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NEW EXPLICIT BOUNDS ON GRONWALL-BELLMAN-BIHARI-GAMIDOV INTEGRAL INEQUALITIES AND THEIR WEAKLY SINGULAR ANALOGUES WITH APPLICATIONS

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ABSTRACT. In this paper we derive some generalizations of certain Gronwall-Bellman-Bihari-Gamidov type integral inequalities and their weakly singular analogues, which provide explicit bounds on unknown functions. To show the feasibility of the obtained inequalities, two illustrative examples are also introduced.

1. INTRODUCTION

The integral inequalities which provide explicit bounds on unknown functions have proved to be very useful in the study of qualitative properties of the solutions of differential and integral equations. During the past few years, many such new inequalities have been discovered, which are motivated by certain applications. For example, see in [1–4, 7–11, 14, 15] and the references therein. In particular, Sh. G. Gamidov [6], while studying the boundary value problem for higher order differential equations, initiated the study of obtaining explicit upper bounds on the integral inequalities of the forms

$$u(t) \leq c + \int_a^t a(s)u(s)ds + \int_a^b b(s)u(s)ds, \tag{1.1}$$

for $t \in [a, b]$, under some suitable conditions on the functions involved in (1.1). In [12], Pachpatte established more general Gamidov inequalities as follows:

$$u(t) \leq a(t) + \int_a^t b(t, s)u(s)ds + \int_a^b c(s)u(s)ds.$$

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On the other hand, Zheng [16] also established a weakly singular version of the Gronwall-Bellman-Gamidov inequality as follows:

$$u(t) \leq c + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)u(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^T (T-s)^{\alpha-1} f(s)u(s)ds.$$

Recently, Kelong Cheng el al. [5] studied the following inequality:

$$\begin{aligned} u^p(t) \leq & a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f\ (s) u^q(s) ds \\ & + c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} n(s) u^r(s) ds, \end{aligned}$$

where $p \geq q \geq 0$, $p \geq r \geq 0$ and $[\alpha_i, \beta_i, \gamma_i]$, $i = 1, 2$, is the ordered parameter group. In this paper, motivated mainly by the work of Kelong Cheng el al. [5], we discuss more general form of nonlinear weakly singular integral inequalities of Gronwall-Bellman-Bihari-Gamidov

$$\begin{aligned} u^p(t) \leq & a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f(s) u^q(s) ds \\ & + c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} n(s) \sqrt[m_1]{g(u^r(s))} ds, \end{aligned}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $\mathbb{R}_0$ with continuous non-increasing first derivative g' on $\mathbb{R}_0$. Our paper is organized as follows. In Section 2 we prepare some tools needed to prove our theorems. Section 3, we discuss some nonlinear Gamidov type integral inequalities and obtain new explicit bounds on these inequalities. Section 4, we give explicit bounds to new nonlinear Gronwall-Bihari-Gamidov integral inequalities with weakly singular integral kernel and in Section 5, we give an examples to show boundedness and uniqueness of solutions of integral equation with weakly singular kernel.

2. Preliminaries

Throughout the paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_0 = (0, \infty)$, $\mathbb{R}_+ = [0, +\infty)$ and $I = [0, T]\,(T \geq 0$ is a constant), $C(X, Y)$ denotes the collection of continuous functions from the set X to the set Y, p, q, r are real constants such that $p \neq 0$, $0 \leq q$, $r \leq p$. For convenience, we give some lemmas which will be used in the proof of the main results.

Lemma 2.1 ([1, page 16]). *Let $q(t)$ and $p(t)$ be continuous functions for $t \geq \alpha$, let $z(t)$ be a differentiable function for $t \geq \alpha$, and suppose*

$$\begin{aligned} & z'(t) \leq p\,(t)\, z\,(t) + q(t), \quad t \geq \alpha, \\ & z(\alpha) \leq z_0. \end{aligned}$$

Then

$$z(t) \leq z(\alpha) \exp\left(\int_\alpha^t p(s) ds \right) + \int_\alpha^t q(s) \exp\left(\int_s^t p(\tau) d\tau \right) ds, \quad t \geq \alpha.$$

Lemma 2.2 ([7]). *Assume that $a \geq 0$, $p \geq q \geq 0$ and $p \neq 0$, then*

$$a^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{p}} a + \frac{p-q}{p} K^{\frac{q}{p}},$$

for any $K > 0$.

