Abstract. In this paper, for holomorphic function $f(z) = z + c_2z^2 + c_3z^3 + \cdots$ belong to the class of $N(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ on the boundary point of the unit disc.

1. Introduction

Let $f$ be a holomorphic function in the unit disc $E = \{ z : |z| < 1 \}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point $z$ in the disc $E$, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = ze^{i\theta}$, where $\theta$ is a real number ([8], p. 329). For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2, 7]).

The basic tool in proving our results is the following lemma due to Jack.

**Lemma 1.1** (Jack’s lemma). Let $f(z)$ be holomorphic function in the unit disc $E$ with $f(0) = 0$. Then if $|f(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in E$, then there exists a real number $k \geq 1$ such that

$$\frac{z_0f'(z_0)}{f(z_0)} = k.$$

Let $\mathcal{A}$ denote the class of functions

$$f(z) = z + c_2z^2 + c_3z^3 + \cdots,$$

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that are holomorphic in the unit disc \( E \). Also, \( N(\lambda) \) be the subclass of \( \mathcal{A} \) consisting of all functions \( f(z) \) which satisfy

\[
|zf'(z)|^\alpha \left| z \left( \frac{zf'(z)}{f(z)} \right)' \right|^\beta < \left( \frac{1}{2} \lambda \right)^\beta,
\]

for some real \( \alpha \geq 0, \beta > 0 \) and \( \lambda = \frac{\beta}{\beta + \alpha} \).

Let \( f(z) \in N(\lambda) \) and define \( \phi(z) \) in \( E \) by

\[
\phi(z) = \frac{(h(z))^{\frac{1}{2}} - 1}{(h(z))^{\frac{1}{2}} + 1},
\]

where \( h(z) = \frac{zf'(z)}{f(z)} \).

Obviously, \( \phi(z) \) is holomorphic function in the unit disc \( E \) and \( \phi(0) = 0 \). We want to prove \( |\phi(z)| < 1 \) for \( |z| < 1 \). Differentiating (1.2) and simplifying, we obtain

\[
\left( \frac{zf'(z)}{f(z)} \right)' = \frac{2\lambda \phi'(z)}{(1 - \phi(z))} \left( 1 + \phi(z) \right) \left( 1 + \phi(z) \right)^{\lambda - 1}
\]

and, so

\[
\left| z f'(z) \right|^{\alpha} \left| z \left( \frac{zf'(z)}{f(z)} \right)' \right|^\beta = \left| \frac{1 + \phi(z)}{1 - \phi(z)} \right|^{\alpha \beta + \beta(\lambda - 1)} \left| \frac{2\lambda \phi'(z)}{(1 - \phi(z))^2} \right|^\beta
\]

\[
= \left| \frac{2\lambda \phi'(z)}{(1 - \phi(z))^2} \right|^\beta < \left( \frac{\lambda}{2} \right)^\beta.
\]

If there exists a point \( z_0 \in E \) such that

\[
\max_{|z| \leq |z_0|} |\phi(z)| = |\phi(z_0)| = 1,
\]

then Jack’s lemma gives us that \( \phi(z_0) = e^{i\theta} \) and \( z_0 \phi'(z_0) = k\phi(z_0), k \geq 1 \).

Thus we have

\[
\left| \frac{zf'(z_0)}{f(z_0)} \right|^{\alpha} \left| \frac{zf'(z_0)}{f(z_0)} \right|^{\beta} = \left| \frac{2\lambda z_0 \phi'(z_0)}{(1 - \phi(z_0))^2} \right|^\beta = \left| \frac{2\lambda k e^{i\theta}}{(1 - e^{i\theta})^2} \right|^\beta
\]

\[
= \frac{(2\lambda k)^\beta}{|1 - e^{i\theta}|^2} \geq \frac{(2\lambda)^\beta}{2^{2\beta}} = \left( \frac{\lambda}{2} \right)^\beta.
\]

This contradict (1.1). So, there is no point \( z_0 \in E \) such that \( \phi(z_0) = 1 \). This means that \( |\phi(z)| < 1 \) for \( |z| < 1 \). Thus, from the Schwarz lemma, we obtain

\[
|c_2| \leq \frac{2\beta}{\beta + \alpha}.
\]

Moreover, the equality \( |c_2| = \frac{2\beta}{\beta + \alpha} \) occurs for the function

\[
f(z) = e^{\int \frac{i}{t} \frac{1}{1 + t} dt}.
\]
That proves the following lemma.

