma_{m, \lambda_n} + \sigma_{mn}) \\
& + \frac{\lambda_m + 1}{\lambda_m - m} (\sigma_{\lambda_m, n} - \sigma_{mn}) + \frac{\lambda_n + 1}{\lambda_n - n} (\sigma_{m, \lambda_n} - \sigma_{mn}) + \sigma_{mn}.
\end{aligned}$$

The use of Lemma 3.2, gives

$$\begin{aligned}
V_{mn}^{\lambda} - S_{mn} =& \sum_{j=m+1}^{\lambda_m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_m - j + 1}{\lambda_m - m} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \cos jx \cos ky \\
& + \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} \cos jx \cos ky \\
& + \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} a_{jk} \cos jx \cos ky.
\end{aligned}$$

□

Lemma 3.4. *For $m, n \geq 0$ and $\lambda > 1$, we have the following representation:*

$$\begin{aligned}
\sum\nolimits_{10}^{\lambda}(m, n; x, y) =& \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} a_{jk} \cos jx \cos ky \\
=& \sum_{j=m+1}^{\lambda_m} \sum_{k=0}^{n} \frac{\lambda_m - j + 1}{\lambda_m - m} \triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y) \\
& + \sum_{j=m+1}^{\lambda_m} \sum_{t=0}^{p-1} \frac{\lambda_m - j + 1}{\lambda_m - m} \triangle_{pt} a_{j,n+1} S_j^{p-1}(x) S_n^{t}(y) \\
& + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \sum_{k=0}^{n} \triangle_{sp} a_{j+1,k} S_j^{s}(x) S_k^{p-1}(y) \\
& + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \triangle_{st} a_{j+1,n+1} S_j^{s}(x) S_n^{t}(y)
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s=0}^{p-1}\sum_{k=0}^{n} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) \\
&-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_m^s(x) S_n^t(y).
\end{aligned}
$$

Proof. We have by summation by parts,

$$
\begin{aligned}
&\textstyle\sum_{10}^{\lambda}(m,n;x,y) \\
&= \sum_{k=0}^{n} \cos ky \left(\sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, a_{jk} \cos jx \right) \\
&= \sum_{k=0}^{n} \cos ky \left(\sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, \triangle_{p0} a_{jk} S_j^{p-1}(x) \right. \\
&\quad \left. + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \triangle_{s0} a_{j+1,k} S_j^s(x) - \sum_{s=0}^{p-1} \triangle_{s0} a_{m+1,k} S_m^s(x) \right) \\
&= \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, S_j^{p-1}(x) \left(\sum_{k=0}^{n} \triangle_{p0} a_{jk} \cos ky \right) \\
&\quad + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{s0} a_{j+1,k} \cos ky \right) S_j^s(x) \\
&\quad - \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{s0} a_{m+1,k} \cos ky \right) S_m^s(x) \\
&= \sum_{j=m+1}^{\lambda_m} \frac{\lambda_m - j + 1}{\lambda_m - m}\, S_j^{p-1}(x) \left(\sum_{k=0}^{n} \triangle_{pp} a_{jk} S_k^{p-1}(y) + \sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S_n^t(y) \right) \\
&\quad + \frac{1}{\lambda_m - m} \sum_{j=m+1}^{\lambda_m} \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{sp} a_{j+1,k} S_k^{p-1}(y) + \sum_{t=0}^{p-1} \triangle_{st} a_{j+1,n+1} S_n^t(y) \right) S_j^s(x) \\
&\quad - \sum_{s=0}^{p-1} \left(\sum_{k=0}^{n} \triangle_{sp} a_{m+1,k} S_k^{p-1}(y) + \sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_n^t(y) \right) S_m^s(x).
\end{aligned}
$$

