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AN APPROXIMATE APPROACH FOR SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BASED ON TAYLOR EXPANSION

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Abstract. The main purpose of this work is to present an efficient approximate approach for solving linear systems of fractional integro-differential equations based on a new application of Taylor expansion. Using the mth-order Taylor polynomial for unknown functions and employing integration method the given system of fractional integro-differential equations will be converted into a system of linear equations with respect to unknown functions and their derivatives. The solutions of this system yield the approximate solutions of fractional integro-differential equations system. The Riemann-Liouville fractional derivative is applied in calculations. An error analysis is discussed as well. The accuracy and the efficiency of the suggested method is illustrated by considering five numerical examples.

1. Introduction

During the past decades, fractional calculus and fractional differential equations have found various applications in sciences and engineering, such as electrical networks, rheology, acoustics, electroanalytical chemistry, neuron modeling, viscoelasticity, material sciences, fluid flow, diffusive transport akin to diffusion, probability, electromagnetic theory, and so on (see [7, 13, 18, 24, 26]).

Since most of FDEs do not have exact solutions, approximate and numerical techniques have received considerable attention to solve fractional differential equations.

Key words and phrases. Fractional differential equation (FDE), systems of fractional integro-differential equations (SFIDE), Riemann-Liouville fractional derivative, Taylor expansion, error analysis.

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So far, several analytical and numerical methods have been proposed to solve fractional differential equations which the interested reader can refer to [1–5, 10–12, 16, 19–23, 25, 27–30, 34] and the references therein.

In this paper, we investigate the approximate solutions of linear fractional integro-differential equations systems based on a new application of Taylor expansion (see [6, 8–10, 14, 15, 17, 31–33]). By expanding unknown functions as an mth-order Taylor polynomial and employing integration method, we can convert the given system of fractional integro-differential equations into a system of linear equations with respect to unknown functions and their derivatives. Approximate solutions can be obtained by solving the resulting system of equations according to a standard rule. The results of the obtained approximations of the suggested method are then compared with the referenced methods for several examples. In the present investigation, the main property of this approximate method besides simplicity and reliability is that an mth-order approximation is equal to exact solution if the exact solution is a polynomial of degree at most m. The present work may be viewed as an extension of the results obtained in [10].

The remainder of this paper is organized as follows. In Section 2, some definitions of fractional calculus are recalled. In Section 3, we describe the proposed method. In Section 4, we give an error analysis. In Section 5, we investigate some examples, which demonstrate the effectiveness of our approach. In Section 6, our findings are concluded.

2. Preliminaries and Basic Definitions

Let's describe some basic concepts, and properties of the fractional calculus, which will be used later.

Definition 2.1. A real function $\phi(x)$, $x > 0$, is said to be in the space $C_\mu,\ \mu \in \mathbb{R}$ if there exists a real number $p\ (> \mu)$, such that $\phi(x) = x^p\phi_1(x)$, where $\phi_1(x) \in C[0, \infty)$, and it is said to be in the space C_μ^n if and only if $\phi^{(n)} \in C_\mu,\ n \in \mathbb{N}$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, of a function $\phi \in C_\mu,\ \mu \geq -1$, is considered as follows

$$\begin{aligned} J^\alpha\phi(x) =& \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\phi(t)dt, \quad \alpha > 0, x > 0, \\ J^0\phi(x) =& \phi(x). \end{aligned}$$

Definition 2.3. The Caputo fractional derivative of $\phi(x)$ is considered as follows

$$D_*^\alpha\phi(x) = J^{n-\alpha}\left(\frac{d^n}{dx^n}\phi(x)\right) = \frac{1}{\Gamma(n-\alpha)} \int_0^x (x-t)^{n-\alpha-1}\phi^{(n)}(t)dt,$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $x > 0$, $\phi \in C_{-1}^n$.

Definition 2.4. The Riemann-Liouville fractional derivative of $\phi(x)$ is considered as follows

$$D^{\alpha}\phi(x) = \frac{d^n}{dx^n}\left(J^{n-\alpha}\phi(x)\right),$$

for $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $x > 0$, $\phi \in C^n_{-1}$.

3. Description of the Method

Consider the following system of linear fractional integro-differential equations

$$\begin{aligned} D^{\alpha_i}\psi_i(x) + \lambda_1 \int_0^1 \sum_{j=1}^{\nu} K_{1_{ij}}(x,t)\psi_j(t)dt + \lambda_2 \int_0^x \sum_{j=1}^{\nu} K_{2_{ij}}(x,t)\psi_j(t)dt \\ = f_i(x), \quad i = 1, \ldots, \nu, \end{aligned} \tag{3.1}$$

with initial conditions

$$\psi_i^{(\kappa)}(0) = 0, \quad \kappa = 0, 1, \ldots, n-1, n-1 < \alpha_i \leq n, n \in \mathbb{N}, \tag{3.2}$$

where $D^{\alpha_i}\psi_i(x)$ indicates Riemann-Liouville fractional derivative of order α_i, and λ_1, λ_2 are constants, $K_{1_{ij}}(x,t)$, $K_{2_{ij}}(x,t)$, $f_i(x)$ are given known functions which satisfy certain conditions so that system (3.2) has a unique solution, and $\psi_i(x)$ are unknown functions.

