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# THE $\overline{\partial}$-CAUCHY PROBLEM ON WEAKLY $q$-CONVEX DOMAINS IN $\mathbb{C}P^n$

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Abstract. Let $D$ be a weakly $q$-convex domain in the complex projective space $\mathbb{C}P^n$. In this paper, the (weighted) $\overline{\partial}$-Cauchy problem with support conditions in $D$ is studied. Specifically, the modified weight function method is used to study the $L^2$ existence theorem for the $\overline{\partial}$-Neumann problem on $D$. The solutions are used to study function theory on weakly $q$-convex domains via the $\overline{\partial}$-Cauchy problem.

## 1. Introduction and Main Results

The $\overline{\partial}$-problem is one of the important central problems of complex variables. A classical result due to Hörmander tells us that the $\overline{\partial}$-problem is solvable in pseudoconvex domains, and hence, pseudoconvex domains has been widely accepted as the standard domain which we can solve the $\overline{\partial}$-problem. In [16], Ho extend this problem to weakly $q$-convex domains. In fact, Ho is the first person to study the $\overline{\partial}$-problem in $q$-convex domains in $\mathbb{C}^n$. This paper is devoted to studying the $L^2$ $\overline{\partial}$ Cauchy problem and the $\overline{\partial}$-closed extension problem for forms on a weakly $q$-convex domain $D$ in the complex projective space $\mathbb{C}P^n$. These problems were first studied by Kohn and Rossi [20] (see also [12]). They proved the holomorphic extension of smooth $CR$ functions and the $\overline{\partial}$-closed extension of smooth forms from the boundary $bD$ of a strongly pseudoconvex domain to the whole domain $D$. The $L^2$ theory of these problems has been obtained for pseudoconvex domains in $\mathbb{C}^n$ or, more generally, for domains in complex manifolds with strongly plurisubharmonic weight functions (see Chapter 9 in [6] and the references therein). The $L^2$ $\overline{\partial}$ Cauchy problem was considered by Derridj [8,9]. In [30,31] Shaw has obtained a solution to this problem on a pseudoconvex domain with $C^1$ boundary in $\mathbb{C}^n$. Also, in the setting of strictly

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$q$-convex (or $q$-concave) domains, this problem has been studied by Sambou in his thesis (see [29]). In [1], Abdelkader-Saber studied this problem on pseudoconvex manifolds satisfing property $B$. In [26, 27], Saber studied this problem on a weakly $q$-convex domain with $C^1$-smooth boundary and on a $q$-pseudoconvex domain $D$ in $\mathbb{C}^n$, $1 \leqslant q \leqslant n$, with Lipschitz boundary. Recently, Saber [28] studied this result to a $q$-pseudoconvex domain $D$ in a Stein manifold. On a pseudoconvex domain in $\mathbb{C}P^n$, Cao-Shaw-Wang [4] (cf. also [5]) obtained the $L^2$ existence theorem for the $\overline{\partial}$-Neumann operator $N$ and obtained the (weighted) $L^2$ $\overline{\partial}$ Cauchy-problem on such domains. The aim of this paper is to extend this result to the situation in which the boundaries are assumed weakly $q$-convex domain $D$ in $\mathbb{C}P^n$. Moreover, the solutions are used to study function theory on such domains via the $\overline{\partial}$-Cauchy problem.

## 2. Notation and Preliminaries

Let $(x_0, x_1, \ldots, x_n)$ be a (fixed) homogeneous coordinates of $\mathbb{C}P^n$. If $U_0$ is the open set in $\mathbb{C}P^n$ defined by $x_0 \neq 0$ and if $(z_1, z_2, \ldots, z_n)$, where $z_i = x_i/x_0$, is the homogeneous coordinates of $U_0$, we assume that

$$\omega = \frac{\sum_{i=1}^{n} |\, dz_i|^2}{1+\sum_{i=1}^{n} |z_i|^2} - \frac{|\sum_{i=1}^{n} z_i\, d\overline{z}_i|^2}{(1+\sum_{i=1}^{n} |z_i|^2)^2} \quad \text{on } U_0.$$

The Fubini-Study metric of $\mathbb{C}P^n$ determined by $(x_0, x_1, \ldots, x_n)$. This is well-known standard Kähler metric of $\mathbb{C}P^n$.

Let $D$ be a bounded domain in $\mathbb{C}P^n$ and let $C^{\infty}_{p,q}(D)$ be the space of complex-valued differential forms of class $C^{\infty}$ and of type $(p,q)$ on $D$. Denote by $L^2(D)$ the space of square integrable functions on $D$ with respect to the Lebesgue measure in $\mathbb{C}P^n$, $L^2_{p,q}(D)$ the space of $(p,q)$-forms with coefficients in $L^2(D)$ and $L^2_{p,q}(D,\phi)$ the space of $(p,q)$-forms with coefficients in $L^2(\,D)$ with respect to the weighted function $e^{-\phi}$. For $u, v \in L^2_{p,q}(D)$, the inner product $\langle u, v\rangle$ and the norm $\parallel u \parallel$ are denoted by:

$$\langle u, v\rangle = \int_D u \wedge \star\, \overline{v} \quad \text{and} \quad \parallel u \parallel^2 = \langle u, u\rangle,$$

where $\star$ is the Hodge star operator. Let $\operatorname{dist}(z, bD)$ be the Fubini distance from $z \in D$ to the boundary $bD$ and let $\delta$ be a $C^2$ defining function for $D$ normalized by $|d\delta| = 1$ on $bD$ such that

$$\delta = \delta(z) = \begin{cases} -\operatorname{dist}(z, bD), & \text{if } z \in D, \\ \operatorname{dist}(z, bD), & \text{if } z \in \mathbb{C}P^n \backslash D. \end{cases}$$

