CERTAIN CLASSES OF BI-UNIVALENT FUNCTIONS OF COMPLEX ORDER ASSOCIATED WITH QUASI-SUBORDINATION INVOLVING \((p, q)\)-DERIVATIVE OPERATOR

Ş. ALTINKAYA\(^1\) AND S. YALÇIN\(^1\)

Abstract. In this present paper, as applications of the post-quantum calculus known as the \((p, q)\)-calculus, we construct a new class \(D^k_{p,q}(\gamma, \zeta, \Psi)\) of bi-univalent functions of complex order defined in the open unit disk. Coefficients inequalities and several special consequences of the results are obtained.

1. Introduction and Preliminaries

The \(q\)-calculus as well as the fractional \(q\)-calculus provide important tools that have been used in the fields of special functions and many other areas. Historically speaking, a firm footing of the usage of the \(q\)-calculus in the context of Geometric Function Theory was actually provided and the basic (or \(q\)-) hypergeometric functions were first used in Geometric Function Theory in a book chapter by Srivastava (see, for details, [30]). In fact, the theory of univalent functions can be described by using the theory of the \(q\)-calculus. Moreover, in recent years, such \(q\)-calculus operators as the fractional \(q\)-integral and fractional \(q\)-derivative operators were used to construct several subclasses of analytic functions (see, for example, [3, 19, 21, 26]). In particular, Purohit and Raina [20] investigated applications of fractional \(q\)-calculus operators to define several classes of functions which are analytic in the open unit disk. On the other hand, Mohammed and Darus [14] studied approximation and geometric properties of these \(q\)-operators in regard to some subclasses of analytic functions in a compact disk.

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Further the possibility of extension of the \(q\)-calculus to post-quantum calculus denoted by the \((p, q)\)-calculus. The \((p, q)\)-calculus which have many applications in areas of science and engineering was introduced in order to generalize the \(q\)-series by Gasper and Rahman [8]. The \((p, q)\)-series is derived as corresponding extensions of \(q\)-identities (for example [2, 6]).

We begin by providing some basic definitions and concept details of the \((p, q)\)-calculus which are used in this paper.

The \((p, q)\)-number is given by
\[
[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad p \neq q,
\]
which is a natural generalization of the \(q\)-number (see [11]), that is
\[
\lim_{p \to 1} [n]_{p,q} = [n]_q = \frac{1 - q^n}{1 - q}, \quad q \neq 1.
\]
It is clear that the notation \([n]_{p,q}\) is symmetric, that is,
\[
[n]_{p,q} = [n]_{q,p}.
\]

Let \(p\) and \(q\) be elements of complex numbers and \(D = D_{p,q} \subset \mathbb{C}\) such that \(x \in D\) implies \(px \in D\) and \(qx \in D\). Here, in this investigation, we give the following two definitions which involve a post-quantum generalization of Sofonea’s work [27].

**Definition 1.1.** Let \(0 < |q| < |p| \leq 1\). A given function \(f : D_{p,q} \to \mathbb{C}\) is called \((p, q)\)-differentiable under the restriction that, if \(0 \in D_{p,q}\), then \(f'(0)\) exists.

**Definition 1.2.** Let \(0 < |q| < |p| \leq 1\). A given function \(f : D_{p,q} \to \mathbb{C}\) is called \((p, q)\)-differentiable of order \(n\), if and only if \(0 \in D_{p,q}\), then \(f^{(n)}(0)\) exists.

**Definition 1.3** ([6]). The \((p, q)\)-derivative of a function \(f\) is defined as
\[
(D_{p,q}f)(x) = \frac{f(px) - f(qx)}{(p - q)x}, \quad x \neq 0,
\]
and \((D_{p,q}f)(0) = f'(0)\), provided \(f'(0)\) exists.