According to definition (2.4), system of fractional integro-differential equation (3.1) can be rewritten as

$$\frac{d^n}{dx^n}\left(J^{n-\alpha_i}\psi_i(x)\right) + \lambda_1 \int_0^1 \sum_{j=1}^{\nu} K_{1_{ij}}(x,t)\psi_j(t)dt + \lambda_2 \int_0^x \sum_{j=1}^{\nu} K_{2_{ij}}(x,t)\psi_j(t)dt = f_i(x),$$

or equivalently by using definition (2.2), we have

$$\begin{aligned} \frac{d^n}{dx^n}\left(\frac{1}{\Gamma(n-\alpha_i)}\int_0^x (x-t)^{n-\alpha_i-1}\psi_i(t)dt\right) + \lambda_1 \int_0^1 \sum_{j=1}^{\nu} K_{1_{ij}}(x,t)\psi_j(t)dt \\ + \lambda_2 \int_0^x \sum_{j=1}^{\nu} K_{2_{ij}}(x,t)\psi_j(t)dt = f_i(x). \end{aligned} \tag{3.3}$$

In the following, by integrating both hand side of (3.3), n times with respect to x from 0 to s and with the help of changing the order of the integrations, we obtain

$$\begin{aligned} \frac{1}{\Gamma(n-\alpha_i)}\int_0^x (x-t)^{n-\alpha_i-1}\psi_i(t)dt + \lambda_1 \sum_{j=1}^{\nu} \int_0^1 \int_0^x \frac{(x-s)^{l-1}}{(l-1)!} K_{1_{ij}}(s,t)\psi_j(t)dsdt \\ + \lambda_2 \sum_{j=1}^{\nu} \int_0^x \int_t^x \frac{(x-s)^{l-1}}{(l-1)!} K_{2_{ij}}(s,t)\psi_j(t)dsdt = F_i(x), \quad l = 1, \ldots, n, \end{aligned} \tag{3.4}$$

where

$$F_i(x) = \int_0^x \frac{(x-t)^{l-1}}{(l-1)!} f_i(t)dt, \quad i = 1, \ldots, \nu,$$

in which the variable s has been replaced by x, for simplicity. Hence we transformed the system of fractional integro-differential equations (3.1) into a system of mixed

Volterra-Fredholm integral equations. To approximately solve the resulting system, we reduce Eq. (3.4) into a system of linear equations with respect to unknown functions and their derivatives. Toward this goal, the method assumes that the desired solutions $\psi_j(t)$ to be $m+1$ times continuously differentiable on the interval I, in other words $\psi_j \in C^{m+1}(I)$. Therefore, for $\psi_j \in C^{m+1}(I)$, $\psi_j(t)$ can be expressed in terms of the mth-order Taylor series at an arbitrary point $x \in I$ as

$$\psi_j(t) = \psi_j(x) + \psi_j'(x)(t-x) + \cdots + \frac{1}{m!}\psi_j^{(m)}(x)(t-x)^m + E_{j,m}(t,x),$$

where $E_{j,m}(t,x)$ indicates the Lagrange error bound

$$E_{j,m}(t,x) = \frac{\psi_j^{(m+1)}(\xi_j)}{(m+1)!}(t-x)^{m+1},$$

for some point ξ_j between x and t. Generally, the Lagrange error bound $E_{j,m}(t,x)$ becomes sufficiently small as m gets great enough. Especially, if the solutions $\psi_j(t)$ are polynomials of degree up to m, then the last Lagrange error bound becomes zero, namely, the obtained approximate solutions of system (3.1) yield the true solutions. With due attention to aforementioned assumption, by omitting the last Lagrange error bound, we consider the truncated Taylor expansion $\psi_j(t)$ as

$$\psi_j(t) \approx \sum_{k=0}^{m} \psi_j^{(k)}(x)\frac{(t-x)^k}{k!}. \tag{3.5}$$

Inserting the approximate relation (3.5), for unknown functions $\psi_j(t)$, into (3.4) we obtain

$$\begin{aligned} &\sum_{k=0}^{m} \frac{(-1)^k}{k!}\psi_j^{(k)}(x)\int_0^x \frac{(x-t)^{k+n-\alpha_i-1}}{\Gamma(n-\alpha_i)}dt \\ &+ \lambda_1 \sum_{j=1}^{\nu}\sum_{k=0}^{m} \frac{\psi_j^{(k)}(x)}{k!}\int_0^1\int_0^x \frac{(x-s)^{l-1}}{(l-1)!}(t-x)^k K_{1_{ij}}(s,t)dsdt \\ &+ \lambda_2 \sum_{j=1}^{\nu}\sum_{k=0}^{m} \frac{\psi_j^{(k)}(x)}{k!}\int_0^x\int_t^x \frac{(x-s)^{l-1}}{(l-1)!}(t-x)^k K_{2_{ij}}(s,t)dsdt \\ =&F_i(x), \quad i = 1,\ldots,\nu. \end{aligned} \tag{3.6}$$