Let $\phi_t = -t \log|\delta|$, $t \geqslant 0$, for $u, v \in L^2_{p,q}(D, \phi_t)$, the inner product $\langle u, v\rangle_{\phi_t}$ and the norm $\parallel u \parallel_{\phi_t}$ are denoted by:

$$\langle u, v\rangle_{\phi_t} = \langle u, v\rangle_t = \int_D u \wedge \star_{(t)}\, \overline{v},$$
$$\|u\|^2_{\phi_t} = \parallel u \parallel^2_t = \langle u, u\rangle_t,$$

where $\star_{(t)} = \delta^t \star = \star\, \delta^t$. Since $\phi_t$ is bounded on $\overline{D}$, the two norms $\|\cdot\|$ and $\|\cdot\|_t$ are equivalent. Let $\overline{\partial} : \operatorname{dom}\overline{\partial} \subset L^2_{p,q}(D,\phi_t) \to L^2_{p,q+1}(D,\phi_t)$ be the maximal closure of the Cauchy-Riemann operator and $\overline{\partial}^*_\phi$ be its Hilbert space adjoint. Let $\Box_t = \overline{\partial}\,\overline{\partial}^*_t + \overline{\partial}^*_t\overline{\partial}$ be the Laplace-Beltrami operator, where $\overline{\partial}^*_t = \overline{\partial}^*_{\phi_t}$.

Denote by $\nabla$ the Levi-Civita connection of $\mathbb{C}P^n$ with the standard Fubini-Study metric $\omega$. Let $\{e_i\}$ be an orthonormal basis of vector fields. For any two vector fields $f, g$, the curvature operator of the connection $\nabla$ is denoted by

$$\mathcal{R}(f,g) = \nabla_f \nabla_g - \nabla_g \nabla_f - \nabla_{[f,g]}.$$

By setting $\mathcal{R}_{ijkl} = \omega(\mathcal{R}(e_i,e_j)e_k, e_l)$, the Ricci tensor $\mathcal{R}_{ij}$ is denoted by

$$\mathcal{R}_{ij} = \sum_k \varepsilon_k \mathcal{R}_{ikkj},$$

which turns out to be self-adjoint with respect to $\omega$ and the scalar curvature

$$\Theta = \sum_i \mathcal{R}_{ii} = \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}_{jiij} \tag{2.1}$$

as the trace of the Ricci tensor.

**Definition 2.1.** Let $D$ be an open set in an $n$-dimensional complex manifold $X$, let $k$ be an integer with $1 \le k \le n-1$ and put $E = X \backslash D$. The set $D$ is said to be pseudoconvex of order $k$ in $X$ if, for every $b \in E$ and for every coordinate neighborhood $(U, (z_1, \ldots, z_n))$ which contains $b$ as the origin, the set

$$\left\{ (z_1, \ldots, z_n) \in U : \; z_i = 0,\; 1 \le i \le k,\; 0 < \sum_{i=k+1}^{n} |z_i|^2 < t \right\}$$

contains no points of $E$ for some $t > 0$, then there exists $\ell > 0$ such that for each $(z'_1, \ldots, z'_k)$ with $|z'_i| < \ell$, $1 \le i \le k$, the set

$$\left\{ (z_1, \ldots, z_n) \in U : \; z_i = z'_i,\; 1 \le i \le k,\; \sum_{i=k+1}^{n} |z_i|^2 < t \right\}$$

contains at least one point of $E$.

**Definition 2.2.** Let $D$ be an $n$-dimensional complex manifold and let $q$ be an integer, $1 \le q \le n$. By Fujita ([13], Proposition 8) a $C^2$ function $\phi : D \to \mathbb{R}$ is pseudoconvex of order $n-q$, if and only if its Levi form $\partial\overline{\partial}\phi$ has at least $n-q+1$ non negative eigenvalues at each point of $D$.

**Definition 2.3.** Let $D$ be an open subset of an $n$-dimensional complex manifold $X$. $D$ is said to have $C^2$ boundary in $X$ if for all $z \in bD$ there exist an open neighborhood $U$ of $z$ and a $C^2$ function $\delta : U \to \mathbb{R}$, called a defining function of $D$ at $z$ such that $d\delta(z) \neq 0$ and $D \cap U = \{z \in U : \delta(z) < 0\}$. Following Ho [16], $D$ is said to be a weakly $q$-convex ($q \geqslant 1$) if at every point $x_0 \in bD$ we have

$$\sum_{|K|}' \sum_{j,k} \frac{\partial^2 \delta}{\partial z_j \partial \overline{z}_k} u_{jK}\, \overline{u_{kK}} \geqslant 0, \quad \text{for every } (0,q)\text{-form},$$

where

$$u = \sum_{|J|=q} u_J\, d\overline{z}^J \text{ such that } \sum_{j=1}^{n} \frac{\partial \delta}{\partial z_j} u_{jK} = 0, \quad \text{for all } |K| = q-1.$$

Moreover, $D$ is weakly $q$-convex if and only if for any $z \in bD$ the sum of any $q$ eigenvalues $\delta_{i_1}, \dots, \delta_{i_q}$, with distinct subscripts, of the Levi-form at $z$ satisfies $\sum_{j=1}^{q} \delta_{i_j} \geqslant 0$ (cf. [15] and Lemma 4.7 in [34]).

