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51

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Contents

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SANDWICH THEOREMS FOR MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH DIFFERENTIAL OPERATOR

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Abstract. The purpose of this paper is to derive subordination and superordination results involving differential operator for multivalent analytic functions in the open unit disk. These results are applied to obtain sandwich results. Our results extend corresponding previously known results.

1. Introduction and Preliminaries

Let $H = H(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, p]$ be the subclass of H consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots, \quad a \in \mathbb{C}, p \in \mathbb{N} = \{1, 2, \ldots\}.$$

Also, let A_p be the subclass of H consisting of functions of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N}.$$

Let $f, g \in H$. The function f is said to be subordinate to g, or g is said to be superordinate to f, if there exists a Schwarz function w analytic in U with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. It is well known that, if the function g is univalent in U, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Key words and phrases. Multivalent functions, differential subordination, differential superoordination, dominant, subordinant, differential operator.

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Let $\xi, h \in H$ and $\psi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If ξ and $\psi(\xi(z), z\xi'(z), z^2\xi''(z); z)$ are univalent functions in U and if ξ satisfies the second-order differential superordination

$$h(z) \prec \psi\left(\xi(z), z\xi'(z), z^2\xi''(z); z\right), \tag{1.1}$$

then ξ is called a solution of the differential superordination (1.1). (If f is subordinate to g, then g is superordinate to f.) An analytic function q is called a subordinant of (1.1), if $q \prec \xi$ for all ξ satisfying (1.1). An univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all the subordinants q of (1.1) is called the best subordinant.

Recently, Miller and Mocanu [11] obtained conditions on the functions h, q and ψ for which the following implication holds:

$$h(z) \prec \psi\left(\xi(z), z\xi'(z), z^2\xi''(z); z\right) \Rightarrow q(z) \prec \xi(z).$$

Using the results due to Miller and Mocanu [11], Bulboacă [4] considered certain classes of first order differential superordination as well as superordination-preserving integral operators [5]. Ali et al. [1] have used the results of Bulboacă [4] to obtain sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functions in U with $q_1(0) = q_2(0) = 1$.

Very recently, Shanmugam et al. [17–19] and Goyal et al. [9] have obtained sandwich results for certain classes of analytic functions.

For $m, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda_1 \geq \lambda_2 \geq 0$ and $f \in A_p$, the differential operator $D^{m,n}_{\lambda_1,\lambda_2,p}$ (see [8]) is defined by

$$D^{m,n}_{\lambda_1,\lambda_2,p} f(z) = z^p + \sum_{k=p+1}^{\infty} \left[\frac{p + (\lambda_1+\lambda_2)(k-p)}{p + \lambda_2(k-p)}\right]^m C(k,n)\, a_k z^k, \tag{1.2}$$

where $C(k,n) = \frac{\Gamma(k+n)}{\Gamma(k)}$.

It follows from (1.2) that

$$\begin{aligned} \lambda_1 z \left(D^{m,n}_{\lambda_1,\lambda_2,p} f(z)\right)' &= (p + \lambda_2(k-p))\, D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z) \\ &\quad - (p + \lambda_2(k-p) - p\lambda_1)\, D^{m,n}_{\lambda_1,\lambda_2,p} f(z), \quad \lambda_1 > 0. \end{aligned} \tag{1.3}$$

Special cases of this operator includes the Ruscheweyh derivative operator [15], the Sălăgean derivative operator [16], the generalized Sălăgean operator [2], the generalized Ruscheweyh derivative operator [3], the generalized Al-Shaqsi and Darus derivative operator [6].

The main object of the present paper is to derive the several subordination and superordination results for multivalent analytic functions involving differential operator $D^{m,n}_{\lambda_1,\lambda_2,p}$.

In order to prove our results, we make use of the following known results.

Definition 1.1 ([10]). Denote by Q the set of all functions f that are analytic and injective on $\overline{U}\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(f)$.

Lemma 1.1 ([10]). *Let q be univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in U;*
(2) $\mathrm{Re}\left(\frac{zh'(z)}{Q(z)}\right) > 0$ *for $z \in U$.*

If ξ is analytic in U, with $\xi(0) = q(0)$, $\xi(U) \subset D$ and

$$\theta(\xi(z)) + z\xi'(z)\phi(\xi(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.4}$$

then $\xi \prec q$ and q is the best dominant of (1.4).

Lemma 1.2 ([11]). *Let q be a convex univalent function in U and let $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}\backslash\{0\}$ with*

$$\mathrm{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

If ξ is analytic in U and

$$\alpha\xi(z) + \beta z\xi'(z) \leq \alpha q(z) + \beta zq'(z), \tag{1.5}$$

then $\xi \prec q$ and q is the best dominant of (1.5).

Lemma 1.3 ([11]). *Let q be convex univalent in U and let $\beta \in \mathbb{C}$. Further assume that* $\mathrm{Re}(\beta) > 0$. *If $\xi \in H[q(0), 1] \cap Q$ and $\xi(z) + \beta z\xi'(z)$ is univalent in U, then*

$$q(z) + \beta zq'(z) \prec \xi(z) + \beta z\xi'(z), \tag{1.6}$$

which implies that $q \prec \xi$ and q is the best subordinant of (1.6).

Lemma 1.4 ([4]). *Let q be convex univalent in the unit disk U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(1) $\mathrm{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))}\right) > 0$ *for $z \in U$;*
(2) *$Q(z) = zq'(z\phi(q(z)))$ is starlike univalent in U.*

If $\xi \in H[q(0), 1] \cap Q$, with $\xi(U) \subset D$, $\phi(\xi(z)) + z\xi'(z)\phi(\xi(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(\xi(z)) + z\xi'(z)\phi(\xi(z)), \tag{1.7}$$

then $q \prec \xi$ and q is the best subordinant of (1.7).

2. Main Results

Theorem 2.1. *Let q be convex univalent in U with $q(0) = 1$, $\sigma \in \mathbb{C}\backslash\{0\}$, $\gamma > 0$ and suppose that q satisfies*

$$\operatorname{Re}\left(1 + \frac{zq''(z)}{q'(z)}\right) > \max\left\{0, -\operatorname{Re}\left(\frac{p\gamma}{\sigma}\right)\right\}. \tag{2.1}$$

If $f \in A_p$ satisfies the subordination

$$\begin{aligned}&\left(1 - \frac{\sigma(p + \lambda_2(k-p))}{\lambda_1 p}\right)\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^{\gamma} \\ &+ \frac{\sigma(p + \lambda_2(k-p))}{\lambda_1 p}\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^{\gamma}\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right) \prec q(z) + \frac{\sigma}{p\gamma} zq'(z),\end{aligned} \tag{2.2}$$

then

$$\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^{\gamma} \prec q(z) \tag{2.3}$$

and q is the best dominant of (2.2).

Proof. Define the function ξ by

$$\xi(z) = \left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^{\gamma}, \quad z \in U. \tag{2.4}$$

Differentiating (2.4) logarithmically with respect to z, we get

$$\frac{z\xi'(z)}{\xi(z)} = \gamma\left(\frac{z\left(D^{m,n}_{\lambda_1,\lambda_2,p} f(z)\right)'}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)} - p\right).$$

Now, in view of (1.3), we obtain the following subordination

$$\frac{z\xi'(z)}{\xi(z)} = \frac{\gamma(p + \lambda_2(k-p))}{\lambda_1}\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)} - 1\right).$$

Therefore,

$$\frac{z\xi'(z)}{p\gamma} = \frac{(p + \lambda_2(k-p))}{\lambda_1 p}\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^{\gamma}\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)} - 1\right).$$

The subordination (2.2) from the hypothesis becomes

$$\xi(z) + \frac{\sigma}{p\gamma} z\xi'(z) \prec q(z) + \frac{\sigma}{p\gamma} zq'(z).$$

Hence, an application of Lemma 1.2 with $\alpha = 1$ and $\beta = \frac{\sigma}{p\gamma}$, we obtain (2.3). ∎

Theorem 2.2. *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $\gamma > 0$, $\delta \in \mathbb{C}\backslash\{0\}$ and q be convex univalent in U with $q(0) = 1$, $q(z) \neq 0$ $(z \in U)$ and assume that q satisfies*

$$\operatorname{Re}\left(1 + \frac{\eta_2}{\delta} q(z) + \frac{2\eta_3}{\delta} q^2(z) + \frac{3\eta_4}{\delta} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0. \tag{2.5}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f \in A_p$ satisfies

$$\Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z) \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)}, \tag{2.6}$$

where

$$\begin{aligned} &\Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z) = \Omega_1(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, m, n, \lambda_1, \lambda_2, p; z) \\ &= \eta_1 + \eta_2 \left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{\gamma} + \eta_3 \left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{2\gamma} + \eta_4 \left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{3\gamma} \\ &\quad + \frac{\gamma\delta(p + \lambda_2(k-p))}{\lambda_1} \left(\frac{D^{m+2,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)} - \frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right), \end{aligned} \tag{2.7}$$

then

$$\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{\gamma} \prec q(z)$$

and q is the best dominant of (2.6).

Proof. Define the function ξ by

$$\xi(z) = \left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{\gamma}, \quad z \in U. \tag{2.8}$$

By a straightforward computation and using (1.3), we have

$$\eta_1 + \eta_2 \xi(z) + \eta_3 \xi^2(z) + \eta_4 \xi^3(z) + \delta \frac{z\xi'(z)}{\xi(z)} = \Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z), \tag{2.9}$$

where $\Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z)$ is given by (2.7).

From (2.6) and (2.9), we obtain

$$\eta_1 + \eta_2 \xi(z) + \eta_3 \xi^2(z) + \eta_4 \xi^3(z) + \delta \frac{z\xi'(z)}{\xi(z)} \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + \delta \frac{zq'(z)}{q(z)}.$$

By setting $\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3$ and $\phi(w) = \frac{\delta}{w}$, $w \neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z) = zq'(z)\,\phi(q(z)) = \delta \frac{zq'(z)}{q(z)}$$

and

$$h(z)=\theta(q(z))+Q(z)=\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\operatorname{Re}\left(\frac{zh'(z)}{Q(z)}\right)=\operatorname{Re}\left(1+\frac{\eta_2}{\delta}q(z)+\frac{2\eta_3}{\delta}q^2(z)+\frac{3\eta_4}{\delta}q^3(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0.$$

Thus, by Lemma 1.1, we get $\xi(z)\prec q(z)$. By using (2.8), we obtain the desired result. □

Theorem 2.3. *Let $\eta_i\in\mathbb{C}$, $i=1,2,3,4$, $\gamma>0$, $\delta\in\mathbb{C}\backslash\{0\}$ and q be convex univalent in U with $q(0)=1$, $q(z)\neq 0$, $z\in U$, and assume that q satisfies* (2.5)*. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f\in A_p$ satisfies*

$$\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)\prec\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}, \tag{2.10}$$

where

$$\begin{aligned}\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)&=\Omega_2(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)\\ =\eta_1+\eta_2\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma}&+\eta_3\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{2\gamma}+\eta_4\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{3\gamma}\\ +\frac{\gamma\delta(p+\lambda_2(k-p))}{\lambda_1}&\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}-\frac{D^{m+2,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right),\end{aligned} \tag{2.11}$$

then

$$\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma}\prec q(z)$$

and q is the best dominant of (2.10)*.*

Proof. The proof is similar to that of Theorem 2.2. □

Theorem 2.4. *Let $\eta_i\in\mathbb{C}$, $i=1,2,3,4$, $\delta\in\mathbb{C}\backslash\{0\}$ and q be convex univalent in U with $q(0)=1$, $q(z)\neq 0$, $z\in U$, and assume that q satisfies* (2.5)*. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. If $f\in A_p$ satisfies*

$$\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)\prec\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}, \tag{2.12}$$

where

$$\begin{aligned}
(2.13)\qquad & \Omega_3\left(\eta_i\right)_1^4\left(\delta, m, n, \lambda_1, \lambda_2, p; z\right)=\Omega_3\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, \delta, m, n, \lambda_1, \lambda_2, p; z\right) \\
= & \eta_1+\eta_2 \frac{\left(D_{\lambda_1, \lambda_2, p}^{m, n} f(z)\right)^2}{z^p D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)}+\eta_3 \frac{\left(D_{\lambda_1, \lambda_2, p}^{m, n} f(z)\right)^4}{z^{2p}\left(D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)\right)^2}+\eta_4 \frac{\left(D_{\lambda_1, \lambda_2, p}^{m, n} f(z)\right)^6}{z^{3p}\left(D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)\right)^3} \\
& +\frac{\delta\left(p+\lambda_2(k-p)\right)}{\lambda_1}\left(\frac{2 D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)}{D_{\lambda_1, \lambda_2, p}^{m, n} f(z)}-\frac{D_{\lambda_1, \lambda_2, p}^{m+2, n} f(z)}{D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)}-1\right),
\end{aligned}$$

then

$$\frac{\left(D_{\lambda_1, \lambda_2, p}^{m, n} f(z)\right)^2}{z^p D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)} \prec q(z)$$

and q is the best dominant of (2.12)*.*

Proof. Define the function ξ by

$$\xi(z)=\frac{\left(D_{\lambda_1, \lambda_2, p}^{m, n} f(z)\right)^2}{z^p D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)}, \quad z \in U. \tag{2.14}$$

By a straightforward computation and using (1.3), we have

$$\eta_1+\eta_2 \xi(z)+\eta_3 \xi^2(z)+\eta_4 \xi^3(z)+\delta \frac{z \xi^{\prime}(z)}{\xi(z)}=\Omega_3\left(\eta_i\right)_1^4\left(\delta, m, n, \lambda_1, \lambda_2, p; z\right), \tag{2.15}$$

where $\Omega_3\left(\eta_i\right)_1^4\left(\delta, m, n, \lambda_1, \lambda_2, p; z\right)$ is given by (2.13). From (2.12) and (2.15), we obtain

$$\eta_1+\eta_2 \xi(z)+\eta_3 \xi^2(z)+\eta_4 \xi^3(z)+\delta \frac{z \xi^{\prime}(z)}{\xi(z)} \prec \eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta \frac{z q^{\prime}(z)}{q(z)}.$$

The remaining part of Theorem 2.4 is similar to that of Theorem 2.2 and hence we omit it. ☐

Theorem 2.5. *Let q be convex univalent in U with $q(0)=1$, $\gamma>0$ and* $\operatorname{Re}(\sigma)>0$*. Let $f \in A_p$ satisfying*

$$\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, n} f(z)}{z^p}\right)^\gamma \in H[q(0), 1] \cap Q$$

and

$$\begin{aligned}
& \left(1-\frac{\sigma\left(p+\lambda_2(k-p)\right)}{\lambda_1 p}\right)\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, n} f(z)}{z^p}\right)^\gamma \\
& +\frac{\sigma\left(p+\lambda_2(k-p)\right)}{\lambda_1 p}\left(\frac{D_{\lambda_1, \lambda_2, p}^{m, n} f(z)}{z^p}\right)^\gamma\left(\frac{D_{\lambda_1, \lambda_2, p}^{m+1, n} f(z)}{D_{\lambda_1, \lambda_2, p}^{m, n} f(z)}\right)
\end{aligned}$$

be univalent in U. If

$$q(z)+\frac{\sigma}{p\gamma}zq'(z)\prec\left(1-\frac{\sigma(p+\lambda_2(k-p))}{\lambda_1 p}\right)\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\gamma}+\frac{\sigma(p+\lambda_2(k-p))}{\lambda_1 p}\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\gamma}\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}\right), \tag{2.16}$$

then

$$q(z)\prec\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\gamma} \tag{2.17}$$

and q is the best subordinant of (2.16)*.*

Proof. Define the function ξ by

$$\xi(z)=\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\gamma}. \tag{2.18}$$

Differentiating (2.18) logarithmically with respect to z, we get

$$\frac{z\xi'(z)}{\xi(z)}=\gamma\left(\frac{z\left(D^{m,n}_{\lambda_1,\lambda_2,p}f(z)\right)'}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}-p\right). \tag{2.19}$$

After some computations and using (1.3), from (2.19), we have

$$\left(1-\frac{\sigma(p+\lambda_2(k-p))}{\lambda_1 p}\right)\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\gamma}+\frac{\sigma(p+\lambda_2(k-p))}{\lambda_1 p}\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{z^p}\right)^{\gamma}\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}\right)=\xi(z)+\frac{\sigma}{p\gamma}z\xi'(z). \tag{2.20}$$

From (2.16) and (2.20), we get

$$q(z)+\frac{\sigma}{p\gamma}zq'(z)\prec\xi(z)+\frac{\sigma}{p\gamma}z\xi'(z).$$

Hence, an application of Lemma 1.3 with $\alpha=1$ and $\beta=\frac{\sigma}{p\gamma}$, we obtain (2.17). □

Theorem 2.6. *Let $\eta_i\in\mathbb{C}$, $i=1,2,3,4$, $\gamma>0$, $\delta\in\mathbb{C}\backslash\{0\}$ and q be convex univalent in U with $q(0)=1$, $q(z)\neq 0$, $z\in U$ and assume that q satisfies*

$$\operatorname{Re}\left(\frac{\eta_2}{\delta}q(z)+\frac{2\eta_3}{\delta}q^2(z)+\frac{3\eta_4}{\delta}q^3(z)\right)>0. \tag{2.21}$$

Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $f\in A_p$ satisfying

$$\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma}\in H[q(0),1]\cap Q$$

and $\Omega_1(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ *be univalent in* U, *where* $\Omega_1(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ *is given by* (2.7). *If*

$$\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}\prec\Omega_1(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z), \tag{2.22}$$

then

$$q(z)\prec\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma}$$

and q *is the best subordinant of* (2.22).

Proof. Define the function ξ by

$$\xi(z)=\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma},\quad z\in U. \tag{2.23}$$

By a straightforward computation, we have

$$\Omega_1(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)=\eta_1+\eta_2\xi(z)+\eta_3\xi^2(z)+\eta_4\xi^3(z)+\delta\frac{z\xi'(z)}{\xi(z)}, \tag{2.24}$$

where $\Omega_1(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ is given by (2.7).

From (2.22) and (2.24), we obtain

$$\begin{aligned}&\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}\\ \prec&\eta_1+\eta_2\xi(z)+\eta_3\xi^2(z)+\eta_4\xi^3(z)+\delta\frac{z\xi'(z)}{\xi(z)}.\end{aligned}$$

By setting $\theta(w)=\eta_1+\eta_2 w+\eta_3 w^2+\eta_4 w^3$ and $\phi(w)=\frac{\delta}{w}$, $w\neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$. Also, we get

$$Q(z)=zq'(z)\phi(q(z))=\delta\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in U,

$$\operatorname{Re}\left(\frac{\theta'(q(z))}{\phi(q(z))}\right)=\operatorname{Re}\left(\frac{\eta_2}{\delta}q(z)+\frac{2\eta_3}{\delta}q^2(z)+\frac{3\eta_4}{\delta}q^3(z)\right)>0.$$

Thus, by Lemma 1.4, we get $q(z)\prec\xi(z)$. By using (2.23), we obtain the desired result. ☐

Theorem 2.7. *Let* $\eta_i\in\mathbb{C}$, $i=1,2,3,4$, $\gamma>0$, $\delta\in\mathbb{C}\backslash\{0\}$ *and* q *be convex univalent in* U *with* $q(0)=1$, $q(z)\neq 0$ $(z\in U)$ *and assume that* q *satisfies* (2.21). *Suppose that* $\frac{zq'(z)}{q(z)}$ *is starlike univalent in* U. *Let* $f\in A_p$ *satisfying*

$$\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma}\in H[q(0),1]\cap Q$$

and $\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ *be univalent in* U*, where* $\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ *is given by* (2.11)*. If*

$$\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}\prec\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z),\tag{2.25}$$

then

$$q(z)\prec\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p}f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\right)^{\gamma}$$

and q is the best subordinant of (2.25)*.*

Proof. The proof is similar to that of Theorem 2.6. □

Theorem 2.8. *Let $\eta_i\in\mathbb{C}$, $i=1,2,3,4$, $\delta\in\mathbb{C}\backslash\{0\}$ and q be convex univalent in U with $q(0)=1$, $q(z)\neq 0$, $z\in U$, and assume that q satisfies* (2.21)*. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in U. Let $f\in A_p$ satisfying*

$$\frac{\left(D^{m,n}_{\lambda_1,\lambda_2,p}f(z)\right)^2}{z^p D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}\in H[q(0),1]\cap Q$$

and $\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)$ *be univalent in* U*, where* $\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)$ *is given by* (2.13)*. If*

$$\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}\prec\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z),\tag{2.26}$$

then

$$q(z)\prec\frac{\left(D^{m,n}_{\lambda_1,\lambda_2,p}f(z)\right)^2}{z^p D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)}$$

and q is the best subordinant of (2.26)*.*

Proof. Define the function ξ by

$$\xi(z)=\frac{\left(D^{m,n}_{\lambda_1,\lambda_2,p}f(z)\right)^2}{z^p D^{m+1,n}_{\lambda_1,\lambda_2,p}f(z)},\quad z\in U.$$

By a straightforward computation and using (1.3), we have

$$\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)=\eta_1+\eta_2\xi(z)+\eta_3\xi^2(z)+\eta_4\xi^3(z)+\delta\frac{z\xi'(z)}{\xi(z)},\tag{2.27}$$

where $\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)$ is given by (2.13).

From (2.26) and (2.27), we obtain

$$\begin{aligned}&\eta_1+\eta_2 q(z)+\eta_3 q^2(z)+\eta_4 q^3(z)+\delta\frac{zq'(z)}{q(z)}\\&\prec\eta_1+\eta_2\xi(z)+\eta_3\xi^2(z)+\eta_4\xi^3(z)+\delta\frac{z\xi'(z)}{\xi(z)}.\end{aligned}$$

The remaining part of Theorem 2.8 is similar to that of Theorem 2.6 and hence we omit it. $\square$

Concluding the results of differential subordination and superordination, we state the following "sandwich results".

Theorem 2.9. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_2 satisfies (2.1), $\gamma > 0$ and $\mathrm{Re}(\sigma) > 0$. Let $f \in A_p$ satisfying*

$$\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^\gamma \in H[1,1] \cap Q$$

and

$$\left(1 - \frac{\sigma(p + \lambda_2(k-p))}{\lambda_1 p}\right)\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^\gamma + \frac{\sigma(p + \lambda_2(k-p))}{\lambda_1 p}\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^\gamma\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right),$$

be univalent in U. If

$$q_1(z) + \frac{\sigma}{p\gamma} z q_1'(z) \prec \left(1 - \frac{\sigma(p + \lambda_2(k-p))}{\lambda_1 p}\right)\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^\gamma + \frac{\sigma(p + \lambda_2(k-p))}{\lambda_1 p}\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^\gamma\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right) \prec q_2(z) + \frac{\sigma}{p\gamma} z q_2'(z),$$

then

$$q_1(z) \prec \left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{z^p}\right)^\gamma \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 2.10. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (2.21) and q_2 satisfies (2.5). Let $f \in A_p$ satisfying*

$$\left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^\gamma \in H[1,1] \cap Q$$

and $\Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z)$ be univalent in U, where $\Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z)$ is given by (2.7). If

$$\eta_1 + \eta_2 q_1(z) + \eta_3 q_1^2(z) + \eta_4 q_1^3(z) + \delta\frac{z q_1'(z)}{q_1(z)} \prec \Omega_1(\eta_i)_1^4(\gamma, \delta, m, n, \lambda_1, \lambda_2, p; z) \prec \eta_1 + \eta_2 q_2(z) + \eta_3 q_2^2(z) + \eta_4 q_2^3(z) + \delta\frac{z q_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \left(\frac{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 2.11. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (2.21) and q_2 satisfies (2.5). Let $f \in A_p$ satisfying*

$$\left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{\gamma} \in H[1,1] \cap Q$$

and $\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ be univalent in U, where $\Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z)$ is given by (2.11). If

$$\begin{aligned}\eta_1 + \eta_2 q_1(z) + \eta_3 q_1^2(z) + \eta_4 q_1^3(z) + \delta\frac{zq_1'(z)}{q_1(z)} \prec & \Omega_2(\eta_i)_1^4(\gamma,\delta,m,n,\lambda_1,\lambda_2,p;z) \\ \prec & \eta_1 + \eta_2 q_2(z) + \eta_3 q_2^2(z) + \eta_4 q_2^3(z) \\ & + \delta\frac{zq_2'(z)}{q_2(z)},\end{aligned}$$

then

$$q_1(z) \prec \left(\frac{D^{m,n}_{\lambda_1,\lambda_2,p} f(z)}{D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)}\right)^{\gamma} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Theorem 2.12. *Let q_1 and q_2 be convex univalent in U with $q_1(0) = q_2(0) = 1$. Suppose q_1 satisfies (2.21) and q_2 satisfies (2.5). Let $f \in A_p$ satisfying*

$$\frac{\left(D^{m,n}_{\lambda_1,\lambda_2,p} f(z)\right)^2}{z^p D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)} \in H[1,1] \cap Q$$

and $\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)$ be univalent in U, where $\Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z)$ is given by (2.13). If

$$\begin{aligned}\eta_1 + \eta_2 q_1(z) + \eta_3 q_1^2(z) + \eta_4 q_1^3(z) + \delta\frac{zq_1'(z)}{q_1(z)} &\prec \Omega_3(\eta_i)_1^4(\delta,m,n,\lambda_1,\lambda_2,p;z) \\ \prec \eta_1 + \eta_2 q_2(z) + \eta_3 q_2^2(z) + \eta_4 q_2^3(z) + \delta\frac{zq_2'(z)}{q_2(z)},&\end{aligned}$$

then

$$q_1(z) \prec \frac{\left(D^{m,n}_{\lambda_1,\lambda_2,p} f(z)\right)^2}{z^p D^{m+1,n}_{\lambda_1,\lambda_2,p} f(z)} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Remark 2.1. By specifying the function ϕ and selecting the particular values of $\eta_1, \eta_2, \eta_3, \eta_4, \gamma, m, n, \lambda_1, \lambda_2$ and p, we can derive a number of known results. Some of them are given below.

