EXISTENCE OF RENORMALIZED SOLUTIONS FOR SOME ANISOTROPIC QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we consider a class of anisotropic quasilinear elliptic equations of the type
\[
\begin{cases}
- \sum_{i=1}^{N} \partial^i a_i(x, u, \nabla u) + |u|^{s(x)-1} u = f(x, u), & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]
where $f(x, s)$ is a Carathéodory function which satisfies some growth condition. We prove the existence of renormalized solutions for our Dirichlet problem, and some regularity results are concluded.

1. Introduction

Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the smooth boundary $\partial \Omega$. Zhao et al. have studied in [17] the quasilinear elliptic problem
\[
\begin{cases}
- \text{div}(a(x, \nabla u)) + |u|^{p-2} u = \lambda f(x, u), & \text{in } \Omega, \\
\int_{\partial \Omega} a(x, \nabla u) \cdot n ds = 0, \\
u = \text{constant} & \text{on } \partial \Omega,
\end{cases}
\]
They have proved the existence of weak solutions under some suitable growth assumptions on $f(x, s)$, (see also [2, 7]). In the framework of Sobolev spaces with variable exponents, Fan and Zhang [11] have considered the following nonlinear elliptic problem
\[
\begin{cases}
\text{div}(|\nabla u|^{p(x)-2} \nabla u) = \lambda f(x, u), & \text{in } \Omega, \\
u = 0 & \text{on } \partial \Omega,
\end{cases}
\]

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where $\lambda > 0$ and $f(x, s)$ satisfies the growth condition $|f(x, s)| \leq \eta + \theta |s|^\delta - 1$, where $1 \leq \delta \leq p^*$ and $\eta, \theta$ are two positive constants (we refer also to [6]). In [3], the authors have proved the existence of weak solutions for the quasilinear $p(x)$-elliptic problem

$$-\text{div} a(x, u, \nabla u) = f(x, u, \nabla u),$$

by using the calculus of variations operators method, where $f(x, s, \xi)$ is a Carathéodory function which satisfies some growth condition.

In the framework of anisotropic Sobolev spaces, Di Nardo, Feo and Guibé have studied in [9] the existence of renormalized solutions for some class of nonlinear anisotropic elliptic problems of the type

$$-\sum_{i=1}^{N} \partial_x a_i(x, u)|\partial_x u|^{p_i-2} \partial_x u = f - \text{div} g, \quad \text{in } \Omega,$$

with $f \in L^1(\Omega)$ and $g \in \prod_{i=1}^{N} L^{p_i}\text{d}(\Omega)$, the uniqueness of renormalized solution was concluded under some local Lipschitz conditions on the function $a_i(x, s)$ with respect to $s$, (see also [1] and [8]).

The aim of this paper is to study the existence and regularity of renormalized solutions for the anisotropic quasilinear elliptic problem

\begin{equation}
\begin{cases}
-\sum_{i=1}^{N} \partial_x a_i(x, u, \nabla u) + |u|^{s(x)-1} u = f(x, u), & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $(a_i(x, s, \xi))_{i=1,...,N}$ are Carathéodory functions, the right-hand side $f(x, s)$ is a Carathéodory function satisfying only some nonstandard growth condition.

One of our motivations for studying (1.1) comes from these applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed in [14,15], also in the robotics and space technology (we refer for example to [16]).

One of the difficulties in proving the existence of renormalized solutions stems from the nonstandard growth of the Carathéodory function $f(x, s)$, to overcome the difficulty, we use the regularizing effect of the term $|u|^{s(x)-1} u$ with some special technics.

The rest of this paper is structured as follows. In Section 2 we recall some definitions and results on the anisotropic variable exponent Sobolev spaces. We introduce in Section 3 some assumptions for which our problem has at least one renormalized solution. Section 4 will be devoted to show the existence of renormalized solutions $u$ for the problem (1.1) in the anisotropic Sobolev space with variable exponents, and we will give some regularity results, that is $|u|^{s(x)-1} u \in L^1(\Omega)$. 
2. Preliminary

Let Ω be a bounded open subset of \( \mathbb{R}^N \), \( N \geq 2 \), we denote

\[ \mathcal{C}_+(\Omega) = \{ \text{measurable function } p(\cdot) : \Omega \to \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < N \}, \]

where

\[ p^- = \text{ess inf}\{ p(x)/x \in \Omega \} \text{ and } p^+ = \text{ess sup}\{ p(x)/x \in \Omega \}. \]

We define the Lebesgue space with variable exponent \( L^{p(\cdot)}(\Omega) \) as the set of all measurable functions \( u : \Omega \to \mathbb{R} \) for which the convex modular

\[ \rho_{p(\cdot)}(u) := \int_{\Omega} |u|^{p(x)} \, dx \]

is finite. If the exponent is bounded, i.e., if \( p^+ < +\infty \), then the expression

\[ \|u\|_{p(\cdot)} = \inf\{ \lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1 \} \]

defines a norm in \( L^{p(\cdot)}(\Omega) \), called the Luxemburg norm. The space \((L^{p(\cdot)}(\Omega), \| \cdot \|_{p(\cdot)})\) is a separable Banach space. Moreover, if \( 1 < p^- \leq p^+ < +\infty \), then \( L^{p(\cdot)}(\Omega) \) is uniformly convex, hence reflexive, and its dual space is isomorphic to \( L^{p'(\cdot)}(\Omega) \), where \( \frac{1}{p(x)} + \frac{1}{p'(x)} = 1 \). Finally, we have the Hölder type inequality:

\[ \left| \int_{\Omega} uv \, dx \right| \leq \left( \frac{1}{p^-} + \frac{1}{p'(x)} \right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)}, \]

for any \( u \in L^{p(\cdot)}(\Omega) \) and \( v \in L^{p'(\cdot)}(\Omega) \).

The Sobolev space with variable exponent \( W^{1,p(\cdot)}(\Omega) \) is defined by

\[ W^{1,p(\cdot)}(\Omega) = \{ u \in L^{p(\cdot)}(\Omega) \text{ and } \nabla u \in L^{p(\cdot)}(\Omega) \}, \]

which is a Banach space, equipped with the following norm

\[ \|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}. \]

The space \((W^{1,p(\cdot)}(\Omega), \| \cdot \|_{1,p(\cdot)})\) is a separable and reflexive Banach space. We define \( W^{1,p(\cdot)}_0(\Omega) \) as the closure of \( \mathcal{C}_0(\Omega) \) in \( W^{1,p(\cdot)}(\Omega) \). For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [10].

Now, we present the anisotropic variable exponent Sobolev space, used in the study of our quasilinear anisotropic elliptic problem.

Let \( p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot) \) be \( N \) variable exponents in \( \mathcal{C}_+(\Omega) \). We denote

\[ \bar{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot)) \]

and define

\[ p^+ = \max\{ p_1^-, \ldots, p_N^- \} \]
\[ p^- = \min\{ p_1^-, \ldots, p_N^- \}, \]

then \( 1 < p^- \leq p^+ \).