Lemma 2.3 (Discrete Jensen inequality). *Let $A_1, A_2, A_3, A_4, \dots, A_n$ be nonnegative real numbers and $r > 1$ a real number. Then*

$$(A_1 + \cdots + A_n)^r \leq n^{r-1}(A_1^r + A_2^r + \cdots + A_n^r).$$

Lemma 2.4 ([8]). *Let α, β, γ and m be positive constants. Then*

$$\int_0^t (t^\alpha - s^\alpha)^{m(\beta-1)} s^{m(\gamma-1)} ds = \frac{t^\theta}{\alpha} \beta \left[\frac{m(\gamma-1)+1}{\alpha}, m(\beta-1)+1 \right], \quad t \in \mathbb{R}_+$$

where

$$B[\zeta, \eta] = \int_0^1 s^{\zeta-1}(1-s)^{\eta-1} ds, \quad \operatorname{Re}\zeta > 0, \operatorname{Re}\eta > 0),$$

is the well-known beta function and

$$\theta = m(\alpha(\beta-1) + \gamma - 1) + 1.$$

Assume that for the parameter group $[\alpha_i, \beta_i, \gamma_i]$

$$\alpha_i \in (0,1], \quad \beta_i \in (0,1), \quad \gamma_i > 1 - \frac{1}{m}, \tag{2.1}$$

such that

$$\frac{1}{m} + \alpha_i(\beta_i - 1) + \gamma_i - 1 \geq 0, \quad m > 1, i = 1, 2. \tag{2.2}$$

Definition 2.1 ([13]). The Riemann-Liouville fractional integral of order α for a function f is defined as

$$I_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,$$

provided that such integral exists.

Now we state the main results of this work.

3. Main Result

Lemma 3.1. *Assume that $u(t), m(t), l(t), n(t) \in C(I, \mathbb{R}_+)$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $\mathbb{R}_0$ with continuous non-increasing first derivative g' on $\mathbb{R}_0$. If*

$$u(t) \leq m(t) + l(t) \int_0^T n(s) g(u(s)) ds, \tag{3.1}$$

then

$$u(t) \leq m(t) + \frac{l(t) \int_0^T n(s)g(m(s))ds}{1 - \int_0^T g'(m(s))n(s)l(s)ds}, \tag{3.2}$$

for all $t \in I$, provided that

$$\int_0^T g'(m(s))n(s)l(s)ds < 1. \tag{3.3}$$

Proof. Let

$$\Pi = \int_0^T n(s)g(u(s))ds.$$

Obviously, Π is a constant. It follows from (3.1) that

$$u(t) \leq m(t) + l(t)\Pi. \tag{3.4}$$

Applying the mean value theorem for the function g, then for every $x \geq y > 0$, there exists $c \in]y, x[$ such that

$$g(x) - g(y) = g'(c)(x - y) \leq g'(y)(x - y),$$

which gives

$$g(u(t)) \leq g(m(t) + l(t)\Pi) \leq g'(m(t))l(t)\Pi + g(m(t)). \tag{3.5}$$

Multiplying both sides of (3.5) by $n(t)$, then integrating the result from 0 to T, it yields

$$\int_0^T n(s)g(u(s))ds \leq \int_0^T n(s)g(m(s))ds + \Pi \int_0^T g'(m(s))n(s)l(s)ds. \tag{3.6}$$

The inequality (3.6) can be restated as

$$\Pi \leq \int_0^T n(s)g(m(s))ds + \Pi \int_0^T g'(m(s))n(s)l(s)ds,$$

that is

$$\Pi \left(1 - \int_0^T g'(m(s))n(s)l(s)ds\right) \leq \int_0^T n(s)g(m(s))ds.$$

From (3.3), we observe that

$$\Pi \leq \frac{\int_0^T n(s)g(m(s))ds}{1 - \int_0^T g'(m(s))n(s)l(s)ds}. \tag{3.7}$$

Therefore, the desired inequality (3.2) follows from (3.7) and (3.4). $\square$

Remark 3.1. If $g(x) = x$, then Lemma 3.1 reduces to [5, Lemma 3].

