ISSN 1450-9628

# KRAGUJEVAC JOURNAL OF MATHEMATICS

Volume 44, Number 4, 2020

University of Kragujevac
Faculty of Science

CIP - Каталогизација у публикацији
Народна библиотека Србије, Београд

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**KRAGUJEVAC Journal of Mathematics** / Faculty of Science, University of Kragujevac ; editor-in-chief Suzana Aleksić. - Vol. 22 (2000)- . - Kragujevac : Faculty of Science, University of Kragujevac, 2000- (Kragujevac : InterPrint). - 24 cm

Tromesečno. - Delimično je nastavak: Zbornik radova Prirodno-matematičkog fakulteta (Kragujevac) = ISSN 0351-6962. - Drugo izdanje na drugom medijumu: Kragujevac Journal of Mathematics (Online) = ISSN 2406-3045
ISSN 1450-9628 = Kragujevac Journal of Mathematics
COBISS.SR-ID 75159042

DOI 10.46793/KgJMat2004

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|---|---|
| **Published By:** | Faculty of Science<br>University of Kragujevac<br>Radoja Domanovića 12<br>34000 Kragujevac<br>Serbia<br>Tel.: +381 (0)34 336223<br>Fax: +381 (0)34 335040<br>Email: `krag_j_math@kg.ac.rs`<br>Website: `http://kjm.pmf.kg.ac.rs` |
| **Designed By:** | Thomas Lampert |
| **Front Cover:** | Željko Mališić |
| **Printed By:** | InterPrint, Kragujevac, Serbia<br>From 2018 the journal appears in one volume and four issues per annum. |

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 495–508.

# RESULTS ON TAUBERIAN THEOREM FOR CESÀRO SUMMABLE DOUBLE SEQUENCES OF FUZZY NUMBERS

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Abstract. The paper aims to establish new results on Tauberian theorem for Cesàro summability of double sequences of fuzzy numbers, and thus to extend and unify several results in the available literature. Further, a number of special cases, corollaries and illustrative example in support of the investigation of this paper are also presented.

## 1. Introduction and Preliminaries

The notion of the fuzzy set was introduced by Zadeh [19]. Matloka [10] has established bounded and convergent sequences of fuzzy numbers and proved that every convergent sequence is bounded. Nanda [12] has studied the spaces of bounded and convergent sequences of fuzzy numbers and proved that every Cauchy sequence of fuzzy numbers is convergent. Subrahmanyam [13] has presented Cesàro summability of sequences of fuzzy numbers and established Tauberian hypotheses identified with the Cesàro summability method. Talo and Çanak [15] introduced necessary and sufficient Tauberian conditions, under which convergence follows from Cesàro convergence of sequences of fuzzy numbers. Altın *et al.* [1] studied the concept of statistical summability by $(C, 1)$-mean for sequences of fuzzy numbers and obtained a Tauberian theorem on that basis. Talo and Başar [14] introduced the concept of slow decreasing sequence for fuzzy numbers and have shown that Cesàro summable sequence $(X_n)$ is convergent, if $(X_n)$ is slowly decreasing. Recently, Çanak [2] has established the concept of the slow oscillation (that is, both slowly decreasing and slowly increasing)

*Key words and phrases.* Double Cesàro summability, slow oscillation, Tauberian condition, sequence of fuzzy numbers.
2010 *Mathematics Subject Classification.* Primary: 40G05, 40E05. Secondary: 40A10, 03E72.
DOI 10.46793/KgJMat2004.495J
*Received*: March 15, 2018.
*Accepted*: June 11, 2018.

sequences for fuzzy numbers and have shown that Cesàro summable sequence $(X_n)$ is convergent if $(X_n)$ is slowly oscillating.

Let $D$ denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R}$. For $X, Y \in D$, we define

$$d(X,Y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},$$

where

$$X = [x_1, x_2], \quad Y = [y_1, y_2].$$

It is surely understood that $(D, d)$ is a complete metric space.

A fuzzy number $X$ is a fuzzy set on $\mathbb{R}$ and is a mapping $X : \mathbb{R} \to [0,1]$ associating each number $t$ with its grade of membership $X(t)$.

A fuzzy number $X$ is said to be convex, if

$$X(t) = \min\{X(s), X(r)\}, \quad s < t < r.$$

If there exists $t_0 \in \mathbb{R}$, such that $X(t_0) = 1$, then the fuzzy number $X$ is called normal. A fuzzy number $X$ is said to be upper semi-continuous if, for each $\epsilon > 0$, we have

$$X^{-1}([0, x + \epsilon))$$

for all $x \in [0,1]$), is open in the usual topology of $\mathbb{R}$. The set of all upper semi-continuous, normal, convex fuzzy number is denoted by $\mathbb{R}([0,1])$. For $\alpha \in (0,1]$, the $\alpha$- level set of fuzzy number $X$ denoted by $X^\alpha$ is defined by

$$X^\alpha = \{t \in \mathbb{R} : X(t) \geqq \alpha\}.$$

The set $X^0$ is defined as the closure of the following set

$$\{t \in \mathbb{R} : X(t) > 0\}.$$

We define,

$$\bar{d} : \mathbb{R}([0,1]) \times \mathbb{R}([0,1]) \to \mathbb{R}_+ \cup \{0\},$$

by

$$\bar{d}(X,Y) = \sup_{0 \leqq \alpha \leqq 1} d(X^\alpha, Y^\alpha).$$

## 2. Definitions and Motivation

A double sequence $(X_{mn})$ of fuzzy numbers is a function, $X : \mathbb{N} \cup \{0\} \times \mathbb{N} \cup \{0\} \to \mathbb{R}([0,1])$ and is said to be convergent to a fuzzy number $X_0$ if, for every $\epsilon > 0$, there exist a positive integer $n_0$ such that

$$\bar{d}\,(X_{mn}, X_0) < \epsilon, \quad \text{for all } m, n \geqq n_0.$$

We define,

$$\Delta_n X_{mn} = \bar{d}(X_{mn}, X_{m,n-1}),$$
$$\Delta_m X_{mn} = \bar{d}(X_{mn}, X_{m-1,n})$$

and

$$\Delta_{m,n} X_{mn} = \bar{d}(X_{mn}, X_{m-1,n}) - \bar{d}(X_{m,n-1}, X_{m-1,n-1}), \quad X_{-1} = 0.$$

A double sequence $(X_{mn})$ of fuzzy numbers is said to be bounded, if there exists a positive number $\mathcal{K} > 0$ such that

$$\bar{d}(X_{mn}, X_0) \leqq \mathcal{K}, \quad \text{for all } m, n \in \mathbb{N} \cup \{0\}.$$

The Cesàro means $(C, 1)$ of sequence $(X_n)$ of fuzzy numbers are defined by

$$\sigma_n = \frac{1}{n+1} \sum_{j=0}^{n} X_j, \quad \text{for all } n \in \mathbb{N} \cup \{0\}.$$

A sequence $(X_n)$ of fuzzy numbers is Cesàro summable to a fuzzy number $L$ if, for every $\epsilon > 0$, we have (see [2])

$$\bar{d}(\sigma_n, L) < \epsilon, \quad n \to \infty.$$

Similarly, the Cesàro means $(C, 1, 1)$ of double sequences $(X_{mn})$ of fuzzy numbers are defined by

$$\sigma_{mn}^{(1,1)}(X) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m} \sum_{q=0}^{n} X_{pq} = \sum_{p=1}^{m} \sum_{q=1}^{n} \frac{Y_{pq}^{(1,1)}}{pq} + X_{00} \tag{2.1}$$

(see [11]). Analogous to equation (2.1), we may define the $(C, 1, 0)$ and $(C, 0, 1)$-means of sequences $(X_{mn})$ are

$$\sigma_{mn}^{(1,0)}(X) = \frac{1}{m+1} \sum_{p=0}^{m} X_{pn} \quad \text{and} \quad \sigma_{mn}^{(0,1)}(X) = \frac{1}{n+1} \sum_{q=0}^{n} X_{mq}, \tag{2.2}$$

respectively.

Then we say that, a double sequence $X = (X_{mn})$ of fuzzy numbers is $(C, 1, 1)$-summable to a fuzzy number $L$ if, for every $\epsilon > 0$, we have

$$\bar{d}\left(\sigma_{mn}^{(1,1)}(X), L\right) < \epsilon, \quad \text{for all } m, n \to \infty.$$

Similarly, we say that it is $(C, 1, 0)$-summable to a fuzzy number $L$ if, for every $\epsilon > 0$, we have

$$\bar{d}\left(\sigma_{mn}^{(1,0)}(X), L\right) < \epsilon, \quad \text{for all } m, n \to \infty$$

and $(C, 0, 1)$-summable to a fuzzy number $L$ if, for every $\epsilon > 0$, we have

$$\bar{d}\left(\sigma_{mn}^{(1,0)}(X), L\right) < \epsilon, \quad \text{for all } m, n \to \infty.$$

Now, for each non-negative integers $k$ and $r$, we may define $\sigma_{mn}^{(kr)}(X)$ as follows:

$$\sigma_{mn}^{(kr)}(X) = \begin{cases} \dfrac{1}{(m+1)(n+1)} \displaystyle\sum_{p=0}^{m} \sum_{q=0}^{n} \sigma_{pq}^{(k-1,r-1)}, & k, r \geq 1, \\ X_{mn}, & k, r = 0. \end{cases}$$

A double sequence $X = (X_{mn})$ of fuzzy numbers is said to be $(C, k, r)$-summable to a fuzzy number $L$ if, for every $\epsilon > 0$, we have

$$\bar{d}\left(\sigma_{mn}^{(kr)}(X), L\right) < \epsilon, \quad \text{for all } m, n \to \infty.$$

*Remark* 2.1. If $k = 1$ and $r = 1$, then $(C, k, r)$-summability reduces to $(C, 1, 1)$-summability. Furthermore, if $k \neq 0$ and $r = 0$ then $(C, k, r)$-summability reduces to $(C, k, 0)$-summability. Finally, if $k = 0$ and $r \neq 0$ then $(C, k, r)$-summability reduces to $(C, 0, r)$-summability.

Note that here, Cesàro summability of $X = (X_{mn})$ refers $(C, 1, 1)$ and $(C, k, r)$-summability of $X = (X_{mn})$.

It may also be noted that, the convergence of a double sequence $X = (X_{mn})$ of fuzzy numbers implies the Cesàro summability of $X = (X_{mn})$, but the converse is not generally true.

For example, consider a function $f(x, y) = e^{2x}\sin(3y)$; the sequence $(X_{mn})$ of fuzzy numbers which is the sequence of coefficients in the Taylor's series expansion of the function $f(x, y)$ about origin is Cesàro summable but not convergent.

For the proof of converse part, certain conditions are presented in terms of oscillatory behavior of double sequence $X = (X_{mn})$ of fuzzy numbers.

Let us define $(X_{mn})$ as

$$X_{mn} = Y_{mn}^{(1,1)} + \sum_{p=1}^{m}\sum_{q=1}^{n} \frac{Y_{pq}^{(1,1)}}{pq} + X_{00}, \quad m, n \in \mathbb{N}, \tag{2.3}$$

where

$$X_{mn} - \sigma_{mn}^{(1,1)}(X) = Y_{mn}^{(1,1)}(\Delta X) = \frac{1}{(m+1)(n+1)} \sum_{p=0}^{m}\sum_{q=0}^{n} pq\,(\Delta_{p,q} X_{pq}) \tag{2.4}$$

(see [9]). Moreover, in analogy to Kronecker identity for a single sequence of fuzzy numbers, we can write

$$Y_{mn}^{(1,0)}(\Delta X) = \frac{1}{(m+1)} \sum_{p=0}^{m} p(\Delta_{p,} X_{p,n}) \tag{2.5}$$

and

$$Y_{mn}^{(0,1)}(\Delta X) = \frac{1}{(n+1)} \sum_{q=0}^{n} q(\Delta_{q,} X_{m,q}), \tag{2.6}$$

as the $(C, 1, 0)$-mean of the sequence $(m\Delta_m X_{mn})$ of fuzzy numbers and the $(C, 0, 1)$-mean of the sequence $(n\Delta_n X_{mn})$ fuzzy number respectively.

Furthermore, as the sequence $Y_{mn}^{(1,1)}(\Delta_{mn} X_{mn})$ of fuzzy numbers is the $(C, 1, 1)$ mean of the sequence $mn(\Delta_{mn} X_{mn})$ of fuzzy number, the sequence $mn(\Delta_{mn} X_{mn})$ is $(C, 1, 1)$-summable to a fuzzy number $L$, whenever

$$\bar{d}\left(Y_{mn}^{(1,1)}(\Delta_{mn} X_{mn}), L\right) < \epsilon, \quad \text{for all } m, n \to \infty.$$

For each non-negative integers $k$ and $r$, let us define $Y_{mn}^{(k,r)}(\Delta X)$ as follows:

$$Y_{mn}^{(k,r)}(\Delta X) = \begin{cases} \dfrac{1}{(m+1)(n+1)} \displaystyle\sum_{p=0}^{m}\sum_{q=0}^{n} Y_{pq}^{(k-1,r-1)}, & k, r \geq 1, \\ mn(\Delta_{m,n} X_{mn}), & k, r = 0. \end{cases}$$

The sequence $mn(\Delta_{mn}X_{mn})$ of fuzzy numbers is said to be $(C,k,r)$-summable to a fuzzy numbers $L$ if, for every $\epsilon > 0$, we have

$$\bar{d}\left(Y_{mn}^{(kr)}(\Delta_{mn}X_{mn}), L\right) < \epsilon, \quad \text{for all } m,n \to \infty.$$

*Remark* 2.2. If $k = 1$ and $r = 1$, then $(C,k,r)$-summabllity reduces to $(C,1,1)$-summability. Furthermore, if $k \neq 0$ and $r = 0$, then $(C,k,r)$-summabllity reduces to $(C,k,0)$-summability. Finally, if $k = 0$ and $r \neq 0$, then $(C,k,r)$-summabllity reduces to $(C,0,r)$-summability.

Next, we present the De la Vallée Poussin mean of double sequence $(X_{mn})$ of fuzzy numbers for sufficiently large nonnegative integers $m,n$ for $\lambda > 1$ and $0 < \lambda < 1$

$$\tau_{mn}(X) = \frac{1}{([\lambda m]-m)([\lambda n]-n)} \sum_{i=m+1}^{[\lambda m]} \sum_{j=n+1}^{[\lambda n]} X_{ij}$$

and

$$\tau_{mn}(X) = \frac{1}{(m-[\lambda m])(n-[\lambda n])} \sum_{i=\lambda m+1}^{m} \sum_{j=\lambda n+1}^{n} X_{ij},$$

respectively.

A single sequence $X = (X_n)$ of fuzzy numbers is slowly oscillating (in the sense of Stanojević) if, (see [18])

$$\lim_{\lambda \to 1^+} \limsup_{n} \max_{n+1 \leqq k \leqq [\lambda n]} \bar{d}(X_k, X_n) = 0. \tag{2.7}$$

Similarly, we may write a double sequence $X = (X_{mn})$ of fuzzy numbers is slowly oscillating (in the sense of Stanojević) if,

$$\lim_{\lambda \to 1^+} \limsup_{m,n} \max_{m+1,n+1 \leqq i,j \leqq [\lambda m],[\lambda n]} \bar{d}\left(\sum_{u=m+1}^{i} \sum_{v=n+1}^{j} \Delta_{u,v}X_{uv}, 0\right) \leqq \epsilon. \tag{2.8}$$

Recently, few researchers have investigated on sequences and sequences of fuzzy numbers for proving Tauberian theorems. Different classes of sequences and sequences of fuzzy numbers have been introduced and studied by Tripathy *et al.* [17], Dutta [3], Dutta [4], Dutta and Bilgin [6], Tripathy and Debnath [16], Dutta and Başar [5], Jena *et al.* [7], Jena *et al.* [8] and many others. Recently, Çanak [2] has introduced Tauberian theorem for Cesàro summability of sequences of fuzzy numbers.

Motivated essentially by the above-mentioned works, here we wish to present the (presumably new) the notion of $(C,1,1)$-summability of a double sequences of fuzzy numbers defined in (2.3).

## 3. Tauberian Theorems for Cesàro Mean

**Theorem 3.1.** *If the double sequence $(X_{mn})$ of fuzzy number is $(C,1,1)$-summable to a fuzzy number L and $(X_{mn})$ is slowly oscillating (in the sense of Stanojević), then*

$$\bar{d}(X_{mn}, L) < \epsilon, \quad \textit{for all } m,n \to \infty.$$

To prove the above theorem, we need the help of the following lemmas.

**Lemma 3.1.** *A double sequence $X = (X_{mn})$ of fuzzy numbers is slowly oscillating if and only if $(Y_{mn}^{(1,1)})$ is slowly oscillating and bounded.*

*Proof.* Let $X = (X_{mn})$ is slowly oscillating. Initially, let us show that

$$\bar{d}(V_{mn}^{(1,1)}, 0) = O(1).$$

We have by definition of slow oscillation, for $\lambda > 1$

$$\lim_{\lambda\to 1^+} \limsup_{m,n} \max_{m+1,n+1 \leqq i,j \leqq [\lambda m, \lambda n]} \bar{d}\left(\sum_{u=m+1}^{i} \sum_{v=n+1}^{j} \Delta_{u,v} X_{uv}, 0\right) \leqq \epsilon$$

and let us rewrite the finite sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta X_{ij},$$

as the series

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \sum_{\frac{m}{2^{u+1}}, \frac{n}{2^{v+1}} \leqq i,j \leqq \frac{m}{2^u}, \frac{n}{2^v}} ij\Delta X_{ij}.$$

Clearly,

$$\begin{aligned}\bar{d}\left(\sum_{i=1}^{m}\sum_{j=1}^{n} ij\Delta X_{ij}, 0\right) &\leqq \bar{d}\left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \sum_{\frac{m}{2^{u+1}}, \frac{n}{2^{v+1}} \leqq i,j < \frac{m}{2^u}, \frac{n}{2^v}} ij\Delta X_{ij}, 0\right) \\ &\leqq \bar{d}\left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{mn}{2^{u+v}}, 0\right)\end{aligned}$$

and

$$\begin{aligned}&\max_{\frac{m}{2^{u+1}}+1, \frac{n}{2^{v+1}}+1 \leqq i,j \leqq \left[\frac{\lambda m}{2^{i+1}}, \frac{\lambda n}{2^{j+1}}\right]} \bar{d}\left(\sum_{u=\frac{m}{2^{u+1}}+1}^{i} \sum_{v=\frac{n}{2^{v+1}}+1}^{j} \Delta_{u,v} X_{uv}, 0\right) \\ \leqq & mnC\left(\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \frac{1}{2^{u+v}}\right) = mnC^*, \quad C^* > 0.\end{aligned}$$

Consequently, we have

$$\bar{d}\left(Y_{mn}^{(1,1)}(\Delta X), 0\right) = \bar{d}\left(\frac{1}{(m+1)(n+1)} \sum_{p=0}^{m}\sum_{q=0}^{n} pq(\Delta_{p,q} X_{pq}), 0\right) = O(1), \quad m,n \to \infty.$$

Since,

$$\left\{\sigma_{mn}^{(1,1)}(X) = \sum_{p=1}^{m}\sum_{q=1}^{n} \frac{Y_{pq}^{(1,1)}}{pq} + X_{00}\right\}$$

is slowly oscillating; so $(Y_{mn}^{(1,1)})$ is oscillating slowly.

To prove the converse part, consider $(Y_{mn}^{(1,1)})$ is bounded and slowly oscillating. Now the boundedness of $(Y_{mn}^{(1,1)})$ implies that $\sigma_{mn}^{(1,1)}(X)$ is slowly oscillating. Furthermore, $(Y_{mn}^{(1,1)})$ being oscillating slowly, so by Kronecker identity (2.4), $(X_{mn})$ is oscillating slowly. Which completes the proof of Lemma 3.1. □

Next, we represent $\bar{d}\left(X_{mn}, \sigma_{mn}^{(11)}(X)\right)$ under two different cases in the following lemma.

**Lemma 3.2.** *Let $X = (X_{mn})$ be a sequence of fuzzy numbers with $m, n$ sufficiently large, then we have the following.*

(i) *For $\lambda > 1$*

$$\begin{aligned}\bar{d}\left(X_{mn}, \sigma_{mn}^{(1,1)}(X)\right) =&\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left\{\bar{d}\left(\sigma_{[\lambda m],[\lambda n]}^{(1,1)}(X), \sigma_{[\lambda m],n}^{(1,1)}(X)\right)\right.\\ &\left.-\bar{d}\left(\sigma_{m,[\lambda n]}^{(1,1)}(X), \sigma_{mn}^{(1,1)}(X)\right)\right\}\\ &+\frac{[\lambda m]+1}{[\lambda n]-m}\bar{d}\left(\sigma_{[\lambda m],n}^{(1,1)}(X), \sigma_{m,n}^{(1,1)}(X)\right)\\ &+\frac{[\lambda n]+1}{[\lambda m]-m}\bar{d}\left(\sigma_{m,[\lambda n]}^{(1,1)}(X), \sigma_{mn}^{(1,1)}(X)\right)\\ &-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}(X_{ij}, X_{mn})\right).\end{aligned} \tag{3.1}$$

(ii) *For $0 < \lambda < 1$*

$$\begin{aligned}\bar{d}\left(X_{mn}, \sigma_{mn}^{(1,1)}(X)\right) =&\frac{([\lambda m]+1)([\lambda n]+1)}{(m-[\lambda m])(n-[\lambda n])}\left\{\bar{d}\left(\sigma_{mn}^{(1,1)}(X), \sigma_{[\lambda m],n}^{(1,1)}(X)\right)\right.\\ &\left.-\bar{d}\left(\sigma_{m,[\lambda n]}^{(1,1)}(X), \sigma_{[\lambda m],[\lambda n]}^{(1,1)}(X)\right)\right\}\\ &+\frac{[\lambda m]+1}{m-[\lambda m]}\bar{d}\left(\sigma_{m,n}^{(1,1)}(X), \sigma_{[\lambda m],n}^{(1,1)}(X)\right)\\ &+\frac{[\lambda n]+1}{n-[\lambda n]}\bar{d}\left(\sigma_{mn}^{(1,1)}(X), \sigma_{m,[\lambda n]}^{(1,1)}(X)\right)\\ &-\frac{1}{(m-[\lambda m])(n-[\lambda n])}\bar{d}\left(\sum_{i=[\lambda m]+1}^{m}\sum_{j=[\lambda n]+1}^{n}(X_{mn}, X_{ij})\right).\end{aligned} \tag{3.2}$$

*Proof.* We have by De la Vallée Poussin mean of double sequence $(X_{mn})$ of fuzzy numbers

$$\tau_{mn}(X) =\frac{1}{([\lambda m]-m)([\lambda n]-n)}\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}X_{ij}$$

$$
\begin{aligned}
=&\frac{1}{([\lambda m]-m)([\lambda n]-n)}\left\{\bar{d}\left(\sum_{i=0}^{[\lambda m]},\sum_{i=0}^{[m]}\right)\bar{d}\left(\sum_{j=0}^{[\lambda n]},\sum_{j=0}^{[n]}\right)\right\}X_{ij}\\
=&\frac{1}{([\lambda m]-m)([\lambda n]-n)}\left\{\bar{d}\left(\sum_{i=0}^{[\lambda m]}\sum_{j=0}^{[\lambda n]},\sum_{i=0}^{[\lambda m]}\sum_{j=0}^{[n]}\right)X_{ij}-\bar{d}\left(\sum_{i=0}^{[m]}\sum_{j=0}^{[\lambda n]},\sum_{i=0}^{[m]}\sum_{j=0}^{[n]}\right)X_{ij}\right\}\\
=&\frac{1}{([\lambda m]-m)([\lambda n]-n)}\Bigg\{([\lambda n]+1)([\lambda m]+1)\sigma^{(1,1)}_{[\lambda m],[\lambda n]}\\
&-([\lambda m]+1)(n+1)\sigma^{(1,1)}_{[\lambda m],n}\Bigg\}-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\\
&-\left\{(m+1)([\lambda n]+1)\sigma^{(1,1)}_{m,[\lambda n]}-(m+1)(n+1)\sigma^{1,1}_{mn}\right\}\\
=&\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\sigma^{(1,1)}_{[\lambda m],[\lambda n]}-\Bigg\{\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\sigma^{(1,1)}_{[\lambda m],n}\\
&-\frac{([\lambda m]+1)}{([\lambda m]-m)}\sigma^{(1,1)}_{[\lambda m],n}\Bigg\}\\
&-\left\{\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\sigma^{(1,1)}_{m,[\lambda n]}-\frac{([\lambda n]+1)}{([\lambda n]-n)}\sigma^{(1,1)}_{m,[\lambda n]}\right\}\\
&+\Bigg\{\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\sigma^{(1,1)}_{m,n}-\frac{([\lambda m]+1)}{([\lambda m]-m)}\sigma^{(1,1)}_{m,n}\\
&-\frac{([\lambda n]+1)}{([\lambda n]-n)}\sigma^{(1,1)}_{m,n}+\sigma^{(1,1)}_{m,n}\Bigg\}.
\end{aligned}
$$

Which implies

$$
\begin{aligned}
\tau_{mn}-\sigma^{(1,1)}_{m,n}=&\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left\{\bar{d}\left(\sigma^{(1,1)}_{[\lambda m],[\lambda n]},\sigma^{(1,1)}_{[\lambda m],n}\right)-\bar{d}\left(\sigma^{(1,1)}_{m,[\lambda n]},\sigma^{(1,1)}_{m,n}\right)\right\}\\
&+\frac{([\lambda m]+1)}{([\lambda m]-m)}\bar{d}\left(\sigma^{(1,1)}_{[\lambda m],n},\sigma^{(1,1)}_{m,n}\right)+\frac{([\lambda n]+1)}{([\lambda n]-n)}\bar{d}\left(\sigma^{(1,1)}_{m,[\lambda n]},\sigma^{(1,1)}_{m,n}\right).
\end{aligned}
$$

Also,

$$
X_{mn}=\tau_{mn}-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}(X_{ij},X_{mn})\right).
$$

Subtracting $\left(\sigma^{(1,1)}_{[\lambda m],[\lambda n]}\right)$ from the above identity, we have

$$
\begin{aligned}
&\bar{d}\left(X_{mn},\sigma^{(1,1)}_{mn}(X)\right)\\
=&\bar{d}\left(\tau_{mn}(X),\sigma^{(1,1)}_{[\lambda m],[\lambda n]}\right)-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}(X_{ij},X_{mn})\right)
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\left\{\bar{d}\left(\sigma^{(1,1)}_{[\lambda m],[\lambda n]}(X),\sigma^{(1,1)}_{[\lambda m],n}(X)\right)-\bar{d}\left(\sigma^{(1,1)}_{m,[\lambda n]}(X),\sigma^{(1,1)}_{mn}(X)\right)\right\}\\
&+\frac{[\lambda m]+1}{[\lambda n]-m}\bar{d}\left(\sigma^{(1,1)}_{[\lambda m],n}(X),\sigma^{(1,1)}_{m,n}(X)\right)+\frac{[\lambda n]+1}{[\lambda m]-m}\bar{d}\left(\sigma^{(1,1)}_{m,[\lambda n]}(X),\sigma^{(1,1)}_{mn}(X)\right)\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}(X_{ij},X_{mn})\right).
\end{aligned}
$$

Which establish (i). Next, the proof of (ii) is similar to (i). □

**Proof of Theorem 1.**

*Proof.* Let $(X_{mn})$ is $(C,1,1)$-summable to a fuzzy number $L$, this implies $\sigma^{(1,1)}_{mn}$ is $(C,1,1)$-summable to a fuzzy number $L$. Now from equation (2.4), we have $(Y^{(1,1)}_{mn})$ is $(C,1,1)$-summable to zero. Thus by Lemma 3.1, $(Y^{(1,1)}_{mn})$ oscillating slowly. Again by Lemma 3.2 (i), we get

$$
\begin{aligned}
\bar{d}\left(Y^{(1,1)}_{mn},\sigma^{(1,1)}_{mn}(Y^{(1,1)}_{mn})\right)=&\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\Bigg\{\bar{d}\left(\sigma^{(1,1)}_{[\lambda m],[\lambda n]}(Y^{(1,1)}_{mn}),\sigma^{(1,1)}_{[\lambda m],n}(Y^{(1,1)}_{mn})\right)\\
&-\bar{d}\left(\sigma^{(1,1)}_{m,[\lambda n]}(Y^{(1,1)}_{mn}),\sigma^{(1,1)}_{mn}(Y^{(1,1)}_{mn})\right)\Bigg\}\\
&+\frac{[\lambda m]+1}{[\lambda n]-m}\bar{d}\left(\sigma^{(1,1)}_{[\lambda m],n}(Y^{(1,1)}_{mn}),\sigma^{(1,1)}_{m,n}(Y^{(1,1)}_{mn})\right)\\
&+\frac{[\lambda n]+1}{[\lambda m]-m}\bar{d}\left(\sigma^{(1,1)}_{m,[\lambda n]}(Y^{(1,1)}_{mn}),\sigma^{(1,1)}_{mn}(Y^{(1,1)}_{mn})\right)\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}(Y^{(1,1)}_{ij},Y^{(1,1)}_{mn})\right).
\end{aligned}\tag{3.3}
$$

It is easy to verify that for $\lambda>1$ and sufficiently large $n$

$$
\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}<\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-1-m)([\lambda n]-1-n)}<\frac{4\lambda^2}{(\lambda-1)^2}.
$$

Next, by (3.3)

$$
\bar{d}\left(Y^{(1,1)}_{mn},\sigma^{(1,1)}_{mn}(Y^{(1,1)}_{mn})\right)\leqq\frac{4\lambda^2}{(\lambda-1)^2}\bar{d}\left(\tau_{mn}(Y^{(1,1)}_{mn}),\sigma^{(1,1)}_{[\lambda m],[\lambda n]}(Y^{(1,1)}_{mn})\right)
$$

$$
-\max_{m+1,n+1\leqq i,j\leqq[\lambda m],[\lambda n]}\bar{d}\left(\sum_{j=n+1}^{[\lambda n]}(Y^{(1,1)}_{ij},Y^{(1,1)}_{mn})\right).\tag{3.4}
$$

Taking lim sup both sides of (3.4), we have

$$\limsup_{m,n} \bar{d}\left(Y_{mn}^{(1,1)}, \sigma_{mn}^{(1,1)}(Y_{mn}^{(1,1)})\right) \leqq \frac{4\lambda^2}{(\lambda-1)^2} \limsup_{m,n} \bar{d}\left(\tau_{mn}(Y_{mn}^{(1,1)}), \sigma_{[\lambda m],[\lambda n]}^{(1,1)}(Y_{mn}^{(1,1)})\right)$$

$$-\limsup_{m,n} \max_{m+1,n+1 \leqq i,j \leqq [\lambda m],[\lambda n]} \bar{d}\left(\sum_{j=n+1}^{[\lambda n]} \left(Y_{ij}^{(1,1)}, Y_{mn}^{(1,1)}\right)\right). \tag{3.5}$$

Furthermore,

$$\sigma_{[\lambda m],[\lambda n]}^{(1,1)}(Y_{mn}^{(1,1)}) \to 0, \quad m,n \to \infty,$$

so first term in the right hand side of equation (3.5), must vanish.
This implies,

$$\limsup_{m,n} \bar{d}\left(Y_{mn}^{(1,1)}, \sigma_{mn}^{(1,1)}(Y_{mn}^{(1,1)})\right)$$

$$\leqq \limsup_{m,n} \max_{m+1,n+1 \leqq i,j \leqq [\lambda m],[\lambda n]} \bar{d}\left(\sum_{j=n+1}^{[\lambda n]} \left(Y_{ij}^{(1,1)}, Y_{mn}^{(1,1)}\right)\right). \tag{3.6}$$

As $\lambda \to 1^+$ in (3.6), so we get

$$\limsup_{m,n} \bar{d}\left(Y_{mn}^{(1,1)}, \sigma_{mn}^{(1,1)}(Y_{mn}^{(1,1)})\right) \leqq 0. \tag{3.7}$$

It implies that,

$$\bar{d}\left(Y_{mn}^{(1,1)}, 0\right) < \epsilon, \quad m,n \to \infty.$$

Since $(X_{mn})$ is summable to a fuzzy number $L$ by $(C,1,1)$ mean and

$$\bar{d}\left(Y_{mn}^{(1,1)}, 0\right) < \epsilon, \quad m,n \to \infty,$$

so

$$\bar{d}\left(X_{mn}, L\right) < \epsilon, \quad m,n \to \infty.$$

Which completes the proof of the Theorem 3.1. □

**Corollary 3.1.** *If $(X_{mn})$ is $(C,k,r)$-summable to a fuzzy number $L$ and $(X_{mn})$ is slowly oscillating (in the sense of Stanojević), then*

$$\bar{d}(X_{mn}, L) < \epsilon, \quad m,n \to \infty.$$

*Proof.* Let $X = (X_{mn})$ be slowly oscillating, then $\sigma_{mn}^{(k,r)}(X)$ is slowly oscillating (by Lemma 1). Furthermore, since $X = (X_{mn})$ is $(C,k,r)$-summable to a fuzzy number $L$, so by Theorem 3.1

$$\bar{d}\left(\sigma_{mn}^{(k,r)}(X), L\right) < \epsilon, \quad m,n \to \infty. \tag{3.8}$$

Next from the definition,

$$\sigma_{mn}^{(k,r)}(X) = \sigma_{mn}^{(1,1)}(X)\left(\sigma_{mn}^{(k-1,r-1)}(X)\right). \tag{3.9}$$

Clearly, equation (3.8) and (3.9) implies $X = (X_{mn})$ is $(C, k-1, r-1)$-summable to a fuzzy number $L$. Again $\left(\sigma_{mn}^{(k-1,r-1)}(X)\right)$ is also slowly oscillating (by Lemma 3.1); thus by Theorem 3.1, we have

$$\bar{d}\left(\sigma_{mn}^{(k-1,r-1)}(X), L\right) < \epsilon, \quad m, n \to \infty.$$

Continuing in this way, we obtain

$$\bar{d}(X_{mn}, L) < \epsilon, \quad m, n \to \infty.$$

Which completes the proof of the Corollary 3.1. $\square$

*Remark* 3.1. If $k = 0$, and $r \neq 0$, then $(C, k, r)$- summability reduces to $(C, 0, r)$-summability. Again for $k \neq 0$ and $r = 0$, $(C, k, r)$-summability reduces to $(C, k, 0)$-summability.

**Theorem 3.2.** *If the double sequence $X = (X_{mn})$ of fuzzy number is $(C, 1, 1)$-summable to a fuzzy number $L$ and $Y_{mn}^{(1,1)}(\Delta_{mn}u_{mn})$ is slowly oscillating, then*

$$\bar{d}(X_{mn}, L) < \epsilon, \quad m, n \to \infty.$$

*Proof.* As $(X_{mn})$ is $(C, 1, 1)$-summable to a fuzzy number $L$, so $(\sigma_{mn}^{(1,1)})$ is $(C, 1, 1)$-summable to a fuzzy number $L$. Therefore, $(Y_{mn}^{(1,1)})$ is $(C, 1, 1)$-summable to zero by equation (2.4). Using identity (2.4) to $(Y_{mn}^{(1,1)})$, we get $Y(Y_{mn}^{(1,1)})$ is Cesàro summable to zero. So that $Y(Y_{mn}^{(1,1)})$ is oscillating slowly by Lemma 3.1. Now by Lemma 3.2(i),

$$\begin{aligned}
&\bar{d}\left(Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)\\
=&\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)}\Bigg[\bar{d}\left(\sigma_{[\lambda m],[\lambda n]}^{(1,1)}Y(Y_{mn}^{(1,1)}), \sigma_{[\lambda m],n}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)\\
&-\bar{d}\left(\sigma_{m,[\lambda n]}^{(1,1)}Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)\Bigg]\\
&+\frac{[\lambda m]+1}{[\lambda n]-m}\bar{d}\left(\sigma_{[\lambda m],n}^{(1,1)}Y(Y_{mn}^{(1,1)}), \sigma_{m,n}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)\\
&+\frac{[\lambda n]+1}{[\lambda m]-m}\bar{d}\left(\sigma_{m,[\lambda n]}^{(1,1)}Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)\\
&-\frac{1}{([\lambda m]-m)([\lambda n]-n)}\bar{d}\left(\sum_{i=m+1}^{[\lambda m]}\sum_{j=n+1}^{[\lambda n]}\left(Y(Y_{ij}^{(1,1)}), Y(Y_{mn}^{(1,1)})\right)\right).
\end{aligned} \tag{3.10}$$

It is easy to verify that for $\lambda > 1$ and sufficiently large $n$

$$\frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-m)([\lambda n]-n)} < \frac{([\lambda m]+1)([\lambda n]+1)}{([\lambda m]-1-m)([\lambda n]-1-n)} < \frac{4\lambda^2}{(\lambda-1)^2}.$$

Next, by (3.10)

$$\bar{d}\left(Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right) \leqq \frac{4\lambda^2}{(\lambda-1)^2}\bar{d}\left(\tau_{mn}Y(Y_{mn}^{(1,1)}), \sigma_{[\lambda m],[\lambda n]}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)$$

$$- \max_{m+1,n+1\leqq i,j\leqq[\lambda m],[\lambda n]} \bar{d}\left(\sum_{j=n+1}^{[\lambda n]}\left(Y(Y_{ij}^{(1,1)}), Y(Y_{mn}^{(1,1)})\right)\right). \tag{3.11}$$

Taking lim sup both sides of (3.11) we have

$$\limsup_{m,n} \bar{d}\left(Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)$$
$$\leqq \frac{4\lambda^2}{(\lambda-1)^2}\limsup_{m,n} \bar{d}\left(\tau_{mn}Y(Y_{mn}^{(1,1)}), \sigma_{[\lambda m],[\lambda n]}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)$$
$$- \limsup_{m,n} \max_{m+1,n+1\leqq i,j\leqq[\lambda m],[\lambda n]} \bar{d}\left(\sum_{j=n+1}^{[\lambda n]}\left(Y(Y_{ij}^{(1,1)}), Y(Y_{mn}^{(1,1)})\right)\right). \tag{3.12}$$

Furthermore, as $\sigma_{[\lambda m],[\lambda n]}^{(1,1)}Y(Y_{mn}^{(1,1)})$ converges, so first term in the right hand side of equation (3.12), must vanish.

This implies,

$$\limsup_{m,n} \bar{d}\left(Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right)$$
$$\leqq \limsup_{m,n} \max_{m+1,n+1\leqq i,j\leqq[\lambda m],[\lambda n]} \bar{d}\left(\sum_{j=n+1}^{[\lambda n]}\left(Y(Y_{ij}^{(1,1)}), Y(Y_{mn}^{(1,1)})\right)\right). \tag{3.13}$$

As $\lambda \to 1^+$ in (3.13), so we get

$$\limsup_{m,n} \bar{d}\left(Y(Y_{mn}^{(1,1)}), \sigma_{mn}^{(1,1)}Y(Y_{mn}^{(1,1)})\right) \leqq 0. \tag{3.14}$$

It implies that,

$$\bar{d}(Y(Y_{mn}^{(1,1)}), 0) < \epsilon, \quad m, n \to \infty.$$

Since $(X_{mn})$ is summable to a fuzzy number $L$ by $(C, 1, 1)$ mean and

$$\bar{d}(Y(Y_{mn}^{(1,1)}), 0) < \epsilon, \quad m, n \to \infty$$

so,

$$\bar{d}(X_{mn}, L) < \epsilon, \quad m, n \to \infty.$$

Which completes the proof of the Theorem 3.2. □

**Corollary 3.2.** *If $(X_{mn})$ is $(C, k, r)$-summable to a fuzzy number $L$ and $Y_{mn}^{(1,1)}(\Delta X)$ is slowly oscillating, then*

$$\bar{d}(X_{mn}, L) < \epsilon, \quad m, n \to \infty.$$

*Proof.* As $Y_{mn}^{(1,1)}(\Delta X)$ is slowly oscillating, setting $X = (X_{mn})$ in place of $Y_{mn}^{(1,1)}(\Delta X)$, then $\sigma_{mn}^{(k,r)}(Y_{mn}^{(1,1)}(\Delta X))$ is slowly oscillating by Lemma 3.1. Again as $Y_{mn}^{(1,1)}(\Delta X)$ is $(C, k, r)$-summable to a fuzzy number $L$, so by Theorem 3.2, we have

$$\bar{d}\left(\sigma_{mn}^{(k,r)}\left(Y_{mn}^{(1,1)}(\Delta X)\right), L\right) < \epsilon, \quad m, n \to \infty. \tag{3.15}$$

By definition,

$$\sigma_{mn}^{(k,r)}\left(Y_{mn}^{(1,1)}(\Delta X)\right) = \sigma_{mn}^{(1,1)}\left(Y_{mn}^{(1,1)}(\Delta X)\right)\left[\sigma_{mn}^{(k-1,r-1)}(Y_{mn}^{(1,1)}(\Delta X))\right]. \tag{3.16}$$

From (3.15) and (3.16) we have $Y_{mn}^{(1,1)}(\Delta X)$ is $(C, k-1, r-1)$-summable to a fuzzy number $L$. Again by Lemma 3.1, since

$$\sigma_{mn}^{(k-1,r-1)}\left(Y_{mn}^{(1,1)}(\Delta X)\right),$$

is slowly oscillating, so we have

$$\bar{d}\left(\sigma_{mn}^{(k-1,r-1)}(Y_{mn}^{(1,1)}(\Delta X)), L\right) < \epsilon \quad \text{(by Theorem 3.2)}.$$

Continuing in this way, we obtain

$$\bar{d}\left((Y_{mn}^{(1,1)}(\Delta X)), L\right) < \epsilon, \quad m, n \to \infty.$$

Which completes the proof of the Corollary 3.2. $\square$

*Remark* 3.2. If $k = 0$, and $r \neq 0$, then $(C, k, r)$- summability reduces to $(C, 0, r)$-summability. Again for $k \neq 0$ and $r = 0$, $(C, k, r)$-summability reduces to $(C, k, 0)$-summability and consequently the following corollaries are generated from the main result.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 509–522.

# SHARP BOUNDS ON THE AUGMENTED ZAGREB INDEX OF GRAPH OPERATIONS

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Abstract. Let $G$ be a finite and simple graph with edge set $E(G)$. The *augmented Zagreb index* of $G$ is

$$AZI(G) = \sum_{uv \in E(G)} \left( \frac{d_G(u)d_G(v)}{d_G(u) + d_G(v) - 2} \right)^3,$$

where $d_G(u)$ denotes the degree of a vertex $u$ in $G$. In this paper, we give some bounds of this index for join, corona, cartesian and composition product of graphs by general sum-connectivity index and general Randić index and compute the sharp amount of that for the regular graphs.

## 1. Introduction

Let $G$ be a finite and simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The integers $n = n(G) = |V(G)|$ and $m = m(G) = |E(G)|$ are the order and the size of the graph $G$, respectively. For a vertex $v \in V(G)$, the *open neighborhood* of $v$, denoted by $N_G(v) = N(v)$, is the set $\{u \in V(G) \mid uv \in E(G)\}$. The *degree* of $v \in V(G)$, denoted by $d_G(v)$, is defined by $d_G(v) = |N_G(v)|$. The maximum (resp. minimum) degree of vertices of $G$ is denoted by $\Delta_G$ (resp. $\delta_G$). We use Bondy and Murty [10] for terminology and notation not defined here.

Several authors defined and studied more vertex degree-based graph invariants such as [16]. One of them is *augmented Zagreb index* of $G$ that is proposed in 2010 by Furtula et al. [15] as

*Key words and phrases.* Augmented Zagreb index, general sum-connectivity index, general Randić index, graph operations.

2010 *Mathematics Subject Classification.* Primary: 05C12. Secondary: 05C07.

DOI 10.46793/KgJMat2004.509D

*Received*: December 02, 2017.

*Accepted*: June 13, 2018.

$$AZI(G) = \sum_{uv\in E(G)} \left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right)^3,$$

where $d_G(u)$ denotes the degree of a vertex $u$ in $G$. The researchers give a good bounds for it by using different graph parameters, investigate the impact of removing and adding the edge for graph on the augmented Zagreb index. For details see [1,18,24,27].

In 2009, Zu and Trijnastić [28] defined the *sum-connectivity index* as

$$\chi(G) = \sum_{uv\in E(G)} d_G(u) + d_G(v)$$

and one year later, they in [29] introduced the *general sum-connectivity index* as

$$\chi_\lambda(G) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))^\lambda, \quad \text{for } \lambda \in \mathbb{R}.$$

There are good results on general sum-connectivity index such as [22, 23]. In 1975, the chemist Milan Randić [21] introduced a topological index $R(G)$ under the name *branching index*. The branching index was renamed the *molecular connectivity index* and is often referred to as the *Randić index* and later named *second Zagreb index*. In 1998, Bollobas and Erdos [9] proposed the generealization state of it named *general Randić index*, $R_\lambda(G)$, as

$$R_\lambda(G) = \sum_{uv\in E(G)} (d_G(u)d_G(v))^\lambda, \quad \text{for } \lambda \in \mathbb{R}$$

later that is named *second general Zagreb index*.

The relation between several indices and operations of graphs were very studied. (see [2–8,11–14,17,19,20,25,26]). In this paper, we calculate bounds of the augmented Zagreb index by two other indices, the general sum-connectivity index and the Randić index for join, corona, cartesian and composition product of graphs and compute the sharp amount of that for the regular graphs.

## 2. The Join of Graphs

The join $G + H$ of graphs $G$ and $H$ with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges joining $V(G)$ and $V(H)$. Obviously, $|V(G + H)| = |V(G)| + |V(H)|$ and $|E(G + H)| = |E(G)| + |E(H)| + |V(G)||V(H)|$.

