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EXISTENCE OF RENORMALIZED SOLUTIONS FOR SOME ANISOTROPIC QUASILINEAR ELLIPTIC EQUATIONS

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Abstract. In this paper, we consider a class of anisotropic quasilinear elliptic equations of the type

$$\begin{cases} -\displaystyle\sum_{i=1}^{N} \partial^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f(x,u), & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(x,s)$ is a Carathéodory function which satisfies some growth condition. We prove the existence of renormalized solutions for our Dirichlet problem, and some regularity results are concluded.

1. Introduction

Let Ω be an open bounded subset of $\mathbb{R}^N$, $N \geq 2$, with the smooth boundary $\partial\Omega$. Zhao et al. have studied in [17] the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}(a(x,\nabla u)) + |u|^{p-2}u = \lambda f(x,u), & \text{in } \Omega, \\ \displaystyle\int_{\partial\Omega} a(x,\nabla u)\cdot n ds = 0, & \\ u = \text{ constant} & \text{on } \partial\Omega, \end{cases}$$

They have proved the existence of weak solutions under some suitable growth assumptions on $f(x,s)$, (see also [2,7]). In the framework of Sobolev spaces with variable exponents, Fan and Zhang [11] have considered the following nonlinear elliptic problem

$$\begin{cases} \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = \lambda f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

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where $\lambda > 0$ and $f(x,s)$ satisfies the growth condition $|f(x,s)| \leq \eta + \theta|s|^{\delta-1}$, where $1 \leq \delta \leq p^-$ and η, θ are two positive constants (we refer also to [6]). In [3], the authors have proved the existence of weak solutions for the quasilinear $p(x)$-elliptic problem

$$-\operatorname{div} a(x,u,\nabla u) = f(x,u,\nabla u),$$

by using the calculus of variations operators method, where $f(x,s,\xi)$ is a Carathéodory function which satisfies some growth condition.

In the framework of anisotropic Sobolev spaces, Di Nardo, Feo and Guibé have studied in [9] the existence of renormalized solutions for some class of nonlinear anisotropic elliptic problems of the type

$$-\sum_{i=1}^{N} \partial_{x_i}(a_i(x,u)|\partial_{x_i}u|^{p_i-2}\partial_{x_i}u) = f - \operatorname{div} g, \quad \text{in } \Omega,$$

with $f \in L^1(\Omega)$ and $g \in \Pi_{i=1}^{N} L^{p_i'}(\Omega)$, the uniqueness of renormalized solution was concluded under some local Lipschitz conditions on the function $a_i(x,s)$ with respect to s, (see also [1] and [8]).

The aim of this paper is to study the existence and regularity of renormalized solutions for the anisotropic quasilinear elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial^i a_i(x,u,\nabla u) + |u|^{s(x)-1}u = f(x,u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $(a_i(x,s,\xi))_{i=1,\dots,N}$ are Carathéodory functions, the right-hand side $f(x,s)$ is a Carathéodory function satisfying only some nonstandard growth condition.

One of our motivations for studying (1.1) comes from these applications to electro-rheological fluids as an important class of non-Newtonian fluids (sometimes referred to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed in [14, 15], also in the robotics and space technology (we refer for example to [16]).

One of the difficulties in proving the existence of renormalized solutions stems from the nonstandard growth of the Carathéodory function $f(x,s)$, to overcome the difficulty, we use the regularizing effect of the term $|u|^{s(x)-1}u$ with some special technics.

The rest of this paper is structured as follows. In Section 2 we recall some definitions and results on the anisotropic variable exponent Sobolev spaces. We introduce in Section 3 some assumptions for which our problem has at least one renormalized solution. Section 4 will be devoted to show the existence of renormalized solutions u for the problem (1.1) in the anisotropic Sobolev space with variable exponents, and we will give some regularity results, that is $|u|^{s(x)-1}u \in L^1(\Omega)$.

2. Preliminary

Let Ω be a bounded open subset of $\mathbb{R}^N$, $N \geq 2$, we denote

$$\mathcal{C}_+(\Omega) = \{\text{measurable function } p(\cdot) : \Omega \to \mathbb{R} \text{ such that } 1 < p^- \leq p^+ < N\},$$

where

$$p^- = \operatorname{ess\,inf}\{p(x)/x \in \Omega\} \text{ and } p^+ = \operatorname{ess\,sup}\{p(x)/x \in \Omega\}.$$

We define the Lebesgue space with variable exponent $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(\cdot)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite. If the exponent is bounded, i.e., if $p^+ < +\infty$, then the expression

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \rho_{p(\cdot)}(u/\lambda) \leq 1\}$$

defines a norm in $L^{p(\cdot)}(\Omega)$, called the Luxemburg norm. The space $(L^{p(\cdot)}(\Omega), \|\cdot\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p^- \leq p^+ < +\infty$, then $L^{p(\cdot)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(\cdot)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. Finally, we have the Hölder type inequality:

$$\left|\int_\Omega uv dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{p(\cdot)} \|v\|_{p'(\cdot)},$$

for any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$.

The Sobolev space with variable exponent $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) \text{ and } |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

which is a Banach space, equipped with the following norm

$$\|u\|_{1,p(\cdot)} = \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

The space $(W^{1,p(\cdot)}(\Omega), \|\cdot\|_{1,p(\cdot)})$ is a separable and reflexive Banach space. We define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$. For more details on variable exponent Lebesgue and Sobolev spaces, we refer the reader to [10].

Now, we present the anisotropic variable exponent Sobolev space, used in the study of our quasilinear anisotropic elliptic problem.

Let $p_1(\cdot), p_2(\cdot), \ldots, p_N(\cdot)$ be N variable exponents in $\mathcal{C}_+(\Omega)$. We denote

$$\vec{p}(\cdot) = (p_1(\cdot), \ldots, p_N(\cdot)) \text{ and } D^i u = \frac{\partial u}{\partial x_i}, \quad \text{for } i = 1, \ldots, N,$$

and we define

$$\underline{p}^+ = \max\{p_1^-, \ldots, p_N^-\} \text{ and } \underline{p}^- = \min\{p_1^-, \ldots, p_N^-\}, \quad \text{then } 1 < \underline{p}^- \leq \underline{p}^+.$$

The anisotropic variable exponent Sobolev space $W^{1,\vec{p}(\cdot)}(\Omega)$ is defined as follow

$$W^{1,\vec{p}(\cdot)}(\Omega) = \{u \in W^{1,1}(\Omega) \text{ and } D^i u \in L^{p_i(\cdot)}(\Omega) \text{ for } i = 1, 2, \ldots, N\},$$

endowed with the norm

$$\|u\|_{1,\vec{p}(\cdot)} = \|u\|_{1,1} + \sum_{i=1}^{N} \|D^i u\|_{p_i(\cdot)}. \tag{2.1}$$

We define also $W_0^{1,\vec{p}(\cdot)}(\Omega)$ as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $W^{1,\vec{p}(\cdot)}(\Omega)$ with respect to the norm (2.1). The space $\left(W_0^{1,\vec{p}(\cdot)}(\Omega), \|u\|_{1,\vec{p}(\cdot)}\right)$ is a reflexive Banach space (cf. [13]).

