THE $\bar{\partial}$-CAUCHY PROBLEM ON WEAKLY $q$-CONVEX DOMAINS IN $\mathbb{C}P^n$

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ABSTRACT. Let $D$ be a weakly $q$-convex domain in the complex projective space $\mathbb{C}P^n$. In this paper, the (weighted) $\bar{\partial}$-Cauchy problem with support conditions in $D$ is studied. Specifically, the modified weight function method is used to study the $L^2$ existence theorem for the $\bar{\partial}$-Neumann problem on $D$. The solutions are used to study function theory on weakly $q$-convex domains via the $\bar{\partial}$-Cauchy problem.

1. INTRODUCTION AND MAIN RESULTS

The $\bar{\partial}$-problem is one of the important central problems of complex variables. A classical result due to Hörmander tells us that the $\bar{\partial}$-problem is solvable in pseudo-convex domains, and hence, pseudoconvex domains has been widely accepted as the standard domain which we can solve the $\bar{\partial}$-problem. In [16], Ho extend this problem to weakly $q$-convex domains. In fact, Ho is the first person to study the $\bar{\partial}$-problem in $q$-convex domains in $\mathbb{C}^n$. This paper is devoted to studying the $L^2 \bar{\partial}$ Cauchy problem and the $\bar{\partial}$-closed extension problem for forms on a weakly $q$-convex domain $D$ in the complex projective space $\mathbb{C}P^n$. These problems were first studied by Kohn and Rossi [20] (see also [12]). They proved the holomorphic extension of smooth CR functions and the $\bar{\partial}$-closed extension of smooth forms from the boundary $bD$ of a strongly pseudoconvex domain to the whole domain $D$. The $L^2$ theory of these problems has been obtained for pseudoconvex domains in $\mathbb{C}^n$ or, more generally, for domains in complex manifolds with strongly plurisubharmonic weight functions (see Chapter 9 in [6] and the references therein). The $L^2 \bar{\partial}$ Cauchy problem was considered by Derridj [8,9]. In [30,31] Shaw has obtained a solution to this problem on a pseudoconvex domain with $C^1$ boundary in $\mathbb{C}^n$. Also, in the setting of strictly

Key words and phrases. $\bar{\partial}$, $\bar{\partial}$-Neumann operator, $q$-convex domains.

2010 Mathematics Subject Classification. Primary: 32F10. Secondary: 32W05.

DOI 10.46793/KgJMat2004.581S

Received: April 23, 2017.

Accepted: July 06, 2018.
$q$-convex (or $q$-concave) domains, this problem has been studied by Sambou in his thesis (see [29]). In [1], Abdelkader-Saber studied this problem on pseudoconvex manifolds satisfying property $B$. In [26,27], Saber studied this problem on a weakly $q$-convex domain with $C^1$-smooth boundary and on a $q$-pseudoconvex domain $D$ in $\mathbb{C}^n$, $1 < q \leq n$, with Lipschitz boundary. Recently, Saber [28] studied this result to a $q$-pseudoconvex domain $D$ in a Stein manifold. On a pseudoconvex domain in $\mathbb{C}P^n$, Cao-Shaw-Wang [4] (cf. also [5]) obtained the $L^2$ existence theorem for the $\bar{\partial}$-Neumann operator $N$ and obtained the (weighted) $L^2 \bar{\partial}$ Cauchy-problem on such domains. The aim of this paper is to extend this result to the situation in which the boundaries are assumed weakly $q$-convex domain $D$ in $\mathbb{C}P^n$. Moreover, the solutions are used to study function theory on such domains via the $\bar{\partial}$-Cauchy problem.

2. Notation and Preliminaries

Let $(x_0, x_1, \ldots, x_n)$ be a (fixed) homogeneous coordinates of $\mathbb{C}P^n$. If $U_0$ is the open set in $\mathbb{C}P^n$ defined by $x_0 \neq 0$ and if $(z_1, z_2, \ldots, z_n)$, where $z_i = x_i/x_0$, is the homogeneous coordinates of $U_0$, we assume that

$$\omega = \frac{\sum_{i=1}^n |dz_i|^2}{1 + \sum_{i=1}^n |z_i|^2} - \frac{|\sum_{i=1}^n z_i d\bar{z}_i|^2}{(1 + \sum_{i=1}^n |z_i|^2)^2} \text{ on } U_0.$$ 

The Fubini-Study metric of $\mathbb{C}P^n$ determined by $(x_0, x_1, \ldots, x_n)$. This is well-known standard Kähler metric of $\mathbb{C}P^n$.