**Theorem 2.1.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned} AZI(G+H) \leq & \frac{(\Delta_G-1)^3 AZI(G)}{(\Delta_G+n_2-1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G+n_2-1)^3} \\ & + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\delta_G+n_2-1)^3} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(\Delta_H-1)^3 AZI(H)}{(\Delta_H+n_1-1)^3}\\
&+\frac{n_1^3\chi_3(H)+(3n_1^2\Delta_H^2+3n_1^4)\chi_2(H)+3n_1^5\chi_1(H)}{8(\delta_H+n_1-1)^3}\\
&+\frac{(6n_1\Delta_H+3n_1^2)R_2(H)+(12n_1^3\Delta_H+3n_1^4)R_1(H)+m_2n_1^6}{8(\delta_H+n_1-1)^3}\\
&+n_1n_2\left(\frac{(\Delta_G+n_2)(\Delta_H+n_1)}{\delta_G+\delta_H+n_1+n_2-2}\right)^3,
\end{aligned}
$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* By definition,

$$AZI(G+H)=\sum_{uv\in E(G+H)}\left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u)+d_{G+H}(v)-2}\right)^3.$$

We partition the edges of $G+H$ in to three subset $E_1$, $E_2$ and $E_3$, as follows:

$$
\begin{aligned}
E_1=&\{e=uv \mid u,v\in V(G)\},\\
E_2=&\{e=uv \mid u,v\in V(H)\},\\
E_3=&\{e=uv \mid u\in V(G), v\in V(H)\}.
\end{aligned}
$$

Let $e=uv\in E_1$. Then $d_{G+H}(u)=d_G(u)+n_2$ and $d_{G+H}(v)=d_G(v)+n_2$. Hence

$$
\begin{aligned}
((d_G(u)+n_2)(d_G(v)+n_2))^3=&(d_G(u)d_G(v))^2[3n_2(d_G(u)+d_G(v))+3n_2^2]\\
&+(d_G(u)d_G(v))^3+d_G(u)d_G(v)\\
&\times[6n_2^3(d_G(u)+d_G(v))+3n_2^4]\\
&+n_2^3(d_G(u)+d_G(v))^3+3n_2^5(d_G(u)+d_G(v))\\
&+(d_G(u)+d_G(v))^2[3n_2^2d_G(u)d_G(v)+3n_2^4]+n_2^6
\end{aligned}
$$

and

$$
\begin{aligned}
&\left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u)+d_{G+H}(v)-2}\right)^3\\
=&\left(1-\frac{2n_2}{d_G(u)+d_G(v)+2n_2-2}\right)^3\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right)^3\\
&+\frac{n_2^3(d_G(u)+d_G(v))^3+[3n_2^2(d_G(u)d_G(v))^2+3n_2^4](d_G(u)+d_G(v))^2}{(d_G(u)+d_G(v)+2n_2-2)^3}\\
&+\frac{3n_2^5(d_G(u)+d_G(v))+[3n_2(d_G(u)+d_G(v))+3n_2^2](d_G(u)d_G(v))^2}{(d_G(u)+d_G(v)+2n_2-2)^3}\\
&+\frac{[6n_2^3(d_G(u)+d_G(v))+3n_2^4]d_G(u)d_G(v)+n_2^6}{(d_G(u)+d_G(v)+2n_2-2)^3}
\end{aligned}
$$

$$\leq \left(\frac{\Delta_G - 1}{\Delta_G + n_2 - 1}\right)^3 \left(\frac{d_G(u)d_G(v)}{d_G(u) + d_G(v) - 2}\right)^3$$
$$+ \frac{n_2^3(d_G(u) + d_G(v))^3 + (3n_2^2\Delta_G^2 + 3n_2^4)(d_G(u) + d_G(v))^2}{8(\delta_G + n_2 - 1)^3}$$
$$+ \frac{3n_2^5(d_G(u) + d_G(v)) + (6n_2\Delta_G + 3n_2^2)(d_G(u)d_G(v))^2}{8(\delta_G + n_2 - 1)^3}$$
$$+ \frac{(12n_2^3\Delta_G + 3n_2^4)d_G(u)d_G(v) + n_2^6}{8(\delta_G + n_2 - 1)^3}.$$

Therefore,

$$\sum_{uv\in E_1} \left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v) - 2}\right)^3 \leq \left(\frac{\Delta_G - 1}{\Delta_G + n_2 - 1}\right)^3 AZI(G) + \frac{m_1 n_2^6}{8(\delta_G + n_2 - 1)^3}$$
$$+ \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3}$$
$$+ \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G)}{8(\delta_G + n_2 - 1)^3}. \tag{2.1}$$

Obviously, equality holds if and only if $\Delta_G = \delta_G$. Similarly

$$\sum_{uv\in E_2} \left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v) - 2}\right)^3 \leq \left(\frac{\Delta_H - 1}{\Delta_H + n_1 - 1}\right)^3 AZI(H) + \frac{m_2 n_1^6}{8(\delta_H + n_1 - 1)^3}$$
$$+ \frac{n_1^3\chi_3(H) + (3n_1^2\Delta_H^2 + 3n_1^4)\chi_2(H) + 3n_1^5\chi_1(H)}{8(\delta_H + n_1 - 1)^3}$$
$$+ \frac{(6n_1\Delta_H + 3n_1^2)R_2(H) + (12n_1^3\Delta_H + 3n_1^4)R_1(H)}{8(\delta_H + n_1 - 1)^3}. \tag{2.2}$$

Equality holds if and only if $\Delta_H = \delta_H$. Let $e = uv \in E_3$ such that $u \in V(G)$ and $v \in V(H)$. Then $d_{G+H}(u) = d_G(u) + n_2$ and $d_{G+H}(v) = d_H(v) + n_1$. Hence for every edge $e = uv \in E_3$,

$$\left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v)}\right)^3 = \left(\frac{(d_G(u) + n_2)(d_H(v) + n_1)}{d_G(u) + d_H(v) + n_1 + n_2 - 2}\right)^3$$
$$\leq \left(\frac{(\Delta_G + n_2)(\Delta_H + n_1)}{\delta_G + \delta_H + n_1 + n_2 - 2}\right)^3.$$

Therefore,

$$\sum_{uv\in E_3} \left(\frac{d_{G+H}(u)d_{G+H}(v)}{d_{G+H}(u) + d_{G+H}(v)}\right)^3 \leq n_1 n_2 \left(\frac{(\Delta_G + n_2)(\Delta_H + n_1)}{\delta_G + \delta_H + n_1 + n_2 - 2}\right)^3, \tag{2.3}$$

with equality if and only if $\Delta_G = \delta_G$ and $\Delta_H = \delta_H$. By Equations (2.1), (2.2) and (2.3), we have:

$$\begin{aligned}
AZI(G+H) \leq & \frac{(\Delta_G - 1)^3 AZI(G)}{(\Delta_G + n_2 - 1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\
& + \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1 n_2^6}{8(\delta_G + n_2 - 1)^3} \\
& + \frac{(\Delta_H - 1)^3 AZI(H)}{(\Delta_H + n_1 - 1)^3} + \frac{n_1^3\chi_3(H) + (3n_1^2\Delta_H^2 + 3n_1^4)\chi_2(H) + 3n_1^5\chi_1(H)}{8(\delta_H + n_1 - 1)^3} \\
& + \frac{(6n_1\Delta_H + 3n_1^2)R_2(H) + (12n_1^3\Delta_H + 3n_1^4)R_1(H) + m_2 n_1^6}{8(\delta_H + n_1 - 1)^3} \\
& + n_1 n_2 \left( \frac{(\Delta_G + n_2)(\Delta_H + n_1)}{\delta_G + \delta_H + n_1 + n_2 - 2} \right)^3 .
\end{aligned}$$

Equality holds if and only if $G$ and $H$ are regular graphs. $\square$

**Theorem 2.2.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned}
AZI(G+H) \geq & \frac{(\delta_G - 1)^3 AZI(G)}{(\delta_G + n_2 - 1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\Delta_G + n_2 - 1)^3} \\
& + \frac{(6n_2\delta_G + 3n_2^2)R_2(G) + (12n_2^3\delta_G + 3n_2^4)R_1(G) + m_1 n_2^6}{8(\Delta_G + n_2 - 1)^3} \\
& + \frac{(\delta_H - 1)^3 AZI(H)}{(\delta_H + n_1 - 1)^3} + \frac{n_1^3\chi_3(H) + (3n_1^2\delta_H^2 + 3n_1^4)\chi_2(H) + 3n_1^5\chi_1(H)}{8(\Delta_H + n_1 - 1)^3} \\
& + \frac{(6n_1\delta_H + 3n_1^2)R_2(H) + (12n_1^3\delta_H + 3n_1^4)R_1(H) + m_2 n_1^6}{8(\Delta_H + n_1 - 1)^3} \\
& + n_1 n_2 \left( \frac{(\delta_G + n_2)(\delta_H + n_1)}{\Delta_G + \Delta_H + n_1 + n_2 - 2} \right)^3 ,
\end{aligned}$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* Using an argument similar to that described in proof of Theorem 2.1, we obtained the result. $\square$

**Corollary 2.1.** *Let $G$ be a $k$-regular graph of order $n_1$ and let $H$ be a $r$-regular graph of order $n_2$. Then*

$$AZI(G+H) = \frac{k(k+n_2)^6}{16(k+n_2-1)^3} + \frac{r(r+n_1)^6}{16(r+n_1-1)^3} + \frac{n_1 n_2 (k+n_2)^3 (r+n_1)^3}{(k+r+n_1+n_2-2)^3}.$$

## 3. The Corona Product of Graphs

The corona product $G \circ H$ of graphs $G$ and $H$ with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is as the graph obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$ and joining the $i$-th vertex of $G$ to every vertex in $i$-th copy of $H$. Obviously, $|V(G\circ H)| = |V(G)| + |V(G)||V(H)|$ and $|E(G\circ H)| = |E(G)| + |V(G)||E(H)| + |V(G)||V(H)|$.

**Theorem 3.1.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned}
AZI(G\circ H) \leq & \frac{(\Delta_G-1)^3 AZI(G)}{(\Delta_G+n_2-1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G+n_2-1)^3}\\
&+ \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\delta_G+n_2-1)^3}\\
&+ \frac{(\Delta_H-1)^3 AZI(H)}{\Delta_H^3} + \frac{\chi_3(H) + (3\Delta_H^2+3)\chi_2(H) + 3\chi_1(H)}{8\delta_H^3}\\
&+ \frac{(6\Delta_H+3)R_2(H) + (12\Delta_H+3)R_1(H) + m_2}{8\delta_H^3}\\
&+ n_1n_2\left(\frac{(\Delta_G+n_2)(\Delta_H+1)}{\delta_G+\delta_H+n_2-1}\right)^3,
\end{aligned}$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* We partition the edges of $G$ in to three subset $E_1$, $E_2$ and $E_3$ such that $E_1 = \{e = uv \mid u, v \in V(G)\}$, $E_2 = \{e = uv \mid u, v \in V(H)\}$ and $E_3 = \{e = uv \mid u \in V(G), v \in V(H)\}$.

If $e = uv \in E_1$, then $d_{G\circ H}(u) = d_G(u) + n_2$ and $d_{G\circ H}(v) = d_G(v) + n_2$ and if $e = uv \in E_2$, then $d_{G\circ H}(u) = d_H(u) + 1$ and $d_{G\circ H}(v) = d_H(v) + 1$. By used of proof of Theorem 2.1, we have,

$$\begin{aligned}
\sum_{uv\in E_1}\left(\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u)+d_{G\circ H}(v)-2}\right)^3 \leq & \frac{(\Delta_G-1)^3 AZI(G)}{(\Delta_G+n_2-1)^3}\\
&+ \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + n_2^5\chi_1(G)}{8(\delta_G+n_2-1)^3}\\
&+ \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G)}{8(\delta_G+n_2-1)^3}\\
&+ \frac{m_1n_2^6}{8(\delta_G+n_2-1)^3}, \qquad (3.1)
\end{aligned}$$

$$\begin{aligned}
\sum_{uv\in E_2}\left(\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u)+d_{G\circ H}(v)-2}\right)^3 \leq & \frac{(\Delta_H-1)^3 AZI(H)}{\Delta_H^3}\\
&+ \frac{\chi_3(H) + (3\Delta_H^2+3)\chi_2(H) + 3\chi_1(H)}{8\delta_H^3}\\
&+ \frac{(6\Delta_{Hh}+3)R_2(H) + (12\Delta_H+3)R_1(H) + m_2}{8\delta_H^3}. \qquad (3.2)
\end{aligned}$$

Obviously, equalities hold if and only if $\Delta_G = \delta_G$ and $\Delta_H = \delta_H$.

Let $e = uv \in E_3$ such that $u \in V(G)$ and $v \in V(H)$. Then $d_{G\circ H}(u) = d_G(u) + n_2$ and $d_{G\circ H}(v) = d_H(v) + 1$. Hence for every edge $e = uv \in E_3$,

$$\begin{aligned}\left(\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u) + d_{G\circ H}(v) - 2}\right)^3 &= \left(\frac{(d_G(u) + n_2)(d_H(v) + 1)}{d_G(u) + d_H(v) + n_2 + 1 - 2}\right)^3 \\ &\le \left(\frac{(\Delta_G + n_2)(\Delta_H + 1)}{\delta_G + \delta_H + n_2 - 1}\right)^3.\end{aligned}$$

Therefore,

$$\sum_{uv\in E_3}\left(\frac{d_{G\circ H}(u)d_{G\circ H}(v)}{d_{G\circ H}(u) + d_{G\circ H}(v) - 2}\right)^3 \le \frac{n_1n_2(\Delta_G + n_2)^3(\Delta_H + 1)^3}{(\delta_G + \delta_H + n_2 - 1)^3}, \tag{3.3}$$

with equality if and only if $\Delta_G = \delta_G$ and $\Delta_H = \delta_H$. By Equations (3.1), (3.2) and (3.3), we have:

$$\begin{aligned}AZI(G\circ H) \le& \frac{(\Delta_G - 1)^3 AZI(G)}{(\Delta_G + n_2 - 1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\Delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\delta_G + n_2 - 1)^3} \\ &+ \frac{(6n_2\Delta_G + 3n_2^2)R_2(G) + (12n_2^3\Delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\delta_G + n_2 - 1)^3} \\ &+ \frac{(\Delta_H - 1)^3 AZI(H)}{\Delta_H^3} + \frac{\chi_3(H) + (3\Delta_H^2 + 3)\chi_2(H) + 3\chi_1(H)}{8\delta_H^3} \\ &+ \frac{(6\Delta_H + 3)R_2(H) + (12\Delta_H + 3)R_1(H) + m_2}{8\delta_H^3} \\ &+ n_1n_2\left(\frac{(\Delta_G + n_2)(\Delta_H + 1)}{\delta_G + \delta_H + n_2 - 1}\right)^3.\end{aligned}$$

Equality holds if and only if $G$ and $H$ are regular graphs. □

**Theorem 3.2.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned}AZI(G\circ H) \ge& \frac{(\delta_G - 1)^3 AZI(G)}{(\delta_G + n_2 - 1)^3} + \frac{n_2^3\chi_3(G) + (3n_2^2\delta_G^2 + 3n_2^4)\chi_2(G) + 3n_2^5\chi_1(G)}{8(\Delta_G + n_2 - 1)^3} \\ &+ \frac{(6n_2\delta_G + 3n_2^2)R_2(G) + (12n_2^3\delta_G + 3n_2^4)R_1(G) + m_1n_2^6}{8(\Delta_G + n_2 - 1)^3} \\ &+ \frac{(\delta_H - 1)^3 AZI(H)}{\delta_H^3} + \frac{\chi_3(H) + (3\delta_H^2 + 3)\chi_2(H) + 3\chi_1(H)}{8\Delta_H^3} \\ &+ \frac{(6\delta_H + 3)R_2(H) + (12\delta_H + 3)R_1(H) + m_2}{8\Delta_H^3} \\ &+ \frac{n_1n_2(\delta_G + n_2)^3(\delta_H + 1)^3}{(\Delta_G + \Delta_H + n_2 - 1)^3},\end{aligned}$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* The proof of the result is similar to this given in Theorem 3.1. □

**Corollary 3.1.** *Let $G$ be a $k$-regular graph of order $n_1$ and let $H$ be a $r$-regular graph of order $n_2$. Then*

$$AZI(G \circ H) = \frac{k(k+n_2)^6}{16(k+n_2-1)^3} + \frac{r(r+1)^6}{16r^3} + \frac{n_1n_2(k+n_2)^3(r+1)^3}{(k+r+n_2-1)^3}.$$

## 4. The Cartesian Product of Graphs

The Cartesian product $G \times H$ of graphs $G$ and $H$ has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \times H$ if $uv \in E(G)$ and $x = y$, or $u = v$ and $xy \in E(H)$. Obviously, $|V(G \times H)| = |V(G)||V(H)|$ and $|E(G \times H)| = |E(G)||V(H)| + |V(G)||E(H)|$.

**Theorem 4.1.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned}
AZI(G \times H) \leq & \frac{n_2(\Delta_G + \Delta_H - \delta_H - 1)^3 AZI(G) + n_1(\Delta_G + \Delta_H - \delta_G - 1)^3 AZI(H)}{(\Delta_G + \Delta_H - 1)^3} \\
& + \frac{n_2\Delta_H^3\chi_3(G) + n_2(3\Delta_H^2\Delta_G^2 + 3\Delta_H^4)\chi_2(G) + 3n_2\Delta_H^5\chi_1(G) + \Delta_G^6 m_2}{8(\delta_G + \delta_H - 1)^3} \\
& + \frac{n_1\Delta_G^3\chi_3(H) + n_1(3\Delta_H^2\Delta_G^2 + 3\Delta_G^4)\chi_2(H) + 3n_1\Delta_G^5\chi_1(H) + \Delta_H^6 m_1}{8(\delta_G + \delta_H - 1)^3} \\
& + \frac{n_2(6\Delta_H\Delta_G + 3\Delta_H^2)R_2(G) + n_2(12\Delta_H^3\Delta_G + 3\Delta_H^4)R_1(G)}{8(\delta_G + \delta_H - 1)^3} \\
& + \frac{n_1(6\Delta_H\Delta_G + 3\Delta_G^2)R_2(H) + n_1(12\Delta_G^3\Delta_H + 3\Delta_G^4)R_1(H)}{8(\delta_G + \delta_H - 1)^3},
\end{aligned}$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* By definition,

$$AZI(G \times H) = \sum_{(u,x)(v,y) \in E(G \times H)} \left( \frac{d_{G\times H}(u,x) d_{G\times H}(v,y)}{d_{G\times H}(u,x) + d_{G\times H}(v,y) - 2} \right)^3.$$

We partition the edges of $G \times H$ in to two subset $E_1$ and $E_2$, as follows:

$$\begin{aligned}
E_1 =& \{e = (u,x)(v,y) \mid uv \in E(G), x = y\}, \\
E_2 =& \{e = (u,x)(v,y) \mid xy \in E(H), u = v\}.
\end{aligned}$$

Let $e = (u,x)(v,x) \in E_1$. Then $d_{G\times H}(u,x) = d_G(u) + d_H(x)$ and $d_{G\times H}(v,x) = d_G(v) + d_H(x)$. By used of proof of Theorem 2.1, we have

$$\left( \frac{d_{G\times H}(u,x) d_{G\times H}(v,x)}{d_{G\times H}(u,x) + d_{G\times H}(v,x) - 2} \right)^3 \leq \frac{(\Delta_G + \Delta_H - \delta_H - 1)^3}{(\Delta_G + \Delta_H - 1)^3} \left( \frac{d_G(u) d_G(v)}{d_G(u) + d_G(v) - 2} \right)^3$$

$$
\begin{aligned}
&+\frac{\Delta_H^3(d_G(u)+d_G(v))^3)(d_G(u)+d_G(v))^2}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{(3\Delta_H^2\Delta_G^2+3\Delta_H^4)(d_G(u)+d_G(v))^2}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{3\Delta_H^5(d_G(u)+d_G(v))}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{(6\Delta_H\Delta_G+3\Delta_H^2)(d_G(u)d_G(v))^2}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{(12\Delta_H^3\Delta_G+3\Delta_H^4)d_G(u)d_G(v)+\Delta_H^6}{8(\delta_G+\delta_H-1)^3}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sum_{(u,x)(v,x)\in E_1}\left(\frac{d_{G\times H}(u,x)d_{G\times H}(v,x)}{d_{G\times H}(u,x)+d_{G\times H}(v,x)-2}\right)^3\\
\leq&\frac{n_2(\Delta_G+\Delta_H-\delta_H-1)^3AZI(G)}{(\Delta_G+\Delta_H-1)^3}+\frac{n_2\Delta_H^3\chi_3(G)+3n_2\Delta_H^5\chi_1(G)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_2(3\Delta_H^2\Delta_G^2+3\Delta_H^4)\chi_2(G)}{8(\delta_G+\delta_H-1)^3}+\frac{n_2(6\Delta_H\Delta_G+3\Delta_H^2)R_2(G)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_2(12\Delta_H^3\Delta_G+3\Delta_H^4)R_1(G)+\Delta_H^6n_2m_1}{8(\delta_G+\delta_H-1)^3}
\end{aligned}
\tag{4.1}
$$

Obviously, equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. Similarly,

$$
\begin{aligned}
\sum_{(u,x)(u,y)\in E_2}\left(\frac{d_{G\times H}(u,x)d_{G\times H}(u,y)}{d_{G\times H}(u,x)+d_{G\times H}(u,y)-2}\right)^3\leq&\frac{n_1(\Delta_G+\Delta_H-\delta_G-1)^3AZI(H)}{(\Delta_G+\Delta_H-1)^3}\\
&+\frac{n_1\Delta_G^3\chi_3(H)+3n_1\Delta_G^5\chi_1(H)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_1(3\Delta_H^2\Delta_G^2+3\Delta_G^4)\chi_2(H)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_1(6\Delta_H\Delta_G+3\Delta_G^2)R_2(H)}{8(\delta_G+\delta_H-1)^3}\\
&+\frac{n_1(12\Delta_G^3\Delta_H+3\Delta_G^4)R_1(H)+\Delta_G^6n_1m_2}{8(\delta_G+\delta_H-1)^3}.
\end{aligned}
\tag{4.2}
$$

Equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. By Equations (4.1) and (4.2), we have:

$$
\begin{aligned}
AZI(G\times H)\leq&\frac{n_2(\Delta_G+\Delta_H-\delta_H-1)^3AZI(G)+n_1(\Delta_G+\Delta_H-\delta_G-1)^3AZI(H)}{(\Delta_G+\Delta_H-1)^3}\\
&+\frac{n_2\Delta_H^3\chi_3(G)+n_2(3\Delta_H^2\Delta_G^2+3\Delta_H^4)\chi_2(G)+3n_2\Delta_H^5\chi_1(G)+\Delta_G^6m_2}{8(\delta_G+\delta_H-1)^3}
\end{aligned}
$$

$$+\frac{n_1\Delta_G^3\chi_3(H)+n_1(3\Delta_H^2\Delta_G^2+3\Delta_G^4)\chi_2(H)+3n_1\Delta_G^5\chi_1(H)+\Delta_H^6m_1}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2(6\Delta_H\Delta_G+3\Delta_H^2)R_2(G)+n_2(12\Delta_H^3\Delta_G+3\Delta_H^4)R_1(G)}{8(\delta_G+\delta_H-1)^3}$$
$$+\frac{n_1(6\Delta_H\Delta_G+3\Delta_G^2)R_2(H)+n_1(12\Delta_G^3\Delta_H+3\Delta_G^4)R_1(H)}{8(\delta_G+\delta_H-1)^3},$$

with equality if and only if $G$ and $H$ are regular graphs. ☐

**Theorem 4.2.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned}AZI(G\times H)\geq&\frac{n_2(\delta_G+\delta_H-\Delta_H-1)^3AZI(G)+n_1(\delta_G+\delta_H-\Delta_G-1)^3AZI(H)}{(\delta_G+\delta_H-1)^3}\\&+\frac{n_2\delta_H^3\chi_3(G)+n_2(3\delta_H^2\delta_G^2+3\delta_H^4)\chi_2(G)+3n_2\delta_H^5\chi_1(G)}{8(\Delta_G+\Delta_H-1)^3}\\&+\frac{n_2(6\delta_H\delta_G+3\delta_H^2)R_2(G)+n_2(12\delta_H^3\delta_G+3\delta_H^4)R_1(G)+\delta_H^6m_1}{8(\Delta_G+\Delta_H-1)^3}\\&+\frac{n_1\delta_G^3\chi_3(H)+n_1(3\delta_H^2\delta_G^2+3\delta_G^4)\chi_2(H)+3n_1\delta_G^5\chi_1(H)}{8(\Delta_G+\Delta_H-1)^3}\\&+\frac{n_1(6\delta_H\delta_G+3\delta_G^2)R_2(H)+n_1(12\delta_G^3\delta_H+3\delta_G^4)R_1(H)+\delta_G^6m_2}{8(\Delta_G+\Delta_H-1)^3},\end{aligned}$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* Using an argument similar to that described in proof of Theorem 4.1, we obtained the result. ☐

**Corollary 4.1.** *Let $G$ be a $k$-regular graph of order $n_1$ and let $H$ be a $r$-regular graph of order $n_2$. Then $AZI(G\times H)=\frac{n_1n_2(k+r)^7}{16(k+r-1)^3}$.*

## 5. The Composition Product of Graphs

The composition $G[H]$ of graphs $G$ and $H$ has the vertex set $V(G[H])=V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G[H]$ if $(uv\in E(G))$ or $(xy\in E(H)$ and $u=v)$. Obviously, $|V(G[H])|=|V(G)||V(H)|$ and $|E(G[H])|=|E(G)||V(H)|^2+|E(H)||V(G)|$.

**Theorem 5.1.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$AZI(G[H])$$
$$\leq\frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3AZI(G)+n_1(\Delta_H+n_2\Delta_G-n_2\delta_G-1)^3AZI(H)}{(n_2\Delta_G+\Delta_H-1)^3}$$

$$+\frac{n_2^5\Delta_H^3\chi_3(G)+n_2^2(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)\chi_2(G)+3n_2^3\Delta_H^5\chi_1(G)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_1n_2^3\Delta_G^3\chi_3(H)+n_1(3n_2^2\Delta_H^2\Delta_G^2+3n_2^4\Delta_G^4)\chi_2(H)+3n_1n_2^5\Delta_G^5\chi_1(H)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2^2(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)R_2(G)+n_2^2(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)R_1(G)+n_2^2m_1\Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_1(6n_2\Delta_H\Delta_G+3n_2^2\Delta_G^2)R_2(H)+n_1(12n_2^3\Delta_G^3\Delta_H+3n_2^4\Delta_G^4)R_1(H)+n_1m_2n_2^6\Delta_G^6}{8(n_2\delta_G+\delta_H-1)^3},$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* We partition the edges of $G[H]$ in to two subset $E_1$ and $E_2$, as follows:

$$E_1=\{e=(u,x)(v,y)\mid uv\in E(G)\},$$
$$E_2=\{e=(u,x)(v,y)\mid xy\in E(H), u=v\}.$$

Let $e=(u,x)(v,y)\in E_1$. Then $d_{G[H]}(u,x)=n_2d_G(u)+d_H(x)$ and $d_{G[H]}(v,y)=n_2d_G(v)+d_H(y)$. By used of proof of Theorem 2.1, we have,

$$\left(\frac{d_{G[H]}(u,x)d_{G[H]}(v,y)}{d_{G[H]}(u,x)+d_{G[H]}(v,y)-2}\right)^3\leq\frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3}{(n_2\Delta_G+\Delta_H-1)^3}\left(\frac{d_G(u)d_G(v)}{d_G(u)+d_G(v)-2}\right)^3$$
$$+\frac{n_2^3\Delta_H^3(d_G(u)+d_G(v))^3+3n_2\Delta_H^5(d_G(u)+d_G(v))}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)(d_G(u)+d_G(v))^2}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)(d_G(u)d_G(v))^2}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)d_G(u)d_G(v)+\Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}.$$

Therefore,

$$\sum_{(u,x)(v,y)\in E_1}\left(\frac{d_{G[H]}(u,x)d_{G[H]}(v,x)}{d_{G[H]}(u,x)+d_{G[H]}(v,x)-2}\right)^3\leq\frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3AZI(G)}{(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_2^2(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)R_2(G)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2^2(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)\chi_2(G)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2^5\Delta_H^3\chi_3(G)}{8(n_2\delta_G+\delta_H-1)^3}+\frac{3n_2^3\Delta_H^5\chi_1(G)}{8(n_2\delta_G+\delta_H-1)^3}$$
$$+\frac{n_2^2(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)R_1(G)}{8(n_2\delta_G+\delta_H-1)^3}$$

$$+\frac{n_2^2 m_1 \Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}. \tag{5.1}$$

Obviously, equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. Similarly,

$$\begin{aligned}\sum_{(u,x)(u,y)\in E_2}\left(\frac{d_{G[H]}(u,x)d_{G[H]}(u,y)}{d_{G[H]}(u,x)+d_{G[H]}(u,y)-2}\right)^3 \leq & \frac{n_1(\Delta_H+n_2\Delta_G-n_2\delta_G-1)^3 AZI(H)}{(n_2\Delta_G+\Delta_H-1)^3}\\ &+\frac{n_1(6n_2\Delta_H\Delta_G+3n_2^2\Delta_G^2)R_2(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1(3n_2^2\Delta_H^2\Delta_G^2+3n_2^4\Delta_G^4)\chi_2(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1n_2^3\Delta_G^3\chi_3(H)}{8(\delta_G+\delta_H-1)^3}+\frac{3n_1n_2^5\Delta_G^5\chi_1(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1(12n_2^3\Delta_G^3\Delta_H+3n_2^4\Delta_G^4)R_1(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1m_2n_2^6\Delta_G^6}{8(n_2\delta_G+\delta_H-1)^3}. \end{aligned} \tag{5.2}$$

Equality holds if and only if $\Delta_G=\delta_G$ and $\Delta_H=\delta_H$. By Equations (5.1) and (5.2), we have:

$$\begin{aligned}&AZI(G[H])\\ \leq&\frac{n_2^5(n_2\Delta_G+\Delta_H-\delta_H-n_2)^3AZI(G)+n_1(\Delta_H+n_2\Delta_G-n_2\delta_G-1)^3AZI(H)}{(n_2\Delta_G+\Delta_H-1)^3}\\ &+\frac{n_2^5\Delta_H^3\chi_3(G)+n_2^2(3n_2^4\Delta_H^2\Delta_G^2+3n_2^2\Delta_H^4)\chi_2(G)+3n_2^3\Delta_H^5\chi_1(G)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1n_2^3\Delta_G^3\chi_3(H)+n_1(3n_2^2\Delta_H^2\Delta_G^2+3n_2^4\Delta_G^4)\chi_2(H)+3n_1n_2^5\Delta_G^5\chi_1(H)}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_2^2(6n_2^5\Delta_H\Delta_G+3n_2^4\Delta_H^2)R_2(G)+n_2^2(12n_2^3\Delta_H^3\Delta_G+3n_2^2\Delta_H^4)R_1(G)+n_2^2m_1\Delta_H^6}{8(n_2\delta_G+\delta_H-1)^3}\\ &+\frac{n_1(6n_2\Delta_H\Delta_G+3n_2^2\Delta_G^2)R_2(H)+n_1(12n_2^3\Delta_G^3\Delta_H+3n_2^4\Delta_G^4)R_1(H)+n_1m_2n_2^6\Delta_G^6}{8(n_2\delta_G+\delta_H-1)^3},\end{aligned}$$

with equality if and only if $G$ and $H$ are regular graphs. $\square$

**Theorem 5.2.** *Let $G$ be a graph of order $n_1$ and of size $m_1$ and let $H$ be a graph of order $n_2$ and of size $m_2$. Then*

$$\begin{aligned}&AZI(G[H])\\ \geq&\frac{n_2^5(n_2\delta_G+\delta_H-\Delta_H-n_2)^3AZI(G)+n_1(\delta_H+n_2\delta_G-n_2\Delta_G-1)^3AZI(H)}{(n_2\delta_G+\delta_H-1)^3}\end{aligned}$$

$$+\frac{n_2^5\delta_H^3\chi_3(G)+n_2^2(3n_2^4\delta_H^2\delta_G^2+3n_2^2\delta_H^4)\chi_2(G)+3n_2^3\delta_H^5\chi_1(G)}{8(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_1n_2^3\delta_G^3\chi_3(H)+n_1(3n_2^2\delta_H^2\delta_G^2+3n_2^4\delta_G^4)\chi_2(H)+3n_1n_2^5\delta_G^5\chi_1(H)}{8(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_2^2(6n_2^5\delta_H\delta_G+3n_2^4\delta_H^2)R_2(G)+n_2^2(12n_2^3\delta_H^3\delta_G+3n_2^2\delta_H^4)R_1(G)+n_2^2m_1\delta_H^6}{8(n_2\Delta_G+\Delta_H-1)^3}$$
$$+\frac{n_1(6n_2\delta_H\delta_G+3n_2^2\delta_G^2)R_2(H)+n_1(12n_2^3\delta_G^3\delta_H+3n_2^4\delta_G^4)R_1(H)+n_1m_2n_2^6\delta_G^6}{8(n_2\Delta_G+\Delta_H-1)^3},$$

*with equality if and only if $G$ and $H$ are regular graphs.*

*Proof.* The proof of the result is similar to this given in Theorem 5.1. □

**Corollary 5.1.** *Let $G$ be a $k$-regular graph of order $n_1$ and let $H$ be a $r$-regular graph of order $n_2$. Then $AZI(G[H])=\frac{n_1n_2(n_2k+r)^7}{16(n_2k+r-1)^3}$.*

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 523–532.

# ON EQUIENERGETIC, HYPERENERGETIC AND HYPOENERGETIC GRAPHS

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Abstract. The eigenvalue of a graph $G$ is the eigenvalue of its adjacency matrix and the energy $E(G)$ is the sum of absolute values of eigenvalues of graph $G$. Two non-isomorphic graphs $G_1$ and $G_2$ of the same order are said to be equienergetic if $E(G_1) = E(G_2)$. The graphs whose energy is greater than that of complete graph are called hyperenergetic and the graphs whose energy is less than that of its order are called hypoenergetic graphs. The natural question arises: Are there any pairs of equienergetic graphs which are also hyperenergetic (hypoenergetic)? We have found an affirmative answer of this question and contribute some new results.

## 1. Introduction

We begin with finite connected and undirected graphs without loops and multiple edges. The terms not defined here are used in sense of Balakrishnan and Ranganathan [1] or Cvetković et al. [5]. The adjacency matrix of a graph $G$ with vertices $v_1, v_2, \ldots, v_n$ is an $n \times n$ matrix $[a_{ij}]$ such that,

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent with } v_j, \\ 0, & \text{otherwise.} \end{cases}$$

The eigenvalues of adjacency matrix of graph is known as eigenvalues of graph. The set of eigenvalues of the graph with their multiplicities is known as spectrum of the graph. Hence,

$$\operatorname{spec}(G) = \begin{pmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ m_1 & m_2 & \cdots & m_n \end{pmatrix}.$$

*Key words and phrases.* Equienergetic, hyperenergetic, hypoenergetic.
2010 *Mathematics Subject Classification.* Primary: 05C50, 05C76.
DOI 10.46793/KgJMat2004.523V
*Received*: June 30, 2017.
*Accepted*: June 15, 2018.

Two non-isomorphic graphs are said to be cospectral if they have same spectra, otherwise they are known as non-cospectral. Let $G$ be a graph on $n$ vertices and $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of $G$. The energy of a graph $G$ is the sum of absolute values of the eigenvalues of graph $G$ and denoted by $E(G)$. Hence,

$$E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

The concept of energy was introduced by Gutman [6]. A brief account of energy of graph can be found in Cvetković *et al.* [5] and Li *et al.* [10]. Two non-isomorphic graphs $G_1$ and $G_2$ of same order are said to be *equienergetic* if $E(G_1) = E(G_2)$.

Ramane et al. [12, 13] have proved that if $G_1$ and $G_2$ are regular graphs of same order then for $k \geq 2$, $L^k(G_1)$ and $L^k(G_2)$, $\overline{L^k(G_1)}$ and $\overline{L^k(G_2)}$ are equienergetic. Here, $L^k(G)$ is called iterated line graph of $G$.

Some equienergetic graphs have been described in Li et al. [10], while a symmetric computer aided study have carried out for equienergetic trees [2, 11]. Some open problem on equienergetic graphs were posted in [8]. To find out non-copspectral equienergetic graphs other than trees is challenging and interesting as well. We take up this problems and construct a pair of graphs which are equienergetic.

In 1978 Gutman [6] conjectured that among all graphs with $n$ vertices, the complete graph $K_n$ has the maximum energy. This was disproved by Walikar et al. [16] and was defined the concept of *hyperenergetic graphs* whose energy is greater than that of complete graphs. Gutman [7] has proved that hyperenergetic graphs on $n$ vertices exist for all $n \geq 8$ and there are no hyperenergetic graphs on less than 8 vertices.

A graph $G$ on order $n$ is said to be *hypoenergetic* [3] if $E(G)$ is less than its order otherwise it is said to be *non-hypoenergetic* [4]. In 2007 Gutman [9] have proved that if the graph $G$ is regular of any non-zero degree, then $G$ is non hypoenergetic.

The present work is aimed to contribute to find families of hyperenergetic and hypoenergetic.

The *splitting graph* $S'(G)$ of a graph $G$ is obtained by adding to each vertex $v$ a new vertex $v'$, such that $v'$ is adjacent to every vertex that is adjacent to $v$ in $G$. The *shadow graph* $D_2(G)$ of a connected graph $G$ is constructed by taking two copies of $G$ say $G'$ and $G''$. Join each vertex $u'$ in $G'$ to the neighbors of the corresponding vertex $u''$ in $G''$. Vaidya and Popat [15] have proved that for any graph $G$, $E(S'(G)) = \sqrt{5}E(G)$ and $E(D_2(G)) = 2E(G)$.

The *m-splitting graph* $\mathrm{Spl}_m(G)$ of a graph $G$ is obtained by adding to each vertex $v$ of $G$ new $m$ vertices, say $v_1, v_2, v_3, \ldots, v_m$, such that $v_i$, $1 \leq i \leq m$, is adjacent to each vertex that is adjacent to $v$ in $G$.

The *m-shadow graph* $D_m(G)$ of a connected graph $G$ is constructed by taking $m$ copies of $G$, say $G_1, G_2, \ldots, G_m$, then join each vertex $u$ in $G_i$ to the neighbors of the corresponding vertex $v$ in $G_j$, $1 \leq i, j \leq m$.

**Proposition 1.1** ([14]). $E(\mathrm{Spl}_m(G)) = \sqrt{1 + 4m}\, E(G)$.

**Proposition 1.2** ([14])**.** $E(D_m(G)) = mE(G)$.

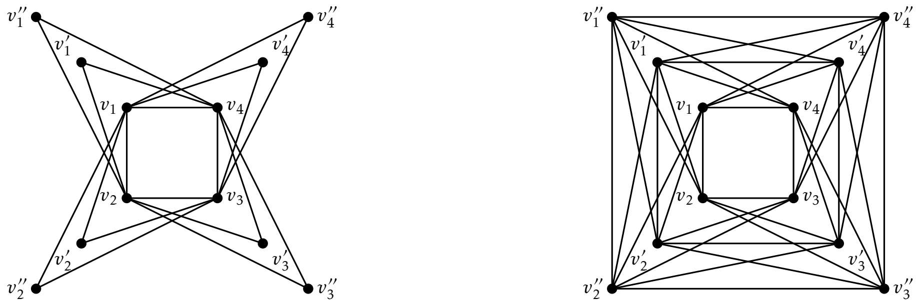
## 2. Equienergetic Graphs

**Theorem 2.1.** $\mathrm{Spl}_2(G)$ *and* $D_3(G)$ *are equienergetic.*

*Proof.* Let $G$ be any graph with $n$ vertices. Then, $D_3(G)$ and $\mathrm{Spl}_2(G)$ are graphs with $3n$ vertices. According to Proposition 1.1 and Proposition 1.2,

$$E(\mathrm{Spl}_2(G)) = \sqrt{1+4(2)}E(G) = 3E(G) = E(D_3(G)). \qquad \square$$

*Example* 2.1. Consider $\mathrm{Spl}_2(C_4)$ and $D_3(C_4)$,




$\mathrm{Spl}_2(C_4)$ $\qquad\qquad$ $D_3(C_4)$

Figure 1

$$A(\mathrm{Spl}_2(C_4)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \\ \boldsymbol{v_1'} \\ \boldsymbol{v_2'} \\ \boldsymbol{v_3'} \\ \boldsymbol{v_4'} \\ \boldsymbol{v_1''} \\ \boldsymbol{v_2''} \\ \boldsymbol{v_3''} \\ \boldsymbol{v_4''} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v_1'} & \boldsymbol{v_2'} & \boldsymbol{v_3'} & \boldsymbol{v_4'} & \boldsymbol{v_1''} & \boldsymbol{v_2''} & \boldsymbol{v_3''} & \boldsymbol{v_4''} \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Therefore, $\operatorname{spec}(\mathrm{Spl}_2(C_4))=\begin{pmatrix}2 & -2 & 4 & -4 & 0\\ 1 & 1 & 1 & 1 & 8\end{pmatrix}$. Here,

$$E(\mathrm{Spl}_2(C_4)) = 12,$$

$$A(D_3(C_4)) = \begin{array}{c} \\ \boldsymbol{v_1}\\ \boldsymbol{v_2}\\ \boldsymbol{v_3}\\ \boldsymbol{v_4}\\ \boldsymbol{v_1'}\\ \boldsymbol{v_2'}\\ \boldsymbol{v_3'}\\ \boldsymbol{v_4'}\\ \boldsymbol{v_1''}\\ \boldsymbol{v_2''}\\ \boldsymbol{v_3''}\\ \boldsymbol{v_4''} \end{array} \!\!\begin{array}{c} \begin{array}{cccccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v_1'} & \boldsymbol{v_2'} & \boldsymbol{v_3'} & \boldsymbol{v_4'} & \boldsymbol{v_1''} & \boldsymbol{v_2''} & \boldsymbol{v_3''} & \boldsymbol{v_4''} \end{array}\\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \end{array}.$$

Therefore, $\operatorname{spec}(D_3(C_4)) = \begin{pmatrix}6 & -6 & 0\\ 1 & 1 & 10\end{pmatrix}$. Here, $E(D_3(C_4)) = 12$. Hence, $\mathrm{Spl}_2(C_4)$ and $D_3(C_4)$ are equienergetic.

## 3. Hyperenergetic Graphs

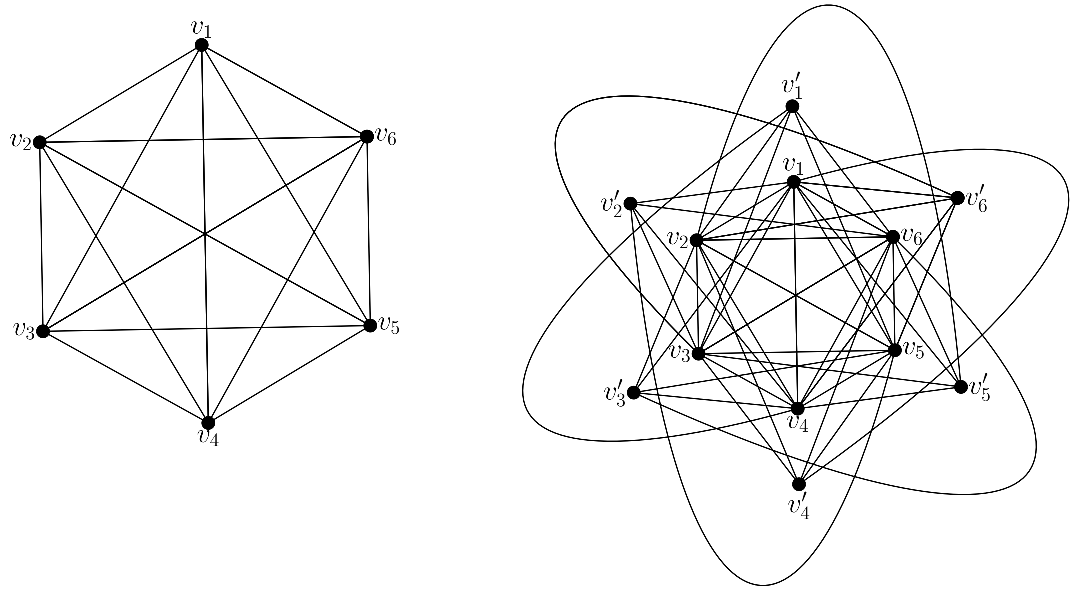
**Theorem 3.1.** *$S'(K_n)$ is hyperenergetic if and only if $n \geq 6$.*

*Proof.* Consider a complete graph $K_n$ on $n$ vertices. Then, $S'(K_n)$ is a graph with $2n$ vertices. It is obvious that energy of complete graph with $2n$ vertices is $2(2n-1)$. Now, if $S'(K_n)$ is hyperenergetic, then

$$\begin{aligned} E(S'(K_n)) > 2(2n-1) &\Leftrightarrow \sqrt{5}(E(K_n)) > 2(2n-1)\\ &\Leftrightarrow \sqrt{5}(2(n-1)) > 2(2n-1)\\ &\Leftrightarrow n > \frac{\sqrt{5}-1}{\sqrt{5}-2}\\ &\Leftrightarrow n \geq 6. \end{aligned}$$

□

*Example* 3.1. Consider complete graph $K_6$ and $S'(K_6)$.




$K_6$

$S'(K_6)$

FIGURE 2

$$A(S'(K_6)) = \begin{array}{c} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \\ \boldsymbol{v_5} \\ \boldsymbol{v_6} \\ \boldsymbol{v_1'} \\ \boldsymbol{v_2'} \\ \boldsymbol{v_3'} \\ \boldsymbol{v_4'} \\ \boldsymbol{v_5'} \\ \boldsymbol{v_6'} \end{array} \begin{array}{c} \begin{array}{cccccccccccc} \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v_5} & \boldsymbol{v_6} & \boldsymbol{v_1'} & \boldsymbol{v_2'} & \boldsymbol{v_3'} & \boldsymbol{v_4'} & \boldsymbol{v_5'} & \boldsymbol{v_6'} \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Hence,

$$\operatorname{spec}(S'(K_6)) = \begin{pmatrix} \dfrac{-1+\sqrt{5}}{2} & \dfrac{-1-\sqrt{5}}{2} & \dfrac{5+5\sqrt{5}}{2} & \dfrac{5-5\sqrt{5}}{2} \\ 5 & 5 & 1 & 1 \end{pmatrix}.$$

Here,

$$E(S'(K_6)) = 10\sqrt{5} \Rightarrow E(S'(K_6)) > 22$$
$$\Rightarrow E(S'(K_6)) > E(K_{12})$$
$$\Rightarrow S'(K_6) \text{ is hyperenergetic.}$$

The following is a graph of $E(S'(K_n))$ and $E(K_{2n})$ which helps to understand that $S'(K_n)$ is hyperenergetic when $n \geq 6$.

