Kragujevac Journal of Mathematics
Volume 44(3) (2020), Pages 423–430.

PARACONTACT METRIC $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-HARMONIC MANIFOLDS

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Abstract. We give classifications of paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ manifolds M^{2n+1} with harmonic curvature for $n > 1$ and $n = 1$.

1. Introduction

Paracontact metric structures were introduced in [5], as a natural odd-dimensional counterpart to para-Hermitian structures, like contact metric structures correspond to the Hermitian ones. Paracontact metric manifolds $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ have been studied by many authors in the recent years, particularly since the appearance of [10]. An important class among paracontact metric manifolds is that of the $(\tilde{\kappa}, \tilde{\mu})$-spaces, which satisfy the nullity condition (see [4])

$$\tilde{R}(X, Y)\xi = \tilde{\kappa}(\eta(Y)X - \eta(X)Y) + \tilde{\mu}(\eta(Y)\tilde{h}X - \eta(X)\tilde{h}Y), \tag{1.1}$$

for all X, Y vector fields on M, where $\tilde{\kappa}$ and $\tilde{\mu}$ are constants and $\tilde{h} = \frac{1}{2}\mathcal{L}_\xi\tilde{\varphi}$.

This class includes the para-Sasakian manifolds (see [5, 10]), the paracontact metric manifolds satisfying $\tilde{R}(X, Y)\xi = 0$, for all X, Y (see [11]), etc.

In [4], the authors showed that while the values of $\tilde{\kappa}$ and $\tilde{\mu}$ change the form of (1.1) remains unchanged under $\mathcal{D}$-homothetic deformations. There are differences between a contact metric (κ, μ)-space $(M^{2n+1}, \varphi, \xi, \eta, g)$ and a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-space $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$. Namely, unlike in the contact Riemannian case, a paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifold such that $\tilde{\kappa} = -1$ in general is not para-Sasakian. In fact, there are paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds such that $\tilde{h}^2 = 0$ (which is equivalent to take $\tilde{\kappa} = -1$) but with $\tilde{h} \neq 0$. For 5-dimensional, Cappelletti Montano and Di Terlizzi gave the first example of paracontact metric $(-1, 2)$-space $(M^{2n+1}, \tilde{\varphi}, \xi, \eta, \tilde{g})$ with $\tilde{h}^2 = 0$ but

Key words and phrases. Paracontact metric manifolds, R-harmonic manifold, (κ, μ)-nullity distribution.

2010 *Mathematics Subject Classification.* Primary: 53B30, 53C25. Secondary: 53D10.

DOI 10.46793/KgJMat2003.423E

Received: February 12, 2018.

Accepted: May 30, 2018.

$\tilde{h} \neq 0$ in [3] and then Cappelletti Montano et al. gave the first paracontact metric structures defined on the tangent sphere bundle and constructed an example with arbitrary n in [4]. Later, for 3-dimensional, the first numerical example was given in [6]. Another important difference with the contact Riemannian case, due to the non-positive definiteness of the metric, is that while for contact metric (κ, μ)-spaces the constant κ can not be greater than 1, paracontact metric $(\tilde{\kappa}, \tilde{\mu})$-space has no restriction for the constants $\tilde{\kappa}$ and $\tilde{\mu}$.

Contact metric R-harmonic manifolds were studied in [1], [9]. But no effort has been made for paracontact $(\tilde{\kappa}, \tilde{\mu})$-manifolds. Hence, in this paper, we give some characterizations for paracontact $(\tilde{\kappa}, \tilde{\mu})$ R-harmonic manifolds, i.e, for paracontact metric manifolds whose characteristic vector ξ belongs to the $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution and whose curvature tensor $\tilde{R}$ satisfies the condition $(div\tilde{R})(X, Y, Z) = 0$.