Similarly we can have representation for $\sum_{01}^{\lambda}(m,n;x,y)$. □

4. Proof of Theorems

Proof of Theorem 1.1. For $m, n \geq 0$ and $p > 1$, we have from Lemma 3.1,

$$
\begin{aligned}
S_{mn}(x,y) = &\sum_{j=0}^{m}\sum_{k=0}^{n} \triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y) + \sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S_j^{p-1}(x) S_n^t(y) \\
&+ \sum_{k=0}^{n}\sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) + \sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_m^s(x) S_n^t(y)
\end{aligned}
$$

$$=\sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3 + \sum\nolimits_4.$$

Using the results as given in [6] that $S_j^p(x) = O\left(\frac{1}{x^p}\right)$, for all $p \geq 2$, $0 < x \leq \pi$, etc, we have for $0 < x, y \leq \pi$,

$$\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |\triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y)| < \infty \quad \text{(by (1.2))}$$

and also by (1.3)–(1.5), we have

$$\begin{aligned}\sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} &\leq \sum_{t=0}^{p-1}\sum_{v=0}^{t} \binom{t}{v} \left(\sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}|\right) \\ &\leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| \\ &\leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty.\end{aligned}$$

Thus,

$$\sum_{j=0}^{m}\sum_{t=0}^{p-1} \triangle_{pt} a_{j,n+1} S_j^{p-1}(x) S_n^t(y) \to 0 \text{ as } \min\{m,n\} \to \infty.$$

And similarly

$$\begin{aligned}\sum_{s=0}^{p-1}\sum_{k=0}^{n} \triangle_{sp} a_{m+1,k} &\leq \sum_{s=0}^{p-1}\sum_{u=0}^{s} \binom{s}{u} (\sum_{k=0}^{n} |\triangle_{0p} a_{m+u+1,k}|) \\ &\leq \sup_{m<j\leq m+p} \sum_{k=0}^{n} |\triangle_{0p} a_{jk}| \\ &\leq \sup_{m<j\leq m+p} \sum_{k=0}^{n} |\triangle_{0p} a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty.\end{aligned}$$

Thus,

$$\sum_{k=0}^{n}\sum_{s=0}^{p-1} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) \to 0,$$

as $\min\{m,n\} \to \infty$. Also

$$\begin{aligned}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} &\leq \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t} \binom{s}{u}\binom{t}{v} |\triangle_{00} a_{m+u+1,n+v+1}| \\ &\leq \sup_{j>m,k>n} |a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty.\end{aligned}$$

So,

$$\sum_{s=0}^{p-1}\sum_{t=0}^{p-1} \triangle_{st} a_{m+1,n+1} S_m^s(x) S_n^t(y) \to 0 \text{ as } \min\{m,n\} \to \infty.$$

Consequently, series (1.1) converges to the function $f(x,y)$ where

$$f(x,y)=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\quad\text{and}\quad\lim_{m,n\to\infty}S_{mn}(x,y)=f(x,y).$$

Now we will calculate $\|\sum_1\|$, $\|\sum_2\|$, $\|\sum_3\|$ and $\|\sum_4\|$ in the following way:

$$\begin{aligned}
\left\|\sum\nolimits_1\right\| =&\left\|\sum_{j=0}^{m}\sum_{k=0}^{n}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right\|\\
\leq&\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|\int_0^{\pi}\int_0^{\pi}|S_j^{p-1}(x)S_k^{p-1}(y)|dxdy\\
\leq&\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|A_j^{p-1}A_k^{p-1}\int_0^{\pi}\int_0^{\pi}|T_j^{p-1}(x)T_k^{p-1}(y)|dxdy\\
\leq&C_p\sum_{j=0}^{m}\sum_{k=0}^{n}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1},\\
\left\|\sum\nolimits_2\right\| =&\left\|\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}S_j^{p-1}(x)S_n^{t}(y)\right\|\\
\leq&\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\left(\sum_{j=0}^{m}|\triangle_{p0}a_{j,n+v+1}|\right)A_j^{p-1}A_n^{t}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}|T_j^{p-1}(x)T_n^{t}(y)|dxdy\\
\leq&C_p\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\;j^{p-1}\left(\sum_{t=0}^{p-1}n^t\right)\\
\leq&C_p\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\;j^{p-1}k^{p-1},\\
\left\|\sum\nolimits_3\right\| =&\left\|\sum_{s=0}^{p-1}\sum_{k=0}^{n}\triangle_{sp}a_{m+1,k}S_m^{s}(x)S_k^{p-1}(y)\right\|\\
\leq&\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\left(\sum_{k=0}^{n}|\triangle_{0p}a_{m+u+1,k}|\right)A_m^{s}A_k^{p-1}\int_0^{\pi}\int_0^{\pi}|T_m^{s}(x)T_k^{p-1}(y)|dxdy\\
\leq&C_p\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\;k^{p-1}\left(\sum_{s=0}^{p-1}m^s\right)\\
\leq&C_p\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\;j^{p-1}k^{p-1},\\
\left\|\sum\nolimits_4\right\| =&\left\|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}S_m^{s}(x)S_n^{t}(y)\right\|
\end{aligned}$$