Remark 2.1. In view of the continuous embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,1}(\Omega)$ and the Poincaré type inequality we conclude that the two norms $\|u\|_{1,\vec{p}(\cdot)}$ and $\sum_{i=1}^N \|D^i u\|_{p_i(\cdot)}$ are equivalent in the anisotropic variable exponent Sobolev spaces.

Lemma 2.1. *We have the following continuous and compact embeddings.*

- *If $\underline{p}^- < N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for $q \in [\underline{p}^-, \underline{p}^*[$, where $\underline{p}^* = \frac{N\underline{p}^-}{N-\underline{p}^-}$.*
- *If $\underline{p}^- = N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$, for all $q \in [\underline{p}^-, +\infty[$.*
- *If $\underline{p}^- > N$, then $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^\infty(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$.*

The proof of this lemma follows from the fact that the embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow W_0^{1,\underline{p}^-}(\Omega)$ is continuous, and in view of the compact embedding theorem for Sobolev spaces.

Proposition 2.1. *The dual of $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is denote by $W^{-1,\vec{p}'(\cdot)}(\Omega)$, where $\vec{p}'(\cdot) = (p_1'(\cdot), \dots, p_N'(\cdot))$ and $\frac{1}{p_i'(x)} + \frac{1}{p_i(x)} = 1$ (cf. [5] for the constant exponent case). For each $F \in W^{-1,\vec{p}'(\cdot)}(\Omega)$ there exists $F_0 \in (L^{\underline{p}^+}(\Omega))'$ and $F_i \in L^{p_i'(\cdot)}(\Omega)$ for $i = 1, 2, \dots, N$, such that $F = F_0 - \sum_{i=1}^N D^i F_i$. Moreover, for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, we have*

$$\langle F, u \rangle = \sum_{i=0}^{N} \int_\Omega F_i D^i u dx.$$

We define a norm on the dual space by

$$\|F\|_{-1,\vec{p}'(\cdot)} = \inf \left\{ \sum_{i=0}^{N} \|F_i\|_{p_i'(\cdot)} \text{ with } F = F_0 - \sum_{i=1}^{N} D^i F_i \text{ such that } F_0 \in (L^{\underline{p}^+}(\Omega))' \text{ and } F_i \in L^{p_i'(\cdot)}(\Omega) \right\}.$$

Definition 2.1. Let $k > 0$, the truncation function $T_k(\cdot) : \mathbb{R} \to \mathbb{R}$ is defined by

$$T_k(s) = \begin{cases} s, & \text{if } |s| \leq k, \\ k\dfrac{s}{|s|}, & \text{if } |s| > k, \end{cases}$$

and we define

$$\mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega) := \{u : \Omega \to \mathbb{R} \text{ measurable, such that } T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ for any } k > 0\}.$$

Proposition 2.2. *Let $u \in \mathcal{T}_0^{1,\vec{p}(\cdot)}(\Omega)$. For any $i \in \{1, \dots, N\}$, there exists a unique measurable function $v_i : \Omega \to \mathbb{R}$ such that*

$$D^i T_k(u) = v_i \cdot \chi_{\{|u|<k\}} \text{ a.e. } x \in \Omega, \text{ for all } k > 0,$$

where χ_A denotes the characteristic function of a measurable set A. The functions v_i are called the weak partial derivatives of u and are still denoted $D^i u$. Moreover, if u belongs to $W_0^{1,1}(\Omega)$, then v_i coincides with the standard distributional derivative of u, that is, $v_i = D^i u$.

3. Essential Assumptions

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 2)$. We consider $\vec{p}(\cdot) = (p_1(\cdot), \dots, p_N(\cdot))$ the vector of exponents $p_i(\cdot) \in C_+(\Omega)$ for $i = 1, \dots, N$, and let $q(\cdot), s(\cdot) \in C_+(\Omega)$ where

$$q(x) < \max(s(x), \underline{p}^+ - 1) \text{ a.e. in } \Omega.$$

We consider the Leray-Lions operator A acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into its dual $W^{-1,\vec{p}'(\cdot)}(\Omega)$, defined by the formula

$$Au = -\sum_{i=1}^{N} \partial^i a_i(x, u, \nabla u),$$

where $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ are Carathéodory function which satisfy the following conditions

$$|a_i(x, s, \xi)| \leq \beta(K_i(x) + |s|^{p_i(x)-1} + |\xi|^{p_i(x)-1}), \quad \text{for any } i = 1, \dots, N, \tag{3.1}$$

$$a_i(x, s, \xi)\xi_i \geq \alpha |\xi_i|^{p_i(x)}, \quad \text{for any } i = 1, \dots, N, \tag{3.2}$$

for all $\xi = (\xi_1, \dots, \xi_N)$ and $\xi' = (\xi'_1, \dots, \xi'_N)$, we have

$$[a_i(x, s, \xi) - a_i(x, s, \xi')](\xi_i - {\xi'}_i) > 0, \quad \text{for } \xi_i \neq \xi'_i, \tag{3.3}$$

for a.e. $x \in \Omega$, all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, where $K_i(x)$ is a positive function lying in $L^{p'_i(\cdot)}(\Omega)$ and $\alpha, \beta > 0$.

As a consequence of (3.2) and the continuity of the function $a_i(x, s, \cdot)$ with respect to ξ, we have

$$a_i(x, s, 0) = 0.$$

In this paper, we consider the following quasilinear anisotropic elliptic problem

$$\begin{cases} -\sum_{i=1}^{N} \partial^i a_i(x, u, \nabla u) + |u|^{s(x)-1} u = f(x, u), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying

$$|f(x, r)| \leq g(x) + |r|^{q(x)} \text{ a.e in } \Omega, \tag{3.5}$$

and $g(\cdot)$ is a measurable positive function in $L^1(\Omega)$.

Remark 3.1. The assumption (3.1) is used here to ensure that $a_i(x,u,\nabla u)$ belongs to $L^{p_i'(\cdot)}(\Omega)$. In the other case where $Au = -\sum_{i=1}^N \partial^i a_i(x,\nabla u)$, the uniqueness of solution can be concluded under some additional conditions on the Carathéodory function $f(x,s)$.

4. Main Results

We begin by recalling some important lemmas useful to prove our main result.

Lemma 4.1 ([3]). *Let $g \in L^{r(\cdot)}(\Omega)$ and $g_n \in L^{r(\cdot)}(\Omega)$ with $\|g_n\|_{r(\cdot)} \le C$ for $1 < r(x) < \infty$. If $g_n(x) \to g(x)$ a.e. on Ω, then $g_n \rightharpoonup g$ in $L^{r(\cdot)}(\Omega)$.*

Lemma 4.2 ([4]). *Assuming that (3.1)-(3.3) hold, and let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u_n \rightharpoonup u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and*

$$\begin{aligned}&\int_\Omega (|u_n|^{p_0(x)-2}u_n - |u|^{p_0(x)-2}u)(u_n-u)dx\\&+\sum_{i=1}^N \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))(D^i u_n - D^i u)dx \to 0,\end{aligned}$$

then $u_n \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ for a subsequence.

Our objective is to prove the existence of renormalized solutions for the quasilinear anisotropic elliptic problem (3.4).