In fact, system (3.1) was converted into a linear system of ordinary differential equations with respect to $\psi_j(x)$ and its derivatives up to order m. In other word, we have obtained ν linear equations in (3.6) with respect to $\nu \times (m+1)$ unknown functions $\psi_j^{(k)}$, for $k = 0,\ldots,m$, $j = 1,\ldots,\nu$. In the following, we want to determine $\psi_j^{(k)}$ by solving a system of linear equations. In order to achieve this goal, other $\nu \times m$ independent linear equations with respect to $\psi_j(x),\ldots,\psi_j^{(m)}(x)$ are needed, which can be achieved by integrating both sides of Eq.(3.4) m times with respect to x. Thus,

we have

$$\int_0^x \frac{(x-t)^{\gamma+n-\alpha_i-1}}{\Gamma(\gamma+n-\alpha_i)}\psi_i(t)dt + \lambda_1 \sum_{j=1}^{\nu}\int_0^1\int_0^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!}K_{1_{ij}}(s,t)\psi_j(t)dsdt \tag{3.7}$$
$$+ \lambda_2 \sum_{j=1}^{\nu}\int_0^x\int_t^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!}K_{2_{ij}}(s,t)\psi_j(t)dsdt = F_i^{(\gamma)}(x), \quad \gamma = 1,\dots,m,$$

where

$$F_i^{(\gamma)}(x) = \int_0^x \frac{(x-t)^{\gamma-1}}{(\gamma-1)!}F_i(t)dt, \quad i = 1,\dots,\nu, \gamma = 1,\dots,m.$$

We apply the Taylor expansion again and substituting (3.5) for $\psi_j(t)$ into E(3.7) leads to

$$\sum_{k=0}^{m}\frac{(-1)^k}{k!}\psi_j^{(k)}(x)\int_0^x \frac{(x-t)^{k+\gamma+n-\alpha_i-1}}{\Gamma(\gamma+n-\alpha_i)}dt$$
$$+ \lambda_1 \sum_{j=1}^{\nu}\sum_{k=0}^{m}\frac{\psi_j^{(k)}(x)}{k!}\int_0^1\int_0^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!}(t-x)^k K_{1_{ij}}(s,t)dsdt+$$
$$\lambda_2 \sum_{j=1}^{\nu}\sum_{k=0}^{m}\frac{\psi_j^{(k)}(x)}{k!}\int_0^x\int_t^x \frac{(x-s)^{\gamma+l-1}}{(\gamma+l-1)!}(t-x)^k K_{2_{ij}}(s,t)dsdt$$
$$= F_i^{(\gamma)}(x), \quad \gamma = 1,\dots,m. \tag{3.8}$$

In this way, (3.4) and (3.8) construct a system of linear equations with resect to unknown functions $\psi_j(x)$ and its derivatives up to order m. The obtained system is indicated as follows

$$\mathbf{Q}(x)\Psi(x) = F(x),$$

where

$$\mathbf{Q}(x) = \begin{bmatrix}
q_{10}^{10}(x) & \cdots & q_{\nu 0}^{10}(x) & \cdots & q_{1k}^{10}(x) & \cdots & q_{\nu k}^{10}(x) & \cdots & q_{1m}^{10}(x) & \cdots & q_{\nu m}^{10}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{\nu 0}(x) & \cdots & q_{\nu 0}^{\nu 0}(x) & \cdots & q_{1k}^{\nu 0}(x) & \cdots & q_{\nu k}^{\nu 0}(x) & \cdots & q_{1m}^{\nu 0}(x) & \cdots & q_{\nu m}^{\nu 0}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{1\gamma}(x) & \cdots & q_{\nu 0}^{1\gamma}(x) & \cdots & q_{1k}^{1\gamma}(x) & \cdots & q_{\nu k}^{1\gamma}(x) & \cdots & q_{1m}^{1\gamma}(x) & \cdots & q_{\nu m}^{1\gamma}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{\nu\gamma}(x) & \cdots & q_{\nu 0}^{\nu\gamma}(x) & \cdots & q_{1k}^{\nu\gamma}(x) & \cdots & q_{\nu k}^{\nu\gamma}(x) & \cdots & q_{1m}^{\nu\gamma}(x) & \cdots & q_{\nu m}^{\nu\gamma}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{1m}(x) & \cdots & q_{\nu 0}^{1m}(x) & \cdots & q_{1k}^{1m}(x) & \cdots & q_{\nu k}^{1m}(x) & \cdots & q_{1m}^{1m}(x) & \cdots & q_{\nu m}^{1m}(x) \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
q_{10}^{\nu m}(x) & \cdots & q_{\nu 0}^{\nu m}(x) & \cdots & q_{1k}^{\nu m}(x) & \cdots & q_{\nu k}^{\nu m}(x) & \cdots & q_{1m}^{\nu m}(x) & \cdots & q_{\nu m}^{\nu m}(x)
\end{bmatrix}, \tag{3.9}$$