**Definition 2.4.** Let $D$ be a smooth domain in $\mathbb{C}^n$, $D$ is said to be a weakly $q$-concave if $\overline{D}^c$ is weakly $q$-convex.

**Lemma 2.1** ([16])**.** *Let $D$ be a smooth domain in $\mathbb{C}^n$ and $\rho$ be its defining function. The following two conditions are equivalent.*

(1) *$D$ is weakly $q$-convex.*
(2) *For any $z \in bD$ the sum of any $q$ eigenvalues $\rho_{i_1}, \dots, \rho_{i_q}$, with distinct subscripts, of the Levi-form at $z$ satisfies $\sum_{j=1}^{q} \rho_{i_j} \geqslant 0$.*

It follows from Lemma 2.1 that $D$ is weakly $q$-concave if and only if for any $q$ eigenvalues $\rho_{i_1}, \dots, \rho_{i_q}$ of the Levi-form at $z \in bD$ with distinct subscripts we have $\sum_{j=1}^{q} \rho_{i_j} \leqslant 0$.

*Example* 2.1. Let $D$ be an open subset of an $n$-dimensional complex manifold $X$ and suppose that the boundary $bD$ is a real hypersurface of class $C^2$ in $X$, that is, there exist, for each $z \in bD$, a neighborhood $U$ of $z$ and a $C^2$ function $\rho : U \to \mathbb{R}$ such that $d\rho(z) \neq 0$ and $D \cap U = \{z \in U : \rho(z) < 0\}$. Then $D$ is pseudoconvex of order $n-q$ in $X$, if and only if the Levi form $\partial\overline{\partial}\rho$ has at least $n-q$ non-negative eigenvalues on $T'_z(bD)$ for each defining function $\rho$ of $D$ near $z$, where $T'_z(bD)(\subset T_z(bD))$ is the holomorphic tangent space of the real hypersurface $bD$ at $z$ (cf. [10, 35] called such a subset $D$ a $(q-1)$-pseudoconvex open subset with $C^2$ boundary).

**Theorem 2.1** ([23])**.** *Let $D \Subset \mathbb{C}P^n$ be a pseudoconvex domain of order $n-q$, $1 \leq q \leq n$. Let $d(z, bD)$ be the Fubini distance from $z \in D$ to the boundary $bD$. Then the function $-\log d(z, bD)$ is $(q-1)$-pluirsubharmonic in $D$.*

**Lemma 2.2** ([17], Lemma 2.6)**.** *Let $\phi$ be a real valued function of class $C^2$ defined in an $n$-dimensional complex manifold $D$. Then $\phi$ is $(q-1)$-plurisubharmonic, $1 \leq q \leq n$, in $D$ if and only if $\phi$ is weakly $q$-convex in $D$.*

*Remark* 2.1. Pseudoconvex open sets in the original sense are pseudoconvex of order $n-1$.

*Remark* 2.2. The pseudoconvexity of order $n-q$ of an open subset $D$ in $X$ is a local property of the boundary $bD \subset X$ of $D$. More precisely, $D$ is pseudoconvex of order

$n-q$ in $X$ if, for each $p \in bD$, there exists a neighborhood $U \subset X$ of $p$ such that $D \cap U$ is pseudoconvex of order $n-q$ in $U$.

*Remark* 2.3. If an open set $D$ in an $n$-dimensional complex manifold $X$ is weakly $q$-convex, $1 \leq q \leq n$, then $D$ is pseudoconvex of order $n-q$ in $X$. However, the converse is not valid even if $X = \mathbb{C}^n$ (see [10] and [22]). By Fujita [13], an open subset $D$ of $\mathbb{C}^n$ is pseudoconvex of order $n-q$ in $\mathbb{C}^n$, if and only if $D$ has an exhaustion function which is pseudoconvex of order $n-q$ on $D$. Thus, by the approximation theorem of Bungart [3], an open subset $D$ of $X$ is pseudoconvex of order $n-q$ in $X$, if and only if $D$ is locally $q$-complete with corners in $X$ in the sense of Peternell [24].