Let $D$ be a bounded domain in $\mathbb{C}P^n$ and let $C^\infty_{p,q}(D)$ be the space of complex-valued differential forms of class $C^\infty$ and of type $(p,q)$ on $D$. Denote by $L^2(D)$ the space of square integrable functions on $D$ with respect to the Lebesgue measure in $\mathbb{C}P^n$, $L^2_{p,q}(D)$ the space of $(p,q)$-forms with coefficients in $L^2(D)$ and $L^2_{p,q}(D,\phi)$ the space of $(p,q)$-forms with coefficients in $L^2(D)$ with respect to the weighted function $e^{-\phi}$. For $u, v \in L^2_{p,q}(D)$, the inner product $\langle u, v \rangle$ and the norm $\| u \|$ are denoted by:

$$\langle u, v \rangle = \int_D u \wedge \ast \bar{v} \quad \text{and} \quad \| u \|^2 = \langle u, u \rangle,$$

where $\ast$ is the Hodge star operator. Let $\text{dist}(z, bD)$ be the Fubini distance from $z \in D$ to the boundary $bD$ and let $\delta$ be a $C^2$ defining function for $D$ normalized by $|d\delta| = 1$ on $bD$ such that

$$\delta = \delta(z) = \begin{cases} -\text{dist}(z, bD), & \text{if } z \in D, \\ \text{dist}(z, bD), & \text{if } z \in \mathbb{C}P^n \setminus D. \end{cases}$$

Let $\phi_t = -t \log |\delta|$, $t \geq 0$, for $u, v \in L^2_{p,q}(D, \phi_t)$, the inner product $\langle u, v \rangle_{\phi_t}$ and the norm $\| u \|_{\phi_t}$ are denoted by:

$$\langle u, v \rangle_{\phi_t} = \langle u, v \rangle_t = \int_D u \wedge \ast(t) \bar{v},$$

$$\| u \|_{\phi_t}^2 = \| u \|_{t}^2 = \langle u, u \rangle_t,$$
where \( *_{(t)} = \delta^* * = * \delta^t \). Since \( \phi_t \) is bounded on \( \overline{D} \), the two norms \( \| \cdot \| \) and \( \| \cdot \|_t \) are equivalent. Let \( \overline{\mathcal{D}} : \text{dom} \overline{\mathcal{D}} \subset L^2_{p,q}(D, \phi_t) \rightarrow L^2_{p,q+1}(D, \phi_t) \) be the maximal closure of the Cauchy-Riemann operator and \( \overline{\mathcal{D}}_\phi \) be its Hilbert space adjoint. Let \( \Box_t = \overline{\mathcal{D}}_t^* + \overline{\mathcal{D}}_t^* \) be the Laplace-Beltrami operator, where \( \overline{\mathcal{D}}_t^* = \overline{\mathcal{D}}_\phi^* \).

Denote by \( \nabla \) the Levi-Civita connection of \( \mathbb{C}P^n \) with the standard Fubini-Study metric \( \omega \). Let \( \{ e_i \} \) be an orthonormal basis of vector fields. For any two vector fields \( f, g \), the curvature operator of the connection \( \nabla \) is denoted by

\[
\mathcal{R}(f, g) = \nabla_f \nabla_g - \nabla_g \nabla_f - \nabla_{[f, g]}
\]

By setting \( \mathcal{R}_{ijkl} = \omega(\mathcal{R}(e_i, e_j)e_k, e_l) \), the Ricci tensor \( \mathcal{R}_{ij} \) is denoted by

\[
\mathcal{R}_{ij} = \sum_k \varepsilon_k \mathcal{R}_{ikkj},
\]

which turns out to be self-adjoint with respect to \( \omega \) and the scalar curvature

\[
(2.1) \quad \Theta = \sum_i \mathcal{R}_{ii} = \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}_{jjj}
\]

as the trace of the Ricci tensor.