The anisotropic variable exponent Sobolev space \( W^{1,\bar{p}(\cdot)}(\Omega) \) is defined as follow

\[ W^{1,\bar{p}(\cdot)}(\Omega) = \{ u \in W^{1,1}(\Omega) \text{ and } D^i u \in L^{p_i(\cdot)}(\Omega) \text{ for } i = 1, 2, \ldots, N \}, \]
endowed with the norm

\[(2.1) \quad \|u\|_{1,p(\cdot)} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}.
\]

We define also \(W_{0}^{1,p(\cdot)}(\Omega)\) as the closure of \(C_0^\infty(\Omega)\) in \(W^{1,p(\cdot)}(\Omega)\) with respect to the norm (2.1). The space \(\left(W_{0}^{1,p(\cdot)}(\Omega), \|u\|_{1,p(\cdot)}\right)\) is a reflexive Banach space (cf. [13]).

**Remark 2.1.** In view of the continuous embedding \(W_{0}^{1,p(\cdot)}(\Omega) \hookrightarrow W_{0}^{1,1}(\Omega)\) and the Poincaré type inequality we conclude that the two norms \(\|u\|_{1,p(\cdot)}\) and \(\sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}\) are equivalent in the anisotropic variable exponent Sobolev spaces.

**Lemma 2.1.** We have the following continuous and compact embeddings.

- If \(\overline{p}^- < N\), then \(W_{0}^{1,p(\cdot)}(\Omega) \hookrightarrow \hookrightarrow L^q(\Omega),\) for \(q \in [\overline{p}^-, \overline{p}^*]\), where \(\overline{p}^* = \frac{N\overline{p}^-}{N-\overline{p}^-}\).
- If \(\overline{p}^- = N\), then \(W_{0}^{1,p(\cdot)}(\Omega) \hookrightarrow \hookrightarrow L^q(\Omega),\) for all \(q \in [\overline{p}^-, +\infty[\).
- If \(\overline{p}^- > N\), then \(W_{0}^{1,p(\cdot)}(\Omega) \hookrightarrow \hookrightarrow L^\infty(\Omega) \cap C^0(\Omega)\).

The proof of this lemma follows from the fact that the embedding \(W_{0}^{1,p(\cdot)}(\Omega) \hookrightarrow W_{0}^{1,\overline{p}^-}(\Omega)\) is continuous, and in view of the compact embedding theorem for Sobolev spaces.

**Proposition 2.1.** The dual of \(W_{0}^{1,p(\cdot)}(\Omega)\) is denote by \(W^{-1,\tilde{p}(\cdot)}(\Omega)\), where \(\tilde{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))\) and \(\frac{1}{p_i(x)} + \frac{1}{\tilde{p}_i(x)} = 1\) (cf. [5] for the constant exponent case). For each \(F \in W^{-1,\tilde{p}(\cdot)}(\Omega)\) there exists \(F_0 \in (L^\overline{p}_i(\Omega))^'\) and \(F_i \in L^{p_i(\cdot)}(\Omega)\) for \(i = 1, 2, \ldots, N\), such that \(F = F_0 - \sum_{i=1}^{N} D^i F_i\). Moreover, for any \(u \in W_{0}^{1,p(\cdot)}(\Omega)\), we have

\[
\langle F, u \rangle = \sum_{i=0}^{N} \int_{\Omega} F_i D^i u dx.
\]

We define a norm on the dual space by

\[
\|F\|_{-1,\tilde{p}(\cdot)} = \inf \left\{ \sum_{i=0}^{N} \|F_i\|_{\tilde{p}_i(\cdot)} \mid \text{with } F = F_0 - \sum_{i=1}^{N} D^i F_i \text{ such that } F_0 \in (L^\overline{p}_i(\Omega))^' \right. \\
\left. \quad \text{and } F_i \in L^{p_i(\cdot)}(\Omega) \right\}.
\]

**Definition 2.1.** Let \(k > 0\), the truncation function \(T_k(\cdot) : \mathbb{R} \to \mathbb{R}\) is defined by

\[
T_k(s) = \begin{cases} 
   s, & \text{if } |s| \leq k, \\
   k \frac{s}{|s|}, & \text{if } |s| > k,
\end{cases}
\]

and we define

\[
W_{0}^{1,p(\cdot)}(\Omega) := \{ u : \Omega \to \mathbb{R} \text{ measurable, such that } T_k(u) \in W_{0}^{1,\tilde{p}(\cdot)}(\Omega) \text{ for any } k > 0 \}.
\]
Proposition 2.2. Let $u \in T_0^{1,p}(\Omega)$. For any $i \in \{1, \ldots, N\}$, there exists a unique measurable function $v_i: \Omega \to \mathbb{R}$ such that

$$D^i T_k(u) = v_i \cdot \chi_{\{|v| < k\}} \text{ a.e. } x \in \Omega, \text{ for all } k > 0,$$

where $\chi_A$ denotes the characteristic function of a measurable set $A$. The functions $v_i$ are called the weak partial derivatives of $u$ and are still denoted $D^i u$. Moreover, if $u$ belongs to $W^{1,1}_0(\Omega)$, then $v_i$ coincides with the standard distributional derivative of $u$, that is, $v_i = D^i u$.

3. Essential Assumptions

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N (N \geq 2)$. We consider $\vec{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot))$ the vector of exponents $p_i(\cdot) \in C_+(\Omega)$ for $i = 1, \ldots, N$, and let $q(\cdot), s(\cdot) \in C_+(\Omega)$ where

$$q(x) < \max(s(x), \vec{p}^+ - 1) \text{ a.e. in } \Omega.$$

We consider the Leray–Lions operator $A$ acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}^/(\cdot)}(\Omega)$, defined by the formula

$$Au = -\sum_{i=1}^N \partial^i a_i(x, u, \nabla u),$$

where $a_i: \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory function which satisfy the following conditions

\begin{align*}
(3.1) \quad |a_i(x, s, \xi)| &\leq \beta(K_i(x) + |s|^{p_i(x)-1} + |\xi|^{p_i(x)-1}), \quad \text{for any } i = 1, \ldots, N, \\
(3.2) \quad a_i(x, s, \xi)\xi_i &\geq \alpha |\xi_i|^{p_i(x)}, \quad \text{for any } i = 1, \ldots, N,
\end{align*}

for all $\xi = (\xi_1, \ldots, \xi_N)$ and $\xi' = (\xi'_1, \ldots, \xi'_N)$, we have

$$|a_i(x, s, \xi) - a_i(x, s, \xi')|(|\xi_i - \xi'_i|) > 0, \quad \text{for } \xi_i \neq \xi'_i,$$

for a.e. $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K_i(x)$ is a positive function lying in $L^{p_i(x)}(\Omega)$ and $\alpha, \beta > 0$.