The outline of the article goes as follows. In Section 2, we recall basic facts which we will need throughout the paper. In Section 3, we deal with some results related with paracontact metric manifolds with characteristic vector field ξ belongs to the $(\tilde{\kappa}, \tilde{\mu})$-nullity distribution. Section 4 is devoted to paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ R-Harmonic manifolds. For such manifolds, our first result is that a paracontact metric R-harmonic manifold $M^{2n+1}$where $n > 1$, for which the characteristic vector field ξ belongs to the $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution is either locally product of a flat $(n+1)$-dimensional manifold and n-dimensional of negative constant curvature equal to -4, or Ricci operator of the manifold has the form $\tilde{Q} = (n^2+n+2)I + (3n+1)\tilde{h} - (3n^2+7n+2)\eta \otimes \xi$ with $\tilde{\kappa} \leq -5$, or the manifold is an Einstein manifold. Our second result is that a paracontact metric R-harmonic manifold M^3, for which the characteristic vector field ξ belongs to the $(\tilde{\kappa} \neq -1, \tilde{\mu})$-nullity distribution is either flat, or Ricci operator of the manifold has the form $\tilde{Q} = 4I + 4\tilde{h} - 12\eta \otimes \xi$ with $\tilde{\kappa} = -4$.

2. Preliminaries

In this section we collect the formulas and results we need on paracontact metric manifolds. All manifolds are assumed to be connected and smooth. We may refer to [5], [10] and references therein for more information about paracontact metric geometry.

An $(2n+1)$-dimensional smooth manifold M is said to have an *almost paracontact structure* if it admits a $(1,1)$-tensor field $\tilde{\varphi}$, a vector field ξ and a 1-form η satisfying the following conditions:

(i) $\eta(\xi) = 1$, $\tilde{\varphi}^2 = I - \eta \otimes \xi$;

(ii) the tensor field $\tilde{\varphi}$ induces an almost paracomplex structure on each fibre of $\mathcal{D} = \ker(\eta)$, i.e., the ± 1-eigendistributions, $\mathcal{D}^{\pm} = \mathcal{D}_{\tilde{\varphi}}(\pm 1)$ of $\tilde{\varphi}$ have equal dimension n.

From the definition it follows that $\tilde{\varphi}\xi = 0$, $\eta \circ \tilde{\varphi} = 0$ and the endomorphism $\tilde{\varphi}$ has rank $2n$. We denote by $[\tilde{\varphi}, \tilde{\varphi}]$ the Nijenhius torsion

$$[\tilde{\varphi}, \tilde{\varphi}](X, Y) = \tilde{\varphi}^2[X, Y] + [\tilde{\varphi}X, \tilde{\varphi}Y] - \tilde{\varphi}[\tilde{\varphi}X, Y] - \tilde{\varphi}[X, \tilde{\varphi}Y].$$

When the tensor field $N_{\tilde{\varphi}} = [\tilde{\varphi}, \tilde{\varphi}] - 2d\eta \otimes \xi$ vanishes identically the almost paracontact manifold is said to be *normal*. If an almost paracontact manifold admits a pseudo-Riemannian metric $\tilde{g}$ such that

$$\tilde{g}(\tilde{\varphi}X, \tilde{\varphi}Y) = -\tilde{g}(X, Y) + \eta(X)\eta(Y),$$

for all $X, Y \in \Gamma(TM)$, then we say that $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is an *almost paracontact metric manifold*. Notice that any such a pseudo-Riemannian metric is necessarily of signature $(n+1, n)$. For an almost paracontact metric manifold, there always exists an orthogonal basis $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi\}$, such that $\tilde{g}(X_i, X_j) = \delta_{ij}$, $\tilde{g}(Y_i, Y_j) = -\delta_{ij}$, $\tilde{g}(X_i, Y_j) = 0$, $\tilde{g}(\xi, X_i) = \tilde{g}(\xi, Y_j) = 0$, and $Y_i = \tilde{\varphi}X_i$, for any $i, j \in \{1, \ldots, n\}$. Such basis is called a $\tilde{\varphi}$-basis.