**Definition 2.1.** Let \( D \) be an open set in an \( n \)-dimensional complex manifold \( X \), let \( k \) be an integer with \( 1 \leq k \leq n - 1 \) and put \( E = X \setminus D \). The set \( D \) is said to be pseudoconvex of order \( k \) in \( X \) if, for every \( b \in E \) and for every coordinate neighborhood \( (U, (z_1, \ldots, z_n)) \) which contains \( b \) as the origin, the set

\[
\left\{ (z_1, \ldots, z_n) \in U : z_i = 0, 1 \leq i \leq k, 0 < \sum_{i=k+1}^{n} |z_i|^2 < t \right\}
\]

contains no points of \( E \) for some \( t > 0 \), then there exists \( \ell > 0 \) such that for each \( (z_1', \ldots, z_k') \) with \( |z_i'| < \ell, 1 \leq i \leq k \), the set

\[
\left\{ (z_1, \ldots, z_n) \in U : z_i = z_i', 1 \leq i \leq k, \sum_{i=k+1}^{n} |z_i|^2 < t \right\}
\]

contains at least one point of \( E \).

**Definition 2.2.** Let \( D \) be an \( n \)-dimensional complex manifold and let \( q \) be an integer, \( 1 \leq q \leq n \). By Fujita ([13], Proposition 8) a \( C^2 \) function \( \phi : D \rightarrow \mathbb{R} \) is pseudoconvex of order \( n - q \), if and only if its Levi form \( \partial \overline{\partial} \phi \) has at least \( n - q + 1 \) non negative eigenvalues at each point of \( D \).

**Definition 2.3.** Let \( D \) be an open subset of an \( n \)-dimensional complex manifold \( X \). \( D \) is said to have \( C^2 \) boundary in \( X \) if for all \( z \in bD \) there exist an open neighborhood \( U \) of \( z \) and a \( C^2 \) function \( \delta : U \rightarrow \mathbb{R} \), called a defining function of \( D \) at \( z \) such that \( d\delta(z) \neq 0 \) and \( D \cap U = \{ z \in U : \delta(z) < 0 \} \). Following Ho [16], \( D \) is said to be a
The following two conditions are equivalent. (cf. [15] and Lemma 4.7 in [34]).

Remark 2.2

2.1

Remark

Example 2.1

Theorem 2.1

Let \( D \) be a smooth domain in \( \mathbb{C}^n \), \( D \) is said to be a weakly \( q \)-concave if \( \overline{D} \) is weakly \( q \)-convex.

Lemma 2.1 ([16]). Let \( D \) be a smooth domain in \( \mathbb{C}^n \) and \( \rho \) be its defining function. The following two conditions are equivalent.

1. \( D \) is weakly \( q \)-convex.

2. For any \( z \in bD \) the sum of any \( q \) eigenvalues \( \rho_{i_1}, \ldots, \rho_{i_q} \), with distinct subscripts, of the Levi-form at \( z \) satisfies \( \sum_{j=1}^{q} \rho_{i_j} \geq 0 \) (cf. [15] and Lemma 4.7 in [34]).

Definition 2.4. Let \( D \) be a smooth domain in \( \mathbb{C}^n \), \( D \) is said to be a weakly \( q \)-concave if and only if for any \( z \in bD \) the sum of any \( q \) eigenvalues \( \delta_{i_1}, \ldots, \delta_{i_q} \), with distinct subscripts, of the Levi-form at \( z \) satisfies \( \sum_{j=1}^{q} \delta_{i_j} \geq 0 \) (cf. [15] and Lemma 4.7 in [34]).

Moreover, \( D \) is weakly \( q \)-convex if and only if for any \( z \in bD \) the sum of any \( q \) eigenvalues \( \delta_{i_1}, \ldots, \delta_{i_q} \), with distinct subscripts, of the Levi-form at \( z \) satisfies \( \sum_{j=1}^{q} \delta_{i_j} \geq 0 \) (cf. [15] and Lemma 4.7 in [34]).

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It follows from Lemma 2.1 that \( D \) is weakly \( q \)-concave if and only if for any \( q \) eigenvalues \( \rho_{i_1}, \ldots, \rho_{i_q} \) of the Levi-form at \( z \in bD \) with distinct subscripts we have \( \sum_{j=1}^{q} \rho_{i_j} \leq 0 \).