**Lemma 1.2.** If \( f(z) \in N(\lambda) \), then we have

\[
|c_2| \leq \frac{2\beta}{\beta + \alpha}.
\]

The equality in (1.3) occurs for the function

\[
f(z) = e^0 \int \frac{1}{t} \left( \frac{1+t}{1-t} \right)^\lambda dt.
\]

The following boundary version of the Schwarz lemma was proved in 1938 by Unkelbach in [21] and then rediscovered and partially improved by Osserman in [17].

**Lemma 1.3.** Let \( f(z) \) be a holomorphic function self-mapping of \( E = \{ z : |z| < 1 \} \), that is \( |f(z)| < 1 \) for all \( z \in E \). Assume that there is a \( b \in \partial E \) so that \( f \) extend continuously to \( b \), \( |f(b)| = 1 \) and \( f'(b) \) exists. Then

\[
|f'(b)| \geq \frac{2}{1 + |f'(0)|}.
\]

The equality in (1.4) holds if and only if \( f \) is of the form

\[
f(z) = -z \frac{a - z}{1 - az}, \quad \text{for all } z \in E,
\]

for some constant \( a \in (-1, 0] \).

**Corollary 1.1.** Under the hypotheses lemma, we have

\[
|f'(b)| \geq 1,
\]

with equality only if \( f \) is of the form

\[
f(z) = ze^{i\theta},
\]

where \( \theta \) is a real number.

The following Lemma 1.4 and Corollary 1.2, known as the Julia-Wolff lemma, is needed in the sequel [15].

**Lemma 1.4 (Julia-Wolff lemma).** Let \( f \) be a holomorphic function in \( E \), \( f(0) = 0 \) and \( f(E) \subset E \). If, in addition, the function \( f \) has an angular limit \( f(b) \) at \( b \in \partial E \), \( |f(b)| = 1 \), then the angular derivative \( f'(b) \) exists and \( 1 \leq |f'(b)| \leq \infty \).

**Corollary 1.2.** The holomorphic function \( f \) has a finite angular derivative \( f'(b) \) if and only if \( f' \) has the finite angular limit \( f'(b) \) at \( b \in \partial E \).

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [8, 18]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1, 2, 5–7, 15–17, 19, 20] and references therein).
Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwar lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, with a zero set $\{z_k\}$ (see [5]).

S. G. Krantz and D. M. Burns [3] and D. Chelst [4] studied the uniqueness part of the Schwarz lemma. According to M. Mateljević’s studies, some other types of results which are related to the subject can be found in ([13,14] and [12]). In addition, [11] was posed on ResearchGate where is discussed concerning results in more general aspects.

Also, M. Jeong [10] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [9] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main Results

In this section, for holomorphic function $f(z) = z + c_2 z^2 + c_3 z^3 + \cdots$ belong to the class of $N(\lambda)$, it has been estimated from below the modulus of the angular derivative of the function $\frac{zf'(z)}{f(z)}$ on the boundary point of the unit disc.

**Theorem 2.1.** Let $f(z) \in N(\lambda)$. Assume that, for some $b \in \partial E$, $f$ has angular limit $f(b)$ at $b$ and $\frac{bf'(b)}{f(b)} = i^\lambda$. Then we have the inequality

$$\left| \left( \frac{z f'(z)}{f(z)} \right)_{z=b} \right| \geq \frac{\beta}{\beta + \alpha}. \tag{2.1}$$

The equality in (2.1) occurs for the function

$$f(z) = e^{\int \frac{1}{(1 + (i^\lambda)^{1/\lambda})^{1/\lambda}} dt},$$

where $\lambda = \frac{\beta}{\beta + \alpha}$.