$$\leq \sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\sum_{u=0}^{s}\sum_{v=0}^{t}\binom{s}{u}\binom{t}{v}|\triangle_{00}a_{m+u+1,n+v+1}|A_m^s A_n^t \int_0^\pi \int_0^\pi |T_m^s(x)T_n^t(y)|dxdy$$
$$\leq C_p \sup_{j>m,k>n}|a_{jk}|\, j^{p-1}k^{p-1}.$$

Now let R_{mn} consists of all (j,k) with $j>m$ or $k>n$, that is,

$$\sum\sum\nolimits_{(j,k)\in R_{mn}} = \sum_{j=m+1}^{\infty}\sum_{k=0}^{n} + \sum_{j=0}^{\infty}\sum_{k=n+1}^{\infty} + \sum_{j=m+1}^{\infty}\sum_{k=n+1}^{\infty}.$$

Then

$$\begin{aligned}
\|f-S_{mn}\|_r =& \left(\int_0^\pi\int_0^\pi |f(x,y)-S_{mn}(x,y)|^r\,dxdy\right)^{1/r}, \quad 1\leq r<\infty,\\
\leq& \left\|\sum_{(j,k)}\sum_{\in R_{mn}} \triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right\|_r\\
&+\left\|\sum_{j=0}^{m}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,n+1}S_j^{p-1}(x)S_n^t(y)\right\|_r\\
&+\left\|\sum_{k=0}^{n}\sum_{s=0}^{p-1}\triangle_{sp}a_{m+1,k}S_m^s(x)S_k^{p-1}(y)\right\|_r\\
&+\left\|\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}S_m^s(x)S_n^t(y)\right\|_r\\
\leq& C_{pr}\Bigg\{\left(\sum_{(j,k)\in R_{mn}}|\triangle_{pp}a_{jk}|j^{p-1}k^{p-1}\right)\\
&+\left(\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\,j^{p-1}k^{p-1}\right)\\
&+\left(\sup_{m<j\leq m+p}\sum_{k=0}^{n}|\triangle_{0p}a_{jk}|\,j^{p-1}k^{p-1}\right)\\
&+\left(\sup_{j>m,k>n}|a_{jk}|\,j^{p-1}k^{p-1}\right)\Bigg\} \quad \text{(as discussed above)}\\
\to& 0 \quad \text{as } \min\{m,n\}\to\infty \quad \text{(by (1.2)-(1.5))},
\end{aligned}$$

which proves (ii) part.

Proof of Theorem 1.2. Using the relation (1.8), we find that (1.9) or (1.10) implies

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits_{11}^{\lambda}(m,n;x,y)\right|\right)=0. \tag{4.1}$$

Assume that $V^{\lambda}_{mn}(x,y)$ converges uniformly on E to $f(x,y)$. Then by Lemma 3.3, we get

$$\overline{\lim_{m,n\to\infty}}\left(\left|\sup_{(x,y)\in E}\left(S_{mn}(x,y)-V^{\lambda}_{mn}(x,y)\right)\right|\right)\leq\overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits^{\lambda}_{10}(m,n;x,y)\right|\right)$$
$$+\overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits^{\lambda}_{01}(m,n;x,y)\right|\right)$$
$$+\overline{\lim_{m,n\to\infty}}\left(\sup_{(x,y)\in E}\left|\sum\nolimits^{\lambda}_{11}(m,n;x,y)\right|\right).$$

After taking $\lambda\downarrow 1$ the result follows from (1.9), (1.10) and (4.1).