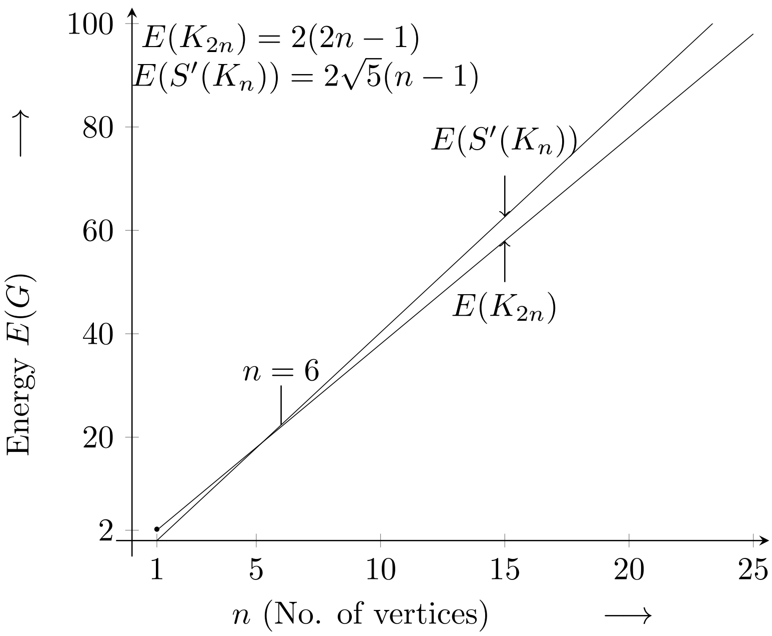



FIGURE 3

The natural question arises: Are there any graphs which are equienergetic and hyperenergetic as well? To answer this question we prove following corollary.

**Corollary 3.1.** *$D_3(S'(K_n))$ and $\mathrm{Spl}_2(S'(K_n))$ are equihyperenergetic graphs for $n \geq 9$.*

*Proof.* As we have discussed in Theorem 3.1, $S'(K_n)$ is a graph with $2n$ vertices. Therefore, $D_3(S'(K_n))$ is a graph with $6n$ vertices. To prove above result we show that $D_3(S'(K_n))$ is hyperenergetic if and only if $n \geq 9$.

If $D_3(S'(K_n))$ is hyperenergetic then

$$E(D_3(S'(K_n))) > 2(6n-1) \Leftrightarrow 3E(S'(K_n)) > 2(6n-1)$$
$$\Leftrightarrow 3\sqrt{5}(E(K_n)) > 2(6n-1)$$
$$\Leftrightarrow 3\sqrt{5}(2(n-1)) > 2(6n-1)$$

$$\Leftrightarrow n > \frac{3\sqrt{5}-1}{3\sqrt{5}-6}$$
$$\Leftrightarrow n \geq 9.$$

Hence, $D_3(S'(K_n))$ is hyperenergetic for $n \geq 9$. Therefore, according to Theorem 2.1, $D_3(S'(K_n))$ and $\mathrm{Spl}_2(S'(K_n))$ are equihyperenergetic for $n \geq 9$. □

## 4. Hypoenergetic Graphs

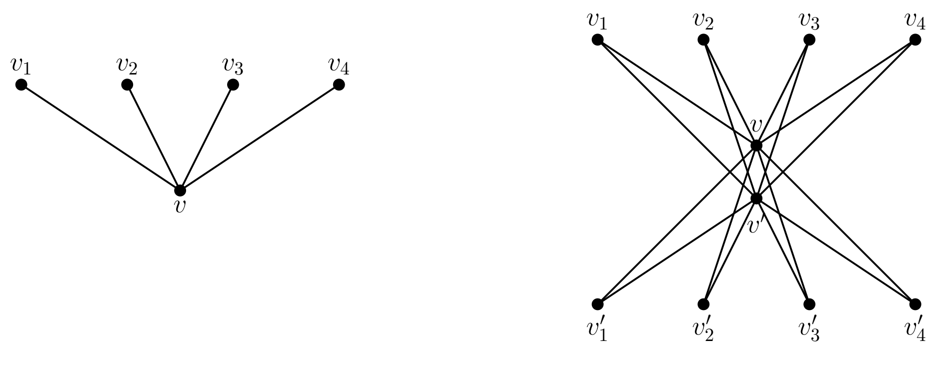
**Theorem 4.1.** *$D_m(K_{1,n})$ is hypoenergetic.*

*Proof.* Consider star graph $K_{1,n}$ on $n$ vertices. Then $E(K_{1,n}) = 2\sqrt{n}$. Now, $D_m(K_{1,n})$ is a graph with $m(n+1)$ vertices. As,

$$\begin{aligned} n > 1 \Rightarrow & (n-1)^2 > 0 \\ \Rightarrow & n^2 - 2n + 1 > 0 \\ \Rightarrow & n^2 + 2n + 1 > 4n \\ \Rightarrow & 4n < (n+1)^2 \\ \Rightarrow & 2\sqrt{n} < (n+1) \\ \Rightarrow & m(2\sqrt{n}) < m(n+1). \end{aligned}$$

According to Proposition 1.2, we have $E(D_m(K_{1,n})) = mE(K_{1,n}) = m(2\sqrt{n}) < m(n+1)$. Hence, $D_m(K_{1,n})$ is hypoenergetic. □

*Example* 4.1. Consider star graph $K_{1,4}$ and $D_2(K_{1,4})$ (see Figure 4). Therefore, $\mathrm{spec}(D_2(K_{1,4})) = \begin{pmatrix} 4 & -4 & 0 \\ 1 & 1 & 8 \end{pmatrix}$. Hence, $E(D_2(K_{1,4})) = 8 < 10$ and $D_2(K_{1,4})$ is hypoenergetic.




$K_{1,4}$ $D_2(K_{1,4})$

Figure 4

$$A(D_2(K_{1,4})) = \begin{array}{cc} & \begin{array}{cccccccccc} \boldsymbol{v} & \boldsymbol{v_1} & \boldsymbol{v_2} & \boldsymbol{v_3} & \boldsymbol{v_4} & \boldsymbol{v'} & \boldsymbol{v'_1} & \boldsymbol{v'_2} & \boldsymbol{v'_3} & \boldsymbol{v'_4} \end{array} \\ \begin{array}{c} \boldsymbol{v} \\ \boldsymbol{v_1} \\ \boldsymbol{v_2} \\ \boldsymbol{v_3} \\ \boldsymbol{v_4} \\ \boldsymbol{v'} \\ \boldsymbol{v'_1} \\ \boldsymbol{v'_2} \\ \boldsymbol{v'_3} \\ \boldsymbol{v'_4} \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

The following graph on Figure 5 is a graph of $n$ and $E(G)$ which helps to understand that $D_2(K_{1,n})$ is hypoenergetic.

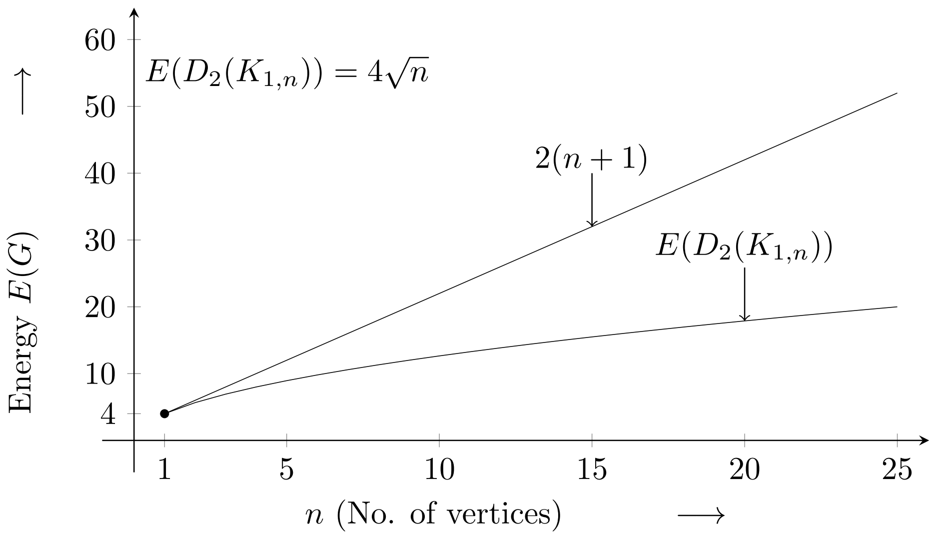



FIGURE 5

The natural question arises: are there any graphs which are equienergetic as well as hypoenergetic? We call such graphs as equihypoenergetic. To answer this question we prove following corollary.

**Corollary 4.1.** *$D_3(K_{1,n})$ and $\mathrm{Spl}_2(K_{1,n})$ are equihypoenergertic graphs.*

*Proof.* It is obvious that from Theorem 4.1, $D_3(K_{1,n})$ is hypoenergetic and from Theorem 2.1, $D_3(K_{1,n})$ and $\mathrm{Spl}_2(K_{1,n})$ are equienergetic. Hence, $D_3(K_{1,n})$ and $\mathrm{Spl}_2(K_{1,n})$ are equihypoenergertic graphs. $\square$

**Acknowledgements.** The present work is a part of the research work carried out under Major Research Project No. IQAC/GJY/MRP/OCT/2016/1670-A, dated: 4th October, 2016 funded by Saurashtra University-Rajkot (Gujarat), India.

The authors thank the anonymous referees for their valuable suggestions leading to the improvement of the original manuscript.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 533–538.

# A GENERALIZED CLASS OF CLOSE-TO-CONVEX FUNCTIONS

PARDEEP KAUR[1] AND SUKHWINDER SINGH BILLING[2]

Abstract. Let $\mathcal{H}_\alpha^\phi(\beta)$ denote the class of functions $f$, analytic in the open unit disk $\mathbb{E}$ which satisfy the condition

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right)>\beta,\quad z\in\mathbb{E},$$

where $\alpha,\ \beta$ are pre-assigned real numbers and $\phi(z)$ is a starlike function. The special cases of the class $\mathcal{H}_\alpha^\phi(\beta)$ have been studied in literature by different authors. In 2007, Singh et al. [5] studied the class $\mathcal{H}_\alpha^z(\beta)$ and they established that functions in $\mathcal{H}_\alpha^z(\beta)$ are univalent for all real numbers $\alpha,\ \beta$ satisfying the condition $\alpha\leq\beta<1$ and the result is sharp in the sense that constant $\beta$ cannot be replaced by a real number smaller than $\alpha$. Singh et al. [7] in 2005, proved that for $0<\alpha<1$ functions in class $\mathcal{H}_\alpha^z(\alpha)$ are univalent. In 1975, Al-Amiri and Reade [2] showed that functions in class $\mathcal{H}_\alpha^z(0)$ are univalent for all $\alpha\leq 0$ and also for $\alpha=1$ in $\mathbb{E}$. In the present paper, we prove that members of the class $\mathcal{H}_\alpha^\phi(\beta)$ are close-to-convex and hence univalent for real numbers $\alpha,\ \beta$ and for a starlike function $\phi$ satisfying the condition $\beta+\alpha-1<\alpha\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)\leq\beta<1$.

## 1. Introduction and Preliminary

Let $\mathcal{A}$ be the class of functions $f$, analytic in the open unit disk $\mathbb{E}=\{z:|z|<1\}$ and normalized by the conditions $f(0)=f'(0)-1=0$. Let $\mathcal{S}^*$ and $\mathcal{K}$ denote the classes of starlike and convex function respectively analytically defined as follows:

$$\mathcal{S}^*=\left\{f\in\mathcal{A}:\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>0,\ z\in\mathbb{E}\right\},$$

*Key words and phrases.* Analytic function, univalent function, close-to-convex function
2010 *Mathematics Subject Classification.* Primary: 30C80. Secondary: 30C45.
DOI 10.46793/KgJMat2004.533K
*Received*: June 30, 2017.
*Accepted*: June 19, 2018.

and
$$\mathcal{K} = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{E} \right\}.$$
It is well-known that
$$f \in \mathcal{K} \Leftrightarrow zf' \in \mathcal{S}^*. \tag{1.1}$$
A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number $\alpha,\ -\pi/2 < \alpha < \pi/2$, and a convex function $g$ (not necessarily normalized) such that
$$\mathrm{Re}\left(e^{i\alpha}\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in \mathbb{E}.$$
In view of the relation (1.1), the above definition takes the following form in case $g$ is normalized. A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a real number $\alpha,\ -\pi/2 < \alpha < \pi/2$, and a starlike function $\phi$ such that
$$\mathrm{Re}\left(e^{i\alpha}\frac{f'(z)}{\phi(z)}\right) > 0, \quad z \in \mathbb{E}.$$
It is well known that every close-to-convex function is univalent. In 1934/35, Noshiro [4] and Warchawski [8] independently obtained a simple but elegant criterion for univalence of analytic functions. They proved that if an analytic function $f$ satisfies $\mathrm{Re}f'(z) > 0$ for all $z$ in $\mathbb{E}$, then $f$ is close-to-convex and hence univalent in $\mathbb{E}$.

For pre-assigned real numbers $\alpha,\ \beta$ and $\phi \in \mathcal{S}^*$, the class $\mathcal{H}_\alpha^\phi(\beta)$ is defined as the class of functions $f \in \mathcal{A}$ as follows:
$$\mathrm{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{E}.$$
The following special cases of the class $\mathcal{H}_\alpha^\phi(\beta)$ have been studied in literature by different authors. In fact, the class $\mathcal{H}_\alpha^z(0)$ was first studied in 1975 by Al-Amiri and Reade [2]. They proved that for $\alpha \le 0$, each function in $\mathcal{H}_\alpha^z(0)$ satisfies $\mathrm{Re}(f'(z)) > 0$ in $\mathbb{E}$ and hence univalent in $\mathbb{E}$. They left the problem of univalence open for $\alpha > 0$ (except for $\alpha = 1$, where $f$ is convex, obviously). Ahuja and Silverman [1] observed that the convex function $f(z) = z/(1-z)$ is not in $\mathcal{H}_\alpha^z(0)$ for any real $\alpha,\ \alpha \ne 1$. Further this problem pursued by Singh et al. [7] and they proved that for $0 < \alpha < 1$, the class $\mathcal{H}_\alpha^z(\alpha)$ consisting univalent functions. In 2007, Singh et al. [5] studied the class $\mathcal{H}_\alpha^z(\beta)$. They proved that if $f \in \mathcal{H}_\alpha^z(\beta)$, then $\mathrm{Re}(f'(z)) > 0$ in $\mathbb{E}$ for all real numbers $\alpha, \beta$ satisfying $\alpha \le \beta < 1$ and the result is best possible one in the sense that $\beta$ cannot be replaced by a real number smaller than $\alpha$. Their result contains the previous result of Singh et al. [7] and improves the result of Al-Amiri and Reade [2].

In the present paper, we study a more general class $\mathcal{H}_\alpha^\phi(\beta)$ and establish that the functions in $\mathcal{H}_\alpha^\phi(\beta)$ are close-to-convex and consequently univalent subject to the condition
$$\beta + \alpha - 1 < \alpha\mathrm{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) \le \beta < 1.$$

where $\alpha, \beta$ are pre-assigned real numbers and $\phi$ is a starlike function. We claim that our results generalize the previous known results in this direction.

To prove our result, we shall need the following lemma of Miller [3].

**Lemma 1.1.** *Let $\mathbb{D}$ be a subset of $\mathbb{C} \times \mathbb{C}$, where $\mathbb{C}$ is the complex plane and let $\Phi : \mathbb{D} \to \mathbb{C}$ be a complex function. For $u = u_1 + iu_2,\ v = v_1 + iv_2$ ($u_1,\ u_2,\ v_1,\ v_2$ are reals), let $\Phi$ satisfies the following conditions:*

(i) $\Phi(u, v)$ *is continuous in* $\mathbb{D}$*;*
(ii) $(1, 0) \in \mathbb{D}$ *and* $\mathrm{Re}(\Phi(1, 0)) > 0$ *and*
(iii) $\mathrm{Re}\ \Phi(iu_2, v_1) \leq 0$ *for all* $((iu_2, v_1) \in \mathbb{D}$ *such that* $v_1 \leq -(1 + u_2^2)/2$.

*Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be regular in the open unit disk $\mathbb{E}$, such that $(p(z), zp'(z)) \in \mathbb{D}$ for all $z \in \mathbb{E}$. If*

$$\mathrm{Re}(\Phi(p(z), zp'(z))) > 0, \quad z \in \mathbb{E},$$

*then* $\mathrm{Re}(p(z)) > 0$*,* $z \in \mathbb{E}$.

## 2. Univalence of Functions in $\mathcal{H}_\alpha^\phi(\beta)$

**Theorem 2.1.** *Let $\phi$ be a starlike function and let $\alpha,\ \beta$ be real numbers such that*

$$\beta + \alpha - 1 < \alpha \mathrm{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) \leq \beta < 1. \tag{2.1}$$

*If $f \in \mathcal{A}$ satisfies*

$$\mathrm{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{E}, \tag{2.2}$$

*then* $\mathrm{Re}\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ *in $\mathbb{E}$. So $f$ is close-to-convex and hence univalent in $\mathbb{E}$.*

*Proof.* Write $p(z) = \frac{zf'(z)}{\phi(z)}$, where $p$ is analytic in $\mathbb{E}$ such that $p(0) = 1$ and $\phi$ is a starlike in $\mathbb{E}$. Then

$$(1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = (1-\alpha)p(z) + \alpha\left(\frac{zp'(z)}{p(z)} + \frac{z\phi'(z)}{\phi(z)}\right).$$

Thus, condition (2.2) is equivalent to

$$\mathrm{Re}\left(\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha\frac{z\phi'(z)}{\phi(z)} - \beta}{1-\beta}\right) > 0, \quad z \in \mathbb{E}. \tag{2.3}$$

Let $\mathbb{D} = \mathbb{C} \setminus \{0\} \times \mathbb{C}$ and define $\Phi(u, v) : \mathbb{D} \to \mathbb{C}$ as under:

$$\Phi(u, v) = \frac{1-\alpha}{1-\beta}u + \frac{\alpha}{1-\beta}\frac{v}{u} + \frac{\alpha\frac{z\phi'(z)}{\phi(z)} - \beta}{1-\beta}.$$

Then $\Phi(u,v)$ is continuous in $\mathbb{D}$, $(1,0)\in\mathbb{D}$ and in view of the given condition, we have

$$\operatorname{Re}(\Phi(1,0)) = \frac{1-\alpha\left(1-\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)\right)-\beta}{1-\beta} > 0.$$

Further, from (2.3), we get $\operatorname{Re}[\Phi(p(z), zp'(z))] > 0$, $z\in\mathbb{E}$. Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ where $u_1$, $u_2$, $v_1$ *and* $v_2$ are all reals. Then, for $(iu_2, v_1)\in\mathbb{D}$, with $v_1 \le -\frac{1+u_2^2}{2}$, we have

$$\begin{aligned}\operatorname{Re}(\Phi(iu_2,v_1)) =& \operatorname{Re}\left(\frac{1-\alpha}{1-\beta}iu_2 + \frac{\alpha}{1-\beta}\frac{v_1}{iu_2} + \frac{\alpha\left(\frac{z\phi'(z)}{\phi(z)}\right)-\beta}{1-\beta}\right)\\ =& \frac{\alpha\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right)-\beta}{1-\beta} \le 0.\end{aligned}$$

The proof now follows from Lemma 1.1. $\square$

To illustrate the above result, we consider the following example.

*Example* 2.1. On selecting $\phi(z) = ze^z$ and $f(z) = z + \frac{z^2}{2}$ in Theorem 2.1, we can easily check that for $\alpha = -0.1$ and $\beta = 0$, the condition (2.1) is satisfied as follows

$$-1.1 < -0.1\operatorname{Re}(1+z) \le 0 < 1$$

and

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right) = \operatorname{Re}\left(1.1e^{-z}(1+z) - \frac{0.1+0.2z}{1+z}\right) > 0.$$

Therefore,

$$\operatorname{Re}\left(\frac{zf'(z)}{\phi(z)}\right) = \operatorname{Re}(1+z)e^{-z} > 0,$$

thus $f$ is close-to-convex and hence univalent in $\mathbb{E}$.

**Theorem 2.2.** *Suppose that $\phi$ is starlike in $\mathbb{E}$ and $\alpha$, $\beta$ are real numbers such that*

$$\beta + \alpha - 1 > \alpha\operatorname{Re}\left(\frac{z\phi'(z)}{\phi(z)}\right) \ge \beta > 1.$$

*If $f\in\mathcal{A}$ satisfies*

$$\operatorname{Re}\left((1-\alpha)\frac{zf'(z)}{\phi(z)} + \alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\right) < \beta, \quad z\in\mathbb{E}, \tag{2.4}$$

*then* $\operatorname{Re}\left(\frac{zf'(z)}{\phi(z)}\right) > 0$ *in $\mathbb{E}$. So, $f$ is close-to-convex and hence univalent in $\mathbb{E}$.*

*Proof.* Write $\frac{zf'(z)}{\phi(z)} = p(z)$, where $p$ is analytic in $\mathbb{E}$ such that $p(0) = 1$ and $\phi$ is starlike in $\mathbb{E}$. Note that $1 - \beta < 0$, thus the condition (2.4) reduces to

$$\operatorname{Re}\left(\frac{1-\alpha}{1-\beta}p(z) + \frac{\alpha}{1-\beta}\frac{zp'(z)}{p(z)} + \frac{\alpha\frac{z\phi'(z)}{\phi(z)} - \beta}{1-\beta}\right) > 0, \quad z \in \mathbb{E}.$$

The proof can now be completed on the same lines as the proof of Theorem 2.1. $\square$

In a special case when $\phi(z) = z$ in Theorem 2.1, we obtain the following result of Singh et al. [5].

**Theorem 2.3.** *Let $\alpha$ and $\beta$ be real numbers such that $\alpha \leq \beta < 1$. Assume that an analytic function $f \in \mathcal{A}$ satisfies the condition*

$$\operatorname{Re}\left((1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) > \beta, \quad z \in \mathbb{E}. \tag{2.5}$$

*Then* $\operatorname{Re} f'(z) > 0$ *in* $\mathbb{E}$*. So, $f$ is close-to-convex and hence univalent in* $\mathbb{E}$*. The result is sharp in the sense that the constant $\beta$ on the right hand side of* (2.5) *cannot be replaced by a constant smaller than $\alpha$.*

Selecting $\phi(z) = z$ in Theorem 2.2, we obtain the following result of Singh et al. [6].

**Theorem 2.4.** *For real numbers $\alpha$ and $\beta$ such that $\alpha \geq \beta > 1$, if $f \in \mathcal{A}$ satisfies the inequality*

$$\operatorname{Re}\left((1-\alpha)f'(z) + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right) < \beta, \quad z \in \mathbb{E}.$$

*Then* $\operatorname{Re} f'(z) > 0$ *in* $\mathbb{E}$*. So, $f$ is close-to-convex and hence univalent in* $\mathbb{E}$*.*

**Acknowledgements.** The authors are thankful to the reviewer for valuable comments.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 539–550.

# A NEW CLASS OF INTEGRALS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION AND MULTIVARIABLE ALEPH-FUNCTION

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Abstract. The aim of this paper is to evaluate an interesting integral involving generalized hypergeometric function and the multivariable Aleph-function. The integral is evaluated with the help of an integral involving generalized hypergeometric function obtained recently by Kim et al. [8]. The integral is further used to evaluate an interesting summation formula concerning the multivariable Aleph-function. A few interesting special cases and corollaries have also been discussed.

## 1. Introduction and Preliminaries

Hypergeometric function is an important and useful tool for special functions that plays an important role in the field of analysis. Transformation theory plays a major role to provide a platform for the development of beautiful transformation. It is important to mention that whenever generalized hypergeometric function reduces to a gamma function, the results are very important from application point of view in mathematics, statistics and mathematical physics [2, 11, 22]. Recently Rohira et al. [17] have evaluated a class of integrals involving generalized hypergeometric function and the $H$-function defined by Fox [5] (see also, [16]). In this paper, we aim to present a class of integrals involving generalized hypergeometric function and the multivariable Aleph-function.

The multivariable Aleph-function is an extension of the multivariable $I$-function defined by Sharma and Ahmad [20], which is a generalization of the multivariable $H$-function defined by Srivastava et al. [24, 25] (see also, [3, 4, 10, 23]). The multivariable Aleph-function is defined by means of the multiple contour integral given by the

*Key words and phrases.* Multivariable $I$-function, multivariable $H$-function, double finite integrals.
2010 *Mathematics Subject Classification.* Primary: 33C60, 33C99. Secondary: 44A20.
DOI 10.46793/KgJMat2004.539K
*Received*: March 13, 2018.
*Accepted*: June 22, 2018.

following manner:

$$\aleph\left(z_1,\ldots,z_r\right)$$

$$= \aleph^{0,\mathfrak{n}:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{array}\left|\begin{array}{l} \left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,\mathfrak{n}}\right], \\ \quad\ldots\ldots\ldots, \end{array}\right.\right.$$

$$\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{\mathfrak{n}+1,p_i}\right]:\left[\left(c_j^{(1)}\right),\left(\gamma_j^{(1)}\right)_{1,n_1}\right],$$
$$\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right]:\left[\left(d_j^{(1)}\right),\left(\delta_j^{(1)}\right)_{1,m_1}\right],$$

$$\left.\begin{array}{l}\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right];\ldots;\left[\left(c_j^{(r)}\right),\left(\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right] \\ \left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right];\ldots;\left[\left(d_j^{(r)}\right),\left(\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]\end{array}\right)$$

(1.1)

$$= \frac{1}{(2\pi\omega)^r}\int_{\mathfrak{L}_1}\cdots\int_{\mathfrak{L}_r}\psi\left(\xi_1,\ldots,\xi_r\right)\prod_{k=1}^{r}\theta_k\left(\xi_k\right)\,z_k^{\xi_k}\,\mathrm{d}\xi_1\ldots\mathrm{d}\xi_r,$$

with $\omega=\sqrt{-1}$,

$$\psi\left(\xi_1,\ldots,\xi_r\right)=\frac{\prod_{j=1}^{\mathfrak{n}}\Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}\xi_k\right)}{\sum_{i=1}^{R}\left[\tau_i\prod_{j=\mathfrak{n}+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}\xi_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}\xi_k\right)\right]}$$

and

$$\theta_k\left(\xi_k\right)=\frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)}-\delta_j^{(k)}\xi_k\right)\prod_{j=1}^{n_k}\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}\xi_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}\xi_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}\xi_k\right)\right]}.$$

For more details, reader can refer to recent works [1, 18]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable $H$-function given as $|\arg z_k|<\frac{1}{2}A_i^{(k)}\pi$, where

$$\begin{aligned}A_i^{(k)}=&\sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)}-\tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)}-\tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)}+\sum_{j=1}^{n_k}\gamma_j^{(k)}-\tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}+\sum_{j=1}^{m_k}\delta_j^{(k)}\\&-\tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)}>0,\quad\text{with } k=1,\ldots,r,\ i=1,\ldots,R,\ i^{(k)}=1,\ldots,R^{(k)},\end{aligned}$$

where $k=1,\ldots,r$, $i=1,\ldots,R$, $i^{(k)}=1,\ldots,R^{(k)}$.

The complex numbers $z_i\neq 0$. Throughout the paper, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. Here and in the following, let $\mathrm{Re}(a)$ be the real part of a complex number $a$. We establish the asymptotic expansion in the convenient form, below

$$\aleph(z_1,\ldots,z_r) = 0\left(|z_1|^{\alpha_1},\ldots,|z_r|^{\alpha_r}\right), \quad \max\left(|z_1|,\ldots,|z_r|\right)\to 0,$$
$$\aleph(z_1,\ldots,z_r) = 0\left(|z_1|^{\beta_1},\ldots,|z_r|^{\beta_r}\right), \quad \min\left(|z_1|,\ldots,|z_r|\right)\to \infty,$$

where $k = 1,\ldots,r$, $\alpha_k = \min\left[\mathrm{Re}\left(d_j^{(k)}/\delta_j^{(k)}\right)\right]$, $j = 1,\ldots,m_k$ and $\beta_k = \max\left[\mathrm{Re}\left(\left(c_j^{(k)}-1\right)/\gamma_j^{(k)}\right)\right]$, $j = 1,\ldots,n_k$. For convenience, we will also use the following notations in this paper:

$$V = m_1,n_1;\ldots;m_r,n_r,$$

$$W = p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}},\tau_{i^{(r)}};R^{(r)}, \tag{1.2}$$

(1.3)
$$A=\left\{\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}\right\},\ \left\{\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right\}\left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\},$$
$$\left\{\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right\};\ldots;\left\{\left(c_j^{(r)};\gamma_j^{(r)}\right)_{1,n_r}\right\},\left\{\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}\right\}.$$

(1.4)
$$B=\left\{\tau_i\left(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right\}:\left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\},$$
$$\left\{\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right\};\ldots;\left\{\left(d_j^{(r)};\delta_j^{(r)}\right)_{1,m_r}\right\},\left\{\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}\right\}.$$

## 2. Required Formula

Recently, Kim et al. [8] have obtained the following integral formula involving generalized hypergeometric function which will be required in our present study. Here and in the following, let $\mathbb{C}$ and $\mathbb{Z}_0^-$ be the sets of complex numbers and non-positive integers, respectively.

**Lemma 2.1.** *For* $\mathrm{Re}\,(2c-a-b) > -1$ *and* $d \in \mathbb{C}\setminus\mathbb{Z}_0^-$, *we have the following integral formula, given by*

$$\int_0^1 x^{c-1}(1-x)^c\ {}_3F_2\left[\begin{matrix} a,b,d+1; \\ \frac{1}{2}(a+b+1),d; \end{matrix}\ x\right]\mathrm{d}x$$
$$=\frac{\pi\Gamma(c)\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}b+\frac{1}{2}\right)}$$
$$+\left(\frac{2c-d}{d}\right)\frac{\pi\Gamma(c)\,4^{-c}\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)\Gamma\left(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)\Gamma\left(c-\frac{1}{2}a+1\right)\Gamma\left(c-\frac{1}{2}b+1\right)}.$$

## 3. Main Integrals

In this section, we evaluate the following interesting integral involving generalized hypergeometric function and the multivariable Aleph-function.

**Theorem 3.1.**
(3.1)

$$\int_0^1 x^{c-1}\,(1-x)^c\;{}_3F_2\left[\begin{matrix} a,b,d+1; \\ \frac{1}{2}(a+b+1),d; \end{matrix}\; x\right]\aleph\begin{pmatrix} z_1\,x^{h_1}\,(1-x)^{h_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{h_r} \end{pmatrix}\mathrm{d}x = A_1$$

$$\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{matrix} z_1\,4^{-h_1} \\ \cdot \\ \cdot \\ z_r\,4^{-h_r}\end{matrix}\;\middle|\;\begin{matrix}(1-c;h_1,\ldots,h_r),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),A \\ \cdot \\ \cdot \\ \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\ldots,h_r\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1,\ldots,h_r\right),B\end{matrix}\right)$$

$$+\,A_2\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{matrix} z_1\,4^{-h_1} \\ \cdot \\ \cdot \\ z_r\,4^{-h_r}\end{matrix}\;\middle|\;\begin{matrix}(1-c;h_1,\ldots,h_r),(d-2c;2h_1,\ldots,2h_r), \\ \cdot \\ \cdot \\ \left(\frac{1}{2}a-c;h_1,\ldots,h_r\right),\left(\frac{1}{2}b-c;h_1,\ldots,h_r\right),\end{matrix}\right.$$

$$\left.\begin{matrix}\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),\,A \\ \cdot \\ \cdot \\ (1+d-2c;2h_1,\ldots,2h_r),\,B\end{matrix}\right),$$

*where $A$ and $B$ are given by* (1.3) *and* (1.4) *respectively. Also,*

$$A_1=\frac{\pi\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}b+\frac{1}{2}\right)} \tag{3.2}$$

*and*

$$A_2=\frac{\pi\,4^{-c}\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2}\right)}{d\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)}. \tag{3.3}$$

*Provided that*

$$h_i>0,\quad \textit{for}\; i=1,\ldots,r,\;\mathrm{Re}\,(c)>0,\;d\in\mathbb{C}\backslash\mathbb{Z}_0^-,$$

$$\mathrm{Re}\,(c)+\sum_{i=1}^{r}h_i\min_{1\le j\le m_i}\mathrm{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)>0,\quad \textit{for}\; i=1,\ldots,r,$$

$$\left|\arg z_k\,x^{h_k}\,(1-x)^{h_k}\right|<\frac{1}{2}A_i^{(k)}\,\pi,$$

*where $A_i^{(k)}$ is defined by* (1.2) *for $k=1,\ldots,r$.*

*Proof.* To prove (3.1), first we assume the left side of (3.1) by the notation $\mathcal{F}_1$, and then express the Aleph-function of several variables involved on the left hand side of (3.1) in terms of Mellin-Barnes contour integral with the help of (1.1), and next change the order of integrations which is permissible under the stated conditions, so we obtain

$$\mathcal{F}_1 = \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^{r} \theta_k(s_k)\, z_k^{s_k} \left[ \int_0^1 x^{c+\sum_{i=1}^r h_i s_i - 1} \right.$$
$$\left. \times\ (1-x)^{c+\sum_{i=1}^r h_i s_i}\ {}_3F_2\left[ \begin{matrix} a, b, d+1; \\ \frac{1}{2}(a+b+1), d; \end{matrix}\ x \right] \mathrm{d}x \right] \mathrm{d}s_1 \ldots \mathrm{d}s_r.$$

Now, we evaluate the inner integral with the help of lemma 2.1, after algebraic manipulations, we have

$$\mathcal{F}_1 = \frac{\pi\, 4^{-c}\, \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(\frac{1}{2}b + \frac{1}{2}\right)} \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^{r} \theta_k(s_k)\ z_k^{s_k}$$
$$\times\ \frac{4^{-\sum_{i=1}^r h_i s_i}\, \Gamma\left(c + \sum_{i=1}^r h_i s_i\right) \Gamma\left(c + \sum_{i=1}^r h_i s_i - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(c + \sum_{i=1}^r h_i s_i - \frac{1}{2}a + \frac{1}{2}\right)\Gamma\left(c + \sum_{i=1}^r h_i s_i - \frac{1}{2}b + \frac{1}{2}\right)} \mathrm{d}s_1 \ldots \mathrm{d}s_r$$
$$+\ \frac{\pi\, 4^{-c}\, \Gamma\left(\frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}\right)}{d\Gamma\left(\frac{1}{2}a\right)\Gamma\left(\frac{1}{2}b\right)} \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_r} \psi(s_1, \ldots, s_r) \prod_{k=1}^{r} \theta_k(s_k)\ z_k^{s_k}$$
$$\times\ \frac{4^{-\sum_{i=1}^r h_i s_i}\Gamma\left(c + \sum_{i=1}^r h_i s_i\right) \Gamma\left(c + \sum_{i=1}^r h_i s_i - \frac{1}{2}a - \frac{1}{2}b + \frac{1}{2}\right)}{\Gamma\left(c + \sum_{i=1}^r h_i s_i - \frac{1}{2}a + 1\right)\Gamma\left(c + \sum_{i=1}^r h_i s_i - \frac{1}{2}b + 1\right)}$$
$$\times\ \frac{\Gamma\left(2c - d + 2\sum_{i=1}^r h_i s_i + 1\right)}{\Gamma\left(2c - d + 2\sum_{i=1}^r h_i s_i\right)} \mathrm{d}s_1 \ldots \mathrm{d}s_r,$$

and reinterpreting the multiple Mellin-Barnes contour integral in terms of Aleph-functions of $r$-variables, we obtain the desired result (3.1). □

**Theorem 3.2.**

$$\int_0^1 x^{\alpha-1}\ (1-x)^{\beta-1}\ \aleph \begin{pmatrix} z_1\, x^{h_1}\ (1-x)^{l_1} \\ \cdot \\ \cdot \\ z_r\, x^{h_r}\ (1-x)^{l_r} \end{pmatrix} \mathrm{d}x$$

$$= \aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W} \left( \begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_r \end{matrix} \left| \begin{matrix} (1-\alpha; h_1, \ldots, h_r),\ (1-\beta; l_1, \ldots, l_r),\ A \\ \cdot \\ \cdot \\ (1-\alpha-\beta; h_1+l_1, \ldots, h_r+l_r),\ B \end{matrix} \right. \right), \tag{3.4}$$

*here provided that*

$$h_i > 0,\ l_i > 0, \quad \textit{for } i = 1, \ldots, r,$$

$$\operatorname{Re}(\alpha) + \sum_{i=1}^{r} h_i \min_{1 \le j \le m_i} \operatorname{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0,$$
$$\operatorname{Re}(\beta) + \sum_{i=1}^{r} l_i \min_{1 \le j \le m_i} \operatorname{Re}\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right) > 0, \quad i = 1, \ldots, r,$$
$$\left|\arg z_k \, x^{h_k} \, (1-x)^{l_k}\right| < \frac{1}{2} A_i^{(k)} \pi,$$

*where* $A_i^{(k)}$ *is given by* (1.2) *for* $k = 1, \ldots, r$.

*Proof.* To prove (3.4), we express the Aleph-function of several variables involved on the left hand side of (3.4) in the terms of Mellin-Barnes contour integral with the help of (1.1), and change the order of integrations which is permissible under the stated conditions and use the formula concerning beta-integral to evaluate the inner integral. Now reinterpreting the multiple Mellin-Barnes contour integrals in terms of Aleph-functions of $r$-variables, we obtain the desired result (3.4). ☐

## 4. Application in Obtaining a New Summation Formula

We have the following summation formula concerning the multivariable Aleph-function, defined as

**Theorem 4.1.**

(4.1)

$$\sum_{s=0}^{\infty} \frac{(a)_s \, (b)_s \, (d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s (d)_s \; s!}$$
$$\times \; \aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} z_1 & (1-c-s; h_1, \ldots, h_r), (-c; h_1, \ldots, h_r), \, A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r & (-2c-s; 2h_1, \ldots, 2h_r), \, B \end{array}\right) = A_1$$
$$\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1 \, 4^{-h_1} & (1-c; h_1, \ldots, h_r), \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c; h_1, \ldots, h_r\right), \, A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r \, 4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a-c; h_1, \ldots, h_r\right), \left(\frac{1}{2}+\frac{1}{2}b-c; h_1, \ldots, h_r\right), \, B \end{array}\right)$$
$$+ \; A_2 \, \aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} z_1 \, 4^{-h_1} & (1-c; h_1, \ldots, h_r), (d-2c; 2h_1, \ldots, 2h_r), \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r \, 4^{-h_r} & \left(\frac{1}{2}a-c; h_1, \ldots, h_r\right), \left(\frac{1}{2}b-c; h_1, \ldots, h_r\right), \end{array}\right.$$

$$\left.\begin{matrix} \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\ldots,h_r\right),\ A \\ \cdot \\ \cdot \\ (1+d-2c;2h_1,\ldots,2h_r),\ B \end{matrix}\right),$$

*where* $A_1$ *and* $A_2$ *are defined in* (3.2) *and* (3.3) *respectively, also the validity conditions can easily be obtained from* (3.1).

*Proof.* We have the following integral denoted by $\mathcal{I}$ (say), given as

$$\mathcal{I}=\int_0^1 x^{c-1}\,(1-x)^c\,{}_3F_2\left[\begin{matrix} a,b,d+1; \\ \frac{1}{2}(a+b+1),d; \end{matrix}\ x\right]\aleph\begin{pmatrix} z_1\,x^{h_1}\,(1-x)^{l_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{l_r} \end{pmatrix}\mathrm{d}x.$$

Expressing the generalized hypergeometric function ${}_3F_2$ as a series, and after algebraic manipulations we have

$$\mathcal{I}=\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s(d)_s\,s!}\int_0^1 x^{c+s-1}\,(1-x)^c\,\aleph\begin{pmatrix} z_1\,x^{h_1}\,(1-x)^{l_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{l_r} \end{pmatrix}\mathrm{d}x.$$

Finally, evaluating the above integral with the help of (3.4), we arrive at

$$\mathcal{I}=\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s(d)_s\,s!}$$

$$\times\ \aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-c-s;h_1,\ldots,h_r),(-c;h_1,\ldots,h_r),\ A \\ \cdot \\ \cdot \\ (-2c-s;2h_1,\ldots,2h_r),\ B \end{matrix}\right). \tag{4.2}$$

Hence, the summation formula (4.1) follows from equating the two integrals (3.1) and (4.2). □

When $d=2c$, then above result reduces to the following interesting relation:

$$\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(2c+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s(2c)_s\,s!}$$

$$\times\,\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{matrix} z_1 \\ \cdot \\ \cdot \\ z_r \end{matrix}\ \middle|\ \begin{matrix} (1-c-s;h_1,\ldots,h_r),(-c;h_1,\ldots,h_r),A \\ \cdot \\ \cdot \\ (-2c-s;2h_1,\ldots,2h_r),B \end{matrix}\right)=A_1,$$

$$\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\dots,h_r)\,,\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\dots,h_r\right),\,A \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\dots,h_r\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1,\dots,h_r\right),\,B \end{array}\right)$$

$$+\,A_2\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+2,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\dots,h_r)\,,(0;2h_1,\dots,2h_r)\,, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}a-c;h_1,\dots,h_r\right),\left(\frac{1}{2}b-c;h_1,\dots,h_r\right), \end{array}\right.$$

$$\left.\begin{array}{l} \left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1,\dots,h_r\right),\,A \\ \cdot \\ \cdot \\ (1;2h_1,\dots,2h_r)\,,\,B \end{array}\right).$$

## 5. Special Cases

In this section, we will see the interesting special cases of integral formula (3.1) and summation formula (4.1).

Let $b=-2s$ and replace $a$ by $a+2s$, where $s$ is zero or a positive integer. In such case, one of the two terms on the right hand side of (3.1) will be vanished and we get the following interesting result, as concerning by the following corollary.

**Corollary 5.1.**

$$\int_0^1 x^{c-1}\,(1-x)^c\;{}_3F_2\left[\begin{array}{c} a+2s,-2s,d+1; \\ \frac{1}{2}(a+1)\,,d; \end{array}\;x\right]\aleph\left(\begin{array}{c} z_1\,x^{h_1}\,(1-x)^{h_1} \\ \cdot \\ \cdot \\ z_r\,x^{h_r}\,(1-x)^{h_r} \end{array}\right)\mathrm{d}x$$

$$=\frac{(-)^s\,\sqrt{\pi}\,\left(\frac{1}{2}\right)_s}{4^c\,\left(\frac{1}{2}a+\frac{1}{2}\right)_s}\;\aleph^{0,\mathfrak{n}+2:V}_{p_i+2,q_i+2,\tau_i;R:W}\left(\begin{array}{l|c} z_1\,4^{-h_1} & (1-c;h_1,\dots,h_r)\,, \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a+s-c;h_1,\dots,h_r\right), \end{array}\right.$$

$$\left.\begin{array}{c} \left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\dots,h_r\right),\,A \\ \cdot \\ \cdot \\ \left(\frac{1}{2}-s-c;h_1,\dots,h_r\right),\,B \end{array}\right),$$

*provided that the condition easily obtainable from* (3.1) *is satisfied.*

Let $b=-2s-1$ and replace $a$ by $a+2s+1$, where $s$ is zero or a positive integer. Then, one of the two terms on the right hand side of (3.1) will vanish and we get the following corollary.

**Corollary 5.2.** *By assuming that the validity condition easily obtainable from* (3.1) *is satisfied, then we have*

$$\int_0^1 x^{c-1}\,(1-x)^c\;{}_3F_2\left[\begin{array}{l} a+2s+1,-2s-1,d+1;\\ \frac{1}{2}(a+1),d; \end{array} x\right]\aleph\left(\begin{array}{c} z_1\,x^{h_1}\,(1-x)^{h_1}\\ \cdot\\ \cdot\\ z_r\,x^{h_r}\,(1-x)^{h_r}\end{array}\right)\mathrm{d}x$$

$$=\frac{(-)^{s-1}\,\sqrt{\pi}\,\left(\frac{3}{2}\right)_s}{d\,2^{2c+1}\,\left(\frac{1}{2}a+\frac{1}{2}\right)_s}\,\aleph^{0,\mathfrak{n}+3:V}_{p_i+3,q_i+3,\tau_i;R:W}\left(\begin{array}{c|c} z_1\,4^{-h_1} & (1-c;h_1,\ldots,h_r),\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\,4^{-h_r} & \left(\frac{1}{2}+\frac{1}{2}a+s-c;h_1,\ldots,h_r\right),\end{array}\right.$$

$$\left.\begin{array}{l}\left(\frac{1}{2}+\frac{1}{2}a-c;h_1,\ldots,h_r\right),(d-2c;2h_1,\ldots,2h_r),\,A\\ \cdot\\ \cdot\\ \left(-\frac{1}{2}-s-c;h_1,\ldots,h_r\right),(1+d-2c;2h_1,\ldots,2h_r),\,B\end{array}\right).$$

Next, we will provide the special cases of the summation formula (4.1).
Concerning the following corollary, we consider the Aleph-function of one variable defined by Südland et al. [26, 27] (see also, Saxena et al. [18]).

**Corollary 5.3.**

$$\sum_{s=0}^{\infty}\frac{(a)_s\,(b)_s\,(d+1)_s}{\left(\frac{1}{2}(a+b+1)\right)_s\,(d)_s\,s!}\,\aleph^{m_1,n_1+2}_{p_1+2,q_1+1,\tau_{i^{(1)}},R^{(1)}}\left(z_1\left|\begin{array}{c}(1-c-s;h_1),(-c;h_1),\,\mathbf{A}\\ \cdot\\ \cdot\\ (-2c-s;2h_1),\,\mathbf{B}\end{array}\right.\right)$$

$$=A_1\,\aleph^{m_1,n_1+2}_{p_1+2,q_1+2,\tau_{i^{(1)}},R^{(1)}}\left(z_1\,4^{-h_1}\left|\begin{array}{c}(1-c;h_1),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\,\mathbf{A}\\ \cdot\\ \cdot\\ \left(\frac{1}{2}+\frac{1}{2}a-c;h_1\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1\right),\,\mathbf{B}\end{array}\right.\right)$$

$$+A_2\,\aleph^{m_1,n_1+3}_{p_1+3,q_1+3,\tau_{i^{(1)}},R^{(1)}}\left(z_1\,4^{-h_1}\left|\begin{array}{c}(1-c;h_1),(d-2c;2h_1),\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\mathbf{A}\\ \cdot\\ \cdot\\ \left(\frac{1}{2}a-c;h_1\right),\left(\frac{1}{2}b-c;h_1\right),(1+d-2c;2h_1),\mathbf{B}\end{array}\right.\right),$$

*where*

$$\mathbf{A}=\left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\},\left\{\tau_{i^{(1)}}\left(c^{(1)}_{ji^{(1)}};\gamma^{(1)}_{ji^{(1)}}\right)_{n_1+1,p_{i^{(1)}}}\right\}$$

*and*

$$\mathbf{B}=\left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\},\left\{\tau_{i^{(1)}}\left(d^{(1)}_{ji^{(1)}};\delta^{(1)}_{ji^{(1)}}\right)_{m_1+1,q_{i^{(1)}}}\right\}.$$

*Provided that:*

$$h_1 > 0,\ \mathrm{Re}\,(c) > 0, \quad d \neq 0, -1, -2, \ldots,$$

$$\mathrm{Re}\,(c) + h_1 \min_{1\leq l\leq m_1} \mathrm{Re}\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right) > 0, \quad \left|\arg z_1 x^{h_1}\ (1-x)^{h_1}\right| < \frac{1}{2}\pi,$$

$$\left(\sum_{j=1}^{n_1} \gamma_j^{(1)} - \tau_{i^{(1)}} \sum_{j=n_1+1}^{p_{i^{(1)}}} \gamma_{ji^{(1)}}^{(1)} + \sum_{j=1}^{m_1} \delta_j^{(1)} - \tau_{i^{(1)}} \sum_{j=m_1+1}^{q_{i^{(1)}}} \delta_{ji^{(1)}}^{(1)}\right) > 0.$$

Now, we consider the $I$-function defined by Saxena [19]. We have the following result.