(1) Taking $n = \lambda_2 = 0$ and $p = 1$ in Theorems 2.1, 2.5, 2.9, we get the results obtained by Răducanu and Nechita [14, Theorem 3.1, Theorem 3.6, Theorem 3.9].
(2) Taking $n = \lambda_2 = 0$ and $\lambda_1 = p = 1$ in Theorems 2.1, 2.5, 2.9, we get the results obtained by Răducanu and Nechita [14, Corollary 3.3, Corollary 3.8, Corollary 3.11].
(3) Putting $n = m = \lambda_2 = 0$ and $\lambda_1 = p = 1$ in Theorem 2.1, we obtain the results obtained by Murugusundaramoorthy and Magesh [12, Corollary 3.3].
(4) Taking $n = m = \lambda_2 = 0$ and $\lambda_1 = p = 1$ in Theorems 2.5, 2.9, we obtain the results obtained by Răducanu and Nechita [14, Corollary 3.7, Corollary 3.10].
(5) For $\lambda_2 = \eta_1 = \eta_2 = \eta_4 = 0$, $\gamma = p = 1$ and $\phi(w) = \delta$ in Theorems 2.2, 2.6, 2.10, we have the results obtained by Darus and Al-Shaqsi [7, Theorem 2.1, Theorem 3.1, Theorem 3.3].
(6) By taking $n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = 0$, $\gamma = \eta_2 = p = 1$ and $\phi(w) = \delta$ in Theorems 2.3, 2.7, 2.11, we get the results obtained by Nechita [13, Theorem 5, Theorem 10, Corollary 13].
(7) Putting $n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = 0$, $\gamma = \lambda_1 = \eta_2 = p = 1$ and $\phi(w) = \delta$ in Theorems 2.3, 2.7, 2.11, we obtain the results obtained by Shanmugan et al. [17, Theorem 5.1, Theorem 5.2, Theorem 5.3].
(8) Putting $n = m = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = 0$, $\gamma = \lambda_1 = \eta_2 = p = 1$ and $\phi(w) = \delta$ in Theorems 2.3, 2.7, 2.11, we get the results obtained by Shanmugam et al. [17, Theorem 3.1, Theorem 3.2, Theorem 3.3].

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CHAIN CONDITION AND FUNDAMENTAL RELATION ON (Δ, G)-SETS DERIVED FROM Γ-SEMIHYPERGROUPS

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Abstract. The aim of this research work is to define a new class of hyperstructure as a generalization of semigroups, semihypergroups and Γ-semihypergroups that we call (Δ, G)-sets. Also, we define fundamental relation on (Δ, G)-sets and prove some results in this respect. Then, we introduce the notions of quotient (Δ, G)-sets by using a congruence relations. Finally, we introduce the concept of complete parts and Noetherian(Artinian) (Δ, G)-sets.

1. Introduction

The hypergroup notion was introduced in 1934 by a French mathematician F. Marty [17], at the 8^{th} Congress of Scandinavian Mathematicians. He published some notes on hypergroups, using them in different contexts: algebraic functions, rational fractions, non commutative groups. Algebraic hyperstructures are a suitable generalization of classical algebraic structures. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Since then, hundreds of papers and several books have been written on this topic, see [4–6].

The concept of Γ-semigroup defined by Sen and Saha [18] in 1986 that is a generalization of a semigroup. Many classical notions of semigroups have been extended to Γ-semigroups and a lot of results on Γ-semigroups are published by a lot of mathematicians, for instance, Chattopadhyay [2, 3], Hila [15, 16] and [18].

Recently, the notion of Γ-hyperstructure introduced and studied by many researchers and represent an intensively studied field of research, for example, see

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[1, 7, 8, 11–14]. The concept of Γ-semihypergroups was introduced by Davvaz et al. [1, 14] and is a generalization of semigroups, a generalization of semihypergroups and a generalization of Γ-semigroups. Also, the concept of (Δ, G)-set was introduced by S. Ostadhadi-Dehkordi [9, 10]. He using them in different contexts such as twist product, flat Γ-semihypergroup, absolutely flat Γ-semihypergroup and direct limit that is important tools in the theory of homological algebra.

In this paper, by using a special scalar hyperoperations on Γ-semihypergroups we denote the notions left(right) (Δ, G)-set, (G_1, Δ, G_2)-biset. Also, we introduced regular and strongly regular relations on (Δ, G)-sets and by using fundamental relation we define quotient (Δ, G)-sets. Finally, we define the concept of complete part and Noetherian(Artinian) (Δ, G)-sets and prove some results in respect.

2. Introduction and preliminaries

In this section, we present some basic notions of Γ-semihypergroup. These definitions and results are necessary for the next sections.

Let H be a non-empty set. Then, the map $\circ : H \times H \to P^*(H)$ is called *hyperoperation* or *join operation* on the set H, where $P^*(H)$ denotes the set of all non-empty subsets of H. A hypergroupoid is a set H together with a (binary)hyperoperation. A hypergroupoid $(H, \circ)$ is called a *semihypergroup* if for all $a, b, c \in H$, we have $a \circ (b \circ c) = (a \circ b) \circ c$. A hypergroupoid $(H, \circ)$ is called *quasihypergroup* if for all $a \in H$, we have $a \circ H = H \circ a = H$. A hypergroupoid $(H, \circ)$ which is both a semihypergroup and a quasihypergroup is called a *hypergroup*.

Definition 2.1 ([14])**.** Let G and Γ be nonempty sets and $\alpha : G \times G \to P^*(G)$ be a hyperoperation, where α is an arbitrary element in the set Γ. Then, G is called Γ-*hypergroupoid*.

For any two nonempty subsets G_1 and G_2 of G, we define

$$G_1 \alpha G_2 = \bigcup_{g_1 \in G_1, g_2 \in G_2} g_1 \alpha g_2, \quad G_1 \alpha \{x\} = G_1 \alpha x, \quad \{x\} \alpha G_2 = x \alpha G_2.$$

A Γ-hypergroupoid G is called Γ-*semihypergroup* if for all $x, y, z \in G$ and $\alpha, \beta \in \Gamma$ we have

$$(x \alpha y) \beta z = x \alpha (y \beta z).$$

Example 2.1. Let $\Gamma \subseteq \mathbb{N}$ be a nonempty set. We define

$$x \alpha y = \{z \in \mathbb{N} : z \geq \max\{x, \alpha, y\}\},$$

where $\alpha \in \Gamma$ and $x, y \in \mathbb{N}$. Then, $\mathbb{N}$ is a Γ-semihypergroup.

Example 2.2. Let $\Gamma = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$. Then, we define hyperoperations $x \alpha_k y = xyk\mathbb{Z}$. Hence, $\mathbb{Z}$ is a Γ-semihypergroup.

Example 2.3. Let G be a nonempty set and Γ be a nonempty set of G. Then, we define $x \alpha y = \{x, \alpha, y\}$. Hence, G is a Γ-semihypergroup.

Example 2.4. Let $(\Gamma, \cdot)$ be a semigroup and $\{A_\alpha\}_{\alpha\in\Gamma}$ be a collection of nonempty disjoint sets and $G = \bigcup_{\alpha\in\Gamma} A_\alpha$, for every $g_1, g_2 \in G$ and $\alpha \in \Gamma$, we define $g_1\widehat{\alpha}g_2 = A_{\alpha_1\alpha\alpha_2}$, where $g_1 \in A_{\alpha_1}$ and $g_2 \in A_{\alpha_2}$. Then, G is a $\widehat{\Gamma}$-semihypergroup, $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$.

Let G be a Γ-semihypergroup. Then, an element $e_\alpha \in G$ is called *α-identity* if for every $x \in G$, we have $x \in e_\alpha\alpha x \cap x\alpha e_\alpha$ and e_α is called *scalar α-identity* if $x = e_\alpha\alpha x = x\alpha e_\alpha$. We note that if for every $\alpha \in \Gamma$, e is a scalar α-identity, then $x\alpha y = x\beta y$, where $\alpha, \beta \in \Gamma$ and $x, y \in G$. Indeed,

$$x\alpha y = (x\beta e)\alpha y = x\beta(e\alpha y) = x\beta y.$$

Let G be a Γ-semihypergroup and for every $\alpha \in \Gamma$ has an α-identity. Then, G is called a Γ-semihypergroup with identity. In a same way, we can define Γ-semihypergroup with scalar identity.

A Γ-semihypergroup G is *commutative* when

$$x\alpha y = y\alpha x,$$

for every $x, y \in G$ and $\alpha \in \Gamma$.

Definition 2.2. Let G be a Γ-semihypergroup and ρ be an equivalence relation on G. Then, ρ is called *right regular relation* if $x\rho y$ and $g \in G$ implies that for every $t_1 \in x\alpha g$ there is $t_2 \in y\alpha g$ such that $t_1\rho t_2$ and for every $s_1 \in y\alpha g$ there is $s_2 \in x\alpha g$ such that $s_1\rho s_2$. In a same way, we can define *left regular relation.* An equivalence relation ρ is called *strong regular* when $x\rho y$ and $g \in G$ implies that for every $t_1 \in x\alpha g$ and $t_2 \in y\alpha g$, $t_1\rho t_2$, for every $\alpha \in \Gamma$.

Example 2.5. Let $\mathbb{R} = \bigcup_{n\in\mathbb{Z}} A_n$, where $A_n = [n, n+1)$ and $x, y \in \mathbb{R}$ such that $x \in A_n$, $y \in A_m$ and $\alpha \in \mathbb{Z}$. Then, $\mathbb{R}$ is a $\widehat{\mathbb{Z}}$-semihypergroup such that $x\widehat{\alpha}y = A_{n\alpha m}$, where $\widehat{\alpha} \in \widehat{\mathbb{Z}} = \{\widehat{\alpha} : \alpha \in \mathbb{Z}\}$. Let

$$x\rho y \leftrightarrow 2|n - m, \quad x \in A_n, y \in A_m.$$

Then, the relation ρ is strong regular. Also, $x \in \mathbb{R}$, implies that

$$\rho(x) = \{z \in \mathbb{R} : z \in \cdots [n-4, n-3) \cup [n-2, n-1) \cup [n, n+1) \cup [n+2, n+3) \cdots\},$$

where $x \in [n, n+1)$.

Proposition 2.1. *Let G be a Γ-semihypergroup and ρ be a regular relation on G. Then, $[G : \rho] = \{\rho(x) : x \in G\}$ is a $\widehat{\Gamma}$-semihypergroup with respect the following hyperoperation:*

$$\rho(x)\widehat{\alpha}\rho(y) = \{\rho(z) : z \in \rho(x)\alpha\rho(y)\},$$

where $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$.

Proof. The proof is straightforward. $\square$

Corollary 2.1. *Let G be a Γ-semihypergroup and ρ be an equivalence relation G. Then, ρ is regular (strong regular) if and only if $[G:\rho]$ is $\widehat{\Gamma}$-semihypergroup ($\widehat{\Gamma}$-semigroup).*

Definition 2.3 ([9])**.** Let G be a Γ-semihypergroup with identity and X, Δ be nonempty sets. Then, we say that X is a *left* (Δ, G)*- set* if there is a scalar hyperaction $\delta : G \times X \to P^*(X)$ with the following properties:

$$(g_1\alpha g_2)\delta x = g_1\delta(g_2\delta x),$$
$$e_\alpha \delta x = x,$$

for every $g_1, g_2 \in G$, $\alpha \in \Gamma$, $x \in X$ and $\delta \in \Delta$.

When $\delta : G \times X \to X$, then X is called *scalar left* (Δ, G)*-set.*

Example 2.6. Let G be a Γ-semihypergroup with scalar identity, X and Δ be nonempty sets such that $x_0 \in X$ is a fixed element and $\delta : G \times X \to P^*(X)$ defined by $\delta(g, x) = \{x_0\}$, where $\delta \in \Delta$ and $x \in X$. Then, G is left (Δ, G)-set.

Example 2.7. Let $(G, \circ)$ be a semihypergroup and H be a subsemihypergroup of G. Then, H is a left (Δ, G)-set where $\Delta = \{\circ\}$.

In a same way, we can define a *right* (Δ, G)*-set.* Let G_1 and G_2 be Γ-semihypergroups and X be a nonempty set. Then, we say that X is a (G_1, Δ, G_2)*-bisets* if it is a left (Δ, G_1)-set, right (Δ, G_2)-set and

$$(g_1\delta_1 x)\delta_2 g_2 = g_1\delta_1(x\delta_2 g_2),$$

for every $\delta_1, \delta_2 \in \Delta$, $g_1 \in G_1$, $g_2 \in G_2$ and $x \in X$. When X is a (G_1, Δ, G_2)-bisets and $G_1 = G_2 = G$, we sat that X is a (Δ, G)-bisets.

If G is a commutative Γ-semihypergroup, then there is no distinction between a left and a right (Δ, G)-sets. A *left* (Δ, G)*-subset* Y of X such that $Y\Delta X \subseteq Y$ is called *left* (Δ, G)*-subset* of X. Let X be a left (Δ, G)-set and $\Gamma \subseteq \Delta$. Then, X is also (Γ, G)-set where $\delta : G \times X \to P^*(X)$ and $\delta \in \Gamma$.

Definition 2.4. Let X be a left (Δ, G)-set and Y be a left (Δ, G)-subset of X. Then, we say that Y *closed*, if for all $y \in Y$ and $g \in G$ from $y \in g\delta b$ implies that $b \in Y$.

Definition 2.5. Let X be a (G, Δ, G)-biset and Y be a (G, Δ, G)-subbiset of X. Then, Y is called *invertible* on a right(on a left) if for all $y_1, y_2 \in Y$ and $g \in G$ from $y_1 \in y_2\delta G (y_1 \in G\delta y_2)$ it follows that $y_2 \in y_1\delta G$ $(y_2 \in G\delta y_1)$.

Proposition 2.2. *Let G be a Γ-semihypergroup and X be a (Δ, G)-biset such that Y be a (Δ, G)-subbiset. Then, Y is invertible on the right if and only if $\{y\delta G\}_{y\in Y}$ is a partition of X, for every $y \in Y$.*

Proof. Suppose that Y is invertible on the right and $y \in y_1\delta G \cap y_2\delta G$. Then, $y_1, y_2 \in y\delta G$. This implies that $y_1\delta G \subseteq y\delta G$ and $y_2\delta G \subseteq y\delta G$. Also,

$$y\delta G \subseteq (y_1\delta G)\delta G \subseteq y_1\delta(G\Gamma G) \subseteq y_1\delta G,$$

and $y\delta G \subseteq (y_2\delta G)\delta G = y_2\delta(G\Gamma G) \subseteq y_2\delta G$. Then, $y\delta G = y_1\delta G = y_2\delta G$. On the other hand, $y \in y_1\delta G = y\delta G$. Then, for every $y \in Y$, we have $y \in y\delta G$.

Conversely, let $\{y\delta G\}_{y\in Y}$ be a partition of Y and $y_1 \in y_2\delta G$. Then,

$$y_1\delta G \subseteq (y_2\delta G)\delta G \subseteq y_2\delta(G\Gamma G) \subseteq y_2\delta G,$$

whence $y_1\delta G = y_2\delta G$ and so $y_1 \in y_2\delta G = y_1\delta G$. Then, for all $y \in Y$ we have $y \in y\delta G$. Therefore, $y_2 \in y_2\delta G = y_1\delta G$. □

Definition 2.6. Let X be a left (Δ, G)-set and Y be a left (Δ, G)-subset of X. Then, Y is called *ultraclosed* if for all $g \in G$ and $\delta \in \Delta$, we have $g\delta Y \cap g\delta(X - Y) = \emptyset$.

Proposition 2.3. *Let X be a left (Δ, G)-set and Y be a invertible (Δ, G)-subset. Then, X is closed.*

Proof. Suppose that $y, x \in Y$, $\delta \in \Delta$ and $g \in G$ such that $y \in g\delta x$. Hence $x \in g\delta y \subseteq Y$ and we obtain $x \in Y$. □

Definition 2.7. Let X be a left (Δ, G)-set and H be a Γ-subsemihypergroup of G. Then, we define the following relation:

$$x_1 \equiv x_2 \Leftrightarrow x_1 \in H\delta x_2.$$

This relation is denoted by $x_1 H^* x_2$.

Definition 2.8. Let X be a left (G, Δ)-set and ρ be a regular relation on X. Then, ρ is called *regular* if $x_1\rho x_2$ implies that for every $s_1 \in g\delta x_1$ there is $s_2 \in g\delta x_2$ such that $s_1\rho s_2$ and for every $t_2 \in g\delta x_2$ there is $t_1 \in g\delta x_1$ such that $t_1\rho t_2$, where $x_1, x_2 \in X$ and $\delta \in \Delta$. Also, an equivalence relation ρ is called *strongly regular*, when for every $s_1 \in g\delta x_1$ and $s_2 \in g\delta x_2$ implies that $s_1\rho s_2$.

Proposition 2.4. *Let X be an invertible left (Δ, G)-set such that G is commutative. Then, the relation H^* is regular.*

Proof. Suppose that $x \in X$. Then, $x = e_\alpha\delta x \in H\delta x$. It follows that xH^*x, i.e., H^* is reflexive. Let $x_1H^*x_2$. Then, there exist $\delta \in \Delta$ and $h \in H$ such that $x_1 \in h\delta x_2$ which implies that $x_2 \in h\delta x_1 \subseteq H\delta x_1$ which meanies that $x_2H^*x_1$ and so H^* is symmetric. Let $x_1, x_2, x_3 \in X$ such that $x_1H^*x_2$ and $x_2H^*x_3$. Then, there exist $h_1, h_2 \in H$ such that $x_1 \in h_1\delta x_2$ and $x_2 \in h_2\delta x_3$. Hence $x_1 \in h_1\delta(h_2\delta x_3) = (h_1\alpha h_2)\delta x_3 \subseteq H\delta x_3$. This implies that $x_1 \in H\delta x_3$ and so H^* is transitive.

Let x_1, x_2 be an arbitrary elements of X such that $x_1H^*x_2$. It follows that $x_1 \in H\delta x_2$. Hence there exist $h_1 \in H$ such that $x_1 \in h_1\delta x_2$. Let $g \in G$ and $t_1 \in g\delta x_1$. Then,

$$t_1 \in g\delta x_1 \subseteq g\delta(h_1\delta x_2) = (g\alpha h_1)\delta x_2 = (h_1\alpha g)\delta x_2 = h_1\delta(g\delta x_2).$$

Hence there exists $t_2 \in g\delta x_2$ such that $t_1 \in h_1\delta t_2 \subseteq H\delta t_2$. Thus, $t_1H^*t_2$. In a same way, we can see for every $s_2 \in g\delta x_2$ there is $s_1 \in g\delta x_1$ such that $s_1H^*s_2$. Therefore, H^* is a regular relation. □

Proposition 2.5. *Let X be a left (Δ, G)-set and H be a Γ-subsemihypergroup of G. Then, $H^*(x) = H\delta x$.*

Proof. The proof is straightforward. □

Theorem 2.1. *Let X be a left (Δ, G)-set and H be a Γ-subsemihypergroup of G. Then, the set of all classes $[X : H^*] = \{H^*(x) : x \in X\}$ is a left $(\widehat{\Delta}, G)$-set by the following scalar hyperoperation:*

$$g\widehat{\delta}H^*(x) = \{H^*(y) : y \in g\delta H^*(x)\}.$$

Proof. Suppose that $H^*(x_1) = H^*(x_2)$, $g \in G$ and $y \in g\delta H^*(x_1)$. This implies that $x_1 \in H\delta x_2$. Hence, there are $h_1, h_2 \in H$ such that $y \in g\delta(h_1\delta x_1)$ and $x_1 \in h_2\delta x_2$. We have

$$y \in g\delta(h_1\delta x_1) \subseteq g\delta(h_1\delta(h_2\delta x_2)) = g\delta(h_1\alpha h_2)\delta x_2 \subseteq g\delta(H\delta x_2) = g\delta H^*(x_2).$$

Then, $g\delta H^*(x_1) \subseteq g\delta H^*(x_2)$. In a same way, we can see, $g\delta H^*(x_2) \subseteq g\delta H^*(x_1)$. Hence,

$$g\widehat{\delta}H^*(x_1) = g\widehat{\delta}H^*(x_2).$$

Therefore, the scalar hyperoperation $\widehat{\alpha}$ is well-defined. It is easy to see that

$$(g_1\alpha g_2)\widehat{\delta}H^*(x) = g_1\widehat{\delta}(g_2\widehat{\delta}H^*(x)).$$ □

Let X be a left (Δ, G)-set. Then, we define an equivalence relation on X such that smallest strongly regular relation on X. Suppose that X be a left (Δ, G)-set and n be a nonzero natural number. We say that

$$a\beta_n b \Leftrightarrow (\exists \delta_1, \delta_2, \ldots, \delta_n \in \Delta, x \in X, g_1, g_2, \ldots, g_n \in G)\ \{a, b\} \subseteq g_1\delta_1 g_2\delta_2, \ldots, g_n\delta_n x.$$

Let $\beta = \bigcup_{n\geq 1} \beta_n$. Clearly, the relation β is reflexive and symmetric. Denote by β^* the transitive closure.

We say that $x\beta_{\delta^n}y$ when

$$a\beta_{\delta^n} b \Leftrightarrow (\exists x \in X, g_1, g_2, \ldots, g_n \in G)\ \{a, b\} \subseteq g_1\delta g_2\delta, \ldots, g_n\delta x.$$

Let $\beta_\delta = \bigcup_{n\geq 1} \beta_{\delta^n}$ and β^*_δ be transitive closure. Obviously, $\beta^*_\delta \subseteq \beta^*$.

Let X be a (Δ, G)-biset. Then, the relation β_n defined on X as follows:

$$a\beta_n b \Leftrightarrow (\exists x \in X, \delta_i, \gamma_i \in \Delta, g_i, s_i \in G)\ \{a, b\} \subseteq \prod_{i=1}^{n} (g_i\delta_i x)\gamma_i s_i.$$

In a same way, we can define β_δ and transitive closure β^*_δ.

Example 2.8. Let $\mathbb{R}$ be a $\widehat{\mathbb{Z}}$-semihyperring Example 2.5, $x, y \in \mathbb{R}$ such that $\beta(x) = \beta(y)$ and $t_1 = [x]$, $t_2 = [y]$. Then, there exist $g_1, g_2, \ldots, g_m \in \mathbb{R}$ and $\widehat{\delta}_1, \widehat{\delta}_2, \ldots, \widehat{\delta}_m \in \widehat{\mathbb{Z}}$ such that $\{x, y\} \subseteq g_1\widehat{\delta}_1 g_2\widehat{\delta}_2 g_3 \ldots g_{m-1}\widehat{\delta}_{m-1}g_m$. This implies that $t_1 = t_2 = \prod_{i=1}^{m} g_i\delta_i g_{i+1}$. Therefore, $\beta(x) = \beta(y)$ if and only there exists $n \in \mathbb{Z}$ such that $x, y \in [n, n+1)$. Hence $\beta^*(x) = \beta^*(y)$ implies that $x, y \in [n, n+1)$ for some $n \in \mathbb{Z}$.

Theorem 2.2. *Let X be a left (Δ, G)-set. Then, β^* is the smallest strongly regular relation on X.*

Proof. Suppose that $a\beta^* b$ be an arbitrary element of X. It follows that there exist $x_0 = a, x_1, \dots, x_n = b$ such that for all $i \in \{0, 1, 2, \dots, n\}$ we have $x_i \beta x_{i+1}$. Let $u_1 \in g\delta a$ and $u_2 \in g\delta b$, where $g \in G$, $\delta \in \Delta$. From $x_i \beta x_{i+1}$ it follows that there exists a hyperproduct P_i, such that $\{x_i, x_{i+1}\} \subseteq P_i$ and so $g\delta x_i \subseteq g\delta P_i$ and $g\delta x_{i+1} \subseteq g\delta P_{i+1}$, which meanies that $g\delta x_i \overline{\overline{\beta}} g\delta x_{i+1}$. Hence for all $i \in \{0, 1, 2, \dots, n-1\}$ and for all $s_i \in g\delta x_i$ we have $s_i \beta s_{i+1}$. We consider $s_0 = u_1$ and $s_n = u_2$ then we obtain $u_1 \beta^* u_2$. Then β^* is strongly regular on a left.

Let ρ be a strongly regular relation on X. Then, we have

$$\beta_1 = \{(x, x) : x \in X\} \subseteq \rho,$$

since ρ is reflexive. Let $\beta_{n-1} \subseteq \rho$ and $a\beta_n b$. Then, there exist $g_1, g_2, \dots, g_n \in G$, $\delta_1, \delta_2, \dots, \delta_n \in \Delta$ and $x \in X$ such that $\{a, b\} \subseteq \prod_{i=1}^{n} g_i\delta_i x = g_1\delta_1 \prod_{i=2}^{n} g_i\delta_i x$. This implies that there exits $u, v \in \prod_{i=2}^{n} g_i\delta_i x$ such that $a \in g_1\delta_1 u$ and $v \in g_1\delta_1 v$. We have $u\beta_{n-1} v$ and according to the hypothesis, we obtain $u\rho v$. Since ρ is regular it follows that $a\rho b$ and $\beta_n \subseteq \rho$. By induction, it follows that $\beta \subseteq \rho$. Therefore, $\beta^* \subseteq \rho$. $\square$

Proposition 2.6. *Let X_1 and X_2 be left (Δ, G)- and right (Δ, G)-sets, respectively and $\beta^*_{X_1}$, $\beta^*_{X_2}$ and $\beta^*_{X_1 \times X_2}$ be relations on X_1, X_2 and $X_1 \times X_2$, respectively. Then,*

$$(a, b)\beta^*_{X_1 \times X_2}(c, d) \Leftrightarrow a\beta^*_{X_1} c, \;\; b\beta^*_{X_2} d.$$

Proof. Suppose that $(a, b)\beta^*_{X_1 \times X_2}(c, d)$. Then,

$$\{(a, b), (c, d)\} \subseteq \prod_{i=1}^{n} g_i \widehat{\delta_i}(x, y)\widehat{\gamma_i} s_i = \left(\prod_{i=1}^{n} g_i\delta_i x, \prod_{i=1}^{n} y\gamma_i s_i \right).$$

This implies that $\{a, c\} \subseteq \prod_{i=1}^{n} g_i\delta_i x$ and $\{b, d\} \subseteq \prod_{i=1}^{n} y\gamma_i s_i$. Then, $a\beta^*_{X_1} c$ and $b\beta^*_{X_2} d$. One can see that $a\beta^*_{X_1} c$ and $b\beta^*_{X_2} d$ implies that $(a, b)\beta^*_{X_1 \times X_2}(c, d)$. $\square$

Corollary 2.2. *Let X_1 and X_2 be left (Δ, G)- and right (Δ, G)-sets, respectively and $\beta^*_{X_1}$, $\beta^*_{X_2}$ and $\beta^*_{X_1 \times X_2}$ be relations on X_1, X_2 and $X_1 \times X_2$, respectively. Then,*

$$[X_1 \times X_2 : \beta^*_{X_1 \times X_2}] \simeq [X_1 : \beta^*_{X_1}] \times [X_2 : \beta^*_{X_2}].$$

Definition 2.9. A map $\varphi : X \to Y$ from a left (Δ, G)-set X into a left (Δ, G)-set Y is called *morphism* (*G-morphism*) if

$$\varphi(g\delta x) = g\delta\varphi(x),$$

for every $x \in X, \delta \in \Delta$ and $g \in G$.