For (ii) part of theorem, we have

$$\left\|\sum\nolimits^{\lambda}_{11}(m,n;x,y)\right\|_r=\frac{1}{\lambda_m-m}\sum_{u=m+1}^{\lambda_m}\left(\left\|\sum\nolimits^{\lambda}_{01}(u,n;x,y)\right\|_r+\left\|\sum\nolimits^{\lambda}_{01}(m,n;x,y)\right\|_r\right)$$
$$\leq 2\left(\sup_{m\leq u\leq\lambda_m}\left(\left\|\sum\nolimits^{\lambda}_{01}(u,n;x,y)\right\|_r\right)\right).$$

Thus (1.11) implies

$$\lim_{\lambda\downarrow 1}\overline{\lim_{m,n\to\infty}}\left\|\sum\nolimits^{\lambda}_{11}(m,n;x,y)\right\|_r=0.$$

Thus, the result of Theorem 1.2 (ii) follows.

Proof of Theorem 1.3. Using the Lemma 3.4, we can write the expression for $\sum^{\lambda}_{01}(m,n;x,y)$ as

$$\begin{aligned}\sum\nolimits^{\lambda}_{01}(m,n;x,y)=&\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\;a_{jk}\cos jx\cos ky\\
=&\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\;\triangle_{pp}a_{jk}S^{p-1}_j(x)S^{p-1}_k(y)\\
&+\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\frac{\lambda_n-k+1}{\lambda_n-n}\;\triangle_{sp}a_{m+1,k}S^{s}_m(x)S^{p-1}_k(y)\\
&+\frac{1}{\lambda_n-n}\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\sum_{t=0}^{p-1}\triangle_{pt}a_{j,k+1}S^{p-1}_j(x)S^{t}_k(y)\\
&+\frac{1}{\lambda_n-n}\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,k+1}S^{s}_m(x)S^{t}_k(y)\\
&-\sum_{t=0}^{p-1}\sum_{j=0}^{m}\triangle_{pt}a_{j,n+1}S^{p-1}_j(x)S^{t}_n(y)\end{aligned}$$

$$
\begin{aligned}
&-\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}\triangle_{st}a_{m+1,n+1}S_m^s(x)S_n^t(y)\\
=&I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
$$

Now by using (1.2)–(1.4) and (1.6) along with estimates of $S_j^{p-1}(x)$ etc., as mentioned in [6], we have the following estimates in brief:

$$
\begin{aligned}
|I_1|=&\left|\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{pp}a_{jk}S_j^{p-1}(x)S_k^{p-1}(y)\right|\\
\le&\sum_{j=0}^{m}\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n-k+1}{\lambda_n-n}\left|\triangle_{pp}a_{jk}\right|\\
\to&0\quad\text{as } \min\{m,n\}\to\infty.
\end{aligned}
$$

Consequently, $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\Big(\sup\limits_{(x,y)\in E}|I_1|\Big)\to 0$ as $\min\{m,n\}\to\infty$. Also,

$$
\begin{aligned}
|I_2|=&\left|\sum_{k=n+1}^{\lambda_n}\sum_{s=0}^{p-1}\frac{\lambda_n-k+1}{\lambda_n-n}\triangle_{sp}a_{m+1,k}S_m^s(x)S_k^{p-1}(y)\right|\\
\le&\sum_{s=0}^{p-1}\sum_{u=0}^{s}\binom{s}{u}\sum_{k=n+1}^{\lambda_n}|\triangle_{0p}a_{m+u+1,k}|\\
\le&\sup_{m<j\le m+p}\sum_{k=n+1}^{\lambda_n}|\triangle_{0p}a_{jk}|\ \to 0\qquad\text{as } \min\{m,n\}\to\infty.
\end{aligned}
$$

So, $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\left(\sup\limits_{(x,y)\in E}|I_2|\right)\to 0$ as $\min\{m,n\}\to\infty$. Also,

$$
\begin{aligned}
|I_3|\le&\sup_{n<k\le\lambda_n}\sum_{t=0}^{p-1}\sum_{j=0}^{m}|\triangle_{pt}a_{j,k+1}|\\
\le&\sup_{n<k\le\lambda_n}\sum_{t=0}^{p-1}\sum_{v=0}^{t}\binom{t}{v}\sum_{j=0}^{m}|\triangle_{pt}a_{j,k+v+1}|\\
\le&\sup_{n<k\le\lambda_n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|\to 0\text{ as } \min\{m,n\}\to\infty,
\end{aligned}
$$