Definition 4.1. A measurable function u is called renormalized solution of the quasilinear elliptic problem (3.4) if $T_k(u) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ for any $k>0$, with $f(x,u) \in L^1(\Omega)$, and

$$\lim_{h\to\infty} \sum_{i=1}^N \int_{\{h<|u|\le h+1\}} a_i(x,u,\nabla u) D^i u dx = 0, \tag{4.1}$$

such that u satisfies the following equality

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x,u,\nabla u)\Big(S'(u)\varphi D^i u + S(u) D^i\varphi\Big)dx + \int_\Omega |u|^{s(x)-1}uS(u)\varphi dx\\&= \int_\Omega f(x,u)S(u)\varphi dx,\end{aligned}$$

for every $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)\cap L^\infty(\Omega)$ and for any smooth function $S(\cdot) \in W^{1,\infty}(\mathbb{R})$ with a compact support.

Theorem 4.1. *Assuming that the conditions (3.1)–(3.3) and (3.5) hold true, then the quasilinear anisotropic elliptic problem (3.4) has at least one renormalized solution. Moreover, we have*

$$|u|^{s(x)} \in L^1(\Omega).$$

4.1. **Proof of Theorem** 4.1.

Step 1: approximate problems. Firstly, we consider the approximate problem

$$\begin{cases} A_n u_n + |T_n(u_n)|^{s(x)-1} T_n(u_n) = f_n(x, T_n(u_n)), & \text{in } \Omega, \\ u_n = 0, & \text{on } \partial\Omega, \end{cases} \tag{4.2}$$

where $A_n v = -\sum_{i=1}^N \partial^i a_i(x, T_n(v), \nabla v)$ and $f_n(x,r) = T_n(f(x,r))$. Thanks to (3.5), it's clear that

$$|f_n(x,r)| \leq n \text{ and } |f_n(x,r)| \leq g(x) + |r|^{q(x)}.$$

We consider the operator $G_n : W_0^{1,\vec{p}(\cdot)}(\Omega) \to W^{-1,\vec{p}'(\cdot)}(\Omega)$ by

$$\langle G_n u, v \rangle = \int_\Omega |T_n(u)|^{s(x)-1} T_n(u) v dx - \int_\Omega f_n(x, T_n(u)) v dx,$$

for any $u, v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$. In view of the generalized Hölder-type inequality, we have

$$\begin{aligned} |\langle G_n u, v \rangle| &\leq \int_\Omega |T_n(u)|^{s(x)}\ |v| dx + \int_\Omega |f_n(x, T_n(u))|\ |v| dx \\ &\leq n^{s^+} \int_\Omega |v| dx + n \int_\Omega |v| dx \\ &= (n^{s^+} + n) \|v\|_1 \\ &\leq C_1 \|v\|_{1,\vec{p}(\cdot)}. \end{aligned} \tag{4.3}$$

Lemma 4.3. *The bounded operator $B_n = A_n + G_n$ acted from $W_0^{1,\vec{p}(\cdot)}(\Omega)$ into $W^{-1,\vec{p}'(\cdot)}(\Omega)$ is pseudo-monotone. Moreover, B_n is coercive in the following sense:*

$$\frac{\langle B_n v, v \rangle}{\|v\|_{1,\vec{p}(\cdot)}} \to +\infty \ \textit{as}\ \|v\|_{1,\vec{p}(\cdot)} \to \infty, \quad \textit{for any}\ v \in W_0^{1,\vec{p}(\cdot)}(\Omega).$$

Proof. In view of the Hölder's inequality and the growth condition (3.1), it's easy to see that the operator A_n is bounded, and by (4.3) we conclude that B_n is bounded. For the coercivity, we have for any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$,

$$\begin{aligned} \langle B_n u, u \rangle = &\ \langle A_n u, u \rangle + \langle G_n u, u \rangle \\ = &\ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u), \nabla u)\ D^i u dx + \int_\Omega |T_n(u)|^{s(x)} |u| dx \\ &- \int_\Omega |f_n(x, T_n(u))| |u| dx \\ \geq &\ \alpha \sum_{i=1}^N \int_\Omega |D^i u|^{p_i(x)} dx + \int_\Omega |T_n(u)|^{s(x)+1} dx - C_2 n \|u\|_{p_0(\cdot)} \\ \geq &\ C_0\ \|u\|_{1,\vec{p}(\cdot)}^{p^-} - \alpha N |\Omega| - C_2 n \|u\|_{1,\vec{p}(\cdot)}, \end{aligned}$$

it follows that

$$\frac{\langle B_n u, u \rangle}{\|u\|_{1,\vec{p}(\cdot)}} \to +\infty \text{ as } \|u\|_{1,\vec{p}(\cdot)} \to \infty.$$

It remains to show that B_n is pseudo-monotone. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\left\{\begin{array}{l} u_k \rightharpoonup u, \quad \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ B_n u_k \rightharpoonup \chi_n, \quad \text{ in } W^{-1,\vec{p}'(\cdot)}(\Omega), \\ \limsup\limits_{k\to\infty} \langle B_n u_k, u_k\rangle \leq \langle \chi_n, u\rangle. \end{array}\right. \tag{4.4}$$

We will prove that

$$\chi_n = B_n u \text{ and } \langle B_n u_k, u_k\rangle \to \langle \chi_n, u\rangle \text{ as } k\to\infty.$$

In view of the compact embedding $W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$, we have $u_k \to u$ in $L^1(\Omega)$ and a.e. Ω, for a subsequence still denoted $(u_k)_{k\in\mathbb{N}}$.

We have $(u_k)_{k\in\mathbb{N}}$ is a bounded sequence in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, using the growth condition (3.1) it's clear that the sequence $(a_i(x,T_n(u_k),\nabla u_k))_{k\in\mathbb{N}}$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists a function $\varphi_i \in L^{p_i'(\cdot)}(\Omega)$ such that

$$a_i(x,T_n(u_k),\nabla u_k) \rightharpoonup \varphi_i \text{ in } L^{p_i'(\cdot)}(\Omega) \text{ as } k\to\infty. \tag{4.5}$$

On the one hand we have

$$|T_n(u_k)|^{s(x)-1}T_n(u_k) \to |T_n(u)|^{s(x)-1}T_n(u) \text{ weak}-* \text{ in } L^\infty(\Omega), \tag{4.6}$$

and since $f_n(x,T_n(s))$ is a Carathéodory function, then

$$f_n(x,T_n(u_k)) \to f_n(x,T_n(u)) \text{ weak}-* \text{ in } L^\infty(\Omega). \tag{4.7}$$

Then, for any $v \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ we have

(4.8)

$$\begin{aligned} \langle \chi_n, v\rangle = & \lim_{k\to\infty}\langle B_n u_k, v\rangle \\ = & \lim_{k\to\infty}\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i v dx + \lim_{k\to\infty}\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)v dx \\ & -\lim_{k\to\infty}\int_\Omega f_n(x,T_n(u_k))v dx \\ = & \sum_{i=1}^N \int_\Omega \varphi_i\ D^i v dx + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)v dx - \int_\Omega f_n(x,T_n(u))v dx. \end{aligned}$$