Corollary 3.1. *Suppose that the hypotheses of Lemma* 3.1 *hold. If*

$$u(t) \leq m(t) + l(t) \int_0^T n(s) \arctan(u(s)) ds.$$

Then

$$u(t) \leq m(t) + \frac{l(t) \int_0^T n(s) \arctan(m(s)) ds}{1 - \int_0^T \frac{n(s)l(s)}{1 + m^2(s)} ds},$$

for all $t \in I$, provided that

$$\int_0^T \frac{n(s)l(s)}{1 + m^2(s)} ds < 1.$$

And if

$$u(t) \leq m(t) + l(t) \int_0^T n(s) \ln(u(s) + 1) ds,$$

then

$$u(t) \leq m(t) + \frac{l(t) \int_0^T n(s) \ln(m(s) + 1) ds}{1 - \int_0^T \frac{n(s)l(s)}{1 + m(s)} ds},$$

for all $t \in I$, provided that

$$\int_0^T \frac{n(s)l(s)}{1 + m(s)} ds < 1.$$

Theorem 3.1. *Assume that $u(t), a(t), b(t), c(t), f(t), n(t) \in C(I, \mathbb{R}_+)$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable increasing function on $\mathbb{R}_0$ with continuous non-increasing first derivative g' on $\mathbb{R}_0$. If $u(t)$ satisfies*

$$u^p(t) \leq a(t) + b(t) \int_0^t f(s) u^q(s) ds + c(t) \int_0^T n(s) g(u^r(s)) ds, \tag{3.8}$$

then, under the condition that

$$\int_0^T g'(m(s)) n(s) l(s) ds < 1,$$

the following explicit estimate

$$u(t) \leq \left(m(t) + \frac{l(t) \int\limits_0^T n(s) g(m(s)) ds}{1 - \int\limits_0^T g'(m(s)) n(s) l(s) ds} \right)^{\frac{1}{r}}, \tag{3.9}$$

holds for all $t \in I$, *where*

$$m(t) = \frac{r}{p} K^{\frac{r-p}{p}} \bar{b}(t) \int_0^t Q(s) \exp\left(\int_s^t P(\tau) d\tau \right) ds + \frac{r}{p} K^{\frac{r-p}{p}} a(t) + \frac{p-r}{p} K^{\frac{r}{p}},$$

$$l(t) = \frac{r}{p} K^{\frac{r-p}{p}} \bar{b}(t) \left(\exp \int_0^t P(s) ds \right),$$

$$\bar{b}(t) = b(t) + c(t), \tag{3.10}$$

and

$$P(t) = \frac{q}{p} K^{\frac{q-p}{p}} f(t)\ \bar{b}(t),$$

$$Q(t) = f(t) \left(\frac{q}{p} K^{\frac{q-p}{p}} a(t) + \frac{p-q}{p} K^{\frac{q}{p}} \right). \tag{3.11}$$

Proof. The inequality (3.8) can be rewritten as

$$u^p(t) \leq a(t) + (b(t) + c(t)) \left(\int\limits_0^t f(s) u^q(s) ds + \int\limits_0^T n(s) g(u^r(s)) ds \right). \tag{3.12}$$

Define a function $z(t)$ by

$$z(t) = \int\limits_0^t f(s) u^q(s) ds + \int\limits_0^T n(s) g(u^r(s)) ds. \tag{3.13}$$