Example 2.9. Let $(G, \circ)$ be a semihypergroup with scalar identity and G_1 be a subsemihypergroup of $(G, \circ)$. Then, G_1 is a (Γ, G_1)-biset in the obvious way, where $\Gamma = \{\circ\}$.

Example 2.10. Let ρ be a left regular relation on Γ-semihypergroup G. Then, there is a well-defined action of G on $[G : \rho]$ given by

$$g\widehat{\alpha}(\rho(x)) = \{\rho(t) : t \in g\alpha x\},$$

where $\widehat{\alpha} \in \widehat{\Gamma}$ such that $\widehat{\Gamma} = \{\widehat{\alpha} : \alpha \in \Gamma\}$. Hence, with this definition $[G : \rho]$ is a left $(\widehat{\Gamma}, G)$-system.

It is easy to see that the cartesian product $X \times Y$ of a left (Δ, G_1)-set X and a right (Δ, G_2)-set Y becomes $(G_1, \widehat{\Delta}, G_2)$-biset if we make the obvious definitions

$$g_1\widehat{\delta}_1(x, y) = \{(t, y) : t \in g_1\delta_1 x\},\ (x, y)\widehat{\delta}_2 g_2 = \{(x, t) : t \in y\delta_2 g_2\},$$

where $\widehat{\delta}_1, \widehat{\delta}_2 \in \widehat{\Delta}$, $x \in X$, $y \in Y$ and $g_1 \in G_1$, $g_2 \in G_2$.

Let X and Y be (G_1, Δ, G_2)- and (G_2, Δ, G_3)-bisets, respectively and Z be a (G_1, Δ, G_3)-biset. Then, the cartesian product $X \times Y$ is (G_1, Δ, G_3)-biset. A (G_1, Δ, G_3)-map $\varphi_\delta : X \times Y \to Z$ is called *δ-bimap* if

$$\varphi(x\delta g_2, y) = \varphi(x, g_2\delta y),$$

where $x \in X$, $y \in Y$, $g_2 \in G_2$ and $\delta \in \Delta$.

Definition 2.10 ([9])**.** A pair (P, ψ) consisting of (G_1, Δ, G_3)-biset P and a δ-bimap $\psi : X \times Y \to P$ will be called a *twist product* of X and Y over G_2 if for every (G_1, Δ, G_3)-biset Z and for every bimap $\omega : X \times Y \to Z$ there exists a unique bimap $\overline{\omega} : P \to Z$ such that $\overline{\omega} \circ \psi = \omega$.

Suppose that ρ is an equivalence relation on $X \times Y$ as follows:

$$\rho = \{(t_1, t_2) : t_1 \in x\delta g, t_2 \in g\delta y, x \in X, y \in Y, g \in G_2\}.$$

Let us define $X \ominus Y$ to be $[X \times Y : \rho^*]$, where ρ^* is a transitive closure of ρ. We denote a typical element $\rho^*(x, y)$ by $x \ominus y$. By definition of ρ^*, we have $x\delta g \ominus y = x \ominus g\delta y$, where $\delta \in \Delta$.

Proposition 2.7 ([9])**.** *Let X and Y be (G_1, Δ, G_2)- and (G_2, Δ, G_3)-bisets, respectively. Then, two element $x \ominus y$ and $x' \ominus y'$ are equal if and only if $(x, y) = (x', y')$ or there exist $x_1, x_2, \ldots, x_{n-1}$ in X, $h_1, h_2, \ldots, h_{n-1} \in G_2$ and $\delta \in \Delta$ such that*

$$\begin{aligned} x \in x_1\delta g_1, x_1\delta h_1 = x_2\delta g_2, \ldots, x_i\delta g_i = x_{i+1}\delta g_{i+1}, x_{n-1}\delta h_{n-1} =& x'\delta g_n, \\ g_1\delta y = h_1\delta y_1, g_2\delta y_1 = h_2\delta y_2, \ldots, g_{i+1}\delta y_i =& h_{i+1}\delta y_{i+1} \\ =& g_n\delta y_{n-1} \\ =& y'. \end{aligned}$$

Theorem 2.3 ([9])**.** *Let X and Y be (G_1, Δ, G_2)- and (G_2, Δ, G_3)-bisets. Then, the twist product X and Y over G_2 is unique up to isomorphism.*

Proposition 2.8. *Let X and Y be a scalar (Δ, G)-bisets. Then, $X \ominus Y$ is a (Δ, G)-biset by following scalar hyperoperations:*

$$g\widehat{\delta}(x \ominus y) = g\delta x \ominus y, \quad (x \ominus y)\widehat{\delta} g = x \ominus y\delta g,$$

where $\widehat{\delta} \in \widehat{\Delta}$ *and* $x \in X$, $y \in Y$.

Proof. Suppose that $x \ominus y = x' \ominus y'$. By Proposition 2.7, there exist $\delta \in \Delta$, $x_1, x_2, \dots, x_{n-1} \in X$ and $h_1, h_2, \dots, h_{n-1} \in G$, such that

$$\begin{aligned} x = x_1\delta g_1, x_1\delta h_1 = x_2\delta g_2, \cdots x_i\delta h_i &= x_{i+1}\delta g_{i+1} \\ x_{n-1}\delta h_{n-1} &= x'\delta g_n, \\ g_1\delta y = h_1\delta y_1,\ g_2\delta y_1 = h_2\delta y_2, \dots, g_{i+1}\delta y_i &= h_{i+1}\delta y_{i+1} \\ &= g_n\delta y_{n-1} \\ &= y'. \end{aligned}$$

Hence,

$$\begin{aligned} g\delta x = g\delta(x_1\delta g_1), g\delta(x_1\delta h_1) = g\delta(x_2\delta g_2), \dots, g\delta(x_i\delta h_i) = &\ \ g\delta(x_{i+1}\delta g_{i+1}) \\ g\delta(x_{n-1}\delta h_{n-1}) = &\ \ g\delta(x'\delta g_n). \end{aligned}$$

We have

$$\begin{aligned} g\delta x \ominus y = t_1 \ominus g_1\delta y = t_1 \ominus h_1\delta y = t_1\delta h_1 \ominus y_1 &= t_2\delta g_2 \ominus y_1 \\ &\ \vdots \\ &= t'\delta g_n \ominus y_{n-1} \\ &= t' \ominus g_n\delta y_{n-1} \\ &= g\delta x' \ominus y', \end{aligned}$$

where $t_i \in X$. Then, the left scalar operation $\widehat{\delta}$ is well-defined. Moreover,

$$(g_1\alpha g_2)\widehat{\delta}(x \ominus y) = (g_1\alpha g_2)\delta x \ominus y = g_1\delta(g_2\delta x) \ominus y = g_1\widehat{\delta}(g_2\widehat{\delta}(x \ominus y)),$$

where $x \in X$, $y \in Y$ and $g \in G$. Hence $X \ominus Y$ is a left $(\widehat{\Delta}, G)$-set. In a same way, we can see $X \ominus Y$ is also right (Δ, G)-set. $\square$

3. Complete Parts and Regular Relations

In this section we define the concept of complete parts and present some results.

Definition 3.1. Let X be a left (Δ, G)-set and Y be a nonempty subset of X. We say that Y is a *complete part* of X if for any nonzero natural number n and $g_1, g_2, \dots, g_n \in G$, $\delta_1, \delta_2, \dots, \delta_n \in \Delta$, $x \in X$, the following implication holds:

$$Y \cap \prod_{i=1}^{n} g_i\delta_i x \neq \emptyset \Rightarrow \prod_{i=1}^{n} g_i\delta_i x_i \subseteq Y.$$

Proposition 3.1. *Let X be a left (Δ, G)-set and ρ be a strongly regular relation on X. Then, the equivalence class x is a complete part of X.*

Proof. Suppose that $g_1, g_2, \ldots, g_n \in G$, $\delta_1, \delta_2, \ldots, \delta_n \in \Delta$ and $x \in X$ such that

$$\rho(x) \cap \prod_{i=1}^{n} g_i \widehat{\delta}_i x \neq \emptyset.$$

Then, there exists $y \in \prod_{i=1}^{n} g_i \delta_i x$ such that $y \rho x$. The morphism $\pi : X \to [X : \rho]$ is good and the scalar hyperoperation $\widehat{\delta}$ defined on $[X : \rho]$ is scalar operation. It follows that

$$\pi(y) = \rho(y) = \rho(x) = \pi\left(\prod_{i=1}^{n} g_i \delta_i x\right) = \prod_{i=1}^{n} \pi(g_i \delta_i x) = \prod_{i=1}^{n} g_i \widehat{\delta}_i \pi(x).$$

This implies that $\prod_{i=1}^{n} g_i \widehat{\delta}_i x \subseteq \rho(x)$. □

Proposition 3.2. *Let X and Y be scalar (Δ, G)-bisets such that $X_1 \subseteq X$ be a complete part. Then, $X_1 \ominus Y$ is also complete part in $X \ominus Y$.*

Proof. The proof is straightforward. □

Let A be a nonempty subset of (Δ, G)-sets X. Then, denoted by $C(A)$ the *complete closure* of A, which is the smallest complete part of X, that contain A.

Denote $K_1(A) = A$ and for all $n \geq 1$ denote

$$K_{n+1}(A) = \left\{ x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^{t} g_i \delta_i x, K_n(A) \cap \prod_{i=1}^{t} g_i \delta_i x \right\}.$$

Let $K(A) = \bigcup_{n \geq 1} K_n(A)$.

Theorem 3.1. *Let X be a left (Δ, G)-set and A be a nonempty subset of A. Then, $C(A) = K(A)$.*

Proof. Suppose that $K(A) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$. Then, there exits $n \geq 1$ such that $K_n(A) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$ which meanies that $\prod_{i=1}^{t} g_i \delta_i x \subseteq K_{n+1}(A)$. This implies that $K(A)$ is a complete part of X.

Let C_1 be a complete pat of X such that $A \subseteq C_1$. Then, by induction we prove that $K(A) \subseteq C_1$. We have $K_1(A) \subseteq C_1$ and suppose that $K_n(A) \subseteq C_1$. Let $x \in K_{n+1}(A)$. Then, there exists $t \in \mathbb{N}$ such that $a \in \prod_{i=1}^{t} g_i \delta_i x$ and $K_n(A) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$. Hence, $C_1 \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset$ implies that $\prod_{i=1}^{t} g_i \delta_i x \subseteq C_1$. We obtain $a \in C_1$. Therefore, $C(A) = K(A)$. □

Proposition 3.3. *Let X be a left (Δ, G)-set and x be an arbitrary element of X. Then,*

(1) *for all $n \geq 2$ we have $K_n(K_2(x)) = K_{n+1}(x)$;*
(2) *for every $x, y \in X$, $x \in K_n(y) \Leftrightarrow y \in K_n(x)$.*

Proof. (1) We prove the equality by induction. We have

$$K_2(K_2(x)) = \left\{ x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^{t} g_i \delta_i x, K_2(x) \cap \prod_{i=1}^{t} g_i \delta_i x \neq \emptyset \right\} = K_3(x).$$

Let $K_{n-1}(K_2(x)) = K_n(x)$. Then,

$$\begin{aligned} K_n(K_2(x)) &= \left\{x \in X : (\exists t \in \mathbb{N})\ x \in \textstyle\prod_{i=1}^t g_i\delta_i x, K_{n-1}(K_2(x)) \cap \prod_{i=1}^t g_i\delta_i x \neq \emptyset\right\} \\ &= K_{n+1}(x). \end{aligned}$$

(2) We check the equivalence by induction. For $n = 2$, we have

$$x \in K_2(y) = \left\{x \in X : (\exists t \in \mathbb{N})\ x \in \prod_{i=1}^t g_i\delta_i x, K_1(y) \cap \prod_{i=1}^t g_i\delta_i x \neq \emptyset\right\}.$$

This implies that $\{y, x\} \subseteq \prod_{i=1}^t g_i\delta_i x$ and $y \in K_2(x)$. Suppose that the following equivalence holds:

$$x \in K_{n-1}(y) \Leftrightarrow y \in K_{n-1}(x).$$

We check that $x \in K_n(y) \Leftrightarrow y \in K_n(x)$. Let $x \in K_n(y)$. Then, there exists $\prod_{i=1}^t g_i\delta_i a$ with $x \in \prod_{i=1}^t g_i\delta_i a$ and there exists $b \in \prod_{i=1}^t g_i\delta_i a \cap K_{n-1}(y)$. It follows that $b \in K_2(x)$ and $y \in K_{n-1}(b)$. Hence, $y \in K_{n-1}(K_2(x)) = K_n(x)$. Similarly, we obtain the converse implication. □

Definition 3.2. Let X be a left (Δ, G)-set. Then, we define the relation ω as follows:

$$(x, y) \in \omega \Leftrightarrow (\exists n \geq 1)\ x \in K_n(y).$$

Theorem 3.2. *Let X be a left (Δ, G)-set. Then, the relation ω is an equivalence and coincide with β^*.*

Proof. By Proposition 3.3, the relation ω is an equivalence. Let $(x_1, x_2) \in \beta$. Then, $\{x_1, x_2\} \subseteq \prod_{i=1}^n g_i\delta_i x$, where $g_i \in G$, $\delta_i \in \Delta$ and $t \in \mathbb{N}$. Hence, x_1, x_2 belong to the same scalar hyperoperation and so, $x_1 \in K_2(x_2) \subseteq K(x_2)$. This implies that $\beta \subseteq \omega$ and $\beta^* \subseteq \omega$. Let $(x, y) \in K$ and $x \neq y$. Then, there exists $n \geq 1$, such that $(x, y) \in K_{n+1}$, which means that there exists a scalar hyperproduct P_1, such that $x \in P_1$ and $P_1 \cap K_n(y) \neq \emptyset$. Let $x_1 \in P_1 \cap K_n(y)$. Then, $\{x, x_1\} \subseteq P_1$. Hence $(x, x_1) \in \beta$. Since $x_1 \in K_n(y)$ it follows that there exists a scalar hyperproduct P_2 such that $x_1 \in P_2$ and $P_2 \cap K_{n-1}(y) \neq \emptyset$. Let $x_2 \in P_2 \cap K_{n-1}(y)$. Then, $x_2 \in K_{n-1}(y)$ and $\{x_1, x_2\} \subseteq P_2$. After finite number of steps, we obtain there exists a scalar hyperoperation P_n such that $\{x_{n-1}, x_n\} \subseteq P_n$ and $x_n \in K_{n-(n-1)}(y) = \{y\}$. □

4. Fundamental, Noetherian and Artinian (Δ, G)-Sets

In this section, we introduce the notion of right Noetherian and Artinian (Δ, G)-sets and define fundamental (Δ, G)-sets.

Let X be a left (Δ, G)-set such that G be a Γ-semihypergroup and $\Gamma \subseteq \Delta$. We define a relation ρ on $\Delta \times X$ as follows:

$$((\delta_1, x_1), (\delta_2, x_2)) \in \rho \Leftrightarrow g\delta_1 x_1 = g\delta_2 x_2, \quad \text{for all } g \in G,$$

where $\delta_1, \delta_2 \in \Delta$ and $x_1, x_2 \in X$. Obviously, ρ is an equivalence.

Let $\Theta[X] = [\Delta \times X : \rho]$ denote the set of all equivalence classes. We denote the equivalence class (δ, x) by $[\delta, x]$. We define a relation ϵ on $\Gamma \times G$ as follows:

$$((\delta_1, g_1), (\delta_2, g_2)) \in \epsilon \Leftrightarrow g\delta_1 g_1 = g\delta_2 g_2, \quad \text{for all } g \in G,$$

where $g_1, g_2 \in G$ and $\delta_1, \delta_2 \in \Gamma$. Obviously, ϵ is an equivalence relation and $[\delta, g]$ denote the equivalence class containing (δ, g). We denote $\Theta[G] = \{[\delta, g] : g \in G, \delta \in \Gamma\}$. We define a hyperoperation $\circ$ on $\Theta[G]$ as follows:

$$[\delta_1, g_1] \circ [\delta_2, g_2] = \{[\delta_1, z] : z \in g_1\delta_2 g_2\},$$

where $\delta_1, \delta_2 \in \Delta$ and $g_1, g_2 \in G$. This hyperoperation is well-defined. Indeed, let $[\delta_1, g_1] = [\gamma_1, h_1]$ and $[\delta_2, g_2] = [\gamma_2, , h_2]$, where $\delta_1, \delta_2, \gamma_1, \gamma_2 \in \Gamma$ and $g_1, g_2, h_1, h_2 \in G$. Then,

$$g\delta_1 g_1 = g\gamma_1 h_1, \quad g\delta_2 g_2 = g\gamma_2 h_2, \quad \text{for all } g \in G.$$

Hence,

$$(g\delta_1 g_1)\delta_2 g_2 = (g\gamma_1 h_1)\gamma_2 h_2, \quad \text{for all } g \in G,$$

and

$$g\delta_1(g_1\delta_2 g_2) = g\gamma_1(h_1\gamma_2 h_2).$$

Thus,

$$[\delta_1, g_1] \circ [\delta_2, g_2] = [\gamma_1, h_1] \circ [\gamma_2, h_2].$$

Also

$$\begin{aligned}
([\delta_1, g_1] \circ [\delta_2, g_2]) \circ [\delta_3, g_3] &= (\{[\delta_1, z] : z \in g_1\delta_2 g_2\}) \circ [\delta_3, g_3] \\
&= \bigcup_{z \in g_1\delta_2 g_2} [\delta_1, z] \circ [\delta_3, x] \\
&= \bigcup_{z \in g_1\delta_2 g_2} \{[\delta_1, t] : t \in z\delta_3 g_3\} \\
&= \bigcup_{t \in (g_1\delta_2 g_2)\delta_3 g_3} [\delta_1, t] \\
&= \bigcup_{t \in g_1\delta_2(g_2\delta_3 g_3)} [\delta_1, t] \\
&= [\delta_1, g_1] \circ ([\delta_2, g_2] \circ [\delta_3, g_3]).
\end{aligned}$$

Therefore, $(\Theta[G], \circ)$ is a semihypergroup.

Let $\circ$ be a scalar hyperoperation $\circ : \Theta[G] \times \Theta[X] \to P^*(\Theta[X])$ such that

$$[\delta_1, g] \circ [\delta_2, x] = \{[\delta_1, z] : z \in g\delta_2 x\}.$$

This scalar hyperoperation is well-defined. Indeed, let $[\delta_1, g_1] = [\delta_2, g_2]$ and $[\delta_3, x_1] = [\delta_4, x_2]$ such that $g_1, g_2 \in G$, $\delta_1, \delta_2 \in \Delta$, $x_1, x_2 \in X$ and $\delta_3, \delta_4 \in \Delta$. Then,

$$g\delta_1 g_1 = g\delta_2 g_2, \quad g\delta_3 x_1 = g\delta_4 x_2, \quad \text{for all } g \in G.$$

This implies that $(g\delta_1 g_1)\delta_3 x_1 = (g\delta_2 g_2)\delta_4 x_2$. Hence,

$$[\delta_1, g_1] \circ [\delta_3, x_1] = [\delta_2, g_2] \circ [\delta_4, x_2].$$

Thus the scalar hyperoperation $\circ$ is well-defined. Let $[\delta_1, g_1], [\delta_2, g_2] \in \Theta[G]$ and $[\delta_3, x] \in \Theta[X]$, where $\delta_1, \delta_2 \in \Gamma$. Then,

$$\begin{aligned}([\delta_1, g_1] \circ [\delta_2, g_2]) \circ [\delta_3, x] &= (\{[\delta_1, z] : z \in g_1\delta_2 g_2\}) \circ [\delta_3, x] \\ &= \bigcup_{z \in g_1\delta_2 g_2} [\delta_1, z] \circ [\delta_3, x] \\ &= \bigcup_{z \in g_1\delta_2 g_2} \{[\delta_1, t] : t \in z\delta_3 x\} \\ &= \bigcup_{t \in (g_1\delta_2 g_2)\delta_3 x} [\delta_1, t] \\ &= \bigcup_{t \in g_1\delta_2 (g_2\delta_3 x)} [\delta_1, t] \\ &= [\delta_1, g_1] \circ ([\delta_2, g_2] \circ [\delta_3, x]).\end{aligned}$$

Therefore, $\Theta[X]$ is a left $\Theta[G]$-set and is called *fundamental left (Δ, G)-set.*

Let $\Theta[X]$ be a fundamental left (Δ, G)-set, $H \subseteq \Theta[X]$ and $T \subseteq X$. Then, we define

$$\begin{aligned}[H] =& \{x \in X : [\delta, x] \in H \text{ for all } \delta \in \Delta\}, \\ [[T]] =& \{[\delta, x] \in \Theta[X] : g\delta x \subseteq T \text{ for all } g \in G\}.\end{aligned}$$

A nonempty subset T of a left (Δ, G)-set X is called left (Δ, G)-subset of X when $G\Delta T \subseteq T$. A nonempty subset H of $\Theta[X]$ is called left $\Theta[G]$-subset if $\Theta[G] \circ H \subseteq H$.

Proposition 4.1. *Let X be a left (Δ, G)-set and $H \subseteq \Theta[X]$ be a complete part. Then, $[H]$ is a complete part of X.*

Proof. Suppose that

$$[H] \cap \prod_{i=1}^{n} g_i\delta_i x \neq \emptyset.$$

This implies that there exists $a \in X$ such that $a \in [H] \cap \prod_{i=1}^{n} g_i\delta_i x$. Then, for every $\delta \in \Delta$, $[\delta, a] \in H$. This implies that

$$[\delta, a] \in H \cap \prod_{i=1}^{n} [\delta, g_i] \circ [\delta_i, x].$$

Since $[H]$ is a complete part, $\prod_{i=1}^{n} [\delta, g_i] \circ [\delta_i, x] \subseteq H$. Then,

$$\left\{ b \in \prod_{i=1}^{n} g_i\delta_i x : \forall \delta \in \Delta, [\delta, b] \right\} \subseteq H.$$

Therefore, $[H]$ is a complete part. □

Proposition 4.2. *Let X be a left (Δ, G)-set and $T \subseteq X$ is a complete part. Then, $[[T]]$ is also a complete part of $\Theta[X]$.*

Proof. Suppose that

$$[[T]] \cap \prod_{i=1}^{n} [\delta_i, g_i] \circ [\delta, x] \neq \emptyset.$$

This implies that

$$\begin{aligned}
\left\{[\delta_1, z] : z \in \prod_{i=1}^{n} g_i\delta x\right\} \cap [[T]] \neq \emptyset &\Rightarrow \left(\exists z \in \prod_{i=1}^{n} g_i\delta x\right)\ [\delta_1, z] \in [[T]] \\
&\Rightarrow \left(\exists z \in \prod_{i=1}^{n} g_i\delta x\right) (\forall g \in G)\ \ g\delta_1 z \subseteq T \\
&\Rightarrow g\delta \prod_{i=1}^{n} g_i\delta x \cap T \neq \emptyset \\
&\Rightarrow (\forall g \in G)\ g\delta \prod_{i=1}^{n} g_i\delta x \subseteq T \\
&\Rightarrow \prod_{i=1}^{n}[\delta_i, g_i] \circ [\delta, x] \neq \emptyset \subseteq [[T]].
\end{aligned}$$

Therefore, $[[T]]$ is also complete part of $\Theta[X]$. □

Proposition 4.3. *Let X be a left (Δ, X)-set such that $T \subseteq X$. Then, $C[[T]] = [[C(T)]]$.*

Proof. Since $C(T)$ is a complete part by Proposition 4.2, $[[C(T)]]$ is also complete part of $\Theta[X]$. Also, $[[T]] \subseteq [[C(T)]]$. Let T_1 be a complete part contain $[[T]]$. Hence, $C[[T]] \subseteq T_1$. Thus, $[[C(T)]]$ is a smallest compte part contain $[[T]]$. Therefore, $C[[T]] = [[C(T)]]$. □

Theorem 4.1. *Let X be a left (Δ, G)-set and $\Theta[X]$ be a fundamental left (Δ, G)-set. Then,*

(i) *If H is a left $\Theta[G]$-subset of $\Theta[X]$, then $[H]$ is a left (Δ, G)-subset of X;*
(ii) *If T is a left (Δ, G)-subset of X, then $[[T]]$ is a left $\Theta[G]$ of $\Theta[X]$.*

Proof. (i) Suppose that $x \in [H]$. Then, for every $\delta \in \Delta$ we have $[\delta, x] \in H$. Since H is a left $\Theta[G]$-set of $\Theta[X]$, thus $[\delta_1, g] \circ [\delta, x] \subseteq H$. So $\{[\delta_1, t] : t \in g\delta x\} \subseteq H$. This implies that $g\delta x \subseteq [H]$. Therefore, $[H]$ is a left (Δ, G)-set of X.
(ii) Let $[\delta, x] \in [[T]]$ and $[\delta_1, g] \in \Theta[G]$. Then, for all $g \in G$, $g\delta x \subseteq T$. Now,

$$[\delta_1, g] \circ [\delta, x] = \{[\delta_1, t] : t \in g\delta x\} \subseteq [[T]].$$

Therefore, $[[T]]$ is a left $\Theta[G]$-subset of $\Theta[X]$. □

Let X be a left (Δ, G)-set and T be a nonempty subset of X. Then,

$$[[[T]]] = \{x \in X : \forall\delta \in \Delta, [\delta, x] \in [[T]]\} = \{x \in X : g\delta x \subseteq T \text{ for all } \delta \in \Delta, g \in G\}.$$

This implies that T is a left (Δ, G)-subset of $[[[T]]]$. Also, when $H \subseteq \Theta[X]$, we have

$$\begin{aligned}
[[[H]]] &= \{[\delta, x] \in \Theta[X] : g\delta x \subseteq [H] \text{ for all } g \in G\} \\
&= \{[\delta, x] \in \Theta[X] : [\delta_1, t] \in H \text{ for all } g \in G, \delta_1 \in \Delta, t \in g\delta x\}.
\end{aligned}$$

Let H be a left $\Theta[G]$-subset of $\Theta[X]$. Then, for every $\delta_1 \in \Gamma$, $g \in G$ and $[\delta, x] \in H$ we have
$$[\delta_1, g] \circ [\delta, x] = \{[\delta_1, t] : t \in g\delta x\} \subseteq H.$$
When H is a left $\Theta[G]$-subset of $\Theta[X]$, we have $H \subseteq [[[H]]]$.