**Proposition 2.1** (Bochner-Hörmander-Kohn-Morrey formula)**.** *Let $D$ be a compact domain with $C^2$-smooth boundary $bD$ and $\delta(x) = -d(x, bD)$. Suppose that $\Theta$ is the curvature term defined in* (2.1) *with respect to the Fubini-Study metric $\omega$. Then, for any $u \in C^\infty_{p,q}(\overline{D}) \cap \mathrm{dom}\overline{\partial}^*_\phi$ with $1 \leqslant q \leqslant n-1$, and $\phi \in C^2(\overline{D})$, we have*

$$\overline{\partial} u\|^2_\phi + \| \overline{\partial}^*_\phi u\|^2_\phi = \langle \Theta u, \overline{u}\rangle_\phi + \left\| \frac{\partial u_{IJ}}{\partial \overline{z}^k} \right\|^2_\phi + \langle (i\partial\overline{\partial}\phi)u, \overline{u}\rangle_\phi + \int_{bD} ((i\partial\overline{\partial}\delta)u, \overline{u})\, e^{-\phi} ds. \tag{2.2}$$

*This formula is known* (*cf.* [2,7,15,18,19,32,36]) *for some special cases, although it has not been stated in the literature in the form* (2.2)*. If $u$ has compact support in the interior of $D$, the* (2.2) *was proved in* [2]*, Chapter* 8 *of* [7] *and* (2.12) *of* [36]*. The boundary term had been computed in* [14]*, Chapter* 3 *by combining the Morrey-Kohn technique on the boundary with non-trivial weight function. If one combines the results of* [15] *and* [37] *with the interior formulae discussed above, one can prove that* (2.2) *holds for the general case with a weight function $e^{-\phi}$ and the curvature term. Specially, for $\phi = 0$,* (2.2) *was proved in* [32].

**Proposition 2.2.** *For any $(p,q)$-form $u$ of $D \Subset \mathbb{C}P^n$ with $q \geqslant 1$,*

$$(\Theta u, \overline{u}) = q(2n+1)|u|^2, \quad \textit{when } u \textit{ is a } (0,q)\textit{-form},$$
$$(\Theta u, \overline{u}) = 0, \quad \textit{for any } (n,q)\textit{-form } u,$$
$$(\Theta u, \overline{u}) \geq 0, \quad \textit{when } p \geq 1 \textit{ and } u \textit{ is a } (p,q)\textit{-form}.$$

*The statement for $(0,q)$-forms and $(n,q)$-forms was computed in* [32] *and* [36]*. Also, following Lemma* 3.3 *of Henkin-Iordan* [14] *and its proof showed that the curvature operator $\Theta$ acting on $L^2_{p,q}(D)$ is a non-negative operator.*

## 3. The $\overline{\partial}$-Cauchy Problem on Weakly $q$-Convex Domains

This section is devoted to showing the existence of the $\overline{\partial}$-Neumann operator on a weakly $q$-convex domain $D$ in $\mathbb{C}P^n$, $1 \leqslant q \leqslant n$, and by applying these existence to solve the $\overline{\partial}$ problem with support conditions on $D$. The boundary integral in (2.2) is non-negative for $q \geqslant 1$ by the assumption on $D$. Also, by taking $\phi \equiv 0$ in (2.2) and using Proposition 2.2, we find the fundamental estimate

$$\|u\|^2 \leqslant c\left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2\right).$$

This means that $\Box$ has closed range and $\ker \Box = \{0\}$. Thus, one can establish the $L^2$-existence theorem of the $\overline{\partial}$-Neumann operator $N$.

**Theorem 3.1.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-convex domain with $C^2$ smooth boundary. Then, for each $0 \leqslant p \leqslant n$, $1 \leqslant q \leqslant n$, there exists a bounded linear operator $N : L^2_{p,q}(D) \to L^2_{p,q}(D)$ with the following properties:*

(i) $\mathrm{Range} N \subset \mathrm{dom}\,\Box$, $\Box\, N = N\, \Box = Id$ *on* $\mathrm{dom}\Box$*;*

(ii) *for* $f \in L^2_{p,q}(D)$*,*

$$f = \overline{\partial}\,\overline{\partial}^* N f \,\oplus\, \overline{\partial}^* \overline{\partial} N f;$$

(iii) $N\,\overline{\partial} = \overline{\partial}\,N$ *on* $\mathrm{dom}\overline{\partial}$, $1 \leqslant q \leqslant n-1$*;*

(iv) $\overline{\partial}^* N = N\,\overline{\partial}^*$ *on* $\mathrm{dom}\overline{\partial}^*$, $2 \leqslant q \leqslant n$*;*

(v) $N$, $\overline{\partial} N$ *and* $\overline{\partial}^* N$ *are bounded linear operators on* $L^2_{p,q}(D)$.

Using the duality relations pertaining to the $\overline{\partial}$-Neumann problem, one solve the $L^2$ $\overline{\partial}$ Cauchy problem on weakly $q$-convex domains in $\mathbb{C}P^n$, $1 \leqslant q \leqslant n$. This method was first used by Kohn-Rossi [20] for smooth forms on strongly pseudoconvex domains. More precisely, we prove the following $L^2$ Cauchy problem for $\overline{\partial}$ in $\mathbb{C}P^n$:

**Theorem 3.2.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-convex domain, $1 \leqslant q \leqslant n$ with $C^2$ smooth boundary. Then, for $f \in L^2_{p,q}(\mathbb{C}P^n)$, $\mathrm{supp}\, f \subset \overline{D}$, $1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{C}P^n$, there exists $u \in L^2_{p,q-1}(\mathbb{C}P^n)$, $\mathrm{supp}\, u \subset \overline{D}$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$.*