**Proof.** Consider the function

$$\phi(z) = \frac{(h(z))^{1/\lambda} - 1}{(h(z))^{1/\lambda} + 1},$$

where $h(z) = \frac{zf'(z)}{f(z)}$ and $\lambda = \frac{\beta}{\beta + \alpha}$. $\phi(z)$ is a holomorphic function in the unit disc $E$ and $\phi(0) = 0$. From the Jack’s lemma and since $f(z) \in N(\lambda)$, we obtain $|\phi(z)| < 1$ for $|z| < 1$. Also, we have $|\phi(b)| = 1$ for $b \in \partial E$.

From (1.5), we obtain

$$1 \leq |\phi'(b)| = \frac{2}{\lambda} \left| \frac{(h(b))^{1/\lambda} - 1}{1 + (h(b))^{1/\lambda}} \right| = \frac{2}{\lambda} \left| \frac{i^\lambda)^{1/\lambda} - 1}{1 + (i^\lambda)^{1/\lambda}} \right| = \frac{2}{\lambda} \left| \frac{(i^\lambda)^{1/\lambda} - 1}{1 + (i^\lambda)^{1/\lambda}} \right|.$$
and
\[ 1 \leq \frac{2}{\lambda} \left| h'(b) \right|^2 = \frac{\left| h'(b) \right|}{\lambda}. \]

So, we take the inequality (2.1).

Now, we shall show that the inequality (2.1) is sharp. Let
\[ f(z) = e^{\frac{z}{4} (\frac{1+i}{1-i})^\lambda} dt. \]

Then, we have
\[
\ln f(z) = \ln e^{\frac{z}{4} (\frac{1+i}{1-i})^\lambda} dt = \frac{z}{4} \left( \frac{1+t}{1-t} \right)^\lambda dt,
\]
\[
f'(z) = \frac{f'(z)}{f(z)} = \frac{1}{z} \left( \frac{1+z}{1-z} \right)^\lambda,
\]
\[
h(z) = \frac{f'(z)}{f(z)} = \left( \frac{1+z}{1-z} \right)^\lambda
\]
and
\[
h'(z) = \lambda \left( \frac{1+z}{1-z} \right)^{\lambda-1} \frac{2}{(1-z)^2}.
\]

Therefore, we obtain
\[
h'(i) = \lambda \left( \frac{1+i}{1-i} \right)^{\lambda-1} \frac{2}{(1-i)^2}
\]
and
\[
|h'(i)| = \lambda = \frac{\beta}{\beta + \alpha}. \]

\[\square\]

**Theorem 2.2.** Under the same assumptions as in Theorem 2.1, we have
\[
(2.2) \quad \left| \frac{zf'(z)}{f(z)} \right|_{z=b} \geq \frac{4\beta^2}{(\beta + \alpha) (2\beta + (\beta + \alpha) |c_2|)}.
\]

The inequality (2.2) is sharp with equality for the function
\[ f(z) = e^{\frac{z}{4} (\frac{1+i}{1-i})^\lambda} dt, \]
where \( \lambda = \frac{\beta}{\beta + \alpha} \).

**Proof.** Let \( \phi(z) \) be as in the proof of Theorem 2.1. Using the inequality (1.4) for the function \( \phi(z) \), we obtain
\[
\frac{2}{1 + |\phi'(0)|} \leq |\phi'(b)| = \frac{2}{\lambda} \left| \frac{(h(b))^{\frac{1}{\lambda}} - h'(b)}{1 + (h(b))^{\frac{1}{\lambda}}} \right| = \frac{2}{\lambda} \left| \frac{h'(b)}{1 + i} \right|^2 = \frac{|h'(b)|}{\lambda}.
\]
Since
\[ \phi'(z) = \frac{2 (h(z))^{\frac{1}{2} - 1} h'(z)}{\lambda (1 + (h(z))^{\frac{1}{2}})^2} \]
and
\[ |\phi'(0)| = \frac{2}{\lambda} \frac{|(h(0))^{\frac{1}{2} - 1} h'(0)|}{(1 + (h(0))^{\frac{1}{2}})^2} = \frac{2}{\lambda} \frac{|c_2|}{4} = \frac{|c_2|}{2\lambda}, \]
we have
\[ \frac{2}{1 + \frac{|c_2|}{2\lambda}} \leq \frac{|h'(b)|}{\lambda} \]
and
\[ |h'(b)| \geq \frac{4\lambda^2}{2\lambda + |c_2|}. \]
So, we obtain the inequality (2.2).