As with ordinary derivative, the action of the \((p, q)\)-derivative of a function is a linear operator. More precisely, for any constants \(a\) and \(b\),
\[
D_{p,q}(af(z) + bg(z)) = aD_{p,q}f(z) + bD_{p,q}g(z).
\]
The \((p, q)\)-derivative fulfills the following product rules
\[
D_{p,q}(f(z)g(z)) = f(px)D_{p,q}g(z) + g(qz)D_{p,q}f(z),
\]
\[
D_{p,q}(f(z)g(z)) = g(px)D_{p,q}f(z) + f(qz)D_{p,q}g(z).
\]
Further, the \((p, q)\)-derivative fulfills the following product rules
\[
\begin{align*}
D_{p,q} \left( \frac{f(z)}{g(z)} \right) &= \frac{g(qz)D_{p,q}f(z) - f(qz)D_{p,q}g(z)}{g(pz)g(qz)}, \\
D_{p,q} \left( \frac{f(z)}{g(z)} \right) &= \frac{g(pz)D_{p,q}f(z) - f(pz)D_{p,q}g(z)}{g(pz)g(qz)}.
\end{align*}
\]

Let \(A\) indicate an analytic function family, which is normalized under the condition of \(f(0) = f'(0) - 1 = 0\) in \(\Delta = \{ z : z \in \mathbb{C} \text{ and } |z| < 1 \}\) and given by the following Taylor-Maclaurin series:
\[
(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.
\]

Further, by \(S\) we shall denote the class of all functions in \(A\) which are univalent in \(\Delta\).

If \(f\) is of the form (1.1), then
\[
(D_{p,q}f)(z) = 1 + \sum_{n=2}^{\infty} [n]_{p,q} a_n z^{n-1}.
\]

With a view to recalling the principle of subordination between analytic functions, let the functions \(f\) and \(g\) be analytic in \(\Delta\). Then we say that the function \(f\) is subordinate to \(g\) if there exists a Schwarz function \(w(z)\), analytic in \(\Delta\) with
\[
w(0) = 0, |w(z)| < 1, \quad z \in \Delta,
\]
such that
\[
f(z) = g(w(z)), \quad z \in \Delta.
\]

We denote this subordination by
\[
f \prec g \quad \text{or} \quad f(z) \prec g(z), \quad z \in \Delta.
\]

In particular, if the function \(g\) is univalent in \(\Delta\), the above subordination is equivalent to
\[
f(0) = g(0), \quad f(\Delta) \subset g(\Delta).
\]

In the year 1970, Robertson [23] introduced the concept of quasi-subordination. For two analytic functions \(f\) and \(g\), the function \(f\) is said to be quasi-subordinate to \(g\) in \(\Delta\) and written as
\[
f(z) \prec_{\rho} g(z), \quad z \in \Delta,
\]
if there exists an analytic function \(|h(z)| \leq 1\) such that \(\frac{f(z)}{h(z)}\) analytic in \(\Delta\) and
\[
\frac{f(z)}{h(z)} \prec g(z), \quad z \in \Delta,
\]
that is, there exists a Schwarz function \(w(z)\) such that \(f(z) = h(z)g(w(z))\). Observe that if \(h(z) = 1\), then \(f(z) = g(w(z))\) so that \(f(z) \prec g(z)\) in \(\Delta\). Also notice that if \(w(z) = z\), then \(f(z) = h(z)g(z)\) and it is said that is majorized by \(g\) and written \(f(z) \ll g(z)\) in \(\Delta\). Hence it is obvious that quasi-subordination is a generalization.
of subordination as well as majorization (see, e.g., [13, 22, 23] for works related to quasi-subordination).

The Koebe-One Quarter Theorem [7] ensures that the image of \( \Delta \) under every univalent function \( f \in A \) contains a disk of radius \( 1/4 \). Thus every univalent function \( f \) has an inverse \( f^{-1} \) satisfying \( f^{-1}(f(z)) = z \) and \( f(f^{-1}(w)) = w \)
\(|w| < r_0(f), r_0(f) \geq \frac{1}{4} \), where

\[
(1.2) \quad f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2a_3 + a_4) w^4 + \cdots.
\]

A function \( f \in A \) is said to be bi-univalent in \( \Delta \) if both \( f \) and \( f^{-1} \) are univalent in \( \Delta \). Let \( \Sigma \) denote the class of bi-univalent functions in \( \Delta \) given by (1.1). For a brief history and interesting examples in the class \( \Sigma \), see [29] (see also [4, 5, 12, 16]). Furthermore, judging by the remarkable flood of papers on the subject (see, for example, [10, 17, 28]). Not much is known about the bounds on the general coefficient \( |a_n| \). In the literature, there are only a few works determining the general coefficient bounds \( |a_n| \) for the analytic bi-univalent functions ([1, 9, 15, 31]). The coefficient estimate problem for each of \( |a_n| (n \in \mathbb{N} \setminus \{1, 2\}, \mathbb{N} = \{1, 2, 3, \ldots\}) \) is still an open problem.