*Proof.* Let $f \in L^2_{p,q}(\mathbb{C}P^n)$, $\mathrm{supp}\, f \subset \overline{D}$, then $f \in L^2_{p,q}(D)$. From Theorem 3.1, $N_{n-p,n-q}$ exists for $n - q \geqslant 1$. Since $N_{n-p,n-q} = \Box^{-1}_{n-p,n-q}$ on $\mathrm{Range}\,\Box_{n-p,n-q}$ and $\mathrm{Range}\, N_{n-p,n-q} \subset \mathrm{dom}\,\Box_{n-p,n-q}$, then $N_{n-p,n-q} \star \overline{f} \in \mathrm{dom}\,\Box_{n-p,n-q} \subset L^2_{n-p,n-q}(D)$, for $q \leqslant n-1$. Thus, we can define $u \in L^2_{p,q-1}(D)$ by

$$u = -\star\ \overline{\overline{\partial}\, N_{n-p,n-q}\ \star \overline{f}}.$$

Thus $\mathrm{supp}\, u \subset \overline{D}$ and $u$ vanishes on $bD$. Now, we extend $u$ to $\mathbb{C}P^n$ by defining $u = 0$ in $\mathbb{C}P^n \setminus D$. It follows from the same arguments of Theorem 9.1.2 in [6] and Theorem 2.2 in [1] that the form $u$ satisfies the equation $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}P^n$. Thus the proof follows. $\Box$

## 4. The Weighted $\overline{\partial}$-Cauchy Problem

In this section, we assume that $D$ is a weakly $q$-convex domain, $1 \leqslant q \leqslant n$, with $C^2$ smooth boundary in $\mathbb{C}P^n$. Also, we will choose $\phi_t = -t \log |\delta|$, $t > 0$ in (2.2), and using Remark 2.3 and by using Proposition 2.2, the inequality (2.2) implies the weighted $L^2$-existence for the $\overline{\partial}$. Also, for $u \in \mathrm{Dom}(\Box_t)$ of degree $q \geqslant 1$ and for $t > 0$, we have

$$\begin{aligned} t\|u\|_t^2 &\leqslant (\|\overline{\partial}u\|_t^2 + \|\overline{\partial}_t^* u\|_t^2) \\ &= \langle \Box_t u, u\rangle_t \\ &\leqslant \|\Box_t f\|_t \|u\|_t, \end{aligned}$$

i.e.,

$$t\|u\|_t \leqslant \|\Box_t u\|_t.$$

Since $\Box_t$ is a linear closed densely defined operator, then, from [15, Theorem 1.1.1], $\mathrm{Range}(\Box_t)$ is closed. Thus, from (1.1.1) in [15] and the fact that $\Box_t$ is self adjoint, we have the Hodge decomposition

$$L^2_{p,q}(D,\phi_t) = \overline{\partial}\,\overline{\partial}_t^*\mathrm{dom}(\Box_t) \oplus \overline{\partial}_t^*\overline{\partial}\,\mathrm{dom}(\Box_t).$$

Since $\Box_t$ is one to one on $\mathrm{dom}(\Box_t)$ from (1.5.3) in [15], then there exists a unique bounded inverse operator

$$N_t : \mathrm{Ran}(\Box_t) \to \mathrm{dom}(\Box_t) \cap (\ker(\Box_t))^{\perp}$$

such that $N_t\Box_t f = f$ on $\mathrm{dom}(\Box_t)$. Therefore, we can establish the existence theorem of the inverse of $\Box_t$ the so called weighted $\overline{\partial}$-Neumann operator $N_t$.

**Theorem 4.1.** *For any* $1 \leqslant q \leqslant n$ *and* $t > 0$, *there exists a bounded linear operator* $N_t : L^2_{p,q}(D,\phi_t) \to L^2_{p,q}(D,\phi_t)$ *satisfies the following properties:*

(i) $\mathrm{Range}\,(N_t) \subset \mathrm{dom}(\Box_t)$, $N_t\Box_t = I$ *on* $\mathrm{dom}(\Box_t)$*;*
(ii) *for* $f \in L^2_{p,q}(D,\phi_t)$, *we have* $u = \overline{\partial}\,\overline{\partial}_t^* N_t f \,\oplus\, \overline{\partial}_t^*\overline{\partial} N_t f$*;*
(iii) $\overline{\partial}N_t = N_t\overline{\partial}$, $1 \leqslant q \leqslant n-1$*;*
(iv) $\overline{\partial}^* N_t = N_t\,\overline{\partial}^*$, $2 \leqslant q \leqslant n$*;*
(v) *for all* $f \in L^2_{p,q}(D,\phi_t)$, *we have the estimates*

$$\begin{aligned} t\|N_t f\|_t &\leqslant \|f\|_t, \\ \sqrt{t}\|\overline{\partial}N_t f\|_t + \sqrt{t}\|\overline{\partial}_t^* N_t f\|_t &\leqslant \|f\|_t; \end{aligned}$$

(vi) *if* $\overline{\partial}f = 0$, *then* $u_t = \overline{\partial}_t^* N_t f$ *solves the equation* $\overline{\partial}u_t = f$.

**Theorem 4.2.** *For* $f \in L^2_{p,q}(D,\phi_t)$, $1 \leqslant q \leqslant n-1$, $\mathrm{supp}\, f \subset \overline{D}$, *satisfying* $\overline{\partial}f = 0$ *in the distribution sense in* $\mathbb{C}P^n$, *there exists* $u \in L^2_{p,q-1}(D,\phi_t)$, $\mathrm{supp}\, u \subset \overline{D}$ *such that* $\overline{\partial}u = f$ *in the distribution sense in* $\mathbb{C}P^n$.