Let X be a left (Δ, G)-set such that e_α is a unit element of G where $\alpha \in \Gamma$. Then,
$$[\delta, e_\alpha] \circ [\delta, x] = [\delta, e_\alpha \delta x] = [\delta, x].$$
This implies that $[\delta, e_\alpha]$ is a left unity of $\Theta[X]$.

Proposition 4.4. *Let X be a left (Δ, G)-set and T be a left (Δ, G)-subset of X. Then, $[[[T]]] = T$.*

Proof. The proof is straightforward. □

Definition 4.1. Let X be a left (Δ, G)-set. Then, X is said Noetherian, when X satisfies the ascending chain condition on left (Δ, G)-subsets and X is said Artinian when X satisfies the descending chain condition.

Theorem 4.2. *Let X be a left (Δ, G)-set such that $\Theta[X]$ is Noetherian (Artinian) $\Theta[G]$-set. Then, X is Noetherian left (Δ, G)-set.*

Proof. Suppose that $X_1 \subseteq X_2 \subseteq X_3 \subseteq \cdots \subseteq X_n \subseteq \cdots$ be an ascending chain of left (Δ, G)-set of X. Hence $[X_1] \subseteq [X_2] \subseteq [X_3] \subseteq \cdots \subseteq [X_n] \cdots$ is an ascending chain in $\Theta[X]$. Since $\Theta[X]$ is Noetherian thus there exists a positive integer n such that $[X_n] = [X_{n+k}]$ for every $k \in \mathbb{N}$. This implies that $X_n = [[[X_n]] = [[X_{n+k}]]] = X_{n+k}$ for every $k \in \mathbb{N}$. Therefore, X is Noetherian left (Δ, G)-set. In a same way, when X is Artinian left (Δ, G)-set, then $\Theta[X]$ is also $\Theta[G]$-set. □

Corollary 4.1. *Let X be a left (Δ, G)-set and $\Theta[X]$ is Artinian $\Theta[G]$-set. Then, X is Artinian left (Δ, G)-set.*

Definition 4.2. Let X be a left (Δ, G)-set and A be a nonempty subset of X. Then, intersection of all ideals of X containing A is a left (Δ, G)-set generated by A and denoted by $< A >$.

Proposition 4.5. *Let X be a left (Δ, G)-set and $A \subseteq X$. Then, $< A >= G\Delta A$.*

Proof. Suppose that $H = G\Delta A$. Obviously, $A \subseteq H$ and H is a left (Δ, G)-set of X. Indeed,
$$G\Delta H = G\Delta(G\Delta A) = (G\Gamma G)\Delta A \subseteq G\Delta A = H.$$
Let C be a left (Δ, G)-subset of X such that $A \subseteq C$. Then,
$$H = G\Delta A \subseteq G\Delta C \subseteq C.$$
Therefore, H is a smallest left (Δ, G)-set contain A and $H =< A >$. □

Let X be a left (Δ, G)-set and every nonempty of left (Δ, G)-subset of X partially ordered by inclusion has a maximal element. Then, we say that maximum condition holds for left (Δ, G)-sets.

Theorem 4.3. *Let X be a left (Δ, G)-set. Then, the following conditions are equivalent:*

(i) *X is Noetherian;*
(ii) *X satisfies the maximum condition for left (Δ, G)-sets;*
(iii) *every left (Δ, G)-subset of X is finitely generated.*

Proof. (i)⇒(ii) Suppose that Λ is a nonempty set of left (Δ, G)-subsets which has no maximal element. Let $\Lambda_1 \in \Lambda$. Then, there exists an element $\Lambda_2 \in \Lambda$ such that $\Lambda_1 \subset \Lambda_2$. Also, there exists an element $\Lambda_3 \in \Lambda$ such that $\Lambda_2 \subset \Lambda_3$. By continuing this process we have the accenting chain $\Lambda_1 \subset \Lambda_2 \subset \Lambda_3 \subset \cdots$. This is impossible.

(ii)⇒(iii) Let X_1 be a left (Δ, G)-set and $\Omega = \{< A >: A$ is a finite subset of $X_1\}$. By (ii), Ω has a maximal element $< A_0 >$. Now, if $x \in X_1$, then $< A_0 \cup \{x\} > \in \Omega$. By Maximality of $< A_0 >$ we have $x \in < A_0 >$. Therefore, X_1 is finite generated.

(iii)⇒(i) Suppose that $X_1 \subseteq X_2 \subseteq \cdots$ is a accenting chain of left (Δ, G)-sets and $T = \bigcup_{n \geq 1} X_n$. One can see that T is a left (Δ, G)-set of X. By (iii), T is finite generated. Then, there exist $x_1, x_2, \ldots, x_n \in X$ such that $T = < x_1, x_2, \ldots, x_n >$. Hence for $1 \leq k \leq n$ there exists X_k such that $x_k \in X_{i_k}$. We put $m := \max\{i_1, i_2, \ldots, i_n\}$. Hence, for every $t \geq m$ we have $I_m = I_t$. □

Theorem 4.4. *Let Ω be a partition (Δ, G)-set such that $\Omega = \bigcup_{t \in X} A_t$. Then, H is a left (Δ, G)-subset of X if and only if $\Omega_H = \bigcup_{t \in H} A_t$ is a left (Δ, G) of Ω.*

Proof. Suppose that H is a left (Δ, G)-set of X. Then,

$$G\widehat{\Delta}\Omega_H = G\widehat{\Delta} \bigcup_{t \in H} A_t = \bigcup_{t \in H} G\widehat{\Delta} A_t = \bigcup_{t \in G\Delta H} A_t \subseteq \bigcup_{t \in H} A_t = \Omega_H.$$

Hence Ω_H is a left (Δ, G)-subset of Ω.

Conversely, suppose that Ω_H is a left (Δ, G)-subset of Ω, $g \in G$, $\delta \in \Delta$ and $h \in H$. Choose $x \in A_h$. Since Ω_H is a left (Δ, G)-subset of Ω_H, we have

$$g\widehat{\delta}x = \{A_z : z \in g\delta h\} \subseteq \Omega_H.$$

Hence ,$g\delta h \subseteq H$. □

Corollary 4.2. *Let Ω be a partition (Δ, G)-set such that X is Noetherian (Artinian) (Δ, G)-set. Then, Ω is Noetherian (Artinian).*

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BEURLING'S THEOREM FOR THE Q-FOURIER-DUNKL TRANSFORM

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Abstract. The Q-Fourier-Dunkl transform satisfies some uncertainty principles in a similar way to the Euclidean Fourier transform. By using the heat kernel associated to the Q-Fourier-Dunkl operator, we establish an analogue of Beurling's theorem for the Q-Fourier-Dunkl transform $\mathcal{F}_Q$ on $\mathbb{R}$.

1. Introduction and Preliminaries

There are many known theorems which state that a function and its classical Fourier transform on $\mathbb{R}$ cannot both be sharply localized. That is, it is impossible for a nonzero function and its Fourier transform to be simultaneously small. This principle has several version which were proved by A. Beurling [3]. The Beurling theorem for the classical Fourier transform on $\mathbb{R}$ which was proved by L. Hörmander [5], says that for any non trivial function f in $L^2(\mathbb{R})$, the function $f(x)\mathcal{F}(y)$ is never integrable on $\mathbb{R}^2$ with respect to the measure $e^{|xy|}dxdy$. A far reaching generalization of this result has been recently proved in [4]. In this paper the author proved that a square integrable function f on $\mathbb{R}$ satisfying for an integer N

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}(y)|}{(1+|x|+|y|)^N} e^{|xy|}dxdy < \infty,$$

has the form $f(x) = P(x)e^{-rx^2}$, where P is a polynomial of degree strictly lower than $\frac{N-1}{2}$ and $r > 0$. Many authors have established the analogous of Beurling's theorem in other various setting of harmonic analysis (see for instance [1,6]). In this paper we study an analogue of Beurling's theorem, in the next we deduce an analogue of Gelfand-Shilov, for the Q-Fourier-Dunkl transform.

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The outline of the content of this paper is as follows. Section 2 is dedicated to some properties and results concerning the Q-Fourier-Dunkl transform. In Section 3 we give an analogue of Beurling's theorem and Gelfand-Shilov theorems for the Q-Fourier-Dunkl transform. Let us now be more precise and describe our results. To do so, we need to introduce some notations. Throughout this paper $\alpha > -\frac{1}{2}$,

- $Q(x) = \exp\left(-\int_0^x q(t)dt\right)$, $x \in \mathbb{R}$, where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$;
- $L^p_\alpha(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,\alpha} < \infty$, where

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p |x|^{2\alpha+1} dx\right)^{\frac{1}{p}}, \quad \text{if } p < \infty,$$

  and $\|f\|_{\infty,\alpha} = \|f\|_\infty = \operatorname{esssup}_{x\in\mathbb{R}} |f(x)|$.
- $L^p_Q(\mathbb{R})$ the class of measurable functions f on $\mathbb{R}$ for which $\|f\|_{p,Q} = \|Qf\|_{p,\alpha} < \infty$, where Q is given by $Q(x) = \exp\left(-\int_0^x q(t)dt\right)$, $x \in \mathbb{R}$.

We consider the first singular differential-difference operator Λ defined on $\mathbb{R}$

$$\Lambda f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x} + q(x)f(x),$$

where q is a $\mathcal{C}^\infty$ real-valued odd function on $\mathbb{R}$. For $q = 0$ we regain the Dunkl operator Λ_α associated with reflection group $\mathbb{Z}_2$ on $\mathbb{R}$ given by

$$\Lambda_\alpha f(x) = f'(x) + \left(\alpha + \frac{1}{2}\right)\frac{f(x) - f(-x)}{x}.$$

1.1. **Q-Fourier-Dunkl Transform.** The following statements are proved in [2].

Lemma 1.1. (a) *For each* $\lambda \in \mathbb{C}$, *the differential-difference equation*

$$\Lambda u = i\lambda u, \quad u(0) = 1,$$

admits a unique $\mathcal{C}^\infty$ *solution on* $\mathbb{R}$, *denoted by* Ψ_λ, *given by*

$$\Psi_\lambda(x) = Q(x)e_\alpha(i\lambda x),$$

where e_α *denotes the one-dimensional Dunkl kernel defined by*

$$e_\alpha(z) = j_\alpha(iz) + \frac{z}{2(\alpha+1)} j_{\alpha+1}(z), \quad z \in \mathbb{C},$$

and j_α *being the normalized spherical Bessel function of index* α *given by*

$$j_\alpha(z) = \Gamma(\alpha+1)\sum_{n=0}^{\infty} \frac{(-1)^n (\frac{z}{2})^{2n}}{n!\ \Gamma(n+\alpha+1)}, \quad z \in \mathbb{C}.$$

(b) *For all* $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ *and* $n = 0, 1, \ldots$, *we have*

$$\left|\frac{\partial^n}{\partial \lambda^n}\Psi_\lambda(x)\right| \leq Q(x)|x|^n e^{|\mathrm{Im}(\lambda)|\cdot|x|}.$$

In particular,

$$|\Psi_\lambda(x)| \leq Q(x)e^{|\mathrm{Im}(\lambda)|\cdot|x|}.$$

(c) *For all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, we have the Laplace type integral representation*

$$\Psi_\lambda(x) = a_\alpha Q(x) \int_{-1}^{1} (1-t^2)^{\alpha-\frac{1}{2}}(1+t)e^{i\lambda xt}dt,$$

where $a_\alpha = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$.

Definition 1.1. The Q-Fourier-Dunkl transform associated with Λ for a function in $L^1_Q(\mathbb{R})$ is defined by

$$\mathcal{F}_Q(f)(\lambda) = \int_{\mathbb{R}} f(x)\Psi_{-\lambda}(x)|x|^{2\alpha+1}dx.$$

Theorem 1.1. (a) *Let $f \in L^1_Q(\mathbb{R})$ such that $\mathcal{F}_Q(f) \in L^1_\alpha(\mathbb{R})$. Then for almost $x \in \mathbb{R}$ we have the inversion formula*

$$f(x)\,(Q(x))^2 = m_\alpha \int_{\mathbb{R}} \mathcal{F}_Q(f)(\lambda)\Psi_\lambda(x)|\lambda|^{2\alpha+1}d\lambda,$$

where

$$m_\alpha = \frac{1}{2^{2(\alpha+1)}(\Gamma(\alpha+1))^2}.$$

(b) *For every $f \in L^2_Q(\mathbb{R})$, we have the Plancherel formula*

$$\int_{\mathbb{R}} |f(x)|^2\,(Q(x))^2\,|x|^{2\alpha+1}dx = m_\alpha \int_{\mathbb{R}} |\mathcal{F}_Q(f)(\lambda)|^2|\lambda|^{2\alpha+1}d\lambda.$$

(c) *The Q-Fourier-Dunkl transform $\mathcal{F}_Q$ extends uniquely to an isomorphism from $L^2_Q(\mathbb{R})$ onto $L^2_\alpha(\mathbb{R})$.*

The heat kernel $N(x,s)$, $x \in \mathbb{R}$, $s > 0$, associated with the Q-Fourier-Dunkl transform is given by

$$N(x,s) = m_\alpha \frac{e^{-\frac{x^2}{4s}}}{(2s)^{\alpha+\frac{1}{2}}Q(x)}.$$

Some basic properties of $N(x,s)$ are the following:

- $N(x,s)Q^2(x) = m_\alpha \int_{\mathbb{R}} e^{-sy^2}\Psi_y(x)|y|^{2\alpha+1}dy;$
- $\mathcal{F}_Q(N(.,s))(x) = e^{-sx^2}$.

We define the heat functions W_l, $l \in \mathbb{N}$, as

$$Q^2(x)W_l(x,s) = \int_{\mathbb{R}} y^l e^{-\frac{y^2}{4s}}\Psi_y(x)|y|^{2\alpha+1}dy, \tag{1.1}$$

$$\mathcal{F}_Q(W_l(.,s)) = i^l y^l e^{-sy^2}. \tag{1.2}$$

The intertwining operators associated with a Q-Fourier-Dunkl transform on the real line is given by

$$X_Q(f)(x) = a_\alpha Q(x)\int_{-1}^{1} f(tx)(1-t^2)^{\alpha-\frac{1}{2}}dt,$$

its dual is given by

$${}^tX_Q(f)(y) = a_\alpha \int_{|x|\geq|y|} f(x)Q(x)\,\mathrm{sgn}(x)(x^2-y^2)^{\alpha-\frac{1}{2}}(x+y)dx. \tag{1.3}$$

Proposition 1.1. *If $f \in L^1_Q(\mathbb{R})$, then ${}^tX_Q(f) \in L^1(\mathbb{R})$ and $\|{}^tX_Q(f)\|_1 \le \|f\|_{1,Q}$.*

For every $f \in L^1_Q(\mathbb{R})$ we have

$$\mathcal{F}_Q = \mathcal{F} \circ^t X_Q(f), \tag{1.4}$$

where $\mathcal{F}$ is the usual Fourier transform defined by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}} f(x)e^{-i\lambda x}dx.$$

2. Beurling's Theorem for the Q-Fourier-Dunkl Transform

Theorem 2.1. *Let $N \in \mathbb{N}$ and $f \in L^2_Q(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|f(x)||\mathcal{F}_Q(f)(y)|Q(x)}{(1+|x|+|y|)^N} e^{|x||y|}|x|^{2\alpha+1}dxdy < \infty. \tag{2.1}$$

If $N > 1$, then $f(y) = \Sigma_{|s|<\frac{N-1}{2}} b_s W_s(r,y)$ a.e. where $r > 0$, $b_s \in \mathbb{C}$ and $W_s(r,\cdot)$ is given by (1.1). *Otherwise, $f(y) = 0$ a.e.*

Proof. We start with the following lemma.

Lemma 2.1. *We suppose that $f \in L^2_Q(\mathbb{R})$ satisfies* (2.1)*. Then $f \in L^1_Q(\mathbb{R})$.*

Proof. We may suppose that $f \neq 0$ in $L^2_Q(\mathbb{R})$. (2.1) and Fubini theorem imply that for almost every $y \in \mathbb{R}$,

$$\frac{|\mathcal{F}_Q(f)(y)|}{(1+|y|)^N}\int_{\mathbb{R}} \frac{Q(x)|f(x)|}{(1+|x|)^N} e^{|x||y|}|x|^{2\alpha+1}dx < \infty.$$

Since $\mathcal{F}_Q(f) \neq 0$, there exist $y_0 \in \mathbb{R}$, $y_0 \neq 0$, such that $\mathcal{F}_Q(f)(y_0) \neq 0$.

Therefore,

$$\int_{\mathbb{R}} \frac{Q(x)|f(x)|}{(1+|x|)^N} e^{|x||y_0|}|x|^{2\alpha+1}dx < \infty. \tag{2.2}$$

Since $\frac{e^{|x||y_0|}}{(1+|x|)^N} \geq 1$ for large $|x|$, it follows that $\int_{\mathbb{R}} Q(x)|f(x)||x|^{2\alpha+1}dx < \infty$. □

This Lemma and Proposition 1.1 imply that ${}^tX_Q(f)$ is well-defined almost everywhere on $\mathbb{R}$. We shall prove that we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{|{}^tX_Q(f)(x)||\mathcal{F}({}^tX_Q)(f)(y)|}{(1+|x|+|y|)^N} e^{|x||y|}dxdy < \infty. \tag{2.3}$$

Take y_0 as in Lemma 2.1, we write the above integral as a sum of the following integrals

$$I = \int_{\mathbb{R}}\int_{|y|\le|y_0|} \frac{e^{|x||y|}}{(1+|x|+|y|)^N}|{}^tX_Qf(x)||\mathcal{F}({}^tX_Q(f))(y)|dydx$$

and

$$J = \int_{\mathbb{R}}\int_{|y|\ge|y_0|} \frac{e^{|x||y|}}{(1+|x|+|y|)^N}|{}^tX_Qf(x)||\mathcal{F}({}^tX_Q(f))(y)|dydx.$$

We will prove that I and J are finite, which implies (2.3).

- As the functions $|\mathcal{F}_Q(f)(y)|$ is continuous in the compact $\{y \in \mathbb{R} \mid |y| \leq |y_0|\}$, so we get
$$I \leq C \int_{\mathbb{R}} \frac{e^{|x||y_0|}|{}^tX_Q f(x)|}{(1+|x|)^N} dx.$$
Writing the integral of the second member as $I_1 + I_2$ with
$$I_1 = \int_{|x| \leq \frac{N}{|y_0|}} \frac{e^{|x||y_0|}|{}^tX_Q f(x)|}{(1+|x|)^N} dx$$
and
$$I_2 = \int_{|x| \geq \frac{N}{|y_0|}} \frac{e^{|x||y_0|}|{}^tX_Q f(x)|}{(1+|x|)^N} dx.$$
Therefore, we have the following results.
  - As the function $x \to \frac{e^{|x||y_0|}}{(1+|x|)^N}$ is continuous in the compact $\left\{x \in \mathbb{R} \mid |x| \leq \frac{N}{|y_0|}\right\}$, and $f \in L^1_Q(\mathbb{R})$, we deduce by using proposition (1.1) that $|{}^tX_Q(f)|$ belongs to $L^1(\mathbb{R})$. Hence, I_1 is finite.
  - On the other hand, for $t > \frac{N}{|y_0|}$, the function $t \mapsto \frac{e^{t|y_0|}}{(1+t)^N}$ is increasing, so we obtain by using Proposition 1.1 that
$$I_2 \leq \int_{\mathbb{R}} \frac{Q(\xi)e^{|\xi||y_0|}}{(1+|\xi|)^N} |f(\xi)||\xi|^{2\alpha+1} d\xi.$$
The inequality (2.2) assert that I_2 is finite. This proves that I is finite.
- We suppose $|y_0| \leq N$. Then $J = J_1 + J_2 + J_3$, with
$$J_1 = \int_{|x| \leq \frac{N}{|y_0|}} \int_{|y_0| \leq |y| \leq N} \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)| dy dx,$$
$$J_2 = \int_{|x| \geq \frac{N}{|y_0|}} \int_{|y_0| \leq |y| \leq N} \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)| dy dx,$$
$$J_3 = \int_{\mathbb{R}} \int_{|y| \geq N} \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |{}^tX_Q(f)(x)||\mathcal{F}_Q(f)(y)| dy dx.$$
  - As the function $(x, y) \mapsto \frac{e^{|x||y|}}{(1+|x|+|y|)^N} |\mathcal{F}_Q(f)(y)|$ is bounded in the compact $\left\{x \in \mathbb{R} \mid |x| \leq \frac{N}{|y_0|}\right\} \times \{\xi \in \mathbb{R} \mid |y_0| \leq |\xi| \leq N\}$ and ${}^tX_Q(|f|)(x)$ is Lebesgue-integrable on $\mathbb{R}$, then J_1 is finite.
  - Let $\lambda > 0$. As the function $t \mapsto \frac{e^{\lambda t}}{(1+t+\lambda)^N}$ is increasing for $t > \frac{N}{\lambda}$. Thus, for all $(x, y) \in C(\xi, y_0, N)$ we have the inequality
$$\frac{e^{|x||y|}}{(1+|x|+|y|)^N} \leq \frac{e^{|\xi||y|}}{(1+|\xi|+|y|)^N},$$
with
$$C(\xi, y_0, N) = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \;\middle|\; \frac{N}{|y_0|} \leq |x| \leq |\xi| \text{ et } |y_0| \leq |y| \leq N\right\}.$$

Therefore, from Fubini-Tonelli's theorem and Proposition 1.1 we get

$$J_2 \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |Q(\xi)f(\xi)||\mathcal{F}_Q(f)(y)| \frac{e^{|\xi||y|}}{(1+|\xi|+|y|)^N} |\xi|^{2\alpha+1} d\xi dy.$$

Taking account of the condition (2.1), we deduce that J_2 is finite.

– For $|y| > N$, the function $t \mapsto \frac{e^{t|y|}}{(1+t+|y|)^N}$ is increasing. We deduce, by using Fubini-Tonelli's theorem and Proposition 1.1, that

$$J_3 \leq \int_{\mathbb{R}} \int_{|y|\geq N} |(f)(\xi)||F_Q(f)(y)| \frac{e^{|\xi||y|}}{(1+|\xi|+|y|)^N} dy |\xi|^{2\alpha+1} d\xi < +\infty.$$

This implies that J_3 is finite. Finally for $|y_0| > N$, we have $J \leq J_3 < \infty$. This completes the proof of the relation (2.3).

According to Corollary 3.1, ii) of [4], we conclude that

$${}^tX_Q(f)(x) = R(x)e^{-\delta x^2}, \quad \text{for all } x \in \mathbb{R},$$

with $\delta > 0$ and R a polynomial of degree strictly lower than $\frac{N-1}{2}$.

Using this relation and (1.4), we deduce that

$$\mathcal{F}_Q(f)(y) = \mathcal{F} \circ^t X_Q(f)(y) = \mathcal{F}(R(x)e^{-\delta x^2})(y), \quad \text{for all } x \in \mathbb{R},$$

but

$$\mathcal{F}(P(x)e^{-\delta x^2})(y) = S(y)e^{\frac{-y^2}{4\delta}}, \quad \text{for all } x \in \mathbb{R},$$

with S a polynomial of degree strictly lower than $\frac{N-1}{2}$.

Thus from (1.2) we obtain

$$\mathcal{F}_Q(f)(y) = \mathcal{F}_Q\left(\sum_{|s|<\frac{N-1}{2}} b_s W_s \left(\frac{1}{4\delta}, \cdot \right) \right)(y), \quad \text{for all } x \in \mathbb{R}.$$

The injectivity of the transform $\mathcal{F}_Q$ implies

$$f(x) = \sum_{|s|<\frac{N-1}{2}} b_s W_s \left(\frac{1}{4\delta}, \cdot \right)(x) \text{ a.e,} \quad \text{for all } x \in \mathbb{R},$$

and the theorem is proved. □

As an application of Beurling's Theorem, we can deduce a Gelfand-Shilov type theorem for the Q-Fourier-Dunkl transform.

Theorem 2.2. *Let $N \in \mathbb{N}$, $a, b > 0$ and $1 < p, q < \infty$, with $\frac{1}{p} + \frac{1}{q} = 1$ and let $f \in L^2_Q(\mathbb{R})$ satisfy*

$$\int_{\mathbb{R}} \frac{Q(x)|f(x)|e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} |x|^{2\alpha+1} dx < \infty \tag{2.4}$$

and

$$\int_{\mathbb{R}} \frac{|\mathcal{F}_Q(f)(y)| e^{\frac{(2b)^q}{q}|y|^q}}{(1+|y|)^N} dy < \infty. \tag{2.5}$$

If $ab > \frac{1}{4}$ or $(p, q) \neq (2, 2)$, then $f(x) = 0$ a.e. If $ab = \frac{1}{4}$ and $(p, q) = (2, 2)$, then $f(x) = \Sigma_{|s| < \frac{N-1}{2}} b_s W_s(r, \cdot))(x)$, whenever $N > 1$ and $r = 2b^2$. Otherwise, $f(x) = 0$ a.e.