Having in mind (4.4) and (4.8), we conclude that

$$\begin{aligned} \limsup_{k\to\infty}\langle B_n(u_k),u_k\rangle = & \limsup_{k\to\infty}\Big(\sum_{i=1}^N \int_\Omega a_i(x,T_n(u_k),\nabla u_k)D^i u_k dx \\ & + \int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k dx - \int_\Omega f_n(x,T_n(u_k))u_k dx\Big) \\ \leq & \sum_{i=1}^N \int_\Omega \varphi_i\ D^i u dx + \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u dx \\ & - \int_\Omega f_n(x,T_n(u))u dx. \end{aligned}$$

Since $u_k \to u$ strongly in $L^1(\Omega)$, and thanks to (4.6)–(4.7) we obtain

$$\int_\Omega |T_n(u_k)|^{s(x)-1}T_n(u_k)u_k dx \to \int_\Omega |T_n(u)|^{s(x)-1}T_n(u)u dx \tag{4.9}$$

and

$$\int_\Omega f_n(x, T_n(u_k))u_k dx \to \int_\Omega f_n(x, T_n(u))u dx. \tag{4.10}$$

Therefore,

$$\limsup_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx \le \sum_{i=1}^N \int_\Omega \varphi_i \ D^i u dx. \tag{4.11}$$

On the other hand, in view of (3.3) we have

$$\sum_{i=1}^N \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) dx \ge 0, \tag{4.12}$$

then

$$\begin{aligned}\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx \ &\ge \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u dx \\ &+ \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u)(D^i u_k - D^i u) dx.\end{aligned}$$

In view of Lebesgue's dominated convergence theorem we have $T_n(u_k) \to T_n(u)$ in $L^{p_i(\cdot)}(\Omega)$, thus $a_i(x, T_n(u_k), \nabla u) \to a_i(x, T_n(u), \nabla u)$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and using (4.5) we get

$$\liminf_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx \ge \sum_{i=1}^N \int_\Omega \varphi_i \ D^i u dx.$$

Having in mind (4.11), we conclude that

$$\lim_{k\to\infty} \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_k), \nabla u_k) D^i u_k dx = \sum_{i=1}^N \int_\Omega \varphi_i \ D^i u dx. \tag{4.13}$$

Therefore, by combining (4.8) and (4.9)–(4.10), we conclude that

$$\langle B_n u_k, u_k \rangle \to \langle \chi_n, u \rangle \text{ as } k \to \infty.$$

Now, by (4.13) we can prove that

$$\lim_{k\to\infty} \Big(\sum_{i=1}^N \int_\Omega (a_i(x, T_n(u_k), \nabla u_k) - a_i(x, T_n(u_k), \nabla u))(D^i u_k - D^i u) dx$$

$$+ \int_\Omega (|u_k|^{\underline{p}^+-2} u_k - |u|^{\underline{p}^+-2} u)(u_k - u) dx \Big) = 0,$$

and so, by virtue of Lemma 4.2, we get

$$u_k \to u \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega) \text{ and } D^i u_k \to D^i u \text{ a.e. in } \Omega,$$

then

$$a_i(x, T_n(u_k), \nabla u_k) \rightharpoonup a_i(x, T_n(u), \nabla u) \text{ in } L^{p_i'(\cdot)}(\Omega), \quad \text{for } i = 1, \dots, N,$$

and thanks to (4.6)–(4.7), we obtain $\chi_n = B_n u$, which conclude the proof of Lemma 4.3. □

In view of Lemma 4.3, there exists at least one weak solution $u_n \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ of the approximate problem (4.2) (cf. [12], Theorem 2.7, page 180).

Step 2: a priori estimates. Choose $1 < \theta < \underline{p}^-$ such that $1 \leq q(x) < \max(s(x), \underline{p}^+ - \theta)$. By taking $\varphi(u_n) = \left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)$ sign $(u_n) \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ as a test function in (4.2), we obtain

$$(\theta - 1)\sum_{i=1}^{N}\int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n)\cdot D^i u_n}{(1+|u_n|)^\theta}dx + \int_\Omega |T_n(u_n)|^{s(x)}\left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)dx$$
$$= \int_\Omega f_n(x, T_n(u_n))\left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)\text{sign } (u_n)dx.$$

By using the coercivity (3.2) and the growth condition (3.5), we obtain

$$\alpha(\theta - 1)\sum_{i=1}^{N}\int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}dx + \int_\Omega |T_n(u_n)|^{s(x)}\left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)dx$$
$$\leq \int_\Omega (|g(x)| + |T_n(u_n)|^{q(x)})\left(1 - \frac{1}{(1+|u_n|)^{\theta-1}}\right)dx. \tag{4.14}$$

For the first term on the left hand side of (4.14), for any $i = 1, \dots, N$, we have

$$\begin{aligned}\int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta}dx &\geq \int_\Omega \frac{|D^i u_n|^{p_i^-}}{(1+|u_n|)^\theta}dx - |\Omega| \\ &= \int_\Omega \left|\frac{D^i u_n}{(1+|u_n|)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - |\Omega| \\ &= \int_\Omega \left|D^i \int_0^{|u_n|}\frac{ds}{(1+s)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - |\Omega| \\ &\geq \frac{1}{C_p}\int_\Omega \left|\int_0^{|u_n|}\frac{ds}{(1+s)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - |\Omega| \\ &\geq \frac{1}{C_p}\int_\Omega \frac{|u_n|^{p_i^-}}{(1+|u_n|)^\theta}dx - |\Omega| \\ &\geq \frac{1}{2^\theta C_p}\int_\Omega |u_n|^{p_i^- - \theta}dx - 2|\Omega|,\end{aligned}$$

and since $\varphi(u_n) \geq \frac{1}{2}$ for $|u_n| \geq R$, with $R = 2^{\frac{1}{1-\theta}} - 1$. Using Young's inequality it follows that

$$\begin{aligned} \frac{\alpha(\theta-1)}{2^\theta C_p} \sum_{i=1}^N \int_\Omega |u_n|^{p_i^- - \theta} dx + \frac{1}{2} \int_{\{|u_n| \geq R\}} |T_n(u_n)|^{s(x)} dx \\ \leq \int_\Omega |g(x)| dx + \int_\Omega |T_n(u_n)|^{q(x)} dx + 2\alpha N(\theta - 1)|\Omega|. \end{aligned} \tag{4.15}$$

Since $1 \leq q(x) < \max(s(x), \underline{p}^+ - \theta)$, by using Young's inequality we conclude that (4.16)

$$\int_\Omega |T_n(u_n)|^{q(x)} dx \leq \frac{\alpha(\theta-1)}{2^{\theta+1} C_p} \sum_{i=1}^N \int_\Omega |u_n|^{p_i^- - \theta} dx + \frac{1}{4} \int_{\{|u_n| \geq R\}} |T_n(u_n)|^{s(x)} dx + C_0.$$

It follows from (4.15) that there exists a constant C_1 that does not depend on n, such that

$$\sum_{i=1}^N \int_\Omega |u_n|^{p_i^- - \theta} dx + \int_\Omega |T_n(u_n)|^{s(x)} dx + \int_\Omega |T_n(u_n)|^{q(x)} dx \leq C_1. \tag{4.17}$$

Let $k \geq 1$, in view of (4.14) we conclude that

$$\frac{1}{(1+k)^\theta} \sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx \leq \sum_{i=1}^N \int_\Omega \frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^\theta} dx + \int_\Omega |T_n(u_n)|^{s(x)} \leq C_2. \tag{4.18}$$