As a consequence of (3.2) and the continuity of the function $a_i(x, s, \cdot)$ with respect to $\xi$, we have

$$a_i(x, s, 0) = 0.$$

In this paper, we consider the following quasilinear anisotropic elliptic problem

\begin{equation}
\begin{cases}
-\sum_{i=1}^N \partial^i a_i(x, u, \nabla u) + |u|^{q(x)-1} u = f(x, u), \quad \text{in } \Omega, \\
\qquad u = 0, \quad \text{on } \partial \Omega,
\end{cases}
\end{equation}

where $f: \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x, r)| \leq g(x) + |r|^q(x) \text{ a.e in } \Omega,$$

and $g(\cdot)$ is a measurable positive function in $L^1(\Omega)$.
Remark 3.1. The assumption (3.1) is used here to ensure that \( a_i(x, u, \nabla u) \) belongs to \( L^{p_i}(\Omega) \). In the other case where \( Au = -\sum_{i=1}^{N} \partial^i a_i(x, \nabla u) \), the uniqueness of solution can be concluded under some additional conditions on the Carathéodory function \( f(x, s) \).

4. Main Results

We begin by recalling some important lemmas useful to prove our main result.

Lemma 4.1 ([3]). Let \( g \in L^{r}(\Omega) \) and \( g_n \in L^{r}(\Omega) \) with \( \|g_n\|_{r(\cdot)} \leq C \) for \( 1 < r(\cdot) < \infty \). If \( g_n(x) \to g(x) \) a.e. on \( \Omega \), then \( g_n \to g \) in \( L^{r}(\Omega) \).

Lemma 4.2 ([4]). Assuming that (3.1)-(3.3) hold, and let \( (u_n)_{n \in \mathbb{N}} \) be a sequence in \( W_0^{1,p(x)}(\Omega) \) such that \( u_n \to u \) in \( W_0^{1,p(x)}(\Omega) \) and

\[
\int_{\Omega} (|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u)(u_n - u)dx \\
+ \sum_{i=1}^{N} \int_{\Omega} (a_i(x, u_n, \nabla u_n) - a_i(x, u_n, \nabla u))(D^i u_n - D^i u)dx \to 0,
\]

then \( u_n \to u \) in \( W_0^{1,p(x)}(\Omega) \) for a subsequence.

Our objective is to prove the existence of renormalized solutions for the quasilinear anisotropic elliptic problem (3.4).

Definition 4.1. A measurable function \( u \) is called renormalized solution of the quasilinear elliptic problem (3.4) if \( T_k(u) \in W_0^{1,p(x)}(\Omega) \) for any \( k > 0 \), with \( f(x, u) \in L^1(\Omega) \), and

\[
(4.1) \quad \lim_{h \to \infty} \sum_{i=1}^{N} \int_{\{h < |u| \leq h+1\}} a_i(x, u, \nabla u) D^i u dx = 0,
\]

such that \( u \) satisfies the following equality

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u)(S'(u)\varphi D^i u + S(u)D^i \varphi)dx + \int_{\Omega} |u|^{p(x)-1} u S(u) \varphi dx \\
= \int_{\Omega} f(x, u) S(u) \varphi dx,
\]

for every \( \varphi \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega) \) and for any smooth function \( S(\cdot) \in W^{1,\infty}(\mathbb{R}) \) with a compact support.

Theorem 4.1. Assuming that the conditions (3.1)-(3.3) and (3.5) hold true, then the quasilinear anisotropic elliptic problem (3.4) has at least one renormalized solution. Moreover, we have

\[
|u|^{p(x)} \in L^1(\Omega).
\]
4.1. Proof of Theorem 4.1.

Step 1: approximate problems. Firstly, we consider the approximate problem

\[
\begin{cases}
A_n u_n + |T_n(u_n)|^{s(x)-1}T_n(u_n) = f_n(x, T_n(u_n)), & \text{in } \Omega, \\
u_n = 0, & \text{on } \partial\Omega,
\end{cases}
\]

where \(A_n v = -\sum_{i=1}^N \partial_i a_i(x, T_n(v), \nabla v)\) and \(f_n(x, r) = T_n(f(x, r))\). Thanks to (3.5), it’s clear that

\[|f_n(x, r)| \leq n \quad \text{and} \quad |f_n(x, r)| \leq g(x) + |r|^{q(x)}.
\]

We consider the operator \(G_n : W_0^{1,p(\cdot)}(\Omega) \to W^{-1,p(\cdot)}(\Omega)\) by

\[
\langle G_n u, v \rangle = \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)vdx - \int_\Omega f_n(x, T_n(u))vdx,
\]

for any \(u, v \in W_0^{1,p(\cdot)}(\Omega)\). In view of the generalized Hölder-type inequality, we have

\[
\langle G_n u, v \rangle \leq \int_\Omega |T_n(u)|^{s(x)} |v|dx + \int_\Omega |f_n(x, T_n(u))| |v|dx \\
\leq n^{s^+} \int_\Omega |v|dx + n \int_\Omega |v|dx \\
= (n^{s^+} + n)\|v\|_1 \\
\leq C_1\|v\|_{1,p(\cdot)}.
\]

Lemma 4.3. The bounded operator \(B_n = A_n + G_n\) acted from \(W_0^{1,p(\cdot)}(\Omega)\) into \(W^{-1,p(\cdot)}(\Omega)\) is pseudo-monotone. Moreover, \(B_n\) is coercive in the following sense:

\[
\frac{\langle B_n v, v \rangle}{\|v\|_{1,p(\cdot)}} \to +\infty \quad \text{as} \quad \|v\|_{1,p(\cdot)} \to \infty, \quad \text{for any} \quad v \in W_0^{1,p(\cdot)}(\Omega).
\]

Proof. In view of the Hölder’s inequality and the growth condition (3.1), it’s easy to see that the operator \(A_n\) is bounded, and by (4.3) we conclude that \(B_n\) is bounded. For the coercivity, we have for any \(u \in W_0^{1,p(\cdot)}(\Omega),\)

\[
\langle B_n u, u \rangle = \langle A_n u, u \rangle + \langle G_n u, u \rangle \\
= \sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u) \, Du^i dx + \int_\Omega |T_n(u)|^{s(x)}|u|dx \\
- \int_\Omega |f_n(x, T_n(u))| |u|dx \\
\geq \alpha \sum_{i=1}^N \int_\Omega |Du^i|^{p(x)}dx + \int_\Omega |T_n(u)|^{s(x)+1}dx - C_2n\|u\|_{p(\cdot)} \\
\geq C_0 \|u\|_{1,p(\cdot)}^p - \alpha N|\Omega| - C_2n\|u\|_{1,p(\cdot)},
\]

it follows that

\[
\frac{\langle B_n u, u \rangle}{\|u\|_{1,p(\cdot)}} \to +\infty \quad \text{as} \quad \|u\|_{1,p(\cdot)} \to \infty.
\]
It remains to show that $B_n$ is pseudo-monotone. Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $W_0^{1,\tilde{p}(\cdot)}(\Omega)$ such that

\begin{equation}
\begin{cases}
u_k \rightharpoonup u, & \text{in } W_0^{1,\tilde{p}(\cdot)}(\Omega), \\
B_n u_k \rightharpoonup \chi_n, & \text{in } W^{-1,\tilde{p}(\cdot)}(\Omega), \\
\limsup_{k \to \infty} \langle B_n u_k, u_k \rangle \leq \langle \chi_n, u \rangle.
\end{cases}
\end{equation}

We will prove that $\chi_n = B_n u$ and $\langle B_n u_k, u_k \rangle \to \langle \chi_n, u \rangle$ as $k \to \infty$.