Recently for \( f \in A \), Selvaraj et al. [25] defined and discussed \((p, q)\)-analogue of Salagean differential operator as given below:

\[
D^0_{p,q} f(z) = f(z)
\]

\[
D^1_{p,q} f(z) = z(D_{p,q} f(z))
\]

\[
: \quad D^k_{p,q} f(z) = zD_{p,q}(D^{k-1}_{p,q} f(z))
\]

\[
D^k_{p,q} f(z) = z + \sum_{n=2}^{\infty} [n]^k_{p,q} a_n z^n, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, z \in \Delta.
\]

If we let \( p = 1 \) and \( q \to 1^- \), then \( D^k_{p,q} f(z) \) reduces to the well-known Salagean differential operator (see [24]).

Making use of the differential operator \( D^k_{p,q} \), we introduce a new class of analytic bi-univalent functions as follows.

**Definition 1.4.** A function \( f \in \Sigma \) given by (1.1) is said to be in the class

\[
D^k_{p,q} (\gamma, \zeta, \Psi), \quad \gamma \in \mathbb{C} \setminus \{0\}, \quad 0 \leq \zeta < 1, k \in \mathbb{N}_0, \quad 0 < q < p \leq 1, \quad z, w \in \Delta,
\]

if the following conditions are satisfied:

\[
\frac{1}{\gamma} \left( \frac{z(D^k_{p,q} f(z))^\prime}{(1-\zeta)D^k_{p,q} f(z) + \zeta z(D^k_{p,q} f(z))^\prime} - 1 \right) \prec_p (\Psi(z) - 1)
\]
and

\[ \frac{1}{\gamma} \left( \frac{w \left( D^k_{p,q}g(w) \right)'}{(1 - \zeta) D^k_{p,q}g(w) + \zeta w \left( D^k_{p,q}g(w) \right)'} - 1 \right) \prec_{\rho} (\Psi(w) - 1), \]

where the function \( g \) is given by (1.2).

**Remark 1.1.** For \( p = 1 \) and \( q \to 1 \), a function \( f \in \Sigma \) given by (1.1) is said to be in the class \( D^k(\gamma, \zeta, \Psi) \), if the following conditions are satisfied:

\[ \frac{1}{\gamma} \left( \frac{z \left( D^k f(z) \right)'}{(1 - \zeta) D^k f(z) + \zeta z \left( D^k f(z) \right)'} - 1 \right) \prec_{\rho} (\Psi(z) - 1), \quad z \in \Delta \]

and

\[ \frac{1}{\gamma} \left( \frac{w \left( D^k g(w) \right)'}{(1 - \zeta) D^k g(w) + \zeta w \left( D^k g(w) \right)'} - 1 \right) \prec_{\rho} (\Psi(w) - 1), \quad z \in \Delta, \]

where \( \gamma \in \mathbb{C}\setminus\{0\} \), \( 0 \leq \zeta < 1 \), \( k \in \mathbb{N}_0 \) and the function \( g \) is given by (1.2).

**Remark 1.2.** For \( \zeta = 0 \) and \( \gamma \in \mathbb{C}\setminus\{0\} \), a function \( f \in \Sigma \) given by (1.1) is said to be in the class \( D^k_{p,q}(\gamma, \Psi) \), if the following conditions are satisfied:

\[ \frac{1}{\gamma} \left( \frac{z \left( D^k_{p,q}f(z) \right)'}{D^k_{p,q}f(z)} - 1 \right) \prec_{\rho} (\Psi(z) - 1), \quad z \in \Delta \]

and

\[ \frac{1}{\gamma} \left( \frac{w \left( D^k_{p,q}g(w) \right)'}{D^k_{p,q}g(w)} - 1 \right) \prec_{\rho} (\Psi(w) - 1), \quad z \in \Delta, \]

where \( k \in \mathbb{N}_0 \), \( 0 < q < p \leq 1 \) and the function \( g \) is given by (1.2).