**Corollary 5.4.**

$$\sum_{s=0}^{\infty} \frac{(a)_s\ (b)_s\ (d+1)_s}{\left(\frac{1}{2}\,(a+b+1)\right)_s\ (d)_s\ s!}\ I_{p_1+2,q_1+1;R^{(1)}}^{m_1,n_1+2}\left(z_1 \left|\begin{array}{c} (1-c-s;h_1)\,,(-c;h_1)\,,\ \mathbf{A}' \\ \cdot \\ \cdot \\ (-2c-s;2h_1)\,,\ \mathbf{B}' \end{array}\right.\right)$$

$$=A_1\ I_{p_1+2,q_1+2;R^{(1)}}^{m_1,n_1+2}\left(z_1\, 4^{-h_1} \left|\begin{array}{c} (1-c;h_1)\,,\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\ \mathbf{A}' \\ \cdot \\ \cdot \\ \left(\frac{1}{2}+\frac{1}{2}a-c;h_1\right),\left(\frac{1}{2}+\frac{1}{2}b-c;h_1\right),\ \mathbf{B}' \end{array}\right.\right)$$

$$+\ A_2\ I_{p_1+3,q_1+3;R^{(1)}}^{m_1,n_1+3}\left(z_1\, 4^{-h_1} \left|\begin{array}{c} (1-c;h_1)\,,(d-2c;2h_1)\,,\left(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b-c;h_1\right),\ \mathbf{A}' \\ \cdot \\ \cdot \\ \left(\frac{1}{2}a-c;h_1\right),\left(\frac{1}{2}b-c;h_1\right),(1+d-2c;2h_1)\,,\ \mathbf{B}' \end{array}\right.\right),$$

*where*

$$\mathbf{A}' = \left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\},\ \left\{\left(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right\}$$

*and*

$$\mathbf{B}' = \left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\},\ \left\{\left(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right\}.$$

*Provided that*

$$h_1 > 0,\ \mathrm{Re}\,(c) > 0, \quad d \neq 0, -1, -2, \ldots,$$

$$\mathrm{Re}\,(c) + h_1 \min_{1\leq l\leq m_1} \mathrm{Re}\left(\frac{d_l^{(1)}}{\delta_l^{(1)}}\right) > 0, \quad \left|\arg z_1 x^{h_1}\ (1-x)^{h_1}\right| < \frac{1}{2}\pi,$$

$$\left(\sum_{j=1}^{n_1} \gamma_j^{(1)} - \sum_{j=n_1+1}^{p_{i^{(1)}}} \gamma_{ji^{(1)}}^{(1)} + \sum_{j=1}^{m_1} \delta_j^{(1)} - \sum_{j=m_1+1}^{q_{i^{(1)}}} \delta_{ji^{(1)}}^{(1)}\right) > 0.$$

*Remark* 5.1. By the similar methods, we can obtain the similar summation formula with the Aleph-function of two variables (see [9]), the $I$-function of two variables (see [12, 21]), the multivariable $I$-function (see [13, 15]), the multivariable $A$-function (see [7]), the $A$-function [6], the modified multivariable $H$-function (see [14]) and the multivariable $H$-function (see [3, 4, 10, 24, 25]).

## 6. Concluding Remarks

In this paper, we have established two integrals formulas and one summation formula involving the generalized hypergeometric function and Aleph-function of $r$-variables. On account of the most general character of the multivariable Aleph-function in Theorems 3.1, 3.2 and 4.1, numerous other special cases associated with potentially useful higher transcendental functions, orthogonal polynomials of one and several variables can be deduced.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 551–562.

# LOWER BOUNDS FOR INVERSE SUM INDEG INDEX OF GRAPHS

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Abstract. Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, be a simple connected graph with $n$ vertices and $m$ edges and let $d_1 \geq d_2 \geq \cdots \geq d_n > 0$, be the sequence of its vertex degrees. With $i \sim j$ we denote the adjacency of the vertices $i$ and $j$ in $G$. The inverse sum indeg index is defined as $ISI = \sum \frac{d_i\, d_j}{d_i + d_j}$ with summation going over all pairs of adjacent vertices. We consider lower bounds for $ISI$. We first analyze some lower bounds reported in the literature. Then we determine some new lower bounds.

## 1. Introduction

Let $G = (V, E)$, $V = \{1, 2, \dots, n\}$, $E = \{e_1, e_2, \dots, e_m\}$, be a simple connected graph with $n$ vertices and $m$ edges, and let $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$, $d_i = d(i)$, and $d(e_1) \geq d(e_2) \geq \cdots \geq d(e_m)$, be sequences of its vertex and edge degrees, respectively. We denote by $\Delta_{e_1} = d(e_1) + 2$ and $\delta_{e_1} = d(e_m) + 2$. If the vertices $i$ and $j$ are adjacent, we write $i \sim j$.

In graph theory, an invariant is a property of graphs that depends only on their abstract structure, not on the labeling of vertices or edges, or on the drawing of the graph. Such quantities are also referred to as topological indices. Topological indices gained considerable popularity because of their applications in chemistry as molecular structure descriptors [2, 24, 25].

An important class of graph invariants are those whose general formula is

$$VDB = VDB(G) = \sum_{i \sim j} \Phi(d_i, d_j),$$

*Key words and phrases.* Degree (of vertex), degree (of edge), inverse sum indeg index, Zagreb index.

2010 *Mathematics Subject Classification.* Primary: 05C12. Secondary: 05C50.

DOI 10.46793/KgJMat2004.551G

*Received*: March 09, 2018.

*Accepted*: June 28, 2018.

which are usually referred to as *vertex–degree based topological indices*. Here $\Phi$ may be any function satisfying the condition $\Phi(x,y) = \Phi(y,x)$. A very large number of particular VDB indices has been considered in the literature, some of which are listed below. There are countless papers reporting relations for VDB indices, which includes bounds (in terms of various graph parameters), characterization of graphs extremal w.r.t. some particular VDB index (in some particular class of graphs), and inequalities between various members of the VDB family. Readers interested in this topic may consult the recent collections of review articles [12–14].

The present paper contributes to the theory of VDB indices, comparing some previously known inequalities and challenging their validity, and offering a few new results of the same kind.

The oldest VDB topological indices, the *first and the second Zagreb indices* are defined as (see [8,9])

$$M_1 = M_1(G) = \sum_{i=1}^{n} d_i^2 \quad \text{and} \quad M_2 = M_2(G) = \sum_{i\sim j} d_i\, d_j \,,$$

where the first Zagreb index can be expressed as

$$M_1 = \sum_{i\sim j} (d_i + d_j)\,. \tag{1.1}$$

Bearing in mind that for the edge $e$ connecting the vertices $i$ and $j$,

$$d(e) = d_i + d_j - 2\,,$$

the index $M_1$ can also be considered as an edge–degree based invariant (see [17])

$$M_1 = \sum_{i=1}^{m} \Big[ d(e_i) + 2 \Big]\,.$$

A so-called *forgotten topological index* is defined as (see [8])

$$F = F(G) = \sum_{i=1}^{n} d_i^3 = \sum_{i\sim j} (d_i^2 + d_j^2)\,.$$

It can be easily observed that for the indices $M_2$ and $F$ the following identities hold:

$$F + 2M_2 = \sum_{i=1}^{m} \Big[ d(e_i) + 2 \Big]^2 \quad \text{and} \quad F - 2M_2 = \sum_{i\sim j} (d_i - d_j)^2\,.$$

Multiplicative versions of the first and second Zagreb indices, denoted by $\Pi_1$ and $\Pi_2$, respectively, were first considered in a paper [10] published in 2011, and were promptly followed by numerous additional studies. These indices are defined as:

$$\Pi_1 = \Pi_1(G) = \prod_{i=1}^{n} d_i^2 \quad \text{and} \quad \Pi_2 = \Pi_2(G) = \prod_{i\sim j} d_i\, d_j\,.$$

One year later, motivated by the identity (1.1), the *multiplicative sum–Zagreb index* was conceived as [3]:

$$\Pi_1^* = \Pi_1^*(G) = \prod_{i\sim j}(d_i + d_j)\,.$$

Probably the most popular and most thoroughly investigated molecular–structure descriptor is the classical *Randić (or connectivity) index*

$$R = R(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i\, d_j}}\,, \tag{1.2}$$

invented by Randić in 1975 [21].

Replacing in (1.2) multiplication by summation, the so-called *sum–connectivity index* was put forward as (see [32])

$$SCI = SCI(G) = \sum_{i\sim j} \frac{1}{\sqrt{d_i + d_j}}\,.$$

In [1] (see also [11, 16]) a topological index called *general Randić index*, $R_\alpha$, was introduced as

$$R_\alpha = R_\alpha(G) = \sum_{i\sim j}(d_i\, d_j)^\alpha\,,$$

where $\alpha$ is an arbitrary real number. For $\alpha = -1/2$ we have $R = R_{-1/2}$, whereas for $\alpha = 1/2$, the reciprocal Randić index, $RR$, [11, 16] is obtained.

In order to improve the predictive power of the Randić index, a large number of additional vertex–degree based topological descriptors was introduced. The *geometric–arithmetic index*, introduced in [30], is defined as

$$GA = GA(G) = \sum_{i\sim j} \frac{2\sqrt{d_i\, d_j}}{d_i + d_j}\,.$$

The *harmonic index*, introduced in [4], is defined as

$$H = H(G) = \sum_{i\sim j} \frac{2}{d_i + d_j}\,.$$

It should be noted that $\Pi_1^*$, $SCI$, and $H$ can be considered as edge–degree based topological indices as well, since the following identities hold:

$$\Pi_1^* = \prod_{i=1}^{m}\Big[d(e_i) + 2\Big], \quad SCI = \sum_{i=1}^{m} \frac{1}{\sqrt{d(e_i) + 2}}, \quad H = \sum_{i=1}^{m} \frac{2}{d(e_i) + 2}\,.$$

In a series of papers [26–28, 31], Vukičević introduced the so-called *Adriatic indices*, providing a general method for constructing vertex–degree based graph invariants; for review see [29]. Vukičević himself restricted the considerations to some 148 such indices, although their possible number would be infinite. One of these Adriatic indices, named *symmetric division deg index*, is

$$SDD = SDD(G) = \sum_{i\sim j} \frac{1}{2}\left(\frac{d_i}{d_j} + \frac{d_j}{d_i}\right).$$

Another Adriatic index, the so-called *inverse sum indeg index*, was singled out in [26] as being a significantly accurate predictor of total surface area of octane isomers. It is defined as

$$ISI = ISI(G) = \sum_{i\sim j} \frac{d_i\, d_j}{d_i + d_j}.$$

In this paper, we are interested in lower bounds on $ISI$. We first perform the analysis of some earlier reported lower bounds for $ISI$ [5, 19, 23]. Then we determine some new lower bounds for it, in terms of some other vertex–degree based graph invariants.

## 2. Preliminary considerations

In this section, we analyze some lower bounds for the inverse sum indeg index reported in [5, 19, 23].

In [23] the following inequality was proven

$$ISI \geq \frac{(n-1)^2}{n}, \tag{2.1}$$

with equality if and only if $G \cong K_{1,n-1}$. This bound is the best possible in its class.

In [5] it was proven

$$ISI \geq \frac{m^2}{n}, \tag{2.2}$$

with equality if and only if the graph $G$ is regular or biregular. This bound depends on the parameters $n$ and $m$, and it is the best one in its class, so far.

The bounds given by (2.1) and (2.2), although simple, are very important and, as we shall demonstrate, are convenient for testing whether other lower bounds, depending on some other parameters, have any sense. Of course, it is of interest to determine other (lower) bounds that establish relationships between $ISI$ and other graph invariants. But, if these inequalities are weaker than inequalities (2.1) and (2.2), the question of their purpose arises. In that sense we will analyze lower bounds for $ISI$ obtained in [5] and [19].

In [5] the following lower bounds for $ISI$ were also established:

$$ISI \geq \frac{m^2\,\delta^2}{M_1}, \tag{2.3}$$

$$ISI \geq \frac{\delta^2 H}{2}, \tag{2.4}$$

$$ISI \geq \frac{M_2}{2\Delta}, \tag{2.5}$$

$$ISI \geq \frac{\delta^2 (SCI)^2}{m}, \tag{2.6}$$

$$ISI \geq H, \tag{2.7}$$

$$ISI \geq \frac{M_1}{2} - \frac{F}{4\delta}, \tag{2.8}$$

$$ISI \geq \frac{m^2 \sqrt{\delta\Delta}}{(\delta+\Delta)R}, \tag{2.9}$$

$$ISI \geq \frac{(SCI)^2}{R_{-1}}, \tag{2.10}$$

$$ISI \geq m \left(\frac{\Pi_2}{\Pi_1^*}\right)^{1/m}, \tag{2.11}$$

whereas in [19] it was proven that

$$ISI \geq \frac{\sqrt{\delta\Delta}\, H\, M_2}{m(\delta+\Delta)}. \tag{2.12}$$

The inequalities (2.3)–(2.12) are all correct. However, it is questionable whether any of the bounds given by (2.3)–(2.10) are worthy. In what follows we discuss this matter.

Since

$$M_1 = \sum_{i=1}^{n} d_i^2 \geq n\delta^2,$$

we have that

$$\frac{m^2}{n} \geq \frac{m^2\,\delta^2}{M_1}.$$

Thus, the inequality (2.3) is a direct consequence of the inequality (2.2).

Since

$$\frac{n\,\delta^2 H}{2} = \frac{\delta^2}{2} \sum_{i\sim j} \left(\frac{1}{d_i} + \frac{1}{d_j}\right) \sum_{i\sim j} \frac{2}{d_i + d_j} \leq \frac{\delta^2}{2}\,\frac{2m}{\delta}\,\frac{m}{\delta},$$

it holds

$$\frac{m^2}{n} \geq \frac{\delta^2 H}{2}.$$

Thus, the inequality (2.4) is a direct consequence of the inequality (2.2).

Using the arithmetic–harmonic mean inequality for real numbers (see for example [18]), we get

$$\frac{1}{2}HM_1 = \frac{1}{2}\sum_{i\sim j}\frac{2}{d_i+d_j}\sum_{i\sim j}(d_i+d_j) \geq m^2$$

that is

$$\frac{\delta^2 H}{2} \geq \frac{m^2\,\delta^2}{M_1},$$

implying that the inequality (2.3) is a consequence of (2.4).

If $m \geq n$, the inequality (2.5) is a consequence of (2.2).

Let $m = n-1$, i.e., $G$ is a tree. In [6] it was proven that

$$M_2(T) \leq \Delta(2n-\Delta-1-k)+k(k-1), \tag{2.13}$$

where

$$k \equiv n-1 \pmod{\Delta-1}, \quad 1 \leq k \leq n-1.$$

From (2.13) it follows

$$M_2(T) \leq \Delta(2n-\Delta-1-k)+k(k-1) \leq \frac{2\Delta(n-1)^2}{n},$$

wherefrom we get

$$\frac{m^2}{n} = \frac{(n-1)^2}{n} \geq \frac{M_2(T)}{2\Delta}.$$

This means that the inequality (2.5) is a consequence of (2.2) for every connected graph $G$.

According to the inequality

$$(SCI)^2 = \left(\sum_{i\sim j}\frac{1}{\sqrt{d_i+d_j}}\right)^2 \leq m\sum_{i\sim j}\frac{1}{d_i+d_j} = \frac{mH}{2},$$

it follows

$$\frac{m^2}{n} \geq \frac{\delta^2 H}{2} \geq \frac{\delta^2 (SCI)^2}{m}.$$

This means that the inequality (2.6) is a consequence of both (2.2) and (2.4).

Let $m = n-1$, i.e., $G$ is a tree of order $n$, and let $n \geq 3$. Then $d_i + d_j \geq 3$ for every $i \sim j$. Therefore,

$$\frac{(n-1)^2}{n} \geq \frac{2}{3}(n-1) \geq H.$$

It follows that in this case the inequality (2.7) is a consequence of both (2.1) and (2.2).

Let $m \geq n$. Then $d_i + d_j \geq 2$ for every $i \sim j$. Then we have

$$\frac{m^2}{n} \geq m \geq H.$$

Therefore, in this case, the inequality (2.7) is also a consequence of (2.2).

The inequality (2.2) is stronger than the inequality (2.8) when $G$ is a biregular graph, or $G \cong P_n$, or $G \cong K_n - e$, or $G \cong K_{n-1} + e$. When $n \geq 3$ and $G$ is not a regular graph, then we could not find any connected graph for which the inequality (2.8) is stronger than the inequality (2.2). Moreover, if $\Delta \geq 2\delta$, then the right-hand side of (2.8) can be negative. Therefore, the right-hand side of (2.8) should be avoided when estimating lower bound for $ISI$.

Since

$$n = \sum_{i \sim j} \left( \frac{1}{d_i} + \frac{1}{d_j} \right) = \sum_{i \sim j} \frac{d_i + d_j}{d_i\, d_j} = \sum_{i \sim j} \frac{d_i + d_j}{\sqrt{d_i\, d_j}} \frac{1}{\sqrt{d_i\, d_j}}$$

and

$$\frac{d_i + d_j}{\sqrt{d_i\, d_j}} = \sqrt{\frac{d_i}{d_j}} + \sqrt{\frac{d_j}{d_i}} \leq \sqrt{\frac{\Delta}{\delta}} + \sqrt{\frac{\delta}{\Delta}}\,,$$

for every edge in the graph $G$, it follows

$$n \leq \frac{(\Delta + \delta)R}{\sqrt{\Delta\delta}}\,.$$

Therefore,

$$\frac{m^2}{n} \geq \frac{m^2\sqrt{\Delta\delta}}{(\Delta + \delta)R}\,.$$

Thus, the inequality (2.9) is a consequence of the inequality (2.2).

The inequality (2.2) is stronger than the inequality (2.10) when $G \cong P_n$, or $G \cong K_n - e$ or $G \cong K_{n-1} + e$, $n \geq 3$. If $n \geq 3$ and $G$ is not a regular or biregular graph, then we could not find any connected graph for which the inequality (2.10) is stronger than the inequality (2.2). However, it remains an open question whether this is the case for every connected graph under given conditions.

The inequality (2.11) is stronger than the inequality (2.2) for $G \cong P_n$, $G \cong K_n - e$ or $G \cong K_{n-1} + e$. Again, we could not find any connected graph which is not regular or biregular for which the inequality (2.2) is stronger than the inequality (2.11). It is still an open question if this is always the case.

The inequalities (2.2) and (2.12) are not comparable. Thus, for example, if the connected graph is biregular or $G \cong K_{n-1} + e$, then the inequality (2.2) is stronger than the inequality (2.12). If, however, $G \cong P_n$ or $G \cong K_n - e$, then the inequality (2.12) is stronger than (2.2).

## 3. Main Results

Before we establish some new lower bounds for $ISI$, we recall some discrete inequalities for real number sequences that will be used subsequently.

Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, m$, be positive real number sequences with the properties $p_1 + p_2 + \cdots + p_m = 1$ and $0 < a \leq a_i \leq A < +\infty$. In [22] the following inequality was proven

$$\sum_{i=1}^{m} p_i\, a_i + aA \sum_{i=1}^{m} \frac{p_i}{a_i} \leq a + A\,. \tag{3.1}$$

Equality holds if and only if $a_i = A$ or $a_i = a$, for every $i = 1, 2, \ldots, m$.

Let $x = (x_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, m$, be positive real number sequences. In [20] it was proven that for any $r \geq 0$ holds

$$\sum_{i=1}^{m} \frac{x_i^{r+1}}{a_i^r} \geq \frac{\left(\sum\limits_{i=1}^{m} x_i\right)^{r+1}}{\left(\sum\limits_{i=1}^{m} a_i\right)^{r}}, \tag{3.2}$$

with equality if and only if $\frac{a_1}{x_1} = \cdots = \frac{a_m}{x_m}$.

If $a = (a_i)$, $i = 1, 2, \ldots, m$, is a positive real number sequence, then [15]

$$\left(\sum_{i=1}^{m} \sqrt{a_i}\right)^2 \geq \sum_{i=1}^{m} a_i + m(m-1)\left(\prod_{i=1}^{m} a_i\right)^{1/m}. \tag{3.3}$$

Equality holds if and only if $a_1 = a_2 = \cdots = a_m$.

**Theorem 3.1.** *Let $G$ be a simple connected graph. Then*

$$ISI \geq \frac{4R_{-1}M_2 + \Delta_{e_1}\delta_{e_1}H^2}{4(\Delta_{e_1} + \delta_{e_1})R_{-1}}. \tag{3.4}$$

*Equality holds if and only if $G$ is regular or biregular.*

*Proof.* For $p_i := \frac{d_i\, d_j}{(d_i+d_j)ISI}$, $a_i := d_i + d_j$, $a = \delta_{e_1}$, $A = \Delta_{e_1}$, where summation is performed over all pairs of adjacent vertices of $G$, the inequality (3.1) becomes

$$\sum_{i\sim j} d_i\, d_j + \Delta_{e_1}\delta_{e_1} \sum_{i\sim j} \frac{d_i\, d_j}{(d_i + d_j)^2} \leq (\Delta_{e_1} + \delta_{e_1})ISI,$$

i.e.,

$$M_2 + \Delta_{e_1}\delta_{e_1} \sum_{i\sim j} \frac{d_i\, d_j}{(d_i + d_j)^2} \leq (\Delta_{e_1} + \delta_{e_1})ISI. \tag{3.5}$$

For $r = 1$, $x_i := \frac{1}{d_i+d_j}$, $a_i := \frac{1}{d_i\, d_j}$, where summation goes over all pairs of adjacent vertices, the inequality (3.2) transforms into

$$\sum_{i\sim j} \frac{d_i\, d_j}{(d_i + d_j)^2} \geq \frac{\left(\sum\limits_{i\sim j} \frac{1}{d_i + d_j}\right)^2}{\sum\limits_{i\sim j} \frac{1}{d_i\, d_j}},$$

that is

$$\sum_{i\sim j} \frac{d_i\, d_j}{(d_i + d_j)^2} \geq \frac{H^2}{4R_{-1}}. \tag{3.6}$$

In view of (3.5) and (3.6), we obtain (3.4).

The equality in (3.6) holds if and only if for any two pairs of adjacent vertices $i \sim j$ and $u \sim v$

$$\frac{1}{d_i} + \frac{1}{d_j} = \frac{1}{d_u} + \frac{1}{d_v}\,. \tag{3.7}$$

Let $j$ and $u$ be two vertices adjacent to $i$, that is $i \sim j$ and $i \sim u$. Then, from the above identity, it follows $d_j = d_u$. Since $G$ is a connected graph, equality in (3.6) holds if and only if $G$ is regular or biregular.

Equality in (3.5) holds if and only if $d_i + d_j = \Delta_{e_1}$ or $d_i + d_j = \delta_{e_1}$, for every edge of $G$. This means that equality in (3.5) holds if and only if $G$ is regular or biregular or for some edges $d_i + d_j = \Delta_{e_1}$ holds whereas for the remaining edges $d_i + d_j = \delta_{e_1}$. This means that equality in (3.4) holds if and only if $G$ is regular or biregular. $\square$

In the next theorem we obtain a lower bound for $ISI$ in terms of the parameters $m$, $\Delta_{e_1}$, $\delta_{e_1}$, and the topological indices $M_2$ and $SDD$.

**Theorem 3.2.** *Let $G$ be a simple connected graph with $m$ edges. Then*

$$ISI \geq \frac{2M_2(SDD+m) + m^2\Delta_{e_1}\delta_{e_1}}{2(SDD+m)(\Delta_{e_1}+\delta_{e_1})}\,. \tag{3.8}$$

*Equality is attained if and only if for any two pairs of adjacent vertices $i \sim j$ and $u \sim v$ the identity*

$$\frac{d_i}{d_j} + \frac{d_j}{d_i} = \frac{d_u}{d_v} + \frac{d_v}{d_u} \tag{3.9}$$

*holds.*

*Proof.* By the arithmetic–harmonic mean inequality (see e.g. [18]), we have

$$\sum_{i\sim j} \frac{d_i\,d_j}{(d_i+d_j)^2} \sum_{i\sim j} \frac{(d_i+d_j)^2}{d_i\,d_j} \geq m^2\,. \tag{3.10}$$

Since

$$\sum_{i\sim j} \frac{(d_i+d_j)^2}{d_i\,d_j} = \sum_{i\sim j} \frac{d_i^2+d_j^2+2d_i\,d_j}{d_i\,d_j} = \sum_{i\sim j} \frac{d_i^2+d_j^2}{d_i\,d_j} + 2m = 2(SDD+m)\,,$$

from (3.10) and the above it follows

$$\sum_{i\sim j} \frac{d_i\,d_j}{(d_i+d_j)^2} \geq \frac{m^2}{2(SDD+m)}\,.$$

From this and inequality (3.5) we obtain (3.8).

Equality in (3.10) is attained if and only if for any two pairs of adjacent vertices $i \sim j$ and $u \sim v$ the equality (3.9) holds. Consequently, equality in (3.8) holds if and only if for any two pairs of adjacent vertices $i \sim j$ and $u \sim v$ the equality (3.9) is valid. $\square$

In the following theorem we determine a lower bound for $ISI$ in terms of the parameters $m$, $\Delta_{e_1}$, $\delta_{e_1}$, and the topological indices $M_2$ and $GA$.

**Theorem 3.3.** *Let $G$ be a simple connected graph with $m$ edges. Then*

$$ISI \geq \frac{4mM_2 + \Delta_{e_1}\delta_{e_1}(GA)^2}{4m(\Delta_{e_1} + \delta_{e_1})} . \tag{3.11}$$

*Equality in* (3.11) *holds if and only if for any two pairs of adjacent vertices $i \sim j$ and $u \sim v$, the equality* (3.9) *is valid.*

*Proof.* Since

$$\sum_{i\sim j} \frac{d_i\, d_j}{(d_i + d_j)^2} = \sum_{i\sim j} \left( \frac{\sqrt{d_i\, d_j}}{d_i + d_j} \right)^2 \geq \frac{1}{m} \left( \sum_{i\sim j} \frac{\sqrt{d_i\, d_j}}{d_i + d_j} \right)^2 ,$$

it follows

$$\sum_{i\sim j} \frac{d_i\, d_j}{(d_i + d_j)^2} \geq \frac{1}{m} \left( \frac{GA}{2} \right)^2 .$$

From this inequality and (3.5) we obtain (3.11).

The equality case in Theorem 3.3 is proved in a same way as in the case of Theorem 3.2. □

In the following theorem we determine a lower bound for $ISI$ in terms of $M_1$ and $RR$.

**Theorem 3.4.** *Let $G$ be a simple connected graph with $m$ edges. Then*

$$ISI \geq \frac{(RR)^2}{M_1} . \tag{3.12}$$

*Equality holds if and only if for any two pairs of adjacent vertices $i \sim j$ and $u \sim v$, the equality* (3.9) *is valid.*

*Proof.* For $r = 1$, $x_i := \sqrt{d_i\, d_j}$, $a_i := d_i + d_j$, where summation goes over all pairs of adjacent vertices of $G$, the inequality (3.2) transforms into

$$\sum_{i\sim j} \frac{\left(\sqrt{d_i\, d_j}\right)^2}{d_i + d_j} \geq \frac{\left( \sum_{i\sim j} \sqrt{d_i\, d_j} \right)^2}{\sum_{i\sim j} (d_i + d_j)} ,$$

that is

$$ISI \geq \frac{(RR)^2}{M_1} .$$

The equality case in (3.12) is proved in a same way as in the case of Theorem 3.2. □

The inequalities (3.4), (3.8), (3.11) and (3.12) are stronger than the inequality (2.2) when $G \cong P_n$, $G \cong K_n - e$ or $G \cong K_{n-1} + e$. We could not find any connected graph for which the inequality (2.2) is stronger than these inequalities. However, it is an open question whether these inequalities are always stronger than (2.2).

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 563–570.

# A NOTE ON THE DEFINITION OF BOUNDED VARIATION OF HIGHER ORDER FOR DOUBLE SEQUENCES

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Abstract. In this study the definition of bounded variation of order $p$ ($p \in \mathbb{N}$) for double sequences is considered. Some inclusion relations are proved and counter examples are provided for ensuring proper inclusions.

## 1. Introduction

While studying convergence properties of double trigonometric and Walsh series, many authors have considered double sequences which are of bounded variation or more generally of bounded variation of order $(p, 0)$, $(0, p)$, and $(p, p)$ (see, e.g., [1, 3]). Also, many results regarding the convergence of trigonometric and Walsh series with coefficients of bounded variation of higher order are proved (see, e.g., [1, 3]). But, it seems that showing the inclusion relations for such classes of sequences and constructing counter examples for showing proper inclusions have not yet been done.

The main goal of this note is to prove such inclusion relations and constructing examples for showing proper inclusions. We start with the one dimensional case. In what follows, by a sequence (or a single sequence), we mean a function from $\mathbb{Z}$ to $\mathbb{C}$, and by a double sequence, we mean a function from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{C}$.

## 2. One Dimensional Case

We recall the definition of bounded variation of order $p$ for a single sequence (see [2, Defintion 1.4]).

*Key words and phrases.* Double sequence of bounded variation, double sequence of bounded variation of order $p$ ($p \in \mathbb{N}$), double sequence of bounded variation of order $(p, 0)$, double sequence of bounded variation of order $(0, p)$, double sequence of bounded variation of order $(p, p)$.

2010 *Mathematics Subject Classification.* Primary: 40B05.

DOI 10.46793/KgJMat2004.563G

*Received*: April 20, 2018.

*Accepted*: July 02, 2018.

**Definition 2.1.** A null sequence $\{a_k : k = \dots, -1, 0, 1, \dots\}$, i.e., $\{a_k\}$ such that $a_k \to 0$ as $|k| \to \infty$, is said to be of bounded variation of order $p$ ($p \in \mathbb{N}$) if

$$\sum_{k=-\infty}^{\infty} |\Delta^p a_k| < \infty,$$

where $\Delta^p a_k = \Delta(\Delta^{p-1} a_k) = \Delta^{p-1} a_k - \Delta^{p-1} a_{k+1}$ and $\Delta^0 a_k = a_k$.

From this definition, it is clear that if $\{a_k\}$ is of bounded variation of order $p$, then it is of bounded variation of order $p+1$ also. Also, in [2] an example of a sequence is given which is of bounded variation of order 2, but not of bounded variation.

## 3. Two Dimensional Case

In this section, we shall consider the definition of a double sequence of bounded variation of order $p$. For that first we have the following definition of differences.

**Definition 3.1.** Let $\{c(j,k) : j,k = \dots, -1, 0, 1, \dots\}$ be a double sequence. Its differences are defined by

$$\begin{aligned}
&\Delta_{00} c(j,k) = c(j,k),\\
&\Delta_{pq} c(j,k) = \Delta_{p-1,q} c(j,k) - \Delta_{p-1,q} c(j+1,k), \quad p \geq 1,\\
&\Delta_{pq} c(j,k) = \Delta_{p,q-1} c(j,k) - \Delta_{p,q-1} c(j,k+1), \quad q \geq 1.
\end{aligned}$$

As is well-known, the two right-hand sides coincide if $\min(p,q) \geq 1$. Also, we mention that that double induction argument gives

$$\Delta_{pq} c(j,k) = \sum_{s=0}^{p} \sum_{t=0}^{q} (-1)^{s+t} \binom{p}{s} \binom{q}{t} c(j+s, k+t).$$

**Definition 3.2.** A double sequence $\{c(j,k) : j,k = \dots, -1, 0, 1, \dots\}$ of complex numbers is called a null sequence, if it satisfies

$$c(j,k) \to 0 \quad \text{as} \quad \max(|j|,|k|) \to \infty. \tag{3.1}$$

**Definition 3.3.** A double null sequence $\{c(j,k) : j,k = \dots, -1, 0, 1, \dots\}$ is said to be of *bounded variation* if

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j,k)| < \infty. \tag{3.2}$$

We shall denote the class of all double sequences of bounded variation by $\mathfrak{BV}$.

Now, we give an analogous definition of bounded variation of order $p$ ($p \geq 2$) for a double sequence.

**Definition 3.4.** A double null sequence $\{c(j,k) : j,k = \dots, -1, 0, 1, \dots\}$ is said to belong to the class $(\mathfrak{BV})^p$, i.e., of bounded variation of order $p \geq 2$, if the following three conditions are satisfied:

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{pp} c(j,k)| < \infty, \tag{3.3}$$

$$\lim_{|k|\to\infty} \sum_{j=-\infty}^{\infty} |\Delta_{p0} c(j,k)| = 0, \tag{3.4}$$

and

$$\lim_{|j|\to\infty} \sum_{k=-\infty}^{\infty} |\Delta_{0p} c(j,k)| = 0. \tag{3.5}$$

Some authors (see, e.g., [1,3]) call conditions (3.3)–(3.5) as conditions of bounded variation of order $(p,p)$, $(p,0)$, and $(0,p)$, respectively.

Our main aim is to prove that the following chain of inclusion relations holds:

$$\mathfrak{BV} = (\mathfrak{BV})^1 \subset (\mathfrak{BV})^2 \subset \cdots \subset (\mathfrak{BV})^p \subset (\mathfrak{BV})^{p+1} \subset \cdots .$$

We also show that each of above relation is proper.

**Theorem 3.1.** *If* $\{c(j,k)\} \in \mathfrak{BV}$, *then* $\{c(j,k)\} \in (\mathfrak{BV})^2$.

*Proof.* Suppose $\{c(j,k)\} \in \mathfrak{BV}$. Then, we write

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j,k)| < \infty. \tag{3.6}$$

Since

$$\begin{aligned}\Delta_{22} c(j,k) &= \Delta_{12} c(j,k) - \Delta_{12} c(j+1,k)\\ &= \Delta_{11} c(j,k) - \Delta_{11} c(j,k+1) - \Delta_{11} c(j+1,k) + \Delta_{11} c(j+1,k+1),\end{aligned}$$

it follows from (3.6) that

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{22} c(j,k)| \le 4 \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j,k)| < \infty.$$

So, $\{c(j,k)\}$ satisfies (3.3) for $p=2$. Now, as $\{c(j,k)\}$ is of bounded variation, that is, $\{c(j,k)\}$ is a double sequence satisfying (3.1) and (3.2), it follows (see, e.g., [4, Proof of Lemma 1]) that

$$\Delta_{10} c(j,k_0) = \sum_{k=k_0}^{\infty} \Delta_{11} c(j,k) \quad \text{and} \quad \Delta_{10} c(j,k_0) = -\sum_{k=-\infty}^{k_0-1} \Delta_{11} c(j,k),$$

for each fixed $k_0 \in \mathbb{Z}$. Therefore, for each fixed $k_0 \in \mathbb{Z}$, we have

$$\begin{aligned}|\Delta_{20} c(j,k_0)| &= |\Delta_{10} c(j,k_0) - \Delta_{10} c(j+1,k_0)|\\ &= \left| \sum_{k=k_0}^{\infty} \Delta_{11} c(j,k) - \sum_{k=k_0}^{\infty} \Delta_{11} c(j+1,k) \right|\\ &\le \sum_{k=k_0}^{\infty} |\Delta_{11} c(j,k)| + \sum_{k=k_0}^{\infty} |\Delta_{11} c(j+1,k)|\end{aligned}$$

and

$$|\Delta_{20} c(j,k_0)| = |\Delta_{10} c(j,k_0) - \Delta_{10} c(j+1,k_0)|$$

$$
\begin{aligned}
&= \left| -\sum_{k=-\infty}^{k_0-1} \Delta_{11}c(j,k) + \sum_{k=-\infty}^{k_0-1} \Delta_{11}c(j+1,k) \right| \\
&\le \sum_{k=-\infty}^{k_0-1} |\Delta_{11}c(j,k)| + \sum_{k=-\infty}^{k_0-1} |\Delta_{11}c(j+1,k)| \,.
\end{aligned}
$$

Therefore, in view of (3.6), we have

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty} |\Delta_{20}c(j,k_0)| &\le \sum_{j=-\infty}^{\infty}\sum_{k=k_0}^{\infty} |\Delta_{11}c(j,k)| + \sum_{j=-\infty}^{\infty}\sum_{k=k_0}^{\infty} |\Delta_{11}c(j+1,k)| \\
&\to 0 \ \text{ as } \ k_0 \to \infty
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty} |\Delta_{20}c(j,k_0)| &\le \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{k_0-1} |\Delta_{11}c(j,k)| + \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{k_0-1} |\Delta_{11}c(j+1,k)| \\
&\to 0 \ \text{ as } \ k_0 \to -\infty.
\end{aligned}
$$

Therefore, $\{c(j,k)\}$ satisfies (3.4) for $p = 2$. Similarly, it satisfies (3.5) for $p = 2$. Thus, $\{c(j,k)\} \in (\mathfrak{BV})^2$. This completes the proof. □

**Theorem 3.2.** *If $\{c(j,k)\} \in (\mathfrak{BV})^p$, $p \ge 2$, then $\{c(j,k)\} \in (\mathfrak{BV})^{p+1}$.*

*Proof.* Suppose $\{c(j,k)\} \in (\mathfrak{BV})^p$. Then, (3.3)–(3.5) hold true. Since

$$
\begin{aligned}
\Delta_{p+1,p+1}c(j,k) &= \Delta_{p,p+1}c(j,k) - \Delta_{p,p+1}c(j+1,k) \\
&= \Delta_{pp}c(j,k) - \Delta_{pp}c(j,k+1) - \Delta_{pp}c(j+1,k) + \Delta_{pp}c(j+1,k+1),
\end{aligned}
$$

it follows from (3.3) that

$$
\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{p+1,p+1}c(j,k)| \le 4 \sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} |\Delta_{pp}c(j,k)| < \infty.
$$

So, $\{c(j,k)\}$ satisfies (3.3) for $p+1$ in place of $p$. Also, for a fixed $k \in \mathbb{Z}$, we have

$$
\begin{aligned}
\sum_{j=-\infty}^{\infty} |\Delta_{p+1,0}c(j,k)| &= \sum_{j=-\infty}^{\infty} |\Delta_{p,0}c(j,k) - \Delta_{p,0}c(j+1,k)| \\
&\le \sum_{j=-\infty}^{\infty} |\Delta_{p,0}c(j,k)| + \sum_{j=-\infty}^{\infty} |\Delta_{p,0}c(j+1,k)| \\
&\to 0 \ \text{ as } \ |k| \to \infty,
\end{aligned}
$$

in view of (3.4). So, $\{c(j,k)\}$ satisfies (3.4) for $p+1$ in place of $p$. Similarly, in view of (3.5), $\{c(j,k)\}$ satisfies (3.5) for $p+1$ in place of $p$. Therefore, $\{c(j,k)\}$ is of bounded variation of order $p+1$. □

Now, we will prove that the inclusion relations proved in above theorems are proper. In the following example, we give an example of a double sequence defined on $\mathbb{Z} \times \mathbb{Z}$, which is of bounded variation of of order 2, but not of bounded variation.

*Example* 3.1. We consider $\{a_j\}$ and $\{b_k\}$ to be single sequences as in [2, Example, p. 424]. That is, for $j,k = 1,2,\dots,$ and $-j \le p < j$, $-k \le q < k$, we put

$$a_{j^2+p} = \frac{j-|p|}{j^2}, \quad b_{k^2+q} = \frac{k-|q|}{k^2}.$$

As argued in [2], the sequences $\{a_j\}$ and $\{b_k\}$ are well-defined on $\mathbb{N}\cup\{0\}$ as $j^2+j = (j+1)^2-(j+1)$ and $k^2+k = (k+1)^2-(k+1)$. We also put $a_j = 0$ if $j \le -1$, and $b_k = 0$ if $k \le -1$. Then, $\{a_j\}$ and $\{b_k\}$ are well-defined sequences on $\mathbb{Z}$.

Now, we put

$$c(j,k) = a_j b_k, \quad j,k \in \mathbb{Z}.$$

Then, $\{c(j,k)\}$ is a well-defined double sequence on $\mathbb{Z}\times\mathbb{Z}$. Actually, it is proved in [2] that these single sequences are of bounded variation of order 2, but not of bounded variation. We claim that the double sequence $\{c(j,k)\}$ is of bounded variation of order 2, but not of bounded variation.

We first observe that

$$\begin{aligned}\Delta_{11}c(j,k) &= c(j,k) - c(j+1,k) - c(j,k+1) + c(j+1,k+1)\\ &= a_j b_k - a_{j+1}b_k - a_j b_{k+1} + a_{j+1}b_{k+1}\\ &= (a_j - a_{j+1})b_k - (a_j - a_{j+1})b_{k+1}\\ &= (\Delta a_j)b_k - (\Delta a_j)b_{k+1}\\ &= (\Delta a_j)(\Delta b_k)\end{aligned}$$

and therefore, we also have

$$\begin{aligned}\Delta_{22}c(j,k) &= \Delta_{12}c(j,k) - \Delta_{12}c(j+1,k)\\ &= \Delta_{11}c(j,k) - \Delta_{11}c(j,k+1) - \Delta_{11}c(j+1,k) + \Delta_{11}c(j+1,k+1)\\ &= (\Delta a_j)(\Delta b_k) - (\Delta a_j)(\Delta b_{k+1}) - (\Delta a_{j+1})(\Delta b_k) + (\Delta a_{j+1})(\Delta b_{k+1})\\ &= (\Delta a_j)(\Delta b_k - \Delta b_{k+1}) - (\Delta a_{j+1})(\Delta b_k - \Delta b_{k+1})\\ &= (\Delta^2 a_j)(\Delta^2 b_k).\end{aligned}$$

Now, by definition of $\{a_j\}$,

$$\begin{aligned}\Delta a_{j^2+p} &= a_{j^2+p} - a_{j^2+p+1} = \frac{j-|p|}{j^2} - \frac{j-|p+1|}{j^2} = \frac{|p+1|-|p|}{j^2}\\ &= \begin{cases}\dfrac{1}{j^2}, & \text{if } p \ge 0,\\ \dfrac{-1}{j^2}, & \text{if } p \le -1.\end{cases}\end{aligned}$$

Therefore, $|\Delta a_{j^2+p}| = 1/j^2$ and similarly $|\Delta b_{k^2+q}| = 1/k^2$. Next, we have

$$\sum_{j=0}^{\infty} |\Delta a_j| = \sum_{j=1}^{\infty}\sum_{p=-j}^{j-1} |\Delta a_{j^2+p}| = \sum_{j=1}^{\infty}\sum_{p=-j}^{j-1} \frac{1}{j^2} = \sum_{j=1}^{\infty} \frac{2j}{j^2} = \sum_{j=1}^{\infty} \frac{2}{j}$$

and similarly

$$\sum_{k=0}^{\infty} |\Delta b_k| = \sum_{k=1}^{\infty} \frac{2}{k}.$$

Therefore, as $\Delta a_j = 0$ for $j \leq -1$, and $\Delta b_k = 0$ for $k \leq -1$, we have

$$\begin{aligned}
\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{11} c(j,k)| &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |(\Delta a_j)(\Delta b_k)| = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} |(\Delta a_j)(\Delta b_k)| \\
&= \left(\sum_{j=0}^{\infty} |\Delta a_j|\right) \left(\sum_{k=0}^{\infty} |\Delta b_k|\right) \\
&= \left(\sum_{j=1}^{\infty} \frac{2}{j}\right) \left(\sum_{k=1}^{\infty} \frac{2}{k}\right) = \infty,
\end{aligned}$$

which proves that $\{c(j,k)\}$ is not of bounded variation.

But, for $-j \leq p \leq -2$,

$$\Delta^2 a_{j^2+p} = \Delta a_{j^2+p} - \Delta a_{j^2+p+1} = \left(\frac{-1}{j^2}\right) - \left(\frac{-1}{j^2}\right) = 0,$$

for $0 \leq p \leq j-1$,

$$\Delta^2 a_{j^2+p} = \Delta a_{j^2+p} - \Delta a_{j^2+p+1} = \left(\frac{1}{j^2}\right) - \left(\frac{1}{j^2}\right) = 0,$$

and, for $p = -1$ we have

$$\Delta^2 a_{j^2-1} = \Delta a_{j^2-1} - \Delta a_{j^2} = \left(\frac{-1}{j^2}\right) - \left(\frac{1}{j^2}\right) = \frac{-2}{j^2}.$$

Similarly, $\Delta^2 b_{k^2+q} = 0$, for $-k \leq q \leq -2$, for $0 \leq q \leq k-1$, and $\Delta^2 b_{k^2-1} = -2/k^2$. Hence, as $\Delta^2 a_j = 0$, for $j \leq -2$, and $\Delta^2 b_k = 0$, for $k \leq -2$, we have

$$\begin{aligned}
&\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |\Delta_{22} c(j,k)| = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \left|\left(\Delta^2 a_j\right)\left(\Delta^2 b_k\right)\right| = \sum_{j=-1}^{\infty} \sum_{k=-1}^{\infty} \left|\left(\Delta^2 a_j\right)\left(\Delta^2 b_k\right)\right| \\
&= \left|(\Delta^2 a_{-1})(\Delta^2 b_{-1})\right| + \left|(\Delta^2 a_{-1})(\Delta^2 b_0)\right| + \left|(\Delta^2 a_0)(\Delta^2 b_{-1})\right| \\
&\quad + \left(\sum_{j=0}^{\infty} \left|\Delta^2 a_j\right|\right) \left(\sum_{k=0}^{\infty} \left|\Delta^2 b_k\right|\right) \\
&= |1 \cdot 1| + |1 \cdot (-2)| + |(-2) \cdot 1| + \left(\sum_{j=1}^{\infty} \sum_{p=-j}^{j-1} \left|\Delta^2 a_{j^2+p}\right|\right) \left(\sum_{k=1}^{\infty} \sum_{q=-k}^{k-1} \left|\Delta^2 b_{k^2+q}\right|\right) \\
&= 5 + \left(\sum_{j=1}^{\infty} \frac{2}{j^2}\right) \left(\sum_{k=1}^{\infty} \frac{2}{k^2}\right) < \infty.
\end{aligned}$$

So, $\{c(j,k)\}$ satisfies (3.3) for $p=2$. Now, for each fixed $k_0 \in \mathbb{Z}$, we have

$$\sum_{j=0}^{\infty} |\Delta_{20} c(j,k_0)| = \sum_{j=0}^{\infty} \left|\Delta^2 a_j b_{k_0}\right| = \left(\sum_{j=0}^{\infty} |\Delta^2 a_j|\right) |b_{k_0}| = \left(\sum_{j=1}^{\infty} \frac{2}{j^2}\right) |b_{k_0}| < \infty$$

and in view of $|b_{k_0}| \to 0$ as $|k_0| \to \infty$, it follows that $\{c(j,k)\}$ satisfies (3.4) for $p=2$. Similarly, $\{c(j,k)\}$ satisfies (3.5) also for $p=2$. Thus, $\{c(j,k)\}$ is of bounded variation of order 2.