In view of the compact embedding $W_0^{1,\tilde{p}(\cdot)}(\Omega) \hookrightarrow L^1(\Omega)$, we have $u_k \to u$ in $L^1(\Omega)$ and a.e. $\Omega$, for a subsequence still denoted $(u_k)_{k \in \mathbb{N}}$.

We have $(u_k)_{k \in \mathbb{N}}$ is a bounded sequence in $W_0^{1,\tilde{p}(\cdot)}(\Omega)$, using the growth condition (3.1) it’s clear that the sequence $(a_i(x, T_n(u_k), \nabla u_k))_{k \in \mathbb{N}}$ is bounded in $L^{p_i(\cdot)}(\Omega)$, then there exists a function $\varphi_i \in L^{p_i(\cdot)}(\Omega)$ such that

\begin{equation}
a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup \varphi_i \text{ in } L^{p_i(\cdot)}(\Omega) \text{ as } k \to \infty.
\end{equation}

On the one hand we have

\begin{equation}
|T_n(u_k)|^{s(x)-1} T_n(u_k) \rightharpoonup |T_n(u)|^{s(x)-1} T_n(u) \text{ weak-* in } L^\infty(\Omega),
\end{equation}

and since $f_n(x, T_n(s))$ is a Carathéodory function, then

\begin{equation}
f_n(x, T_n(u_k)) \rightharpoonup f_n(x, T_n(u)) \text{ weak-* in } L^\infty(\Omega).
\end{equation}

Then, for any $v \in W_0^{1,\tilde{p}(\cdot)}(\Omega)$ we have

\begin{equation}
\langle \chi_n, v \rangle = \lim_{k \to \infty} \langle B_n u_k, v \rangle = \lim_{k \to \infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i v dx + \lim_{k \to \infty} \int_\Omega |T_n(u_k)|^{s(x)-1} T_n(u_k) v dx - \lim_{k \to \infty} \int_\Omega f_n(x, T_n(u_k)) v dx
\end{equation}

\begin{equation}
\begin{aligned}
&= \sum_{i=1}^N \int_\Omega \varphi_i D^i v dx + \int_\Omega |T_n(u)|^{s(x)-1} T_n(u) v dx - \int_\Omega f_n(x, T_n(u)) v dx.
\end{aligned}
\end{equation}

Having in mind (4.4) and (4.8), we conclude that

\begin{equation}
\limsup_{k \to \infty} \langle B_n(u_k), u_k \rangle = \limsup_{k \to \infty} \left( \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx + \int_\Omega |T_n(u_k)|^{s(x)-1} T_n(u_k) u_k dx - \int_\Omega f_n(x, T_n(u_k)) u_k dx \right)
\end{equation}

\begin{equation}
\leq \sum_{i=1}^N \int_\Omega \varphi_i D^i u dx + \int_\Omega |T_n(u)|^{s(x)-1} T_n(u) u dx - \int_\Omega f_n(x, T_n(u)) u dx.
\end{equation}
Since \( u_k \to u \) strongly in \( L^1(\Omega) \), and thanks to (4.6)–(4.7) we obtain
\[
\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k \, dx \to \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u \, dx
\]
and
\[
\int_\Omega f_n(x, T_n(u_k))u_k \, dx \to \int_\Omega f_n(x, T_n(u))u \, dx.
\]
Therefore,
\[
\limsup_{k \to \infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k)D^i u_k \, dx \leq \sum_{i=1}^N \int_\Omega \varphi_i \, D^i u \, dx.
\]
On the other hand, in view of (3.3) we have
\[
\sum_{i=1}^N \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx \geq 0,
\]
then
\[
\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k)D^i u_k \, dx \geq \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u)D^i u \, dx + \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u) \, dx.
\]
In view of Lebesgue’s dominated convergence theorem we have \( T_n(u_k) \to T_n(u) \) in \( L^{p(x)}(\Omega) \), thus \( a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u) \) strongly in \( L^{p(x)}(\Omega) \), and using (4.5) we get
\[
\liminf_{k \to \infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k)D^i u_k \, dx \geq \sum_{i=1}^N \int_\Omega \varphi_i \, D^i u \, dx.
\]
Having in mind (4.11), we conclude that
\[
\lim_{k \to \infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k)D^i u_k \, dx = \sum_{i=1}^N \int_\Omega \varphi_i \, D^i u \, dx.
\]
Therefore, by combining (4.8) and (4.9)–(4.10), we conclude that
\[
\langle B_n u_k, u_k \rangle \to \langle \chi_n, u \rangle \text{ as } k \to \infty.
\]
Now, by (4.13) we can prove that
\[
\lim_{k \to \infty} \left( \sum_{i=1}^N \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) \, dx \right.
\]
\[
+ \int_\Omega (|u_k|^{s+2} - |u|^{s+2})u_k - |u|^{s+2}u)(u_k - u) \, dx \right) = 0,
\]
and so, by virtue of Lemma 4.2, we get
\[
u_k \to u \text{ in } W^{1, s(x)}_0(\Omega) \text{ and } D^i u_k \to D^i u \text{ a.e. in } \Omega,
\]
we obtain
\[ a_i(x, T_n(u_k), \nabla u_k) \to a_i(x, T_n(u), \nabla u) \in L^p(\Omega), \quad \text{for } i = 1, \ldots, N, \]
and thanks to (4.6)–(4.7), we obtain \( \chi_n = B_n u \), which conclude the proof of Lemma 4.3.

In view of Lemma 4.3, there exists at least one weak solution \( u_n \in W^{1,p}(\Omega) \) of the approximate problem (4.2) (cf. [12], Theorem 2.7, page 180).

**Step 2: a priori estimates.** Choose \( 1 < \theta < p^- \) such that \( 1 \leq q(x) < \max(s(x), p^+ - \theta) \).

By taking \( \varphi(u_n) = \left(1 - \frac{1}{(1 + |u_n|)^{p^-}}\right) \text{sign}(u_n) \in W^{1,p}(\Omega) \) as a test function in (4.2), we obtain
\[
(\theta - 1) \sum_{i=1}^{N} \int_{\Omega} \frac{a_i(x, T_n(u_n), \nabla u_n)}{(1 + |u_n|)^{\theta}} \, dx + \int_{\Omega} |T_n(u_n)|^{s(x)} \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) \, dx = \int_{\Omega} f_n(x, T_n(u_n)) \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) \text{sign}(u_n) \, dx.
\]

By using the coercivity (3.2) and the growth condition (3.5), we obtain
\[
\alpha(\theta - 1) \sum_{i=1}^{N} \int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} \, dx + \int_{\Omega} |T_n(u_n)|^{s(x)} \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) \, dx \leq \int_{\Omega} (|g(x)| + |T_n(u_n)|^{g(x)}) \left(1 - \frac{1}{(1 + |u_n|)^{\theta-1}}\right) \, dx.
\]