**Remark 1.3.** For \( \zeta = k = 0 \) and \( \gamma \in \mathbb{C}\setminus\{0\} \), a function \( f \in \Sigma \) given by (1.1) is said to be in the class \( S_{\Sigma}(\gamma, \Psi) \), if the following conditions are satisfied:

\[ \frac{1}{\gamma} \left( \frac{zf'(z)}{f(z)} - 1 \right) \prec_{\rho} (\Psi(z) - 1), \quad z \in \Delta \]

and

\[ \frac{1}{\gamma} \left( \frac{wg'(w)}{g(w)} - 1 \right) \prec_{\rho} (\Psi(w) - 1), \quad z \in \Delta, \]

where the function \( g \) is given by (1.2).
2. Main Result and its Consequences

Firstly, we will state the Lemma 2.1 to obtain our result.

Lemma 2.1 ([18]). If \( s \in P \), then \( |s_i| \leq 2 \) for each \( i \), where \( P \) is the family of all functions \( s \), analytic in \( \Delta \), for which

\[
\text{Re} (s(z)) > 0,
\]

where

\[
s(z) = 1 + s_1 z + s_2 z^2 + \cdots.
\]

Through out this paper it is assumed that \( \Psi \) is analytic in \( \Delta \) with \( \Psi(0) = 1 \) and let

\[
(2.1) \quad \Psi(z) = 1 + C_1 z + C_2 z^2 + \cdots, \quad C_1 > 0.
\]

Also let

\[
(2.2) \quad h(z) = D_0 + D_1 z + D_2 z^2 + \cdots, \quad |h(z)| \leq 1, \quad z \in \Delta.
\]

We begin this section by finding the estimates on the coefficients \( |a_2| \) and \( |a_3| \) for functions in the class \( D_{p,q}^k (\gamma, \zeta, \Psi) \) proposed by Definition 1.4.

Theorem 2.1. Let \( f \) of the form (1.1) be in the class \( D_{p,q}^k (\gamma, \zeta, \Psi) \). Then

\[
|a_2| \leq \frac{|\gamma| |D_0| C_1 \sqrt{C_1}}{\sqrt{(1 - \zeta) [2^k]_{p,q} \gamma C_1^2 D_0 - [2^k]_{p,q} [(1 - \zeta)(C_2 - C_1) \gamma C_1^2 D_0]}}
\]

and

\[
|a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1 - \zeta)^2 [2^k]_{p,q}} + \frac{|\gamma D_1| C_1}{2(1 - \zeta) [3^k]_{p,q}} + \frac{|\gamma D_0| C_1}{2(1 - \zeta) [3^k]_{p,q}}.
\]

Proof. If \( f \in D_{p,q}^k (\gamma, \zeta, \Psi) \) then, there are two analytic functions \( u, v : \Delta \rightarrow \Delta \) with \( u(0) = v(0) = 0, |u(z)| < 1, |v(w)| < 1 \) and a function \( h \) given by (2.2), such that

\[
(2.3) \quad \frac{1}{\gamma} \left( \frac{z \left( D_{p,q}^k f(z) \right)'}{(1 - \zeta) D_{p,q}^k f(z) + \zeta z \left( D_{p,q}^k f(z) \right)^2} - 1 \right) = h(z) (\Psi(u(z)) - 1)
\]

and

\[
(2.4) \quad \frac{1}{\gamma} \left( \frac{w \left( D_{p,q}^k g(w) \right)'}{(1 - \zeta) D_{p,q}^k g(w) + \zeta w \left( D_{p,q}^k g(w) \right)^2} - 1 \right) = h(w) (\Psi(v(w)) - 1).
\]

Determine the functions \( s_1 \) and \( s_2 \) in \( P \) given by

\[
s_1(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + t_1 z + t_2 z^2 + \cdots.
\]

and

\[
s_2(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + q_1 w + q_2 w^2 + \cdots.
\]
Thus,
\[ u(z) = \frac{s_1(z) - 1}{s_1(z) + 1} = \frac{1}{2} \left( t_1 z + \left( t_2 - \frac{t_1^2}{2} \right) z^2 + \cdots \right) \]
and
\[ v(w) = \frac{s_2(w) - 1}{s_2(w) + 1} = \frac{1}{2} \left( q_1 w + \left( q_2 - \frac{q_1^2}{2} \right) w^2 + \cdots \right). \]