$$\mathbf{F}(x) = \left[\ F_1(x),\dots,F_\nu(x),\dots,F_1^{(\gamma)}(x),\dots,F_\nu^{(\gamma)}(x),\dots,F_1^{(m)}(x),\dots,F_\nu^{(m)}(x)\ \right]^T,$$

$$\mathbf{\Psi}(x) = \left[\ \psi_1(x), \ldots, \psi_\nu(x), \ldots, \psi_1^{(k)}(x), \ldots, \psi_\nu^{(k)}(x), \ldots, \psi_1^{(m)}(x), \ldots, \psi_\nu^{(m)}(x)\ \right]^T.$$

In coefficient matrix (3.9), the first ν rows refer to coefficients of $\psi_j^{(k)}(x)$ in (3.4) for $k = 0, \ldots, m$, $j = 1, \ldots, \nu$ and the other rows refer to coefficients of $\psi_j^{(k)}(x)$ in (3.8) for $\gamma = 1, \ldots, m$. Application of a standard rule to the resulting new system yields an mth-order approximate solution of (3.1) as $\psi_{im}(x)$. It is to be noted that not only $\psi_j(x)$ but also $\psi_j^{(k)}(x)$, for $k = 1, \ldots, m$, are determined by solving the resulting new system but in point of fact, it is $\psi_j(x)$ that we need.

4. Error Analysis

In this section, we expand the error analysis proposed in [9] for derived mth-order approximate solution of fractional integro-differential equations system (3.1). We assume that the exact solutions $\psi_j(t)$ are infinitely differentiable on the interval I; so $\psi_j(t)$ can be expressed as an uniformly convergent Taylor series in I as follows

$$\psi_j(t) = \sum_{k=0}^{\infty} \psi_j^{(k)}(x) \frac{(t-x)^k}{k!}.$$

Using the proposed method in the previous section, system of fractional integro-differential equations (3.1) can be converted into an equivalent system of linear equations with respect to unknown functions $\psi_i^{(k)}(x)$, $k = 0, 1, \ldots$ as

$$\mathbf{Q}\mathbf{\Psi} = \mathbf{F},$$

where

$$\mathbf{Q} = \lim_{\nu \longrightarrow \infty} \mathbf{Q}_{\nu\nu}^{\nu\nu}, \qquad \mathbf{\Psi} = \lim_{\nu \longrightarrow \infty} \mathbf{\Psi}_\nu, \qquad \mathbf{F} = \lim_{\nu \longrightarrow \infty} \mathbf{F}_\nu,$$

in which $\mathbf{Q}_{\nu\nu}^{\nu\nu}$, $\mathbf{\Psi}_\nu$, and $\mathbf{F}_\nu$, as shown in the previous section, are defined as follows

$$\mathbf{Q}_{\nu\nu}^{\nu\nu} = \left[q_{ij}^{pq}(x)\right]_{\nu(m+1)\times\nu(m+1)}, \ \mathbf{\Psi}_\nu = \left[\psi_i^{(k)}(x)\right]_{\nu(m+1)\times 1}, \ \mathbf{F}_\nu = \left[f_i^{(l)}(x)\right]_{\nu(m+1)\times 1}.$$

Hence, under the solvability conditions for the above system and letting $\mathbf{B} = \mathbf{Q}^{-1}$, the unique solution is represented as

$$\mathbf{\Psi} = \mathbf{B}\mathbf{F}. \tag{4.1}$$

We rewrite relation (4.1) in an alternative matrix form as

$$\begin{bmatrix} \mathbf{\Psi}_\nu \\ \mathbf{\Psi}_\infty \end{bmatrix} = \begin{bmatrix} \mathbf{B}_{\nu\nu}^{\nu\nu} & \mathbf{B}_{\nu\infty}^{\nu\infty} \\ \mathbf{B}_{\infty\nu}^{\infty\nu} & \mathbf{B}_{\infty\infty}^{\infty\infty} \end{bmatrix} \begin{bmatrix} \mathbf{F}_\nu \\ \mathbf{F}_\infty \end{bmatrix}. \tag{4.2}$$

Accordingly, we can find out that the vector $\mathbf{\Psi}_\nu$ consists of the first $\nu(m+1)$ elements of the exact solution vector $\mathbf{\Psi}$ must satisfy the following relation

$$\mathbf{\Psi}_\nu = \mathbf{B}_{\nu\nu}^{\nu\nu}\mathbf{F}_\nu + \mathbf{B}_{\nu\infty}^{\nu\infty}\mathbf{F}_\infty. \tag{4.3}$$

According to the proposed process, the unique solution of SFIDE (3.1) can be denoted as

$$\widetilde{\mathbf{\Psi}}_\nu = \mathbf{Q}_{\nu\nu}^{\nu\nu^{-1}}\mathbf{F}_\nu, \tag{4.4}$$

where $\mathbf{\Psi}_\nu$ is replaced by $\widetilde{\mathbf{\Psi}}_\nu$ as its approximate solution.