Proof. Since

$$4ab|x||y| \leq \frac{(2a)^p}{p}|x|^p + \frac{(2b)^q}{q}|y|^q,$$

it follows from (2.4) and (2.5) that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{Q(x)|f(x)||\mathcal{F}_Q(f)(y)|}{(1+|x|+|y|)^{2N}} e^{4ab|x||y|} |x|^{2\alpha+1} dxdy < \infty.$$

Then (2.1) is satisfied, because $4ab \leq 1$. Especially, according to the proof of Theorem 2.1, we can deduce that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|{}^t X_Q(f)(x)||\mathcal{F}_Q(f)(y)|}{(1+|x|+|y|)^{2N}} e^{4ab|x||y|} dxdy < \infty,$$

and ${}^t X_Q(f)$ and f are of the forms ${}^t X_Q(f) = R(x) e^{-\frac{x^2}{4r}}$ and $\mathcal{F}_Q(f)(y) = S(y) e^{-ry^2}$, where $r > 0$ and S, R are polynomials of the same degree strictly lower than $\frac{2N-1}{2}$. Therefore, substituting these, we can deduce that

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{e^{-(\sqrt{r}|y| - \frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1)|x||y|} R(x) S(y)}{(1+|x|+|y|)^{2N}} e^{4ab|x||y|} dxdy < \infty. \tag{2.6}$$

When $4ab > 1$, this integral is not finite unless $f = 0$ almost everywhere. Indeed, as $ab > \frac{1}{4}$, there exists $\varepsilon > 0$ such that $4ab - 1 - \varepsilon > 0$. If R is non null, S is also non null and we have

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|R(x)||S(y)|}{(1+|x|+|y|)^{2N}} e^{-(\sqrt{r}|y| - \frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1)|x||y|} dxdy \\ \geq &C \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-(\sqrt{r}|y| - \frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1-\varepsilon)|x||y|} dxdy, \end{aligned}$$

where C is a positive constant. But the function

$$e^{-(\sqrt{r}|y| - \frac{1}{2\sqrt{r}}|x|)^2} e^{(4ab-1-\varepsilon)|x||y|}$$

is not integrable, (2.6) does not hold. Hence, $f(x) = 0$ a.e.

Moreover, it follows from (2.4) and (2.5) that

$$\int_{\mathbb{R}} \frac{|f(x)| Q(x) e^{\frac{(2a)^p}{p}|x|^p}}{(1+|x|)^N} |x|^{2\alpha+1} dx = \int_{\mathbb{R}} \frac{e^{-\frac{1}{4}x^2} e^{\frac{(2a)^p}{p}|x|^p} R(x) Q(x)}{(1+|x|)^N} |x|^{2\alpha+1} dx < \infty \tag{2.7}$$

and

$$\int_{\mathbb{R}} \frac{|\mathcal{F}_Q(f)(y)| e^{\frac{(2b)^q}{q}|y|^q}}{(1+|y|)^N} dy = \int_{\mathbb{R}} \frac{e^{-ry^2} e^{\frac{(2b)^q}{q}|y|^q} S(y)}{(1+|y|)^N} dy < \infty. \tag{2.8}$$

Hence, one of these integrals is not finite unless $(p, q) = (2, 2)$. When $4ab = 1$ and $(p, q) = (2, 2)$, the finiteness of above integrals implies that $r = 2b^2$ and the rest follows from Theorem 2.1. □

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ρ-ATTRACTIVE ELEMENTS IN MODULAR FUNCTION SPACES

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Abstract. In this paper, we introduce the notion of ρ-attractive elements in modular function spaces. A new class of mappings called ρ-k-nonspreading mappings is also introduced. Making a good use of the two notions, we first prove existence results and then some approximation results in the setup of modular function spaces. An example is presented to support the results proved herein.

1. Introduction and preliminaries

The notion of attractive points of nonlinear mappings in Hilbert spaces was coined by Takahashi and Takeuchi [16] in 2011.

Let E be a nonempty subset of a Hilbert space H and $T : E \to H$ then the set of attractive points $A(T)$ is given by,

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\| \text{ for all } x \in E\}.$$

They proved an existence result on attractive points for the so-called hybrid mappings in a Hilbert space. They went on to prove a weak convergence theorem of Mann-type without closedness.

Motivated by the idea of Takahashi et al. [17], study of attractive points gained momentum. Several different classes of mappings were introduced. Kohsaka et al. [8] presented a new class of mappings called nonspreading mappings.

A mapping $T : E \to E$ is said to be nonspreading mapping if for any $x, y \in E$,

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|Tx - y\|^2 .$$

Suantai et al. [15], using Hausdorff metric, introduced the class of generalized nonspreading mappings, known as k-nonspreading multivalued mappings. Kaewkhao et

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al. [1] studied the attractive points and convergence theorems for normally generalized hybrid mappings in $CAT(0)$ spaces in 2015.

In the same year, Zheng [18] proved strong and weak convergence theorem of the Ishikawa iteration for an (α,β)-generalized hybrid mapping in a uniformly convex Banach space. Kunwai et al. [5] proved an attractive point theorem for normally generalized hybrid mappings in $CAT(0)$ spaces under certain conditions. Recently, fixed point theory in modular function spaces has gained interest of many mathematicians. The idea of modular function spaces was established by Nakano in [13] and was improved and generalized by Musielak and Orlicz [12]. Later on, Khamsi et al. [11] introduced the fixed point theory in modular function spaces and proved Banach contraction principle in modular function spaces (also see [6]). Kuaket and Kumam [10] established some fixed point for generalized contraction mappings in modular function spaces. Dehaish and Kozolwoski [2], proved results on approximating fixed points in modular function spaces for the first time. Recently, Khan et al. [14] successfully handled the problem of approximating fixed points for multivalued ρ-quasi nonexpansive mappings in modular function spaces. Ilchev and Zlatanov [3] presented some sufficient conditions for the existence and uniqueness of best proximity points and fixed points for cyclic Kannan maps in modular function spaces. For further discussion in modular spaces see [4, 9, 19].

The above efforts stimulate us to define attractive elements in the setting of modular function spaces. Another purpose of this paper is to define a class of $\rho-k$-nonspreading mappings. This will lead us proving existence and approximation results for attractive elements in modular function spaces. Towards the end of this paper, our results will be vindicated using some examples.

Let us recall some basic definitions and notions which can be found in [7]. Let Ω be a nonempty set and Σ be a nontrivial σ-algebra of subsets of Ω. Let P be a nontrivial δ-ring of subsets of Ω which means that P is closed with respect to forming of countable intersections, and finite unions and differences. Assume further that $E \cap A \in P$ for any $E \in P$ and $A \in \sum$. Let us assume that there exists an increasing sequence of sets $K_n \in P$ such that $\Omega = \bigcup K_n$. By $\mathscr{E}$ we denote the linear space of all simple functions with supports from P. $\mathcal{M}_\infty$ represents the space of all extended measurable functions, that is, all functions $f : \Omega \to [-\infty, \infty]$ such that there exists a sequence $\{g_n\} \subset \mathscr{E}$, $|g_n| \leq |f|$ and $g_n(\omega) \to f(\omega)$ for all $\omega \in \Omega$.

Definition 1.1. Let $\rho : \mathcal{M}_\infty \to [0, \infty]$ be a nontrivial, convex, and even function. We say that ρ is a regular convex function pseudomodular if

(a) $\rho(0) = 0$;
(b) ρ is monotone, i.e., $|f(\omega)| \leq |g(\omega)|$ for any $\omega \in \Omega$ implies $\rho(f) \leq \rho(g)$, where $f, g \in \mathcal{M}_\infty$;
(c) ρ is orthogonally subadditive, i.e., $\rho(f1_{A\cup B}) \leq \rho(f1_A) + \rho(f1_B)$ for any $A, B \in \Sigma$ such that $A \cup B \neq \phi$, $f \in \mathcal{M}_\infty$;

(d) ρ has Fatou property, that is, $|f_n(\omega)| \uparrow |f(\omega)|$ for all $\omega \in \Omega$ implies $\rho(f_n) \uparrow \rho(f)$, where $f \in \mathcal{M}_\infty$;
(e) ρ is order continuous in $\mathscr{E}$, i.e., $g_n \in \mathscr{E}$, and $|g_n(\omega)| \downarrow 0$ implies $\rho(g_n) \downarrow 0$.

We say that a set $A \in \Sigma$ is ρ-null if $\rho(g1_A) = 0$ for every $g \in \mathscr{E}$. A property holds ρ-almost everywhere (ρ-a.e.) if the set $\{\omega \in \Omega : p(\omega)$ does not hold$\}$ is ρ-null. We identify any pair of measurable sets whose symmetric difference is ρ-null as well as any pair of measurable functions differing only on a ρ-null set. With this in mind we define $\mathcal{M} = \{f \in \mathcal{M}_\infty : |f(\omega)| < \infty \ \rho$–a.e.$\}$ where each $f \in \mathcal{M}$ is actually an equivalence class of functions equal ρ–a.e. rather than an individual function.

Definition 1.2. Let ρ be a regular convex function pseudomodular. Then, we say that ρ is a regular convex function modular if $\rho(f) = 0$ implies that $f = 0 \ \rho$ – a.e.

The class of all nonzero regular convex function modular defined on Ω is denoted by $\Re$.

Definition 1.3. The convex function modular ρ defines the modular function space $\mathbb{L}_\rho$ as

$$\mathbb{L}_\rho = \{f \in \mathcal{M}_\infty : \rho(\lambda f) \to 0 \text{ as } \lambda \to 0\}.$$

Generally, the modular ρ is not subadditive and hence doesn't behave like a norm. However, the modular space $\mathbb{L}_\rho$ can be equipped with an F-norm defined by

$$\|f\|_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le \alpha\right\}.$$

If ρ is a convex modular,

$$||f||_\rho = \inf\left\{\alpha > 0 : \rho\left(\frac{f}{\alpha}\right) \le 1\right\}$$

defines a norm on the modular space $\mathbb{L}_\rho$, and is called the Luxemburg norm. The following definitions will be needed in this paper.

Definition 1.4. Let $\mathbb{L}_\rho$ be a modular space. Then
(a) the sequence $\{f_n\} \subset \mathbb{L}_\rho$ is said to be ρ-convergent to $f \in \mathbb{L}_\rho$ if $\rho(f_n - f) \to 0$ as $n \to \infty$;
(b) the sequence $\{f_n\} \subset \mathbb{L}_\rho$ is said to be ρ-Cauchy if $\rho(f_n - f_m) \to 0$ as n and m approach ∞;
(c) we say that $\mathbb{L}_\rho$ is ρ-complete if and only if any ρ-Cauchy sequence in $\mathbb{L}_\rho$ is ρ-convergent.

Definition 1.5. A subset E of $\mathbb{L}_\rho$ is called
(a) ρ-closed if the ρ-limit of a ρ-convergent sequence of E always belongs to E;
(b) ρ-compact if every sequence in E has a ρ-convergent subsequence in E;
(c) ρ-bounded if $\delta_\rho(E) = \sup\{\rho(f - g) : f, g \in E\} < \infty$;

(d) the ρ-distance between f and E is defined as:

$$d_\rho(f, E) = \inf\{\rho(f - j) : j \in E\}.$$

The terminology defined for ρ is similar to metric spaces but ρ does not satisfy triangle inequality. Hence, if a sequence in $\mathbb{L}_\rho$ is ρ-convergent it does not imply ρ-Cauchy. This is only true if and only if ρ satisfies Δ_2-condition.

Definition 1.6. The modular function ρ is said to satisfy the Δ_2-condition if $\rho(2f_n) \to 0$ as n approaches ∞, whenever $\rho(f_n) \to 0$ as n approaches ∞.

The modular ρ satisfies some uniform convexity type properties. A few of those are given below which can be found in [7].

Definition 1.7. Let $\rho \in \Re$.

(a) Let $r > 0$, $\epsilon > 0$. Define,

$$D_1(r, \epsilon) = \{(f, h) : f, h \in \mathbb{L}_p, \rho(f) \le r, \rho(h) \le r, \rho(f - h) \ge \epsilon r\}.$$

Let

$$\delta_1(r, \epsilon) = \inf\left\{1 - \frac{1}{r}\rho\left(\frac{f+h}{2}\right) : (f, h) \in D_1(r, \epsilon)\right\}, \quad \text{if } D_1(r, \epsilon) \ne \phi,$$

and $\delta_1(r, \epsilon) = 1$ if $D_1(r, \epsilon) = \phi$. We say that ρ satisfies $(UC1)$ if for every $r > 0$, $\epsilon > 0$, $\delta_1(r, \epsilon) > 0$. Note that for every $r > 0$, $D_1(r, \epsilon) \ne \phi$ for every $\epsilon > 0$ small enough.

(b) We say that ρ satisfies $(UUC1)$ if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_1(s, \epsilon) > 0$ depending only upon s and ϵ such that $\delta_1(r, \epsilon) > \eta_1(s, \epsilon) > 0$ for any $r > s$.

(c) We say that ρ satisfies $(UUC2)$ if for every $s \ge 0$, $\epsilon > 0$, there exists $\eta_2(s, \epsilon) > 0$ depending upon s and ϵ such that $\delta_2(r, \epsilon) > \eta_2(s, \epsilon) > 0$ for any $r > s$.

Note that $(UC1)$ implies $(UUC1)$ and $(UCC1)$ implies $(UUC2)$. If $\rho \in \Re$ satisfies Δ_2, then $(UUC2)$ and $(UCC1)$ are equivalent (see [9]).

Definition 1.8. We will say that ρ is uniformly continuous if for every $\epsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that

$$|\rho(g) - \rho(g + h)| < \epsilon \quad \text{if} \quad \rho(h) \le \delta, \rho(g) \le R.$$

A sequence $\{t_n\} \subset (0, 1)$ is called bounded away from 0 if there exists $a > 0$ such that $t_n \ge a$ for every $n \in \mathbb{N}$. Similarly, $\{t_n\} \subset (0, 1)$ is called bounded away from 1 if there exists $b < 1$ such that $t_n \le b$ for every $n \in \mathbb{N}$. The following lemma helpful in studying the convergence of fixed points as well as attractive elements in the $(UUC1)$ modular function spaces.

Lemma 1.1. *Let $\rho \in \Re$ satisfy $(UUC1)$ and let $\{t_n\} \subset (0, 1)$ be bounded away from 0 and 1. If there exists $R \ge 0$ such that*

$$\limsup_{n\to\infty} \rho(f_n) \le R, \quad \limsup_{n\to\infty} \rho(g_n) \le R \quad \textit{and} \quad \lim_{n\to\infty} \rho(t_n f_n + (1 - t_n)g_n) = R,$$

then $\lim_{n\to\infty} \rho(f_n - g_n) = 0$.

Since the modular function space doesn't satisfy the triangle inequality so, the following theorem is useful.

Theorem 1.1. *Let $\rho \in \Re$ satisfy Δ_2-condition. Let $\{f_n\}$ and $\{g_n\}$ be two sequences in $\mathbb{L}_\rho$. Then,*

$$\lim_{n\to\infty} \rho(g_n) = 0 \quad \textit{implies} \quad \limsup_{n\to\infty} \rho(f_n + g_n) = \lim_{n\to\infty} \rho(f_n)$$

and

$$\lim_{n\to\infty} \rho(g_n) = 0 \quad \textit{implies} \quad \liminf_{n\to\infty} \rho(f_n + g_n) = \lim_{n\to\infty} \rho(f_n).$$

The notion of a ρ-type is a powerful tool that will be used in our result.

Definition 1.9. Let $E \subset \mathbb{L}_p$ be convex and ρ-bounded. A function $\tau : E \to [0, \infty]$ is called a ρ-type (or shortly a type) if there exists a sequence $\{g_k\}$ of elements of E such that for any $f \in E$ there holds $\tau(f) = \limsup_{k\to\infty} \rho(g_k - f)$.

The following lemma [7] establishes an important minimizing sequence property of uniformly convex modular function spaces which is used proving existence of fixed points.

Lemma 1.2. *Assume that $\rho \in \Re$ is $(UUC1)$. Let E be a ρ-closed ρ-bounded convex nonempty subset of $\mathbb{L}_\rho$. Let τ be a ρ-type defined on E. Then, any minimizing sequence of τ is ρ-convergent. Its ρ-limit is independent of the minimizing sequence.*

Since our goal is to prove existence of attractive elements without the condition of ρ-closedness. The following is the modified version of the above lemma which can be proved exactly on lines of [7].

Lemma 1.3. *Assume that $\rho \in \Re$ is $(UUC1)$. Let E be a ρ-bounded convex nonempty subset of $\mathbb{L}_\rho$. Let τ be a ρ-type defined on E. Then, any minimizing sequence of τ is ρ-convergent in $\mathbb{L}_\rho$. Its ρ-limit is independent of the minimizing sequence.*

Definition 1.10. Let $\rho \in \Re$. The growth function ω_ρ of a function modular ρ is defined as:

$$\omega_\rho(\beta) = \sup\left\{\frac{\rho(\beta f)}{\rho(f)}, 0 \le \rho(f) < \infty\right\}, \quad \text{for all } 0 \le \beta < \infty.$$

Notice that whenever $\beta \in [0, 1]$, $\omega_\rho(\beta) \le 1$.

Let $T : E \to E$ be a mapping then a point $x \in E$ is said to be a fixed point of T if $x = Tx$. We denote the set of fixed points by $F(T)$. A mapping T is said to be

(a) ρ-nonexpansive if $\rho(Tf - Tg) \le \rho(f - g)$ for all $f, g \in E$;
(b) ρ-quasi-nonexpansive mapping if $\rho(Tf - g) \le \rho(f - g)$ for all $f \in E$ and $g \in F(T)$.

2. Main Results

In this section, we introduce a new class of $\rho-k$-nonspreading mappings and present the concept ρ-attractive elements. Then, we prove an existence and some convergence results.

Definition 2.1. Let $\rho \in \Re$. Let $T : E \to \mathbb{L}_\rho$ then T is a $\rho - k$-nonspreading mapping if there exists a $k > 0$ such that

$$\rho^2(Tf - Tg) \leq k(\rho^2(f - Tg) + \rho^2(Tf - g)),$$

for all $f, g \in E$.

A $\rho - \frac{1}{2}$-nonspreading mapping with $F(T) \neq \phi$ is ρ-quasi nonexpansive. In fact, if g is a fixed point of T, then in Definition 2.1, with $k = \frac{1}{2}$, we have

$$2\rho^2(g - Tf) \leq \rho^2(g - f) + \rho^2(g - Tf),$$

and, hence

$$\rho^2(Tf - g) \leq \rho^2(f - g).$$

This implies,

$$\rho(Tf - g) \leq \rho(f - g).$$

Now, we give an example of a $\rho - k-$nonspreading mapping which is not a ρ-nonexpansive mapping.

Example 2.1. Let the real number system $\mathbb{R}$ be the space modulared as $\rho(f) = |f|^k$ for $k \geq 1$. Let $E = \{f \in \mathbb{L}_\rho : -3 < f < 2\}$ and

$$Tf = \begin{cases} \dfrac{|f| - 1}{2}, & -2 < f < 2, \\ \dfrac{-|f|}{|f| + 1}, & -3 < f \leq -2. \end{cases}$$

It is easy to see that T is a $\rho - \frac{1}{2}$-nonspreading mapping. However, T is not a ρ-nonexpansive mapping since if $f = -2$ and $g = -1.5$, then

$$\rho(Tf - Tg) = \left|\frac{-2}{3} - \frac{1}{4}\right|^k = \left|\frac{11}{12}\right|^k > \rho(f - g) = |-2 + 1.5|^k = \left|\frac{1}{2}\right|^k.$$

Definition 2.2. Let ρ be a convex function modular. Let E be a nonempty subset of $\mathbb{L}_\rho$ and $T : E \to E$ be a mapping then a function $g \in \mathbb{L}_\rho$ is called a $\rho-$attractive element of T if for all $f \in E$, we have $\rho(Tf - g) \leq \rho(f - g)$. Let $A_\rho(T)$ denote the set of ρ-attractive elements, i.e., $A_\rho(T) = \{g \in \mathbb{L}_\rho : \rho(Tf - g) \leq \rho(f - g) \text{ for all } f \in E\}$.

First of all, we will give some useful properties of $A_\rho(T)$.

Lemma 2.1. *Let $\rho \in \Re$ and be uniformly continuous. Let E be a nonempty subset of $\mathbb{L}_\rho$ and $T : E \to \mathbb{L}_\rho$, with $A_\rho(T) \neq \phi$. Then $A_\rho(T)$ is closed.*

Proof. Let $\{g_n\} \subset A_\rho(T)$ such that $\lim_{n\to\infty} \rho(g_n - g) = 0$. Then for any $f \in E$, we have

$$\rho(Tf - g) = \rho((Tf - g_n) - (g - g_n)). \tag{2.1}$$

Then taking limit as $n \to \infty$ in (2.1) and using the uniform continuity of ρ, we get

$$\rho(Tf - g) \le \lim_{n\to\infty} \rho(Tf - g_n) \le \lim_{n\to\infty} \rho(f - g_n) = \rho(f - g).$$

This shows $g \in A_\rho(T)$. Hence, $A_\rho(T)$ is closed. $\square$

An attractive point need not be a fixed point. However, if a mapping $T : E \to E$ is ρ-quasi nonexpansive then the ρ-attractive elements lying in E are also its fixed points.

Lemma 2.2. *Let $\rho \in \Re$. Let E be a nonempty subset of $\mathbb{L}_\rho$ and $T : E \to \mathbb{L}_\rho$ be a ρ-quasi nonexpansive mapping. Then $A_\rho(T) \cap E = F(T)$.*

Proof. Let $g \in A_\rho(T) \cap E$, then $\rho(Tf - g) \le \rho(f - g)$ for all $f \in E$. In particular, let $f = g \in E$, then we have $\rho(Tf - f) \le \rho(f - f) = \rho(0) = 0$. Hence, $Tf = f$ showing $f \in F(T)$. Conversely, since T is ρ-quasi nonexpansive, then for any $h \in F(T)$ and $f \in E$, we get $\rho(Tf - h) \le \rho(f - h)$. Then, clearly $h \in A_\rho(T)$. $\square$

Now, we will prove existence of a ρ-attractive point for $\rho - k$-nonspreading mapping for $k \in (0, \frac{1}{2}]$.

Theorem 2.1. *Assume that $\mathbb{L}_\rho$ is complete, $\rho \in \Re$ is $(UUC1)$ and uniformly continuous. Let E be a nonempty ρ-bounded convex subset of $\mathbb{L}_\rho$. Let $T : E \to E$ be a $\rho - k$-nonspreading mapping with $k \in (0, \frac{1}{2}]$. Then T has a ρ-attractive point.*

Proof. Let $\{f_0\} \in E$. Define the ρ-type, $\tau : E \to [0, \infty]$ by

$$\tau(f) = \limsup_{n\to\infty} \rho(f - T^n(f_0)).$$

Then by Lemma 1.3, there exists a minimizing sequence, say, $\{g_n\}$, of τ such that $\tau(g_n) = \inf_{f\in E} \tau(f)$. Since $\{T^n(f_0)\} \subset E$ and E is ρ-bounded we have

$$\tau(f) \le \delta_\rho(E) < \infty, \quad \text{for every } f \in E,$$

and

$$\tau(Tf) = \limsup_{n\to\infty} \rho(Tf - T^n(f_0)).$$

Now,

$$\rho^2(T^n(f_0) - Tf) \le k(\rho^2(Tf - T^{n-1}(f_0)) + \rho^2(f - T^n(f_0))).$$

Taking $n \to \infty$ implies,

$$\limsup_{n\to\infty} \rho^2(T^n(f_0) - Tf) \le k(\limsup_{n\to\infty}(\rho^2(Tf - T^{n-1}(f_0)) + \limsup_{n\to\infty} \rho^2(T^n(f_0) - f)).$$

Thus we have

$$\tau^2(Tf) \le k\tau^2(Tf) + k\tau^2(f),$$

which implies

$$\tau^2(Tf) \le \frac{k}{1-k}\tau^2(f).$$

Since $\frac{k}{1-k} < 1$, we obtain $\tau(Tf) \le \tau(f)$. Thus, $\tau(Tg_n) \le \tau(g_n)$. Hence, $\{T(g_n)\}$ is also a minimizing sequence of τ.

Again, according to Lemma 1.3, $\{g_n\}$ converges to some g in $\mathbb{L}_\rho$ and if there is any other minimizing sequence it also converges to g then $\lim_{n\to\infty} Tg_n = g$. Next, we show that g is the ρ-attractive element of T.

From Definition 2.1 and uniform continuity of ρ, we have

$$\lim_{n\to\infty} \rho^2(Tg_n - Tf) \le k \lim_{n\to\infty} \rho^2(Tf - g_n) + k \lim_{n\to\infty} \rho^2(f - Tg_n).$$

Therefore,

$$\rho^2(g - Tf) \le k\rho^2(Tf - g) + k\rho^2(f - g),$$

which implies

$$(1-k)\rho^2(g - Tf) \le k\rho^2(f - g).$$

Consequently,

$$\rho(Tf - g) \le \rho(f - g).$$

Hence, g is a ρ-attractive element of T. □

As an immediate consequence of Theorem 2.1, we obtain the next result.

Theorem 2.2. *Assume that $\mathbb{L}_\rho$ is complete, $\rho \in \Re$ is $(UUC1)$ and uniformly continuous. Let E be a nonempty ρ-bounded, ρ-closed and convex subset of $\mathbb{L}_\rho$. Let $T : E \to E$ be a $\rho - k$-nonspreading mapping with $k \in \left(0, \frac{1}{2}\right]$. Then T has a fixed point.*

Theorem 2.3. *Let $\rho \in \Re$ satisfy $(UUC2)$ and Δ_2-condition. Let E be a nonempty convex subset of $\mathbb{L}_\rho$ and $T : E \to \mathbb{L}_\rho$ be a $\rho - k$-nonspreading mapping with $k \in (0, \frac{1}{2}]$. Suppose $A_\rho(T)$ is nonempty and let $\{f_n\}$ be defined by*

$$\begin{aligned} f_{n+1} &= \alpha_n Tf_n + (1-\alpha_n)Tg_n, \\ g_n &= \beta_n f_n + (1-\beta_n)Tf_n, \end{aligned} \tag{2.2}$$

with $0 < \alpha_n, \beta_n < 1$, then $\lim_{n\to\infty} \rho(f_n - h)$ exists for $h \in A_\rho(T)$ and $\lim_{n\to\infty} \rho(f_n - Tf_n) = 0$.