Therefore, we obtain

$$\sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx \leq C_2(1+k)^\theta, \quad \text{for } k \geq 1.$$

Thus, the sequence $(T_k(u_n))_n$ is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, and there exists a subsequence still denoted $(T_k(u_n))_n$ and $\eta_k \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

$$\begin{cases} T_k(u_n) \rightharpoonup \eta_k \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega), \\ T_k(u_n) \to \eta_k \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega. \end{cases} \tag{4.19}$$

On the other hand, in view of Poincaré type inequality, for any $i \in \{1, \ldots, N\}$ we have

$$\begin{aligned} k^{p_i^-} \operatorname{meas}\{|u_n| > k\} &= \int_{\{|u_n|>k\}} |T_k(u_n)|^{p_i^-} dx \leq \int_\Omega |T_k(u_n)|^{p_i^-} dx \\ &\leq C_p^{p_i^-} \int_\Omega |D^i T_k(u_n)|^{p_i^-} dx \\ &\leq C_p^{p_i^-} \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx + C_p^{p_i^-} |\Omega| \\ &\leq \max_{1 \leq i \leq N} (C_p^{p_i^-}) \Big(\sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx + |\Omega| \Big) \\ &\leq C_3(1+k)^\theta, \end{aligned}$$

where C_3 is a constant that does not depend on k and n. Since $1 < \theta < \underline{p}^-$, we conclude that

$$\text{meas}\{|u_n| > k\} \leq \frac{C_3(1+k)^\theta}{k^{\underline{p}^+}} \to 0 \text{ as } k \to \infty. \tag{4.20}$$

Now, we will show that $(u_n)_n$ is a Cauchy sequence in measure. Indeed, we have for every $\delta > 0$,

$$\begin{aligned}\text{meas}\{|u_n - u_m| > \delta\} \leq\ & \text{meas}\{|u_n| > k\} + \text{meas}\{|u_m| > k\} \\ & + \text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\}.\end{aligned}$$

Let $\varepsilon > 0$, in view of (4.20) we may choose $k = k(\varepsilon)$ large enough such that

$$\text{meas}\{|u_n| > k\} \leq \frac{\varepsilon}{3} \text{ and } \text{meas}\{|u_m| > k\} \leq \frac{\varepsilon}{3}. \tag{4.21}$$

Moreover, thanks to (4.19) we have

$$T_k(u_n) \to \eta_k \text{ in } L^1(\Omega) \text{ and a.e. in } \Omega.$$

Thus $(T_k(u_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in measure, and for any $k > 0$ and $\delta, \varepsilon > 0$, there exists $n_0 = n_0(k, \delta, \varepsilon)$ such that

$$\text{meas}\{|T_k(u_n) - T_k(u_m)| > \delta\} \leq \frac{\varepsilon}{3}, \quad \text{for all } m, n \geq n_0(k, \delta, \varepsilon). \tag{4.22}$$

By combining (4.21) and (4.22), we conclude that for all $\delta, \varepsilon > 0$, there exists $n_0 = n_0(\delta, \varepsilon)$ such that

$$\text{meas}\{|u_n - u_m| > \delta\} \leq \varepsilon, \quad \text{for any } n, m \geq n_0.$$

Thus $(u_n)_n$ is a Cauchy sequence in measure, and converges almost everywhere, for a subsequence, to some measurable function u. Thanks to (4.19) we conclude that

$$T_k(u_n) \rightharpoonup T_k(u) \text{ in } W_0^{1,\vec{p}(\cdot)}(\Omega).$$

In view of Lebesgue dominated convergence theorem, we obtain

$$T_k(u_n) \to T_k(u) \text{ in } L^{p_i(\cdot)}(\Omega), \quad \text{for } i = 1, \ldots, N.$$

Moreover, by taking $T_k(u_n)$ as a test function in the approximate problem (4.2), we have

$$\begin{aligned}&\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n) D^i T_k(u_n)dx + \int_\Omega |T_n(u_n)|^{s(x)} |T_k(u_n)|dx \\ &= \int_\Omega f_n(x, T_n(u_n))\ T_k(u_n)dx.\end{aligned}$$

In view of (3.2), (3.5), and using (4.17) we obtain

$$\begin{aligned}\alpha \sum_{i=1}^N \int_\Omega |D^i T_k(u_n)|^{p_i(x)}dx &\leq \int_\Omega g(x)\ |T_k(u_n)|dx + \int_\Omega |T_n(u_n)|^{q(x)}\ |T_k(u_n)|dx \\ &\leq k\|g(x)\|_{L^1(\Omega)} + k\||T_n(u_n)|^{q(x)}\|_{L^1(\Omega)} \\ &\leq k(\|g(x)\|_{L^1(\Omega)} + C_1).\end{aligned}$$

It follows, for any $i = 1, \dots, N$, that

$$\begin{aligned} k^{p_i^-} \text{ meas}\{|u_n| > k\} &\le \int_\Omega |T_k(u_n)|^{p_i^-} dx \\ &\le C_p^{p_i^-} \int_\Omega |D^i T_k(u_n)|^{p_i^-} dx \\ &\le C_p^{p_i^-} \int_\Omega |D^i T_k(u_n)|^{p_i(x)} dx + C_p^{p_i^-} |\Omega| \\ &\le C_4 k. \end{aligned}$$

Thus, we conclude that

$$k^{\underline{p}^+ - 1} \cdot \text{meas}\{|u_n| > k\} \le C_4, \quad \text{for any } k \ge 1, \tag{4.23}$$

where C_4 is a constant that doesn't depend on k and n.

Step 3: the equi-integrability of $(|T_n(u_n)|^{s(x)-1} T_n(u_n))_n$ *and* $(f_n(x, T_n(u_n)))_n$. In the sequel, we denote by $\varepsilon_i(n)$, $i = 1, 2, \dots$, various real-valued functions of real variables that converge to 0 as n tends to infinity. Similarly, we define $\varepsilon_i(h)$ and $\varepsilon_i(n, h)$.

In order to pass to the limit in the approximate equation, we shall show that

$$|T_n(u_n)|^{s(x)-1} T_n(u_n) \to |u|^{s(x)-1} u \text{ strongly in } L^1(\Omega) \tag{4.24}$$

and

$$f_n(x, T_n(u_n)) \to f(x, u) \text{ strongly in } L^1(\Omega). \tag{4.25}$$

We have $|T_n(u_n)|^{s(x)-1} T_n(u_n) \to |u|^{s(x)-1} u$ and $f_n(x, T_n(u_n)) \to f(x, u)$ a.e. in Ω. Thus, in view of Vitali's theorem, to show the convergence $(4.24) - (4.25)$, it is suffices to prove that$(f_n(x, T_n(u_n)))_n$ and $(|T_n(u_n)|^{s(x)-1} T_n(u_n))_n$ are uniformly equi-integrable. Let $h \ge R$, by taking $v_n = \varphi(u_n) |T_{h+1}(u_n) - T_h(u_n)|$ as a test function in (4.2), and since v_n have the same sign as u_n, we have