Example 2.1. Let \( D \) be an open subset of an \( n \)-dimensional complex manifold \( X \) and suppose that the boundary \( bD \) is a real hypersurface of class \( C^2 \) in \( X \), that is, there exist, for each \( z \in bD \), a neighborhood \( U \) of \( z \) and a \( C^2 \) function \( \rho : U \to \mathbb{R} \) such that \( d\rho(z) \neq 0 \) and \( D \cap U = \{ z \in U : \rho(z) < 0 \} \). Then \( D \) is pseudoconvex of order \( n - q \) in \( X \), if and only if the Levi form \( \partial \partial \rho \) has at least \( n - q \) non-negative eigenvalues on \( T'_z(bD) \) for each defining function \( \rho \) of \( D \) near \( z \), where \( T'_z(bD)(\subset T_z(bD)) \) is the holomorphic tangent space of the real hypersurface \( bD \) at \( z \) (cf. [10,35] called such a subset \( D \) a \( (q-1) \)-pseudoconvex open subset with \( C^2 \) boundary).

Theorem 2.1 ([23]). Let \( D \subset \mathbb{C}P^n \) be a pseudoconvex domain of order \( n - q \), \( 1 \leq q \leq n \). Let \( d(z,bD) \) be the Fubini distance from \( z \in D \) to the boundary \( bD \). Then the function \( -\log d(z,bD) \) is \( (q-1) \)-plurisubharmonic in \( D \).

Lemma 2.2 ([17], Lemma 2.6). Let \( \phi \) be a real valued function of class \( C^2 \) defined in an \( n \)-dimensional complex manifold \( D \). Then \( \phi \) is \( (q-1) \)-plurisubharmonic, \( 1 \leq q \leq n \), in \( D \) if and only if \( \phi \) is weakly \( q \)-convex in \( D \).

Remark 2.1. Pseudoconvex open sets in the original sense are pseudoconvex of order \( n - 1 \).

Remark 2.2. The pseudoquasiconvexity of order \( n - q \) of an open subset \( D \) in \( X \) is a local property of the boundary \( bD \subset X \) of \( D \). More precisely, \( D \) is pseudoconvex of order...
n – q in X if, for each p ∈ bD, there exists a neighborhood U ⊂ X of p such that D ∩ U is pseudoconvex of order n – q in U.

Remark 2.3. If an open set D in an n-dimensional complex manifold X is weakly q-convex, 1 ≤ q ≤ n, then D is pseudoconvex of order n – q in X. However, the converse is not valid even if X = \mathbb{C}^n (see [10] and [22]). By Fujita [13], an open subset D of \mathbb{C}^n is pseudoconvex of order n – q in \mathbb{C}^n, if and only if D has an exhaustion function which is pseudoconvex of order n – q on D. Thus, by the approximation theorem of Bungart [3], an open subset D of X is pseudoconvex of order n – q in X, if and only if D is locally q-complete with corners in X in the sense of Peternell [24].

Proposition 2.1 (Bochner-Hörmander-Kohn-Morrey formula). Let D be a compact domain with C^2-smooth boundary bD and δ(x) = −d(x, bD). Suppose that Θ is the curvature term defined in (2.1) with respect to the Fubini-Study metric ω. Then, for any u ∈ C^{p,q}_{bD}(\overline{D}) ∩ dom\overline{\partial}_φ with 1 ≤ q ≤ n – 1, and φ ∈ C^2(\overline{D}), we have

\begin{equation}
(2.2) \quad ||\overline{\partial}u||_φ^2 + ||\overline{\partial}_φ^*u||_φ^2 = (\Theta u, \overline{u})_φ + \left\langle \left\langle \frac{\partial u_{IJ}}{\partial \overline{z}_k} \right\rangle_φ \right\rangle^2 + \left\langle (i\partial \overline{\partial} \phi)u, \overline{u} \right\rangle_φ + \int_{bD} ((i\partial \overline{\partial} \delta)u, \overline{u}) e^{-\phi} ds.
\end{equation}

This formula is known (cf. [2, 7, 15, 18, 19, 32, 36]) for some special cases, although it has not been stated in the literature in the form (2.2). If u has compact support in the interior of D, the (2.2) was proved in [2], Chapter 8 of [7] and (2.12) of [36]. The boundary term had been computed in [14], Chapter 3 by combining the Morrey-Kohn technique on the boundary with non-trivial weight function. If one combines the results of [15] and [37] with the interior formulae discussed above, one can prove that (2.2) holds for the general case with a weight function e^{−\phi} and the curvature term. Specially, for φ = 0, (2.2) was proved in [32].

Proposition 2.2. For any (p.q) -form u of D ⊂ \mathbb{C}P^n with q ≥ 1,

(\Theta u, \overline{u}) = q(2n + 1)|u|^2, \quad \text{when u is a (0,q)-form},

(\Theta u, \overline{u}) = 0, \quad \text{for any (n,q)-form u},

(\Theta u, \overline{u}) ≥ 0, \quad \text{when p ≥ 1 and u is a (p,q)-form}.