We can now define the *fundamental form* of the almost paracontact metric manifold by $F(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$. If $d\eta(X, Y) = \tilde{g}(X, \tilde{\varphi}Y)$, then $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be *paracontact metric manifold*. In a paracontact metric manifold one defines a symmetric, trace-free operator $\tilde{h} = \frac{1}{2}\mathcal{L}_\xi \tilde{\varphi}$, where $\mathcal{L}_\xi$, denotes the Lie derivative. It is known [10] that $\tilde{h}$ anti-commutes with $\tilde{\varphi}$ and satisfies $\tilde{h}\xi = 0$, tr$\tilde{h}$ =tr$\tilde{h}\tilde{\varphi} = 0$ and

$$\tilde{\nabla}\xi = -\tilde{\varphi} + \tilde{\varphi}\tilde{h}, \tag{2.1}$$

where $\tilde{\nabla}$ is the Levi-Civita connection of the pseudo-Riemannian manifold $(M, \tilde{g})$. Let $\tilde{R}$ be Riemannian curvature operator

$$\tilde{R}(X, Y)Z = (\tilde{\nabla}^2_{X,Y}Z) - (\tilde{\nabla}^2_{Y,X}Z) = [\tilde{\nabla}_X, \tilde{\nabla}_Y]Z - \tilde{\nabla}_{[X,Y]}Z.$$

Moreover $\tilde{h} = 0$ if and only if ξ is Killing vector field. In this case $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ is said to be a *K-paracontact manifold*. A normal paracontact metric manifold is called a *para-Sasakian manifold*. Also in this context the para-Sasakian condition implies the K-paracontact condition and the converse holds only in dimension 3. We also recall that any para-Sasakian manifold satisfies

$$\tilde{R}(X, Y)\xi = -(\eta(Y)X - \eta(X)Y).$$

3. Paracontact Metric $(\tilde{\kappa}, \tilde{\mu})$-Manifolds

In this section we recall several notions and results which will be needed throughout the paper.

Let $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ be a paracontact manifold. The $(\tilde{\kappa}, \tilde{\mu})$-nullity distribution of a $(M, \tilde{\varphi}, \xi, \eta, \tilde{g})$ for the pair $(\tilde{\kappa}, \tilde{\mu})$ is a distribution

$$\begin{aligned} N(\tilde{\kappa}, \tilde{\mu}) : p \to N_p(\tilde{\kappa}, \tilde{\mu}) = \{Z \in T_pM \mid \tilde{R}(X, Y)Z &= \tilde{\kappa}(\tilde{g}(Y, Z)X - \tilde{g}(X, Z)Y) \\ &+ \tilde{\mu}(\tilde{g}(Y, Z)\tilde{h}X - \tilde{g}(X, Z)\tilde{h}Y)\}, \end{aligned}$$

for some real constants $\tilde{\kappa}$ and $\tilde{\mu}$. If the characteristic vector field ξ belongs to the $(\tilde{\kappa}, \tilde{\mu})$-nullity distribution we have (1.1). [4] is a complete study of paracontact metric manifolds for which the Reeb vector field of the underlying contact structure satisfies a nullity condition (the condition (1.1), for some real numbers $\tilde{\kappa}$ and $\tilde{\mu}$).

Lemma 3.1 ([4]). *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a paracontact metric $(\tilde\kappa,\tilde\mu)$-manifold of dimension $2n+1$. Then the following identity holds:*

$$(\tilde\nabla_X\tilde h)Y-(\tilde\nabla_Y\tilde h)X=-(1+\tilde\kappa)(2\tilde g(X,\tilde\varphi Y)\xi+\eta(X)\tilde\varphi Y-\eta(Y)\tilde\varphi X)+(1-\tilde\mu)(\eta(X)\tilde\varphi\tilde hY-\eta(Y)\tilde\varphi\tilde hX), \tag{3.1}$$

for any vector fields X, Y on M.