*Example* 3.2. Consider the sequences $\{a_j\}$ and $\{b_k\}$ defined in Example 3.1. Let $\{a'_j\}$ and $\{b'_k\}$ be sequences defined on $\mathbb{Z}$ such that $a'_0 = 0$, $b'_0 = 0$, and $\Delta a'_j = a_j$ and $\Delta b'_k = b_k$, for $j, k \in \mathbb{Z}$. These sequences $\{a'_j\}$ and $\{b'_k\}$ can be constructed as follows. By our definition, we have $a_i = 0$, if $i \le -1$, and the elements $a_0, a_1, a_2, \ldots$ are:

$$\frac{0}{1}, \frac{1}{1}; \frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{1}{4}; \frac{0}{9}, \frac{1}{9}, \frac{2}{9}, \frac{3}{9}, \frac{2}{9}, \frac{1}{9}, \ldots.$$

In view of $0 = a_0 = \Delta a'_0 = a'_0 - a'_1$ and $a'_0 = 0$, we calculate $a'_1 = 0$. Then, from $1 = a_1 = \Delta a'_1 = a'_1 - a'_2$ and from $a'_1 = 0$, we calculate $a'_2 = -1$. Similarly, we can calculate all other elements of $\{a'_j\}$ and $\{b'_k\}$.

Now, we put $c'(j,k) = a'_j b'_k$, for $j, k \in \mathbb{Z}$. Then, as in Example 3.1, we can easily see that $\{c'(j,k)\}$ is of bounded variation of order 3 but not of bounded variation of order 2.

Continuing in this way, for each $p \in \mathbb{N}$, we can construct a sequence of bounded variation of order $p+1$ which not of bounded variation of order $p$.

This shows that $(\mathfrak{BV})^p$ is a proper subset of $(\mathfrak{BV})^{p+1}$ for each $p \in \mathbb{N}$.

**Acknowledgements.** This research was completed while the first author was under a visit of Bolyai Institute, University of Szeged, Szeged, Hungary, under the Hungarian State Scholarship Grant Award during the academic year 2016–2017 between May 16, 2017 to June 15, 2017. For this visit, the travel expense was covered by the Financial Assistance under U.G.C. “Travel Grant for the year 2017-2018”, The Maharaja Sayajirao University of Baroda, Vadodara, Gujarat, India.

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KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 44(4) (2020), PAGES 571–579.

# SOME GRÜSS TYPE INEQUALITIES FOR FRÉCHET DIFFERENTIABLE MAPPINGS

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ABSTRACT. Let $X$ be a Hilbert $C^*$-module on $C^*$-algebra $A$ and $p \in A$. We denote by $D_p(A, X)$ the set of all continuous functions $f : A \to X$, which are Fréchet differentiable on a open neighborhood $U$ of $p$. Then, we introduce some generalized semi-inner products on $D_p(A, X)$, and using them some Grüss type inequalities in semi-inner product $C^*$-module $D_p(A, X)$ and $D_p(A, X^n)$ are established.

## 1. INTRODUCTION

Let $A, X$ be two normed vector spaces over $\mathbb{K}(\mathbb{K} = \mathbb{C}, \mathbb{R})$, we recall that a function $f : A \to X$ is Fréchet differentiable in $p \in A$, if there exists a bounded linear mapping $u : A \to X$ such that

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - u(h)\|_X}{\|h\|_A} = 0,$$

and in this case, we denote $u$ by $Df(p)$. Let $D_p(A, X)$ denotes the set of all continuous functions $f : A \to X$, which are Fréchet differentiable on a open neighborhood (say $U$) of $p$. The main purpose of differential calculus consists in getting some information using an affine approximation to a given nonlinear map around a given point. In many applications it is important to have Fréchet derivatives of $f$, since they provide genuine local linear approximation to $f$. For instance, let $U$ be an open subset of $A$ containing the segment $[x, y] = \{(1-\theta)x + \theta y : 0 \le \theta \le 1\}$, and let $f : A \to X$ be Fréchet differentiable on $U$, then the following mean value formula holds

$$\|f(x) - f(y) \le \|x - y\| \sup_{0<\theta<1} \|Df((1-\theta)x + \theta y)\| .$$

*Key words and phrases.* Fréchet differentiable mappings, $C^*$-modules, Grüss inequality.
2010 *Mathematics Subject Classification.* Primary: 26D10. Secondary: 46C05, 46L08.
DOI 10.46793/KgJMat2004.571T
*Received*: March 07, 2018.
*Accepted*: July 02, 2018.

For two Lebesgue integrable functions $f, g : [a, b] \to \mathbb{R}$, consider the Čebyšev functional:

$$T(f,g) := \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \frac{1}{b-a}\int_a^b g(t)dt.$$

In 1934, G. Grüss [4] showed that

$$|T(f,g)| \leq \frac{1}{4}(M-m)(N-n), \tag{1.1}$$

provided $m, M, n, N$ are real numbers with the property $-\infty < m \leq f \leq M < \infty$ and $-\infty < n \leq g \leq N < \infty$ a.e. on $[a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller quantity and is achieved for

$$f(x) = g(x) = \operatorname{sgn}\left(x - \frac{a+b}{2}\right).$$

The discrete version of (1.1) states that: if $a \leq a_i \leq A,\ b \leq b_i \leq B,\ i = 1, ..., n$, where $a, A, b, B, a_i, b_i$ are real numbers, then

$$\left|\frac{1}{n}\sum_{i=1}^n a_i b_i - \frac{1}{n}\sum_{i=1}^n a_i . \frac{1}{n}\sum_{i=1}^n b_i\right| \leq \frac{1}{4}(A-a)(B-b), \tag{1.2}$$

where the constant $\frac{1}{4}$ is the best possible for an arbitrary $n \geq 1$. Some refinements of the discrete version of Grüss inequality (1.2) for inner product spaces are available in [1, 6].

**Theorem 1.1.** ([2, Theorem 2]). *Let $(H; \langle\cdot,\cdot\rangle)$ and $\mathbb{K}$ be as above and $\overline{x} = (x_1, \ldots, x_n) \in H^n$, $\overline{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{K}^n$ and $\overline{p} = (p_1, \ldots, p_n)$ a probability vector. If $x, X \in H$ are such that*

$$Re \langle X - x_i, x_i - x\rangle \geq 0, \quad \textit{for all } i \in \{1, \ldots, n\},$$

*or, equivalently,*

$$\left\|x_i - \frac{x+X}{2}\right\| \leq \frac{1}{2}\|X - x\|, \quad \textit{for all } i \in \{1, \ldots, n\},$$

*holds, then the following inequality holds*

$$\begin{aligned}\left\|\sum_{i=1}^n p_i\alpha_i x_i - \sum_{i=1}^n p_i\alpha_i \sum_{i=1}^n p_i x_i\right\| &\leq \frac{1}{2}\|X - x\| \sum_{i=1}^n p_i \left|\alpha_i - \sum_{j=1}^n p_j\alpha_j\right| \\ &\leq \frac{1}{2}\|X - x\| \left[\sum_{i=1}^n p_i|\alpha_i|^2 - \left|\sum_{i=1}^n p_i\alpha_i\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

*The constant $\frac{1}{2}$ in the first and second inequalities is the best possible.*

In recent years several refinements and generalizations have been considered for the Grüss inequality. We would like to refer the reader to [2–6, 8, 9] and references therein for more information.

In this paper, for every Hilbert $C^*$-module $X$ over a $C^*$-algebra $A$, some Grüss type inequalities in semi-inner product $C^*$-module $D_p(A, X^n)$ are established. We also for two arbitrary Banach $*$-algebras, define a norm and an involution map on $D_p(A, B)$ and prove that $D_p(A, B)$ is a Banach $*$-algebra.

## 2. Grüss Type Inequalities for Differentiable Mappings

Let $A$ be a $C^*$-algebra. A semi-inner product module over $A$ is a right module $X$ over $A$ together with a generalized semi-inner product, that is with a mapping $\langle ., . \rangle$ on $X \times X$, which is $A$-valued and has the following properties:

(i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in X$;
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for $x, y \in X, a \in A$;
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in X$;
(iv) $\langle x, x \rangle \geq 0$ for $x \in X$.

We will say that $X$ is a semi-inner product $C^*$-module. If, in addition,

(v) $\langle x, x \rangle = 0$ implies $x = 0$,

then $\langle \cdot, \cdot \rangle$ is called a generalized inner product and $X$ is called an inner product module over $A$ or an inner product $C^*$-module. An inner product $C^*$-module which is complete with respect to its norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$, is called a Hilbert $C^*$-module.

As we can see, an inner product module obeys the same axioms as an ordinary inner product space, except that the inner product takes values in a more general structure rather than in the field of complex numbers. If $A$ is a $C^*$-algebra and $X$ is a semi-inner product $A$-module, then the following Schwarz inequality holds:

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle x, x \rangle\| \langle y, y \rangle, \quad x, y \in X$$

(e.g., [7, Proposition 1.1]).

**Theorem 2.1** ([3])**.** *Let $A$ be a $C^*$- Algebra, $X$ a Hilbert $C^*$- module. If $x, y, e \in X$, $\langle e, e \rangle$ is an idempotent in $A$ and $\alpha, \beta, \lambda, \mu$ are complex numbers such that*

$$\left\| x - \frac{\alpha+\beta}{2} e \right\| \leq \frac{1}{2} |\alpha - \beta|, \quad \left\| y - \frac{\lambda+\mu}{2} e \right\| \leq \frac{1}{2} |\lambda - \mu|,$$

*hold, then one has the following inequality:*

$$\|\langle x, y \rangle - \langle x, e \rangle \langle e, y \rangle\| \leq \frac{1}{4} |\alpha - \beta| \, |\lambda - \mu|.$$

*Example* 2.1. Let $A$ be a $C^*$-algebra and $X$ be a semi-inner product $C^*$-module on a $C^*$-algebra $B$. If functions $f, g \in D_p(A, X)$, then function $k : A \to B$ as $k(a) = \langle f(a), g(a) \rangle$ is differentiable in $p \in A$ and derivative of that is a linear mapping $Dk(p) : A \to B$ defined by

$$Dk(p)(a) = \langle Df(p)(a^*), g(p) \rangle + \langle f(p), Dg(p)(a) \rangle.$$

Because

$$\begin{aligned}&\langle f(p+h), g(p+h)\rangle - \langle f(p), g(p)\rangle - \langle Df(p)(h^*), g(p)\rangle - \langle f(p), Dg(p)(h)\rangle \\ =&\langle f(p+h), g(p+h) - g(p) - Dg(p)(h)\rangle\rangle + \langle f(p+h) - f(p), Dg(p)(h)\rangle \\ &+ \langle f(p+h^*) - f(p) - Df(p)(h^*), g(p)\rangle + \langle f(p+h) - f(p+h^*), g(p)\rangle.\end{aligned}$$

Let $A$ be a $C^*$-algebra and $X$ a semi-inner product $A$-module. If $f \in D_p(A, X)$ and $a \in A$, we define the function $f_a : A \to X$ by $f_a(t) = f(t)a$.

**Theorem 2.2.** *Let $X$ be a semi-inner product $C^*$-module on $C^*$-algebra $A$, and $p \in A, e \in X$. If $\langle e, e\rangle$ is an idempotent element in $A$, and $f, g \in D_p(A, X)$, then for every $a \in A$, the map $[\cdot, \cdot]_a : D_p(A, X) \times D_p(A, X) \to A$ with*

$$[f, g]_a := \Big\langle Df(p)(a), Dg(p)(a)\Big\rangle_1 + \Big\langle f(p), g(p)\Big\rangle_1 - D\Big\langle f(\cdot), g(\cdot)\Big\rangle_1(p)(a),$$

*is a generalized semi-inner product on $D_p(A, X)$, where*

$$\langle f(a), g(a)\rangle_1 = \langle f(a), g(a)\rangle - \langle f(a), e\rangle\langle e, g(a)\rangle.$$

*Proof.* First, we show that $f_a \in D_p(A, X)$ and $Df_a(p) = (Df(p))a$. There exists a bounded convex set $V(= B(p, r))$ containing $p$ such that $V \subseteq U$. Let $p, h \in V$, $a \in A$, then

$$\begin{aligned}\|f_a(p+h) - f_a(p) - (Df(p)(h))a\| &= \|[f(p+h) - f(p) - Df(p)(h)]a\| \\ &\le \|f(p+h) - f(p) - Df(p)(h)\|\|a\|.\end{aligned}$$

This implies that $f_a \in D_p(A, X)$.

A simple calculation shows

$$\begin{aligned}[f, g]_a =&\Big\langle Df(p)(a) - f(p), Dg(p)(a) - g(p)\Big\rangle \\ &- \Big\langle Df(p)(a) - f(p), e\rangle\langle e, Dg(p)(a) - g(p)\Big\rangle \\ =&\Big\langle (Df(p)(a) - f(p)) - e\Big\langle e, (Df(p)(a) - f(p))\Big\rangle, \\ &(Dg(p)(a) - g(p)) - e\Big\langle e, (Dg(p)(a) - g(p))\Big\rangle\Big\rangle.\end{aligned}$$

Therefore,

$$\begin{aligned}[f, f]_a =&\Big\langle (Df(p)(a) - f(p)) - e\Big\langle e, (Df(p)(a) - f(p))\Big\rangle, \\ &(Df(p)(a) - f(p)) - e\Big\langle e, (Df(p)(a) - f(p))\Big\rangle\Big\rangle \ge 0.\end{aligned}$$

It is easy to show that $[\cdot, \cdot]_a$ is a generalized semi-inner product on $D_p(A, X)$. $\square$

**Lemma 2.1.** *Let $X$ be a semi-inner product $C^*$-module on $C^*$-algebra $A$, and $p, a \in A, e \in X$. If $\langle e, e\rangle$ is an idempotent element in $A$, $f, g \in D_p(A, X)$ and $\alpha, \beta, \alpha', \beta', \mu,$*

*$\lambda, \mu', \lambda'$ are complex numbers such that*

$$\left\|f(p) - \frac{\alpha+\beta}{2}e\right\| \leq \frac{1}{2}|\alpha-\beta|,$$
$$\left\|Df(p)(a) - \frac{\alpha'+\beta'}{2}e\right\| \leq \frac{1}{2}|\alpha'-\beta'|,$$
$$\left\|g(p) - \frac{\lambda+\mu}{2}e\right\| \leq \frac{1}{2}|\lambda-\mu|,$$
$$\left\|Dg(p)(a) - \frac{\mu'+\lambda'}{2}e\right\| \leq \frac{1}{2}|\mu'-\lambda'|,$$

*then the following inequality holds*

$$\begin{aligned}&\|\langle Df(p)(a), Dg(p)(a)\rangle_1 + \langle f(p), g(p)\rangle_1 - D\Big\langle f(\cdot), g(\cdot)\Big\rangle_1(p)(a)\| \\ \leq&\frac{1}{2}(|\alpha-\beta| + |\alpha'-\beta'|)(|\lambda-\mu| + |\lambda'-\mu'|).\end{aligned}$$

*Proof.* Since $[\cdot,\cdot]_a$ is a generalized semi-inner product on $D_p(A, X)$, the Schwartz inequality holds, i.e,

$$\|[f,g]_a\|^2 \leq \|[f,f]_a\|\ \|[g,g]_a\|.$$

We know that

$$\begin{aligned}\|[f,f]_a\| \leq&\Big\|\langle Df(p)(a), Df(p)(a)\rangle - \langle Df(p)(a), e\rangle\langle e, Df(p)(a)\rangle\Big\| \\ &+ \Big\|\langle f(p), f(p)\rangle - \langle f(p), e\rangle\langle e, f(p)\rangle\Big\| \\ &+ \Big\|\langle Df(p)(a), f(p)\rangle - \langle Df(p)(a), e\rangle\langle e, f(p)\rangle\Big\| \\ &+ \Big\|\langle f(p), Df(p)(a)\rangle - \langle f(p), e\rangle\langle e, Df(p)(a)\rangle\Big\|.\end{aligned}$$

This inequality and Theorem 2.1 imply that

$$\begin{aligned}\|[f,f]_a\| \leq&\frac{1}{4}|\alpha'-\beta'|^2 + \frac{1}{4}|\alpha-\beta|^2 + \frac{1}{2}|\alpha'-\beta'||\alpha-\beta| \\ =&\frac{1}{4}(|\alpha-\beta| + |\alpha'-\beta'|)^2.\end{aligned}$$

Similarly,

$$\begin{aligned}\|[g,g]_a\| \leq&\frac{1}{4}|\lambda'-\mu'|^2 + \frac{1}{4}|\lambda-\mu|^2 + \frac{1}{2}|\lambda'-\mu'||\lambda-\mu| \\ =&\frac{1}{4}(|\lambda-\mu| + |\lambda'-\mu'|)^2.\end{aligned}$$

□

Let $X$ be a semi-inner product $C^*$-module over $C^*$-algebra $A$. For every $x \in X$, we define the map $\hat{x} : A \to X^n$ by $\hat{x}(a) = (xa, \ldots, xa)$, $a \in A$.

**Lemma 2.2.** *Let $X$ be a semi-inner product $C^*$-module, $x_0, y_0, x_1, y_1 \in X$ and $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ a probability vector. If $p \in A$ and $f = (f_1, \ldots, f_n)$, $g = (g_1, \ldots, g_n) \in D_p(A, X^n)$ such that*

$$\left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \leq \left\| \frac{x_0 - y_0}{2} \right\|$$

*and*

$$\left\| Dg(p) - \widehat{\frac{x_1 + y_1}{2}} \right\| \leq \left\| \frac{x_1 - y_1}{2} \right\|,$$

*then for all $a \in A$, we have*

$$\begin{aligned} &\left\| \sum_{i=1}^{n} r_i \left\langle Df_i(p)(a), Dg_i(p)(a) \right\rangle - \left\langle \sum_{i=1}^{n} r_i Df_i(p)(a), \sum_{i=1}^{n} r_i Dg_i(p)(a) \right\rangle \right\| \\ \leq& \frac{1}{4} \|x_0 - y_0\| \|x_1 - y_1\| \|a\|^2. \end{aligned}$$

*Proof.* For every $a \in A$, we define the map $\Big(\cdot, \cdot\Big)_a : D_p(A, X^n) \times D_p(A, X^n) \to A$ with

$$\Big(f, g\Big)_a = \sum_{i=1}^{n} r_i \Big\langle Df_i(p)(a), Dg_i(p)(a) \Big\rangle - \Big\langle \sum_{i=1}^{n} r_i Df_i(p)(a), \sum_{i=1}^{n} r_i Dg_i(p)(a) \Big\rangle.$$

The following Korkine type inequality for differentiable mappings holds:

$$\Big(f, g\Big)_a = \frac{1}{2} \sum_{i=1, j=1}^{n} r_i r_j \Big\langle Df_i(p)(a) - Df_j(p)(a), Dg_i(p)(a) - Dg_j(p)(a) \Big\rangle,$$

Therefore, $\Big(f, f\Big)_a \geq 0$. It is easy to show that $\Big(\cdot, \cdot\Big)_a$ is a generalized semi-inner product on $D_p(A, X^n)$.

A simple calculation shows that

$$\begin{aligned} \Big(f, g\Big)_a =& \sum_{i=1}^{n} r_i \Big\langle Df_i(p)(a) - \frac{x_0 + y_0}{2} a,\ Dg_i(p)(a) - \frac{x_1 + y_1}{2} a \Big\rangle \\ &- \Big\langle \sum_{i=1}^{n} r_i Df_i(p)(a) - \frac{x_0 + y_0}{2} a,\ \sum_{i=1}^{n} r_i Dg_i(p)(a) - \frac{x_1 + y_1}{2} a \Big\rangle. \end{aligned}$$

From Schwartz inequality, we have

$$\begin{aligned} \left\| \Big(f, g\Big)_a \right\|^2 &\leq \sum_{i=1}^{n} r_i \left\| Df_i(p)(a) - \frac{x_0 + y_0}{2} a \right\|^2 \sum_{i=1}^{n} r_i \left\| Dg_i(p)(a) - \frac{x_1 + y_1}{2} a \right\|^2 \\ &\leq \left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\|^2 \left\| Dg(p) - \widehat{\frac{x_1 + y_1}{2}} \right\|^2 \|a\|^4 \\ &\leq \frac{1}{16} \|x_0 - y_0\|^2 \|x_1 - y_1\|^2 \|a\|^4. \end{aligned}$$

□

**Corollary 2.1.** *Let $X$ be a semi-inner product $C^*$-module, $x_0, y_0 \in X$, $(\alpha_1, \ldots, \alpha_n) \in \mathbb{C}^n$ and $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ a probability vector. If $p \in A$ and $f = (f_1, \ldots, f_n) \in D_p(A, X^n)$ such that*

$$\left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \leq \left\| \frac{x_0 - y_0}{2} \right\|,$$

*then for all $a \in A$, we have*

$$\left\| \sum_{i=1}^{n} r_i \alpha_i Df_i(p)(a) - \sum_{i=1}^{n} r_i \alpha_i \sum_{i=1}^{n} r_i Df_i(p)(a) \right\|$$

$$\leq \|a\| \left\| \frac{x_0 - y_0}{2} \right\| \left[ \sum_{i=1}^{n} r_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} r_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \tag{2.1}$$

*Proof.* We have

$$\left\| \sum_{i=1}^{n} r_i \alpha_i Df_i(p)(a) - \sum_{i=1}^{n} r_i \alpha_i \sum_{i=1}^{n} r_i Df_i(p)(a) \right\|$$

$$= \left| \sum_{i=1}^{n} r_i \Big( \alpha_i - \sum_{j=1}^{n} r_j \alpha_j \Big) \right| \left\| Df_i(p)(a) - \frac{x_0 + y_0}{2} a \right\|$$

$$\leq \sum_{i=1}^{n} r_i \left| \alpha_i - \sum_{j=1}^{n} r_j \alpha_j \right| \left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \|a\|$$

$$\leq \|a\| \left\| \frac{x_0 - y_0}{2} \right\| \left[ \sum_{i=1}^{n} r_i |\alpha_i|^2 - \left| \sum_{i=1}^{n} r_i \alpha_i \right|^2 \right]^{\frac{1}{2}}. \qquad \square$$

**Corollary 2.2.** *Let $X$ be a semi-inner product $C^*$-module, $x_0, y_0 \in X$. If $p \in A$ and $f = (f_1, \ldots, f_n) \in D_p(A, X^n)$ such that*

$$\left\| Df(p) - \widehat{\frac{x_0 + y_0}{2}} \right\| \leq \left\| \frac{x_0 - y_0}{2} \right\|,$$

*then for all $a \in A$, we have*

$$\left\| \sum_{k=1}^{n} k Df_k(p)(a) - \frac{n+1}{2} . \sum_{k=1}^{n} Df_k(p)(a) \right\| \leq \frac{\|a\| \|x_0 - y_0\| n}{4} \left[ \frac{(n-1)(n+1)}{3} \right]^{\frac{1}{2}}, \tag{2.2}$$

*and*

$$\left\| \sum_{k=1}^{n} k^2 Df_k(p)(a) - \frac{(n+1)(2n+1)}{6} . \sum_{k=1}^{n} Df_k(p)(a) \right\| \tag{2.3}$$

$$\leq \frac{\|a\| \|x_0 - y_0\| n}{12\sqrt{5}} \sqrt{(n-1)(n+1)(2n+1)(8n+11)}.$$

*Proof.* If we put $ri = \frac{1}{n}$, $\alpha_i = k$ in inequality (2.1), then we get (2.2), and if $ri = \frac{1}{n}, \alpha_i = k^2$ in inequality (2.1), then we get (2.3). $\square$

## 3. Differentiable Mappings on Banach *-Algebras

**Theorem 3.1.** *Let $A, B$ be two Banach $*$-algebras and $p \in A$, then $D_p(A, B)$ is a Banach $*$-algebra with the point-wise operations and the involution $f^*(a) = (f(a))^*$, $a \in A$, and the norm*

$$\|f\| := \max\left\{\sup_{x\in U}\|Df(x)\|,\ \sup_{a\in A}\|f(a)\|\right\} < \infty.$$

*Proof.* First we show that the involution $f \mapsto f^*$ is differentiable and $Df^*(x)(h) = (Df(x)(h^*))^*$, $x, h \in U$. It is trivial that $Df^*(x)$ is a bounded linear map with $\|Df^*(x)\| = \|Df(x)\|$ and

$$\begin{aligned}
&\|f^*(x+h) - f^*(x) - Df^*(x)(h)\| \\
=&\|\big(f(x+h) - f(x) - Df(x)(h^*)\big)^*\| \\
=&\|f(x+h) - f(x) - Df(x)(h^*)\| \\
=&\|f(x+h) - f(x) - Df(x)(h) + Df(x)(h) - Df(x)(h^*)\| \\
\le&\epsilon\|h\| + \|Df(x)(h-h^*)\| \le \epsilon\|h\| + 2\|Df(x)\|\|h\|.
\end{aligned}$$

From $\|Df^*(x)\| = \|Df(x)\|$ and $\|f^*(a)\| = \|f(a)\|$, we obtain

$$\begin{aligned}
\|f^*\| &= \max\left\{\sup_{x\in U}\|Df^*(x)\|,\ \sup_{a\in A}\|f^*(a)\|\right\} \\
&= \max\left\{\sup_{x\in U}\|Df(x)\|,\ \sup_{a\in A}\|f(a)\|\right\} = \|f\|.
\end{aligned}$$

Now, we show that $D_p(A, B)$ is complete. There exists a bounded convex set $V(= B(p, r))$ containing $p$ such that $V \subseteq U$. Suppose that $(f_n)$ is a Cauchy sequence in $D_p(A, B)$, i.e.,

$$\|f_n(a) - f_m(a)\| \to 0, \quad a \in A, \qquad \text{and} \qquad \|Df_n(x) - Df_m(x)\| \to 0, \quad x \in V.$$

Since $B$ is complete, therefore $L(A, B)$ the space of all bounded linear maps from $A$ into $B$, is complete. So, there are functions $f, g$ such that $\sup_{a\in A}\|f_n(a) - f(a)\| \to 0$ and $\sup_{x\in V}\|Df_n(x) - g(x)\| \to 0$. Given $\varepsilon > 0$, we can find $N \in \mathbb{N}$ such that for $m > n \ge N$ one has

$$\|Df_m - Df_n\|_\infty = \sup_{x\in V}\|Df_m(x) - Df_n(x)\| < \frac{\varepsilon}{3},$$

$$\|g - Df_n\|_\infty = \sup_{x\in V}\|g(x) - Df_n(x)\| < \frac{\varepsilon}{3}. \tag{3.1}$$

We may suppose that there exist $a \in A$ such that $p + a \in V$. Using Lipschitzian functions $f_m - f_n$, we obtain that

$$\begin{aligned}
&\|f_m(p+a) - f_m(p) - (f_n(p+a) - f_n(p))\| \\
\le &\sup_{0<\theta<1}\|Df_m(p+\theta a) - Df_n(p+\theta a)\|\|a\| \le \frac{\varepsilon}{3}\|a\|.
\end{aligned}$$

Passing to the limit on $m$, we get

$$\|f(p+a)-f(p)-(f_n(p+a)-f_n(p))\| \leq \frac{\varepsilon}{3}\|a\|. \tag{3.2}$$

Utilizing differentiability $f_N$ and (3.1), we have

$$\begin{aligned}\|f_N(p+a)-f_N(p)-g(p)(a)\| \leq & \|f_N(p+a)-f_N(p)-Df_N(p)(a)\| \\ & + \|Df_N(p)(a)-g(p)(a)\| \leq \frac{\varepsilon}{3}\|a\|+\frac{\varepsilon}{3}\|a\|.\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we obtain

$$\|f(p+a)-f(p)-g(p)(a)\| \leq \varepsilon\|a\|.$$

Therefore, $D_p(A,B)$ is a Banach $*$-algebra. □

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KRAGUJEVAC JOURNAL OF MATHEMATICS
VOLUME 44(4) (2020), PAGES 581–591.

# THE $\overline{\partial}$-CAUCHY PROBLEM ON WEAKLY $q$-CONVEX DOMAINS IN $\mathbb{C}P^n$

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ABSTRACT. Let $D$ be a weakly $q$-convex domain in the complex projective space $\mathbb{C}P^n$. In this paper, the (weighted) $\overline{\partial}$-Cauchy problem with support conditions in $D$ is studied. Specifically, the modified weight function method is used to study the $L^2$ existence theorem for the $\overline{\partial}$-Neumann problem on $D$. The solutions are used to study function theory on weakly $q$-convex domains via the $\overline{\partial}$-Cauchy problem.

## 1. INTRODUCTION AND MAIN RESULTS

The $\overline{\partial}$-problem is one of the important central problems of complex variables. A classical result due to Hörmander tells us that the $\overline{\partial}$-problem is solvable in pseudoconvex domains, and hence, pseudoconvex domains has been widely accepted as the standard domain which we can solve the $\overline{\partial}$-problem. In [16], Ho extend this problem to weakly $q$-convex domains. In fact, Ho is the first person to study the $\overline{\partial}$-problem in $q$-convex domains in $\mathbb{C}^n$. This paper is devoted to studying the $L^2$ $\overline{\partial}$ Cauchy problem and the $\overline{\partial}$-closed extension problem for forms on a weakly $q$-convex domain $D$ in the complex projective space $\mathbb{C}P^n$. These problems were first studied by Kohn and Rossi [20] (see also [12]). They proved the holomorphic extension of smooth $CR$ functions and the $\overline{\partial}$-closed extension of smooth forms from the boundary $bD$ of a strongly pseudoconvex domain to the whole domain $D$. The $L^2$ theory of these problems has been obtained for pseudoconvex domains in $\mathbb{C}^n$ or, more generally, for domains in complex manifolds with strongly plurisubharmonic weight functions (see Chapter 9 in [6] and the references therein). The $L^2$ $\overline{\partial}$ Cauchy problem was considered by Derridj [8,9]. In [30,31] Shaw has obtained a solution to this problem on a pseudoconvex domain with $C^1$ boundary in $\mathbb{C}^n$. Also, in the setting of strictly

*Key words and phrases.* $\overline{\partial}$, $\overline{\partial}$-Neumann operator, $q$-convex domains.
2010 *Mathematics Subject Classification.* Primary: 32F10. Secondary: 32W05.
DOI 10.46793/KgJMat2004.581S
*Received*: April 23, 2017.
*Accepted*: July 06, 2018.

$q$-convex (or $q$-concave) domains, this problem has been studied by Sambou in his thesis (see [29]). In [1], Abdelkader-Saber studied this problem on pseudoconvex manifolds satisfing property $B$. In [26, 27], Saber studied this problem on a weakly $q$-convex domain with $C^1$-smooth boundary and on a $q$-pseudoconvex domain $D$ in $\mathbb{C}^n$, $1 \leqslant q \leqslant n$, with Lipschitz boundary. Recently, Saber [28] studied this result to a $q$-pseudoconvex domain $D$ in a Stein manifold. On a pseudoconvex domain in $\mathbb{C}P^n$, Cao-Shaw-Wang [4] (cf. also [5]) obtained the $L^2$ existence theorem for the $\overline{\partial}$-Neumann operator $N$ and obtained the (weighted) $L^2$ $\overline{\partial}$ Cauchy-problem on such domains. The aim of this paper is to extend this result to the situation in which the boundaries are assumed weakly $q$-convex domain $D$ in $\mathbb{C}P^n$. Moreover, the solutions are used to study function theory on such domains via the $\overline{\partial}$-Cauchy problem.

## 2. Notation and Preliminaries

Let $(x_0, x_1, \ldots, x_n)$ be a (fixed) homogeneous coordinates of $\mathbb{C}P^n$. If $U_0$ is the open set in $\mathbb{C}P^n$ defined by $x_0 \neq 0$ and if $(z_1, z_2, \ldots, z_n)$, where $z_i = x_i/x_0$, is the homogeneous coordinates of $U_0$, we assume that

$$\omega = \frac{\sum_{i=1}^n |\, dz_i|^2}{1 + \sum_{i=1}^n |z_i|^2} - \frac{|\sum_{i=1}^n z_i \, d\overline{z}_i|^2}{(1 + \sum_{i=1}^n |z_i|^2)^2} \quad \text{on } U_0.$$

The Fubini-Study metric of $\mathbb{C}P^n$ determined by $(x_0, x_1, \ldots, x_n)$. This is well-known standard Kähler metric of $\mathbb{C}P^n$.

Let $D$ be a bounded domain in $\mathbb{C}P^n$ and let $C^\infty_{p,q}(D)$ be the space of complex-valued differential forms of class $C^\infty$ and of type $(p, q)$ on $D$. Denote by $L^2(D)$ the space of square integrable functions on $D$ with respect to the Lebesgue measure in $\mathbb{C}P^n$, $L^2_{p,q}(D)$ the space of $(p, q)$-forms with coefficients in $L^2(D)$ and $L^2_{p,q}(D, \phi)$ the space of $(p, q)$-forms with coefficients in $L^2(\, D)$ with respect to the weighted function $e^{-\phi}$. For $u, v \in L^2_{p,q}(D)$, the inner product $\langle u, v\rangle$ and the norm $\| \, u \, \|$ are denoted by:

$$\langle u, v\rangle = \int_D u \wedge \star \overline{v} \quad \text{and} \quad \| \, u \, \|^2 = \langle u, u\rangle,$$

where $\star$ is the Hodge star operator. Let $\operatorname{dist}(z, bD)$ be the Fubini distance from $z \in D$ to the boundary $bD$ and let $\delta$ be a $C^2$ defining function for $D$ normalized by $|d\delta| = 1$ on $bD$ such that

$$\delta = \delta(z) = \begin{cases} -\operatorname{dist}(z, bD), & \text{if } z \in D, \\ \operatorname{dist}(z, bD), & \text{if } z \in \mathbb{C}P^n \backslash D. \end{cases}$$

Let $\phi_t = -t \log |\delta|$, $t \geqslant 0$, for $u, v \in L^2_{p,q}(D, \phi_t)$, the inner product $\langle u, v\rangle_{\phi_t}$ and the norm $\| \, u \, \|_{\phi_t}$ are denoted by:

$$\langle u, v\rangle_{\phi_t} = \langle u, v\rangle_t = \int_D u \, \wedge \star_{(t)} \overline{v},$$
$$\|u\|^2_{\phi_t} = \| \, u \, \|^2_t = \langle u, u\rangle_t,$$

where $\star_{(t)} = \delta^t \star = \star\, \delta^t$. Since $\phi_t$ is bounded on $\overline{D}$, the two norms $\|\cdot\|$ and $\|\cdot\|_t$ are equivalent. Let $\overline{\partial} : \operatorname{dom} \overline{\partial} \subset L^2_{p,q}(D, \phi_t) \to L^2_{p,q+1}(D, \phi_t)$ be the maximal closure of the Cauchy-Riemann operator and $\overline{\partial}^*_\phi$ be its Hilbert space adjoint. Let $\Box_t = \overline{\partial}\,\overline{\partial}^*_t + \overline{\partial}^*_t \overline{\partial}$ be the Laplace-Beltrami operator, where $\overline{\partial}^*_t = \overline{\partial}^*_{\phi_t}$.

Denote by $\nabla$ the Levi-Civita connection of $\mathbb{C}P^n$ with the standard Fubini-Study metric $\omega$. Let $\{e_i\}$ be an orthonormal basis of vector fields. For any two vector fields $f, g$, the curvature operator of the connection $\nabla$ is denoted by

$$\mathcal{R}(f, g) = \nabla_f \nabla_g - \nabla_g \nabla_f - \nabla_{[f,g]}.$$

By setting $\mathcal{R}_{ijkl} = \omega(\mathcal{R}(e_i, e_j)e_k, e_l)$, the Ricci tensor $\mathcal{R}_{ij}$ is denoted by

$$\mathcal{R}_{ij} = \sum_k \varepsilon_k \mathcal{R}_{ikkj},$$

which turns out to be self-adjoint with respect to $\omega$ and the scalar curvature

$$\Theta = \sum_i \mathcal{R}_{ii} = \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}_{jiij} \tag{2.1}$$

as the trace of the Ricci tensor.

**Definition 2.1.** Let $D$ be an open set in an $n$-dimensional complex manifold $X$, let $k$ be an integer with $1 \le k \le n-1$ and put $E = X \backslash D$. The set $D$ is said to be pseudoconvex of order $k$ in $X$ if, for every $b \in E$ and for every coordinate neighborhood $(U, (z_1, \ldots, z_n))$ which contains $b$ as the origin, the set

$$\left\{ (z_1, \ldots, z_n) \in U : \; z_i = 0,\; 1 \le i \le k,\; 0 < \sum_{i=k+1}^{n} |z_i|^2 < t \right\}$$

contains no points of $E$ for some $t > 0$, then there exists $\ell > 0$ such that for each $(z'_1, \ldots, z'_k)$ with $|z'_i| < \ell$, $1 \le i \le k$, the set

$$\left\{ (z_1, \ldots, z_n) \in U : \; z_i = z'_i,\; 1 \le i \le k,\; \sum_{i=k+1}^{n} |z_i|^2 < t \right\}$$

contains at least one point of $E$.

**Definition 2.2.** Let $D$ be an $n$-dimensional complex manifold and let $q$ be an integer, $1 \le q \le n$. By Fujita ([13], Proposition 8) a $C^2$ function $\phi : D \to \mathbb{R}$ is pseudoconvex of order $n-q$, if and only if its Levi form $\partial\overline{\partial}\phi$ has at least $n-q+1$ non negative eigenvalues at each point of $D$.

**Definition 2.3.** Let $D$ be an open subset of an $n$-dimensional complex manifold $X$. $D$ is said to have $C^2$ boundary in $X$ if for all $z \in bD$ there exist an open neighborhood $U$ of $z$ and a $C^2$ function $\delta : U \to \mathbb{R}$, called a defining function of $D$ at $z$ such that $d\delta(z) \neq 0$ and $D \cap U = \{z \in U : \delta(z) < 0\}$. Following Ho [16], $D$ is said to be a weakly $q$-convex ($q \geqslant 1$) if at every point $x_0 \in bD$ we have

$$\sum_{|K|}' \sum_{j,k} \frac{\partial^2 \delta}{\partial z_j \partial \overline{z}_k} u_{jK} \, \overline{u_{kK}} \geqslant 0, \quad \text{for every } (0,q)\text{-form},$$

where

$$u = \sum_{|J|=q} u_J \, d\overline{z}^J \text{ such that } \sum_{j=1}^{n} \frac{\partial \delta}{\partial z_j} u_{jK} = 0, \quad \text{for all } |K| = q-1.$$

Moreover, $D$ is weakly $q$-convex if and only if for any $z \in bD$ the sum of any $q$ eigenvalues $\delta_{i_1}, \ldots, \delta_{i_q}$, with distinct subscripts, of the Levi-form at $z$ satisfies $\sum_{j=1}^{q} \delta_{i_j} \geqslant 0$ (cf. [15] and Lemma 4.7 in [34]).

**Definition 2.4.** Let $D$ be a smooth domain in $\mathbb{C}^n$, $D$ is said to be a weakly $q$-concave if $\overline{D}^c$ is weakly $q$-convex.

**Lemma 2.1** ([16])**.** *Let $D$ be a smooth domain in $\mathbb{C}^n$ and $\rho$ be its defining function. The following two conditions are equivalent.*

(1) *$D$ is weakly $q$-convex.*
(2) *For any $z \in bD$ the sum of any $q$ eigenvalues $\rho_{i_1}, \ldots, \rho_{i_q}$, with distinct subscripts, of the Levi-form at $z$ satisfies $\sum_{j=1}^{q} \rho_{i_j} \geqslant 0$.*

It follows from Lemma 2.1 that $D$ is weakly $q$-concave if and only if for any $q$ eigenvalues $\rho_{i_1}, \ldots, \rho_{i_q}$ of the Levi-form at $z \in bD$ with distinct subscripts we have $\sum_{j=1}^{q} \rho_{i_j} \leqslant 0$.

*Example* 2.1. Let $D$ be an open subset of an $n$-dimensional complex manifold $X$ and suppose that the boundary $bD$ is a real hypersurface of class $C^2$ in $X$, that is, there exist, for each $z \in bD$, a neighborhood $U$ of $z$ and a $C^2$ function $\rho : U \to \mathbb{R}$ such that $d\rho(z) \neq 0$ and $D \cap U = \{z \in U : \rho(z) < 0\}$. Then $D$ is pseudoconvex of order $n-q$ in $X$, if and only if the Levi form $\partial\overline{\partial}\rho$ has at least $n-q$ non-negative eigenvalues on $T'_z(bD)$ for each defining function $\rho$ of $D$ near $z$, where $T'_z(bD)(\subset T_z(bD))$ is the holomorphic tangent space of the real hypersurface $bD$ at $z$ (cf. [10, 35] called such a subset $D$ a $(q-1)$-pseudoconvex open subset with $C^2$ boundary).

**Theorem 2.1** ([23])**.** *Let $D \Subset \mathbb{C}P^n$ be a pseudoconvex domain of order $n-q$, $1 \leq q \leq n$. Let $d(z, bD)$ be the Fubini distance from $z \in D$ to the boundary $bD$. Then the function $-\log d(z, bD)$ is $(q-1)$-pluirsubharmonic in $D$.*

**Lemma 2.2** ([17], Lemma 2.6)**.** *Let $\phi$ be a real valued function of class $C^2$ defined in an $n$-dimensional complex manifold $D$. Then $\phi$ is $(q-1)$-plurisubharmonic, $1 \leq q \leq n$, in $D$ if and only if $\phi$ is weakly $q$-convex in $D$.*

*Remark* 2.1. Pseudoconvex open sets in the original sense are pseudoconvex of order $n-1$.

*Remark* 2.2. The pseudoconvexity of order $n-q$ of an open subset $D$ in $X$ is a local property of the boundary $bD \subset X$ of $D$. More precisely, $D$ is pseudoconvex of order

$n-q$ in $X$ if, for each $p \in bD$, there exists a neighborhood $U \subset X$ of $p$ such that $D \cap U$ is pseudoconvex of order $n-q$ in $U$.

*Remark* 2.3. If an open set $D$ in an $n$-dimensional complex manifold $X$ is weakly $q$-convex, $1 \leq q \leq n$, then $D$ is pseudoconvex of order $n-q$ in $X$. However, the converse is not valid even if $X = \mathbb{C}^n$ (see [10] and [22]). By Fujita [13], an open subset $D$ of $\mathbb{C}^n$ is pseudoconvex of order $n-q$ in $\mathbb{C}^n$, if and only if $D$ has an exhaustion function which is pseudoconvex of order $n-q$ on $D$. Thus, by the approximation theorem of Bungart [3], an open subset $D$ of $X$ is pseudoconvex of order $n-q$ in $X$, if and only if $D$ is locally $q$-complete with corners in $X$ in the sense of Peternell [24].

**Proposition 2.1** (Bochner-Hörmander-Kohn-Morrey formula)**.** *Let $D$ be a compact domain with $C^2$-smooth boundary $bD$ and $\delta(x) = -d(x, bD)$. Suppose that $\Theta$ is the curvature term defined in* (2.1) *with respect to the Fubini-Study metric $\omega$. Then, for any $u \in C^\infty_{p,q}(\overline{D}) \cap \mathrm{dom}\overline{\partial}^*_\phi$ with $1 \leqslant q \leqslant n-1$, and $\phi \in C^2(\overline{D})$, we have*

$$\overline{\partial} u\|^2_\phi + \| \overline{\partial}^*_\phi u\|^2_\phi = \langle \Theta u, \overline{u}\rangle_\phi + \left\| \frac{\partial u_{IJ}}{\partial \overline{z}^k} \right\|^2_\phi + \langle (i\partial\overline{\partial}\phi) u, \overline{u}\rangle_\phi + \int_{bD} ((i\partial\overline{\partial}\delta)u, \overline{u})\, e^{-\phi} ds. \tag{2.2}$$

*This formula is known* (*cf.* [2, 7, 15, 18, 19, 32, 36]) *for some special cases, although it has not been stated in the literature in the form* (2.2)*. If $u$ has compact support in the interior of $D$, the* (2.2) *was proved in* [2]*, Chapter* 8 *of* [7] *and* (2.12) *of* [36]*. The boundary term had been computed in* [14]*, Chapter* 3 *by combining the Morrey-Kohn technique on the boundary with non-trivial weight function. If one combines the results of* [15] *and* [37] *with the interior formulae discussed above, one can prove that* (2.2) *holds for the general case with a weight function $e^{-\phi}$ and the curvature term. Specially, for $\phi = 0$,* (2.2) *was proved in* [32].

**Proposition 2.2.** *For any $(p, q)$-form $u$ of $D \Subset \mathbb{C}P^n$ with $q \geqslant 1$,*

$$(\Theta u, \overline{u}) = q(2n+1)|u|^2, \quad \text{when } u \text{ is a } (0,q)\text{-form},$$
$$(\Theta u, \overline{u}) = 0, \quad \text{for any } (n,q)\text{-form } u,$$
$$(\Theta u, \overline{u}) \geq 0, \quad \text{when } p \geq 1 \text{ and } u \text{ is a } (p,q)\text{-form}.$$

*The statement for $(0,q)$-forms and $(n,q)$-forms was computed in* [32] *and* [36]*. Also, following Lemma* 3.3 *of Henkin-Iordan* [14] *and its proof showed that the curvature operator $\Theta$ acting on $L^2_{p,q}(D)$ is a non-negative operator.*

## 3. The $\overline{\partial}$-Cauchy Problem on Weakly $q$-Convex Domains

This section is devoted to showing the existence of the $\overline{\partial}$-Neumann operator on a weakly $q$-convex domain $D$ in $\mathbb{C}P^n$, $1 \leqslant q \leqslant n$, and by applying these existence to solve the $\overline{\partial}$ problem with support conditions on $D$. The boundary integral in (2.2) is non-negative for $q \geqslant 1$ by the assumption on $D$. Also, by taking $\phi \equiv 0$ in (2.2) and using Proposition 2.2, we find the fundamental estimate

$$\|u\|^2 \leqslant c\left(\|\overline{\partial}u\|^2 + \|\overline{\partial}^* u\|^2\right).$$

This means that $\Box$ has closed range and $\ker \Box = \{0\}$. Thus, one can establish the $L^2$-existence theorem of the $\overline{\partial}$-Neumann operator $N$.