The fact that \( s_1 \) and \( s_2 \) are analytic in \( \Delta \) with \( s_1(0) = s_2(0) = 1 \). Since \( u, v : \Delta \to \Delta \), the functions \( s_1, s_2 \) have a positive real part in \( \Delta \), and the relations \(|t_i| \leq 2\) and \(|q_i| \leq 2\) are true. Using (2.5) and (2.6) together with (2.1) and (2.2) in the right hands of the relations (2.3) and (2.4), we obtain
\[ h(z) \left( \Psi (u(z)) - 1 \right) = D_{0} C_{1} t_1 z \]
\[ + \left( \frac{1}{2} D_{1} C_{1} t_1 + \frac{1}{2} D_{0} C_{1} \left( t_2 - \frac{t_1^2}{2} \right) + \frac{1}{4} D_{0} C_{2} t_1^2 \right) z^2 + \cdots \]
and
\[ h(w) \left( \Psi (v(w)) - 1 \right) = D_{0} C_{1} q_1 w \]
\[ + \left( \frac{1}{2} D_{1} C_{1} q_1 + \frac{1}{2} D_{0} C_{1} \left( q_2 - \frac{q_1^2}{2} \right) + \frac{1}{4} D_{0} C_{2} q_1^2 \right) w^2 + \cdots. \]

In the light of (2.3) and (2.4), we get
\[ \frac{(1 - \zeta) [2]_{p,q}^k a_2}{\gamma} = D_{0} C_{1} t_1 \]
and
\[ \frac{2(1 - \zeta) [3]_{p,q}^k a_3 - (1 - \zeta^2) [2]_{p,q}^k a_2^2}{\gamma} = D_{1} C_{1} t_1 \frac{2}{2} + D_{0} C_{1} \left( t_2 - \frac{t_1^2}{2} \right) + \frac{D_{0} C_{2} t_1^2}{4} \]
and
\[ \frac{(1 - \zeta) [2]_{p,q}^k a_2}{\gamma} = D_{0} C_{1} q_1 \]
\[ \frac{2(1 - \zeta) [3]_{p,q}^k (2a_2^2 - a_3) - (1 - \zeta^2) [2]_{p,q}^k a_2^2}{\gamma} = D_{1} C_{1} q_1 \frac{2}{2} + D_{0} C_{1} \left( q_2 - \frac{q_1^2}{2} \right) + \frac{D_{0} C_{2} q_1^2}{4}. \]

Now, (2.9) and (2.11) give
\[ t_1 = -q_1 \]
and
\[ 8(1 - \zeta)^2 [2]_{p,q}^{2k} a_2^2 = \gamma^2 D_{0}^2 C_{1}^2 \left( t_1^2 + q_1^2 \right). \]
Adding (2.10) and (2.12), we get
\[ (2.15) \quad \frac{4(1-\zeta) [3]_{p,q}^{k} - 2(1-\zeta^2) [2]_{p,q}^{2k}}{\gamma} a_2 = \frac{D_0 C_1 (t_2 + q_2)}{2} + \frac{D_0 (C_2 - C_1) (t_1^2 + q_1^2)}{4}. \]

By using (2.13), (2.14) and Lemma 2.1 in (2.15), we obtain
\[ |a_2| \leq \frac{|\gamma||D_0| C_1 \sqrt{C_1}}{(1-\zeta) [2]_{p,q}^{2k} \gamma C_1^2 D_0 - [2]_{p,q}^{2k} [(1-\zeta)(C_2 - C_1) + (1+\zeta)\gamma C_1^2 D_0]|}. \]

Next, to find the bound on \(|a_3|\), by subtracting (2.12) from (2.10), we have
\[ (2.16) \quad \frac{4(1-\zeta) [3]_{p,q}^{k}}{\gamma} (a_3 - a_2^2) = \frac{D_0 C_1 (t_2 - q_2)}{2} + \frac{D_1 C_1 (t_1 - q_1)}{2}. \]