The statement for (0,q)-forms and (n,q)-forms was computed in [32] and [36]. Also, following Lemma 3.3 of Henkin-Iordan [14] and its proof showed that the curvature operator Θ acting on L^2_{p,q}(D) is a non-negative operator.

3. THE \overline{\partial}-CAUCHY PROBLEM ON WEAKLY q-CONVEX DOMAINS

This section is devoted to showing the existence of the \overline{\partial}-Neumann operator on a weakly q-convex domain D in \mathbb{C}P^n, 1 ≤ q ≤ n, and by applying these existence to solve the \overline{\partial} problem with support conditions on D. The boundary integral in (2.2) is
non-negative for \( q \geq 1 \) by the assumption on \( D \). Also, by taking \( \phi \equiv 0 \) in (2.2) and using Proposition 2.2, we find the fundamental estimate
\[
\|u\|^2 \leq c \left( \|\mathcal{D}u\|^2 + \|\mathcal{D}^+ u\|^2 \right).
\]
This means that \( \Box \) has closed range and \( \ker \Box = \{0\} \). Thus, one can establish the \( L^2 \)-existence theorem of the \( \overline{\partial} \)-Neumann operator \( N \).

**Theorem 3.1.** Let \( D \Subset \mathbb{C}^p \) be a weakly \( q \)-convex domain with \( C^2 \) smooth boundary. Then, for each \( 0 \leq p \leq n \), \( 1 \leq q \leq n \), there exists a bounded linear operator \( N : L^2_{p,q}(D) \rightarrow L^2_{p,q}(D) \) with the following properties:

(i) \( \text{Range} N \subset \text{dom} \Box, \Box N = N \Box = \text{Id} \) on \( \text{dom} \Box \);
(ii) for \( f \in L^2_{p,q}(D) \),
\[
f = \mathcal{D} \mathcal{D}^+ N f + \mathcal{D}^+ \mathcal{D} N f;
\]
(iii) \( N \mathcal{D} = \mathcal{D} N \) on \( \text{dom} \mathcal{D} \), \( 1 \leq q \leq n - 1 \);
(iv) \( \mathcal{D}^+ N = N \mathcal{D}^+ \) on \( \text{dom} \mathcal{D}^+ \), \( 2 \leq q \leq n \);
(v) \( N, \mathcal{D} N \) and \( \mathcal{D}^+ N \) are bounded linear operators on \( L^2_{p,q}(D) \).

Using the duality relations pertaining to the \( \overline{\partial} \)-Neumann problem, one solve the \( L^2 \) \( \overline{\partial} \) Cauchy problem on weakly \( q \)-convex domains in \( \mathbb{C}^p \), \( 1 \leq q \leq n \). This method was first used by Kohn-Rossi [20] for smooth forms on strongly pseudoconvex domains. More precisely, we prove the following \( L^2 \) Cauchy problem for \( \overline{\partial} \) in \( \mathbb{C}^p \):

**Theorem 3.2.** Let \( D \Subset \mathbb{C}^p \) be a weakly \( q \)-convex domain, \( 1 \leq q \leq n \) with \( C^2 \) smooth boundary. Then, for \( f \in L^2_{p,q}(\mathbb{C}^p) \), \( \text{supp} f \subset \overline{D} \), \( 1 \leq q \leq n - 1 \), satisfying \( \overline{\partial} f = 0 \) in the distribution sense in \( \mathbb{C}^p \), there exists \( u \in L^2_{p,q-1}(\mathbb{C}^p) \), \( \text{supp} u \subset \overline{D} \) such that \( \overline{\partial} u = f \) in the distribution sense in \( \mathbb{C}^p \).