For the first term on the left hand side of (4.14), for any \( i = 1, \ldots, N \), we have
\[
\int_{\Omega} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} \, dx \geq \int_{\Omega} \frac{|D^i u_n|^{p_i^-}}{(1 + |u_n|)^{\theta}} \, dx - |\Omega|
\]
\[
= \int_{\Omega} \left| \frac{D^i u_n}{(1 + |u_n|)^{\theta}} \right|^{p_i^-} \, dx - |\Omega|
\]
\[
= \int_{\Omega} \left| \frac{D^i \int_{0}^{u_n} \frac{ds}{(1 + s)^{r_i}}}{(1 + |u_n|)^{\theta}} \right|^{p_i^-} \, dx - |\Omega|
\]
\[
\geq \frac{1}{C_p} \int_{\Omega} \left| \frac{u_n}{(1 + |u_n|)^{\theta}} \right|^{p_i^-} \, dx - |\Omega|
\]
\[
\geq \frac{1}{C_p} \int_{\Omega} \left| u_n \right|^{p_i^- - \theta} \, dx - 2|\Omega|
\]

\[ \geq \frac{1}{2^{\theta-1}} \int_{\Omega} \left| u_n \right|^{p_i^- - \theta} \, dx - 2|\Omega|, \]
Thus, the sequence \( \varphi(u_n) \geq \frac{1}{2} \) for \(|u_n| \geq R\), with \( R = 2^{\frac{1}{s}} - 1 \). Using Young’s inequality it follows that

\[
\frac{\alpha(\theta - 1)}{2^\theta C_p} \sum_{i=1}^{N} \int_{\Omega} |u_n|^{p_i^- - \theta} dx + \frac{1}{2} \int_{\{|u_n| \geq R\}} |T_n(u_n)|^{s(x)} dx \leq \int_{\Omega} |g(x)| dx + \int_{\Omega} |T_n(u_n)|^{q(x)} dx + 2\alpha N(\theta - 1)|\Omega|.
\]

Since \( 1 \leq q(x) < \max(s(x), p_i^+ - \theta) \), by using Young’s inequality we conclude that

\[
\int_{\Omega} |T_n(u_n)|^{q(x)} dx \leq \frac{\alpha(\theta - 1)}{2^\theta C_p} \sum_{i=1}^{N} \int_{\Omega} |u_n|^{p_i^- - \theta} dx + \frac{1}{4} \int_{\{|u_n| \geq R\}} |T_n(u_n)|^{s(x)} dx + C_0.
\]

It follows from (4.15) that there exists a constant \( C_1 \) that does not depend on \( n \), such that

\[
\sum_{i=1}^{N} \int_{\Omega} |u_n|^{p_i^- - \theta} dx + \int_{\Omega} |T_n(u_n)|^{s(x)} dx + \int_{\Omega} |T_n(u_n)|^{q(x)} dx \leq C_1.
\]

Let \( k \geq 1 \), in view of (4.14) we conclude that

\[
\frac{1}{(1 + k)^\theta} \sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx \leq \sum_{i=1}^{N} \int_{\Omega} \left| \frac{D^i u_n}{1 + |u_n|} \right|^{p_i} dx + \int_{\Omega} |T_n(u_n)|^{s(x)} dx \leq C_2.
\]

Therefore, we obtain

\[
\sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx \leq C_2(1 + k)^\theta, \quad \text{for } k \geq 1.
\]

Thus, the sequence \( (T_k(u_n))_n \) is bounded in \( W^{1, p_i(\cdot)}(\Omega) \), and there exists a subsequence still denoted \( (T_k(u_n))_n \) and \( \eta_k \in W^{1, p_i(\cdot)}(\Omega) \) such that

\[
\begin{align*}
T_k(u_n) &\to \eta_k \text{ in } W^{1, p_i(\cdot)}(\Omega), \\
T_k(u_n) &\to \eta_k \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega.
\end{align*}
\]

On the other hand, in view of Poincaré type inequality, for any \( i \in \{1, \ldots, N\} \) we have

\[
k^{p_i^-} \text{ meas}\{|u_n| > k\} = \int_{\{|u_n| > k\}} |T_k(u_n)|^{p_i^-} dx \leq \int_{\Omega} |T_k(u_n)|^{p_i^-} dx \leq C_{p_i^-} \int_{\Omega} |D^i T_k(u_n)|^{p_i^-} dx \leq C_{p_i^-} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx + C_{p_i^-} |\Omega| \leq \max_{1 \leq i \leq N}(C_{p_i^-}) \left( \sum_{i=1}^{N} \int_{\Omega} |D^i T_k(u_n)|^{p_i(x)} dx + |\Omega| \right) \leq C_3(1 + k)^\theta,
\]
Thus
\[
\text{(4.20)} \quad \text{meas}\{|u_n| > k\} \leq \frac{C_3(1+k)^\theta}{k^{\theta/2}} \to 0 \text{ as } k \to \infty.
\]

Moreover, by taking a subsequence, to some measurable function \(\eta_k\) we have
\[
\text{meas}\{|u_n - u_m| > \delta\} \leq \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.
\]

Let \(\varepsilon > 0\), in view of (4.20) we may choose \(k = k(\varepsilon)\) large enough such that
\[
\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \quad \text{and} \quad \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}.
\]

Moreover, thanks to (4.19) we have
\[
T_k(u_n) \to \eta_k \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega.
\]

Thus \((T_k(u_n))_{n \in \mathbb{N}}\) is a Cauchy sequence in measure, and for any \(k > 0\) and \(\delta, \varepsilon > 0\), there exists \(n_0 = n_0(k, \delta, \varepsilon)\) such that
\[
\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3}, \quad \text{for all } m, n \geq n_0(k, \delta, \varepsilon).
\]

By combining (4.21) and (4.22), we conclude that for all \(\delta, \varepsilon > 0\), there exists \(n_0 = n_0(\delta, \varepsilon)\) such that
\[
\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon, \quad \text{for any } n, m \geq n_0.
\]

Thus \((u_n)\) is a Cauchy sequence in measure, and converges almost everywhere, for a subsequence, to some measurable function \(u\). Thanks to (4.19) we conclude that
\[
T_k(u_n) \rightharpoonup T_k(u) \text{ in } W^{1,\tilde{p}(\cdot)}_0(\Omega).
\]

In view of Lebesgue dominated convergence theorem, we obtain
\[
T_k(u_n) \to T_k(u) \text{ in } L^{p_i(\cdot)}(\Omega), \quad \text{for } i = 1, \ldots, N.
\]

Moreover, by taking \(T_k(u_n)\) as a test function in the approximate problem (4.2), we have
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n)) \nabla u_n \cdot \nabla T_k(u_n) \, dx + \int_{\Omega} T_n(u_n)|^{p_i(x)}|T_k(u_n)| \, dx
\]
\[
= \int_{\Omega} f_n(x, T_n(u_n)) \, T_k(u_n) \, dx.
\]