Proof. Let h be a ρ-attractive point of T. Then by convexity of ρ we have

$$\begin{aligned} \rho(f_{n+1} - h) &= \rho(\alpha_n Tf_n + (1-\alpha_n)Tg_n - h) \\ &\le \rho(\alpha_n(Tf_n - h) + (1-\alpha_n)(Tg_n - h)) \\ &\le \alpha_n\rho(Tf_n - h) + (1-\alpha_n)\rho(Tg_n - h) \\ &\le \alpha_n\rho(f_n - h) + (1-\alpha_n)\rho(g_n - h). \end{aligned} \tag{2.3}$$

Also,

$$\begin{aligned}\rho(g_n - h) &= \rho(\beta_n f_n + (1-\beta_n)Tf_n - h)\\ &\le \rho(\beta_n(f_n - h) + (1-\beta_n)(Tf_n - h))\\ &\le \beta_n\rho(f_n - h) + (1-\beta_n)\rho(f_n - h)\\ &\le \rho(f_n - h). \qquad (2.4)\end{aligned}$$

Thus, from (2.3) and (2.4) we have

$$\rho(f_{n+1} - h) \le \rho(f_n - h).$$

Hence, $\{f_n\}$ is ρ-bounded and $\rho(f_n - h)$ is a nonincreasing sequence. Then $\lim_{n\to\infty} \rho(f_n - h)$ exists for each $h \in A_\rho(T)$.

Now we show that $\lim_{n\to\infty} \rho(f_n - Tf_n) = 0$. Suppose that

$$\lim_{n\to\infty} \rho(f_n - h) = L. \qquad (2.5)$$

Since $h \in A_\rho(T)$, we have $\rho(Tf_n - h) \le \rho(f_n - h)$. Thus,

$$\limsup_{n\to\infty} \rho(Tf_n - h) \le \limsup_{n\to\infty} \rho(f_n - h).$$

It follows,

$$\limsup_{n\to\infty} \rho(Tf_n - h) \le L. \qquad (2.6)$$

Also,

$$\begin{aligned}\rho(Tg_n - h) &\le \rho(g_n - h)\\ &\le \rho(f_n - h)\end{aligned}$$

implies

$$\rho(Tg_n - h) \le \limsup_{n\to\infty} \rho(Tg_n - h) \le L \qquad (2.7)$$

and

$$\rho(g_n - h) \le \rho(f_n - h).$$

Thus

$$\limsup_{n\to\infty} \rho(g_n - h) \le L. \qquad (2.8)$$

Therefore,

$$\begin{aligned}L &= \lim_{n\to\infty} \rho(f_{n+1} - h)\\ &= \lim_{n\to\infty} \rho(\alpha_n Tf_n + (1-\alpha_n)Tg_n - h)\\ &= \lim_{n\to\infty} \rho(\alpha_n(Tf_n - h) + (1-\alpha_n)(Tg_n - h)). \qquad (2.9)\end{aligned}$$

Then using (2.6), (2.7), (2.9) and Lemma 1.1 we have $\lim_{n\to\infty} \rho(Tf_n - Tg_n) = 0$.

Fix $\epsilon > 0$. Then there exists $n_0 \in \mathbb{N}$ such that

$$\rho(Tf_n - Tg_n) < \epsilon, \quad \text{for all } n \ge n_0.$$

Now, using the definition of growth function,

$$\begin{aligned}\rho(\alpha_n(Tf_n - Tg_n)) &\leq \omega_\rho(\alpha_n)\rho(Tf_n - Tg_n)\\ &\leq \rho(Tf_n - Tg_n)\\ &< \epsilon.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \rho(\alpha_n(Tf_n - Tg_n)) = 0. \tag{2.10}$$

Next,

$$\begin{aligned}\rho(f_{n+1} - h) &= \rho(\alpha_n Tf_n + (1-\alpha_n)Tg_n - h)\\ &= \rho(\alpha_n(Tf_n - Tg_n) + (Tg_n - h)).\end{aligned}$$

By using Theorem 1.1 and (2.10), we get

$$\begin{aligned}\liminf_{n\to\infty} \rho(f_{n+1} - h) &= \liminf_{n\to\infty} \rho(\alpha_n(Tf_n - Tg_n) + (Tg_n - h))\\ &= \liminf_{n\to\infty} \rho(Tg_n - h).\end{aligned}$$

Thus,

$$\liminf_{n\to\infty} \rho(Tg_n - h) = L.$$

Now,

$$\liminf_{n\to\infty} \rho(Tg_n - h) \leq \liminf_{n\to\infty} \rho(g_n - h),$$

$$\Rightarrow L \leq \liminf_{n\to\infty} \rho(g_n - h). \tag{2.11}$$

Again, from (2.8) and (2.11),

$$\lim_{n\to\infty} \rho(g_n - h) = L.$$

Consequently,

$$\begin{aligned}\lim_{n\to\infty} \rho(g_n - h) &= \lim_{n\to\infty} \rho(\beta(f_n - h) + (1-\beta)(Tf_n - h))\\ &= L.\end{aligned} \tag{2.12}$$

Hence, using (2.5), (2.6), (2.12) and Lemma 1.1 we get

$$\lim_{n\to\infty} \rho(f_n - Tf_n) = 0.$$

□

Our next result discusses the ρ-convergence of the iterative process (2.2) to attractive elements of the mapping T where T satisfies condition (I).

Definition 2.3. Let E be a nonempty subset of $\mathbb{L}_\rho$. A mapping $T : E \to E$ is said to satisfy condition (I) if there exists a nondecreasing function $\ell : [0,\infty) \to [0,\infty)$ with $\ell(0) = 0$, $\ell(r) > 0$ for all $r \in (0,\infty)$ such that $\rho(f - Tf) \geq \ell(\mathrm{dist}_\rho(f, A_\rho(T)))$ where $\mathrm{dist}_\rho(f, A_\rho(T)) = \inf\{\rho(f - g) : g \in A_\rho(T)\}$.

We give an example of a mapping that satisfies the condition (I).

Example 2.2. Let the set of real numbers $\mathbb{R}$ be the space modulared as $\rho(f) = |f|$. Let $E = \{f \in \mathbb{L}_\rho : 0 < f < 1\}$, define $T : E \to E$ as $Tf = \frac{f}{2}$. Clearly, T is $\rho - \frac{1}{4}$-nonspreading mapping. We know that an element $g \in \mathbb{L}_\rho$ is an attractive point of T if $\rho(Tf - g) \leq \rho(f - g)$ for all $f \in E$. Assume that $g \in A_\rho(T)$, then

$$
\begin{aligned}
\left|\frac{f}{2} - g\right| &\leq \left|f - g\right|, \qquad (2.13)\\
\left|\frac{f}{2} - g\right|^2 &\leq \left|f - g\right|^2,\\
\left|\frac{f}{2} - g\right|^2 - \left|f - g\right|^2 &\leq 0,\\
\left(\frac{f}{2} - g + f - g\right)\left(\frac{f}{2} - g - f + g\right) &\leq 0,\\
\left(\frac{3f}{2} - 2g\right)\left(\frac{-f}{2}\right) &\leq 0.
\end{aligned}
$$

Hence, we have $g \leq \frac{3f}{4}$. Since g must satisfy (2.13) for all f such that $0 < f < 1$, g must be less or equal to 0. Hence, $A_\rho(T) = (-\infty, 0]$. Define a continuous nondecreasing function $\ell : [0, \infty) \to [0, \infty)$ by $\ell(r) = \frac{r}{8}$. Then,

$$\ell(d_\rho(f, A_\rho(T))) = \ell(d_\rho(f, (-\infty, 0])) = \ell(|f|) = \frac{|f|}{8} < \left|\frac{f}{2} - f\right|.$$

Hence, $\rho(f - Tf) \geq \ell(d_\rho(f, A_\rho(T)))$ for all $f \in E$.

Theorem 2.4. *Let $\rho \in \Re$ satisfies $(UUC2)$ and Δ_2-condition. In addition, ρ is uniformly continuous. Let E be a nonempty convex subset of $\mathbb{L}_\rho$ and $T : E \to E$ be a $\rho - k$-nonspreading mapping with $k \in \left(0, \frac{1}{2}\right]$. Assume $A_\rho(T) \neq \phi$ and T satisfies condition (I). Let $\{f_n\}$ be defined as in* (2.2)*, with $0 < \alpha_n, \beta_n < 1$. Then $\{f_n\}$ ρ-converges to a ρ-attractive point of T.*

Proof. We already know $\rho(f_{n+1} - h) \leq \rho(f_n - h)$ and $\lim_{n\to\infty} \rho(f_n - Tf_n) = 0$. Then by condition (I) and Theorem (2.3), we have

$$
\begin{aligned}
\liminf_{n\to\infty} \rho(f_n - Tf_n) &\geq \liminf_{n\to\infty} \ell(d_\rho(f_n, A_\rho(T)),\\
0 &\geq \liminf_{n\to\infty} \ell(d_\rho(f_n, A_\rho(T)).
\end{aligned}
$$

This implies $\lim_{n\to\infty} \ell(d_\rho(f_n, A_\rho(T)) = 0$. It follows $\lim_{n\to\infty} d_\rho(f_n, A_\rho(T)) = 0$, since $\ell(0) = 0$.

Now, we show that $\{f_n\}$ is ρ-Cauchy. Since $\lim_{n\to\infty} d_\rho(f_n, A_\rho(T)) = 0$, let $\epsilon > 0$, then there exists a constant n_0 such that for $n \geq n_0$

$$d_\rho(f_n, A_\rho(T)) < \frac{\epsilon}{2},$$

$$\{\inf \rho(f_n - h) : h \in A_\rho(T)\} < \frac{\epsilon}{2}.$$

Then there must exist some $h^* \in A_\rho(T)$ such that $\rho(f_{n_0} - h^*) < \epsilon$. Now for $m, n \geq n_0$, we have by convexity of ρ and the fact that $\rho(\{f_n - h\})$ is non increasing,

$$\begin{aligned}\rho\left(\frac{f_{n+m} - f_n}{2}\right) &\leq \rho\left(\frac{(f_{n+m} - h) - (f_n - h)}{2}\right)\\ &\leq \frac{1}{2}(\rho(f_{n+m} - h)) + \frac{1}{2}(\rho(f_n - h))\\ &< \frac{1}{2}(\rho(f_{n_0} - h^*)) + \frac{1}{2}(\rho(f_{n_0} - h^*))\\ &= \rho(f_{n_0} - h^*)\\ &< \epsilon.\end{aligned}$$

Hence, by Δ_2-condition, $\{f_n\}$ is a ρ-Cauchy sequence. Since $\mathbb{L}_\rho$ is complete, the sequence $\{f_n\}$ ρ-converges to some q in $\mathbb{L}_\rho$.

Let $\lim_{n\to\infty} \rho(f_n - q) = 0$. Then, by convexity of ρ and Theorem 2.3,

$$\lim_{n\to\infty} \rho(Tf_n - q) = 0.$$

Further, by definition (2.1) and uniform convexity of ρ, we get the following

$$\lim_{n\to\infty} \rho^2(Tf_n - Tf) \leq k \lim_{n\to\infty} \rho^2(f_n - Tf) + k \lim_{n\to\infty} \rho^2(Tf_n - f).$$

This implies

$$\rho^2(q - Tf) \leq k\rho^2(q - Tf) + k\rho^2(q - f).$$

This results

$$\rho(q - Tf) \leq \frac{k}{1-k}\rho(q - f) \leq \rho(q - f).$$

Hence, $q \in A_\rho(T)$ and $\lim_{n\to\infty} \rho(f_n - q) = 0$. □

Let E be a subset of $\mathbb{L}_\rho$. A mapping $T : E \to \mathbb{L}_\rho$ is said to be ρ-demicompact if it has the property that whenever a sequence $\{f_n\} \in E$ is ρ-bounded and the sequence $\{f_n - Tf_n\}$ ρ-converges, then there exists a subsequence $\{f_{n_k}\}$ which is ρ-convergent.

Theorem 2.5. *Let $\rho \in \Re$ satisfies $(UUC2)$ and Δ_2-condition. In addition, let ρ is uniformly continuous. Let E be a nonempty convex subset of $\mathbb{L}_\rho$ and $T : E \to E$ be a $\rho - k$-nonspreading with $k \in (0, \frac{1}{2}]$ and ρ-demicompact mapping with $A_\rho(T) \neq \phi$. Let $\{f_n\}$ be defined as in (2.2) with $0 < \alpha_n, \beta_n < 1$. Then $\{f_n\}$ ρ-converges to a ρ-attractive point of T.*

Proof. From Theorem 2.3 we already know that $\{f_n\}$ is a bounded sequence and $\lim_{n\to\infty} \rho(f_n - Tf_n) = 0$. Then by demicompactness of operator T there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and $g \in \mathbb{L}_\rho$ such that $\lim_{n\to\infty} \rho(f_{n_k} - g) = 0$. Also, by uniform continuity of ρ and since $\lim_{n\to\infty} \rho(f_n - Tf_n) = 0$, we have

$$\lim_{n\to\infty} \rho(Tf_{n_k} - g) = 0.$$

Now, by definition of $\rho - k$-nonspreading mapping and uniform continuity of ρ we have,

$$\lim_{n\to\infty} \rho^2(Tf_{n_k} - Tf) \leq k \lim_{n\to\infty} \rho^2(f_{n_k} - Tf) + k \lim_{n\to\infty} \rho^2(Tf_{n_k} - f).$$

Consequently,

$$\rho^2(g - Tf) \leq k\rho^2(g - Tf) + k\rho^2(g - f).$$

That is,

$$\rho(g - Tf) \leq \rho(g - f).$$

So, $g \in A_\rho(T)$. By Theorem 2.3, if $\lim_{n\to\infty} \rho(f_n - g)$ exists for any $g \in A_\rho(T)$, then we have, $\lim_{n\to\infty} \rho(f_n - g) = 0$. □

3. Numerical Results

Now the following examples verify the results in Theorems 2.4 and 2.5.

Example 3.1. Let the set of real numbers $\mathbb{R}$ be the space modulared as $\rho(f) = |f|$. Let $E = \{f \in \mathbb{L}_\rho : 0 < f < 1\}$, define $T : E \to E$ as $Tf = \frac{f}{2}$. Obviously, E is a nonempty convex subset of $\mathbb{R}$ which satisfies $(UC1)$ condition. Also $\rho(f) = |f|$ is uniformly continuous and $(UUC2)$ holds. We have already seen $A_\rho(T)$ is nonempty. Finally, we generate the sequence (2.2) and show that it converges to its attractive point. Choose $f_1 = 0.3125$ and $\alpha = \beta = \frac{1}{2}$, then we have the results in Table 1.

Table 1. Numerical results of Example 3.1

n	f_n
1	0.312500000000000
2	0.136718750000000
3	0.059814453125000
4	0.026168823242188
5	0.011448860168457
6	0.005008876323700
7	0.002191383391619
⋮	⋮
48	4.176559929877658e-18
49	1.827244969321475e-18
50	7.994196740781455e-19

This shows that $\{f_n\}$ converges to $0 \in A_\rho(T)$. This is worth mentioning here that T does not have any fixed point in D.

Example 3.2. Let the set of real numbers $\mathbb{R}$ be the space modulared as $\rho(f) = |f|^k$. Let $E = \{f \in \mathbb{L}_\rho : -3 < f < 2\}$, define $T : E \to E$ as:

$$Tf = \begin{cases} \dfrac{|f| - 1}{2}, & -2 < f < 2, \\ \dfrac{-|f|}{|f| + 1}, & -3 < f \leq -2. \end{cases}$$

Obviously, E is a nonempty convex subset of $\mathbb{R}$ which satisfies (UC1) condition. Also $\rho(f) = |f|^k$ is uniformly continuous and $(UUC2)$ holds. Since the mapping is ρ-quasi nonexpansive and $F(T) = \{-\frac{1}{3}\}$ then $A_\rho(T) \neq \phi$. T is ρ-demicompact since any sequence $\{f_n\} \in (-3, -2)$ is bounded, i.e., $|f_n| < 3$ and any bounded sequence in $\mathbb{R}$ has a convergent subsequence. Now finally, we generate the sequence (2.2) and show that it converges to its attractive point. Choose $f_1 = 1.5$ and $\alpha = \beta = \frac{1}{2}$, then we have the results in Table 2. This shows that $\{f_n\}$ converges to $-\frac{1}{3} \in A_\rho(T)$.

Table 2. Numerical results of Example 3.2

n	f_n
1	1.5
2	0.093750000000000
3	-0.431640625000000
4	-0.302612304687500
5	-0.342933654785156
6	-0.330333232879639
7	-0.334270864725113
⋮	⋮
28	-0.333333333333310
29	-0.333333333333341
30	-0.333333333333331

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ON PERFECT CO-ANNIHILATING-IDEAL GRAPH OF A COMMUTATIVE ARTINIAN RING

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Abstract. Let R be a commutative ring with identity. The co-annihilating-ideal graph of R, denoted by A_R, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = (0)$. In this paper, we characterize all Artinian rings for which both of the graphs A_R and $\overline{A_R}$ (the complement of A_R), are chordal. Moreover, all Artinian rings whose A_R (and thus $\overline{A_R}$) is perfect are characterized.

1. Introduction

Assigning a graph to a ring gives us the ability to translate algebraic properties of rings into graph-theoretic language and vice versa. It leads to arising interesting algebraic and combinatorics problems. Therefore, the study of graphs associated with rings has attracted many researches. There are a lot of papers which apply combinatorial methods to obtain algebraic results in ring theory; for instance see [2, 3, 5, 6, 10, 11] and [12].

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\mathrm{Max}(R)$, $\mathrm{Nil}(R)$ and $J(R)$ the set of all zero-divisor elements of R, the set of all maximal ideals of R, the set of all nilpotent elements of R and jacobson radical of R, respectively. We call an ideal I of R, an *annihilating-ideal* if there exists $r \in R \setminus \{0\}$ such that $Ir = (0)$. The set of all annihilating-ideals of R is denote by $A(R)$. Let I be an ideal of R. We denote by $A(I)$ the set of all ideals of R contained in I. The ring R is said to be *reduced* if it has no non-zero nilpotent element. For every ideal I of R, we denote the *annihilator* of I by $\mathrm{Ann}(I)$. We let $A^* = A \setminus \{0\}$. For any undefined notation or terminology in ring theory, we refer the reader to [4, 7].

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We use the standard terminology of graphs following [13]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\overline{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph of* G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph by* V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{\{u, v\} \in E \mid u, v \in V_0\}$. Also G is called a *null graph* if it has no edge. A complete graph of n vertices is denoted by K_n. An *n-part graph* is one whose vertex set can be partitioned into n subsets, so that no edge has both ends in any one subset. A *complete n-partite graph* is an n-part graph such that every pair of graph vertices in the n sets are adjacent. In a graph G, a vertex x is *isolated*, if no vertices of G is adjacent to x. Let G_1 and G_2 be two disjoint graphs. The *join* of G_1 and G_2, denoted by $G_1 \vee G_2$, is a graph with the vertex set $V(G_1 \vee G_2) = V(G_1) \cup V(G_2)$ and edge set $E(G_1 \vee G_2) = E(G_1) \cup E(G_2) \cup \{uv \mid u \in V(G_1), v \in V(G_2)\}$. For a graph G, $S \subseteq V(G)$ is called a *clique* if the subgraph induced on S is complete. The number of vertices in the largest clique of graph G is called the *clique number* of G and is often denoted by $\omega(G)$. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Clearly, for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect graph* G is a graph in which every induced subgraph is weakly perfect. A *chord* of a cycle C is an edge which is not in C but has both its endvertices in C. A graph G is *chordal* if every cycle of length at least 4 has a chord.

Let R be a commutative ring with identity. The co-annihilating-ideal graph of R, denoted by A_R, is a graph whose vertex set is the set of all non-zero proper ideals of R and two distinct vertices I and J are adjacent whenever $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = (0)$. This graph was first introduced and studied in [1] and many interesting properties of this graph were explored by the authors. In [1, Theorem 17], it was proved A_R is a weakly perfect graph, if R is an Artinian ring. In this paper, we continue study the perfectness of A_R. Indeed, we characterize all Artinian rings for which both of the graphs A_R and $\overline{A_R}$, are chordal. Moreover, all Artinian rings whose A_R is perfect are given.

2. When A_R and $\overline{A_R}$ are Chordal?

In this section, we characterize all Artinian rings R, for which A_R and $\overline{A_R}$ are chordal. We begin with the following lemmas.

Lemma 2.1. *Let R be an Artinian ring. Then there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$.*

Proof. See [4, Theorem 8.7]. □

Lemma 2.2. *Let R be an Artinian ring and I be a non-zero ideal of R. Then I is a nilpotent ideal of R if and only if I is an isolated vertex in A_R.*

Proof. Assume that I is a non-zero nilpotent ideal of R. First, we show that $\mathrm{Ann}(I)$ is an essential ideal of R. Suppose to the contrary, there exists an ideal J such that $J \cap \mathrm{Ann}(I) = (0)$. Thus $KI \neq (0)$, for every $K \subseteq J$. Obviously, $KI \subseteq J$ and so $(KI)I = KI^2 \neq (0)$. By continuing this procedure, $KI^n \neq 0$, for every positive integer n, a contradiction. Hence $\mathrm{Ann}(I)$ is an essential ideal of R and so $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) \neq (0)$, for every $J \in A(R)^*$. Therefore, I is an isolated vertex in A_R.

Conversely, suppose that I is an isolated vertex in A_R. If I is not a nilpotent ideal of R, then $I \nsubseteq J(R)$, i.e, there exists $\mathfrak{m} \in \mathrm{Max}(R)$ such that $I + \mathfrak{m} = R$, and so I is adjacent to $\mathfrak{m}$, a contradiction. Thus I is a nilpotent ideal of R. □

Next we need to study the structure of A_R, where R is an Artinian ring with at most two maximal ideals.

Theorem 2.1. *Let R be an Artinian ring. Then the following statements are equivalent:*

(1) $|\mathrm{Max}(R)| = 1$*;*

(2) $A_R = \overline{K_n}$*, where* $n = |A(R)^*|$.

Proof. (1) ⇒ (2) Since R is an Artinian local ring, every ideal of $A(R)^*$ is a nilpotent ideal of R and thus by Lemma 2.2, A_R is a null graph.

(2) ⇒ (1) is obtained by Lemma 2.2. □

Theorem 2.2. *Let R be an Artinian ring. Then the following statements are equivalent:*

(1) $|\mathrm{Max}(R)| = 2$*;*

(2) $A_R = \overline{K_{n_1}} + K_{n_2,n_3}$*, where* $n_1 = |A(\mathrm{Nil}(R))^*|$*,* $n_2 = |A(\mathfrak{m}_1)^*| - n_1$*,* $n_3 = |A(\mathfrak{m}_2)^*| - n_1$ *and* $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$*.*

Proof. (1) ⇒ (2) Let $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$. Since $\mathfrak{m}_1 \cap \mathfrak{m}_2 = \mathrm{Nil}(R)$, Lemma 2.2 implies that $A_R[A(\mathrm{Nil}(R))^*]$ is a null graph. Let $A = \{I \in A(\mathfrak{m}_1) \setminus A(\mathrm{Nil}(R))\}$ and $B = \{I \in A(\mathfrak{m}_2) \setminus A(\mathrm{Nil}(R))\}$. If $I \in A$ and $J \in B$, then $I + J = R$, and thus I is adjacent to J. Moreover, $A_R[A]$ and $A_R[B]$ are null graphs. This means that $A_R[A \cup B] = K_{|A|,|B|}$. Since $A \cup B \cup A(\mathrm{Nil}(R))^* = A(R)^*$, we deduce that $A_R = \overline{K_{n_1}} + K_{n_2,n_3}$, where $n_1 = |A(\mathrm{Nil}(R))^*|$, $n_2 = |A(\mathfrak{m}_1)^*| - n_1$, $n_3 = |A(\mathfrak{m}_2)^*| - n_1$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$.

(2) ⇒ (1) By Theorem 2.1, $|\mathrm{Max}(R)| \geq 2$. If $|\mathrm{Max}(R)| \geq 3$, then A_R has a cycle of length 3, as $A_R[\mathrm{Max}(R)]$ is a complete graph, a contradiction. Thus $|\mathrm{Max}(R)| = 2$. □

We are now in a position to characterize all Artinian rings for which both of the graphs A_R and $\overline{A_R}$ are chordal.

Theorem 2.3. *Let R be an Artinian ring. Then*

(1) A_R *is chordal if and only if one of the following statements holds:*

(i) R *is local;*

(ii) $R \cong F \times S$*, where F is a field and S is local;*

(iii) $R \cong F_1 \times F_2 \times F_3$, *where* F_i *is a field for every* $1 \leq i \leq 3$;

(2) $\overline{A_R}$ *is chordal if and only if* $|\mathrm{Max}(R)| \leq 3$.

Proof. (1) Let A_R be chordal. First we show that $|\mathrm{Max}(R)| \leq 3$. If $|\mathrm{Max}(R)| \geq 4$, then Figure 1 is a cycle of length 4,

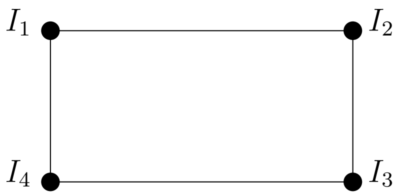

FIGURE 1. A cycle of length 4 in A_R

where

$$\begin{aligned}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n,\\
I_2 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times \cdots \times R_n,\\
I_3 &= R_1 \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n,\\
I_4 &= R_1 \times (0) \times R_3 \times R_4 \times R_5 \times \cdots \times R_n.
\end{aligned}$$

Thus $|\mathrm{Max}(R)| \leq 3$. If $|\mathrm{Max}(R)| = 3$, then $R \cong R_1 \times R_2 \times R_3$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. If R_1 is not field, then consider $I \in A(\mathrm{Nil}(R_1))^*$ and thus Figure 2 is a cycle of length 4,

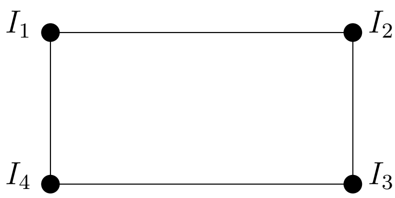

FIGURE 2. A cycle of length 4 in A_R

where

$$\begin{aligned}
I_1 &= R_1 \times (0) \times (0),\\
I_2 &= (0) \times R_2 \times R_3,\\
I_3 &= R_1 \times R_2 \times (0),\\
I_4 &= I \times R_2 \times R_3.
\end{aligned}$$

Hence R_1 is a field. Similarly, R_2 and R_3 are fields. Let $|\mathrm{Max}(R)| = 2$. Then $R \cong R_1 \times R_2$, where R_i is an Artinian local ring, for every $1 \leq i \leq 2$. We show that one of the rings R_1 and R_2 is a field. If I, J are non-zero proper ideals of R_1 and R_2, respectively, then Figure 3 is a cycle of length 4, where

$$\begin{aligned} I_1 &= I \times R_2, \\ I_2 &= R_1 \times J, \\ I_3 &= (0) \times R_2, \\ I_4 &= R_1 \times (0). \end{aligned}$$

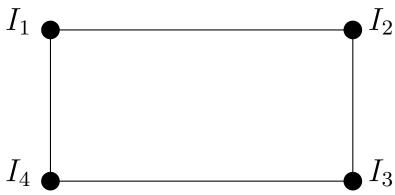

FIGURE 3. A cycle of length 4 in A_R

This means that one of the rings R_1 and R_2 is a field. Thus in this case $R \cong F \times S$, where F is a field and S is local. Clearly, if $|\mathrm{Max}(R)| = 1$, R is local.