Lemma 3.2 ([4]). *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a paracontact $(\tilde\kappa,\tilde\mu)$-manifold such that $\tilde\kappa\neq-1$. Then the operator $\tilde h$ in the case $\tilde\kappa>-1$ and the operator $\tilde\varphi\tilde h$ in the case $\tilde\kappa<-1$ are diagonalizable and admit three eigenvalues: 0, associated to the eigenvector ξ, $\tilde\lambda$ and -$\tilde\lambda$, of multiplicity n, where $\tilde\lambda:=\sqrt{|1+\tilde\kappa|}$. The corresponding eigendistributions $\mathcal{D}_{\tilde h}(0)=\mathbb{R}\xi$, $\mathcal{D}_{\tilde h}(\tilde\lambda)$, $\mathcal{D}_{\tilde h}(-\tilde\lambda)$ and $\mathcal{D}_{\tilde\varphi\tilde h}(0)=\mathbb{R}\xi$, $\mathcal{D}_{\tilde\varphi\tilde h}(\tilde\lambda)$, $\mathcal{D}_{\tilde\varphi\tilde h}(-\tilde\lambda)$ are mutually orthogonal and one has $\tilde\varphi\mathcal{D}_{\tilde h}(\pm\tilde\lambda)=\mathcal{D}_{\tilde h}(\mp\tilde\lambda)$ and $\tilde\varphi\mathcal{D}_{\tilde\varphi\tilde h}(\pm\tilde\lambda)=\mathcal{D}_{\tilde\varphi\tilde h}(\mp\tilde\lambda)$. Furthermore,*

$$\mathcal{D}_{\tilde h}(\pm\tilde\lambda)=\left\{X\pm\frac{1}{\sqrt{1+\tilde\kappa}}\tilde hX\mid X\in\Gamma(\mathcal{D}^{\mp})\right\},$$

in the case $\tilde\kappa>-1$, and

$$\mathcal{D}_{\tilde\varphi\tilde h}(\pm\tilde\lambda)=\left\{X\pm\frac{1}{\sqrt{-1-\tilde\kappa}}\tilde\varphi\tilde hX\mid X\in\Gamma(\mathcal{D}^{\mp})\right\},$$

in the case $\tilde\kappa<-1$, where $\mathcal{D}^+$ and $\mathcal{D}^-$ denote the eigendistributions of $\tilde\varphi$ corresponding to the eigenvalues 1 and -1, respectively. Finally any two among the four distributions $\mathcal{D}^+$, $\mathcal{D}^-$, $\mathcal{D}_{\tilde h}(\tilde\lambda)$, $\mathcal{D}_{\tilde h}(-\tilde\lambda)$ in the case $\tilde\kappa>-1$ or $\mathcal{D}^+$, $\mathcal{D}^-$, $\mathcal{D}_{\tilde\varphi\tilde h}(\tilde\lambda)$, $\mathcal{D}_{\tilde\varphi\tilde h}(-\tilde\lambda)$ in the case $\tilde\kappa<-1$ are mutually transversal.

Theorem 3.1 ([4]). *Any positive or negative definite paracontact $(\tilde\kappa,\tilde\mu)$-manifold such that $\tilde\kappa<-1$ carries a canonical contact Riemannian structure (ϕ,ξ,η,g) given by*

$$\phi:=\pm\frac{1}{\sqrt{-1-\tilde\kappa}}\tilde h,\qquad g:=-d\eta(\cdot,\phi\cdot)+\eta\otimes\eta,$$

where the sign $\pm$ depends on the positive or negative definiteness of the paracontact $(\tilde\kappa,\tilde\mu)$-manifold. Moreover, (ϕ,ξ,η,g) is a contact metric (κ,μ)-structure, where

$$\kappa=\tilde\kappa+2-\left(1-\frac{\tilde\mu}{2}\right)^2,\quad \mu=2.$$

Lemma 3.3 ([4]). *In any $(2n+1)$-dimensional paracontact $(\tilde\kappa,\tilde\mu)$-manifold $(M,\tilde\varphi,\xi,\eta,\tilde g)$ such that $\tilde\kappa\neq-1$, the Ricci operator $\tilde Q$ is given by*

$$\tilde Q=(2(1-n)+n\tilde\mu)I+(2(n-1)+\tilde\mu)\tilde h+(2(n-1)+n(2\tilde\kappa-\tilde\mu))\eta\otimes\xi. \tag{3.2}$$