In view of (3.2), (3.5), and using (4.17) we obtain
\[
\alpha \sum_{i=1}^{N} \int_{\Omega} |D^{i}T_k(u_n)|^{p_i(x)} \, dx \leq \int_{\Omega} g(x) \, |T_k(u_n)| \, dx + \int_{\Omega} T_n(u_n)|^{q(x)}|T_k(u_n)| \, dx
\]
\[
\leq k\|g(x)\|_{L^1(\Omega)} + k\|T_n(u_n)|^{q(x)}\|_{L^1(\Omega)}
\]
\[
\leq k\|g(x)\|_{L^1(\Omega)} + C_1.
\]
It follows, for any $i = 1, \ldots, N$, that
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u), \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n))\varphi(u_n)dx
\]
\[
\quad + (\theta - 1) \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i u_n |T_{h+1}(u_n) - T_h(u_n)| |\varphi(u_n)|dx
\]
\[
\quad + \int_{\Omega} |T_n(u_n)|^s |T_{h+1}(u_n) - T_h(u_n)| |\varphi(u_n)|dx
\]
\[
\leq \int_{\Omega} |f_n(x, T_n(u_n))| |T_{h+1}(u_n) - T_h(u_n)| |\varphi(u_n)|dx.
\]
We have $|\varphi(u_n)| \geq \frac{1}{2}$ on the set $\{h \leq |u_n|\}$, and thanks to (3.2) we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n)) |\varphi(u_n)|dx
\]
\[
\quad + (\theta - 1) \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) D^i u_n |T_{h+1}(u_n) - T_h(u_n)| |\varphi(u_n)|dx
\]
\[
\begin{aligned}
\geq & \frac{1}{4} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx + \frac{\alpha}{4} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq h+1\}} |D^i u_n|^{p_i(x)} \, dx \\
& + \alpha (\theta - 1) \sum_{i=1}^{N} \int_{\{h+1 \leq |u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} \, dx \\
\geq & \frac{1}{4} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx + C_5 \sum_{i=1}^{N} \int_{\{h+1 \leq |u_n|\}} |D^i u_n|^{p_i(x)} \, dx,
\end{aligned}
\]

with \( C_5 = \alpha \cdot \min \left\{ \frac{1}{4}, (\theta - 1) \right\} \). Having in mind (3.5) we conclude that

\[
(4.26) \quad \frac{1}{4} \sum_{i=1}^{N} \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx + C_5 \sum_{i=1}^{N} \int_{\{h+1 \leq |u_n|\}} |D^i u_n|^{p_i(x)} \, dx \\
+ \int_{\Omega} |T_n(u_n)|^{p(x)} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) \, dx \\
\leq \int_{\{h<|u_n|\}} |g(x)||T_{h+1}(u_n) - T_h(u_n)| \, dx \\
+ \int_{\{h<|u_n|\}} |T_n(u_n)|^{p(x)} |T_{h+1}(u_n) - T_h(u_n)| \varphi(u_n) \, dx.
\]

For the second term on the left-hand side of (4.26), thanks to Poincaré’s inequality we have

\[
\begin{aligned}
C_5 \sum_{i=1}^{N} \int_{\{h<|u_n|\}} \frac{|D^i u_n|^{p_i(x)}}{(1 + |u_n|)^{\theta}} \, dx \\
\geq C_5 \sum_{i=1}^{N} \int_{\{h<|u_n|\}} \frac{|D^i u_n|^{p_i\text{meas}}}{(1 + |u_n|)^{\theta}} \, dx - C_5 N \text{meas}\{h \leq |u_n|\} \\
= C_6 \sum_{i=1}^{N} \int_{\Omega} \left| \frac{|u_n|}{|T_n(u_n)|} \right|^{p_i\text{meas}} \, dx - C_5 N \text{meas}\{h \leq |u_n|\} \\
\geq C_6 \sum_{i=1}^{N} \int_{\{h<|u_n|\}} \frac{|u_n| - |T_h(u_n)|}{(1 + |u_n|)^{\theta}} \, dx - C_5 N \text{meas}\{h \leq |u_n|\} \\
\geq C_7 \sum_{i=1}^{N} \int_{\{h<|u_n|\}} |u_n|^{p_i - \theta} \, dx - C_6 \sum_{i=1}^{N} \int_{\{h<|u_n|\}} \frac{|h^{p_i}}{(1 + |u_n|)^{\theta}} \, dx \\
- C_5 N \text{meas}\{h \leq |u_n|\}.
\end{aligned}
\]
Having in mind (4.26), we conclude that
\[
\frac{1}{4} \sum_{i=1}^{N} \int_{\{h < |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx + C_7 \sum_{i=1}^{N} \int_{\{h \leq |u_n|\}} |u_n|^{p_i - \theta} \, dx \\
+ \int_{\{h < |u_n|\}} |T_n(u_n)|^{s(x)} |T_{h+1}(u_n) - T_h(u_n)| \, \varphi(u_n) \, dx
\]
\[
\leq \int_{\{h < |u_n|\}} |g(x)| \, dx + \int_{\{h < |u_n|\}} |T_n(u_n)|^{q(x)} |T_{h+1}(u_n) - T_h(u_n)| \, |\varphi(u_n)| \, dx \\
+ C_6 \sum_{i=1}^{N} \int_{\{h < |u_n|\}} \frac{h^{p_i}}{(1 + |u_n|)^\theta} \, dx + C_5 N \operatorname{meas}\{h \leq |u_n|\}.
\]

Since \(q(x) < \max(s(x), P^+ - \theta)\), and in view of Young’s inequality we have
\[
\int_{\{h < |u_n|\}} |T_n(u_n)|^{q(x)} |T_{h+1}(u_n) - T_h(u_n)| \, |\varphi(u_n)| \, dx
\]
\[
\leq \frac{C_7}{2} \sum_{i=1}^{N} \int_{\{h \leq |u_n|\}} |u_n|^{p_i - \theta} \, dx + C_8 \int_{\{h < |u_n|\}} |T_{h+1}(u_n) - T_h(u_n)| \, dx \\
+ \frac{1}{2} \int_{\{h < |u_n|\}} |T_n(u_n)|^{s(x)} |T_{h+1}(u_n) - T_h(u_n)| \, |\varphi(u_n)| \, dx,
\]
and thanks to (4.23), we have
\[
\varepsilon_1(h) = \frac{N}{4} \sum_{i=1}^{N} \int_{\{h < |u_n|\}} \frac{h^{p_i}}{(1 + |u_n|)^\theta} \, dx \leq \frac{N}{4} \sum_{i=1}^{N} h^{p_i - \theta} \operatorname{meas}\{h < |u_n|\}
\]
\[
\leq N h^{p_\theta - \theta} \operatorname{meas}\{h < |u_n|\} \\
= \frac{N h^{p_\theta - 1} \operatorname{meas}\{h < |u_n|\}}{h^{\theta - 1}} \\
\leq \frac{NC_4}{h^{\theta - 1}} \to 0 \text{ as } h \to \infty.
\]

Also, we have \(\operatorname{meas}\{|u_n| > h\}\) goes to zero, as \(h\) tends to infinity, and since \(g(x) \in L^1(\Omega)\) we conclude that
\[
\varepsilon_2(h) = \int_{\{h < |u_n|\}} |g(x)| \, dx + C_5 N \operatorname{meas}\{h \leq |u_n|\} \to 0 \text{ as } h \to \infty.
\]