Conversely, suppose that one of the conditions (i), (ii), (ii) is satisfied. Condition (i) implies that A_R is a null graph by Theorem 2.1, and thus A_R is chordal. If (ii) holds, then by Theorem 2.2, $A_R = \overline{K_n} + K_{1,n+1}$ where $n = |A(\mathrm{Nil}(R))^*|$. This implies that A_R is chordal. If (iii) holds, then Figure 4 shows that A_R is chordal where

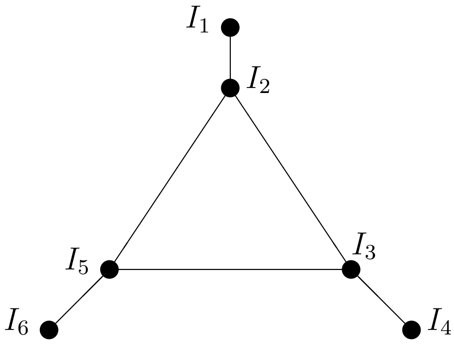

FIGURE 4. $A_{F_1 \times F_2 \times F_3}$

$$\begin{aligned} I_1 &= (0) \times (0) \times F_3, \\ I_2 &= F_1 \times F_2 \times (0), \\ I_3 &= F_1 \times (0) \times F_3, \\ I_4 &= (0) \times F_2 \times (0), \\ I_5 &= (0) \times F_2 \times F_3, \end{aligned}$$

$$I_6 = F_1 \times (0) \times (0).$$

(2) First suppose that $\overline{A_R}$ is chordal. If $|\mathrm{Max}(R)| \geq 4$, then we put

$$\begin{aligned}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times \cdots \times R_n, \\
I_2 &= (0) \times R_2 \times (0) \times R_4 \times R_5 \times \cdots \times R_n, \\
I_3 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times \cdots \times R_n, \\
I_4 &= R_1 \times (0) \times R_3 \times (0) \times R_5 \times \cdots \times R_n.
\end{aligned}$$

Now, it is not hard to see that $I_1 - I_2 - I_3 - I_4 - I_1$ is a cycle of length 4, a contradiction. Thus $|\mathrm{Max}(R)| \leq 3$.

Conversely, suppose that $|\mathrm{Max}(R)| \leq 3$. We show that $\overline{A_R}$ is chordal. To see this, we consider the following cases.

Case 1. $|\mathrm{Max}(R)| = 1$. In this case, R is local and thus by Theorem 2.1, $\overline{A_R}$ is a complete graph. Hence $\overline{A_R}$ is chordal.

Case 2. $|\mathrm{Max}(R)| = 2$. By Theorem 2.2, $\overline{A_R} = K_{n_1} \bigvee (K_{n_2} + K_{n_3})$, where $n_1 = |A(\mathrm{Nil}(R))^*|$, $n_2 = |A(\mathfrak{m}_1)^*| - n_1$, $n_3 = |A(\mathfrak{m}_2)^*| - n_1$ and $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Max}(R)$. Thus every cycle is a triangle, i.e, $\overline{A_R}$ is chordal.

Case 3. $|\mathrm{Max}(R)| = 3$. In this case, $R \cong R_1 \times R_2 \times R_3$. Let I_i be an ideal of R_i, for every $1 \leq i \leq 3$. Suppose that

$$\begin{aligned}
A_1 &= \{I_1 \times I_2 \times I_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 1, 2, 3\} \setminus \{(0) \times (0) \times (0)\}, \\
A_2 &= \{R_1 \times I_2 \times I_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 2, 3\}, \\
A_3 &= \{I_1 \times R_2 \times I_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 1, 3\}, \\
A_4 &= \{I_1 \times I_2 \times R_3 \mid I_i \subseteq \mathrm{Nil}(R_i), \text{ for } i = 1, 2\}, \\
B_1 &= \{R_1 \times R_2 \times I_3 \mid I_3 \subseteq \mathrm{Nil}(R_3)\}, \\
B_2 &= \{R_1 \times I_2 \times R_3 \mid I_2 \subseteq \mathrm{Nil}(R_2)\}, \\
B_3 &= \{I_1 \times R_2 \times R_3 \mid I_1 \subseteq \mathrm{Nil}(R_1)\}.
\end{aligned}$$

Let $A = \cup_{i=1}^4 A_i$ and $B = \cup_{i=1}^3 B_i$. One may check that $A \cap B = \emptyset$ and $V(\overline{A_R}) = A \cup B$ and so $\{A, B\}$ is a partition of $V(\overline{A_R})$. We claim that $\overline{A_R}$ contains no induced cycle of length at least 4. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced cycle of length at least 4 in $\overline{A_R}$. We show that

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Suppose to the contrary (and with no loss of generality), $a_1 \in B_1$. Thus $a_1 = R_1 \times R_2 \times I_3$, where $I_3 \subseteq \mathrm{Nil}(R_3)$. Since a_2 and a_n are adjacent to a_1, we conclude that the third components of a_2 and a_n must be nilpotent ideals of R_3. This implies that a_2 and a_n are adjacent, a contradiction. Hence,

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Similarly,

$$\{a_1, a_2, \ldots, a_n\} \cap B_2 = \{a_1, a_2, \ldots, a_n\} \cap B_3 = \emptyset.$$

This means that

$$\{a_1, a_2, \dots, a_n\} \subseteq A.$$

But this contradicts the fact that $\overline{A_R}[A]$ is a complete graph, and so $\overline{A_R}$ contains no induced cycle of length at least 4. Thus $\overline{A_R}$ is chordal. □

3. When A_R is Perfect?

In this section, we characterize all Artinian rings rings R whose A_R is Perfect. First, we need two celebrate results.

Theorem 3.1 (The Strong Perfect Graph Theorem [8])**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

In light of Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let G be a graph. Then the following statements hold.*
(1) *G is a perfect graph if and only if $\overline{G}$ is a perfect graph.*
(2) *If G is a complete bipartite graph, then G is a perfect graph.*

Theorem 3.2. [9] *Every chordal graph is perfect.*

Lemma 3.1. *Let n be a positive integer and $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian ring for every $1 \le i \le n$. Let $I = I_1 \times \cdots \times I_n, J = J_1 \times \cdots \times J_n$ be two distinct ideals of R and $n \ge 2$. Then $I - J$ is an edge of A_R if and only if for every $1 \le i \le n$, $I_i \notin A(\mathrm{Nil}(R_i))$ or $J_i \notin A(\mathrm{Nil}(R_i))$.*

Proof. Let $I - J$ be an edge of A_R. If there exists $1 \le i \le n$ such that $I_i, J_i \in A(\mathrm{Nil}(R_i))$, then by Lemma 2.2, $\mathrm{Ann}(I_i) \cap \mathrm{Ann}(J_i) \neq (0)$. So if $0 \neq a_i \in \mathrm{Ann}(I_i) \cap \mathrm{Ann}(J_i)$, then $(0) \times \cdots \times (0) \times R_i a_i \times (0) \times \cdots \times (0) \subseteq \mathrm{Ann}(I) \cap \mathrm{Ann}(J)$ and thus $I - J$ is not an edge of A_R, a contradiction.

Conversely, suppose that $I_i \notin A(\mathrm{Nil}(R_i))$ or $J_i \notin A(\mathrm{Nil}(R_i))$, for every $1 \le i \le n$. Thus $I_i = R_i$ or $J_i = R_i$, for every $1 \le i \le n$. This implies that $\mathrm{Ann}(I) \cap \mathrm{Ann}(J) = (0)$. Hence $I - J$ is an edge of A_R. □

We are now in a position to state our main result in this paper.

Theorem 3.3. *Let R be an Artinian rings. Then $\overline{A_R}$ is a perfect graph if and only if $|\mathrm{Max}(R)| \le 4$.*

Proof. First suppose $\overline{A_R}$ is perfect. Since R is an Artinian ring, there exists a positive integer $n = |\mathrm{Max}(R)|$ such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \le i \le n$, by Lemma 2.1. If $n \ge 5$, then we put

$$\begin{aligned}
I_1 &= (0) \times R_2 \times R_3 \times (0) \times R_5 \times R_6 \times \cdots \times R_n,\\
I_2 &= (0) \times R_2 \times (0) \times R_4 \times R_5 \times R_6 \times \cdots \times R_n,\\
I_3 &= R_1 \times (0) \times (0) \times R_4 \times R_5 \times R_6 \times \cdots \times R_n,\\
I_4 &= R_1 \times (0) \times R_3 \times R_4 \times (0) \times R_6 \times \cdots \times R_n,
\end{aligned}$$

$$I_5 = R_1 \times R_2 \times R_3 \times (0) \times (0) \times R_6 \times \cdots \times R_n.$$

Then it is easily seen that

$$I_1 - I_2 - I_3 - I_4 - I_5 - I_1$$

is a cycle of length 5 in $\overline{A_R}$, a contradiction (by Theorem 3.1). So $n \leq 4$.

Conversely, suppose that $|\mathrm{Max}(R)| \leq 4$. We show that $\overline{A_R}$ is a perfect graph. If $|\mathrm{Max}(R)| \leq 3$, then by part (2) of Theorem 2.3, $\overline{A_R}$ is chordal and thus by Theorem 3.2, $\overline{A_R}$ is a perfect graph. Therefore, we need only to check the case $|\mathrm{Max}(R)| = 4$. Let $R \cong R_1 \times R_2 \times R_3 \times R_4$. We have the following claims.

Claim 1. $\overline{A_R}$ contains no induced odd cycle of length at least 5. We consider the following partition for $V(\overline{A_R})$:

$$\begin{aligned}
A &= \{I_1 \times I_2 \times I_3 \times I_4 \mid I_i \in A(R_i) \text{ for every } 1 \leq i \leq 4 \text{ and } I_4 \in A(\mathrm{Nil}(R_4))\},\\
B &= \{I_1 \times I_2 \times I_3 \times R_4 \mid I_i \in A(R_i) \text{ for every } 1 \leq i \leq 3 \text{ and } I_3 \in A(\mathrm{Nil}(R_3))\},\\
C &= \{I_1 \times I_2 \times R_3 \times R_4 \mid I_i \in A(R_i) \text{ for every } 1 \leq i \leq 2 \text{ and } I_2 \in A(\mathrm{Nil}(R_2))\},\\
D &= \{R_1 \times I_2 \times R_3 \times R_4, I_1 \times R_2 \times R_3 \times R_4 \mid \text{for every } 1 \leq i \leq 2 \ I_i \in A(\mathrm{Nil}(R_i))\}.
\end{aligned}$$

Now, assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $\overline{A_R}$. We consider the following cases.

Case 1. $\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset$. Let $a_i \in \{a_1, a_2, \ldots, a_n\} \cap D$, for some $1 \leq i \leq n$. Then we can let $a_i = I_1 \times R_2 \times R_3 \times R_4$ or $a_i = R_1 \times I_2 \times R_3 \times R_4$. If $a_i = I_1 \times R_2 \times R_3 \times R_4$, then the first components of a_{i-1} and a_{i+1} must be in $A(\mathrm{Nil}(R_i))$ and $A(\mathrm{Nil}(R_i))$, respectively. So by Lemma 3.1, a_{i-1} is adjacent to a_{i+1}, a contradiction. Thus, $a_i \neq I_1 \times R_2 \times R_3 \times R_4$. Similarly, $a_i \neq R_1 \times I_2 \times R_3 \times R_4$. This means that $\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset$.

Case 2. $\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset$. First we show that $|\{a_1, a_2, \ldots, a_n\} \cap C| \leq 1$. Let $a, b \in \{a_1, a_2, \ldots, a_n\} \cap C$. Then we can easily check that if there exits $x \in V(\overline{A_R})$ such that $\mathrm{Ann}(x) \cap \mathrm{Ann}(a) \neq (0)$, then $\mathrm{Ann}(x) \cap \mathrm{Ann}(b) \neq (0)$. This means that if x is adjacent to a, then x is adjacent to b, a contradiction. So $|\{a_1, a_2, \ldots, a_n\} \cap C| \leq 1$. This together with the fact that $\overline{A_R}[A]$ and $\overline{A_R}[B]$ are complete subgraphs, imply that $n = 5$ and $|\{a_1, a_2, \ldots, a_n\} \cap B| = |\{a_1, a_2, \ldots, a_n\} \cap A| = 2$. Hence $|\{a_1, a_2, \ldots, a_n\} \cap C| = 1$, and thus we can let $a \in \{a_1, a_2, \ldots, a_n\} \cap C$. Since a is adjacent to all vertices of $B \setminus \{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \subseteq \mathrm{Nil}(R_3)\}$ and $\overline{A_R}[B]$ is a complete subgraph, $a_i \in \{a_1, a_2, \ldots, a_n\} \cap \{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \subseteq \mathrm{Nil}(R_3)\}$, for some $1 \leq i \leq n$. We can let $a_i = R_1 \times R_2 \times I_3 \times R_4$. Since only one of the components of a_i is a nilpotent ideal of R_i, by a similar argument to that of case 1, we get a contradiction. Hence, $\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset$.

By the above cases, $\{a_1, a_2, \ldots, a_n\} \subseteq A \cup B$, but this contradicts the fact $\overline{A_R}[A]$ and $\overline{A_R}[B]$ are complete graphs, and thus $\overline{A_R}$ contains no induced odd cycle of length at least 5.

Claim 2. A_R contains no induced odd cycle of length at least 5. We consider the following partition for $V(A_R)$:

$$
\begin{aligned}
A_1 =&\{I_1 \times R_2 \times R_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1))\},\\
A_2 =&\{R_1 \times I_2 \times R_3 \times R_4 \mid I_2 \in A(\mathrm{Nil}(R_2))\},\\
A_3 =&\{R_1 \times R_2 \times I_3 \times R_4 \mid I_3 \in A(\mathrm{Nil}(R_3))\},\\
A_4 =&\{R_1 \times R_2 \times R_3 \times I_4 \mid I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_1 =&\{I_1 \times I_2 \times R_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2))\},\\
B_2 =&\{R_1 \times R_2 \times I_3 \times I_4 \mid I_3 \in A(\mathrm{Nil}(R_3)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_3 =&\{I_1 \times R_2 \times I_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_3 \in A(\mathrm{Nil}(R_3))\},\\
B_4 =&\{R_1 \times I_2 \times R_3 \times I_4 \mid I_2 \in A(\mathrm{Nil}(R_2)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_5 =&\{I_1 \times R_2 \times R_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
B_6 =&\{R_1 \times I_2 \times I_3 \times R_4 \mid I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3))\},\\
C_1 =&\{R_1 \times I_2 \times I_3 \times I_4 \mid I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
C_2 =&\{I_1 \times R_2 \times I_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_3 \in A(\mathrm{Nil}(R_3)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
C_3 =&\{I_1 \times I_2 \times R_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2)), I_4 \in A(\mathrm{Nil}(R_4))\},\\
C_4 =&\{I_1 \times I_2 \times I_3 \times R_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3))\},\\
D =&\{I_1 \times I_2 \times I_3 \times I_4 \mid I_1 \in A(\mathrm{Nil}(R_1)), I_2 \in A(\mathrm{Nil}(R_2)), I_3 \in A(\mathrm{Nil}(R_3)),\\
&I_4 \in A(\mathrm{Nil}(R_4))\}.
\end{aligned}
$$

If we put $A = \cup_{i=1}^{4} A_i$, $B = \cup_{i=1}^{6} B_i$ and $C = \cup_{i=1}^{4} C_i$, then one may check that $\{A, B, C, D\}$ is a partition of $V(A_R)$. We show that A_R contains no induced odd cycle of length at least 5. Assume to the contrary, $a_1 - a_2 - \cdots - a_n - a_1$ is a induced odd cycle of length at least 5 in A_R. By Lemma 2.2, every vertex in D is an isolated vertex in A_R and thus $\{a_1, a_2, \ldots, a_n\} \cap D = \emptyset$. Next, we show that

$$\{a_1, a_2, \ldots, a_n\} \cap C_1 = \emptyset.$$

To see this, if $a_i \in \{a_1, a_2, \ldots, a_n\} \cap C_1$, for some $1 \leq i \leq n$, then with no loss of generality, assume that $a_1 \in C_1$. Since every vertex of C_1 is adjacent only to vertices of A_1, $a_2, a_n \in A_1$. This is impossible, as every vertex of A_R is adjacent to a_2 if and only if it is adjacent to a_n. Therefore

$$\{a_1, a_2, \ldots, a_n\} \cap C_1 = \emptyset.$$

Similarly,

$$\{a_1, a_2, \ldots, a_n\} \cap C_2 = \{a_1, a_2, \ldots, a_n\} \cap C_3 = \{a_1, a_2, \ldots, a_n\} \cap C_4 = \emptyset.$$

Thus

$$\{a_1, a_2, \ldots, a_n\} \cap C = \emptyset.$$

Finally, we show that

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Assume to the contrary and with no loss of generality, $a_1 \in B_1$. As a_1 is adjacent only to vertices of $B_2 \cup A_3 \cup A_4$, $\{a_2, a_n\} \subseteq B_2 \cup A_3 \cup A_4$. If $a_2 \in B_2$, then a_3 is adjacent to a_n (since if a is adjacent to a_2 and b is adjacent to a_1, a is adjacent to b), a contradiction. Thus $a_2 \notin B_2$. Similarly, $a_n \notin B_2$ and so $\{a_2, a_n\} \subseteq A_3 \cup A_4$. Since $A_R[A_3 \cup A_4]$ is a complete bipartite graph, we conclude that $\{a_2, a_n\} \subseteq A_3$ or $\{a_2, a_n\} \subseteq A_4$. With no loss of generality, we may assume that $\{a_2, a_n\} \subseteq A_3$. This implies that a_3 is adjacent to a_2 and a_n (since a vertex is adjacent to a_2 if and only if it is adjacent to a_n), a contradiction. Hence,

$$\{a_1, a_2, \ldots, a_n\} \cap B_1 = \emptyset.$$

Similarly, for every $2 \leq i \leq 6$

$$\{a_1, a_2, \ldots, a_n\} \cap B_i = \emptyset.$$

This means that

$$\{a_1, a_2, \ldots, a_n\} \subseteq A.$$

But $A_R[A]$ is a complete 4-partite graph with parts A_i for $1 \leq i \leq 4$, a contradiction. Therefore, A_R contains no induced odd cycle of length at least 5 and thus by Claim 1, Claim 2 and Theorem 3.1, we have A_R is a perfect graph. □

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SOME IDENTITIES IN RINGS AND NEAR-RINGS WITH DERIVATIONS

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Abstract. In the present paper we investigate commutativity in prime rings and 3-prime near-rings admitting a generalized derivation satisfying certain algebraic identities. Some well-known results characterizing commutativity of prime rings and 3-prime near-rings have been generalized.

1. Introduction

In this paper, $\mathcal{N}$ will denote a right near-ring with center $Z(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $0x = 0$). A non empty subset U of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $U\mathcal{N} \subseteq U$) and if U is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. As usual for all x, y in $\mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a near-ring, $-(x + y) = -y - x$. Recall that $\mathcal{N}$ is 3-prime if for a, b in $\mathcal{N}$, $a\mathcal{N}b = \{0\}$ implies that $a = 0$ or $b = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. An additive mapping $d : \mathcal{N} \to \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [20], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The concept of derivation in rings has been generalized in several ways by various authors. Generalized derivation has been introduced already in rings by M. Brešar [10]. Also the notions of generalized derivation has been introduced in near-rings by Öznur Gölbasi [14]. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is called a right generalized derivation with associated derivation d if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in \mathcal{N}$ and $\mathcal{F}$ is called a left generalized derivation with associated derivation d if

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$\mathcal{F}(xy) = d(x)y + x\mathcal{F}(y)$, for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is called a generalized derivation with associated derivation d if it is both a left as well as a right generalized derivation with associated derivation d. An additive mapping $\mathcal{F} : \mathcal{N} \to \mathcal{N}$ is said to be a left (resp. right) multiplier (or centralizer) if $\mathcal{F}(xy) = \mathcal{F}(x)y$ (resp. $\mathcal{F}(xy) = x\mathcal{F}(y)$) holds for all $x, y \in \mathcal{N}$. $\mathcal{F}$ is said to be a multiplier if it is both left as well as right multiplier. Notice that a right (resp. left) generalized derivation with associated derivation $d = 0$ is a left (resp. right) multiplier. Over the past few years, many authors have investigated commutativity of prime and semi-prime rings admitting suitably constrained derivations [3, 11–13, 16, 18] and [19]. Some comparable results on near-rings have also been derived, see e.g. [1, 2, 4, 7, 9, 15] and [17]. In [11] the authors showed that a prime ring $\mathcal{R}$ must be commutative if it admits a derivation d such that either $d([x, y]) = [x, y]$ for all $x, y \in K$ or $d([x, y]) = -[x, y]$ for all $x, y \in K$, where K is a nonzero ideal of $\mathcal{R}$.

In 2002, Rehman [18] established that if a prime ring of a characteristic not 2 admits a generalized derivation F associated with a nonzero derivation such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) = -[x, y]$) for all x, y in a nonzero square closed Lie ideal U, then $U \subseteq Z(\mathcal{R})$. Quadri, Khan and Rehman [16], without the characteristic assumption on the ring, proved that a prime ring must be commutative if it admits a generalized derivation F, associated with a nonzero derivation, such that $F([x, y]) = [x, y]$ (resp. $F([x, y]) = -[x, y]$) for all x, y in a nonzero ideal I. Motivated by the above results, in the following theorem we explore the commutativity of a prime ring, provided with a generalized derivation F and left multiplier G satisfying the following conditions: $F([x, y]_{\alpha,\beta}) = [x, y]_{u,v}$, $F([x, y]_{\alpha,\beta}) = G([\beta(x), y])$ for all $x, y \in \mathcal{R}$, where α, β, u, v automorphisms of $\mathcal{R}$ and $[x, y]_{\alpha,\beta} = \alpha(x)y - y\beta(x)$.

2. Some Preliminaries

For the proofs of our main theorems, we need the following lemmas. The first lemmas appear in [7] and [20] in the context of left near-rings, and it is easy to see that they hold for right near-rings as well.

Lemma 2.1. *Let $\mathcal{N}$ be a 3-prime near-ring and U be a nonzero semigroup ideal of $\mathcal{N}$. Let d be a nonzero derivation on $\mathcal{N}$.*

(i) *If $x, y \in \mathcal{N}$ and $xUy = \{0\}$, then $x = 0$ or $y = 0$.*
(ii) *If $x \in \mathcal{N}$ and $xU = \{0\}$ or $Ux = \{0\}$, then $x = 0$.*
(iii) *If $z \in Z(\mathcal{N})$, then $d(z) \in Z(\mathcal{N})$.*

Lemma 2.2. *Let d be an arbitrary derivation of a near-ring $\mathcal{N}$. Then $\mathcal{N}$ satisfies the following partial distributive laws:*

(i) *$z(xd(y) + d(x)y) = zxd(y) + zd(x)y$ for all $x, y, z \in \mathcal{N}$;*
(ii) *$z(d(x)y + xd(y)) = zd(x)y + zxd(y)$ for all $x, y, z \in \mathcal{N}$.*

Lemma 2.3. ([5, Theorem 2.1]). *Let $\mathcal{N}$ be a 3-prime near-ring, U a nonzero semigroup left ideal or semigroup right ideal. If $\mathcal{N}$ admits a nonzero derivation d such that $d(U) \subseteq Z(\mathcal{N})$, then $\mathcal{N}$ is a commutative ring.*

3. Some Results Involving Prime Rings

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring, I a nonzero ideal of $\mathcal{R}$ and α, β, u, v automorphisms of $\mathcal{R}$ such that $\beta(I) = I$. If F is a generalized derivation of $\mathcal{R}$ associated with a derivation d and G is a left multiplier of $\mathcal{R}$ which satisfy one of the following conditions:*

(i) *$F([x, y]_{\alpha,\beta}) = [x, y]_{u,v}$ for all $x, y \in I$;*
(ii) *$F([x, y]_{\alpha,\beta}) = G([\beta(x), y])$ for all $x, y \in I$,*

then $\mathcal{R}$ is commutative.