Then, from (3.12), we have

$$u^p(t) \leq a(t) + \bar{b}(t) z(t),$$

$$\bar{b}(t) = b(t) + c(t),$$

$$u(t) \leq (a(t) + \bar{b}(t) z(t))^{\frac{1}{p}}. \tag{3.14}$$

Applying Lemma 2.2 to inequality (3.14), for any $K > 0$, we obtain

$$u^r(t) \leq (a(t) + \bar{b}(t) z(t))^{\frac{r}{p}} \leq \frac{r}{p} K^{\frac{r-p}{p}} (a(t) + \bar{b}(t) z(t)) + \frac{p-r}{p} K^{\frac{r}{p}} = w(t),$$

$$u^q(t) \leq (a(t) + \bar{b}(t) z(t))^{\frac{q}{p}} \leq \frac{q}{p} K^{\frac{q-p}{P}} (a(t) + \bar{b}(t) z(t)) + \frac{p-q}{p} K^{\frac{q}{p}}, \tag{3.15}$$

$$z(0) = \int\limits_0^T n(s) g(u^r(s)) ds \leq \int\limits_0^T n(s) g(w(s)) ds.$$

From (3.13) and (3.15), we get

$$z'(t) \leq f(t)\left(\frac{q}{p}K^{\frac{q-p}{p}}(a(t)+\bar{b}(t)z(t)) + \frac{p-q}{p}K^{\frac{q}{p}}\right).$$

Then

$$z'(t) \leq \frac{q}{p}K^{\frac{q-p}{p}}f(t)\bar{b}(t)z(t) + f(t)\left(\frac{q}{p}K^{\frac{q-p}{p}}a(t) + \frac{p-q}{p}K^{\frac{q}{p}}\right), \tag{3.16}$$

the inequality (3.16) can be restated as

$$z'(t) \leq P(t)z(t) + Q(t), \tag{3.17}$$

where P and Q are defined as in (3.11). Applying Lemma 2.1 to the inequality (3.17), we have

$$z(t) \leq z(0)\exp\left(\int_0^t P(s)ds\right) + \int_0^t Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds. \tag{3.18}$$

Substituting (3.15) in (3.18), we get

$$\begin{aligned} z(t) \leq &\int_0^t Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds + \\ &+ \left(\exp\int_0^t P(s)ds\right)\int_0^T n(s)g(w(s))ds. \end{aligned} \tag{3.19}$$

Then we can write the inequality (3.19) in the following form

$$\begin{aligned} w(t) \leq &\frac{r}{p}K^{\frac{r-p}{p}}\bar{b}(t)\int_{\alpha}^t\int_0 Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds + \frac{r}{p}K^{\frac{r-p}{p}}a(t) + \frac{p-r}{p}K^{\frac{r}{p}} \\ &+ \frac{r}{p}K^{\frac{r-p}{p}}\bar{b}(t)\left(\exp\int_0^t P(s)ds\right)\int_0^T n(s)g(w(s))ds, \end{aligned} \tag{3.20}$$

where $w(t)$ is defined as (3.15). The inequality (3.20) can be restated as

$$w(t) \leq m(t) + l(t)\int_0^T n(s)g(w(s))ds, \tag{3.21}$$

where m, l are defined as in (3.10).

Applying Lemma 3.1 to the inequality (3.21) and using (3.15), we get the required inequality in (3.9). □

Remark 3.2. If $g(x) = x$, inequality (3.8) can be reduced to the case discussed by Kelong Cheng el al. [5, Theorem 7].

4. Nonlinear Weakly Singular Integral Inequalities

Theorem 4.1. *Let $a(t)$, $b(t)$, $c(t)$, $f(t)$, $n(t)$ and g be as in Theorem 3.1. Suppose that $u(t) \in C(I, \mathbb{R}_+)$ satisfies*

$$\begin{aligned} u^p(t) \leq & a(t) + b(t) \int_0^t (t^{\alpha_1} - s^{\alpha_1})^{\beta_1 - 1} s^{\gamma_1 - 1} f(s) u^q(s) ds \\ & + c(t) \int_0^T (T^{\alpha_2} - s^{\alpha_2})^{\beta_2 - 1} s^{\gamma_2 - 1} n(s) \sqrt[m_1]{g(u^r(s))} ds, \end{aligned} \tag{4.1}$$