**Theorem 3.1.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-convex domain with $C^2$ smooth boundary. Then, for each $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, there exists a bounded linear operator $N : L^2_{p,q}(D) \to L^2_{p,q}(D)$ with the following properties:*

(i) $\mathrm{Range} N \subset \mathrm{dom}\,\Box$, $\Box N = N \Box = Id$ *on* $\mathrm{dom}\Box$*;*

(ii) *for* $f \in L^2_{p,q}(D)$*,*

$$f = \overline{\partial}\,\overline{\partial}^* N f \,\oplus\, \overline{\partial}^* \overline{\partial} N f;$$

(iii) $N\overline{\partial} = \overline{\partial} N$ *on* $\mathrm{dom}\overline{\partial}$*,* $1 \leqslant q \leqslant n-1$*;*

(iv) $\overline{\partial}^* N = N\overline{\partial}^*$ *on* $\mathrm{dom}\overline{\partial}^*$*,* $2 \leqslant q \leqslant n$*;*

(v) $N$*,* $\overline{\partial}N$ *and* $\overline{\partial}^* N$ *are bounded linear operators on* $L^2_{p,q}(D)$*.*

Using the duality relations pertaining to the $\overline{\partial}$-Neumann problem, one solve the $L^2$ $\overline{\partial}$ Cauchy problem on weakly $q$-convex domains in $\mathbb{C}P^n$, $1 \leqslant q \leqslant n$. This method was first used by Kohn-Rossi [20] for smooth forms on strongly pseudoconvex domains. More precisely, we prove the following $L^2$ Cauchy problem for $\overline{\partial}$ in $\mathbb{C}P^n$:

**Theorem 3.2.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-convex domain, $1 \leqslant q \leqslant n$ with $C^2$ smooth boundary. Then, for $f \in L^2_{p,q}(\mathbb{C}P^n)$, $\mathrm{supp}\, f \subset \overline{D}$, $1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{C}P^n$, there exists $u \in L^2_{p,q-1}(\mathbb{C}P^n)$, $\mathrm{supp}\, u \subset \overline{D}$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$.*

*Proof.* Let $f \in L^2_{p,q}(\mathbb{C}P^n)$, $\mathrm{supp}\, f \subset \overline{D}$, then $f \in L^2_{p,q}(D)$. From Theorem 3.1, $N_{n-p,n-q}$ exists for $n-q \geqslant 1$. Since $N_{n-p,n-q} = \Box^{-1}_{n-p,n-q}$ on $\mathrm{Range}\,\Box_{n-p,n-q}$ and $\mathrm{Range}\, N_{n-p,n-q} \subset \mathrm{dom}\,\Box_{n-p,n-q}$, then $N_{n-p,n-q} \star \overline{f} \in \mathrm{dom}\,\Box_{n-p,n-q} \subset L^2_{n-p,n-q}(D)$, for $q \leqslant n-1$. Thus, we can define $u \in L^2_{p,q-1}(D)$ by

$$u = -\star\ \overline{\overline{\partial}\, N_{n-p,n-q}\ \star \overline{f}}.$$

Thus $\mathrm{supp}\, u \subset \overline{D}$ and $u$ vanishes on $bD$. Now, we extend $u$ to $\mathbb{C}P^n$ by defining $u = 0$ in $\mathbb{C}P^n \setminus D$. It follows from the same arguments of Theorem 9.1.2 in [6] and Theorem 2.2 in [1] that the form $u$ satisfies the equation $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$. Thus the proof follows. $\Box$

## 4. The Weighted $\overline{\partial}$-Cauchy Problem

In this section, we assume that $D$ is a weakly $q$-convex domain, $1 \leqslant q \leqslant n$, with $C^2$ smooth boundary in $\mathbb{C}P^n$. Also, we will choose $\phi_t = -t \log|\delta|$, $t > 0$ in (2.2), and using Remark 2.3 and by using Proposition 2.2, the inequality (2.2) implies the weighted $L^2$-existence for the $\overline{\partial}$. Also, for $u \in \mathrm{Dom}(\Box_t)$ of degree $q \geqslant 1$ and for $t > 0$, we have

$$\begin{aligned} t\|u\|_t^2 &\leqslant (\|\overline{\partial}u\|_t^2 + \|\overline{\partial}_t^* u\|_t^2) \\ &= \langle \Box_t u, u\rangle_t \\ &\leqslant \|\Box_t f\|_t \|u\|_t, \end{aligned}$$

i.e.,

$$t\|u\|_t \leqslant \|\Box_t u\|_t.$$

Since $\Box_t$ is a linear closed densely defined operator, then, from [15, Theorem 1.1.1], $\mathrm{Range}(\Box_t)$ is closed. Thus, from (1.1.1) in [15] and the fact that $\Box_t$ is self adjoint, we have the Hodge decomposition

$$L^2_{p,q}(D, \phi_t) = \overline{\partial}\,\overline{\partial}_t^* \mathrm{dom}(\Box_t) \oplus \overline{\partial}_t^* \overline{\partial}\, \mathrm{dom}(\Box_t).$$

Since $\Box_t$ is one to one on $\mathrm{dom}(\Box_t)$ from (1.5.3) in [15], then there exists a unique bounded inverse operator

$$N_t : \mathrm{Ran}(\Box_t) \to \mathrm{dom}(\Box_t) \cap (\ker(\Box_t))^{\perp}$$

such that $N_t \Box_t f = f$ on $\mathrm{dom}(\Box_t)$. Therefore, we can establish the existence theorem of the inverse of $\Box_t$ the so called weighted $\overline{\partial}$-Neumann operator $N_t$.

**Theorem 4.1.** *For any $1 \leqslant q \leqslant n$ and $t > 0$, there exists a bounded linear operator $N_t : L^2_{p,q}(D, \phi_t) \to L^2_{p,q}(D, \phi_t)$ satisfies the following properties:*

(i) $\mathrm{Range}\,(N_t) \subset \mathrm{dom}(\Box_t)$, $N_t \Box_t = I$ *on* $\mathrm{dom}(\Box_t)$*;*

(ii) *for $f \in L^2_{p,q}(D, \phi_t)$, we have $u = \overline{\partial}\,\overline{\partial}_t^* N_t f \;\oplus\; \overline{\partial}_t^* \overline{\partial} N_t f$;*

(iii) $\overline{\partial} N_t = N_t \overline{\partial}$, $1 \leqslant q \leqslant n-1$*;*

(iv) $\overline{\partial}^* N_t = N_t\, \overline{\partial}^*$, $2 \leqslant q \leqslant n$*;*

(v) *for all $f \in L^2_{p,q}(D, \phi_t)$, we have the estimates*

$$\begin{aligned} t\|N_t f\|_t &\leqslant \|f\|_t, \\ \sqrt{t}\|\overline{\partial} N_t f\|_t + \sqrt{t}\|\overline{\partial}_t^* N_t f\|_t &\leqslant \|f\|_t; \end{aligned}$$

(vi) *if $\overline{\partial} f = 0$, then $u_t = \overline{\partial}_t^* N_t f$ solves the equation $\overline{\partial} u_t = f$.*

**Theorem 4.2.** *For $f \in L^2_{p,q}(D, \phi_t)$, $1 \leqslant q \leqslant n-1$, $\mathrm{supp}\, f \subset \overline{D}$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{C}P^n$, there exists $u \in L^2_{p,q-1}(D, \phi_t)$, $\mathrm{supp}\, u \subset \overline{D}$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$.*

*Proof.* Following Theorem 4.1, $N_t$ exists for forms in $L^2_{n-p,n-q}(D, \phi_t)$. Thus, one can defines $u_t \in L^2_{p,q-1}(D, \phi_t)$ by

$$u_{(t)} = - \star_{(t)}\; \overline{\overline{\partial}\, N_{n-p,n-q}\;\; \star_{(-t)}\; \overline{f}}. \tag{4.1}$$

Thus $\mathrm{supp}\, u_t \subset \overline{D}$ and $u_t$ vanishes on $bD$. Now, we extend $u_t$ to $\mathbb{C}P^n$ by defining $u_t = 0$ in $\mathbb{C}P^n \setminus D$. We want to prove that the extended form $u_t$ satisfies the equation

$\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. For $\eta \in L^2_{n-p,n-q-1}(D,-\phi_t) \cap \operatorname{dom} \overline{\partial}$, we have

$$\begin{aligned}
\langle \overline{\partial}\,\eta, \star_{(t)} f\rangle_D &= \int_D \overline{\partial}\,\eta \wedge \star_{(-t)}(\star_{(t)} f)\\
&= \int_D \overline{\partial}\,\eta \wedge \star_{(-t)} \star_{(t)}\; f\\
&= (-1)^{p+q} \int_D \overline{\partial}\,\eta \wedge\; f\\
&= (-1)^{p+q} \langle f, \star_{(-t)} \overline{\partial}\,\eta\rangle_D\\
&= (-1)^{p+q} \langle f, \star_{(-t)} \overline{\partial}\,\eta\rangle_{\mathbb{C}P^n},
\end{aligned}$$

because $\operatorname{supp} f \subset \overline{D}$. Since $\vartheta|_D = \overline{\partial}^*|_D$, when $\vartheta$ acts in the distribution sense (see [15]), then we obtain

$$\begin{aligned}
\langle \overline{\partial}\eta, \star_{(t)} f\rangle_D &= \langle f, \vartheta\; \star_{(-t)}\; \eta\rangle_{\mathbb{C}P^n}\\
&= \langle \overline{\partial}\, f, \star_{(-t)} \eta\rangle_{\mathbb{C}P^n}\\
&= 0.
\end{aligned}$$

It follows that $\overline{\partial}^*_t(\star_{(t)} f) = 0$ on $D$. Using Theorem 4.1 (iv), we have

$$\overline{\partial}^*_t N_t(\star_{(t)} f) = N_t \overline{\partial}^*_t(\star_{(t)} f) = 0. \tag{4.2}$$

Thus, from (4.1) and (4.2), one obtains

$$\begin{aligned}
\overline{\partial} u_t &= -\,\overline{\partial\; \star_{-t}\;\; \overline{\partial} N_{n-p,n-q}\;\; \star_t\, \overline{f}}\\
&= (-1)^{p+q+1}\, \overline{\star\star\; \partial\; \star\;\; \overline{\partial} N_{n-p,n-q}\;\; \star\, \overline{f}}\\
&= (-1)^{p+q} \overline{\star \overline{\partial}^* \overline{\partial} N_{n-p,n-q} \star \overline{f}}\\
&= (-1)^{p+q} \overline{\star\, (\overline{\partial}^* \overline{\partial} + \overline{\partial}\, \overline{\partial}^*)\, N_{n-p,n-q} \star \overline{f}}\\
&= (-1)^{p+q} \overline{\star \star \overline{f}}\\
&= f,
\end{aligned}$$

in the distribution sense in $D$. Since $u = 0$ in $\mathbb{C}P^n \backslash D$, then for $u \in L^2_{p,q}(\mathbb{C}P^n) \cap \operatorname{dom} \overline{\partial}^*$, one obtains

$$\begin{aligned}
< u, \overline{\partial}^* u >_{\mathbb{C}P^n} &=< u, \overline{\partial}^* u >_D\\
&=< \star \overline{\partial}^* u, \star_{(-t)} u >_{(t)D}\\
&= (-1)^{p+q} < \overline{\partial} \star u, \star_{(-t)} u >_{(t)D}\\
&= (-1)^{p+q} < \star u, \overline{\partial}^*\, \star_{(-t)}\; u >_{(t)D}\\
&=< \star u, \star_{(-t)} \overline{\partial} u >_{(t)D}\\
&=< f, u >_D\\
&=< f, u >_{\mathbb{C}P^n},
\end{aligned}$$

where the third equality holds since $\star u = (-1)^{q+1}\, \overline{\partial} N_{n-p,n-q} \star\; f \in \operatorname{dom} \overline{\partial}^*$. Thus $\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. □

As in [5], we prove the following results.

**Proposition 4.1.** *Let $D$ be the same as in Theorem 3.1. Put $\Omega = \mathbb{C}P^n \backslash \overline{D}$. Then, for any $f \in W^{1+\varepsilon}_{p,q}(\Omega)$, $\overline{\partial} f = 0$, $0 \le \varepsilon < \frac{1}{2}$, there exists $F \in W^{\varepsilon}_{p,q}(\mathbb{C}P^n)$ such that $F|_\Omega = f$ and $\overline{\partial} F = 0$ in $\mathbb{C}P^n$.*

*Proof.* Since $D$ has $C^2$ smooth boundary, there exists a bounded extension operator from $W^s_{p,q}(\Omega)$ to $W^s_{p,q}(\mathbb{C}P^n)$ for all $s \geqslant 0$ (cf. e.g. [33]). Let $\tilde{f} \in W^{1+\varepsilon}_{p,q}(\mathbb{C}P^n)$ be the extension of $f$ so that $\tilde{f}|_\Omega = f$ with

$$\|\tilde{f}\|_{W^{1+\varepsilon}(\mathbb{C}P^n)} \leqslant C\|f\|_{W^{1+\varepsilon}(\Omega)}.$$

Furthermore, we can choose an extension such that $\overline{\partial}\tilde{f} \in W^\varepsilon(D) \cap L^2(D, \phi_{2\varepsilon})$.

One defines $T\tilde{f}$ by $T\tilde{f} = - \star_{2\varepsilon} \overline{\partial} N_{2\varepsilon}(\star_{-2\varepsilon}\overline{\partial}\tilde{f})$ in $\Omega$. As in Theorem 4.2, $T\tilde{f} \in L^2(D, \phi_{2\varepsilon})$. But for a $C^2$-smooth domain, we have that $T\tilde{f} \in L^2(D, \phi_{2\varepsilon})$ is comparable to $W^\varepsilon(\Omega)$ for $0 \leqslant \varepsilon < \frac{1}{2}$. This gives that $T\tilde{f} \in W^\varepsilon_{p,q}(\Omega)$ and $T\tilde{f}$ satisfies $\overline{\partial} T\tilde{f} = \overline{\partial}\tilde{f}$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside $\Omega$.

Since $0 \leqslant \varepsilon < \frac{1}{2}$, the extension by 0 outside $\Omega$ is a continuous operator from $W^\varepsilon(\Omega)$ to $W^\varepsilon(\mathbb{C}P^n)$ (cf. e.g. [21]). Thus we have $T\tilde{f} \in W^\varepsilon(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & \text{if } z \in \overline{D}, \\ \tilde{f} - T\tilde{f}, & \text{if } z \in \Omega. \end{cases}$$

Then $F \in W^\varepsilon_{p,q}(\mathbb{C}P^n)$ and $F$ is $\overline{\partial}$-closed extension of $f$ to $\mathbb{C}P^n$. □

**Corollary 4.1.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-concave domain, $n \geqslant 2$ with $C^2$ smooth boundary. Then $W^{1+\varepsilon}_{p,0}(D) \cap \ker \overline{\partial} = \{0\}$, $1 \le p \le n$ and $W^{1+\varepsilon}_{0,0}(D) \cap \ker \overline{\partial} = \mathbb{C}$.*

*Proof.* Using Proposition 4.1 for $q = 0$, we have that any holomorphic $(p, 0)$-form on $D$ extends to be a holomorphic $(p, 0)$ in $\mathbb{C}P^n$, which are zero (when $p > 0$) or constants (when $p = 0$). □

**Corollary 4.2.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-concave domain, $n \geqslant 2$ with $C^2$ smooth boundary. Then, for any $f \in W^{1+\varepsilon}_{p,q}(D)$, where $0 \le p \le n$, $1 \leqslant q \leqslant n-2$, $p \neq q$, and $0 \leqslant \varepsilon < \frac{1}{2}$, such that $\overline{\partial} f = 0$ in $D$, there exists $u \in W^{1+\varepsilon}_{p,q-1}(D)$ such that $\overline{\partial} u = f$ in $D$.*

*Proof.* If $p \neq q$, we have that $F = \overline{\partial} u$ for some $U \in W^1_{p,q-1}(\mathbb{C}P^n)$. Let $u = U$ on $D$, we have $u \in W^1_{p,q-1}(D)$ satisfying $\overline{\partial} u = f$ in $D$. □

**Acknowledgements.** The author is grateful to the referee for several helpful remarks and comments.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 593–601.

# JOHNSON PSEUDO-CONTRACTIBILITY AND PSEUDO-AMENABILITY OF $\theta$-LAU PRODUCT

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Abstract. Given Banach algebras $A$ and $B$ and $\theta \in \Delta(B)$. We shall study the Johnson pseudo-contractibility and pseudo-amenability of the $\theta$-Lau product $A \times_\theta B$. We show that if $A \times_\theta B$ is Johnson pseudo-contractible, then both $A$ and $B$ are Johnson pseudo-contractible and $A$ has a bounded approximate identity. In some particular cases, a complete characterization of Johnson pseudo-contractibility of $A \times_\theta B$ is given. Also, we show that pseudo-amenability of $A \times_\theta B$ implies the approximate amenability of $A$ and pseudo-amenability of $B$.

## 1. Introduction

Let $A$ and $B$ be two Banach algebras and $\theta \in \Delta(B)$, where $\Delta(B)$ is the character space of $B$. Then the Banach space $A \times B$ with the product

$$(a, b)(c, d) = (ac + \theta(d)a + \theta(b)c, bd), \qquad a, c \in A,\ b, d \in B,$$

and $\ell^1$-norm becomes a Banach algebra, which is called the $\theta$-Lau product of $A$ and $B$ which is denoted by $A \times_\theta B$. The $\theta$-Lau product was first introduce by A. T. Lau [14] for $F$-algebras. Recently, this product was extended to general Banach algebras by M. Monfared [15] for every Banach algebras $A$ and $B$ and every character $\theta \in \Delta(B)$. One may regard $A$ ($B$) as a closed two sided ideal (Banach subalgebra) of $A \times_\theta B$ by identifying it with $A \times \{0\}$ ($\{0\} \times B$), respectively. Therefore, if there is no ambiguity, we may simply write $a$ ($b$) instead of $(a, 0)$ ($(0, b)$) for every $a \in A$ ($b \in B$), respectively. Monfared studied several properties of $A \times_\theta B$ including semisimpility, Arens regularity, existence of approximate identity and amenability. We recall that the concept of an amenable Banach algebra was introduced by Johnson in 1972. Indeed,

*Key words and phrases.* $\theta$-Lau product, Johnson pseudo-contractibility, pseudo-amenability.
2010 *Mathematics Subject Classification.* Primary: 46H05, 46H20. Secondary: 43A20.
DOI 10.46793/KgJMat2004.593A
*Received*: May 20, 2018.
*Accepted*: July 21, 2018.

a Banach algebra $A$ is called amenable if there is an element $M \in (A \otimes_p A)^{**}$ such that $a \cdot M = M \cdot a$ and $\pi_A^{**}(M)a = a$ for every $a \in A$, where $\pi : A \otimes_p A \to A$ is the product morphism and $A \otimes_p A$ is the projective tensor product of $A$. Motivated by this construction of Johnson, some authors introduce several modifications of this notion by relaxing some conditions in different versions of definitions of amenability. The notion of pseudo-amenability was introduced by F. Ghahramani and Y. Zhang [13]. A Banach algebra $A$ is called pseudo-amenable if there is a net $(m_\alpha) \subseteq A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \to 0$ and $\pi_A(m_\alpha)a \to a$ for every $a \in A$. The concept of approximately amenable Banach algebras was introduced by F. Ghahramani and R. J. Loy in [11], see also [12]. A Banach algebra $A$ is called approximately amenable if there are nets $(M_\alpha) \subseteq A \otimes_p A$, $(F_\alpha) \subseteq A$ and $(G_\alpha) \subseteq A$ such that for every $a \in A$

(i) $a \cdot M_\alpha - M_\alpha \cdot a + F_\alpha \otimes a - a \otimes G_\alpha \to 0$;
(ii) $aF_\alpha \to a$, $G_\alpha a \to a$ and
(iii) $\pi_A(M_\alpha)a - F_\alpha a - G_\alpha a \to 0$.

Recently the second and third authors [19] have defined a new concept related to amenability called Johnson pseudo-contractibility. Indeed, a Banach algebra $A$ is called Johnson pseudo-contractible if there is a not necessarily bounded net $(M_\alpha) \subseteq (A \otimes_p A)^{**}$ such that $a \cdot M_\alpha = M_\alpha \cdot a$ and $\pi_A^{**}(M_\alpha)a - a \to 0$ for every $a \in A$.

In the Section 2 we deal with Johnson pseudo-contractible Banach algebras. We show that if $A \times_\theta B$ is Johnson pseudo-contractible, then $A$ is Johnson pseudo-contractible and has a bounded approximate identity and $B$ is Johnson pseudo-contractible. Moreover, we show that in particular cases, for example when $A$ is Arens regular and weakly sequentially complete or when $A$ is a dual Banach algebra, Johnson pseudo-contractibility of $A \times_\theta B$ is equivalent with amenability of $A$ and Johnson pseudo-contractibility of $B$. Some example are given at the end of the section.

In the Section 3 we focus on pseudo-amenability of $A \times_\theta B$. Pseudo-amenability of $A \times_\theta B$ was studied by E. Ghaderi *et al.* [10]. They showed that pseudo-amenability of $A \times_\theta B$ implies pseudo-amenability of $B$, and implies pseudo-amenability of $A$ whenever $A$ has a bounded approximate identity. We show that the existence of bounded approximate identity in this result is not a necessary condition. Indeed, we show that if $A \times_\theta B$ is pseudo-amenable, then $A$ is approximately amenable and $B$ is pseudo-amenable.

## 2. Johnson Pseudo-Contractibility of $A \times_\theta B$

We state a result from [2] that will be used frequently in this section.

**Theorem 2.1.** *Let $A$ be a Johnson pseudo-contractible Banach algebra with an identity. Then $A$ is amenable.*

**Lemma 2.1.** *Let $A$ be a Johnson pseudo-contractible Banach algebra and let $I$ be a two sided closed ideal of $A$. If $I$ has a bounded approximate identity, then $I$ is Johnson pseudo-contractible.*

*Proof.* By hypothesis there is a net $(M_\alpha) \subseteq (A \otimes_p A)^{**}$ such that $a \cdot M_\alpha = M_\alpha \cdot a$ and $\pi_A^{**}(M_\alpha)a - a \to 0$ for every $a \in A$. Let $(e_\beta)$ be a bounded approximate identity for $I$ and let $E$ be a weak* cluster point of $(e_\beta)$ in $I^{**}$. Then by setting $(N_\alpha) = (E \cdot M_\alpha \cdot E) \subseteq (I \otimes_p I)^{**}$, we have

$$x \cdot N_\alpha = N_\alpha \cdot x,$$

and

$$\pi_I^{**}(N_\alpha)x = \pi_A^{**}(E \cdot M_\alpha \cdot E)x = \pi_A^{**}(M_\alpha)x \to x,$$

for every $x \in I$. It follows that $I$ is Johnson pseudo-contractible. □

**Theorem 2.2.** *Let $A$ and $B$ be two Banach algebras and $\theta \in \Delta(B)$. If $A \times_\theta B$ is Johnson pseudo-contractible, then the following statements hold.*

(a) *$A$ is Johnson pseudo-contractible and has a bounded approximate identity.*
(b) *$B$ is Johnson pseudo-contractible.*

*Proof.* Suppose that $\Phi : (A \times_\theta B) \otimes_p (A \times_\theta B) \to A \times_\theta B$ is the linear map determined by

$$\Phi((a, b) \otimes (c, d)) = \theta(d)(a, b), \qquad a, c \in A,\ b, d \in B.$$

Let $(U_\alpha) \subseteq ((A \times_\theta B) \otimes_p (A \times_\theta B))^{**}$ be such that

$$(a, b) \cdot U_\alpha = U_\alpha \cdot (a, b), \qquad \pi_{A\times_\theta B}^{**}(U_\alpha)(a, b) \to (a, b),$$

for every $a \in A$ and $b \in B$. Then by Goldstine's theorem for every $\alpha$ there exists a net $(u_{\alpha_\beta})$ in $(A \times_\theta B) \otimes_p (A \times_\theta B)$ such that $w^* - \lim_\beta u_{\alpha_\beta} = U_\alpha$. Suppose that $u_{\alpha_\beta} = \sum_{i=1}^{\infty} (a_i^{\alpha_\beta}, b_i^{\alpha_\beta}) \otimes (c_i^{\alpha_\beta}, d_i^{\alpha_\beta})$ for sequences $(a_i^{\alpha_\beta}), (c_i^{\alpha_\beta}) \subseteq A$ and $(b_i^{\alpha_\beta}), (d_i^{\alpha_\beta}) \subseteq B$, where $\sum_{i=1}^{\infty} ||(a_i^{\alpha_\beta}, b_i^{\alpha_\beta})\| \cdot ||(c_i^{\alpha_\beta}, d_i^{\alpha_\beta})|| < \infty$. Note that $\theta$ has an extension $\tilde\theta \in \Delta(B^{**})$ given by $\tilde\theta(F) = F(\theta)$ for every $F \in B^{**}$. Since $\Phi$ and $\theta$ are bounded, $\Phi^{**}$ and $\tilde\theta$ are weak* continuous maps. Now we have

$$\begin{aligned}\langle (0, \tilde\theta), \Phi^{**}(U_\alpha)\rangle &= w^* - \lim_\beta \langle (0, \theta), \Phi(u_{\alpha_\beta})\rangle \\ &= w^* - \lim_\beta \sum_{i=1}^{\infty} \theta(b_i^{\alpha_\beta})\theta(b_i^{\alpha_\beta}) \\ &= w^* - \lim_\beta \langle (0, \theta), \pi_{A\times_\theta B}(u_{\alpha_\beta})\rangle \\ &= \langle (0, \tilde\theta), \pi_{A\times_\theta B}^{**}(U_\alpha)\rangle \to 1.\end{aligned}$$

Set $\Phi^{**}(U_\alpha) = (\phi_\alpha, \psi_\alpha)$, where $\phi_\alpha \in A^{**}$ and $\psi_\alpha \in B^{**}$. We can see that $\tilde\theta(\psi_\alpha) \to 1$. Take $\alpha_0$ such that $\tilde\theta(\psi_{\alpha_0}) \neq 0$, for every $a \in A$ we have

$$a\Phi^{**}(U_{\alpha_0}) = \Phi^{**}(a \cdot U_{\alpha_0}) = \Phi^{**}(U_{\alpha_0} \cdot a) = 0.$$

Also, we have

$$a\Phi^{**}(U_{\alpha_0}) = (a, 0)(\phi_{\alpha_0}, \psi_{\alpha_0}) = (a\phi_{\alpha_0} + \tilde\theta(\psi_{\alpha_0})a, 0).$$

Therefore $a\phi_{\alpha_0} + \tilde{\theta}(\psi_{\alpha_0})a = 0$, so $a(-\tilde{\theta}(\psi_{\alpha_0})^{-1}\phi_{\alpha_0}) = a$, where $-\tilde{\theta}(\psi_{\alpha_0})^{-1}\phi_{\alpha_0} \in A^{**}$. This shows that $A$ has a bounded right approximate identity. A similar argument shows that $A$ has a bounded left approximate identity. It follows that $A$ has a bounded approximate identity. Since $A$ is a two sided closed ideal of $(A\times_\theta B)$ and has a bounded approximate identity, by Lemma 2.1 it is Johnson pseudo-contractible.

It is well known that $(A \times_\theta B)/A \cong B$ and there is a surjective homomorphism from $A \times_\theta B$ onto $(A \times_\theta B)/A$. So, [19, Proposition 2.9] implies Johnson pseudo-contractibility of $B$. $\square$

We remark that the converse of the previous theorem does not hold in general. For example, $A(H)$, the Fourier algebra on the integer Heisenberg group $H$, is Johnson pseudo-contractible and has a bounded approximate identity and $M(H)$, the measure algebra over $H$, is Johnson pseudo-contractible ($H$ is discrete and amenable). But $A(H) \times_\theta M(H)$ is not Johnson pseudo-contractible for every $\theta \in \Delta(M(H))$. Indeed, $A(H) \times_\theta M(H)$ has an identity [15, Proposition 2.3]. If $A(H) \times_\theta M(H)$ is Johnson pseudo-contractible, then, by Theorem 2.1, $A(H) \times_\theta M(H)$ is amenable and [15, page 285] implies the amenability of $A(H)$. It gives a contradiction that $H$ has an abelian subgroup of finite index, see [9, Theorem 2.3].

From [15, page 285] and Theorem 2.1, we have the following corollary.

**Corollary 2.1.** *If $B$ has an identity, then the following statements are equivalent:*

(a) *$A \times_\theta B$ is Johnson pseudo-contractible;*
(b) *$A \times_\theta B$ is amenable;*
(c) *$A$ and $B$ are amenable.*

**Corollary 2.2.** *If $A$ has an identity, then $A \times_\theta B$ is Johnson pseudo-contractible if and only if $A$ is amenable and $B$ is Johnson pseudo-contractible.*

*Proof.* In view of [3] $A\times_\theta B$ is nothing but the $\ell^1$-direct sum $A\oplus B$ with coordinatewise product whenever $A$ has an identity. If $A$ is amenable and $B$ is Johnson pseudo-contractible, then $A \oplus B$ is Johnson pseudo-contractible by [19, Theorem 2.11]. The converse comes immediately from Theorem 2.2 and Theorem 2.1. $\square$

A Banach algebra $A$ is called dual if it is a dual space such that multiplication in $A$ is separately $w^*$-continuous. It is well known that a dual Banach algebra with a bounded approximate identity has an identity [18, Proposition 1.2], so we have the following corollary from Theorem 2.2 and Corollary 2.2.

**Corollary 2.3.** *Let $B$ be a Banach algebra and let $A$ be a dual Banach algebra and $\theta \in \Delta(B)$. Then $A \times_\theta B$ is Johnson pseudo-contractible if and only if $A$ is amenable and $B$ is Johnson pseudo-contractible.*

A Banach algebra $A$ is called Arens regular if the first and the second Arens products on $A^{**}$ coincide. Also, a Banach algebra $A$ is called weakly sequentially complete if every weakly Cauchy sequence in $A$ is weakly convergent.

**Proposition 2.1.** *Suppose that $A$ and $B$ are two Banach algebras and $\theta \in \Delta(B)$. If $A$ is Arens regular and weakly sequentially complete, then $A \times_\theta B$ is Johnson pseudo-contractible if and only if*

(a) *$A$ is amenable and has an identity;*
(b) *$B$ is Johnson pseudo-contractible.*

*Proof.* If $A \times_\theta B$ is Johnson pseudo-contractible, then, by Theorem 2.1, $A$ has a bounded approximate identity. Using Ülger theorem [4, Theorem 2.9.39], $A$ has an identity. Now apply Corollary 2.2. □

It seems that Johnson pseudo-contractibility of $A \times_\theta B$ is related with amenability of $A$. We believe that Corollary 2.2 holds without the assumption that $A$ has an identity. However, it remains as a conjecture. We left it as an open problem in the following questions.

*Question* 1. Does Johnson pseudo-contractibility of $A \times_\theta B$ implies the amenability of $A$?

*Question* 2. Suppose that $A$ is an amenable Banach algebra and $B$ is a Johnson pseudo-contractible Banach algebra and $\theta \in \Delta(B)$. Is $A \times_\theta B$ a Johnson pseudo-contractible Banach algebra?

We finish this section with some examples. First we recall some concepts and notations from semigroup theory. A semigroup $S$ is called regular if for every $s \in S$ there exists an element $t \in S$ such that $sts = s$ and $tst = t$. A semigroup $S$ is an inverse semigroup if for every $s \in S$ there exists a unique element $t \in S$ such that $sts = s$ and $tst = t$. The set of idempotents of a semigroup $S$ is denoted by $E(S)$, which is a partially ordered set with the following order

$$p \leq q \Leftrightarrow p = pq = qp, \qquad p, q \in E(S).$$

For $p \in E(S)$, we set $(p] = \{x : x \leq p\}$. An inverse semigroup $S$ is called uniformly locally finite if $\sup\{|(p]| : p \in E(S)\} < \infty$. It is well known that the discrete semigroup algebra $\ell^1(S)$ is weakly sequentially complete [4, Theorem A.4.4]. Our main reference for semigroup theory is [5].

*Example* 2.1. Suppose that $B$ is a Banach algebra and $\theta \in \Delta(B)$.

(i) Let $S$ be a uniformly locally finite inverse semigroup. Then Johnson pseudo-contractibility of $\ell^1(S) \times_\theta B$ implies that $\ell^1(S)$ is Johnson pseudo-contractible and has a bounded approximate identity. From [16, Proposition 2.1] $E(S)$ must be finite and from [20, Theorem 2.3] every maximal subgroup of $S$ is amenable, in other word $\ell^1(S)$ is amenable, see [7].
(ii) Suppose that $S$ is regular and $\ell^1(S)$ is Arens regular. If $\ell^1(S) \times_\theta B$ is Johnson pseudo-contractible, then, by Proposition 2.1, $\ell^1(S)$ is amenable and has an identity. So, by [7], $E(S)$ is finite. Now [5, Theorem 12.2] implies that $S$ is a unital finite semigroup. Indeed, $\ell^1(S) \times_\theta B$ is Johnson pseudo-contractible if and only if $S$ is a unital finite semigroup and $B$ is Johnson pseudo-contractible.

*Example* 2.2. Using [8, Theorem 3.1] one can see that $M_I(\mathbb{C})$ (the Banach algebra of $I \times I$-matrices over $\mathbb{C}$, with finite $\ell^1$-norm and matrix multiplication) has no bounded approximate identity unless $I$ is finite, but in this case $M_I(\mathbb{C})$ is amenable and has an identity. So, for Banach algebra $B$ and $\theta \in \Delta(B)$, $M_I(\mathbb{C}) \times_\theta B$ is Johnson pseudo-contractible if and only if $I$ is finite and $B$ is Johnson pseudo-contractible.

A linear subspace $S^1(G)$ of $L^1(G)$ is said to be a Segal algebra on $G$ if it satisfies the following conditions:

(i) $S^1(G)$ is dense in $L^1(G)$;
(ii) $S^1(G)$ with a norm $\|\cdot\|_{S^1(G)}$ is a Banach space and $\|f\|_{L^1(G)} \leq \|f\|_{S^1(G)}$ for every $f \in S^1(G)$;
(iii) $S^1(G)$ is left translation invariant (that is, $L_y f \in S^1(G)$ for every $f \in S^1(G)$ and $y \in G$) and the map $y \mapsto L_y(f)$ from $G$ into $S^1(G)$ is continuous, where $L_y(f)(x) = f(y^{-1}x)$;
(iv) $\|L_y(f)\|_{S^1(G)} = \|f\|_{S^1(G)}$, for every $f \in S^1(G)$ and $y \in G$.

*Example* 2.3. Suppose that $B$ is a Banach algebra and $\theta \in \Delta(B)$. Let $S^1(G)$ be a Segal algebra on $G$. If $S^1(G) \times_\theta B$ is Johnson pseudo-contractible, then $S^1(G) = L^1(G)$.

## 3. Pseudo-Amenability of $A \times_\theta B$

*Remark* 3.1. Note that if $U \in (A \times_\theta B) \otimes_p (A \times_\theta B)$, then there are $M \in A \otimes_p A$, $N \in A \otimes_p B$, $L \in B \otimes_p A$ and $H \in B \otimes_p B$ such that

$$U = M + N + L + H$$

and

$$\|U\|_{(A\times_\theta B)\otimes_p(A\times_\theta B)} = \|M\|_{A\otimes_p A} + \|N\|_{A\otimes_p B} + \|L\|_{B\otimes_p A} + \|H\|_{B\otimes_p B}\,.$$

**Theorem 3.1.** *Suppose that $A$ and $B$ are Banach algebras and $\theta \in \Delta(B)$. If $A \times_\theta B$ is pseudo-amenable, then*

(a) *$A$ is approximate amenable and*
(b) *$B$ is pseudo-amenable.*

*Proof.* It is well known that $(A \times_\theta B)/A \cong B$ and there is a surjective homomorphism from $A \times_\theta B$ onto $(A \times_\theta B)/A$. So [13, Proposition 2.2] implies pseudo-amenability of $B$.

By assumption there is a net $(U_\alpha) \subseteq (A \times_\theta B) \otimes_p (A \times_\theta B)$ such that

$$(x, y) \cdot U_\alpha - U_\alpha \cdot (x, y) \to 0, \qquad \pi(U_\alpha)(x, y) \to (x, y),$$

for every $x \in A$, $y \in B$. Particularly for every $x \in A$ we have

$$x \cdot U_\alpha - U_\alpha \cdot x \to 0, \qquad \pi(U_\alpha)x \to x. \tag{3.1}$$

Suppose that $U_\alpha = \sum_{i=1}^{\infty}(a_i^\alpha, b_i^\alpha) \otimes (c_i^\alpha, d_i^\alpha)$ for sequences $(a_i^\alpha), (c_i^\alpha) \subseteq A$ and $(b_i^\alpha), (d_i^\alpha) \subseteq B$, where $\sum_{i=1}^{\infty} ||(a_i^\alpha, b_i^\alpha)|| \cdot ||(c_i^\alpha, d_i^\alpha)|| < \infty$. Set $M_\alpha = \sum_{i=1}^{\infty} a_i^\alpha \otimes c_i^\alpha$, $F_\alpha = -\sum_{i=1}^{\infty} \theta(d_i^\alpha) a_i^\alpha$,

$G_\alpha = -\sum_{i=1}^{\infty} \theta(b_i^\alpha)c_i^\alpha$ and $H_\alpha = \sum_{i=1}^{\infty} b_i^\alpha \otimes d_i^\alpha$. One can easily see that

$$\pi_{A\times_\theta B}(U_\alpha) = (\pi_A(M_\alpha) - F_\alpha - G_\alpha, \pi_B(H_\alpha)).$$

For an arbitrary element $b$ in $B$, we have

$$\pi_{A\times_\theta B}(U_\alpha)(0,b) = (\theta(b)(\pi_A(M_\alpha) - F_\alpha - G_\alpha), \pi_B(H_\alpha)b) \to (0,b),$$

so

$$\pi_A(M_\alpha) - F_\alpha - G_\alpha \to 0, \qquad \theta(\pi_B(H_\alpha)) \to 1.$$

Note that

(3.2)

$$\begin{aligned} x \cdot U_\alpha =& \sum_{i=1}^{\infty}(x,0)(a_i^\alpha,0) \otimes (c_i^\alpha,0) + \sum_{i=1}^{\infty}(x,0)(0,b_i^\alpha) \otimes (c_i^\alpha,0) \\ &+ \sum_{i=1}^{\infty}(x,0)(a_i^\alpha,0) \otimes (0,d_i^\alpha) + \sum_{i=1}^{\infty}(x,0)(0,b_i^\alpha) \otimes (0,d_i^\alpha) \\ =& x \cdot \left(\sum_{i=1}^{\infty}(a_i^\alpha \otimes c_i^\alpha)\right) + \sum_{i=1}^{\infty}(x \otimes \theta(b_i^\alpha)c_i^\alpha) + \sum_{i=1}^{\infty}(xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty}(\theta(b_i^\alpha)x \otimes d_i^\alpha) \\ =& x \cdot M_\alpha - x \otimes G_\alpha + \sum_{i=1}^{\infty}(xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty}(\theta(b_i^\alpha)x \otimes d_i^\alpha). \end{aligned}$$

Similarly we have

$$U_\alpha \cdot x = M_\alpha \cdot x - F_\alpha \otimes x + \sum_{i=1}^{\infty}(b_i^\alpha \otimes c_i^\alpha x) + \sum_{i=1}^{\infty}(b_i^\alpha \otimes \theta(d_i^\alpha)x). \tag{3.3}$$

From (3.2), (3.3) and (3.1), by using Remark 3.1 we obtain

(a) $x \cdot M_\alpha - M_\alpha \cdot x + F_\alpha \otimes x - x \otimes G_\alpha \to 0$;

(b) $\sum_{i=1}^{\infty}(xa_i^\alpha \otimes d_i^\alpha) + \sum_{i=1}^{\infty}(\theta(b_i^\alpha)x \otimes d_i^\alpha) \to 0$;

(c) $\sum_{i=1}^{\infty}(b_i^\alpha \otimes c_i^\alpha x) + \sum_{i=1}^{\infty}(b_i^\alpha \otimes \theta(d_i^\alpha)x) \to 0$.

Define a bounded linear map $\phi : A \otimes_p B \to A$ by $\phi(a \otimes b) = \theta(b)a$. From (b) we have

$$\begin{aligned} -xF_\alpha + \theta(\pi_B(H_\alpha))x =& x\sum_{i=1}^{\infty}\theta(d_i^\alpha)a_i^\alpha + \sum_{i=1}^{\infty}\theta(b_i^\alpha d_i^\alpha)x \\ =& \phi\left(\sum_{i=1}^{\infty}(xa_i^\alpha \otimes d_i^\alpha\right) + \sum_{i=1}^{\infty}(\theta(b_i^\alpha)x \otimes d_i^\alpha)) \to 0, \end{aligned}$$

now $\theta(\pi_B(H_\alpha)) \to 1$ implies that $xF_\alpha \to x$. Similarly, by using (c) we have $G_\alpha x \to x$. So we find $(M_\alpha) \subseteq A \otimes_p A$, $(F_\alpha) \subseteq A$ and $(G_\alpha) \subseteq A$ such that

(a) $x \cdot M_\alpha - M_\alpha \cdot x + F_\alpha \otimes x - x \otimes G_\alpha \to 0$;

(b) $xF_\alpha \to x, \quad G_\alpha x \to x$;

(c) $\pi_A(M_\alpha)x - F_\alpha x - G_\alpha x \to 0$, for every $x \in A$. It follows that $A$ is approximately amenable. $\square$

*Example* 3.1. Let $S$ be a uniformly locally finite inverse semigroup and let $B$ be a Banach algebra and $\theta \in \Delta(B)$. If $\ell^1(S) \times_\theta B$ is pseudo-amenable, then by Theorem 3.1 $\ell^1(S)$ is approximately amenable. Theorem 4.3 of [17] shows that $\ell^1(S)$ is amenable.

*Example* 3.2. Let $G = SU(2)$ be the $2 \times 2$ unitary group, and suppose that $S^1(G) \neq L^1(G)$ is a Segal algebra on $G$. In [1] Alaghmandan showed that $S^1(G)$ is not approximately amenable. Thus, by Theorem 3.1, $S^1(G) \times_\theta B$ is not pseudo-amenable for every Banach algebra $B$ and $\theta \in \Delta(B)$.

*Example* 3.3. Let $G$ be an infinite abelian compact group and let $B$ be a Banach algebra and $\theta \in \Delta(B)$. We claim that $L^2(G) \times_\theta B$ is not pseudo-amenable. To see this, suppose that $L^2(G) \times_\theta B$ is pseudo-amenable. Then Theorem 3.1 implies that $L^2(G)$ is approximately amenable. But by the Plancherel theorem $L^2(G)$ is isometrically isomorphism to $\ell^2(\hat{G})$, where $\hat{G}$ is the dual group of $G$ and $\ell^2(\hat{G})$ is equipped with the pointwise product. So $\ell^2(\hat{G})$ is approximately amenable which is a contradiction with the main result of [6].

**Acknowledgements.** The authors are grateful to the referee for carefully reading the paper, pointing out a number of misprints and for some helpful comments.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 603–615.

# NEW EXPLICIT BOUNDS ON GRONWALL-BELLMAN-BIHARI-GAMIDOV INTEGRAL INEQUALITIES AND THEIR WEAKLY SINGULAR ANALOGUES WITH APPLICATIONS

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Abstract. In this paper we derive some generalizations of certain Gronwall-Bellman-Bihari-Gamidov type integral inequalities and their weakly singular analogues, which provide explicit bounds on unknown functions. To show the feasibility of the obtained inequalities, two illustrative examples are also introduced.

## 1. Introduction

The integral inequalities which provide explicit bounds on unknown functions have proved to be very useful in the study of qualitative properties of the solutions of differential and integral equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain applications. For example, see in [1–4, 7–11, 14, 15] and the references therein. In particular, Sh. G. Gamidov [6], while studying the boundary value problem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms

$$u(t) \leq c + \int_a^t a(s)u(s)ds + \int_a^b b(s)u(s)ds, \tag{1.1}$$

for $t \in [a, b]$, under some suitable conditions on the functions involved in (1.1). In [12], Pachpatte established more general Gamidov inequalities as follows:

$$u(t) \leq a(t) + \int_a^t b(t, s)u(s)ds + \int_a^b c(s)u(s)ds.$$

*Key words and phrases.* Integral inequality, weakly Singular inequality, explicit bounds, Bihari-Gamidov inequality.

2010 *Mathematics Subject Classification.* 26D10, 26D15, 26D20.

DOI 10.46793/KgJMat2004.603M

*Received*: April 10, 2018.

*Accepted*: August 13, 2018.

On the other hand, Zheng [16] also established a weakly singular version of the Gronwall-Bellman-Gamidov inequality as follows:

$$u(t) \leq c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)u(s)ds.$$

Recently, Kelong Cheng el al. [5] studied the following inequality:

$$\begin{aligned} u^p(t) \leq & a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f\ (s) u^q(s) ds \\ & + c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} n(s) u^r(s) ds, \end{aligned}$$

where $p \geq q \geq 0$, $p \geq r \geq 0$ and $[\alpha_i, \beta_i, \gamma_i]$, $i = 1, 2$, is the ordered parameter group. In this paper, motivated mainly by the work of Kelong Cheng el al. [5], we discuss more general form of nonlinear weakly singular integral inequalities of Gronwall-Bellman-Bihari-Gamidov

$$\begin{aligned} u^p(t) \leq & a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f(s) u^q(s) ds \\ & + c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} n(s) \sqrt[m_1]{g(u^r(s))} ds, \end{aligned}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $\mathbb{R}_0$ with continuous non-increasing first derivative $g'$ on $\mathbb{R}_0$. Our paper is organized as follows. In Section 2 we prepare some tools needed to prove our theorems. Section 3, we discuss some nonlinear Gamidov type integral inequalities and obtain new explicit bounds on these inequalities. Section 4, we give explicit bounds to new nonlinear Gronwall-Bihari-Gamidov integral inequalities with weakly singular integral kernel and in Section 5, we give an examples to show boundedness and uniqueness of solutions of integral equation with weakly singular kernel.

## 2. Preliminaries

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_0 = (0, \infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $I = [0, T]\,(T \geq 0$ is a constant), $C(X, Y)$ denotes the collection of continuous functions from the set $X$ to the set $Y$, $p$, $q$, $r$ are real constants such that $p \neq 0$, $0 \leq q$, $r \leq p$. For convenience, we give some lemmas which will be used in the proof of the main results.

**Lemma 2.1** ([1, page 16]). *Let $q(t)$ and $p(t)$ be continuous functions for $t \geq \alpha$, let $z(t)$ be a differentiable function for $t \geq \alpha$, and suppose*

$$\begin{aligned} & z'(t) \leq p\,(t)\,z\,(t) + q(t), \quad t \geq \alpha, \\ & z(\alpha) \leq z_0. \end{aligned}$$

*Then*

$$z(t) \leq z(\alpha) \exp\left(\int_\alpha^t p(s)ds\right) + \int_\alpha^t q(s) \exp\left(\int_s^t p(\tau)d\tau\right) ds, \quad t \geq \alpha.$$

**Lemma 2.2** ([7])**.** *Assume that* $a \geq 0$, $p \geq q \geq 0$ *and* $p \neq 0$, *then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}},$$

*for any* $K > 0$.