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n)) \varphi(u_n) dx \\ &+ (\theta - 1) \sum_{i=1}^N \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| dx \\ &+ \int_\Omega |T_n(u_n)|^{s(x)} |T_{h+1}(u_n) - T_h(u_n)| \, |\varphi(u_n)| dx \\ \le &\int_\Omega |f_n(x, T_n(u_n))| \, |T_{h+1}(u_n) - T_h(u_n) \, |\varphi(u_n)| dx. \end{aligned}$$

We have $|\varphi(u_n)| \ge \dfrac{1}{2}$ on the set $\{h \le |u_n|\}$, and thanks to (3.2) we obtain

$$\begin{aligned} &\sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_{h+1}(u_n) - D^i T_h(u_n)) \, |\varphi(u_n)| dx \\ &+ (\theta - 1) \sum_{i=1}^N \int_\Omega \frac{a_i(x, T_n(u_n), \nabla u_n) D^i u_n}{(1 + |u_n|)^\theta} |T_{h+1}(u_n) - T_h(u_n)| dx \end{aligned}$$

$$
\begin{aligned}
\geq & \frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n dx + \frac{\alpha}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} |D^i u_n|^{p_i(x)}dx \\
& + \alpha(\theta-1)\sum_{i=1}^{N}\int_{\{h+1\leq|u_n|\}}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx \\
\geq & \frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n dx + C_5\sum_{i=1}^{N}\int_{\{h+1\leq|u_n|\}}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx,
\end{aligned}
$$

with $C_5 = \alpha\cdot\min\left\{\frac{1}{4},(\theta-1)\right\}$. Having in mind (3.5) we conclude that

$$
\begin{aligned}
& \frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\leq h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n dx + C_5\sum_{i=1}^{N}\int_{\{h+1\leq|u_n|\}}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx \\
& + \int_{\Omega}|T_n(u_n)|^{s(x)}|T_{h+1}(u_n)-T_h(u_n)|\varphi(u_n)dx \\
\leq & \int_{\{h<|u_n|\}}|g(x)||T_{h+1}(u_n)-T_h(u_n)|dx \\
& + \int_{\{h<|u_n|\}}|T_n(u_n)|^{q(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx.
\end{aligned}
\tag{4.26}
$$

For the second term on the left-hand side of (4.26), thanks to Poincaré's inequality we have

$$
\begin{aligned}
& C_5\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{|D^i u_n|^{p_i(x)}}{(1+|u_n|)^{\theta}}dx \\
\geq & C_5\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{|D^i u_n|^{p_i^-}}{(1+|u_n|)^{\theta}}dx - C_5 N\text{meas}\{h\leq|u_n|\} \\
= & C_5\sum_{i=1}^{N}\int_{\Omega}\left|D^i\int_{|T_h(u_n)|}^{|u_n|}\frac{ds}{(1+s)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - C_5 N\text{meas}\{h\leq|u_n|\} \\
\geq & C_6\sum_{i=1}^{N}\int_{\Omega}\left|\int_{|T_h(u_n)|}^{|u_n|}\frac{ds}{(1+s)^{\frac{\theta}{p_i^-}}}\right|^{p_i^-}dx - C_5 N\text{meas}\{h\leq|u_n|\} \\
\geq & C_6\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{(|u_n|-|T_h(u_n)|)^{p_i^-}}{(1+|u_n|)^{\theta}}dx - C_5 N\text{meas}\{h\leq|u_n|\} \\
\geq & C_7\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}|u_n|^{p_i^- -\theta}dx - C_6\sum_{i=1}^{N}\int_{\{h\leq|u_n|\}}\frac{h^{p_i^-}}{(1+|u_n|)^{\theta}}dx \\
& - C_5 N\text{meas}\{h\leq|u_n|\}.
\end{aligned}
$$

Having in mind (4.26), we conclude that

$$\begin{aligned}
&\frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\le h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n dx + C_7\sum_{i=1}^{N}\int_{\{h\le|u_n|\}}|u_n|^{p_i^- -\theta}dx\\
&+\int_{\{h<|u_n|\}}|T_n(u_n)|^{s(x)}|T_{h+1}(u_n)-T_h(u_n)|\varphi(u_n)dx\\
\le&\int_{\{h<|u_n|\}}|g(x)|dx+\int_{\{h<|u_n|\}}|T_n(u_n)|^{q(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx\\
&+C_6\sum_{i=1}^{N}\int_{\{h<|u_n|\}}\frac{h^{p_i^-}}{(1+|u_n|)^{\theta}}dx + C_5 N\text{meas}\{h\le|u_n|\}.
\end{aligned}$$

Since $q(x)<\max(s(x),\underline{p}^+-\theta)$, and in view of Young's inequality we have

$$\begin{aligned}
&\int_{\{h<|u_n|\}}|T_n(u_n)|^{q(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx\\
\le&\frac{C_7}{2}\sum_{i=1}^{N}\int_{\{h\le|u_n|\}}|u_n|^{p_i^- -\theta}dx + C_8\int_{\{h<|u_n|\}}|T_{h+1}(u_n)-T_h(u_n)|dx\\
&+\frac{1}{2}\int_{\{h<|u_n|\}}|T_n(u_n)|^{s(x)}|T_{h+1}(u_n)-T_h(u_n)|\ |\varphi(u_n)|dx,
\end{aligned}$$

and thanks to (4.23), we have

$$\begin{aligned}
\varepsilon_1(h)=&\sum_{i=1}^{N}\int_{\{h<|u_n|\}}\frac{h^{p_i^-}}{(1+|u_n|)^{\theta}}dx\le\sum_{i=1}^{N}h^{p_i^- -\theta}\ \text{meas}\{h<|u_n|\}\\
\le&Nh^{\underline{p}^+-\theta}\ \text{meas}\{h<|u_n|\}\\
=&\frac{Nh^{\underline{p}^+-1}\ \text{meas}\{h<|u_n|\}}{h^{\theta-1}}\\
\le&\frac{NC_4}{h^{\theta-1}}\to 0 \text{ as } h\to\infty.
\end{aligned}$$

Also, we have meas$\{|u_n|>h\}$ goes to zero, as h tends to infinity, and since $g(x)\in L^1(\Omega)$ we conclude that

$$\varepsilon_2(h)=\int_{\{h<|u_n|\}}|g(x)|dx + C_5N\ \text{meas}\{h\le|u_n|\}\to 0 \text{ as } h\to\infty.$$

It follows that

$$\begin{aligned}
&\frac{1}{4}\sum_{i=1}^{N}\int_{\{h<|u_n|\le h+1\}} a_i(x,T_n(u_n),\nabla u_n)D^i u_n dx\\
&+\frac{C_7}{2}\sum_{i=1}^{N}\int_{\{h\le|u_n|\}}|u_n|^{p_i^- -\theta}dx+\frac{1}{2}\int_{\{h+1<|u_n|\}}|T_n(u_n)|^{s(x)}dx\\
\le&C_8\int_{\{h<|u_n|\}}|T_{h+1}(u_n)-T_h(u_n)|dx+\varepsilon_3(h)\\
\le&\varepsilon_4(h).
\end{aligned}\tag{4.27}$$

We conclude that

$$\lim_{h\to\infty}\Big(\int_{\{h+1<|u_n|\}}|T_n(u_n)|^{s(x)}dx+\int_{\{h+1<|u_n|\}}|T_n(u_n)|^{q(x)}dx\Big)=0, \tag{4.28}$$

therefore, thanks to (4.28) we have for any $\delta>0$, there exists $h(\delta)>1$ such that

$$\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{s(x)}dx+\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{q(x)}dx\leq\frac{\delta}{2}. \tag{4.29}$$