**Proof.** Let \( f \in L^2_{p,q}(\mathbb{C}^p) \), \( \text{supp} f \subset \overline{D} \), then \( f \in L^2_{p,q}(D) \). From Theorem 3.1, \( N_{n-p,n-q} \) exists for \( n - q \geq 1 \). Since \( N_{n-p,n-q} = \Box_{n-p,n-q} \) on \( \text{Range} \Box_{n-p,n-q} \) and \( \text{Range} N_{n-p,n-q} \subset \text{dom} \Box_{n-p,n-q} \), then \( N_{n-p,n-q} \Box f \in \text{dom} \Box_{n-p,n-q} \subset L^2_{n-p,n-q}(D) \), for \( q \leq n - 1 \). Thus, we can define \( u \in L^2_{p,q-1}(D) \) by
\[
u = -\star \overline{\partial} N_{n-p,n-q} \star f.
\]
Thus \( \text{supp} u \subset \overline{D} \) and \( u \) vanishes on \( bD \). Now, we extend \( u \) to \( \mathbb{C}^p \) by defining \( u = 0 \) in \( \mathbb{C}^p \setminus D \). It follows from the same arguments of Theorem 9.1.2 in [6] and Theorem 2.2 in [1] that the form \( u \) satisfies the equation \( \overline{\partial} u = f \) in the distribution sense in \( \mathbb{C}^p \). Thus the proof follows.

4. **The Weighted \( \overline{\partial} \)-Cauchy Problem**

In this section, we assume that \( D \) is a weakly \( q \)-convex domain, \( 1 \leq q \leq n \), with \( C^2 \) smooth boundary in \( \mathbb{C}^p \). Also, we will choose \( \phi_t = -t \log |\delta| \), \( t > 0 \) in (2.2), and using Remark 2.3 and by using Proposition 2.2, the inequality (2.2) implies the
weighted $L^2$-existence for the $\overline{\partial}$. Also, for $u \in \text{Dom}(\Box_t)$ of degree $q \geq 1$ and for $t > 0$, we have

$$t\|u\|_t^2 \leq (\|\overline{\partial}u\|_t^2 + \|\overline{\partial}_tu\|_t^2)$$

$$= \langle \Box_t u, u \rangle_t$$

$$\leq \|\Box_t f\|_t\|u\|_t,$$

i.e.,

$$t\|u\|_t \leq \|\Box_t u\|_t.$$ 

Since $\Box_t$ is a linear densely defined operator, then, from [15, Theorem 1.1.1], $\text{Range}(\Box_t)$ is closed. Thus, from (1.1.1) in [15] and the fact that $\Box_t$ is self adjoint, we have the Hodge decomposition

$$L^2_{p,q}(D, \phi_t) = \overline{\partial}\overline{\partial}^*\text{dom}(\Box_t) \oplus \overline{\partial}_t^*\overline{\partial}\text{dom}(\Box_t).$$

Since $\Box_t$ is one to one on $\text{dom}(\Box_t)$ from (1.5.3) in [15], then there exists a unique bounded inverse operator

$$N_t : \text{Range}(\Box_t) \rightarrow \text{dom}(\Box_t) \cap (\ker(\Box_t))^\perp$$

such that $N_t \Box_t f = f$ on $\text{dom}(\Box_t)$. Therefore, we can establish the existence theorem of the inverse of $\Box_t$ the so called weighted $\overline{\partial}$-Neumann operator $N_t$.

**Theorem 4.1.** For any $1 \leq q \leq n$ and $t > 0$, there exists a bounded linear operator $N_t : L^2_{p,q}(D, \phi_t) \rightarrow L^2_{p,q}(D, \phi_t)$ satisfies the following properties:

(i) $\text{Range}(N_t) \subset \text{dom}(\Box_t)$, $N_t \Box_t = I$ on $\text{dom}(\Box_t)$;

(ii) for $f \in L^2_{p,q}(D, \phi_t)$, we have $u = \overline{\partial}\overline{\partial}^*N_tf \oplus \overline{\partial}_t^*\overline{\partial}N_tf$;

(iii) $\overline{\partial}N_t = N_t\overline{\partial}$, $1 \leq q \leq n - 1$;

(iv) $\overline{\partial}N_t = N_t\overline{\partial}$, $2 \leq q \leq n$;

(v) for all $f \in L^2_{p,q}(D, \phi_t)$, we have the estimates

$$t\|N_t f\|_t \leq \|f\|_t,$$

$$\sqrt{t}\|\overline{\partial}N_t f\|_t + \sqrt{t}\|\overline{\partial}_t^* N_t f\|_t \leq \|f\|_t;$$

(vi) if $\overline{\partial}f = 0$, then $u_t = \overline{\partial}_t^*N_t f$ solves the equation $\overline{\partial}u_t = f$.