*Proof.* Following Theorem 4.1, $N_t$ exists for forms in $L^2_{n-p,n-q}(D,\phi_t)$. Thus, one can defines $u_t \in L^2_{p,q-1}(D,\phi_t)$ by

$$u_{(t)} = -\star_{(t)}\ \overline{\overline{\partial}\, N_{n-p,n-q}\ \star_{(-t)}\ \overline{f}}. \tag{4.1}$$

Thus $\mathrm{supp}\, u_t \subset \overline{D}$ and $u_t$ vanishes on $bD$. Now, we extend $u_t$ to $\mathbb{C}P^n$ by defining $u_t = 0$ in $\mathbb{C}P^n \setminus D$. We want to prove that the extended form $u_t$ satisfies the equation

$\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. For $\eta \in L^2_{n-p,n-q-1}(D, -\phi_t) \cap \operatorname{dom} \overline{\partial}$, we have

$$\begin{aligned}\langle \overline{\partial}\, \eta, \star_{(t)} f\rangle_D &= \int_D \overline{\partial}\, \eta \wedge \star_{(-t)}(\star_{(t)} f)\\ &= \int_D \overline{\partial}\, \eta \wedge \star_{(-t)} \star_{(t)}\ f\\ &= (-1)^{p+q} \int_D \overline{\partial}\, \eta \wedge\ f\\ &= (-1)^{p+q} \langle f, \star_{(-t)} \overline{\partial}\, \eta\rangle_D\\ &= (-1)^{p+q} \langle f, \star_{(-t)} \overline{\partial}\, \eta\rangle_{\mathbb{C}P^n},\end{aligned}$$

because supp $f \subset \overline{D}$. Since $\vartheta|_D = \overline{\partial}^*|_D$, when $\vartheta$ acts in the distribution sense (see [15]), then we obtain

$$\begin{aligned}\langle \overline{\partial} \eta, \star_{(t)} f\rangle_D &= \langle f, \vartheta\ \star_{(-t)}\ \eta\rangle_{\mathbb{C}P^n}\\ &= \langle \overline{\partial}\, f, \star_{(-t)} \eta\rangle_{\mathbb{C}P^n}\\ &= 0.\end{aligned}$$

It follows that $\overline{\partial}^*_t(\star_{(t)} f) = 0$ on $D$. Using Theorem 4.1 (iv), we have

$$\overline{\partial}^*_t N_t(\star_{(t)} f) = N_t \overline{\partial}^*_t(\star_{(t)} f) = 0. \tag{4.2}$$

Thus, from (4.1) and (4.2), one obtains

$$\begin{aligned}\overline{\partial} u_t &= -\overline{\partial\ \star_{-t}\ \overline{\partial} N_{n-p,n-q}\ \ \star_t \overline{f}}\\ &= (-1)^{p+q+1} \overline{\star \star\ \partial\ \star\ \ \overline{\partial} N_{n-p,n-q}\ \ \star \overline{f}}\\ &= (-1)^{p+q} \overline{\star \overline{\partial}^* \overline{\partial} N_{n-p,n-q} \star \overline{f}}\\ &= (-1)^{p+q} \overline{\star\, (\overline{\partial}^* \overline{\partial} + \overline{\partial}\, \overline{\partial}^*)\, N_{n-p,n-q} \star \overline{f}}\\ &= (-1)^{p+q} \overline{\star \star \overline{f}}\\ &= f,\end{aligned}$$

in the distribution sense in $D$. Since $u = 0$ in $\mathbb{C}P^n \backslash D$, then for $u \in L^2_{p,q}(\mathbb{C}P^n) \cap \operatorname{dom} \overline{\partial}^*$, one obtains

$$\begin{aligned}< u, \overline{\partial}^* u >_{\mathbb{C}P^n} &=< u, \overline{\partial}^* u >_D\\ &=< \star \overline{\partial}^* u, \star_{(-t)} u >_{(t)D}\\ &= (-1)^{p+q} < \overline{\partial} \star u, \star_{(-t)} u >_{(t)D}\\ &= (-1)^{p+q} < \star u, \overline{\partial}^*\ \star_{(-t)}\ u >_{(t)D}\\ &=< \star u, \star_{(-t)} \overline{\partial} u >_{(t)D}\\ &=< f, u >_D\\ &=< f, u >_{\mathbb{C}P^n},\end{aligned}$$

where the third equality holds since $\star u = (-1)^{q+1}\, \overline{\partial} N_{n-p,n-q} \star\ f \in \operatorname{dom} \overline{\partial}^*$. Thus $\overline{\partial} u_t = f$ in the distribution sense in $\mathbb{C}P^n$. □

As in [5], we prove the following results.