which implies $\lim\limits_{\lambda\downarrow 1}\ \overline{\lim\limits_{m,n\to\infty}}\left(\sup\limits_{(x,y)\in E}|I_3|\right)\to 0$ as $\min\{m,n\}\to\infty$. Now,

$$
\begin{aligned}
|I_4|\le&\sup_{n<k\le\lambda_n}\sum_{s=0}^{p-1}\sum_{t=0}^{p-1}|\triangle_{st}a_{m+1,k+1}|\\
\le&\sup_{j>m,k>n}|a_{jk}|\to 0\text{ as } \min\{m,n\}\to\infty.
\end{aligned}
$$

Thus $\lim\limits_{\lambda\downarrow 1} \overline{\lim\limits_{m,n\to\infty}} \left(\sup\limits_{(x,y)\in E} |I_4| \right) \to 0$ as $\min\{m,n\} \to \infty$. Also,

$$|I_5| \leq \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}| \leq \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| \to \quad 0 \text{ as } \min\{m,n\} \to \infty,$$

which implies $\lim\limits_{\lambda\downarrow 1} \overline{\lim\limits_{m,n\to\infty}} \left(\sup\limits_{(x,y)\in E} |I_5| \right) \to 0$ as $\min\{m,n\} \to \infty$. Also,

$$\begin{aligned} |I_6| &\leq \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{s} \sum_{v=0}^{t} \binom{s}{u} \binom{t}{v} |\triangle_{00} a_{m+u+1,n+v+1}| \\ &\leq \sup_{j>m,k>n} |a_{jk}| \to 0 \text{ as } \min\{m,n\} \to \infty, \end{aligned}$$

and

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \left(\sup_{(x,y)\in E} |I_6| \right) \to 0 \text{ as } \min\{m,n\} \to \infty.$$

Thus, combining all these, we have

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \left(\sup_{(x,y)\in E} \left|\sum\nolimits_{01}^{\lambda} (m,n;x,y)\right| \right) = 0.$$

Similarly (1.2)–(1.4) and (1.7) results in

$$\lim_{\lambda\downarrow 1} \overline{\lim_{m,n\to\infty}} \left(\sup_{(x,y)\in E} \left|\sum\nolimits_{10}^{\lambda} (m,n;x,y)\right| \right) = 0.$$

Thus, first part of theorem follows from Theorem 1.2.

Proof of (ii). We have

$$\|S_{mn} - f\|_r \leq \|S_{mn} - V_{mn}^{\lambda}\|_r + \|V_{mn}^{\lambda} - f\|_r.$$

By assumption $\|V_{mn}^{\lambda} - f\|_r \to 0$, so it is sufficient to show that

$$\|S_{mn} - V_{mn}^{\lambda}\|_r \to 0 \text{ as } \min\{m,n\} \to \infty.$$

By Lemma 3.3, we have

$$\begin{aligned} \|S_{mn} - V_{mn}^{\lambda}\|_r \leq& \left\|\sum\nolimits_{10}^{\lambda} (m,n;x,y)\right\|_r + \left\|\sum\nolimits_{01}^{\lambda} (m,n;x,y)\right\|_r \\ &+ \left\|\sum\nolimits_{11}^{\lambda} (m,n;x,y)\right\|_r. \end{aligned}$$

Now in order to estimate $\|\sum_{01}^{\lambda} (m,n;x,y)\|_r$, we first find $\|I_1\|$, $\|I_2\|$, $\|I_3\|$, $\|I_4\|$, $\|I_5\|$ and $\|I_6\|$, so we have

$$\begin{aligned} \|I_1\| &= \left\| \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{pp} a_{jk} S_j^{p-1}(x) S_k^{p-1}(y) \right\| \\ &\leq \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{pp} a_{jk} A_j^{p-1} A_k^{p-1} \int_0^{\pi} \int_0^{\pi} |T_j^{p-1}(x) T_k^{p-1}(y)| dx dy \end{aligned}$$