**Theorem 4.2.** For $f \in L^2_{p,q}(D, \phi_t)$, $1 \leq q \leq n - 1$, supp $f \subset \overline{D}$, satisfying $\overline{\partial}f = 0$ in the distribution sense in $\mathbb{C}P^n$, there exists $u \in L^2_{p,q-1}(D, \phi_t)$, supp $u \subset \overline{D}$ such that $\overline{\partial}u = f$ in the distribution sense in $\mathbb{C}P^n$.

**Proof.** Following Theorem 4.1, $N_t$ exists for forms in $L^2_{n-p,n-q}(D, \phi_t)$. Thus, one can defines $u_t \in L^2_{p,q-1}(D, \phi_t)$ by

$$(4.1) \quad u_{(t)} = -\ast(t) \overline{\partial}N_{n-p,n-q} \ast(-t) f.$$ 

Thus $\text{supp} u_t \subset \overline{D}$ and $u_t$ vanishes on $bD$. Now, we extend $u_t$ to $\mathbb{C}P^n$ by defining $u_t = 0$ in $\mathbb{C}P^n \setminus D$. We want to prove that the extended form $u_t$ satisfies the equation
\(\overline{\partial} u_t = f\) in the distribution sense in \(\mathbb{C}P^n\). For \(\eta \in L^2_{n,p,n-q-1}(D, -\phi_t) \cap \text{dom} \overline{\partial}\), we have

\[
\langle \overline{\partial} \eta, \star(t) f \rangle_D = \int_D \overline{\partial} \eta \wedge \star(-t) (\star(t) f)
= \int_D \overline{\partial} \eta \wedge \star(-t) \star(t) f
= (-1)^{p+q} \int_D \overline{\partial} \eta \wedge f
= (-1)^{p+q} \langle f, \star(-t) \overline{\partial} \eta \rangle_D
= (-1)^{p+q} \langle f, \star(-t) \overline{\partial} \eta \rangle_{\mathbb{C}P^n},
\]

because \(\text{supp} f \subset \overline{D}\). Since \(\vartheta|_D = \overline{\partial}^\ast |_D\), when \(\vartheta\) acts in the distribution sense (see [15]), then we obtain

\[
\langle \overline{\partial} \eta, \star(t) f \rangle_D = \langle f, \vartheta \star(-t) \eta \rangle_{\mathbb{C}P^n}
= \langle \overline{\partial} f, \star(-t) \eta \rangle_{\mathbb{C}P^n}
= 0.
\]

It follows that \(\overline{\partial}_t' (\star(t) f) = 0\) on \(D\). Using Theorem 4.1 (iv), we have

(4.2) \[\overline{\partial}_t' N_t (\star(t) f) = N_t \overline{\partial}_t' (\star(t) f) = 0.\]

Thus, from (4.1) and (4.2), one obtains

\[
\overline{\partial} u_t = -\partial \star_{-t} \overline{\partial} N_{n-p,n-q} \star_{-t} \overline{f}
= (-1)^{p+q+1} \star \partial \star \overline{\partial} N_{n-p,n-q} \star \overline{f}
= (-1)^{p+q} \star \overline{\partial} \star \overline{\partial} N_{n-p,n-q} \star \overline{f}
= (-1)^{p+q} \star (\overline{\partial}^2 + \overline{\partial} \overline{\partial}^\ast) N_{n-p,n-q} \star \overline{f}
= (-1)^{p+q} \star \overline{f}
= f,
\]

in the distribution sense in \(D\). Since \(u = 0\) in \(\mathbb{C}P^n \setminus D\), then for \(u \in L^2_{p,q} (\mathbb{C}P^n) \cap \text{dom} \overline{\partial}^\ast\), one obtains

\[
< u, \overline{\partial}' u >_{\mathbb{C}P^n} =< u, \overline{\partial}' u >_D
= < \star \overline{\partial}' u, \star(-t) u >_{(t)D}
= (-1)^{p+q} < \overline{\partial} \star u, \star(-t) u >_{(t)D}
= (-1)^{p+q} < \star u, \overline{\partial}' \star(-t) u >_{(t)D}
= < \star u, \star(-t) \overline{\partial} u >_{(t)D}
= < f, u >_{(t)D}
= < f, u >_{\mathbb{C}P^n},
\]

where the third equality holds since \(\star u = (-1)^q \overline{\partial} N_{n-p,n-q} \star f \in \text{dom} \overline{\partial}^\ast\). Thus \(\overline{\partial} u_t = f\) in the distribution sense in \(\mathbb{C}P^n\). \(\square\)
As in [5], we prove the following results.