if

$$\int_0^T g'(m(s)) n^{m_1}(s) l(s) ds < 1,$$

then

$$u(t) \leq \left(m(t) + \frac{l(t) \int_0^T n^{m_1}(s) g(m(s)) ds}{1 - \int_0^T g'(m(s)) n^{m_1}(s) l(s) ds} \right)^{\frac{1}{r}}, \tag{4.2}$$

for $t \in I$, where $p \geq q \geq 0$, $p \geq r \geq 0, m_1, m_2, p, q$ and r are constants, such that $\frac{1}{m_1} + \frac{1}{m_2} = 1$, and

$$\begin{aligned} m(t) =& \frac{r}{pm_1} K^{\frac{\frac{r}{m_1} - p}{p}} \overline{b^*}(t) \int_0^t Q(s) \exp(\int_s^t P(\tau) d\tau) ds + \\ & + \frac{r}{pm_1} K^{\frac{\frac{r}{m_1} - p}{p}} a^*(t) + \frac{p - \frac{r}{m_1}}{p} K^{\frac{r}{pm_1}}, \\ l(t) =& \frac{r}{pm_1} K^{\frac{\frac{r}{m_1} - p}{p}} \overline{b^*}(t) \exp(\int_0^t P(s) ds), \\ \overline{b^*}(t) =& b^*(t) + c^*(t), \\ P(t) =& \frac{q}{p} K^{\frac{q-p}{P}} f^{m_1}(t) \overline{b^*}(t), \\ Q(t) =& f^{m_1}(t) \left(\frac{q}{p} K^{\frac{q-p}{p}} a^*(t) + \frac{p-q}{p} K^{\frac{q}{p}} \right). \end{aligned} \tag{4.3}$$

$$\begin{aligned} a^*(t) =& 3^{m_1 - 1} a^{m_1}(t), \\ b^*(t) =& 3^{m_1 - 1} b(t)^{m_1} (M_1 t^{\theta 1})^{\frac{m_1}{m_2}}, \\ c^*(t) =& 3^{m_1 - 1} c(t)^{m_1} (M_2 T^{\theta_2})^{\frac{m_1}{m_2}}, \end{aligned} \tag{4.4}$$

$$\begin{aligned} M_i =& \frac{1}{\alpha_i} B \left[\frac{m_2(\gamma_i - 1) + 1}{\alpha_i}, m_2(\beta_i - 1) + 1 \right], \\ \theta_i =& m_2 \left[\alpha_i(\beta_i - 1) + \gamma_i - 1 \right] + 1, \quad i = 1, 2, \end{aligned} \tag{4.5}$$

where the parameter group $[\alpha_i, \beta_i, \gamma_i]$ satisfies (2.1)-(2.2).

Proof. From assumptions (2.1)-(2.2), using the Hölder inequality with indices m_1, m_2 to (4.1), we get

$$\begin{aligned} u^p(t) \leq & a(t) + b(t)\left(\int_0^t (t^{\alpha_1} - s^{\alpha_1})^{m_2(\beta_1-1)} s^{m_2(\gamma_1-1)} ds\right)^{\frac{1}{m_2}} \\ & \times \left(\int_0^t f^{m_1}(s) u^{qm_1}(s) ds\right)^{\frac{1}{m_1}} \\ & + c(t)\left(\int_0^T (T^{\alpha_2} - s^{\alpha_2})^{m_2(\beta_2-1)} s^{m_2(\gamma_2-1)} ds\right)^{\frac{1}{m_2}} \\ & \times \left(\int_0^T n^{m_1}(s) g(u^r(s)) ds\right)^{\frac{1}{m_1}}. \end{aligned} \tag{4.6}$$