It follows from (2.13), (2.14) and (2.16) that
\[ a_3 = \frac{\gamma^2 D_0^2 C_1^2 (t_1^2 + q_1^2)}{8(1-\zeta^2) [2]_{p,q}^{2k}} + \frac{\gamma D_1 C_1 (t_1 - q_1)}{8(1-\zeta) [3]_{p,q}^k} + \frac{\gamma D_0 C_1 (t_2 - q_2)}{8(1-\zeta) [3]_{p,q}^k}. \]

Applying Lemma 2.1 once again for the coefficients \(t_1, t_2, q_1\) and \(q_2\), we readily get
\[ |a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-\zeta^2) [2]_{p,q}^{2k}} + \frac{|\gamma D_1| C_1}{2(1-\zeta) [3]_{p,q}^k} + \frac{|\gamma D_0| C_1}{2(1-\zeta) [3]_{p,q}^k}. \]

This completes the proof of Theorem 2.1. \(\square\)

**Corollary 2.1.** Let \(f\) of the form (1.1) be in the class \(D^k(\gamma, \zeta, \Psi)\). Then
\[ |a_2| \leq \frac{|\gamma||D_0| C_1 \sqrt{C_1}}{(1-\zeta) [2]_{p,q}^{2k} \gamma C_1^2 D_0 - [2]_{p,q}^{2k} [(1-\zeta)(C_2 - C_1) + (1+\zeta)\gamma C_1^2 D_0]|}. \]

and
\[ |a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{(1-\zeta)^2 [2]_{p,q}^{2k}} + \frac{|\gamma D_1| C_1}{2(1-\zeta) [3]_{p,q}^k} + \frac{|\gamma D_0| C_1}{2(1-\zeta) [3]_{p,q}^k}. \]

**Corollary 2.2.** Let \(f\) of the form (1.1) be in the class \(D^k_{p,q}(\gamma, \Psi)\). Then
\[ |a_2| \leq \frac{|\gamma||D_0| C_1 \sqrt{C_1}}{\sqrt{[2]_{p,q}^{2k} \gamma C_1^2 D_0 - [2]_{p,q}^{2k} [(C_2 - C_1) + \gamma C_1^2 D_0]|}}. \]

and
\[ |a_3| \leq \frac{|\gamma D_0|^2 C_1^2}{[2]_{p,q}^{2k}} + \frac{|\gamma D_1| C_1}{2 [3]_{p,q}^k} + \frac{|\gamma D_0| C_1}{2 [3]_{p,q}^k}. \]

**Corollary 2.3.** Let \(f\) of the form (1.1) be in the class \(S^2(\gamma, \Psi)\). Then
\[ |a_2| \leq \frac{|\gamma D_0| C_1 \sqrt{C_1}}{\sqrt{|C_1 - C_2 + \gamma C_1^2 D_0}|}. \]
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and

\[ |a_3| \leq |\gamma D_0|^2 C_1^2 + \frac{(|D_1| + |D_0|) |\gamma| C_1}{2}. \]

3. Concluding Remark

Various choices of \(\Psi\) as mentioned above and suitably choosing the values of \(C_1\) and \(C_2\), we state some interesting results analogous to Theorem 2.1 and the Corollaries 2.1 to 2.3. For example, the function \(\Psi\) is given by

\[ \Psi(z) = \left( \frac{1+z}{1-z} \right)^\theta = 1 + 2\theta z + 2\theta^2 z^2 + \cdots, \quad 0 < \theta \leq 1, \]

which gives

\[ C_1 = 2\theta \text{ and } C_2 = 2\theta^2. \]

By taking

\[ \Psi(z) = \frac{1 + (1 - 2\mu) z}{1 - z} = 1 + 2(1 - \mu) z + 2(1 - \mu)^2 z^2 + \cdots, \quad 0 \leq \mu < 1, \]

we have

\[ C_1 = C_2 = 2(1 - \mu). \]

On the other hand, for \(-1 \leq B < A < 1\), if we let

\[ \Psi(z) = \frac{1 + A z}{1 + B z} = 1 + (A - B) z - B(A - B) z^2 + \cdots, \quad 0 < \theta \leq 1, \]

then we have

\[ C_1 = (A - B) \text{ and } C_2 = -B(A - B). \]

The details involved may be left as an exercise for the interested reader.

References

1Department of Mathematics,
Bursa Uludag University,
16059, Bursa, Turkey

Email address: sahsenealtinkaya@gmail.com
Email address: syalcin@uludag.edu.tr