Subtracting (4.4) from (4.3) leads to

$$\mathbf{\Psi}_\nu - \widetilde{\mathbf{\Psi}}_\nu = \mathbf{D}^{\nu\nu}_{\nu\nu}\mathbf{F}_\nu + \mathbf{B}^{\nu\infty}_{\nu\infty}\mathbf{F}_\infty, \tag{4.5}$$

where

$$\mathbf{D}^{\nu\nu}_{\nu\nu} = \mathbf{B}^{\nu\nu}_{\nu\nu} - \mathbf{Q}^{\nu\nu^{-1}}_{\nu\nu}.$$

In the following, we expand the right-hand side of (4.5) and the first ν elements of the vector at the left-hand side of (4.5) can be expressed as

$$\psi^\nu(x) - \widetilde{\psi}^\nu(x) = \sum_{j=0}^{m}\sum_{i=1}^{\nu} d^{p0}_{ij}(x) f^{(j)}_i(x) + \sum_{j=m+1}^{\infty}\sum_{i=1}^{\nu} b^{p0}_{ij}(x) f^{(j)}_i(x), \quad p = 1, \ldots, \nu,$$

where

$$\psi^\nu(x) = \begin{bmatrix} \psi_1(x) \\ \psi_2(x) \\ \vdots \\ \psi_\nu(x) \end{bmatrix}, \qquad \widetilde{\psi}^\nu(x) = \begin{bmatrix} \widetilde{\psi}_1(x) \\ \widetilde{\psi}_2(x) \\ \vdots \\ \widetilde{\psi}_\nu(x) \end{bmatrix},$$

and $d^{p0}_{ij}(x)$, $b^{p0}_{ij}(x)$ are the elements of $\mathbf{D}^{\nu\nu}_{\nu\nu}$ and $\mathbf{B}^{\nu\infty}_{\nu\infty}$, respectively. Thus, according to the Cauchy-Schwarz inequality we have

$$\begin{aligned}\left|\psi^\nu(x) - \widetilde{\psi}^\nu(x)\right| \le & \left(\sum_{j=0}^{m}\sum_{i=1}^{\nu}\left|d^{p0}_{ij}(x)\right|^2\right)^{\frac{1}{2}}\left(\sum_{j=0}^{m}\sum_{i=1}^{\nu}\left|f^{(j)}_i(x)\right|^2\right)^{\frac{1}{2}} \\ & + \left(\sum_{j=m+1}^{\infty}\sum_{i=1}^{\nu}\left|b^{p0}_{ij}(x)\right|^2\right)^{\frac{1}{2}}\left(\sum_{j=m+1}^{\infty}\sum_{i=1}^{\nu}\left|f^{(j)}_i(x)\right|^2\right)^{\frac{1}{2}}.\end{aligned}$$

It is to be noted that as $\lim_{\nu\longrightarrow\infty}\mathbf{D}^{\nu\nu}_{\nu\nu} = 0$ and $\lim_{\nu\longrightarrow\infty}\mathbf{B}^{\nu\infty}_{\nu\infty} = 0$, we have

$$\lim_{\nu\longrightarrow\infty}\left|\psi^\nu(x) - \widetilde{\psi}^\nu(x)\right| = 0.$$

5. Illustrative Examples

In this section, the efficiency and the accuracy of the proposed approach is illustrated by considering some numerical problems. The obtained numerical results are compared with some existing approaches and it was found that the proposed approximate approach produces acceptable results and even more accurate results in comparison with some existing methods. All computations are performed using Mathematica 8.

Example 5.1. Consider the following system of fractional integro-differential equations (see [5, 29]):

$$\begin{cases} D^{\frac{1}{2}}\psi_1(x) - \int_0^1 \left(\psi_1(t) + \psi_2(t)\right) dt = \dfrac{2\sqrt{x}}{\sqrt{\pi}} - \dfrac{5}{6}, \\ D^{\frac{3}{2}}\psi_1(x) - \int_0^1 \left(\psi_1(t) + \psi_2(t)\right) dt = \dfrac{4\sqrt{x}}{\sqrt{\pi}} - \dfrac{x}{6}, \end{cases} \tag{5.1}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x$ and $\psi_2(x) = x^2$.

Using the present method, the first-order and the second-order approximate solutions at equidistant points are computed. The obtained results and the results given in [5, 29] are listed in Tables 1 and 2. From Tables 1 and 2, we observe that the second-order approximate solution yields the exact solution as expected, since the exact solution is a polynomial function of degree 2.