To show that the inequality (2.2) is sharp, take the holomorphic function
\[ f(z) = e^{\int_{1}^{z} \left( \frac{1}{1+t} \right)^{\lambda} \, dt}. \]
Then
\[ h(z) = z \frac{f'(z)}{f(z)} = \left( \frac{1+z}{1-z} \right) \lambda \]
and
\[ |h'(i)| = \lambda. \]
Since \(|c_2| = 2\lambda\) is satisfied with equality. That is;
\[ \frac{4\lambda^2}{2\lambda + |c_2|} = \frac{4\lambda^2}{2\lambda + 2\lambda} = \lambda. \]

**Theorem 2.3.** Let \( f(z) \in N(\lambda). \) Assume that, for some \( b \in \partial E, \) \( f \) has angular limit \( f(b) \) at \( b \) and \( \frac{bf'(b)}{f(b)} = i^\lambda. \) Then we have the inequality
\[ (2.3) \quad \left| \left( \frac{zf'(z)}{f(z)} \right)' \right|_{z=b} \geq \lambda \left( 1 + \frac{2 (2\lambda - |c_2|)^2}{4\lambda^2 - |c_2|^2 + 4|\lambda c_3 - c_2^2 (2\lambda - 1) + (1 - \lambda) c_2|} \right), \]
where \( \lambda = \frac{\beta}{\beta + \alpha}. \) The inequality (2.3) is sharp with equality for the function
\[ f(z) = e^{\int_{1}^{z} \left( \frac{1}{1+t} \right)^{\lambda} \, dt}. \]

**Proof.** Let \( \phi(z) \) be as in the proof of Theorem 2.1. By the maximum principle for each \( z \in E, \) we have \(|\phi(z)| \leq |z|. \) So,
\[ \psi(z) = \frac{\phi(z)}{z} \]
is a holomorphic function in \( E \) and \( |\psi(z)| < 1 \) for \( |z| < 1 \). For any real number \( \mu = \frac{1}{\lambda} \) that is not a non-negative integer

\[
k^\mu = \sum_{n=0}^{\infty} \left( \frac{\mu}{n} \right) (k-1)^n,
\]

where \( k = \frac{zf'(z)}{f(z)} = 1 + c_2z + (2c_3 - c_2^2)z^2 + \ldots \).

From equality of \( \psi(z) \), we have

\[
\psi(z) = \frac{\phi(z)}{z} = \frac{1}{z} \frac{h(z)}{k \mu - 1} = \frac{1}{z} \frac{(k \mu - 1)}{h(z)}.
\]

Thus, we take

\[
(2.4) \quad |\psi(0)| = \left| \frac{c_2}{2\lambda} \right| \leq 1
\]

and

\[
|\psi'(0)| = \left| \frac{4\lambda c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2}{4\lambda^2} \right|.
\]

Moreover, it can be seen that

\[
\frac{b\phi'(b)}{\phi(b)} = |\phi'(b)| \geq \left| (b^p)' \right| = \frac{b'(b^p)'}{b^p}.
\]

The function

\[
\Phi(z) = \frac{\psi(z) - \psi(0)}{1 - \psi(0)\psi(z)}
\]

is a holomorphic in the unit disc \( E \), \( |\Phi(z)| < 1 \) for \( |z| < 1 \), \( \Phi(0) = 0 \) and \( |\Phi(b)| = 1 \) for \( b \in \partial E \).