Lemma 3.4. *Let $(M,\tilde\varphi,\xi,\eta,\tilde g)$ be a paracontact $(\tilde\kappa,\tilde\mu)$-manifold such that $\tilde\kappa\neq-1$. Then the following identity holds:*

$$(\tilde\nabla_X\tilde S)(Y,Z)=[2(n-1)+\tilde\mu]\,(\tilde g(\tilde\nabla_X\tilde h)Y,Z)+[2(n-1)+n(2\tilde\kappa-\tilde\mu)]\,(\tilde g(\tilde\nabla_X\xi,Y)\eta(Z)+\tilde g(Z,\tilde\nabla_X\xi)\eta(Y)), \tag{3.3}$$

for any vector fields X, Y, Z on M.

Proof. Differentiating $\tilde{S}$ covariantly with respect to X, we have

$$(\tilde{\nabla}_X \tilde{S})(Y, Z) = \tilde{\nabla}_X \tilde{S}(Y, Z) - \tilde{S}(\tilde{\nabla}_X Y, Z) - \tilde{S}(Y, \tilde{\nabla}_X Z). \tag{3.4}$$

By means of $\tilde{S}(Y, Z) = \tilde{g}(\tilde{Q}Y, Z)$ and (3.2), we find

$$\begin{aligned}\tilde{\nabla}_X \tilde{S}(Y, Z) =&(2(1-n) + n\tilde{\mu})(\tilde{g}(\tilde{\nabla}_X Y, Z) + \tilde{g}(Y, \tilde{\nabla}_X Z))\\ &+ (2(n-1) + \tilde{\mu})(\tilde{g}(\tilde{\nabla}_X \tilde{h}Y, Z) + \tilde{g}(\tilde{h}Y, \tilde{\nabla}_X Z))\\ &+ (2(n-1) + n(2\tilde{\kappa} - \tilde{\mu}))(\tilde{g}(\tilde{\nabla}_X Y, \xi) + \tilde{g}(Y, \tilde{\nabla}_X \xi))\eta(Z)\\ &+ (2(n-1) + n(2\tilde{\kappa} - \tilde{\mu}))(\tilde{g}(\tilde{\nabla}_X Z, \xi) + \tilde{g}(Z, \tilde{\nabla}_X \xi))\eta(Y).\end{aligned} \tag{3.5}$$

Taking into account again (3.2), we get

$$\begin{aligned}-\tilde{S}(\tilde{\nabla}_X Y, Z) =& - (2(1-n) + n\tilde{\mu})\tilde{g}(\tilde{\nabla}_X Y, Z)\\ &- (2(n-1) + \tilde{\mu})(\tilde{g}(\tilde{h}\tilde{\nabla}_X Y, Z)\\ &- (2(n-1) + n(2\tilde{\kappa} - \tilde{\mu}))\eta(\tilde{\nabla}_X Y)\eta(Z)\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}-\tilde{S}(Y, \tilde{\nabla}_X Z) =& - (2(1-n) + n\tilde{\mu})\tilde{g}(\tilde{\nabla}_X Z, Y)\\ &- (2(n-1) + \tilde{\mu})\tilde{g}(\tilde{h}Y, \tilde{\nabla}_X Z)\\ &- (2(n-1) + n(2\tilde{\kappa} - \tilde{\mu}))\eta(\tilde{\nabla}_X Z)\eta(Y).\end{aligned} \tag{3.7}$$

Using (3.5)-(3.7) in (3.4), we obtain the requested equation. □

4. Paracontact Metric $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-Harmonic Manifolds

In this section, we will investigate harmonicity of the curvature tensor of a pseudo-Riemannian manifold. It is well known that, if the divergence of the curvature tensor of a pseudo-Riemannian manifold is equal to zero, then this curvature tensor is called harmonic.