**Lemma 2.3** (Discrete Jensen inequality)**.** *Let* $A_1, A_2, A_3, A_4, \ldots, A_n$ *be nonnegative real numbers and* $r > 1$ *a real number. Then*

$$(A_1 + \cdots + A_n)^r \leq n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r).$$

**Lemma 2.4** ([8])**.** *Let* $\alpha$, $\beta$, $\gamma$ *and* $m$ *be positive constants. Then*

$$\int_0^t (t^\alpha - s^\alpha)^{m(\beta-1)} s^{m(\gamma-1)} ds = \frac{t^\theta}{\alpha} \beta \left[ \frac{m(\gamma-1)+1}{\alpha}, m(\beta-1)+1 \right], \quad t \in \mathbb{R}_+$$

*where*

$$B\left[\zeta, \eta\right] = \int_0^1 s^{\zeta-1}(1-s)^{\eta-1} ds, \quad \operatorname{Re}\zeta > 0, \operatorname{Re}\eta > 0),$$

*is the well-known beta function and*

$$\theta = m\left(\alpha(\beta-1) + \gamma - 1\right) + 1.$$

Assume that for the parameter group $[\alpha_i,\, \beta_i,\, \gamma_i]$

$$\alpha_i \in (0,1]\,, \quad \beta_i \in (0,1), \quad \gamma_i > 1 - \frac{1}{m}, \tag{2.1}$$

such that

$$\frac{1}{m} + \alpha_i\left(\beta_i - 1\right) + \gamma_i - 1 \geq 0, \quad m > 1, i = 1, 2. \tag{2.2}$$

**Definition 2.1** ([13])**.** The Riemann-Liouville fractional integral of order $\alpha$ for a function $f$ is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,$$

provided that such integral exists.

Now we state the main results of this work.

## 3. Main Result

**Lemma 3.1.** *Assume that* $u(t), m(t), l(t), n(t) \in C\left(I, \mathbb{R}_+\right)$ *and* $g : \mathbb{R}_+ \to \mathbb{R}_+$ *is a differentiable increasing function on* $\mathbb{R}_0$ *with continuous non-increasing first derivative* $g'$ *on* $\mathbb{R}_0$. *If*

$$u(t) \leq m(t) + l(t) \int\limits_0^T n(s) g(u(s)) ds, \tag{3.1}$$

*then*

$$u(t) \le m(t) + \frac{l(t)\int_0^T n(s)g(m(s))ds}{1 - \int_0^T g'(m(s))n(s)l(s)ds}, \tag{3.2}$$

*for all $t \in I$, provided that*

$$\int_0^T g'(m(s))n(s)l(s)ds < 1. \tag{3.3}$$

*Proof.* Let

$$\Pi = \int_0^T n(s)g(u(s))ds.$$

Obviously, $\Pi$ is a constant. It follows from (3.1) that

$$u(t) \le m(t) + l(t)\Pi. \tag{3.4}$$

Applying the mean value theorem for the function $g$, then for every $x \ge y > 0$, there exists $c \in ]y, x[$ such that

$$g(x) - g(y) = g'(c)(x - y) \le g'(y)(x - y),$$

which gives

$$g(u(t)) \le g(m(t) + l(t)\Pi) \le g'(m(t))l(t)\Pi + g(m(t)). \tag{3.5}$$

Multiplying both sides of (3.5) by $n(t)$, then integrating the result from 0 to $T$, it yields

$$\int_0^T n(s)g(u(s))ds \le \int_0^T n(s)g(m(s))ds + \Pi \int_0^T g'(m(s))n(s)l(s)ds. \tag{3.6}$$

The inequality (3.6) can be restated as

$$\Pi \le \int_0^T n(s)g(m(s))ds + \Pi \int_0^T g'(m(s))n(s)l(s)ds,$$

that is

$$\Pi \left(1 - \int_0^T g'(m(s))n(s)l(s)ds\right) \le \int_0^T n(s)g(m(s))ds.$$

From (3.3), we observe that

$$\Pi \le \frac{\int_0^T n(s)g(m(s))ds}{1 - \int_0^T g'(m(s))n(s)l(s)ds}. \tag{3.7}$$

Therefore, the desired inequality (3.2) follows from (3.7) and (3.4). □

*Remark* 3.1. If $g(x) = x$, then Lemma 3.1 reduces to [5, Lemma 3].

**Corollary 3.1.** *Suppose that the hypotheses of Lemma* 3.1 *hold. If*

$$u(t) \leq m(t) + l(t) \int_0^T n(s) \arctan(u(s)) ds.$$

*Then*

$$u(t) \leq m(t) + \frac{l(t) \int_0^T n(s) \arctan(m(s)) ds}{1 - \int_0^T \frac{n(s)l(s)}{1+m^2(s)} ds},$$

*for all $t \in I$, provided that*

$$\int_0^T \frac{n(s)l(s)}{1+m^2(s)} ds < 1.$$

*And if*

$$u(t) \leq m(t) + l(t) \int_0^T n(s) \ln(u(s)+1) ds,$$

*then*

$$u(t) \leq m(t) + \frac{l(t) \int_0^T n(s) \ln(m(s)+1) ds}{1 - \int_0^T \frac{n(s)l(s)}{1+m(s)} ds},$$

*for all $t \in I$, provided that*

$$\int_0^T \frac{n(s)l(s)}{1+m(s)} ds < 1.$$

**Theorem 3.1.** *Assume that $u(t), a(t), b(t), c(t), f(t), n(t) \in C(I, \mathbb{R}_+)$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $\mathbb{R}_0$ with continuous non-increasing first derivative $g'$ on $\mathbb{R}_0$. If $u(t)$ satisfies*

$$u^p(t) \leq a(t) + b(t) \int_0^t f(s) u^q(s) ds + c(t) \int_0^T n(s) g(u^r(s)) ds, \tag{3.8}$$

*then, under the condition that*

$$\int_0^T g'(m(s)) n(s) l(s) ds < 1,$$

*the following explicit estimate*

$$u(t) \leq \left( m(t) + \frac{l(t) \int_0^T n(s)g(m(s))ds}{1 - \int_0^T g'(m(s))n(s)l(s)ds} \right)^{\frac{1}{r}}, \tag{3.9}$$

*holds for all* $t \in I$*, where*

$$m(t) = \frac{r}{p} K^{\frac{r-p}{p}} \bar{b}(t) \int_0^t Q(s) \exp\left( \int_s^t P(\tau)d\tau \right) ds + \frac{r}{p} K^{\frac{r-p}{p}} a(t) + \frac{p-r}{p} K^{\frac{r}{p}},$$

$$l(t) = \frac{r}{p} K^{\frac{r-p}{p}} \bar{b}(t) \left( \exp \int_0^t P(s)ds \right),$$

$$\bar{b}(t) = b(t) + c(t), \tag{3.10}$$

*and*

$$P(t) = \frac{q}{p} K^{\frac{q-p}{p}} f(t)\ \bar{b}(t),$$

$$Q(t) = f(t) \left( \frac{q}{p} K^{\frac{q-p}{p}} a(t) + \frac{p-q}{p} K^{\frac{q}{p}} \right). \tag{3.11}$$

*Proof.* The inequality (3.8) can be rewritten as

$$u^p(t) \leq a(t) + (b(t) + c(t)) \left( \int_0^t f(s)u^q(s)ds + \int_0^T n(s)g(u^r(s))ds \right). \tag{3.12}$$

Define a function $z(t)$ by

$$z(t) = \int_0^t f(s)u^q(s)ds + \int_0^T n(s)g(u^r(s))ds. \tag{3.13}$$

Then, from (3.12), we have

$$u^p(t) \leq a(t) + \bar{b}(t)z(t),$$

$$\bar{b}(t) = b(t) + c(t),$$

$$u(t) \leq (a(t) + \bar{b}(t)z(t))^{\frac{1}{p}}. \tag{3.14}$$

Applying Lemma 2.2 to inequality (3.14), for any $K > 0$, we obtain

$$u^r(t) \leq (a(t) + \bar{b}(t)z(t))^{\frac{r}{p}} \leq \frac{r}{p} K^{\frac{r-p}{p}} (a(t) + \bar{b}(t)z(t)) + \frac{p-r}{p} K^{\frac{r}{p}} = w(t),$$

$$u^q(t) \leq (a(t) + \bar{b}(t)z(t))^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{P}} (a(t) + \bar{b}(t)z(t)) + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{3.15}$$

$$z(0) = \int_0^T n(s)g(u^r(s))ds \leq \int_0^T n(s)g(w(s))ds.$$

From (3.13) and (3.15), we get

$$z'(t) \leq f(t)\left(\frac{q}{p}K^{\frac{q-p}{p}}(a(t)+\bar{b}(t)z(t))+\frac{p-q}{p}K^{\frac{q}{p}}\right).$$

Then

$$z'(t) \leq \frac{q}{p}K^{\frac{q-p}{p}}f(t)\bar{b}(t)z(t)+f(t)\left(\frac{q}{p}K^{\frac{q-p}{p}}a(t)+\frac{p-q}{p}K^{\frac{q}{p}}\right), \tag{3.16}$$

the inequality (3.16) can be restated as

$$z'(t) \leq P(t)z(t)+Q(t), \tag{3.17}$$

where $P$ and $Q$ are defined as in (3.11). Applying Lemma 2.1 to the inequality (3.17), we have

$$z(t) \leq z(0)\exp\left(\int_0^t P(s)ds\right)+\int_0^t Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds. \tag{3.18}$$

Substituting (3.15) in (3.18), we get

$$\begin{aligned} z(t) \leq & \int_0^t Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds+ \\ & +\left(\exp\int_0^t P(s)ds\right)\int_0^T n(s)g(w(s))ds. \end{aligned} \tag{3.19}$$

Then we can write the inequality (3.19) in the following form

$$\begin{aligned} w(t) \leq & \frac{r}{p}K^{\frac{r-p}{p}}\bar{b}(t)\int_\alpha^t\!\!\int_0 Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds+\frac{r}{p}K^{\frac{r-p}{p}}a(t)+\frac{p-r}{p}K^{\frac{r}{p}} \\ & +\frac{r}{p}K^{\frac{r-p}{p}}\bar{b}(t)\left(\exp\int_0^t P(s)ds\right)\int_0^T n(s)g(w(s))ds, \end{aligned} \tag{3.20}$$

where $w(t)$ is defined as (3.15). The inequality (3.20) can be restated as

$$w(t) \leq m(t)+l(t)\int_0^T n(s)g(w(s))ds, \tag{3.21}$$

where $m, l$ are defined as in (3.10).

Applying Lemma 3.1 to the inequality (3.21) and using (3.15), we get the required inequality in (3.9). $\square$

*Remark* 3.2. If $g(x) = x$, inequality (3.8) can be reduced to the case discussed by Kelong Cheng el al. [5, Theorem 7].

## 4. Nonlinear Weakly Singular Integral Inequalities

**Theorem 4.1.** *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, $n(t)$ and $g$ be as in Theorem 3.1. Suppose that $u(t) \in C(I, \mathbb{R}_+)$ satisfies*

$$\begin{aligned} u^p(t) \leq & a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f(s) u^q(s) ds \\ & + c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} n(s) \sqrt[m_1]{g(u^r(s))} ds, \end{aligned} \tag{4.1}$$

*if*

$$\int_0^T g'(m(s)) n^{m_1}(s) l(s) ds < 1,$$

*then*

$$u(t) \leq \left( m(t) + \frac{l(t) \int_0^T n^{m_1}(s) g(m(s)) ds}{1 - \int_0^T g'(m(s)) n^{m_1}(s) l(s) ds} \right)^{\frac{1}{r}}, \tag{4.2}$$

*for $t \in I$, where $p \geq q \geq 0$, $p \geq r \geq 0, m_1, m_2, p, q$ and $r$ are constants, such that $\frac{1}{m_1} + \frac{1}{m_2} = 1$, and*

$$\begin{aligned} m(t) = & \frac{r}{pm_1} K^{\frac{\frac{r}{m_1} - p}{p}} \overline{b^*}(t) \int_0^t Q(s) \exp\left(\int_s^t P(\tau) d\tau\right) ds + \\ & + \frac{r}{pm_1} K^{\frac{\frac{r}{m_1} - p}{p}} a^*(t) + \frac{p - \frac{r}{m_1}}{p} K^{\frac{r}{pm_1}}, \\ l(t) = & \frac{r}{pm_1} K^{\frac{\frac{r}{m_1} - p}{p}} \overline{b^*}(t) \exp\left(\int_0^t P(s) ds\right), \\ \overline{b^*}(t) = & b^*(t) + c^*(t), \\ P(t) = & \frac{q}{p} K^{\frac{q-p}{p}} f^{m_1}(t) \overline{b^*}(t), \\ Q(t) = & f^{m_1}(t) \left( \frac{q}{p} K^{\frac{q-p}{p}} a^*(t) + \frac{p-q}{p} K^{\frac{q}{p}} \right). \end{aligned} \tag{4.3}$$

$$\begin{aligned} a^*(t) = & 3^{m_1 - 1} a^{m_1}(t), \\ b^*(t) = & 3^{m_1 - 1} b(t)^{m_1} (M_1 t^{\theta 1})^{\frac{m_1}{m_2}}, \\ c^*(t) = & 3^{m_1 - 1} c(t)^{m_1} (M_2 T^{\theta_2})^{\frac{m_1}{m_2}}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} M_i = & \frac{1}{\alpha_i} B \left[ \frac{m_2(\gamma_i - 1) + 1}{\alpha_i}, m_2(\beta_i - 1) + 1 \right], \\ \theta_i = & m_2 \left[ \alpha_i(\beta_i - 1) + \gamma_i - 1 \right] + 1, \quad i = 1, 2, \end{aligned} \tag{4.5}$$

*where the parameter group $[\alpha_i, \beta_i, \gamma_i]$ satisfies (2.1)-(2.2).*

*Proof.* From assumptions (2.1)-(2.2), using the Hölder inequality with indices $m_1$, $m_2$ to (4.1), we get

$$\begin{aligned} u^p(t) \le & a(t) + b(t)\left(\int_0^t \left(t^{\alpha_1} - s^{\alpha_1}\right)^{m_2(\beta_1-1)} s^{m_2(\gamma_1-1)} ds\right)^{\frac{1}{m_2}} \\ & \times \left(\int_0^t f^{m_1}(s) u^{qm_1}(s) ds\right)^{\frac{1}{m_1}} \\ & + c(t)\left(\int_0^T \left(T^{\alpha_2} - s^{\alpha_2}\right)^{m_2(\beta_2-1)} s^{m_2(\gamma_2-1)} ds\right)^{\frac{1}{m_2}} \\ & \times \left(\int_0^T n^{m_1}(s) g(u^r(s)) ds\right)^{\frac{1}{m_1}}. \end{aligned} \tag{4.6}$$

By using Lemmas 2.3 and 2.4, the inequality (4.6) can be rewritten as

$$\begin{aligned} u^{pm_1}(t) \le & 3^{m_1-1} a^{m_1}(t) \\ & + 3^{m_1-1} b^{m_1}(t) \times \left(\int_0^t \left(t^{\alpha_1} - s^{\alpha_1}\right)^{m_2(\beta_1-1)} s^{m_2(\gamma_1-1)} ds\right)^{\frac{m_1}{m_2}} \\ & \times \left(\int_0^t f^{m_1}(s) u^{qm_1}(s) ds\right) \\ & + 3^{m_1-1} c^{m_1}(t)\left(\int_0^T \left(T^{\alpha_2} - s^{\alpha_2}\right)^{m_2(\beta_2-1)} s^{m_2(\gamma_2-1)} ds\right)^{\frac{m_1}{m_2}} \\ & \times \left(\int_0^T n^{m_1}(s) g(u^r(s)) ds\right) \\ = & 3^{m_1-1} a^{m_1}(t) + 3^{m_1-1} b^{m_1}(t)\left(M_1 t^{\theta_1}\right)^{\frac{m_1}{m_2}} \\ & \times \left(\int_0^t f^{m_1}(s) u^{qm_1}(s) ds\right) \\ & + 3^{m_1-1} c^{m_1}(t)\left(M_2 T^{\theta_2}\right)^{\frac{m_1}{m_2}} \left(\int_0^T n^{m_1}(s) g(u^r(s)) ds\right), \end{aligned}$$

where $M_i, \theta_i$, $i = 1, 2$, are given in (4.5).

Letting $u^{m_1}(t) = w(t)$, we have

$$w^p(t) \le a^*(t) + b^*(t)\int_0^t f^{m_1}(s) w^q(s) ds + c^*(t)\int_0^T n^{m_1}(s) g(w^{r_1}(s)) ds,$$

where $r_1 = \frac{r}{m_1}$, which is similar to inequality (3.8), where $a^*(t)$, $b^*(t)$ and $c^*(t)$ are given in (4.4). An application of Theorem 3.1 to the inequality above gives that

$$w(t) \le \left(m(t) + \frac{l(t)\int_0^T n^{m_1}(s) g(m(s)) ds}{1 - \int_0^T g'(m(s)) n^{m_1}(s) l(s) ds}\right)^{\frac{m_1}{r}},$$

holds for $t \in I$, where $m(t)$ and $l(t)$ are given in (4.3). Since $u^{m_1}(t) = w(t)$, we can get (4.2). $\square$

*Remark* 4.1. If $g(x) = x$, inequality (4.1) can be reduced to the case discussed by Kelong Cheng el al. [5, Theorem 12].

## 5. Applications

In this section, we present applications of the inequalities (4.1) in Theorem 4.1 for studying the boundedness of certain fractional integral equation with the Riemann-Liouville (R-L) fractional operator. Consider the following fractional integral equation:

$$u(t) = a(t) + I_0^\alpha(F(t, u(t))) + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} N(s, u(s)) ds, \tag{5.1}$$

where $0 < \alpha < 1$ and $F, N \in C(R \times R, R)$, $a(t) \in C(I, R_+)$.

**Theorem 5.1.** *Consider the fractional integral equation* (5.1) *and suppose that* $F$ *and* $N$ *satisfy the following conditions*

$$\begin{aligned} |F(t,z)| &\leq f(t) \, |z|^q \,, \\ |N(t,z)| &\leq n(t) \sqrt[m_1]{g(z^r)}, \end{aligned} \tag{5.2}$$

*where* $f, n \in C(I, \mathbb{R}_+)$ *and* $g$ *is defined as in Theorem* 3.1, $m_1 > 1 \geq q$, $r \geq 0$. *Under the condition*

$$\int_0^T g'(m(s)) n^{m_1}(s) l(s) ds < 1,$$

*the following estimate*

$$u(t) \leq \left( m(t) + \frac{l(t) \int_0^T n^{m_1}(s) g(m(s)) ds}{1 - \int_0^T g'(m(s)) n^{m_1}(s) l(s) ds} \right)^{\frac{1}{r}}, \tag{5.3}$$

*holds, where*

$$\begin{aligned} m(t) =& \frac{r}{m_1} K^{\frac{r}{m_1}-1} \overline{b^*}(t) \int_0^t Q(s) \exp\left( \int_s^t P(\tau) d\tau \right) ds + \frac{r}{m_1} K^{\frac{r}{m_1}-1} a^*(t) \\ & \alpha \\ & + \left(1 - \frac{r}{m_1}\right) K^{\frac{r}{m_1}}, \\ l(t) =& \frac{r}{m_1} K^{\frac{r}{m_1}-1} \overline{b^*}(t) \exp\left( \int_0^t P(s) ds \right), \\ \overline{b^*}(t) =& b^*(t) + c^*(t), \\ P(t) =& q K^{q-1} f^{m_1}(t) \overline{b^*}(t), \\ Q(t) =& f^{m_1}(t) (q K^{q-1} a^*(t) + (1-q) K^q) \end{aligned}$$

*and*

$$
\begin{aligned}
a^*(t) &= 3^{m_1-1}a^{m_1}(t),\\
b^*(t) &= \frac{3^{m_1-1}}{\Gamma^{m_1}(\alpha)}(M_1 t^{\theta 1})^{\frac{m_1}{m_2}},\\
c^*(t) &= \frac{3^{m_1-1}}{\Gamma^{m_1}(\alpha)}(M_2 T^{\theta_2})^{\frac{m_1}{m_2}},\\
M_1 &= M_2 = B\left[1, m_2(\alpha-1)+1\right],\\
\theta_1 &= \theta_2 = m_2(\alpha-1)+1.
\end{aligned}
$$

*Proof.* According to Definition 2.1, from (5.1)-(5.2), we have

$$
u(t) = a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(F(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}N(s,u(s))ds,
$$

for $t \in I$. Hence,

$$
\begin{aligned}
|u(t)| \le & a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|(F(s,u(s))\right| ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|N(s,u(s))\right| ds\\
\le & a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\left|u(s)\right|^q ds\\
& + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} n(s)\sqrt[m_1]{g(u^r(s))}ds.
\end{aligned}
$$

Letting $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\beta_1 = \beta_2 = \alpha$, $p = 1, b(t) = \frac{1}{\Gamma(\alpha)}$ and $c(t) = \frac{1}{\Gamma(\alpha)}$, and applying Theorem 4.1, we get the desired estimate in (5.3). □

**Proposition 5.1.** *Assume that the functions $F$ and $N$ in* (5.2) *satisfy the conditions*

$$
\begin{aligned}
&\left|F(t,z) - F(t,\overline{z})\right| \le f(t)\left|z-\overline{z}\right|,\\
&\left|N(t,z)\right| - N(t,\overline{z}) \le n(t)\sqrt[1+\epsilon]{\left|z-\overline{z}\right|},
\end{aligned} \tag{5.4}
$$

*where $f(t)$ and $n(t)$ are defined as in Theorem* 4.1*, $\epsilon > 0$ and $z(t)$ is a solution of* (5.1)*. Then* (5.1) *has at most one solution.*

*Proof.* Let $z(t)$ and $\overline{z}(t)$ be two solutions of (5.1), it is easy to see from (5.4) that

$$
\begin{aligned}
\left|z(t) - \overline{z}(t)\right| \le & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\left|z(s)-\overline{z}(s)\right| ds\\
& + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} n(s)\sqrt[1+\varepsilon]{\left|z(s)-\overline{z}(s)\right|}ds.
\end{aligned}
$$

Letting $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\beta_1 = \beta_2 = \alpha$, $p = q = r = 1$, $m_1 = 1+\epsilon$, $a(t) = 0$, $g(t) = t$ and applying Theorem 4.1, we obtain that

$$|z(t) - \overline{z}(t)| \leq \left( \left(1 - \frac{1}{1+\epsilon}\right) K^{\frac{1}{1+\epsilon}} + \frac{\left(1 - \frac{1}{1+\epsilon}\right) K^{\frac{1}{1+\epsilon}} l(t) \int\limits_0^T n^{1+\epsilon}(s) ds}{1 - \int\limits_0^T n^{1+\epsilon}(s) l(s) ds} \right),$$

letting $\epsilon \to 0$, we obtain the uniqueness of solution of equation (5.1). □

## Acknowledgments

The authors thanks the referees very much for their careful comments and valuable suggestions on this paper.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 617–637.

# EXISTENCE OF RENORMALIZED SOLUTIONS FOR SOME ANISOTROPIC QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we consider a class of anisotropic quasilinear elliptic equations of the type

$$\begin{cases} -\sum_{i=1}^{N} \partial^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,s)$ is a Carathéodory function which satisfies some growth condition. We prove the existence of renormalized solutions for our Dirichlet problem, and some regularity results are concluded.

## 1. Introduction

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the smooth boundary $\partial\Omega$. Zhao et al. have studied in [17] the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{p-2}u = \lambda f(x,u), & \text{in } \Omega, \\ \displaystyle\int_{\partial\Omega} a(x,\nabla u)\cdot n ds = 0, & \\ u = \text{ constant} & \text{on } \partial\Omega, \end{cases}$$

They have proved the existence of weak solutions under some suitable growth assumptions on $f(x,s)$, (see also [2, 7]). In the framework of Sobolev spaces with variable exponents, Fan and Zhang [11] have considered the following nonlinear elliptic problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

*Key words and phrases.* Anisotropic Sobolev spaces, variable exponents, quasilinear elliptic equations, renormalized solutions.

2010 *Mathematics Subject Classification.* Primary: 35J62. Secondary: 35J20.
DOI 10.46793/KgJMat2004.617A
*Received*: June 06, 2018.
*Accepted*: August 21, 2018.

where $\lambda > 0$ and $f(x,s)$ satisfies the growth condition $|f(x,s)| \leq \eta + \theta |s|^{\delta-1}$, where $1 \leq \delta \leq p^-$ and $\eta, \theta$ are two positive constants (we refer also to [6]). In [3], the authors have proved the existence of weak solutions for the quasilinear $p(x)$-elliptic problem

$$-\operatorname{div} a(x,u,\nabla u) = f(x,u,\nabla u),$$

by using the calculus of variations operators method, where $f(x,s,\xi)$ is a Carathéodory function which satisfies some growth condition.

In the framework of anisotropic Sobolev spaces, Di Nardo, Feo and Guibé have studied in [9] the existence of renormalized solutions for some class of nonlinear anisotropic elliptic problems of the type

$$-\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u)|\partial_{x_i} u|^{p_i-2}\partial_{x_i} u) = f - \operatorname{div} g, \quad \text{in } \Omega,$$

with $f \in L^1(\Omega)$ and $g \in \Pi_{i=1}^{N} L^{p_i'}(\Omega)$, the uniqueness of renormalized solution was concluded under some local Lipschitz conditions on the function $a_i(x,s)$ with respect to $s$, (see also [1] and [8]).

The aim of this paper is to study the existence and regularity of renormalized solutions for the anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $(a_i(x,s,\xi))_{i=1,\ldots,N}$ are Carathéodory functions, the right-hand side $f(x,s)$ is a Carathéodory function satisfying only some nonstandard growth condition.

One of our motivations for studying (1.1) comes from these applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed in [14, 15], also in the robotics and space technology (we refer for example to [16]).

One of the difficulties in proving the existence of renormalized solutions stems from the nonstandard growth of the Carathéodory function $f(x,s)$, to overcome the difficulty, we use the regularizing effect of the term $|u|^{s(x)-1}u$ with some special technics.

The rest of this paper is structured as follows. In Section 2 we recall some definitions and results on the anisotropic variable exponent Sobolev spaces. We introduce in Section 3 some assumptions for which our problem has at least one renormalized solution. Section 4 will be devoted to show the existence of renormalized solutions $u$ for the problem (1.1) in the anisotropic Sobolev space with variable exponents, and we will give some regularity results, that is $|u|^{s(x)-1}u \in L^1(\Omega)$.

## 2. Preliminary

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function } p(\cdot) : \Omega \to \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < N\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x)/x \in \Omega\} \text{ and } p^+ = \operatorname{ess\,sup}\{p(x)/x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left| \int_\Omega uv dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{(p')^-} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space, equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [10].

Now, we present the anisotropic variable exponent Sobolev space, used in the study of our quasilinear anisotropic elliptic problem.

Let $p_1(\cdot), p_2(\cdot), \dots, p_N(\cdot)$ be $N$ variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot)) \text{ and } D^i u = \frac{\partial u}{\partial x_i}, \quad \text{for } i = 1, \dots, N,$$

and we define

$$\underline{p}^+ = \max\{p_1^-, \dots, p_N^-\} \text{ and } \underline{p}^- = \min\{p_1^-, \dots, p_N^-\}, \quad \text{then } 1 < \underline{p}^- \leq \underline{p}^+.$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) \text{ and } D^i u \in L^{p_i(\cdot)}(\Omega) \text{ for } i = 1, 2, \dots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}. \tag{2.1}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.1). The space $\left(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\right)$ is a reflexive Banach space (cf. [13]).

*Remark* 2.1. In view of the continuous embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ and the Poincaré type inequality we conclude that the two norms $\|u\|_{1,\vec{p}(\cdot)}$ and $\sum_{i=1}^N \|D^i u\|_{p_i(\cdot)}$ are equivalent in the anisotropic variable exponent Sobolev spaces.

**Lemma 2.1.** *We have the following continuous and compact embeddings.*

- *If $\underline{p}^- < N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for $q \in [\underline{p}^-, \underline{p}^*[$, where $\underline{p}^* = \frac{N\underline{p}^-}{N-\underline{p}^-}$.*
- *If $\underline{p}^- = N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for all $q \in [\underline{p}^-, +\infty[$.*
- *If $\underline{p}^- > N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.*

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}^-}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

**Proposition 2.1.** *The dual of $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is denote by $W^{-1,\vec{p}'(\cdot)}(\Omega)$, where $\vec{p}'(\cdot) = (p_1'(\cdot), \ldots, p_N'(\cdot))$ and $\frac{1}{p_i'(x)} + \frac{1}{p_i(x)} = 1$ (cf. [5] for the constant exponent case). For each $F \in W^{-1,\vec{p}'(\cdot)}(\Omega)$ there exists $F_0 \in (L^{\underline{p}^+}(\Omega))'$ and $F_i \in L^{p_i'(\cdot)}(\Omega)$ for $i = 1, 2, \ldots, N$, such that $F = F_0 - \sum_{i=1}^N D^i F_i$. Moreover, for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have*

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega F_i D^i u dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'(\cdot)} = \inf \left\{ \sum_{i=0}^{N} \|F_i\|_{p_i'(\cdot)} \text{ with } F = F_0 - \sum_{i=1}^{N} D^i F_i \text{ such that } F_0 \in (L^{\underline{p}^+}(\Omega))' \right.$$
$$\left. \text{and } F_i \in L^{p_i'(\cdot)}(\Omega) \right\}.$$

**Definition 2.1.** Let $k > 0$, the truncation function $T_k(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\dfrac{s}{|s|}, & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

**Proposition 2.2.** *Let $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. For any $i \in \{1,\dots,N\}$, there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that*

$$D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \ \text{a.e. } x \in \Omega, \ \text{for all } k > 0,$$

*where $\chi_A$ denotes the characteristic function of a measurable set $A$. The functions $v_i$ are called the weak partial derivatives of $u$ and are still denoted $D^i u$. Moreover, if $u$ belongs to $W_0^{1,1}(\Omega)$, then $v_i$ coincides with the standard distributional derivative of $u$, that is, $v_i = D^i u$.*

## 3. Essential Assumptions

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$. We consider $\vec{p}(\cdot) = (p_1(\cdot),\dots,p_N(\cdot))$ the vector of exponents $p_i(\cdot) \in C_+(\Omega)$ for $i = 1,\dots,N$, and let $q(\cdot), s(\cdot) \in C_+(\Omega)$ where

$$q(x) < \max(s(x), \underline{p}^+ - 1) \text{ a.e. in } \Omega.$$

We consider the Leray-Lions operator $A$ acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$, defined by the formula

$$Au = -\sum_{i=1}^N \partial^i a_i(x,u,\nabla u),$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory function which satisfy the following conditions

$$|a_i(x,s,\xi)| \leq \beta(K_i(x) + |s|^{p_i(x)-1} + |\xi|^{p_i(x)-1}), \quad \text{for any } i = 1,\dots,N, \tag{3.1}$$

$$a_i(x,s,\xi)\xi_i \geq \alpha|\xi_i|^{p_i(x)}, \quad \text{for any } i = 1,\dots,N, \tag{3.2}$$

for all $\xi = (\xi_1,\dots,\xi_N)$ and $\xi' = (\xi'_1,\dots,\xi'_N)$, we have

$$[a_i(x,s,\xi) - a_i(x,s,\xi')](\xi_i - {\xi'}_i) > 0, \quad \text{for } \xi_i \neq \xi'_i, \tag{3.3}$$

for a.e. $x \in \Omega$, all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K_i(x)$ is a positive function lying in $L^{p'_i(\cdot)}(\Omega)$ and $\alpha, \beta > 0$.

As a consequence of (3.2) and the continuity of the function $a_i(x,s,\cdot)$ with respect to $\xi$, we have

$$a_i(x,s,0) = 0.$$

In this paper, we consider the following quasilinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^N \partial^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x,r)| \leq g(x) + |r|^{q(x)} \text{ a.e in } \Omega, \tag{3.5}$$

and $g(\cdot)$ is a measurable positive function in $L^1(\Omega)$.

*Remark* 3.1. The assumption (3.1) is used here to ensure that $a_i(x, u, \nabla u)$ belongs to $L^{p_i'(\cdot)}(\Omega)$. In the other case where $Au = -\sum_{i=1}^{N} \partial^i a_i(x, \nabla u)$, the uniqueness of solution can be concluded under some additional conditions on the Carathéodory function $f(x, s)$.

## 4. Main Results

We begin by recalling some important lemmas useful to prove our main result.

**Lemma 4.1** ([3])**.** *Let $g \in L^{r(\cdot)}(\Omega)$ and $g_n \in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{r(\cdot)} \leq C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. on $\Omega$, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

**Lemma 4.2** ([4])**.** *Assuming that (3.1)-(3.3) hold, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$\begin{aligned}&\int_\Omega (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n - u)dx \\ &+\sum_{i=1}^{N} \int_\Omega (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D^i u_n - D^i u)dx \to 0,\end{aligned}$$

*then $u_n \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ for a subsequence.*

Our objective is to prove the existence of renormalized solutions for the quasilinear anisotropic elliptic problem (3.4).

**Definition 4.1.** A measurable function $u$ is called renormalized solution of the quasilinear elliptic problem (3.4) if $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for any $k > 0$, with $f(x, u) \in L^1(\Omega)$, and

$$\lim_{h\to\infty} \sum_{i=1}^{N} \int_{\{h<|u|\leq h+1\}} a_i(x, u, \nabla u)D^i u dx = 0, \tag{4.1}$$

such that $u$ satisfies the following equality

$$\begin{aligned}&\sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla u)\Big(S'(u)\varphi D^i u + S(u)D^i\varphi\Big)dx + \int_\Omega |u|^{s(x)-1}uS(u)\varphi dx \\ &= \int_\Omega f(x, u)S(u)\varphi dx,\end{aligned}$$

for every $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and for any smooth function $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ with a compact support.

**Theorem 4.1.** *Assuming that the conditions (3.1)–(3.3) and (3.5) hold true, then the quasilinear anisotropic elliptic problem (3.4) has at least one renormalized solution. Moreover, we have*

$$|u|^{s(x)} \in L^1(\Omega).$$

4.1. **Proof of Theorem** 4.1.

*Step 1: approximate problems.* Firstly, we consider the approximate problem

$$\begin{cases} A_n u_n + |T_n(u_n)|^{s(x)-1}T_n(u_n) = f_n(x, T_n(u_n)), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $A_n v = -\sum_{i=1}^{N} \partial^i a_i(x, T_n(v), \nabla v)$ and $f_n(x,r) = T_n(f(x,r))$. Thanks to (3.5), it's clear that

$$|f_n(x,r)| \le n \text{ and } |f_n(x,r)| \le g(x) + |r|^{q(x)}.$$

We consider the operator $G_n : W_0^{1,\vec{p}(\cdot)}(\Omega) \to W^{-1,\vec{p}'(\cdot)}(\Omega)$ by

$$\langle G_n u, v\rangle = \int_\Omega |T_n(u)|^{s(x)-1}T_n(u) v dx - \int_\Omega f_n(x, T_n(u)) v dx,$$

for any $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. In view of the generalized Hölder-type inequality, we have

$$\begin{aligned} |\langle G_n u, v\rangle| &\le \int_\Omega |T_n(u)|^{s(x)}\ |v| dx + \int_\Omega |f_n(x, T_n(u))|\ |v| dx \\ &\le n^{s^+} \int_\Omega |v| dx + n \int_\Omega |v| dx \\ &= (n^{s^+} + n)\|v\|_1 \\ &\le C_1 \|v\|_{1,\vec{p}(\cdot)}. \end{aligned} \tag{4.3}$$

**Lemma 4.3.** *The bounded operator $B_n = A_n + G_n$ acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, $B_n$ is coercive in the following sense:*

$$\frac{\langle B_n v, v\rangle}{\|v\|_{1,\vec{p}(\cdot)}} \to +\infty \ \text{as } \|v\|_{1,\vec{p}(\cdot)} \to \infty, \quad \text{for any } v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

*Proof.* In view of the Hölder's inequality and the growth condition (3.1), it's easy to see that the operator $A_n$ is bounded, and by (4.3) we conclude that $B_n$ is bounded. For the coercivity, we have for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned} \langle B_n u, u\rangle = &\ \langle A_n u, u\rangle + \langle G_n u, u\rangle \\ = &\ \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u), \nabla u)\ D^i u dx + \int_\Omega |T_n(u)|^{s(x)}|u| dx \\ &- \int_\Omega |f_n(x, T_n(u))||u| dx \\ \ge &\ \alpha \sum_{i=1}^{N} \int_\Omega |D^i u|^{p_i(x)} dx + \int_\Omega |T_n(u)|^{s(x)+1} dx - C_2 n \|u\|_{p_0(\cdot)} \\ \ge &\ C_0\ \|u\|_{1,\vec{p}(\cdot)}^{p^-} - \alpha N|\Omega| - C_2 n \|u\|_{1,\vec{p}(\cdot)}, \end{aligned}$$

it follows that

$$\frac{\langle B_n u, u\rangle}{\|u\|_{1,\vec{p}(\cdot)}} \to +\infty \text{ as } \|u\|_{1,\vec{p}(\cdot)} \to \infty.$$

It remains to show that $B_n$ is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} u_k \rightharpoonup u, \quad \text{in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ B_n u_k \rightharpoonup \chi_n, \quad \text{in } W^{-1,\vec{p}'(\cdot)}(\Omega),\\ \limsup\limits_{k\to\infty}\langle B_n u_k, u_k\rangle \le \langle \chi_n, u\rangle. \end{cases} \tag{4.4}$$

We will prove that

$$\chi_n = B_n u \text{ and } \langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \text{ as } k\to\infty.$$

In view of the compact embedding $W_0^{1,\vec{p}(\cdot)}(\Omega)\hookrightarrow\hookrightarrow L^1(\Omega)$, we have $u_k\to u$ in $L^1(\Omega)$ and a.e. $\Omega$, for a subsequence still denoted $(u_k)_{k\in\mathbb{N}}$.

We have $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, using the growth condition (3.1) it's clear that the sequence $(a_i(x,T_n(u_k),\nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\varphi_i\in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x,T_n(u_k),\nabla u_k)\rightharpoonup \varphi_i \text{ in } L^{p_i'(\cdot)}(\Omega) \text{ as } k\to\infty. \tag{4.5}$$

On the one hand we have

$$|T_n(u_k)|^{s(x)-1}T_n(u_k)\to |T_n(u)|^{s(x)-1}T_n(u) \text{ weak}-* \text{ in } L^\infty(\Omega), \tag{4.6}$$

and since $f_n(x,T_n(s))$ is a Carathéodory function, then

$$f_n(x,T_n(u_k))\to f_n(x,T_n(u)) \text{ weak}-* \text{ in } L^\infty(\Omega). \tag{4.7}$$

Then, for any $v\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ we have

(4.8)

$$\begin{aligned}\langle\chi_n,v\rangle = {} & \lim_{k\to\infty}\langle B_n u_k, v\rangle\\ = {} & \lim_{k\to\infty}\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v\,dx + \lim_{k\to\infty}\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)v\,dx\\ & - \lim_{k\to\infty}\int_\Omega f_n(x,T_n(u_k))v\,dx\\ = {} & \sum_{i=1}^N\int_\Omega \varphi_i\ D^i v\,dx + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)v\,dx - \int_\Omega f_n(x,T_n(u))v\,dx.\end{aligned}$$

Having in mind (4.4) and (4.8), we conclude that

$$\begin{aligned}\limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle = {} & \limsup_{k\to\infty}\Big(\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k\,dx\\ & + \int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k\,dx - \int_\Omega f_n(x,T_n(u_k))u_k\,dx\Big)\\ \le {} & \sum_{i=1}^N\int_\Omega\varphi_i\ D^i u\,dx + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u\,dx\\ & - \int_\Omega f_n(x,T_n(u))u\,dx.\end{aligned}$$

Since $u_k \to u$ strongly in $L^1(\Omega)$, and thanks to (4.6)–(4.7) we obtain

$$\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k dx \to \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u dx \tag{4.9}$$

and

$$\int_\Omega f_n(x,T_n(u_k))u_k dx \to \int_\Omega f_n(x,T_n(u))u dx. \tag{4.10}$$

Therefore,

$$\limsup_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k dx \le \sum_{i=1}^N \int_\Omega \varphi_i\ D^i u dx. \tag{4.11}$$

On the other hand, in view of (3.3) we have

$$\sum_{i=1}^N \int_\Omega (a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)dx \ge 0, \tag{4.12}$$

then

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k dx \quad &\ge \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u dx \\ &+ \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u)(D^i u_k - D^i u)dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x,T_n(u_k),\nabla u) \to a_i(x,T_n(u),\nabla u)$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and using (4.5) we get

$$\liminf_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k dx \ge \sum_{i=1}^N \int_\Omega \varphi_i\ D^i u dx.$$

Having in mind (4.11), we conclude that

$$\lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k dx = \sum_{i=1}^N \int_\Omega \varphi_i\ D^i u dx. \tag{4.13}$$

Therefore, by combining (4.8) and (4.9)–(4.10), we conclude that

$$\langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \text{ as } k \to \infty.$$

Now, by (4.13) we can prove that

$$\lim_{k\to\infty} \Big( \sum_{i=1}^N \int_\Omega (a_i(x,T_n(u_k),\nabla u_k) - a_i(x,T_n(u_k),\nabla u))(D^i u_k - D^i u)dx$$

$$+ \int_\Omega (|u_k|^{\underline{p}^+-2}u_k - |u|^{\underline{p}^+-2}u)(u_k - u)dx\Big) = 0,$$

and so, by virtue of Lemma 4.2, we get

$$u_k \to u \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ and } D^i u_k \to D^i u \text{ a.e. in } \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \text{ in } L^{p_i'(\cdot)}(\Omega), \quad \text{for } i = 1, \ldots, N,$$

and thanks to (4.6)–(4.7), we obtain $\chi_n = B_n u$, which conclude the proof of Lemma 4.3. □

In view of Lemma 4.3, there exists at least one weak solution $u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the approximate problem (4.2) (cf. [12], Theorem 2.7, page 180).

*Step 2: a priori estimates.* Choose $1 < \theta < \underline{p}^-$ such that $1 \leq q(x) < \max(s(x), \underline{p}^+ - \theta)$. By taking $\varphi(u_n) = \left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)$ sign $(u_n) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ as a test function in (4.2), we obtain

$$(\theta - 1) \sum_{i=1}^{N} \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) \cdot D^i u_n}{(1 + |u_n|)^\theta} dx + \int_\Omega |T_n(u_n)|^{s(x)} \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) dx$$
$$= \int_\Omega f_n(x, T_n(u_n)) \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) \text{sign } (u_n) dx.$$

By using the coercivity (3.2) and the growth condition (3.5), we obtain

$$\alpha(\theta - 1) \sum_{i=1}^{N} \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^\theta} dx + \int_\Omega |T_n(u_n)|^{s(x)} \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) dx$$
$$\leq \int_\Omega (|g(x)| + |T_n(u_n)|^{q(x)}) \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) dx. \tag{4.14}$$

For the first term on the left hand side of (4.14), for any $i = 1, \ldots, N$, we have

$$\begin{aligned}
\int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^\theta} dx &\geq \int_\Omega \frac{|D^i u_n|^{p_i^-}}{(1 + |u_n|)^\theta} dx - |\Omega| \\
&= \int_\Omega \left| \frac{D^i u_n}{(1 + |u_n|)^{\frac{\theta}{p_i^-}}} \right|^{p_i^-} dx - |\Omega| \\
&= \int_\Omega \left| D^i \int_0^{|u_n|} \frac{ds}{(1 + s)^{\frac{\theta}{p_i^-}}} \right|^{p_i^-} dx - |\Omega| \\
&\geq \frac{1}{C_p} \int_\Omega \left| \int_0^{|u_n|} \frac{ds}{(1 + s)^{\frac{\theta}{p_i^-}}} \right|^{p_i^-} dx - |\Omega| \\
&\geq \frac{1}{C_p} \int_\Omega \frac{|u_n|^{p_i^-}}{(1 + |u_n|)^\theta} dx - |\Omega| \\
&\geq \frac{1}{2^\theta C_p} \int_\Omega |u_n|^{p_i^- - \theta} dx - 2|\Omega|,
\end{aligned}$$

and since $\varphi(u_n) \geq \frac{1}{2}$ for $|u_n| \geq R$, with $R = 2^{\frac{1}{1-\theta}} - 1$. Using Young's inequality it follows that

$$\begin{aligned}\frac{\alpha(\theta-1)}{2^\theta C_p}\sum_{i=1}^N\int_\Omega |u_n|^{p_i^- -\theta}dx + \frac{1}{2}\int_{\{|u_n|\geq R\}}|T_n(u_n)|^{s(x)}dx \\ \leq \int_\Omega |g(x)|dx + \int_\Omega |T_n(u_n)|^{q(x)}dx + 2\alpha N(\theta-1)|\Omega|.\end{aligned} \tag{4.15}$$

Since $1 \leq q(x) < \max(s(x), \underline{p}^+ - \theta)$, by using Young's inequality we conclude that
(4.16)

$$\int_\Omega |T_n(u_n)|^{q(x)}dx \leq \frac{\alpha(\theta-1)}{2^{\theta+1}C_p}\sum_{i=1}^N\int_\Omega |u_n|^{p_i^- -\theta}dx + \frac{1}{4}\int_{\{|u_n|\geq R\}}|T_n(u_n)|^{s(x)}dx + C_0.$$

It follows from (4.15) that there exists a constant $C_1$ that does not depend on $n$, such that

$$\sum_{i=1}^N\int_\Omega |u_n|^{p_i^- -\theta}dx + \int_\Omega |T_n(u_n)|^{s(x)}dx + \int_\Omega |T_n(u_n)|^{q(x)}dx \leq C_1. \tag{4.17}$$

Let $k \geq 1$, in view of (4.14) we conclude that

$$\frac{1}{(1+k)^\theta}\sum_{i=1}^N\int_\Omega |D^iT_k(u_n)|^{p_i(x)}dx \leq \sum_{i=1}^N\int_\Omega \frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^\theta}dx + \int_\Omega |T_n(u_n)|^{s(x)} \leq C_2. \tag{4.18}$$

Therefore, we obtain

$$\sum_{i=1}^N\int_\Omega |D^iT_k(u_n)|^{p_i(x)}dx \leq C_2(1+k)^\theta, \quad \text{for } k \geq 1.$$

Thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ and $\eta_k \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \eta_k \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \to \eta_k \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{4.19}$$

On the other hand, in view of Poincaré type inequality, for any $i \in \{1, \ldots, N\}$ we have

$$\begin{aligned} k^{p_i^-}\ \text{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}}|T_k(u_n)|^{p_i^-}dx \leq \int_\Omega |T_k(u_n)|^{p_i^-}dx \\ &\leq C_p^{p_i^-}\int_\Omega |D^iT_k(u_n)|^{p_i^-}dx \\ &\leq C_p^{p_i^-}\int_\Omega |D^iT_k(u_n)|^{p_i(x)}dx + C_p^{p_i^-}|\Omega| \\ &\leq \max_{1\leq i\leq N}(C_p^{p_i^-})\Big(\sum_{i=1}^N\int_\Omega |D^iT_k(u_n)|^{p_i(x)}dx + |\Omega|\Big) \\ &\leq C_3(1+k)^\theta, \end{aligned}$$

where $C_3$ is a constant that does not depend on $k$ and $n$. Since $1 < \theta < \underline{p}^-$, we conclude that

$$\operatorname{meas}\{|u_n| > k\} \leq \frac{C_3(1+k)^\theta}{k^{\underline{p}^+}} \to 0 \text{ as } k \to \infty. \tag{4.20}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$,

$$\begin{aligned}\operatorname{meas}\{|u_n - u_m| > \delta\} \leq\ & \operatorname{meas}\{|u_n| > k\} + \operatorname{meas}\{|u_m| > k\} \\ & + \operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.\end{aligned}$$

Let $\varepsilon > 0$, in view of (4.20) we may choose $k = k(\varepsilon)$ large enough such that

$$\operatorname{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \text{ and } \operatorname{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.21}$$

Moreover, thanks to (4.19) we have

$$T_k(u_n) \to \eta_k \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega.$$

Thus $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\operatorname{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3}, \quad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{4.22}$$

By combining (4.21) and (4.22), we conclude that for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\operatorname{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon, \quad \text{for any } n, m \geq n_0.$$

Thus $(u_n)_n$ is a Cauchy sequence in measure, and converges almost everywhere, for a subsequence, to some measurable function $u$. Thanks to (4.19) we conclude that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega).$$

In view of Lebesgue dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \text{ in } L^{p_i(\cdot)}(\Omega), \quad \text{for } i = 1, \dots, N.$$

Moreover, by taking $T_k(u_n)$ as a test function in the approximate problem (4.2), we have

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n) dx + \int_\Omega |T_n(u_n)|^{s(x)} |T_k(u_n)| dx \\ &= \int_\Omega f_n(x, T_n(u_n))\ T_k(u_n) dx.\end{aligned}$$

In view of (3.2), (3.5), and using (4.17) we obtain

$$\begin{aligned}\alpha \sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx &\leq \int_\Omega g(x)\ |T_k(u_n)| dx + \int_\Omega |T_n(u_n)|^{q(x)}\ |T_k(u_n)| dx \\ &\leq k\|g(x)\|_{L^1(\Omega)} + k\||T_n(u_n)|^{q(x)}\|_{L^1(\Omega)} \\ &\leq k(\|g(x)\|_{L^1(\Omega)} + C_1).\end{aligned}$$

It follows, for any $i = 1, \dots, N$, that

$$\begin{aligned} k^{p_i^-} \text{ meas}\{|u_n| > k\} &\le \int_\Omega |T_k(u_n)|^{p_i^-}\,dx \\ &\le C_p^{p_i^-} \int_\Omega |D^i T_k(u_n)|^{p_i^-}\,dx \\ &\le C_p^{p_i^-} \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx + C_p^{p_i^-} |\Omega| \\ &\le C_4 k. \end{aligned}$$

Thus, we conclude that

$$k^{\underline{p}^+ - 1} \cdot \text{meas}\{|u_n| > k\} \le C_4, \quad \text{for any } k \ge 1, \tag{4.23}$$

where $C_4$ is a constant that doesn't depend on $k$ and $n$.