It follows that
\[
\frac{1}{4} \sum_{i=1}^{N} \int_{\{h < |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx \\
+ \frac{C_7}{2} \sum_{i=1}^{N} \int_{\{h \leq |u_n|\}} |u_n|^{p_i - \theta} \, dx + \frac{1}{2} \int_{\{h+1 < |u_n|\}} |T_n(u_n)|^{s(x)} \, dx
\]
\[
\leq C_8 \int_{\{h < |u_n|\}} |T_{h+1}(u_n) - T_h(u_n)| \, dx + \varepsilon_3(h)
\]
\[
\leq \varepsilon_4(h).
\]
We conclude that
\[
\lim_{h \to \infty} \left( \int_{\{h+1 < |u_n|\}} |T_n(u_n)|^{s(x)} dx + \int_{\{|h+1 < |u_n|\}} |T_n(u_n)|^{q(x)} dx \right) = 0,
\]
therefore, thanks to (4.28) we have for any $\delta > 0$, there exists $h(\delta) > 1$ such that
\[
\int_{\{h(\delta) < |u_n|\}} |T_n(u_n)|^{s(x)} dx + \int_{\{|h(\delta) < |u_n|\}} |T_n(u_n)|^{q(x)} dx \leq \frac{\delta}{2}.
\]
On the other hand, for any measurable subset $E \subseteq \Omega$ we have
\[
\int_E |T_n(u_n)|^{s(x)} dx + \int_E |T_n(u_n)|^{q(x)} dx 
\leq \int_{\{h(\delta) < |u_n|\}} |T_n(u_n)|^{s(x)} dx + \int_{\{|h(\delta) < |u_n|\}} |T_n(u_n)|^{q(x)} dx
\]
\[
+ \int_E |T_{h(\delta)}(u_n)|^{s(x)} dx + \int_E |T_{h(\delta)}(u_n)|^{q(x)} dx.
\]
It’s clear that, there exists $\beta(\delta) > 0$ such that for any $E \subseteq \Omega$ with $\text{meas}(E) \leq \beta(\delta)$ we have
\[
\int_E |T_{h(\delta)}(u_n)|^{s(x)} dx + \int_E |T_{h(\delta)}(u_n)|^{q(x)} dx \leq \frac{\delta}{2}.
\]
Finally, by combining (4.29), (4.30) and (4.31), we obtain
\[
\int_E |T_n(u_n)|^{s(x)} dx + \int_E |T_n(u_n)|^{q(x)} dx \leq \delta \text{ for any } E \subset \Omega \text{ such that } \text{meas}(E) \leq \beta(\delta).
\]
Consequently, $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ and $(|T_n(u_n)|^{q(x)-1}T_n(u_n))_n$ are uniformly equi-integrable, and in view of the growth condition (3.5) we have
\[
|f_n(x, T_n(u_n))| \leq g(x) + |T_n(u_n)|^{q(x)},
\]
with $g(x) \in L^1(\Omega)$, then $(f_n(x, T_n(u_n)))_n$ is also uniformly equi-integrable. According to Vitali’s theorem, the statements (4.24) and (4.25) are concluded. Moreover, in view of (4.27) we have
\[
\lim_{h \to \infty} \limsup_{n \to \infty} \sum_{i=1}^{N} \int_{\{h < |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n dx = 0.
\]

**Step 4: strong convergence of truncations.** Let $h > k \geq 1$, and we set $\psi_h(u_n) = (1 - |T_k(u_n) - T_k(u_n))|).$ By taking $(T_k(u_n) - T_k(u))\psi_h(u_n) \in W^{1,\beta(1)}_0(\Omega)$ as a test function in (4.2) we obtain
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u)) \psi_h(u_n) dx
\]
\[- \sum_{i=1}^{N} \int_{\{h < |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n|T_k(u_n) - T_k(u)| dx
\]
\[+ \int_{\Omega} |T_n(u_n)|^{s(x)-1}T_n(u_n)(T_k(u_n) - T_k(u)) \psi_h(u_n) dx.
\]
\[
\int_{\Omega} f_n(x, T_n(u_n))(T_k(u_n) - T_k(u))\psi_h(u_n)dx.
\]

It follows that
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n)(D^iT_k(u_n) - D^iT_k(u))\psi_h(u_n)dx
\]
\[
\leq \int_{\Omega} |f_n(x, T_n(u_n))| |T_k(u_n) - T_k(u)|dx + \int_{\Omega} |T_n(u_n)|_{s(x)} |T_k(u_n) - T_k(u)|dx
\]
\[
+ \sum_{i=1}^{N} \int_{\{h \leq |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)|D^i u_n|T_k(u_n) - T_k(u)|dx.
\]

For the first and second terms on the right-hand side of (4.34), we have \(T_k(u_n) \rightharpoonup T_k(u)\) weak-\(\ast\) in \(L^\infty(\Omega)\), and thanks to (4.24)–(4.25) we have \(|T_n(u_n)|_{s(x)} \to |u|_{s(x)}\) and \(f_n(x, T_n(u_n)) \to f(x, u)\) strongly in \(L^1(\Omega)\), then
\[
\varepsilon_5(n) = \int_{\Omega} |T_n(u_n)|_{s(x)} |T_k(u_n) - T_k(u)|dx \to 0 \text{ as } n \to \infty
\]
and
\[
\varepsilon_6(n) = \int_{\Omega} |f_n(x, T_n(u_n))| |T_k(u_n) - T_k(u)|dx \to 0 \text{ as } n \to \infty.
\]

On the other hand, according to (4.33) we have
\[
\varepsilon_7(h) = \sum_{i=1}^{N} \int_{\{h \leq |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)|D^i u_n|T_k(u_n) - T_k(u)|dx
\]
\[
\leq 2k \sum_{i=1}^{N} \int_{\{h \leq |u_n| \leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)|D^i u_n|dx \to 0 \text{ as } h \to \infty.
\]

By combining (4.34) and (4.35)–(4.37) we conclude that
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n)(D^iT_k(u_n) - D^iT_k(u))\psi_h(u_n)dx \leq \varepsilon_7(n, h).
\]

For the term on the left-hand side of (4.38), since \(a_i(x, s, 0) = 0\), it follows that
\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n)(D^iT_k(u_n) - D^iT_k(u))\psi_h(u_n)dx
\]
\[
= \sum_{i=1}^{N} \int_{\{|u_n| \leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))(D^iT_k(u_n) - D^iT_k(u))dx
\]
\[
- \sum_{i=1}^{N} \int_{|k < |u_n| \leq k+1\}} a_i(x, T_{h+1}(u_n), \nabla T_{h+1}(u_n))D^iT_k(u)\psi_h(u_n)dx
\]
\[
= \sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right)(D^iT_k(u_n) - D^iT_k(u))dx
\]
In view of Lebesgue dominated convergence theorem, we have

\[ T_k(u_n) \to T_k(u) \]

in \( L^p(\Omega) \), then, \( a_i(x, T_k(u_n), \nabla T_k(u)) \to a_i(x, T_k(u), \nabla T_k(u)) \) strongly in \( L^{p'}(\Omega) \), and since \( D^iT_k(u_n) \) converges to \( D^iT_k(u) \) weakly in \( L^{p'}(\Omega) \), we obtain

\[
(4.40) \quad \varepsilon_k(n) = \sum_{i=1}^{N} \int_{\Omega} a_i(x, T_k(u_n), \nabla T_k(u)) (D^iT_k(u_n) - D^iT_k(u)) dx \to 0 \quad \text{as} \quad n \to \infty.
\]