TABLE 1. Absolute errors of Example 5.1 for $\psi_1(x)$.

x	Method in [29]	Method in [5]	Suggested method	
			$m = 1$	$m = 2$
0.1	8.75559×10^{-2}	2.78470×10^{-3}	1.73688×10^{-1}	0
0.2	1.23823×10^{-1}	3.93816×10^{-3}	5.59324×10^{-1}	0
0.3	1.51651×10^{-1}	4.82324×10^{-3}	1.98751	0
0.4	1.75112×10^{-1}	5.56940×10^{-3}	4.08095	0
0.5	1.95781×10^{-1}	6.22678×10^{-3}	1.10827	0
0.6	2.14467×10^{-1}	6.82110×10^{-3}	5.81370×10^{-1}	0
0.7	2.31651×10^{-1}	7.36763×10^{-3}	3.21226×10^{-1}	0
0.8	2.47646×10^{-1}	7.87633×10^{-3}	1.50704×10^{-1}	0
0.9	2.62668×10^{-1}	8.35411×10^{-3}	2.74544×10^{-2}	0
1.0	2.76876×10^{-1}	8.80600×10^{-3}	6.20423×10^{-2}	0

TABLE 2. Absolute errors of Example 5.1 for $\psi_2(x)$.

x	Method in [5]	Method in [30]	Suggested method	
			$m = 1$	$m = 2$
0.1	1.93140×10^{-4}	1.29824×10^{-4}	3.56504×10^{-5}	0
0.2	1.09257×10^{-3}	3.77788×10^{-4}	3.25545×10^{-3}	0
0.3	3.01076×10^{-3}	7.13496×10^{-4}	3.28085×10^{-2}	0
0.4	6.18049×10^{-3}	1.12845×10^{-3}	1.35422×10^{-1}	0
0.5	1.07969×10^{-2}	1.61892×10^{-3}	6.60271×10^{-2}	0
0.6	1.70314×10^{-2}	2.18315×10^{-3}	6.09208×10^{-2}	0
0.7	2.50391×10^{-2}	2.82043×10^{-3}	6.26674×10^{-2}	0
0.8	3.49621×10^{-2}	3.53063×10^{-3}	6.66494×10^{-2}	0
0.9	4.69331×10^{-2}	4.31399×10^{-3}	7.19976×10^{-2}	0
1.0	6.10763×10^{-2}	5.17100×10^{-3}	7.88615×10^{-2}	0

It is important to note that after converting system (5.1) into a system of linear equations, the Mathematica command 'LinearSolve' is used for the new system.

Example 5.2. Consider the following system of fractional integro-differential equations (see [29]):

$$\begin{cases} D^{\frac{1}{2}}\psi_1(x) - \int_0^1 x\psi_2(t)dt = \dfrac{2\sqrt{x}}{\sqrt{\pi}} - \dfrac{x}{2}, \\ D^{\frac{1}{2}}\psi_2(x) - \int_0^1 x\psi_1(t)dt = \dfrac{2\sqrt{x}}{\sqrt{\pi}} - \dfrac{1}{3}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x$ and $\psi_2(x) = x$.

We employ the approach described in Section 3 to evaluate the approximate solutions. For this case, we can find that $\psi_m(x)$ yields the exact solution only by setting $m = 1$. Moreover, we present the results given in [29] in Table 3.

TABLE 3. Absolute errors of Example 5.2 in [29] for $(\psi_1(x), \psi_2(x))$.

x	Methode in [29]
0.1	$(5.02704 \times 10^{-5}, \quad 5.02704 \times 10^{-4})$
0.2	$(1.42186 \times 10^{-4}, \quad 7.10931 \times 10^{-4})$
0.3	$(2.61213 \times 10^{-4}, \quad 8.70709 \times 10^{-4})$
0.4	$(4.02163 \times 10^{-4}, \quad 1.00541 \times 10^{-3})$
0.5	$(5.62040 \times 10^{-4}, \quad 1.12408 \times 10^{-3})$
0.6	$(7.38821 \times 10^{-4}, \quad 1.23137 \times 10^{-3})$
0.7	$(9.31021 \times 10^{-4}, \quad 1.33003 \times 10^{-3})$
0.8	$(1.13749 \times 10^{-3}, \quad 1.42186 \times 10^{-3})$
0.9	$(1.35730 \times 10^{-3}, \quad 1.50811 \times 10^{-3})$
1.0	$(1.58969 \times 10^{-3}, \quad 1.58969 \times 10^{-3})$

Example 5.3. Consider the following system of fractional integro-differential equations (see [16, 30]):

$$\begin{cases} D^{\frac{3}{4}}\psi_1(x) - \displaystyle\int_0^1 (x+t)\left[\psi_1(t) + \psi_2(t)\right] dt = -\frac{1}{20} - \frac{x}{12} + \frac{4x^{\frac{1}{4}}}{\Gamma(\frac{1}{4})} - \frac{128x^{\frac{9}{4}}}{15\Gamma(\frac{1}{4})}, \\ D^{\frac{3}{4}}\psi_2(x) - \displaystyle\int_0^1 \sqrt{x}t^2\left[\psi_1(t) - \psi_2(t)\right] dt = -\frac{2\sqrt{x}}{15} - \frac{4x^{\frac{1}{4}}}{\Gamma(\frac{1}{4})} + \frac{32x^{\frac{5}{4}}}{5\Gamma(\frac{1}{4})}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x - x^3$ and $\psi_2(x) = x^2 - x$.