*Step 3: the equi-integrability of* $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ *and* $(f_n(x, T_n(u_n)))_n$. In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$, various real-valued functions of real variables that converge to 0 as $n$ tends to infinity. Similarly, we define $\varepsilon_i(h)$ and $\varepsilon_i(n, h)$.

In order to pass to the limit in the approximate equation, we shall show that

$$|T_n(u_n)|^{s(x)-1}T_n(u_n) \to |u|^{s(x)-1}u \text{ strongly in } L^1(\Omega) \tag{4.24}$$

and

$$f_n(x, T_n(u_n)) \to f(x, u) \text{ strongly in } L^1(\Omega). \tag{4.25}$$

We have $|T_n(u_n)|^{s(x)-1}T_n(u_n) \to |u|^{s(x)-1}u$ and $f_n(x, T_n(u_n)) \to f(x, u)$ a.e. in $\Omega$. Thus, in view of Vitali's theorem, to show the convergence (4.24) – (4.25), it is suffices to prove that$(f_n(x, T_n(u_n)))_n$ and $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ are uniformly equi-integrable. Let $h \ge R$, by taking $v_n = \varphi(u_n)|T_{h+1}(u_n) - T_h(u_n)|$ as a test function in (4.2), and since $v_n$ have the same sign as $u_n$, we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n))\varphi(u_n) dx \\ &+ (\theta - 1) \sum_{i=1}^N \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| dx \\ &+ \int_\Omega |T_n(u_n)|^{s(x)} |T_{h+1}(u_n) - T_h(u_n)|\ |\varphi(u_n)| dx \\ \le &\int_\Omega |f_n(x, T_n(u_n))|\ |T_{h+1}(u_n) - T_h(u_n)\ |\varphi(u_n)| dx. \end{aligned}$$

We have $|\varphi(u_n)| \ge \dfrac{1}{2}$ on the set $\{h \le |u_n|\}$, and thanks to (3.2) we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n))\ |\varphi(u_n)| dx \\ &+ (\theta - 1) \sum_{i=1}^N \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| dx \end{aligned}$$

$$
\begin{aligned}
\geq & \frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n dx + \frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} |D^iu_n|^{p_i(x)}dx \\
& + \alpha(\theta-1)\sum_{i=1}^{N}\int_{\{h+1\leq|u_n|\}}\frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx \\
\geq & \frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n dx + C_5\sum_{i=1}^{N}\int_{\{h+1\leq|u_n|\}}\frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx,
\end{aligned}
$$

with $C_5 = \alpha\cdot\min\left\{\frac{1}{4},(\theta-1)\right\}$. Having in mind (3.5) we conclude that

$$
\begin{aligned}
& \frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_n dx + C_5\sum_{i=1}^{N}\int_{\{h+1\leq|u_n|\}}\frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx \\
& + \int_{\Omega}|T_n(u_n)|^{s(x)}|T_{h+1}(u_n)-T_h(u_n)|\varphi(u_n)dx \\
\leq & \int_{\{h<|u_n|\}}|g(x)||T_{h+1}(u_n)-T_h(u_n)|dx \\
& + \int_{\{h<|u_n|\}}|T_n(u_n)|^{q(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx.
\end{aligned}
\tag{4.26}
$$

For the second term on the left-hand side of (4.26), thanks to Poincaré's inequality we have

$$
\begin{aligned}
& C_5\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{|D^iu_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx \\
\geq & C_5\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{|D^iu_n|^{p_i^-}}{(1+|u_n|)^{\theta}}dx - C_5N\text{meas}\{h\leq|u_n|\} \\
= & C_5\sum_{i=1}^{N}\int_{\Omega}\left|D^i\int_{|T_h(u_n)|}^{|u_n|}\frac{ds}{(1+s)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - C_5N\text{meas}\{h\leq|u_n|\} \\
\geq & C_6\sum_{i=1}^{N}\int_{\Omega}\left|\int_{|T_h(u_n)|}^{|u_n|}\frac{ds}{(1+s)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - C_5N\text{meas}\{h\leq|u_n|\} \\
\geq & C_6\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{(|u_n|-|T_h(u_n)|)^{p_i^-}}{(1+|u_n|)^{\theta}}dx - C_5N\text{meas}\{h\leq|u_n|\} \\
\geq & C_7\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}|u_n|^{p_i^- -\theta}dx - C_6\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{h^{p_i^-}}{(1+|u_n|)^{\theta}}dx \\
& - C_5N\text{meas}\{h\leq|u_n|\}.
\end{aligned}
$$

Having in mind (4.26), we conclude that

$$\begin{aligned}&\frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_ndx + C_7\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}|u_n|^{p_i^--\theta}dx\\ &+\int_{\{h<|u_n|\}}|T_n(u_n)|^{s(x)}|T_{h+1}(u_n)-T_h(u_n)|\varphi(u_n)dx\\ \leq&\int_{\{h<|u_n|\}}|g(x)|dx+\int_{\{h<|u_n|\}}|T_n(u_n)|^{q(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx\\ &+C_6\sum_{i=1}^{N}\int_{\{h<|u_n|\}}\frac{h^{p_i^-}}{(1+|u_n|)^\theta}dx+C_5N\text{meas}\{h\leq|u_n|\}.\end{aligned}$$

Since $q(x)<\max(s(x),\underline{p}^+-\theta)$, and in view of Young's inequality we have

$$\begin{aligned}&\int_{\{h<|u_n|\}}|T_n(u_n)|^{q(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx\\ \leq&\frac{C_7}{2}\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}|u_n|^{p_i^--\theta}dx+C_8\int_{\{h<|u_n|\}}|T_{h+1}(u_n)-T_h(u_n)|dx\\ &+\frac{1}{2}\int_{\{h<|u_n|\}}|T_n(u_n)|^{s(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx,\end{aligned}$$

and thanks to (4.23), we have

$$\begin{aligned}\varepsilon_1(h)=&\sum_{i=1}^{N}\int_{\{h<|u_n|\}}\frac{h^{p_i^-}}{(1+|u_n|)^\theta}dx\leq\sum_{i=1}^{N}h^{p_i^--\theta}\ \text{meas}\{h<|u_n|\}\\ \leq&Nh^{\underline{p}^+-\theta}\ \text{meas}\{h<|u_n|\}\\ =&\frac{Nh^{\underline{p}^+-1}\ \text{meas}\{h<|u_n|\}}{h^{\theta-1}}\\ \leq&\frac{NC_4}{h^{\theta-1}}\to0\text{ as }h\to\infty.\end{aligned}$$

Also, we have meas$\{|u_n|>h\}$ goes to zero, as $h$ tends to infinity, and since $g(x)\in L^1(\Omega)$ we conclude that

$$\varepsilon_2(h)=\int_{\{h<|u_n|\}}|g(x)|dx+C_5N\ \text{meas}\{h\leq|u_n|\}\to0\text{ as }h\to\infty.$$

It follows that

$$\begin{aligned}&\frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_ndx\\ &+\frac{C_7}{2}\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}|u_n|^{p_i^--\theta}dx+\frac{1}{2}\int_{\{h+1<|u_n|\}}|T_n(u_n)|^{s(x)}dx\\ \leq&C_8\int_{\{h<|u_n|\}}|T_{h+1}(u_n)-T_h(u_n)|dx+\varepsilon_3(h)\\ \leq&\varepsilon_4(h).\end{aligned}\tag{4.27}$$

We conclude that

$$\lim_{h\to\infty}\Big(\int_{\{h+1<|u_n|\}}|T_n(u_n)|^{s(x)}dx+\int_{\{h+1<|u_n|\}}|T_n(u_n)|^{q(x)}dx\Big)=0, \tag{4.28}$$

therefore, thanks to (4.28) we have for any $\delta>0$, there exists $h(\delta)>1$ such that

$$\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{s(x)}dx+\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{q(x)}dx\le\frac{\delta}{2}. \tag{4.29}$$

On the other hand, for any measurable subset $E\subseteq\Omega$ we have

$$\begin{aligned}&\int_E|T_n(u_n)|^{s(x)}dx+\int_E|T_n(u_n)|^{q(x)}dx\\&\le\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{s(x)}dx+\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{q(x)}dx\\&\quad+\int_E|T_{h(\delta)}(u_n)|^{s(x)}dx+\int_E|T_{h(\delta)}(u_n)|^{q(x)}dx.\end{aligned} \tag{4.30}$$

It's clear that, there exists $\beta(\delta)>0$ such that for any $E\subseteq\Omega$ with meas$(E)\le\beta(\delta)$ we have

$$\int_E|T_{h(\delta)}(u_n)|^{s(x)}dx+\int_E|T_{h(\delta)}(u_n)|^{q(x)}dx\le\frac{\delta}{2}. \tag{4.31}$$

Finally, by combining (4.29), (4.30) and (4.31), we obtain

$$\int_E|T_n(u_n)|^{s(x)}dx+\int_E|T_n(u_n)|^{q(x)}dx\le\delta \text{ for any } E\subset\Omega \text{ such that meas}(E)\le\beta(\delta). \tag{4.32}$$

Consequently, $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ and $(|T_n(u_n)|^{q(x)-1}T_n(u_n))_n$ are uniformly equi-integrable, and in view of the growth condition (3.5) we have

$$|f_n(x,T_n(u_n))|\le g(x)+|T_n(u_n)|^{q(x)},$$

with $g(x)\in L^1(\Omega)$, then $(f_n(x,T_n(u_n)))_n$ is also uniformly equi-integrable. According to Vitali's theorem, the statements (4.24) and (4.25) are concluded. Moreover, in view of (4.27) we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N\int_{\{h<|u_n|\le h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_ndx=0. \tag{4.33}$$

*Step 4: strong convergence of truncations.* Let $h>k\ge1$, and we set $\psi_h(u_n)=(1-|T_1(u_n-T_h(u_n))|)$. By taking $(T_k(u_n)-T_k(u))\psi_h(u_n)\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ as a test function in (4.2) we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\psi_h(u_n)dx\\&-\sum_{i=1}^N\int_{\{h\le|u_n|\le h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n|T_k(u_n)-T_k(u)|dx\\&+\int_\Omega|T_n(u_n)|^{s(x)-1}T_n(u_n)(T_k(u_n)-T_k(u))\psi_h(u_n)dx\end{aligned}$$

$$= \int_\Omega f_n(x, T_n(u_n))(T_k(u_n) - T_k(u))\psi_h(u_n)dx.$$

It follows that

$$\begin{aligned}(4.34)\qquad &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u))\psi_h(u_n)dx \\ &\leq \int_\Omega |f_n(x, T_n(u_n))|\ |T_k(u_n) - T_k(u)|dx + \int_\Omega |T_n(u_n)|^{s(x)}\ |T_k(u_n) - T_k(u)|dx \\ &+ \sum_{i=1}^N \int_{\{h\leq|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n|T_k(u_n) - T_k(u)|dx.\end{aligned}$$

For the first and second terms on the right-hand side of (4.34), we have $T_k(u_n) \rightharpoonup T_k(u)$ weak$-\star$ in $L^\infty(\Omega)$, and thanks to (4.24)–(4.25) we have $|T_n(u_n)|^{s(x)} \to |u|^{s(x)}$ and $f_n(x, T_n(u_n)) \to f(x, u)$ strongly in $L^1(\Omega)$, then

$$(4.35)\qquad \varepsilon_5(n) = \int_\Omega |T_n(u_n)|^{s(x)}|T_k(u_n) - T_k(u)|dx \to 0 \text{ as } n \to \infty$$

and

$$(4.36)\qquad \varepsilon_6(n) = \int_\Omega |f_n(x, T_n(u_n))|\ |T_k(u_n) - T_k(u)|dx \to 0 \text{ as } n \to \infty.$$

On the other hand, according to (4.33) we have

$$\begin{aligned}(4.37)\qquad \varepsilon_7(h) =&\sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n|T_k(u_n) - T_k(u)|dx \\ \leq&2k\sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n dx \to 0 \text{ as } h \to \infty.\end{aligned}$$

By combining (4.34) and (4.35)–(4.37) we conclude that

$$(4.38)\qquad \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u))\psi_h(u_n)dx \leq \varepsilon_7(n, h).$$

For the term on the left-hand side of (4.38), since $a_i(x, s, 0) = 0$, it follows that

(4.39)

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u))\psi_h(u_n)dx \\ =&\sum_{i=1}^N \int_{\{|u_n|\leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))(D^i T_k(u_n) - D^i T_k(u))dx \\ &- \sum_{i=1}^N \int_{\{k<|u_n|\leq h+1\}} a_i(x, T_{h+1}(u_n), \nabla T_{h+1}(u_n))D^i T_k(u)\ \psi_h(u_n)dx \\ =&\sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u))dx\end{aligned}$$

$$+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))dx$$
$$-\sum_{i=1}^{N}\int_{\{k<|u_n|\le h+1\}} a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D^iT_k(u)\ \psi_h(u_n)dx.$$

For the second term on the right-hand side of (4.39), we have $T_k(u_n)\to T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, then, $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and since $D^iT_k(u_n)$ converges to $D^iT_k(u)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\varepsilon_8(n)=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))dx\to 0 \text{ as } n\to\infty. \tag{4.40}$$

Concerning the third term on the right-hand side of (4.39), we have $(|a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|)_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists $\nu_i\in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|\rightharpoonup \nu_i$ weakly in $L^{p_i'(\cdot)}(\Omega)$ for any $i=1,\dots,N$. Therefore,

$$\begin{aligned}\varepsilon_9(n)\le&\Big|\sum_{i=1}^{N}\int_{\{k<|u_n|\le h+1\}} a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D^iT_k(u)\ \psi_h(u_n)dx\Big|\\ \le&\sum_{i=1}^{N}\int_{\{k<|u_n|\le h+1\}} |a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|\ |D^iT_k(u)|dx\\ \to&\sum_{i=1}^{N}\int_{\{k<|u|\le h+1\}} \nu_i\ |D^iT_k(u)|dx=0 \text{ as } n\to\infty.\end{aligned} \tag{4.41}$$

By combining (4.38)–(4.41), we conclude that

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))dx$$
$$\le\varepsilon_{10}(n,h).$$

In view of Lebesgue dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ strongly in $L^{\underline{p}^+}(\Omega)$. Thus, by letting $n$ then $h$ tend to infinity we deduce that

$$\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))dx$$
$$+\int_{\Omega}(|T_k(u_n)|^{\underline{p}^+-2}T_k(u_n)-|T_k(u)|^{\underline{p}^+-2}T_k(u))(T_k(u_n)-T_k(u))dx\to 0 \ \text{ as } \ n\to\infty.$$

In view of Lemma 4.2, we conclude that

$$\begin{cases} T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ D^iu_n\to D^iu \text{ a.e. in } \Omega \text{ for } i=1,\dots,N.\end{cases} \tag{4.42}$$

Moreover, we have $a_i(x,T_n(u_n),\nabla u_n)D^iu_n$ tends to $a_i(x,u,\nabla u)D^iu$ almost everywhere in $\Omega$, and in view of Fatou's lemma and (4.33), we conclude that

$$\begin{aligned}
&\lim_{h\to\infty}\sum_{i=1}^{N}\int_{\{h<|u|<h+1\}} a_i(x,u,\nabla u)D^iudx\\
&\leq \lim_{h\to\infty}\liminf_{n\to\infty}\sum_{i=1}^{N}\int_{\{h<|u_n|<h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_ndx\\
&\leq \lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^{N}\int_{\{h<|u_n|<h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^iu_ndx=0,
\end{aligned}$$

which prove (4.1).

*Step 5: passage to the limit.* Let $\varphi\in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$, and choosing $S(\cdot)$ be a smooth function in $C_0^1(\mathbb{R})$ such that supp $(S(\cdot))\subseteq[-M,M]$ for some $M\geq 0$.

By taking $S(u_n)\varphi\in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$ as a test function in the approximate problem (4.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\left(D^iu_nS'(u_n)\varphi+S(u_n)D^i\varphi\right)dx\\
&\quad+\int_\Omega |T_n(u_n)|^{s(x)-1}T_n(u_n)S(u_n)\varphi dx=\int_\Omega f_n(x,T_n(u_n))S(u_n)\varphi dx.
\end{aligned}\tag{4.43}$$

For the first term on the left-hand side of (4.43), we have

$$\begin{aligned}
&\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\left(D^iu_nS'(u_n)\varphi+S(u_n)D^i\varphi\right)dx\\
&=\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\left(S'(u_n)\varphi D^iT_M(u_n)+S(T_M(u_n))D^i\varphi\right)dx,
\end{aligned}$$

in view of (4.42), we have $(a_i(x,T_M(u_n),\nabla T_M(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, and since $a_i(x,T_M(u_n),\nabla T_M(u_n))$ tends to $a_i(x,T_M(u),\nabla T_M(u))$ almost everywhere in $\Omega$, it follows that

$$a_i(x,T_M(u_n),\nabla T_M(u_n))\rightharpoonup a_i(x,T_M(u),\nabla T_M(u)) \text{ in } L^{p_i'(\cdot)}(\Omega),$$

and since $(S'(u_n)\varphi D^iT_M(u_n)+S(T_M(u_n))D^i\varphi)\to(S'(u)\varphi D^iT_M(u)+S(T_M(u))D^i\varphi)$ strongly in $L^{p_i(\cdot)}(\Omega)$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_n(u_n),\nabla u_n)\left(D^iu_nS'(u_n)\varphi+S(u_n)D^i\varphi\right)dx\\
&=\lim_{n\to\infty}\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u_n),\nabla T_M(u_n))\left(D^iT_M(u_n)S'(u_n)\varphi+S(T_M(u_n))D^i\varphi\right)dx\\
&=\sum_{i=1}^{N}\int_\Omega a_i(x,T_M(u),\nabla T_M(u))\left(D^iT_M(u)S'(u)\varphi+S(T_M(u))D^i\varphi\right)dx
\end{aligned}\tag{4.44}$$

$$=\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)\left(D^i u S'(u)\varphi + S(u)D^i\varphi\right)dx.$$

Concerning the second term on the right-hand side of (4.43), we have $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\Omega)$, and thanks to (4.24), we have $|T_n(u_n)|^{s(x)-1}T_n(u_n) \to |u|^{s(x)-1}u$ strongly in $L^1(\Omega)$, it follows that

$$\lim_{n\to\infty}\int_{\Omega}|T_n(u_n)|^{s(x)-1}T_n(u_n)S(T_M(u_n))\varphi dx = \int_{\Omega}|u|^{s(x)-1}uS(T_M(u))\varphi dx \tag{4.45}$$
$$=\int_{\Omega}|u|^{s(x)-1}uS(u)\varphi dx.$$

Similarly, thanks to (4.25) we have $f_n(x,T_n(u_n)) \to f(x,u)$ strongly in $L^1(\Omega)$ then

$$\lim_{n\to\infty}\int_{\Omega} f_n(x,T_n(u_n))S(T_M(u_n))\varphi dx = \int_{\Omega} f(x,u)S(T_M(u))\varphi dx = \int_{\Omega} f(x,u)S(u)\varphi dx. \tag{4.46}$$

By combining (4.43) and (4.44)–(4.46), we conclude that

$$\sum_{i=1}^{N}\int_{\Omega} a_i(x,u,\nabla u)\left(D^i u S'(u)\varphi + S(u)D^i\varphi\right)dx + \int_{\Omega}|u|^{s(x)-1}uS(u)\varphi dx$$
$$=\int_{\Omega} f(x,u)S(u)\varphi dx.$$

which complete the proof of the Theorem 4.1.

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Kragujevac Journal of Mathematics
Volume 44(4) (2020), Pages 639–649.

# CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH QUASI-SUBORDINATION INVOLVING $(p,q)$-DERIVATIVE OPERATOR

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Abstract. In this present paper, as applications of the post-quantum calculus known as the $(p,q)$-calculus, we construct a new class $\boldsymbol{D}_{p,q}^{k}(\gamma,\zeta,\Psi)$ of bi-univalent functions of complex order defined in the open unit disk. Coefficients inequalities and several special consequences of the results are obtained.

## 1. Introduction and Preliminaries

The $q$-calculus as well as the fractional $q$-calculus provide important tools that have been used in the fields of special functions and many other areas. Historically speaking, a firm footing of the usage of the $q$-calculus in the context of Geometric Function Theory was actually provided and the basic (or $q$-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [30]). In fact, the theory of univalent functions can be described by using the theory of the $q$-calculus. Moreover, in recent years, such $q$-calculus operators as the fractional $q$-integral and fractional $q$-derivative operators were used to construct several subclasses of analytic functions (see, for example, [3, 19, 21, 26]). In particular, Purohit and Raina [20] investigated applications of fractional $q$-calculus operators to define several classes of functions which are analytic in the open unit disk. On the other hand, Mohammed and Darus [14] studied approximation and geometric properties of these $q$-operators in regard to some subclasses of analytic functions in a compact disk.

*Key words and phrases.* Coefficient bounds, Bi-univalent functions, Quasi-subordination, $q$-calculus, $(p,q)$-derivative operator.

2010 *Mathematics Subject Classification.* Primary: 30C45. Secondary: 33D15.

DOI 10.46793/KgJMat2004.639A

*Received*: December 28, 2017.

*Accepted*: August 22, 2018.

Further the possibility of extension of the $q$-calculus to post-quantum calculus denoted by the $(p,q)$-calculus. The $(p,q)$-calculus which have many applications in areas of science and engineering was introduced in order to generalize the $q$-series by Gasper and Rahman [8]. The $(p,q)$-series is derived as corresponding extensions of $q$-identities (for example [2, 6]).

We begin by providing some basic definitions and concept details of the $(p,q)$-calculus which are used in this paper.

The $(p,q)$-number is given by

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad p \neq q,$$

which is a natural generalization of the $q$-number (see [11]), that is

$$\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.$$

It is clear that the notation $[n]_{p,q}$ is symmetric, that is,

$$[n]_{p,q} = [n]_{q,p}\,.$$

Let $p$ and $q$ be elements of complex numbers and $D = D_{p,q} \subset \mathbb{C}$ such that $x \in D$ implies $px \in D$ and $qx \in D$. Here, in this investigation, we give the following two definitions which involve a post-quantum generalization of Sofonea's work [27].

**Definition 1.1.** Let $0 < |q| < |p| \leq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called $(p,q)$-differentiable under the restriction that, if $0 \in D_{p,q}$, then $f'(0)$ exists.

**Definition 1.2.** Let $0 < |q| < |p| \leq 1$. A given function $f : D_{p,q} \to \mathbb{C}$ is called $(p,q)$-differentiable of order $n$, if and only if $0 \in D_{p,q}$, then $f^{(n)}(0)$ exists.

**Definition 1.3** ([6])**.** The $(p,q)$-derivative of a function $f$ is defined as

$$(D_{p,q}f)(x) = \frac{f(px) - f(qx)}{(p-q)x}, \quad x \neq 0,$$

and $(D_{p,q}f)(0) = f'(0)$, provided $f'(0)$ exists.

As with ordinary derivative, the action of the $(p,q)$-derivative of a function is a linear operator. More precisely, for any constants $a$ and $b$,

$$D_{p,q}(af(z) + bg(z)) = aD_{p,q}f(z) + bD_{p,q}g(z).$$

The $(p,q)$-derivative fulfils the following product rules

$$\begin{aligned} D_{p,q}(f(z)g(z)) =& f(pz)D_{p,q}g(z) + g(qz)D_{p,q}f(z), \\ D_{p,q}(f(z)g(z)) =& g(pz)D_{p,q}f(z) + f(qz)D_{p,q}g(z). \end{aligned}$$

Further, the $(p,q)$-derivative fulfils the following product rules

$$D_{p,q}\left(\frac{f(z)}{g(z)}\right) = \frac{g(qz)D_{p,q}f(z) - f(qz)D_{p,q}g(z)}{g(pz)g(qz)},$$
$$D_{p,q}\left(\frac{f(z)}{g(z)}\right) = \frac{g(pz)D_{p,q}f(z) - f(pz)D_{p,q}g(z)}{g(pz)g(qz)}.$$

Let $A$ indicate an analytic function family, which is normalized under the condition of $f(0) = f'(0) - 1 = 0$ in $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and given by the following Taylor-Maclaurin series:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by $S$ we shall denote the class of all functions in $A$ which are univalent in $\Delta$.

If $f$ is of the form (1.1), then

$$(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q}\, a_n z^{n-1}.$$

With a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\Delta$. Then we say that the function $f$ is subordinate to $g$ if there exists a Schwarz function $w(z)$, analytic in $\Delta$ with

$$w(0) = 0, |w(z)| < 1, \quad z \in \Delta,$$

such that

$$f(z) = g(w(z)), \quad z \in \Delta.$$

We denote this subordination by

$$f \prec g \ \text{ or } \ f(z) \prec g(z), \quad z \in \Delta.$$

In particular, if the function $g$ is univalent in $\Delta$, the above subordination is equivalent to

$$f(0) = g(0), \quad f(\Delta) \subset g(\Delta).$$

In the year 1970, Robertson [23] introduced the concept of quasi-subordination. For two analytic functions $f$ and $g$, the function $f$ is said to be quasi-subordinate to $g$ in $\Delta$ and written as

$$f(z) \prec_\rho g(z), \quad z \in \Delta,$$

if there exists an analytic function $|h(z)| \le 1$ such that $\frac{f(z)}{h(z)}$ analytic in $\Delta$ and

$$\frac{f(z)}{h(z)} \prec g(z), \quad z \in \Delta,$$

that is, there exists a Schwarz function $w(z)$ such that $f(z) = h(z)g(w(z))$. Observe that if $h(z) = 1$, then $f(z) = g(w(z))$ so that $f(z) \prec g(z)$ in $\Delta$. Also notice that if $w(z) = z$, then $f(z) = h(z)g(z)$ and it is said that is majorized by $g$ and written $f(z) \ll g(z)$ in $\Delta$. Hence it is obvious that quasi-subordination is a generalization of subordination as well as majorization (see, e.g., [13, 22, 23] for works related to quasi-subordination).

The Koebe-One Quarter Theorem [7] ensures that the image of $\Delta$ under every univalent function $f \in A$ contains a disk of radius $1/4$. Thus every univalent function $f$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w$ $\left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right)$, where

$$f^{-1}(w) = w\ - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{1.2}$$

A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ denote the class of bi-univalent functions in $\Delta$ given by (1.1). For a brief history and interesting examples in the class $\Sigma$, see [29] (see also [4, 5, 12, 16]). Furthermore, judging by the remarkable flood of papers on the subject (see, for example, [10, 17, 28]). Not much is known about the bounds on the general coefficient $|a_n|$. In the literature, there are only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([1, 9, 15, 31]). The coefficient estimate problem for each of $|a_n|$ ( $n \in \mathbb{N}\backslash\{1,2\}$ , $\mathbb{N} = \{1,2,3,...\}$) is still an open problem.

Recently for $f \in A$, Selvaraj et al. [25] defined and discussed $(p,q)$-analogue of Salagean differential operator as given below:

$$\begin{aligned}
\boldsymbol{D}^0_{p,q} f(z) &= f(z)\\
\boldsymbol{D}^1_{p,q} f(z) &= z\left(\boldsymbol{D}_{p,q} f(z)\right)\\
&\ \vdots\\
\boldsymbol{D}^k_{p,q} f(z) &= z\boldsymbol{D}_{p,q}(\boldsymbol{D}^{k-1}_{p,q} f(z))\\
\boldsymbol{D}^k_{p,q} f(z) &= z + \sum_{n=2}^{\infty} [n]^k_{p,q} a_n z^n, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \Delta.
\end{aligned}$$

If we let $p = 1$ and $q \to 1^-$, then $\boldsymbol{D}^k_{p,q} f(z)$ reduces to the well-known Salagean differential operator (see [24]).

Making use of the differential operator $\boldsymbol{D}^k_{p,q}$, we introduce a new class of analytic bi-univalent functions as follows.

**Definition 1.4.** A function $f \in \Sigma$ given by (1.1) is said to be in the class

$$\boldsymbol{D}^k_{p,q}(\gamma, \zeta, \Psi), \quad \gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq \zeta < 1, k \in \mathbb{N}_0,\ 0 < q < p \leq 1,\ z, w \in \Delta,$$

if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}^k_{p,q} f(z)\right)'}{(1-\zeta)\boldsymbol{D}^k_{p,q} f(z) + \zeta z\left(\boldsymbol{D}^k_{p,q} f(z)\right)'} - 1\right) \prec_\rho (\Psi(z) - 1)$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}_{p,q}^{k}g(w)\right)'}{(1-\zeta)\boldsymbol{D}_{p,q}^{k}g(w)+\zeta w\left(\boldsymbol{D}_{p,q}^{k}g(w)\right)'}-1\right)\prec_{\rho}\left(\Psi(w)-1\right),$$

where the function $g$ is given by (1.2).

*Remark* 1.1. For $p = 1$ and $q \to 1$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\boldsymbol{D}^{k}(\gamma, \zeta, \Psi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}^{k}f(z)\right)'}{(1-\zeta)\boldsymbol{D}^{k}f(z)+\zeta z\left(\boldsymbol{D}^{k}f(z)\right)'}-1\right)\prec_{\rho}\left(\Psi(z)-1\right),\quad z\in\Delta$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}^{k}g(w)\right)'}{(1-\zeta)\boldsymbol{D}^{k}g(w)+\zeta w\left(\boldsymbol{D}^{k}g(w)\right)'}-1\right)\prec_{\rho}\left(\Psi(w)-1\right),\quad z\in\Delta,$$

where $\gamma \in \mathbb{C}\backslash\{0\},\ 0 \leq \zeta < 1,\ k \in \mathbb{N}_0$ and the function $g$ is given by (1.2).

*Remark* 1.2. For $\zeta = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\boldsymbol{D}_{p,q}^{k}(\gamma, \Psi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}_{p,q}^{k}f(z)\right)'}{\boldsymbol{D}_{p,q}^{k}f(z)}-1\right)\prec_{\rho}\left(\Psi(z)-1\right),\quad z\in\Delta$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}_{p,q}^{k}g(w)\right)'}{\boldsymbol{D}_{p,q}^{k}g(w)}-1\right)\prec_{\rho}\left(\Psi(w)-1\right),\quad z\in\Delta,$$

where $k \in \mathbb{N}_0,\ 0 < q < p \leq 1$ and the function $g$ is given by (1.2).

*Remark* 1.3. For $\zeta = k = 0$ and $\gamma \in \mathbb{C}\backslash\{0\}$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $S_{\Sigma}(\gamma, \Psi)$, if the following conditions are satisfied:

$$\frac{1}{\gamma}\left(\frac{zf'(z)}{f(z)}-1\right)\prec_{\rho}\left(\Psi(z)-1\right),\quad z\in\Delta$$

and

$$\frac{1}{\gamma}\left(\frac{wg'(w)}{g(w)}-1\right)\prec_{\rho}\left(\Psi(w)-1\right),\quad z\in\Delta,$$

where the function $g$ is given by (1.2).

## 2. Main Result and its Consequences

Firstly, we will state the Lemma 2.1 to obtain our result.

**Lemma 2.1** ([18])**.** *If $s \in P$, then $|s_i| \leq 2$ for each $i$, where $P$ is the family of all functions $s$, analytic in $\Delta$, for which*

$$\operatorname{Re}(s(z)) > 0,$$

*where*

$$s(z) = 1 + s_1 z + s_2 z^2 + \cdots .$$

Through out this paper it is assumed that $\Psi$ is analytic in $\Delta$ with $\Psi(0) = 1$ and let

$$\Psi(z) = 1 + C_1 z + C_2 z^2 + \cdots, \quad C_1 > 0. \tag{2.1}$$

Also let

$$h(z) = D_0 + D_1 z + D_2 z^2 + \cdots, \quad |h(z)| \leq 1, \ z \in \Delta. \tag{2.2}$$

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\boldsymbol{D}_{p,q}^{k}(\gamma, \zeta, \Psi)$ proposed by Definition 1.4.

**Theorem 2.1.** *Let $f$ of the form* (1.1) *be in the class $\boldsymbol{D}_{p,q}^{k}(\gamma, \zeta, \Psi)$. Then*

$$|a_2| \leq \frac{|\gamma|\,|D_0|\,C_1\sqrt{C_1}}{\sqrt{(1-\zeta)\left|2\,[3]_{p,q}^{k}\,\gamma C_1^2 D_0 - [2]_{p,q}^{2k}\left[(1-\zeta)(C_2 - C_1) + (1+\zeta)\gamma C_1^2 D_0\right]\right|}}$$

*and*

$$|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-\zeta)^2\,[2]_{p,q}^{2k}} + \frac{|\gamma D_1|\,C_1}{2(1-\zeta)\,[3]_{p,q}^{k}} + \frac{|\gamma D_0|\,C_1}{2(1-\zeta)\,[3]_{p,q}^{k}}.$$

*Proof.* If $f \in \boldsymbol{D}_{p,q}^{k}(\gamma, \zeta, \Psi)$ then, there are two analytic functions $u, v : \Delta \to \Delta$ with $u(0) = v(0) = 0$, $|u(z)| < 1, |v(w)| < 1$ and a function $h$ given by (2.2), such that

$$\frac{1}{\gamma}\left(\frac{z\left(\boldsymbol{D}_{p,q}^{k} f(z)\right)'}{(1-\zeta)\boldsymbol{D}_{p,q}^{k} f(z) + \zeta z\left(\boldsymbol{D}_{p,q}^{k} f(z)\right)'} - 1\right) = h(z)\left(\Psi(u(z)) - 1\right) \tag{2.3}$$

and

$$\frac{1}{\gamma}\left(\frac{w\left(\boldsymbol{D}_{p,q}^{k} g(w)\right)'}{(1-\zeta)\boldsymbol{D}_{p,q}^{k} g(w) + \zeta w\left(\boldsymbol{D}_{p,q}^{k} g(w)\right)'} - 1\right) = h(w)\left(\Psi(v(w)) - 1\right). \tag{2.4}$$

Determine the functions $s_1$ and $s_2$ in $P$ given by

$$s_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + t_1 z + t_2 z^2 + \cdots$$

and

$$s_2(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_1 w + q_2 w^2 + \cdots .$$

Thus,

$$u(z)=\frac{s_1(z)-1}{s_1(z)+1}=\frac{1}{2}\left(t_1z+\left(t_2-\frac{t_1^2}{2}\right)z^2+\cdots\right) \tag{2.5}$$

and

$$v(w)=\frac{s_2(w)-1}{s_2(w)+1}=\frac{1}{2}\left(q_1w+\left(q_2-\frac{q_1^2}{2}\right)w^2+\cdots\right). \tag{2.6}$$

The fact that $s_1$ and $s_2$ are analytic in $\Delta$ with $s_1(0)=s_2(0)=1$. Since $u,v:\Delta\to\Delta$, the functions $s_1$, $s_2$ have a positive real part in $\Delta$, and the relations $|t_i|\leq 2$ and $|q_i|\leq 2$ are true. Using (2.5) and (2.6) together with (2.1) and (2.2) in the right hands of the relations (2.3) and (2.4), we obtain

$$\begin{aligned} h(z)\left(\Psi\left(u(z)\right)-1\right)=&\frac{1}{2}D_0C_1t_1z\\ &+\left(\frac{1}{2}D_1C_1t_1+\frac{1}{2}D_0C_1\left(t_2-\frac{t_1^2}{2}\right)+\frac{1}{4}D_0C_2t_1^2\right)z^2+\cdots \end{aligned} \tag{2.7}$$

and

(2.8)

$$\begin{aligned} h(w)\left(\Psi\left(v(w)\right)-1\right)=&\frac{1}{2}D_0C_1q_1w\\ &+\left(\frac{1}{2}D_1C_1q_1+\frac{1}{2}D_0C_1\left(q_2-\frac{q_1^2}{2}\right)+\frac{1}{4}D_0C_2q_1^2\right)w^2+\cdots. \end{aligned}$$

In the light of (2.3) and (2.4), we get

$$\frac{(1-\zeta)\,[2]_{p,q}^k}{\gamma}a_2=\frac{D_0C_1t_1}{2}, \tag{2.9}$$

$$\frac{2(1-\zeta)\,[3]_{p,q}^k\,a_3-(1-\zeta^2)\,[2]_{p,q}^{2k}\,a_2^2}{\gamma}=\frac{D_1C_1t_1}{2}+\frac{D_0C_1}{2}\left(t_2-\frac{t_1^2}{2}\right)+\frac{D_0C_2t_1^2}{4} \tag{2.10}$$

and

$$-\frac{(1-\zeta)\,[2]_{p,q}^k}{\gamma}a_2=\frac{D_0C_1q_1}{2}, \tag{2.11}$$

(2.12)

$$\frac{2(1-\zeta)\,[3]_{p,q}^k\,(2a_2^2-a_3)-(1-\zeta^2)\,[2]_{p,q}^{2k}\,a_2^2}{\gamma}=\frac{D_1C_1q_1}{2}+\frac{D_0C_1}{2}\left(q_2-\frac{q_1^2}{2}\right)+\frac{D_0C_2q_1^2}{4}.$$

Now, (2.9) and (2.11) give

$$t_1=-q_1 \tag{2.13}$$

and

$$8(1-\zeta)^2\,[2]_{p,q}^{2k}\,a_2^2=\gamma^2D_0^2C_1^2\left(t_1^2+q_1^2\right). \tag{2.14}$$

Adding (2.10) and (2.12), we get

$$\frac{4(1-\zeta)\,[3]_{p,q}^{k}-2(1-\zeta^{2})\,[2]_{p,q}^{2k}}{\gamma}a_{2}^{2}=\frac{D_{0}C_{1}\,(t_{2}+q_{2})}{2}+\frac{D_{0}\,(C_{2}-C_{1})\,(t_{1}^{2}+q_{1}^{2})}{4}. \tag{2.15}$$

By using (2.13), (2.14) and Lemma 2.1 in (2.15), we obtain

$$|a_{2}|\leq\frac{|\gamma|\,|D_{0}|\,C_{1}\sqrt{C_{1}}}{\sqrt{(1-\zeta)\left|2\,[3]_{p,q}^{k}\,\gamma C_{1}^{2}D_{0}-[2]_{p,q}^{2k}\left[(1-\zeta)(C_{2}-C_{1})+(1+\zeta)\gamma C_{1}^{2}D_{0}\right]\right|}}.$$

Next, to find the bound on $|a_{3}|$, by subtracting (2.12) from (2.10), we have

$$\frac{4(1-\zeta)\,[3]_{p,q}^{k}}{\gamma}\left(a_{3}-a_{2}^{2}\right)=\frac{D_{0}C_{1}\,(t_{2}-q_{2})}{2}+\frac{D_{1}C_{1}\,(t_{1}-q_{1})}{2}. \tag{2.16}$$

It follows from (2.13), (2.14) and (2.16) that

$$a_{3}=\frac{\gamma^{2}D_{0}^{2}C_{1}^{2}\,(t_{1}^{2}+q_{1}^{2})}{8(1-\zeta^{2})\,[2]_{p,q}^{2k}}+\frac{\gamma D_{1}C_{1}\,(t_{1}-q_{1})}{8(1-\zeta)\,[3]_{p,q}^{k}}+\frac{\gamma D_{0}C_{1}\,(t_{2}-q_{2})}{8(1-\zeta)\,[3]_{p,q}^{k}}.$$

Applying Lemma 2.1 once again for the coefficients $t_{1},t_{2},q_{1}$ and $q_{2}$, we readily get

$$|a_{3}|\leq\frac{|\gamma D_{0}|^{2}\,C_{1}^{2}}{(1-\zeta)^{2}\,[2]_{p,q}^{2k}}+\frac{|\gamma D_{1}|\,C_{1}}{2(1-\zeta)\,[3]_{p,q}^{k}}+\frac{|\gamma D_{0}|\,C_{1}}{2(1-\zeta)\,[3]_{p,q}^{k}}.$$

This completes the proof of Theorem 2.1. □

**Corollary 2.1.** *Let $f$ of the form* (1.1) *be in the class $\boldsymbol{D}^{k}\,(\gamma,\zeta,\Psi)$. Then*

$$|a_{2}|\leq\frac{|\gamma|\,|D_{0}|\,C_{1}\sqrt{C_{1}}}{\sqrt{(1-\zeta)\left|2\gamma C_{1}^{2}D_{0}3^{k}-2^{2k}\left[(1-\zeta)(C_{2}-C_{1})+(1+\zeta)\gamma C_{1}^{2}D_{0}\right]\right|}}$$

*and*

$$|a_{3}|\leq\frac{|\gamma D_{0}|^{2}\,C_{1}^{2}}{(1-\zeta)^{2}2^{2k}}+\frac{|\gamma D_{1}|\,C_{1}}{2(1-\zeta)3^{k}}+\frac{|\gamma D_{0}|\,C_{1}}{2(1-\zeta)3^{k}}.$$

**Corollary 2.2.** *Let $f$ of the form* (1.1) *be in the class $\boldsymbol{D}_{p,q}^{k}\,(\gamma,\Psi)$. Then*

$$|a_{2}|\leq\frac{|\gamma|\,|D_{0}|\,C_{1}\sqrt{C_{1}}}{\sqrt{\left|2\,[3]_{p,q}^{k}\,\gamma C_{1}^{2}D_{0}-[2]_{p,q}^{2k}\left[(C_{2}-C_{1})+\gamma C_{1}^{2}D_{0}\right]\right|}}$$

*and*

$$|a_{3}|\leq\frac{|\gamma D_{0}|^{2}\,C_{1}^{2}}{[2]_{p,q}^{2k}}+\frac{|\gamma D_{1}|\,C_{1}}{2\,[3]_{p,q}^{k}}+\frac{|\gamma D_{0}|\,C_{1}}{2\,[3]_{p,q}^{k}}.$$

**Corollary 2.3.** *Let $f$ of the form* (1.1) *be in the class $S_{\Sigma}\,(\gamma,\Psi)$. Then*

$$|a_{2}|\leq\frac{|\gamma D_{0}|\,C_{1}\sqrt{C_{1}}}{\sqrt{|C_{1}-C_{2}+\gamma C_{1}^{2}D_{0}|}}$$

*and*

$$|a_3| \leq |\gamma D_0|^2 C_1^2 + \frac{(|D_1| + |D_0|) |\gamma| C_1}{2}.$$

## 3. Concluding Remark

Various choices of $\Psi$ as mentioned above and suitably choosing the values of $C_1$ and $C_2$, we state some interesting results analogous to Theorem 2.1 and the Corollaries 2.1 to 2.3. For example, the function $\Psi$ is given by

$$\Psi(z) = \left(\frac{1+z}{1-z}\right)^{\theta} = 1 + 2\theta z + 2\theta^2 z^2 + \cdots, \quad 0 < \theta \leq 1,$$

which gives

$$C_1 = 2\theta \text{ and } C_2 = 2\theta^2.$$

By taking

$$\Psi(z) = \frac{1 + (1-2\mu) z}{1-z} = 1 + 2(1-\mu) z + 2(1-\mu) z^2 + \cdots, \quad 0 \leq \mu < 1,$$

we have

$$C_1 = C_2 = 2(1-\mu).$$

On the other hand, for $-1 \leq B \leq A < 1$, if we let

$$\Psi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + \cdots, \quad 0 < \theta \leq 1,$$

then we have

$$C_1 = (A-B) \text{ and } C_2 = -B(A-B).$$

The details involved may be left as an exercise for the interested reader.

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