**Proposition 4.1.** *Let $D$ be the same as in Theorem 3.1. Put $\Omega = \mathbb{C}P^n \backslash \overline{D}$. Then, for any $f \in W^{1+\varepsilon}_{p,q}(\Omega)$, $\overline{\partial} f = 0$, $0 \le \varepsilon < \frac{1}{2}$, there exists $F \in W^{\varepsilon}_{p,q}(\mathbb{C}P^n)$ such that $F|_\Omega = f$ and $\overline{\partial} F = 0$ in $\mathbb{C}P^n$.*

*Proof.* Since $D$ has $C^2$ smooth boundary, there exists a bounded extension operator from $W^s_{p,q}(\Omega)$ to $W^s_{p,q}(\mathbb{C}P^n)$ for all $s \geqslant 0$ (cf. e.g. [33]). Let $\tilde{f} \in W^{1+\varepsilon}_{p,q}(\mathbb{C}P^n)$ be the extension of $f$ so that $\tilde{f}|_\Omega = f$ with

$$\|\tilde{f}\|_{W^{1+\varepsilon}(\mathbb{C}P^n)} \leqslant C \|f\|_{W^{1+\varepsilon}(\Omega)}.$$

Furthermore, we can choose an extension such that $\overline{\partial}\tilde{f} \in W^{\varepsilon}(D) \cap L^2(D, \phi_{2\varepsilon})$.

One defines $T\tilde{f}$ by $T\tilde{f} = - \star_{2\varepsilon} \overline{\partial} N_{2\varepsilon}(\star_{-2\varepsilon} \overline{\partial}\tilde{f})$ in $\Omega$. As in Theorem 4.2, $T\tilde{f} \in L^2(D, \phi_{2\varepsilon})$. But for a $C^2$-smooth domain, we have that $T\tilde{f} \in L^2(D, \phi_{2\varepsilon})$ is comparable to $W^{\varepsilon}(\Omega)$ for $0 \leqslant \varepsilon < \frac{1}{2}$. This gives that $T\tilde{f} \in W^{\varepsilon}_{p,q}(\Omega)$ and $T\tilde{f}$ satisfies $\overline{\partial} T\tilde{f} = \overline{\partial}\tilde{f}$ in $\mathbb{C}P^n$ in the distribution sense if we extend $T\tilde{f}$ to be zero outside $\Omega$.

Since $0 \leqslant \varepsilon < \frac{1}{2}$, the extension by 0 outside $\Omega$ is a continuous operator from $W^{\varepsilon}(\Omega)$ to $W^{\varepsilon}(\mathbb{C}P^n)$ (cf. e.g. [21]). Thus we have $T\tilde{f} \in W^{\varepsilon}(\mathbb{C}P^n)$.

Define

$$F = \begin{cases} f, & \text{if } z \in \overline{D}, \\ \tilde{f} - T\tilde{f}, & \text{if } z \in \Omega. \end{cases}$$

Then $F \in W^{\varepsilon}_{p,q}(\mathbb{C}P^n)$ and $F$ is $\overline{\partial}$-closed extension of $f$ to $\mathbb{C}P^n$. □

**Corollary 4.1.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-concave domain, $n \geqslant 2$ with $C^2$ smooth boundary. Then $W^{1+\varepsilon}_{p,0}(D) \cap \ker \overline{\partial} = \{0\}$, $1 \le p \le n$ and $W^{1+\varepsilon}_{0,0}(D) \cap \ker \overline{\partial} = \mathbb{C}$.*

*Proof.* Using Proposition 4.1 for $q = 0$, we have that any holomorphic $(p, 0)$-form on $D$ extends to be a holomorphic $(p, 0)$ in $\mathbb{C}P^n$, which are zero (when $p > 0$) or constants (when $p = 0$). □

**Corollary 4.2.** *Let $D \Subset \mathbb{C}P^n$ be a weakly $q$-concave domain, $n \geqslant 2$ with $C^2$ smooth boundary. Then, for any $f \in W^{1+\varepsilon}_{p,q}(D)$, where $0 \le p \le n$, $1 \leqslant q \leqslant n-2$, $p \neq q$, and $0 \leqslant \varepsilon < \frac{1}{2}$, such that $\overline{\partial} f = 0$ in $D$, there exists $u \in W^{1+\varepsilon}_{p,q-1}(D)$ such that $\overline{\partial} u = f$ in $D$.*

*Proof.* If $p \neq q$, we have that $F = \overline{\partial} u$ for some $U \in W^1_{p,q-1}(\mathbb{C}P^n)$. Let $u = U$ on $D$, we have $u \in W^1_{p,q-1}(D)$ satisfying $\overline{\partial} u = f$ in $D$. □

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## References


[1] O. Abdelkader and S. Saber, *Solution to $\overline{\partial}$-equations with exact support on pseudoconvex manifolds*, Int. J. Geom. Methods Mod. Phys. **4** (2007), 339–348.

[2] A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace-Beltrami equation on complex manifolds*, Publ. Math. Inst. Hautes Etudes Sci. (1965), 81–150.

[3] L. Bungart, *Piecewise smooth approximations to q-plurisubharmonic functions*, Pacific J. Math. **142** (1990), 227–244.

[4] J. Cao, M. C.-Shaw and L. Wang, *Estimates for the $\overline{\partial}$-Neumann problem and nonexistence of $C^2$ Levi-flat hypersurfaces in $X$*, Math. Z **248** (2004), 183–221.