From (1.4), we obtain

\[
\frac{2}{1 + |\Phi'(0)|} \leq |\Phi'(b)| = \frac{1 - |\psi(0)|^2}{2} \left| \frac{1 - |\psi(0)|^2}{1 - \psi(0)|\psi(b)|^2} \right| |\psi'(b)| \leq \frac{1 + |\psi(0)|}{1 - |\psi(0)|} |\psi'(b)|
\]

\[
= \frac{1 + |\psi(0)|}{1 - |\psi(0)|} \left( |\phi'(b)| - 1 \right).
\]

Since

\[
\Phi'(z) = \frac{1 - |\psi(0)|^2}{1 - \psi(0)\psi(z)} \psi'(z),
\]

\[
|\Phi'(0)| = \frac{|\psi'(0)|}{1 - |\psi(0)|^2} = \frac{|4\lambda c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2|}{4\lambda^2 - |c_2|^2} = \frac{|4\lambda c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2|}{4\lambda^2 - |c_2|^2}.
\]
we take
\[ \frac{2}{1 + \frac{4c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2}{4\lambda^2 - |c_2|^2}} \leq \frac{1 + \frac{|c_2|}{2\lambda}}{1 - \frac{|c_2|}{2\lambda}} \left\{ \frac{|h'(b)|}{\lambda} - 1 \right\} = \frac{2\lambda + |c_2|}{2\lambda - |c_2|} \left\{ \frac{|h'(b)|}{\lambda} - 1 \right\}. \]

Therefore, we obtain
\[ 1 + \frac{2 \left( 4\lambda^2 - |c_2|^2 \right)}{4\lambda^2 - |c_2|^2 + 4\lambda c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2} 2\lambda + |c_2| \leq \frac{|h'(b)|}{\lambda} \]
and
\[ |h'(b)| \geq \lambda \left( 1 + \frac{2 \left( 2\lambda - |c_2| \right)^2}{4\lambda^2 - |c_2|^2 + 4\lambda c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2} \right). \]

So, we obtain the inequality (2.3).

To show that the inequality (2.3) is sharp, take the holomorphic function
\[ f(z) = e^{\int \frac{\beta}{\beta + \alpha} \lambda \, dt}. \]
Then
\[ h(z) = z f'(z) \frac{f'(z)}{f(z)} = \left( \frac{1 + z}{1 - z} \right)^\lambda \]
and
\[ |h'(i)| = \lambda. \]
Since \( |c_2| = 2\lambda \), (2.3) is satisfied with equality. \( \square \)

If \( \left( \frac{z^f(z)}{f(z)} \right)^{\frac{1}{\beta + \alpha}} - 1 \) has no zeros different from \( z = 0 \) in Theorem 2.3, the inequality (2.3) can be further strengthened. This is given by the following theorem.

**Theorem 2.4.** Let \( f(z) \in N(\lambda) \) and \( \left( \frac{z^f(z)}{f(z)} \right)^{\frac{1}{\beta + \alpha}} - 1 \) has no zeros in \( E \) except \( z = 0 \) and \( c_2 > 0 \). Assume that, for some \( b \in \partial E \), \( f \) has angular limit \( f(b) \) at \( b \) and \( \frac{bf'(b)}{f(b)} = i^\lambda \). Then we have the inequality
\[ \left( \frac{z^f(z)}{f(z)} \right)'_{z=b} \geq \lambda \left( 1 - \frac{2\lambda |c_2|}{2\lambda |c_2| \ln \left( \frac{|c_2|}{2\lambda} \right) - \left( 4\lambda c_3 - c_2^2(2\lambda - 1) + (1 - \lambda)c_2 \right)} \right), \]
where \( \lambda = \frac{\beta}{\beta + \alpha} \). In addition, the equality in (2.5) occurs for the function
\[ f(z) = e^{\int \frac{\beta}{\beta + \alpha} \lambda \, dt}, \]
where \( \lambda = \frac{\beta}{\beta + \alpha} \).
Proof. Let $c_2 > 0$ in the expression of the function $f(z)$. Having in mind the inequality (2.4) and the function \( \left( \frac{zf'(z)}{f(z)} \right) - 1 \) has no zeros in $E$ except $E - \{0\}$, we denote by \( \ln \psi(z) \) the holomorphic branch of the logarithm normed by the condition
\[
\ln \psi(0) = \ln \left( \frac{|c_2|}{2\lambda} \right) < 0.
\]
The auxiliary function
\[
\Delta(z) = \ln \psi(z) - \ln \psi(0)
\]
is a holomorphic in the unit disc $E$, $|\Delta(z)| < 1$, $\Delta(0) = 0$ and $|\Delta(b)| = 1$ for $b \in \partial E$.