**Proposition 4.1.** Let $D$ be the same as in Theorem 3.1. Put $Ω = \mathbb{C}P^n \setminus \overline{D}$. Then, for any $f \in W^{1,ε}_{p,q}(Ω)$, $\overline{∂}f = 0$, $0 ≤ ε < \frac{1}{2}$, there exists $F \in W^{ε}_{p,q}(\mathbb{C}P^n)$ such that $F|_Ω = f$ and $∂F = 0$ in $\mathbb{C}P^n$.

**Proof.** Since $D$ has $C^2$ smooth boundary, there exists a bounded extension operator from $W^{s}_{p,q}(Ω)$ to $W^{s}_{p,q}(\mathbb{C}P^n)$ for all $s ≥ 0$ (cf. e.g. [33]). Let $f \in W^{1,ε}_{p,q}(\mathbb{C}P^n)$ be the extension of $f$ so that $\tilde{f}|_Ω = f$ with

$$∥\tilde{f}∥_{W^{1,ε}_{p,q}(\mathbb{C}P^n)} ≤ C∥f∥_{W^{1,ε}_{p,q}(Ω)}.$$ 

Furthermore, we can choose an extension such that $\overline{∂}\tilde{f} ∈ W^{ε}(D) \cap L^2(D, ϕ_{2ε}).$

One defines $T\tilde{f}$ by $T\tilde{f} = -\star_{2ε} \overline{∂}N_{2ε}(\star_{-2ε} \overline{∂}\tilde{f})$ in $Ω$. As in Theorem 4.2, $T\tilde{f} ∈ L^2(D, ϕ_{2ε})$. But for a $C^2$-smooth domain, we have that $T\tilde{f} ∈ L^2(D, ϕ_{2ε})$ is comparable to $W^{ε}(Ω)$ for $0 ≤ ε < \frac{1}{2}$. This gives that $\overline{∂}T\tilde{f} = \overline{∂}f$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside $Ω$.

Since $0 ≤ ε < \frac{1}{2}$, the extension by 0 outside $Ω$ is a continuous operator from $W^{ε}(Ω)$ to $W^{ε}(\mathbb{C}P^n)$ (cf. e.g. [21]). Thus we have $T\tilde{f} ∈ W^{ε}(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & \text{if } z ∈ \overline{D}, \\ \tilde{f} - T\tilde{f}, & \text{if } z ∈ Ω. \end{cases}$$

Then $F ∈ W^{ε}_{p,q}(\mathbb{C}P^n)$ and $F$ is $\overline{∂}$-closed extension of $f$ to $\mathbb{C}P^n$. □

**Corollary 4.1.** Let $D ⊂ \mathbb{C}P^n$ be a weakly $q$-concave domain, $n ≥ 2$ with $C^2$ smooth boundary. Then $W^{1,ε}_{p,0}(D) \cap \text{ker } \overline{∂} = \{0\}$, $1 ≤ p ≤ n$ and $W^{1,ε}_{0,0}(D) \cap \text{ker } \overline{∂} = \mathbb{C}$.

**Proof.** Using Proposition 4.1 for $q = 0$, we have that any holomorphic $(p,0)$-form on $D$ extends to be a holomorphic $(p,0)$ in $\mathbb{C}P^n$, which are zero (when $p > 0$) or constants (when $p = 0$). □

**Corollary 4.2.** Let $D ⊂ \mathbb{C}P^n$ be a weakly $q$-concave domain, $n ≥ 2$ with $C^2$ smooth boundary. Then, for any $f ∈ W^{1,ε}_{p,q}(D)$, where $0 ≤ p ≤ n$, $1 ≤ q ≤ n - 2$, $p ≠ q$, and $0 ≤ ε < \frac{1}{2}$, such that $\overline{∂}f = 0$ in $D$, there exists $u ∈ W^{1,ε}_{p,q-1}(D)$ such that $\overline{∂}u = f$ in $D$.

**Proof.** If $p ≠ q$, we have that $F = \overline{∂}u$ for some $U ∈ W^{1}_{p,q-1}(\mathbb{C}P^n)$. Let $u = U$ on $D$, we have $u ∈ W^{1}_{p,q-1}(D)$ satisfying $\overline{∂}u = f$ in $D$. □

**Acknowledgements.** The author is grateful to the referee for several helpful remarks and comments.

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