By using Lemmas 2.3 and 2.4, the inequality (4.6) can be rewritten as

$$\begin{aligned} u^{pm_1}(t) \leq & 3^{m_1-1} a^{m_1}(t) \\ & + 3^{m_1-1} b^{m_1}(t) \times \left(\int_0^t (t^{\alpha_1} - s^{\alpha_1})^{m_2(\beta_1-1)} s^{m_2(\gamma_1-1)} ds\right)^{\frac{m_1}{m_2}} \\ & \times \left(\int_0^t f^{m_1}(s) u^{qm_1}(s) ds\right) \\ & + 3^{m_1-1} c^{m_1}(t)\left(\int_0^T (T^{\alpha_2} - s^{\alpha_2})^{m_2(\beta_2-1)} s^{m_2(\gamma_2-1)} ds\right)^{\frac{m_1}{m_2}} \\ & \times \left(\int_0^T n^{m_1}(s) g(u^r(s)) ds\right) \\ = & 3^{m_1-1} a^{m_1}(t) + 3^{m_1-1} b^{m_1}(t)\left(M_1 t^{\theta_1}\right)^{\frac{m_1}{m_2}} \\ & \times \left(\int_0^t f^{m_1}(s) u^{qm_1}(s) ds\right) \\ & + 3^{m_1-1} c^{m_1}(t)\left(M_2 T^{\theta_2}\right)^{\frac{m_1}{m_2}}\left(\int_0^T n^{m_1}(s) g(u^r(s)) ds\right), \end{aligned}$$

where M_i, θ_i, $i = 1, 2$, are given in (4.5).

Letting $u^{m_1}(t) = w(t)$, we have

$$w^p(t) \leq a^*(t) + b^*(t)\int_0^t f^{m_1}(s) w^q(s) ds + c^*(t)\int_0^T n^{m_1}(s) g(w^{r_1}(s)) ds,$$

where $r_1 = \frac{r}{m_1}$, which is similar to inequality (3.8), where $a^*(t)$, $b^*(t)$ and $c^*(t)$ are given in (4.4). An application of Theorem 3.1 to the inequality above gives that

$$w(t) \leq \left(m(t) + \frac{l(t)\int\limits_0^T n^{m_1}(s) g(m(s)) ds}{1 - \int\limits_0^T g'(m(s)) n^{m_1}(s) l(s) ds}\right)^{\frac{m_1}{r}},$$

holds for $t \in I$, where $m(t)$ and $l(t)$ are given in (4.3). Since $u^{m_1}(t) = w(t)$, we can get (4.2). □

Remark 4.1. If $g(x) = x$, inequality (4.1) can be reduced to the case discussed by Kelong Cheng el al. [5, Theorem 12].

5. Applications

In this section, we present applications of the inequalities (4.1) in Theorem 4.1 for studying the boundedness of certain fractional integral equation with the Riemann-Liouville (R-L) fractional operator. Consider the following fractional integral equation:

$$u(t) = a(t) + I_0^{\alpha}(F(t,u(t))) + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} N(s,u(s))ds, \tag{5.1}$$

where $0 < \alpha < 1$ and $F, N \in C(R \times R, R)$, $a(t) \in C(I, R_+)$.

Theorem 5.1. *Consider the fractional integral equation* (5.1) *and suppose that* F *and* N *satisfy the following conditions*

$$\begin{aligned} |F(t,z)| &\le f(t)\,|z|^q\,, \\ |N(t,z)| &\le n(t)\,\sqrt[m_1]{g(z^r)}, \end{aligned} \tag{5.2}$$

where $f, n \in C(I, \mathbb{R}_+)$ *and* g *is defined as in Theorem* 3.1, $m_1 > 1 \ge q$, $r \ge 0$. *Under the condition*

$$\int_0^T g'(m(s))n^{m_1}(s)l(s)ds < 1,$$

the following estimate

$$u(t) \le \left(m(t) + \frac{l(t)\int_0^T n^{m_1}(s)g(m(s))ds}{1 - \int_0^T g'(m(s))n^{m_1}(s)l(s)ds} \right)^{\frac{1}{r}}, \tag{5.3}$$