On the other hand, for any measurable subset $E\subseteq\Omega$ we have

$$\begin{aligned}&\int_E|T_n(u_n)|^{s(x)}dx+\int_E|T_n(u_n)|^{q(x)}dx\\ &\leq\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{s(x)}dx+\int_{\{h(\delta)<|u_n|\}}|T_n(u_n)|^{q(x)}dx\\ &\quad+\int_E|T_{h(\delta)}(u_n)|^{s(x)}dx+\int_E|T_{h(\delta)}(u_n)|^{q(x)}dx.\end{aligned} \tag{4.30}$$

It's clear that, there exists $\beta(\delta)>0$ such that for any $E\subseteq\Omega$ with $\text{meas}(E)\leq\beta(\delta)$ we have

$$\int_E|T_{h(\delta)}(u_n)|^{s(x)}dx+\int_E|T_{h(\delta)}(u_n)|^{q(x)}dx\leq\frac{\delta}{2}. \tag{4.31}$$

Finally, by combining (4.29), (4.30) and (4.31), we obtain

$$\int_E|T_n(u_n)|^{s(x)}dx+\int_E|T_n(u_n)|^{q(x)}dx\leq\delta \text{ for any } E\subset\Omega \ \text{ such that } \text{meas}(E)\leq\beta(\delta). \tag{4.32}$$

Consequently, $(|T_n(u_n)|^{s(x)-1}T_n(u_n))_n$ and $(|T_n(u_n)|^{q(x)-1}T_n(u_n))_n$ are uniformly equi-integrable, and in view of the growth condition (3.5) we have

$$|f_n(x,T_n(u_n))|\leq g(x)+|T_n(u_n)|^{q(x)},$$

with $g(x)\in L^1(\Omega)$, then $(f_n(x,T_n(u_n)))_n$ is also uniformly equi-integrable. According to Vitali's theorem, the statements (4.24) and (4.25) are concluded. Moreover, in view of (4.27) we have

$$\lim_{h\to\infty}\limsup_{n\to\infty}\sum_{i=1}^N\int_{\{h<|u_n|\leq h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n dx=0. \tag{4.33}$$

Step 4: strong convergence of truncations. Let $h>k\geq1$, and we set $\psi_h(u_n)=(1-|T_1(u_n-T_h(u_n))|)$. By taking $(T_k(u_n)-T_k(u))\psi_h(u_n)\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ as a test function in (4.2) we obtain

$$\begin{aligned}&\sum_{i=1}^N\int_\Omega a_i(x,T_n(u_n),\nabla u_n)(D^iT_k(u_n)-D^iT_k(u))\psi_h(u_n)dx\\ &-\sum_{i=1}^N\int_{\{h\leq|u_n|\leq h+1\}}a_i(x,T_n(u_n),\nabla u_n)D^iu_n|T_k(u_n)-T_k(u)|dx\\ &+\int_\Omega|T_n(u_n)|^{s(x)-1}T_n(u_n)(T_k(u_n)-T_k(u))\psi_h(u_n)dx\end{aligned}$$

$$= \int_\Omega f_n(x, T_n(u_n))(T_k(u_n) - T_k(u))\psi_h(u_n)dx.$$

It follows that

$$\begin{aligned}(4.34)\quad & \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u))\psi_h(u_n)dx \\ & \leq \int_\Omega |f_n(x, T_n(u_n))|\ |T_k(u_n) - T_k(u)|dx + \int_\Omega |T_n(u_n)|^{s(x)}\ |T_k(u_n) - T_k(u)|dx \\ & + \sum_{i=1}^N \int_{\{h\leq|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n |T_k(u_n) - T_k(u)|dx.\end{aligned}$$

For the first and second terms on the right-hand side of (4.34), we have $T_k(u_n) \rightharpoonup T_k(u)$ weak$-\star$ in $L^\infty(\Omega)$, and thanks to (4.24)–(4.25) we have $|T_n(u_n)|^{s(x)} \to |u|^{s(x)}$ and $f_n(x, T_n(u_n)) \to f(x, u)$ strongly in $L^1(\Omega)$, then

$$(4.35)\qquad \varepsilon_5(n) = \int_\Omega |T_n(u_n)|^{s(x)}|T_k(u_n) - T_k(u)|dx \to 0 \text{ as } n \to \infty$$

and

$$(4.36)\qquad \varepsilon_6(n) = \int_\Omega |f_n(x, T_n(u_n))|\ |T_k(u_n) - T_k(u)|dx \to 0 \text{ as } n \to \infty.$$

On the other hand, according to (4.33) we have

$$\begin{aligned}(4.37)\qquad \varepsilon_7(h) = & \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n |T_k(u_n) - T_k(u)|dx \\ \leq & 2k \sum_{i=1}^N \int_{\{h<|u_n|\leq h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n dx \to 0 \text{ as } h \to \infty.\end{aligned}$$

By combining (4.34) and (4.35)–(4.37) we conclude that

$$(4.38)\qquad \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u))\psi_h(u_n)dx \leq \varepsilon_7(n, h).$$

For the term on the left-hand side of (4.38), since $a_i(x, s, 0) = 0$, it follows that

(4.39)

$$\begin{aligned}& \sum_{i=1}^N \int_\Omega a_i(x, T_n(u_n), \nabla u_n)(D^i T_k(u_n) - D^i T_k(u))\psi_h(u_n)dx \\ = & \sum_{i=1}^N \int_{\{|u_n|\leq k\}} a_i(x, T_k(u_n), \nabla T_k(u_n))(D^i T_k(u_n) - D^i T_k(u))dx \\ & - \sum_{i=1}^N \int_{\{k<|u_n|\leq h+1\}} a_i(x, T_{h+1}(u_n), \nabla T_{h+1}(u_n))D^i T_k(u)\ \psi_h(u_n)dx \\ = & \sum_{i=1}^N \int_\Omega \Big(a_i(x, T_k(u_n), \nabla T_k(u_n)) - a_i(x, T_k(u_n), \nabla T_k(u))\Big)(D^i T_k(u_n) - D^i T_k(u))dx\end{aligned}$$

$$+\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))dx$$
$$-\sum_{i=1}^{N}\int_{\{k<|u_n|\leq h+1\}} a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D^iT_k(u)\ \psi_h(u_n)dx.$$

For the second term on the right-hand side of (4.39), we have $T_k(u_n)\to T_k(u)$ in $L^{p_i(\cdot)}(\Omega)$, then, $a_i(x,T_k(u_n),\nabla T_k(u))\to a_i(x,T_k(u),\nabla T_k(u))$ strongly in $L^{p_i'(\cdot)}(\Omega)$, and since $D^iT_k(u_n)$ converges to $D^iT_k(u)$ weakly in $L^{p_i(\cdot)}(\Omega)$, we obtain

$$\varepsilon_8(n)=\sum_{i=1}^{N}\int_{\Omega} a_i(x,T_k(u_n),\nabla T_k(u))(D^iT_k(u_n)-D^iT_k(u))dx\to 0 \text{ as } n\to\infty. \tag{4.40}$$