$$
\begin{aligned}
&\leq C_p \sum_{j=0}^{m} \sum_{k=n+1}^{\lambda_n} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| j^{p-1} k^{p-1},\\
\|I_2\| &= \left\| \sum_{k=n+1}^{\lambda_n} \sum_{s=0}^{p-1} \frac{\lambda_n - k + 1}{\lambda_n - n} \triangle_{sp} a_{m+1,k} S_m^s(x) S_k^{p-1}(y) \right\|\\
&\leq C_p \sum_{s=0}^{p-1} \sum_{u=0}^{s} \binom{s}{u} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{m+u+1,k}| k^{p-1} m^s\\
&\leq C_p \sup_{m<j\leq m+p} \left(\sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| k^{p-1} \right) \left(\sum_{s=0}^{p-1} m^s \right)\\
&\leq C_p \sup_{m<j\leq m+p} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| j^{p-1} k^{p-1},\\
\|I_3\| &\leq C_p \sup_{n<k\leq \lambda_n} \sum_{t=0}^{p-1} \sum_{j=0}^{m} |\triangle_{pt} a_{j,k+1}| j^{p-1} k^t\\
&\leq C_p \sup_{n<k\leq \lambda_n} \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{j=0}^{m} |\triangle_{pt} a_{j,k+v+1}| j^{p-1} k^t\\
&\leq C_p \sup_{n<k\leq \lambda_n + p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1},\\
\|I_4\| &\leq C_p \sup_{n<k\leq \lambda_n} \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} |\triangle_{st} a_{m+1,k+1}| m^s k^t\\
&\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1},\\
\|I_5\| &\leq C_p \sum_{t=0}^{p-1} \sum_{v=0}^{t} \binom{t}{v} \sum_{j=0}^{m} |\triangle_{p0} a_{j,n+v+1}| j^{p-1} n^t\\
&\leq C_p \sup_{n<k\leq n+p} \sum_{j=0}^{m} |\triangle_{p0} a_{jk}| j^{p-1} k^{p-1},\\
\|I_6\| &\leq C_p \sum_{s=0}^{p-1} \sum_{t=0}^{p-1} \sum_{u=0}^{s} \sum_{v=0}^{t} \binom{s}{u} \binom{t}{v} |\triangle_{00} a_{m+u+1,n+v+1}| m^s n^t\\
&\leq C_p \sup_{j>m,k>n} |a_{jk}| j^{p-1} k^{p-1}.
\end{aligned}
$$

Thus, we can estimate

$$
\begin{aligned}
\left\| \sum\nolimits_{01}^{\lambda}(m,n;x,y) \right\|_r \leq & C_{pr} \sum_{k=n+1}^{\lambda_n} \sum_{j=0}^{m} \frac{\lambda_n - k + 1}{\lambda_n - n} |\triangle_{pp} a_{jk}| j^{p-1} k^{p-1}\\
& + C_{pr} \left(\sup_{m<j\leq m+p} \sum_{k=n+1}^{\lambda_n} |\triangle_{0p} a_{jk}| j^{p-1} k^{p-1} \right)
\end{aligned}
$$

$$+ C_{pr}\left(\sup_{n<k\leq\lambda_n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|j^{p-1}k^{p-1}\right)$$
$$+ C_{pr}\left(\sup_{j>m,k>n}|a_{jk}|j^{p-1}k^{p-1}\right)$$
$$+ C_{pr}\left(\sup_{n<k\leq n+p}\sum_{j=0}^{m}|\triangle_{p0}a_{jk}|j^{p-1}k^{p-1}\right)$$
$$+ C_{pr}\left(\sup_{j>m,k>n}|a_{jk}|j^{p-1}k^{p-1}\right).$$

By (1.2)–(1.4) and (1.6), we conclude that

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\|\textstyle\sum_{01}^{\lambda}(m,n;x,y)\|_r\right) = 0.$$

Similarly, by conditions (1.2)–(1.4) and (1.7), we get

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\|\textstyle\sum_{10}^{\lambda}(m,n;x,y)\|_r\right) = 0.$$

Also, by (1.8), we have

$$\lim_{\lambda\downarrow 1}\ \overline{\lim_{m,n\to\infty}}\left(\|\textstyle\sum_{11}^{\lambda}(m,n;x,y)\|_r\right) = 0.$$

Thus, $\|S_{mn} - V_{mn}^{\lambda}\|_r \to 0$ as $\min\{m,n\}\to\infty$.

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