Proposition 4.1. *Let $\tilde{R}$ be a curvature tensor field which satisfies the second Bianchi identity. If $\tilde{S}$ is the associated Ricci tensor field, then*

$$(\mathrm{div}\tilde{R})(X, Y, Z) = (\tilde{\nabla}_X \tilde{S})(Y, Z) - (\tilde{\nabla}_Y \tilde{S})(X, Z).$$

Definition 4.1 ([7])**.** A curvature tensor field $\tilde{R}$ is *harmonic if*

$$(\mathrm{div}\tilde{R})(X, Y, Z) = 0.$$

A pseudo-Riemannian manifold M is said to be *$\tilde{R}$-harmonic* if its curvature tensor field $\tilde{R}$ is harmonic. Following [8], a pseudo- Riemannian manifold has harmonic curvature tensor if and only if the Ricci operator Q, which is given by $\tilde{S}(X, Y) = \tilde{g}(\tilde{Q}X, Y)$ where S is the Ricci tensor, satisfies

$$(\tilde{\nabla}_X \tilde{Q})Y - (\tilde{\nabla}_Y \tilde{Q})X = 0, \tag{4.1}$$

for any vector fields X, Y on M .

Theorem 4.1 ([11]). *Let M^{2n+1} be a paracontact metric manifold and suppose that $\tilde{R}(X,Y)\xi = 0$ for all vector fields X and Y. Then locally M^{2n+1} is the product of a flat $(n+1)$-dimensional manifold and n-dimensional manifold of negative constant curvature equal to -4.*

Theorem 4.2. *Let M^{2n+1} be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-harmonic manifold where $n > 1$. If $\tilde{\kappa} \neq -1$, then M is either*

i) *locally product of a flat $(n+1)$-dimensional manifold and n-dimensional of negative constant curvature equal to -4, or*

ii) *the Ricci operator of the manifold has the form*

$$\tilde{Q} = (n^2 + n + 2)I + (3n+1)\tilde{h} - (3n^2 + 7n + 2)\eta \otimes \xi,$$

with $\tilde{\kappa} \leq -5$, or

iii) *M is an Einstein manifold.*

Proof. Using (3.3) and (4.1), we obtain

$$\begin{aligned}(\tilde{\nabla}_X \tilde{Q})Y - (\tilde{\nabla}_Y \tilde{Q})X = & [2(n-1) + \tilde{\mu}]\,((\tilde{\nabla}_X \tilde{h})Y - (\tilde{\nabla}_Y \tilde{h})X) \\ & + [2(n-1) + n(2\tilde{\kappa} - \tilde{\mu})]\,(\tilde{g}(\tilde{\nabla}_X \xi, Y)\xi + \eta(Y)\tilde{\nabla}_X \xi \\ & - \tilde{g}(\tilde{\nabla}_Y \xi, X)\xi - \eta(X)\tilde{\nabla}_Y \xi). \end{aligned} \tag{4.2}$$

With the help of (3.1) and *$\tilde{R}$-harmonic manifold definition*, (4.2) returns to

$$\begin{aligned}(\tilde{\nabla}_X \tilde{Q})Y - (\tilde{\nabla}_Y \tilde{Q})X = & [2(n-1) + \tilde{\mu}]\,[-(1+\tilde{\kappa})(2\tilde{g}(X, \tilde{\varphi}Y)\xi + \eta(X)\tilde{\varphi}Y - \eta(Y)\tilde{\varphi}X) \\ & + (1-\tilde{\mu})(\eta(X)\tilde{\varphi}\tilde{h}Y - \eta(Y)\tilde{\varphi}\tilde{h}X)] \\ & + [2(n-1) + n(2\tilde{\kappa} - \tilde{\mu})]\,[\tilde{g}(\tilde{\nabla}_X \xi, Y)\xi + \eta(Y)\tilde{\nabla}_X \xi \\ & - \tilde{g}(\tilde{\nabla}_Y \xi, X)\xi - \eta(X)\tilde{\nabla}_Y \xi] \\ = & 0 \end{aligned} \tag{4.3}$$