We apply the approach described in Section 3 to determine the approximate solutions. For this case, we can find that $\psi_m(x)$ yields the exact solution only by setting $m = 3$. We present our results when $m = 1, 2, 3$, and the results given in [30] in Tables 4 and 5.

Example 5.4. Consider the following system of fractional integro-differential equations (see [16, 30])

$$\begin{cases} D^{\frac{4}{5}}\psi_1(x) - \displaystyle\int_0^1 2xt\left[\psi_1(t) - \psi_2(t)\right] dt = \frac{83}{80}x - \frac{25x^{\frac{6}{5}}}{3\Gamma(\frac{1}{5})} + \frac{125x^{\frac{11}{5}}}{11\Gamma(\frac{1}{5})}, \\ D^{\frac{4}{5}}\psi_2(x) - \displaystyle\int_0^1 (x+t)\left[\psi_1(t) + \psi_2(t)\right] dt = -\frac{67}{160} - \frac{13}{24}x + \frac{125x^{\frac{6}{5}}}{8\Gamma(\frac{1}{5})}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x) = x^3 - x^2$ and $\psi_2(x) = \frac{15}{8}x^2$.

TABLE 4. Absolute errors of Example 5.3 for $\psi_1(x)$

x	Method in [30]		Suggested method	
		$m=1$	$m=2$	$m=3$
0.1	1.86460×10^{-3}	2.33950×10^{-2}	4.37610×10^{-3}	0
0.2	3.38103×10^{-3}	6.86709×10^{-2}	1.69027×10^{-3}	0
0.3	4.91496×10^{-3}	1.21870×10^{-1}	1.70008×10^{-3}	0
0.4	6.51082×10^{-3}	1.73108×10^{-1}	3.93799×10^{-3}	0
0.5	8.18437×10^{-3}	2.11497×10^{-1}	4.52983×10^{-3}	0
0.6	9.94249×10^{-3}	2.25976×10^{-1}	3.55933×10^{-3}	0
0.7	1.17883×10^{-2}	2.06732×10^{-1}	1.36667×10^{-3}	0
0.8	1.37235×10^{-2}	1.47035×10^{-1}	1.59402×10^{-3}	0
0.9	1.57484×10^{-2}	4.52912×10^{-2}	4.82795×10^{-3}	0
1.0	1.78631×10^{-2}	9.29796×10^{-2}	7.84433×10^{-3}	0

TABLE 5. Absolute errors of Example 5.3 for $\psi_2(x)$

x	Method in [30]		Suggested method	
		$m=1$	$m=2$	$m=3$
0.1	1.99879×10^{-4}	1.46339×10^{-2}	3.62132×10^{-3}	0
0.2	4.75397×10^{-4}	3.25600×10^{-2}	1.64100×10^{-2}	0
0.3	7.89170×10^{-4}	4.88261×10^{-2}	2.95774×10^{-2}	0
0.4	1.13069×10^{-3}	6.04406×10^{-2}	3.63960×10^{-2}	0
0.5	1.49445×10^{-3}	6.45455×10^{-2}	3.60909×10^{-2}	0
0.6	1.87697×10^{-3}	5.84157×10^{-2}	3.03835×10^{-2}	0
0.7	2.27584×10^{-2}	3.97032×10^{-2}	2.15300×10^{-2}	0
0.8	2.68925×10^{-2}	6.79901×10^{-3}	1.17235×10^{-2}	0
0.9	3.11582×10^{-2}	4.07493×10^{-2}	2.96446×10^{-3}	0
1.0	3.55442×10^{-2}	1.01834×10^{-1}	2.95048×10^{-3}	0

Applying the approach described in this paper, we determine the approximate solutions. For this case, we can find that $\psi_m(x)$ yields the exact solution only by setting $m=3$. We present our numerical results obtained by proposed Taylor expansion method for $m=1,2,3$ and the results obtained in [30] in Tables 6 and 7.

Example 5.5. Consider the following system of fractional integro-differential equations

$$\begin{cases} D^{\frac{3}{4}}\psi_1(x) - \displaystyle\int_0^x \frac{\psi_1(t)+\psi_2(t)}{\sqrt{x-t}}dt = -\frac{16x^{\frac{5}{2}}}{15} - \frac{32x^{\frac{7}{2}}}{35} + \frac{32x^{\frac{5}{4}}}{5\Gamma(\frac{1}{4})}, \\ D^{\frac{1}{2}}\psi_2(x) - \displaystyle\int_0^x \frac{\psi_1(t)+\psi_2(t)}{(x-t)^{\frac{2}{3}}}dt = -\frac{27x^{\frac{7}{3}}}{14} + \frac{16x^{\frac{5}{2}}}{5\sqrt{\pi}} - \frac{243x^{\frac{10}{3}}}{140}, \end{cases}$$

in which the initial conditions are chosen all to be zero and the exact solutions are $\psi_1(x)=x^2$ and $\psi_2(x)=x^3$.