[5] J. Cao and M.-C. Shaw, *The $\overline{\partial}$-Cauchy problem and nonexistence of Lipschitz Levi-flat hypersurfaces in $\mathbb{C}P^n$ with $n \geq 3$*, Math. Z. **256** (2007), 175–192.

[6] S.-C. Chen and M.-C. Shaw, *Partial Differential Equations in Several Complex Variables*, AMS/IP Stud. Adv. Math. **19**, AMS, Providence, Rhodes Island, 2001.

[7] J.-P. Demailly, *Complex analytic and differential geometry*, American Math. Society (to apperar).

[8] M. Derridj, *Regularité pour $\overline{\partial}$ dans quelques domaines faiblement pseudo-convexes*, J. Differential Geom. **13** (1978), 559–576.

[9] M. Derridj, *Inégatés de Carleman et extension locale des fonctions holomorphes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **15** (1981), 645–669.

[10] K. Diederich and J. E. Fornaess, *Smoothing q-convex functions and vanishing theorems*, Invent. Math. **82** (1985), 291–305.

[11] M. G. Eastwood and G. V. Suria, *Cohomologically complete and pseudoconvex domains*, Comment. Math. Helv. **55** (1980), 413–426.

[12] G. B. Folland and J. J. Kohn, *The Neumann problem for the Cauchy-Riemann complex*, Princeton Math. Ser. **75** (1972).

[13] O. Fujita, *Domaines pseudoconvexes d'ordre général et fonctions pseudoconvexes d'ordre général*, Kyoto J. Math. **30** (1990), 637–649.

[14] G. M. Henkin and A. Iordan, *Regularity of $\overline{\partial}$ on pseudococave compacts and applications*, Asian J. Math. **4** (2000), 855–884.

[15] L. Hörmander, *$L^2$-estimates and existence theorems for the $\overline{\partial}$-operator*, Acta Math. **113** (1965), 89–152.

[16] L. Ho, *$\overline{\partial}$-problem on weakly q-convex domains*, Math. Ann. **290** (1991), 3–18.

[17] L. R. Hunt and J. J. Murray, *q-plurisubharmonic functions and a generalized Dirichlet problem*, Michigan Math. J. **25** (1978), 299–316.

[18] J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds I*, Ann. of Math. **78** (1963), 112–148.

[19] J. J. Kohn, *Harmonic integrals on strongly pseudoconvex manifolds II*, Ann. of Math. **79** (1964), 450–472.

[20] J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. **81** (1965), 451–472.

[21] J.-L. Lions and E. Magenes, *Non-Homogeneous Boundary Value Problems and Applications*, Springer-Verlag, Berlin Heidelberg, New York, 1972.

[22] K. Matsumoto, *Pseudoconvex domains of general order in Stein manifolds*, Memoirs of the Faculty of Science, Kyushu University, Series A, Mathematics **43**(2) (1989), 67–76.

[23] K. Matsumoto, *Pseudoconvex domains of general order and q-convex domains in the complex projective space*, Kyoto J. Math. **33** (1993), 685–695.

[24] M. Peternell, *Continuous q-convex exhaustion functions*, Invent. Math. **85** (1986), 249–262

[25] R. M. Range, *Holomorphic Functions and Integral Representations in Several Complex Variables*, Springer, Berlin, Heidelberg, New York, 1986.

[26] S. Saber, *Solution to $\overline{\partial}$ problem with exact support and regularity for the $\overline{\partial}$-Neumann operator on weakly q-convex domains*, Int. J. Geom. Methods Mod. Phys. **7**(1) (2010), 135–142.

[27] S. Saber, *The $\overline{\partial}$ problem on q-pseudoconvex domains with applications*, Math. Slovaca **63**(3) (2013), 521–530.

[28] S. Saber, *The $L^2$ $\overline{\partial}$-Cauchy problem on weakly q-pseudoconvex domains in Stein manifolds*, Czechoslovak Math. J. **65**(3) (2015), 739–745.

[29] S. Sambou, *Résolution du $\overline{\partial}$ pour les courants prolongeables définis dans un anneau*, Annales de la Faculté des sciences de Toulouse: Mathématiques **11**(1) (2002), 105–129.
[30] M. C. Shaw, *Local existence theorems with estimates for $\overline{\partial}_b$ on weakly pseudo-convex boundaries*, Math. Ann. **294** (1992), 677–700.
[31] M. C. Shaw, *The closed range property for $\overline{\partial}$ on domains with pseudoconcave boundary*, Complex Analysis Trends in Mathematics (2010), 307–320.
[32] Y. T. Siu, *Complex-analyticity of harmonic maps, vanishing and Lefschetz theorems*, J. Differential Geom. **17** (1982), 55–138.
[33] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Math. Series **30**, Princeton University Press, Princeton, New Jersey, 1970.
[34] E. J. Straube, *Lectures on the $L^2$-Sobolev Theory of the $\overline{\partial}$-Neumann Problem*, ESI Lectures in Mathematics and Physics, Freiburg, Germany, 2010.
[35] G. V. Suria, *q-pseudoconvex and q-complete domains*, Compos. Math. **53** (1984), 105–111.
[36] H. H. Wu, *The Bochner Technique in Differential Geometry*, Harwood Academic, New York, 1988.
[37] G. Zampier, *Complex Analysis and CR Geometry*, AMS **43**, Providence, Rhode Island, 2008.

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