holds, where

$$\begin{aligned} m(t) =& \frac{r}{m_1}K^{\frac{r}{m_1}-1}\overline{b^*}(t)\int_0^t Q(s)\exp\left(\int_s^t P(\tau)d\tau\right)ds + \frac{r}{m_1}K^{\frac{r}{m_1}-1}a^*(t) \\ & \alpha \\ &+ \left(1 - \frac{r}{m_1}\right)K^{\frac{r}{m_1}}, \\ l(t) =& \frac{r}{m_1}K^{\frac{r}{m_1}-1}\overline{b^*}(t)\exp\left(\int_0^t P(s)ds\right), \\ \overline{b^*}(t) =& b^*(t) + c^*(t), \\ P(t) =& qK^{q-1}f^{m_1}(t)\overline{b^*}(t), \\ Q(t) =& f^{m_1}(t)(qK^{q-1}a^*(t) + (1-q)K^q) \end{aligned}$$

and

$$\begin{aligned}
a^*(t) &= 3^{m_1-1}a^{m_1}(t),\\
b^*(t) &= \frac{3^{m_1-1}}{\Gamma^{m_1}(\alpha)}(M_1 t^{\theta 1})^{\frac{m_1}{m_2}},\\
c^*(t) &= \frac{3^{m_1-1}}{\Gamma^{m_1}(\alpha)}(M_2 T^{\theta_2})^{\frac{m_1}{m_2}},\\
M_1 &= M_2 = B\left[1, m_2(\alpha-1)+1\right],\\
\theta_1 &= \theta_2 = m_2(\alpha-1)+1.
\end{aligned}$$

Proof. According to Definition 2.1, from (5.1)-(5.2), we have

$$u(t) = a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(F(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}N(s,u(s))ds,$$

for $t \in I$. Hence,

$$\begin{aligned}
|u(t)| \le & a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left|(F(s,u(s))\right| ds + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}\left|N(s,u(s))\right| ds\\
\le & a(t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\left|u(s)\right|^q ds\\
& + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} n(s) \sqrt[m_1]{g(u^r(s))}ds.
\end{aligned}$$

Letting $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\beta_1 = \beta_2 = \alpha$, $p = 1, b(t) = \frac{1}{\Gamma(\alpha)}$ and $c(t) = \frac{1}{\Gamma(\alpha)}$, and applying Theorem 4.1, we get the desired estimate in (5.3). □

Proposition 5.1. *Assume that the functions F and N in* (5.2) *satisfy the conditions*

$$\begin{aligned}
&\left|F(t,z) - F(t,\overline{z})\right| \le f(t)\left|z-\overline{z}\right|,\\
&\left|N(t,z)\right| - N(t,\overline{z}) \le n(t) \sqrt[1+\epsilon]{|z-\overline{z}|},
\end{aligned} \tag{5.4}$$

where $f(t)$ and $n(t)$ are defined as in Theorem 4.1*, $\epsilon > 0$ and $z(t)$ is a solution of* (5.1)*. Then* (5.1) *has at most one solution.*

Proof. Let $z(t)$ and $\overline{z}(t)$ be two solutions of (5.1), it is easy to see from (5.4) that

$$\begin{aligned}
|z(t) - \overline{z}(t)| \le & \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s)\left|z(s) - \overline{z}(s)\right| ds\\
& + \frac{1}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1} n(s) \sqrt[1+\varepsilon]{|z(s) - \overline{z}(s)|}ds.
\end{aligned}$$

Letting $\alpha_1 = \alpha_2 = 1$, $\gamma_1 = \gamma_2 = 1$, $\beta_1 = \beta_2 = \alpha$, $p = q = r = 1$, $m_1 = 1+\epsilon$, $a(t) = 0$, $g(t) = t$ and applying Theorem 4.1, we obtain that

$$|z(t) - \overline{z}(t)| \leq \left(\left(1 - \frac{1}{1+\epsilon}\right) K^{\frac{1}{1+\epsilon}} + \frac{\left(1 - \frac{1}{1+\epsilon}\right) K^{\frac{1}{1+\epsilon}} l(t) \int\limits_0^T n^{1+\epsilon}(s) ds}{1 - \int\limits_0^T n^{1+\epsilon}(s) l(s) ds} \right),$$

letting $\epsilon \to 0$, we obtain the uniqueness of solution of equation (5.1). $\square$

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