From (1.4), we obtain
\[
\frac{2}{1 + |\Delta'(0)|} \leq |\Delta'(b)| = \frac{|2 \ln \psi(0)|}{|\ln \psi(b) + \ln \psi(0)|^2} \frac{\psi'(b)}{\psi(b)}
\]
\[
= \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \left\{ |\phi'(b)| - 1 \right\}.
\]

Since
\[
|\Delta'(0)| = \frac{-1}{\ln \left( \frac{|c_2|}{2\lambda} \right)} \frac{|4\lambda c_3 - c_2^2 (2\lambda - 1) + (1 - \lambda)c_2|}{4\lambda^2}
\]
\[
= \frac{-1}{\ln \left( \frac{|c_2|}{2\lambda} \right)} \frac{|4\lambda c_3 - c_2^2 (2\lambda - 1) + (1 - \lambda)c_2|}{2\lambda |c_2|}
\]
and replacing $\arg^2 \psi(b)$ by zero, then we have
\[
\frac{1}{1 - \frac{4\lambda c_3 - c_2^2 (2\lambda - 1) + (1 - \lambda)c_2}{2\lambda |c_2| \ln \left( \frac{|c_2|}{2\lambda} \right)}} \leq \frac{-1}{\ln \left( \frac{|c_2|}{2\lambda} \right)} \left\{ \frac{|h'(b)|}{\lambda} - 1 \right\}
\]
and
\[
1 - \frac{2\lambda |c_2| \ln \left( \frac{|c_2|}{2\lambda} \right) - |4\lambda c_3 - c_2^2 (2\lambda - 1) + (1 - \lambda)c_2|}{2\lambda |c_2| \ln \left( \frac{|c_2|}{2\lambda} \right)} \leq \frac{|h'(b)|}{\lambda}.
\]
Thus, we obtain the inequality (2.5) with an obvious equality case. \(\square\)

The following inequality (2.6) is weaker, but is simpler than (2.5) and does not contain the coeffient $c_3$.

**Theorem 2.5.** Under the hypotheses of Theorem 2.4, we have the inequality
\[
(2.6) \quad \left| \left( \frac{zf'(z)}{f(z)} \right)'_{z=b} \right| \geq \frac{\beta}{\beta + \alpha} \left[ 1 - \ln \left( \frac{|c_2|}{2\beta} \right) \right].
\]
Moreover, the result is sharp and the extremal function is
\[ f(z) = e^{\int_0^1 \left( \frac{1}{1+z^2} \right) \, dt}, \]
where \( \lambda = \frac{\beta}{\beta + \alpha} \).

Proof. Let \( c_2 > 0 \). Using the inequality (1.5) for the function \( \Phi(z) \), we obtain
\[ 1 \leq |\Delta'(b)| = \left| \frac{2 \ln \psi(0)}{\ln \psi(b) + \ln \psi(0)} \right| \frac{|\psi'(b)|}{\psi(b)} = \frac{-2 \ln \psi(0)}{\ln^2 \psi(0) + \arg^2 \psi(b)} \{ |\varphi'(b)| - 1 \}. \]
Replacing \( \arg^2 \varphi(b) \) by zero, then we have
\[ 1 \leq \frac{-1}{\ln \left( \frac{|c_2|}{2\lambda} \right)} \left[ \frac{|h'(b)|}{\lambda} - 1 \right] \]
and
\[ |h'(b)| \geq \lambda \left[ 1 - \ln \left( \frac{|c_2|}{2\lambda} \right) \right]. \]
Thus, we obtain the inequality (2.6) with an obvious equality case.

References


1 Department of Computer Engineering
Amasya University,
Merkez-Amasya 05100, Turkey
Email address: nafiornek@gmail.com, nafi.ornek@amasya.edu.tr