Concerning the third term on the right-hand side of (4.39), we have \( \{a_i(x, T_{h+1}(u_n), \nabla T_{h+1}(u_n))\}_n \) is bounded in \( L^{p'}(\Omega) \), then there exists \( \nu_i \in L^{p'}(\Omega) \) such that \( |a_i(x, T_{h+1}(u_n), \nabla T_{h+1}(u_n))| \to \nu_i \) weakly in \( L^{p'}(\Omega) \) for any \( i = 1, \ldots, N \). Therefore,

\[
(4.41) \quad \varepsilon_9(n) \leq \sum_{i=1}^{N} \int_{\{k \in |u_n| \leq h+1\}} a_i(x, T_{h+1}(u_n), \nabla T_{h+1}(u_n)) |D^iT_k(u)| \psi(h) dx.
\]

By combining (4.38)–(4.41), we conclude that

\[
\sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) (D^iT_k(u_n) - D^iT_k(u)) dx \leq \varepsilon_{10}(n, h).
\]

In view of Lebesgue dominated convergence theorem, we have \( T_k(u_n) \to T_k(u) \) strongly in \( L^{p^i}(\Omega) \). Thus, by letting \( n \) then \( h \) tend to infinity we deduce that

\[
\sum_{i=1}^{N} \int_{\Omega} \left( a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u)) \right) (D^iT_k(u_n) - D^iT_k(u)) dx \\
+ \int_{\Omega} |\nabla T_k(u_n)|^{p^i-2}T_k(u_n) - |\nabla T_k(u)|^{p^i-2}T_k(u) (T_k(u_n) - T_k(u)) dx \to 0 \quad \text{as} \quad n \to \infty.
\]

In view of Lemma 4.2, we conclude that

\[
(4.42) \begin{cases} 
T_k(u_n) \to T_k(u) \text{ strongly in } W_0^{1,\bar{p}^i}(\Omega), \\
D^i u_n \to D^i u \text{ a.e. in } \Omega \text{ for } i = 1, \ldots, N.
\end{cases}
\]
Moreover, we have $a_i(x, T_n(u_n), \nabla u_n) D^i u_n$ tends to $a_i(x, u, \nabla u) D^i u$ almost everywhere in $\Omega$, and in view of Fatou’s lemma and (4.33), we conclude that

$$
\lim_{h \to \infty} \sum_{i=1}^N \int_{\{h < |u| < h + 1\}} a_i(x, u, \nabla u) D^i u \, dx
$$

$$
\leq \lim_{h \to \infty} \liminf_{n \to \infty} \sum_{i=1}^N \int_{\{h < |u_n| < h + 1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx
$$

$$
\leq \lim_{h \to \infty} \limsup_{n \to \infty} \sum_{i=1}^N \int_{\{h < |u_n| < h + 1\}} a_i(x, T_n(u_n), \nabla u_n) D^i u_n \, dx = 0,
$$

which prove (4.1).

**Step 5: passage to the limit.** Let $\varphi \in W_0^{1, p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and choosing $S(\cdot)$ be a smooth function in $C_0^1(\mathbb{R})$ such that $\text{supp} (S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.

By taking $S(u_n) \varphi \in W_0^{1, p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximate problem (4.2), we obtain

$$
\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) \left( D^i u_n S'(u_n) \varphi + S(u_n) D^i \varphi \right) \, dx
$$

$$
+ \int_{\Omega} |T_n(u_n)|^{s(x)-1} T_n(u_n) S(u_n) \varphi \, dx = \int_{\Omega} f_n(x, T_n(u_n)) S(u_n) \varphi \, dx.
$$

For the first term on the left-hand side of (4.43), we have

$$
\sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) \left( D^i u_n S'(u_n) \varphi + S(u_n) D^i \varphi \right) \, dx
$$

$$
= \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \left( S'(u_n) \varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi \right) \, dx,
$$

in view of (4.42), we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p(\cdot)}(\Omega)$, and since $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in $\Omega$, it follows that

$$
a_i(x, T_M(u_n), \nabla T_M(u_n)) \to a_i(x, T_M(u), \nabla T_M(u)) \text{ in } L^{p(\cdot)}(\Omega),
$$

and since $(S'(u_n) \varphi D^i T_M(u_n) + S(T_M(u_n)) D^i \varphi) \to (S'(u) \varphi D^i T_M(u) + S(T_M(u)) D^i \varphi)$ strongly in $L^{p(\cdot)}(\Omega)$, we deduce that

$$
\lim_{n \to \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_n(u_n), \nabla u_n) \left( D^i u_n S'(u_n) \varphi + S(u_n) D^i \varphi \right) \, dx
$$

$$
= \lim_{n \to \infty} \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u_n), \nabla T_M(u_n)) \left( D^i T_M(u_n) S'(u_n) \varphi + S(T_M(u_n)) D^i \varphi \right) \, dx
$$

$$
= \sum_{i=1}^N \int_{\Omega} a_i(x, T_M(u), \nabla T_M(u)) \left( D^i T_M(u) S'(u) \varphi + S(T_M(u)) D^i \varphi \right) \, dx
$$
\[ \sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u) \left( \nabla^2 u S'(u) \phi + S(u) \nabla^2 \phi \right) \, dx. \]

Concerning the second term on the right-hand side of (4.43), we have \( S(T_M(u_n)) \phi \rightharpoonup S(T_M(u)) \phi \) weak-* in \( L^\infty(\Omega) \), and thanks to (4.24), we have \( |T_n(u_n)|^{s(x)-1} T_n(u_n) \to |u|^{s(x)-1} u \) strongly in \( L^1(\Omega) \), it follows that

\[
\lim_{n \to \infty} \int_{\Omega} |T_n(u_n)|^{s(x)-1} T_n(u_n) S(T_M(u_n)) \phi \, dx = \int_{\Omega} |u|^{s(x)-1} u S(T_M(u)) \phi \, dx
\]

Similarly, thanks to (4.25) we have \( f_n(x, T_n(u_n)) \to f(x, u) \) strongly in \( L^1(\Omega) \) then

\[
\lim_{n \to \infty} \int_{\Omega} f_n(x, T_n(u_n)) S(T_M(u_n)) \phi \, dx = \int_{\Omega} f(x, u) S(T_M(u)) \phi \, dx = \int_{\Omega} f(x, u) S(u) \phi \, dx.
\]

By combining (4.43) and (4.44)–(4.46), we conclude that

\[
\sum_{i=1}^{N} \int_{\Omega} a_i(x, u, \nabla u) \left( \nabla^2 u S'(u) \phi + S(u) \nabla^2 \phi \right) \, dx + \int_{\Omega} |u|^{s(x)-1} u S(u) \phi \, dx = \int_{\Omega} f(x, u) S(u) \phi \, dx.
\]

which complete the proof of the Theorem 4.1.

References


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