If we take the inner product of (4.3) with ξ and use (2.1), one can easily show that

$$0 = 2\tilde{g}(X, \tilde{\varphi}Y)\left[\tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1)\right].$$

Taking into account that $\tilde{g}(X, \tilde{\varphi}Y) = d\eta(X,Y) \neq 0$, we can conclude that

$$\tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1) = 0. \tag{4.4}$$

Replacing X by ξ in (4.3), by direct computations we get

$$[\tilde{\kappa}(2-\tilde{\mu}) - \tilde{\mu}(n+1)]\,\tilde{\varphi}Y + [-2n\tilde{\kappa} + \tilde{\mu}(3-n-\tilde{\mu})]\,\tilde{\varphi}\tilde{h}Y = 0.$$

In virtue of (4.4), we have

$$[-2n\tilde{\kappa} + \tilde{\mu}(3-n-\tilde{\mu})]\,\tilde{\varphi}\tilde{h}Y = 0. \tag{4.5}$$

From the last equation, precisely following cases occurs

$$\begin{aligned}\tilde{\kappa}(2-\tilde{\mu})-\tilde{\mu}(n+1) &=0 \quad \text{and} \quad -2n\tilde{\kappa}+\tilde{\mu}(3-n-\tilde{\mu})=0, \\ \tilde{\varphi}\tilde{h}Y &=0.\end{aligned} \tag{4.6}$$

We now check, case by case, whether (4.5) give rise to a local classifcation.

First of all, solving the system of (4.6), we have following possibilities:

(i) $\tilde{\kappa}=\tilde{\mu}=0$;
(ii) $\tilde{\kappa}=-(n+3)=-\tilde{\mu}$;
(iii) $\tilde{\kappa}=\frac{(1-n)(1+n)}{n}$, $\tilde{\mu}=2-2n$.

If the first (i) equality holds, then using Theorem 4.1, we conclude that M is locally product of a flat $(n+1)$-dimensional manifold and n-dimensional of negative constant curvature equal to -4. If the second (ii) equality holds, then we can deduce that the Ricci operator of the manifold has the form $\tilde{Q}=(n^2+n+2)I+(3n+1)\tilde{h}-(3n^2+7n+2)\eta\otimes\xi$ with $\tilde{\kappa}\leq-5$.

If the third (iii) equality holds, using (3.2), we obtain M is an Einstein manifold.

Secondly, suppose $\tilde{\varphi}\tilde{h}Y=0$. By (2.1), we have $\tilde{\nabla}_Y\xi=-\tilde{\varphi}Y$ which means that M is K-paracontact and hence $\tilde{h}=0$. Using the fact that $h^2=(1+\tilde{\kappa})\tilde{\varphi}^2$, we obtain $\tilde{\kappa}=-1$. But this contradicts with the chosen of $\tilde{\kappa}$. So, we omit this case. ☐

Using the same method for the proof, we can give following result.

Theorem 4.3. *Let M^3 be a paracontact metric $(\tilde{\kappa}, \tilde{\mu})$ $\tilde{R}$-harmonic manifold. If $\tilde{\kappa}\neq-1$, then M is either*

i) *flat, or*
ii) *the Ricci operator of the manifold has the form $\tilde{Q}=4I+4\tilde{h}-12\eta\otimes\xi$ with $\tilde{\kappa}=-4$.*

Remark 4.1. Using Theorem 3.1 and Theorem 4.2, we can say that if M^{2n+1} be a paracontact metric $\tilde{R}$*-harmonic manifold with* ξ belonging to $(\tilde{\kappa}\neq-1, \tilde{\mu})$-nullity distribution ,then M^{2n+1} carries a canonical contact metric (κ, μ)-structure where either $\kappa=1$, $\mu=2$ or $\kappa=\frac{-n^2-6n-5}{4}$, $\mu=2$ or $\kappa=\frac{1-n^2+2n-n^3}{n}$, $\mu=2$.

Acknowledgements. The author is grateful to the referee for the valuable suggestions and comments towards the improvement of the paper.

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