Based on the proposed method in Section 3, we obtain the approximate results by setting $m=1,2,3$ and we observe that the third-order approximate solution yields the

TABLE 6. Absolute errors of Example 5.4 for $\psi_1(x)$.

x	Method in [30]		Suggested method	
		$m=1$	$m=2$	$m=3$
0.1	1.96792×10^{-4}	1.66987×10^{-2}	4.37610×10^{-3}	0
0.2	6.85268×10^{-4}	4.54650×10^{-2}	1.69027×10^{-3}	0
0.3	1.42175×10^{-3}	7.48952×10^{-2}	1.70008×10^{-3}	0
0.4	2.38624×10^{-3}	9.69101×10^{-2}	3.93799×10^{-3}	0
0.5	3.56576×10^{-3}	1.05439×10^{-1}	4.52983×10^{-3}	0
0.6	4.95084×10^{-3}	9.62850×10^{-2}	3.55933×10^{-3}	0
0.7	6.53406×10^{-3}	6.71607×10^{-2}	1.36667×10^{-3}	0
0.8	8.30938×10^{-3}	1.77783×10^{-2}	1.59402×10^{-3}	0
0.9	1.02717×10^{-2}	5.00357×10^{-2}	4.82795×10^{-3}	0
1.0	1.24167×10^{-2}	1.32209×10^{-1}	7.84433×10^{-3}	0

TABLE 7. Absolute errors of Example 5.4 for $\psi_2(x)$.

x	Method in [30]		Suggested method	
		$m=1$	$m=2$	$m=3$
0.1	8.20450×10^{-4}	1.35222×10^{-1}	4.98795×10^{-2}	0
0.2	1.58553×10^{-3}	1.88478×10^{-1}	8.22827×10^{-2}	0
0.3	2.41026×10^{-3}	2.17328×10^{-1}	9.64328×10^{-2}	0
0.4	3.30743×10^{-3}	2.25836×10^{-1}	9.56954×10^{-2}	0
0.5	4.28071×10^{-3}	2.16061×10^{-1}	8.41589×10^{-2}	0
0.6	5.33111×10^{-3}	1.89798×10^{-1}	6.57542×10^{-2}	0
0.7	6.45864×10^{-3}	1.49181×10^{-1}	4.42508×10^{-2}	0
0.8	7.66286×10^{-3}	9.71051×10^{-2}	2.32810×10^{-2}	0
0.9	8.94313×10^{-3}	3.76493×10^{-2}	6.32948×10^{-3}	0
1.0	1.02987×10^{-2}	2.34213×10^{-2}	3.30327×10^{-3}	0

exact solution as expected. In the following, our results for $m = 1, 2, 3$ at equidistant points in $[0, 1]$ are tabulated in Tables 8 and 9.

TABLE 8. Absolute errors of Example 5.5 for $\psi_1(x)$.

x	$m=1$	$m=2$	$m=3$
0.1	4.39572×10^{-4}	5.63735×10^{-8}	0
0.2	2.02649×10^{-3}	1.49505×10^{-6}	0
0.3	6.38129×10^{-3}	1.61418×10^{-5}	0
0.4	1.85611×10^{-2}	1.16368×10^{-4}	0
0.5	4.69815×10^{-2}	6.32737×10^{-4}	0
0.6	9.46103×10^{-2}	2.86770×10^{-3}	0
0.7	1.53109×10^{-1}	1.22967×10^{-2}	0
0.8	2.14122×10^{-1}	7.00457×10^{-2}	0
0.9	2.76101×10^{-1}	2.65058×10^{-1}	0
1.0	3.40830×10^{-1}	1.19614×10^{-1}	0

TABLE 9. Absolute errors of Example 5.5 for $\psi_2(x)$.

x	$m=1$	$m=2$	$m=3$
0.1	1.17689×10^{-4}	2.02948×10^{-5}	0
0.2	1.61962×10^{-3}	1.53357×10^{-4}	0
0.3	9.65962×10^{-3}	4.69785×10^{-4}	0
0.4	3.89089×10^{-2}	8.58738×10^{-4}	0
0.5	1.13095×10^{-1}	4.68815×10^{-4}	0
0.6	2.40454×10^{-1}	4.31091×10^{-3}	0
0.7	3.98040×10^{-1}	2.88928×10^{-2}	0
0.8	5.63382×10^{-1}	1.89377×10^{-1}	0
0.9	7.33038×10^{-1}	7.60116×10^{-1}	0
1.0	9.12716×10^{-1}	3.53191×10^{-1}	0

6. Conclusion

In this paper, we have proposed an approximate method for solving systems of fractional integro-differential equations. In the proposed technique, the SFIDE to be solved, has been converted into integral equations. Then Taylor expansion for unknown functions and integration method have employed to convert the resulting integral equations into a system of linear equations with respect to unknown functions and their derivatives. By applying a standard method the resulting system has been solved. In particular for such cases when the exact solutions are polynomial functions of degree up to m, the derived mth-order approximations are exact.

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