Concerning the third term on the right-hand side of (4.39), we have $(|a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|)_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, then there exists $\nu_i\in L^{p_i'(\cdot)}(\Omega)$ such that $|a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|\rightharpoonup\nu_i$ weakly in $L^{p_i'(\cdot)}(\Omega)$ for any $i=1,\dots,N$. Therefore,

$$\begin{aligned}\varepsilon_9(n)\leq&\Big|\sum_{i=1}^{N}\int_{\{k<|u_n|\leq h+1\}} a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))D^iT_k(u)\ \psi_h(u_n)dx\Big|\\ \leq&\sum_{i=1}^{N}\int_{\{k<|u_n|\leq h+1\}} |a_i(x,T_{h+1}(u_n),\nabla T_{h+1}(u_n))|\ |D^iT_k(u)|dx\\ \to&\sum_{i=1}^{N}\int_{\{k<|u|\leq h+1\}}\nu_i\ |D^iT_k(u)|dx=0 \text{ as } n\to\infty.\end{aligned}\tag{4.41}$$

By combining (4.38)–(4.41), we conclude that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))dx\\ \leq&\varepsilon_{10}(n,h).\end{aligned}$$

In view of Lebesgue dominated convergence theorem, we have $T_k(u_n)\to T_k(u)$ strongly in $L^{\underline{p}^+}(\Omega)$. Thus, by letting n then h tend to infinity we deduce that

$$\begin{aligned}&\sum_{i=1}^{N}\int_{\Omega}\Big(a_i(x,T_k(u_n),\nabla T_k(u_n))-a_i(x,T_k(u_n),\nabla T_k(u))\Big)(D^iT_k(u_n)-D^iT_k(u))dx\\ &+\int_{\Omega}(|T_k(u_n)|^{\underline{p}^+-2}T_k(u_n)-|T_k(u)|^{\underline{p}^+-2}T_k(u))(T_k(u_n)-T_k(u))dx\to 0 \ \text{ as } \ n\to\infty.\end{aligned}$$

In view of Lemma 4.2, we conclude that

$$\begin{cases} T_k(u_n)\to T_k(u) \text{ strongly in } W_0^{1,\vec{p}(\cdot)}(\Omega),\\ D^iu_n\to D^iu \text{ a.e. in } \Omega \text{ for } i=1,\dots,N.\end{cases}\tag{4.42}$$

Moreover, we have $a_i(x, T_n(u_n), \nabla u_n)D^i u_n$ tends to $a_i(x, u, \nabla u)D^i u$ almost everywhere in Ω, and in view of Fatou's lemma and (4.33), we conclude that

$$\begin{aligned}
&\lim_{h\to\infty} \sum_{i=1}^{N} \int_{\{h<|u|<h+1\}} a_i(x, u, \nabla u)D^i u dx \\
\leq &\lim_{h\to\infty} \liminf_{n\to\infty} \sum_{i=1}^{N} \int_{\{h<|u_n|<h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n dx \\
\leq &\lim_{h\to\infty} \limsup_{n\to\infty} \sum_{i=1}^{N} \int_{\{h<|u_n|<h+1\}} a_i(x, T_n(u_n), \nabla u_n)D^i u_n dx = 0,
\end{aligned}$$

which prove (4.1).

Step 5: passage to the limit. Let $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$, and choosing $S(\cdot)$ be a smooth function in $C_0^1(\mathbb{R})$ such that supp $(S(\cdot)) \subseteq [-M, M]$ for some $M \geq 0$.

By taking $S(u_n)\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \cap L^\infty(\Omega)$ as a test function in the approximate problem (4.2), we obtain

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \Big(D^i u_n S'(u_n)\varphi + S(u_n)D^i \varphi\Big) dx \\
&+ \int_\Omega |T_n(u_n)|^{s(x)-1} T_n(u_n)S(u_n)\varphi dx = \int_\Omega f_n(x, T_n(u_n))S(u_n)\varphi dx.
\end{aligned} \tag{4.43}$$

For the first term on the left-hand side of (4.43), we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \Big(D^i u_n S'(u_n)\varphi + S(u_n)D^i \varphi\Big) dx \\
=&\sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \Big(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i \varphi\Big) dx,
\end{aligned}$$

in view of (4.42), we have $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p_i'(\cdot)}(\Omega)$, and since $a_i(x, T_M(u_n), \nabla T_M(u_n))$ tends to $a_i(x, T_M(u), \nabla T_M(u))$ almost everywhere in Ω, it follows that

$$a_i(x, T_M(u_n), \nabla T_M(u_n)) \rightharpoonup a_i(x, T_M(u), \nabla T_M(u)) \text{ in } L^{p_i'(\cdot)}(\Omega),$$

and since $(S'(u_n)\varphi D^i T_M(u_n) + S(T_M(u_n))D^i \varphi) \to (S'(u)\varphi D^i T_M(u) + S(T_M(u))D^i \varphi)$ strongly in $L^{p_i(\cdot)}(\Omega)$, we deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_n(u_n), \nabla u_n) \Big(D^i u_n S'(u_n)\varphi + S(u_n)D^i \varphi\Big) dx \\
=&\lim_{n\to\infty} \sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u_n), \nabla T_M(u_n)) \Big(D^i T_M(u_n)S'(u_n)\varphi + S(T_M(u_n))D^i \varphi\Big) dx \\
=&\sum_{i=1}^{N} \int_\Omega a_i(x, T_M(u), \nabla T_M(u)) \Big(D^i T_M(u)S'(u)\varphi + S(T_M(u))D^i \varphi\Big) dx
\end{aligned} \tag{4.44}$$

$$= \sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\left(D^i u S'(u)\varphi + S(u) D^i\varphi\right) dx.$$

Concerning the second term on the right-hand side of (4.43), we have $S(T_M(u_n))\varphi \rightharpoonup S(T_M(u))\varphi$ weak$-*$ in $L^\infty(\Omega)$, and thanks to (4.24), we have $|T_n(u_n)|^{s(x)-1}T_n(u_n) \to |u|^{s(x)-1}u$ strongly in $L^1(\Omega)$, it follows that

$$\lim_{n\to\infty}\int_\Omega |T_n(u_n)|^{s(x)-1}T_n(u_n)S(T_M(u_n))\varphi dx = \int_\Omega |u|^{s(x)-1}uS(T_M(u))\varphi dx = \int_\Omega |u|^{s(x)-1}uS(u)\varphi dx. \tag{4.45}$$

Similarly, thanks to (4.25) we have $f_n(x,T_n(u_n)) \to f(x,u)$ strongly in $L^1(\Omega)$ then

$$\lim_{n\to\infty}\int_\Omega f_n(x,T_n(u_n))S(T_M(u_n))\varphi dx = \int_\Omega f(x,u)S(T_M(u))\varphi dx = \int_\Omega f(x,u)S(u)\varphi dx. \tag{4.46}$$

By combining (4.43) and (4.44)–(4.46), we conclude that

$$\sum_{i=1}^{N} \int_\Omega a_i(x,u,\nabla u)\left(D^i u S'(u)\varphi + S(u)D^i\varphi\right) dx + \int_\Omega |u|^{s(x)-1}uS(u)\varphi dx = \int_\Omega f(x,u)S(u)\varphi dx.$$

which complete the proof of the Theorem 4.1.

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