Proof. (i) Suppose that

$$F([x, y]_{\alpha,\beta}) = [x, y]_{u,v}, \quad \text{for all } x, y \in I. \tag{3.1}$$

Replacing y by $y\beta(x)$ in (3.1), and using the fact that $[x, y\beta(x)]_{\alpha,\beta} = [x, y]_{\alpha,\beta}\beta(x)$ and $[x, y\beta(x)]_{u,v} = [x, y]_{u,v}\beta(x) + y[v(x), \beta(x)]$ for all $x, y \in I$, we arrive at

$$F([x, y]_{\alpha,\beta})\beta(x) + [x, y]_{\alpha,\beta}d(\beta(x)) = [x, y]_{u,v}\beta(x) + y[v(x), \beta(x)], \quad \text{for all } x, y \in I. \tag{3.2}$$

Using (3.1), (3.2) implies that

$$[x, y]_{\alpha,\beta}d(\beta(x)) = y[v(x), \beta(x)], \quad \text{for all } x, y \in I. \tag{3.3}$$

Substituting ry instead of y in (3.3) where $r \in \mathcal{R}$, we arrive at

$$[\alpha(x), r]Id(\beta(x)) = \{0\}, \quad \text{for all } x \in I, r \in \mathcal{R}.$$

By Lemma 2.1 (i), we get $[\alpha(x), r] = 0$ or $d(\beta(x)) = 0$ for all $x \in I$, $r \in \mathcal{R}$ which gives $\alpha(x) \in Z(\mathcal{R})$ or $d(\beta(x)) = 0$ for all $x \in I$. Since α and β are automorphisms of $\mathcal{R}$, we get $x \in Z(\mathcal{R})$ or $d(\beta(x)) = 0$ for all $x \in I$. Using Lemma 2.1 (iii), we obtain $d(\beta(I)) \subseteq Z(\mathcal{R})$ i.e, $d(I) \subseteq Z(\mathcal{R})$ which forces that $\mathcal{R}$ is commutative by Lemma 2.3.
(ii) Assume that

$$F([x, y]_{\alpha,\beta}) = G([\beta(x), y]), \quad \text{for all } x, y \in I. \tag{3.4}$$

Putting $y\beta(x)$ instead of y in (3.4), we get

$$F([x, y]_{\alpha,\beta})\beta(x) + [x, y]_{\alpha,\beta}d(\beta(x)) = G([\beta(x), y])\beta(x), \quad \text{for all } x, y \in I.$$

Using (3.4), we obtain $[x, y]_{\alpha,\beta}d(\beta(x)) = 0$ for all $x, y \in I$, which implies that

$$\alpha(x)yd(\beta(x)) = y\beta(x)d(\beta(x)), \quad \text{for all } x, y \in I. \tag{3.5}$$

Taking ry in place of y in (3.5) where $r \in \mathcal{R}$ and using it again, we conclude that

$$[\alpha(x), r]Id(\beta(x)) = \{0\}, \quad \text{for all } x \in I, r \in \mathcal{R}.$$

By Lemma 2.1 (i), we get $\alpha(x) \in Z(\mathcal{R})$ or $d(\beta(x)) = 0$ for all $x \in \mathcal{R}$ and using the same techniques as used above, we conclude that $\mathcal{R}$ is commutative. ☐

For $\alpha = \beta = u = v = id_{\mathcal{R}}$, we get the following result.

Corollary 3.1. ([16, Theorem 2.1]). *Let $\mathcal{R}$ be a prime ring and I a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation F associated with a nonzero derivation d such that $F([x,y] = [x,y]$ for all $x, y \in I$, then $\mathcal{R}$ is commutative.*

For $\alpha = \beta = u = id_{\mathcal{R}}$ and $v = -id_{\mathcal{R}}$, we get the following result.

Corollary 3.2. ([16, Theorem 2.2]). *Let $\mathcal{R}$ be a prime ring and I a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation F associated with a nonzero derivation d such that $F([x,y] + [x,y] = 0$ for all $x, y \in I$, then $\mathcal{R}$ is commutative.*

4. Some Results Involving 3-Prime Near-Rings

In this section, we will present a very important result that generalizes several theorems that are well known in the literature. More precisely, we will show that a 2-torsion prime near-ring $\mathcal{N}$ is a commutative ring if and only if $\mathcal{N}$ admits a derivation d and a left multiplier G such that $G([x,y]) = [d(x),y] - [x,d(y)]$ for all $x, y \in U$.

Theorem 4.1. *Let $\mathcal{N}$ be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d and left multiplier G, then the following assertions are equivalents:*

(i) $G([x,y]) = [d(x),y] - [x,d(y)]$ *for all* $x, y \in U$;
(ii) $\mathcal{N}$ *is a commutative ring.*

Proof. It is easy to notice that (ii) implies (i).

(i)⇒(ii) Suppose that

$$G([x,y]) = [d(x),y] - [x,d(y)], \quad \text{for all } x,y \in U. \tag{4.1}$$

Replacing x by xy in (4.1) and using the fact that $[xy,y] = [x,y]y$, we obtain

$$[d(xy),y] - [xy,d(y)] = G([x,y])y, \quad \text{for all } x,y \in U.$$

Which implies that

$$[d(xy),y] - [xy,d(y)] = ([d(x),y] - [x,d(y)])y, \quad \text{for all } x,y \in U.$$

Using Lemma 2.2 and by developing the last expression, we arrive at

$$d(x)y^2 + xd(y)y - yxd(y) - yd(x)y + d(y)xy - xyd(y) = d(x)y^2 - yd(x)y + d(y)xy - xd(y)y.$$

For $x = y$, the equation (4.1) and 2-torsion freeness we give easily $d(y)y = yd(y)$ for all $y \in U$. In this case, by a simplification of last equation, we find that

$$xd(y)y = yxd(y), \quad \text{for all } x,y \in U. \tag{4.2}$$

Substituting tx in place of x, where $t \in \mathcal{N}$ in (4.2) and using it again, we arrive at

$$[y,t]Ud(y) = \{0\}, \quad \text{for all } y \in U, t \in \mathcal{N}.$$

Using Lemma 2.1 (i), we obtain

$$y \in Z(\mathcal{N}) \text{ or } d(y) = 0, \quad \text{for all } y \in U. \tag{4.3}$$

If there exists $y_0 \in Z(\mathcal{N}) \cap U$, then by (4.1), we get $xd(y_0) = d(y_0)x$ for all $x \in U$, in this case, (4.3) gives $xd(y) = d(y)x$ for all $x, y \in U$. Replace x by tx, where $t \in \mathcal{N}$, we get $[d(y), t]x = 0$ for all $x, y \in U$, $t \in \mathcal{N}$ which implies that $[d(y), t]U = \{0\}$ for all $y \in U$, $t \in \mathcal{N}$. Since $U \neq \{0\}$, by Lemma 2.1 (ii), we obtain $d(U) \subseteq Z(\mathcal{N})$ and Lemma 2.3 assures that $\mathcal{N}$ is a commutative ring. □

If we replace G by the null application or the identical application $id_{\mathcal{N}}$, we get the following results.

Corollary 4.1. ([8, Theorem 2.1]). *Let $\mathcal{N}$ be a 2-torsion free prime near-ring. If $\mathcal{N}$ admits a derivation d such that $[d(x), y] = [x, d(y)]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Corollary 4.2. *Let $\mathcal{N}$ be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation d, then the following assertions are equivalent:*

(i) $[x, y] = [d(x), y] - [x, d(y)]$ *for all* $x, y \in U$;
(ii) $[d(x), y] = [x, d(y)]$ *for all* $x, y \in U$;
(iii) $\mathcal{N}$ *is a commutative ring.*

When $d = 0$, we have the following result.

Corollary 4.3. *Let $\mathcal{N}$ be a 2-torsion free prime near-ring and U a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplier G, then the following assertions are equivalent:*

(i) $G([x, y]) = 0$ *for all* $x, y \in U$;
(ii) $\mathcal{N}$ *is a commutative ring.*

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TRIANGULAR SYSTEM OF HIGHER ORDER SINGULAR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we introduce a high dimensional system of singular fractional differential equations. Using Schauder fixed point theorem, we prove an existence result. We also investigate the uniqueness of solution using the Banach contraction principle. Moreover, we study the Ulam-Hyers stability and the generalized-Ulam-Hyers stability of solutions. Some illustrative examples are also presented.

1. Introduction and Preliminaries

Recently, the fractional calculus has attracted the attention of researchers in various fields of applied sciences. For details, see [12,16,19,20] and the references therein. It is important to note that some research studies deal with the existence and uniqueness of solutions for some fractional differential equations are obtained in [1,6–9]. Other studies in [2,3,5,17,23] have been done for the singular fractional differential equations. On the other hand, the Ulam stability of fractional differential equations is quite significant in more realistic problems, numerical analysis, biology and economics. Considerable work has been done in this area, for instance, see [10,11,13–15,18,22,24].

Let us now present some important research papers that inspired our work: We begin by [4], where C. Bai and J. Fang established the existence of solutions for the following singular fractional coupled system:

$$\begin{cases} D^{\delta}u\left(t\right)=f\left(t,v\left(t\right)\right), & 0<t<1, \\ D^{\rho}v\left(t\right)=g\left(t,u\left(t\right)\right), & 0<t<1, \end{cases}$$

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where $0 < \delta,\ \rho < 1$, D^{δ}, D^{ρ} are two standard Riemann-Liouville fractional derivatives, $f,\ g : [0,1) \times [0,\infty) \to [0,\infty)$ are two given continuous functions, $\lim\limits_{t\to 0^+} f(t) = \infty$ and $\lim_{t\to 0^+} g(t) = \infty$.

In [25], A. Yang and W. Ge considered the following fractional coupled system

$$\begin{cases} D^{\alpha_1} u_1(t) + f_1(t, u_2(t), D^{\mu_1} u_2(t)) = 0, \\ \vdots \\ D^{\alpha_{n-1}} u_{n-1}(t) + f_{n-1}(t, u_n(t), D^{\mu_{n-1}} u_n(t)) = 0, \\ D^{\alpha_n} u_n(t) + f_n(t, u_1(t), D^{\mu_n} u_1(t)) = 0, \end{cases}$$

associated with the boundary conditions

$$\begin{cases} u_1(0) = u_2(0) = \cdots = u_n(0) = 0, \\ u_1(1) = u_2(1) = \cdots = u_n(1) = 0, \end{cases}$$

for $1 < \alpha_j < 2$, $\mu_j > 0$, $\alpha_j - \mu_{j-1} > 1$, $j = 1, 2, \ldots, n$, $\mu_0 = \mu_n$ and $f_j : [0,1] \times [0,\infty) \times \mathbb{R} \to [0,\infty)$ is continuous function. Some existence and multiplicity results of solutions are obtained.

In [21], A. Taïeb and Z. Dahmani established new existence and uniqueness results for the following problem:

$$\begin{cases} D^{\alpha} u(t) + \sum\limits_{i=1}^{m} f_i(t, u(t), v(t), D^{\gamma} u(t), D^{\rho} v(t)) = 0, \quad t \in J, \\ D^{\beta} v(t) + \sum\limits_{i=1}^{m} g_i(t, u(t), v(t), D^{\gamma} u(t), D^{\rho} v(t)) = 0, \quad t \in J, \\ u(0) = u_0^*,\ v(0) = v_0^*, \\ u'(0) = u''(0) = v'(0) = v''(0) = 0, \\ u'''(0) = J^{r} u(\tau),\ v'''(0) = J^{\varphi} v(\varsigma), \quad r > 0, \varphi > 0, \end{cases}$$

where $\alpha, \beta \in (3,4)$, $\gamma, \rho \in (0,3)$, $\tau, \varsigma \in (0,1)$, D^{α}, D^{ρ}, D^{β} and D^{γ} denote the Caputo fractional derivatives and J^{r}, J^{φ} denote the Riemann-Liouville fractional integrals, $J := [0,1]$, $u_0^*, v_0^* \in \mathbb{R}$. For each $i = 1, \ldots, m$, f_i and $g_i : J \times \mathbb{R}^4 \to \mathbb{R}$ are specific functions.

In this paper, we discuss the existence, uniqueness and Ulam stability of solutions for the following singular fractional coupled system:

$$\begin{cases} D^{\alpha_1} x_1(t) = f_1(t, x_1(t)), \\ D^{\alpha_2} x_2(t) = f_2(t, x_1(t), x_2(t)), \\ \vdots \\ D^{\alpha_n} x_n(t) = f_n(t, x_1(t), x_2(t) \ldots, x_n(t)), \\ 0 < t \leq 1, \quad k-1 < \alpha_k < k, \quad k = 1, 2, \ldots, n, \\ x_1(0) = a_0^1, \quad k = 1, \\ x_k^{(j)}(0) = a_j^k, \quad j = 0, 1, \ldots, k-2, \quad k = 2, 3, \ldots, n, \\ D^{\delta_{k-1}} x_k(1) = 0, \quad k-2 < \delta_{k-1} < k-1, \quad k = 2, 3, \ldots, n, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N} - \{0,1\}$. For all $k = 1,2,\ldots,n$, the functions $f_k : (0,1] \times \mathbb{R}^k \to \mathbb{R}$ are continuous, singular at $t = 0$, $\lim_{t\to 0^+} f_k(t) = \infty$ and there exist $\beta_k \in (0,1)$, $k = 1,2,\ldots,n$, such that $t^{\beta_k} f_k$, $k = 1,2,\ldots,n$, are continuous on $[0,1]$.

To the best of our knowledge, there are no papers that have considered this kind of singular fractional coupled system.

We present some basic definitions and lemmas that we need to prove our main results. It can be found in [16].

Definition 1.1. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a continuous function f on $[0,\infty)$ is defined as:

$$J^\alpha f(t) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds, & \alpha > 0, \\ f(t), & \alpha = 0, \end{cases}$$

where $t \geq 0$ and $\Gamma(\alpha) := \int_0^{+\infty} e^{-u} u^{\alpha-1} du$.

Definition 1.2. The Caputo derivative of order α for a function $x : [0,+\infty) \to \mathbb{R}$, which is at least k-times differentiable can be defined as the following:

$$D^\alpha x(t) = \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} x^{(k)}(s)\, ds = J^{k-\alpha} x^{(k)}(t),$$

for $k-1 < \alpha < k$, $k \in \mathbb{N} - \{0\}$.

Lemma 1.1. *Let $\alpha, \beta > 0$, and $k-1 < \alpha < k$, $k \in \mathbb{N} - \{0\}$, and let j be a positive integer. Then*

$$D^\alpha t^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} t^{\beta-\alpha-1}, \quad \beta > k,$$

and

$$D^\alpha t^j = 0, \quad j = 0,1,\ldots,k-1.$$

Lemma 1.2. *Let $q > p > 0$ and $f \in L^1([a,b])$. Then for all $t \in [a,b]$, we have*

$$D^p J^q f(t) = J^{q-p} f(t), \quad t \in [a,b].$$

Lemma 1.3. *Let $k-1 < \alpha < k$, $k \in \mathbb{N} - \{0\}$, and let j be a positive integer. Then, the general solution of the fractional differential equation $D^\alpha x(t) = 0$, is given by:*

$$x(t) = \sum_{j=0}^{k-1} c_j t^j, \quad (c_j)_{j=0,1,\ldots,k-1} \in \mathbb{R}.$$

Lemma 1.4. *Let $k \in \mathbb{N} - \{0\}$, $k-1 < \alpha < k$, and let j be a positive integer. Then,*

$$J^\alpha D^\alpha x(t) = x(t) + \sum_{j=0}^{k-1} c_j t^j, \quad (c_j)_{j=0,1,\ldots,k-1} \in \mathbb{R}.$$

Lemma 1.5 (Shauder fixed point theorem)**.** *Let (E,d) be a complete metric space, let U be a closed convex subset of E, and let $T : E \to E$ be a mapping such that the set $V := \{Tx : x \in U\}$ is relatively compact in E. Then T has at least one fixed point.*

We also prove the following auxiliary result to give the integral representation of (1.1).

Lemma 1.6. *Assume that* $k-1<\alpha_k<k,\ k=1,2,\ldots,n,\ n\in\mathbb{N}-\{0,1\}$ *and* $F_k\in C([0,1],\mathbb{R})$. *Then, the following system*

$$\begin{cases} D^{\alpha_1}x_1(t)=F_1(t),\\ D^{\alpha_2}x_2(t)=F_2(t),\\ \vdots\\ D^{\alpha_n}x_n(t)=F_n(t),\end{cases}$$

associated with the conditions:

$$\begin{cases} x_1(0)=a_0^1,\\ x_k^{(j)}(0)=a_j^k, \quad k=2,3,\ldots,n, j=0,1,\ldots,k-2,\\ D^{\delta_{k-1}}x_k(1)=0, \quad k-2<\delta_{k-1}<k-1,\end{cases} \tag{1.2}$$

has a unique solution $(x_1,x_2,\ldots,x_n)$, *where*

$$x_k(t)=\begin{cases} \displaystyle\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}F_1(s)\,ds+a_0^1, \quad k=1,\\ \\ \displaystyle\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}F_k(s)\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j\\ \displaystyle-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}F_k(s)\,ds, \quad k=2,3,\ldots,n.\end{cases} \tag{1.3}$$

Proof. Using Lemma 1.4, we obtain the following integral equation:

$$x_k(t)=\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}F_k(s)\,ds-\sum_{j=0}^{k-1}c_j^k t^j, \quad k=1,2,\ldots,n, \tag{1.4}$$

where

$$\begin{pmatrix} c_0^1 & 0 & \ldots & \ldots & \ldots & 0\\ c_0^2 & c_1^2 & 0 & \ldots & \ldots & 0\\ c_0^3 & c_1^3 & c_2^3 & 0 & \ldots & 0\\ \vdots & \vdots & \vdots & \ddots & 0 & 0\\ c_0^{n-1} & c_1^{n-1} & c_2^{n-1} & \ldots & c_{n-2}^{n-1} & 0\\ c_0^n & c_1^n & c_2^n & \ldots & c_{n-2}^n & c_{n-1}^n\end{pmatrix}\in M_n(\mathbb{R}).$$

Applying the conditions given in (1.2), we observe that

$$x_1(0)=-c_0^1=a_0^1,$$

and for all $k=2,3,\ldots,n$, we get

$$\begin{cases} x_k^{(j)}(0)=-j!c_j^k=a_j^k, \quad j=0,1,\ldots,k-2,\\ D^{\delta_{k-1}}x_k(1)=\displaystyle\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}F_k(s)\,ds-\frac{\Gamma(k)}{\Gamma(k-\delta_{k-1})}c_{k-1}^k=0,\\ k-2<\delta_{k-1}<k-1,\end{cases}$$

which implies that

$$c_0^1 = -a_0^1, \tag{1.5}$$

and

$$c_j^k = \begin{cases} -\dfrac{a_j^k}{j!}, \quad j = 0, 1, \ldots, k-2, \\ \dfrac{\Gamma(k-\delta_{k-1})}{\Gamma(k)} \displaystyle\int_0^1 \dfrac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})} F_k(s)\, ds, \quad j = k-1, \end{cases} \tag{1.6}$$

where $k = 2, 3, \ldots, n$.

Substituting (1.5) and (1.6) in (1.4), we find (1.3). The proof of Lemma 1.6 is thus achieved. □

Now, we introduce the Banach space

$$S := \{(x_1, x_2, \ldots, x_n) : x_k \in C([0,1], \mathbb{R}), k = 1, 2, \ldots, n\},$$

endowed with the norm:

$$\|(x_1, x_2, \ldots, x_n)\|_S = \max_{1\leq k\leq n} \|x_k\|_\infty, \quad \|x_k\|_\infty = \max_{t\in[0,1]} |x_k(t)|.$$

2. Existence and Uniqueness

In this section, we try to establish sufficient conditions for the existence and uniqueness of solutions to the problem (1.1).

Define the nonlinear operator $A : S \to S$ by

$$A(x_1, x_2, \ldots, x_n)(t) := (A_1(x_1)(t), A_2(x_1, x_2)(t), \ldots, A_n(x_1, x_2, \ldots, x_n)(t)),$$

such that, for all $t \in [0,1]$,

$$A_k(x_1, \ldots, x_k)(t) := \begin{cases} \displaystyle\int_0^t \dfrac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} f_1(s)\, ds + a_0^1, \quad k = 1, \\ \displaystyle\int_0^t \dfrac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} f_k(s, \ldots)\, ds + \sum_{j=0}^{k-2} \dfrac{a_j^k}{j!} t^j - \dfrac{\Gamma(k-\delta_{k-1})}{(k-1)!} t^{k-1} \\ \times \displaystyle\int_0^1 \dfrac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})} f_k(s, \ldots)\, ds, \quad k = 2, 3, \ldots, n. \end{cases}$$

Lemma 2.1. *Let* $k-1 < \alpha_k < k$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N} - \{0, 1\}$, $0 < \beta_k < 1$, $T_k : (0,1] \to \mathbb{R}$ *be continuous function and* $\lim_{t\to 0^+} T_k(t) = \infty$. *Assume that* $t^{\beta_k} T_k(t)$

is continuous on $[0,1]$. *Then*

$$x_k(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}T_1(s)\,ds+a_0^1, \quad k=1,\\ \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}T_k(s)\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\\ \times\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}T_k(s)\,ds, \quad k=2,3,\ldots,n,\end{cases}$$

is continuous on $[0,1]$.

Proof. By the continuity of $t^{\beta_k}T_k$ and

$$x_k(t)=\begin{cases}\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}s^{-\beta_1}s^{\beta_1}T_1(s)\,ds+a_0^1, \quad k=1,\\ \int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\\ \times\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds, \quad k=2,3,\ldots,n,\end{cases}$$

we get $x_k(0)=a_0^k$, $k=1,2,\ldots,n$. Then, we will divide the proof into three cases.

Case 1. For $t_0=0$ and for all $t\in(0,1]$, by the continuity of $t^{\beta_k}T_k$, there exist $M_1,\ldots,M_n>0$, such that for all $t\in[0,1]$, $\left|t^{\beta_k}T_k(t)\right|\leq M_k$. Therefore, we get

$$\begin{aligned}&|x_k(t)-x_k(0)|\\ &=\begin{cases}\left|\int_0^t \frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}s^{-\beta_1}s^{\beta_1}T_1(s)\,ds\right|, \quad k=1,\\ \left|\int_0^t \frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds+\sum_{j=1}^{k-2}\frac{a_j^k}{j!}t^j\right.\\ \left.-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1}\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}s^{-\beta_k}s^{\beta_k}T_k(s)\,ds\right|, \quad k=2,3,\ldots,n,\end{cases}\\ &\leq\begin{cases}\frac{M_1}{\Gamma(\alpha_1)}\int_0^t (t-s)^{\alpha_1-1}s^{-\beta_1}ds, \quad k=1,\\ \frac{M_k}{\Gamma(\alpha_k)}\int_0^t (t-s)^{\alpha_k-1}s^{-\beta_k}ds+\sum_{j=1}^{k-2}\frac{\left|a_j^k\right|}{j!}t^j+\frac{\Gamma(k-\delta_{k-1})M_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}t^{k-1}\\ \times\int_0^1 (1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds, \quad k=2,3,\ldots,n.\end{cases}\end{aligned}$$

Using Beta Euler function denoted by B, we obtain

$$\begin{aligned}
&\left|x_k(t)-x_k(0)\right| \\
&\leq \begin{cases}
\dfrac{M_1 t^{\alpha_1-\beta_1}}{\Gamma(\alpha_1)} \displaystyle\int_0^1 (1-u)^{\alpha_1-1} u^{-\beta_1} du, \quad k=1, \\
\dfrac{M_k t^{\alpha_k-\beta_k}}{\Gamma(\alpha_k)} \displaystyle\int_0^1 (1-u)^{\alpha_k-1} u^{-\beta_k} du + \sum_{j=1}^{k-2} \dfrac{\left|a_j^k\right|}{j!} t^j \\
+\dfrac{\Gamma(k-\delta_{k-1}) M_k B(\alpha_k-\delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} t^{k-1}, \quad k=2,3,\ldots,n,
\end{cases} \\
&\leq \begin{cases}
\dfrac{M_1 B(\alpha_1, 1-\beta_1) t^{\alpha_1-\beta_1}}{\Gamma(\alpha_1)}, \quad k=1, \\
\dfrac{M_k B(\alpha_k, 1-\beta_k) t^{\alpha_k-\beta_k}}{\Gamma(\alpha_k)} + \displaystyle\sum_{j=1}^{k-2} \dfrac{\left|a_j^k\right|}{j!} t^j \\
+\dfrac{\Gamma(k-\delta_{k-1}) M_k B(\alpha_k-\delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} t^{k-1}, \quad k=2,3,\ldots,n,
\end{cases} \\
&\rightarrow 0 \ \text{ as } t\rightarrow 0, \quad k=1,2,\ldots,n.
\end{aligned}$$

Case 2. For $t_0 \in (0,1)$ and for all $t \in (t_0, 1]$, we have

$$\begin{aligned}
&\left|x_k(t)-x_k(t_0)\right| \\
&\leq \begin{cases}
\left|\displaystyle\int_0^t \dfrac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} s^{-\beta_1} s^{\beta_1} T_1(s)\, ds - \int_0^{t_0} \dfrac{(t_0-s)^{\alpha_1-1}}{\Gamma(\alpha_1)} s^{-\beta_1} s^{\beta_1} T_1(s)\, ds\right|, \quad k=1, \\
\left|\displaystyle\int_0^t \dfrac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} s^{-\beta_k} s^{\beta_k} T_k(s)\, ds - \int_0^{t_0} \dfrac{(t_0-s)^{\alpha_k-1}}{\Gamma(\alpha_k)} s^{-\beta_k} s^{\beta_k} T_k(s)\, ds\right| \\
+\displaystyle\sum_{j=0}^{k-2} \dfrac{\left|a_j^k\right|}{j!} \left(t^j - t_0^j\right) + \dfrac{\Gamma(k-\delta_{k-1})}{(k-1)!} \left(t^{k-1} - t_0^{k-1}\right) \\
\times \left|\displaystyle\int_0^1 \dfrac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})} s^{-\beta_k} s^{\beta_k} T_k(s)\, ds\right|, \quad k=2,3,\ldots,n,
\end{cases} \\
&\leq \begin{cases}
\dfrac{M_1}{\Gamma(\alpha_1)} \left(\displaystyle\int_0^t (t-s)^{\alpha_1-1} s^{-\beta_1} ds - \int_0^{t_0} (t_0-s)^{\alpha_1-1} s^{-\beta_1} ds\right), \quad k=1, \\
\dfrac{M_k}{\Gamma(\alpha_k)} \left(\displaystyle\int_0^t (t-s)^{\alpha_k-1} s^{-\beta_k} ds - \int_0^{t_0} (t_0-s)^{\alpha_k-1} s^{-\beta_k} ds\right) \\
+\displaystyle\sum_{j=1}^{k-2} \dfrac{\left|a_j^k\right|}{j!} \left(t^j - t_0^j\right) + \dfrac{\Gamma(k-\delta_{k-1}) M_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} \left(t^{k-1} - t_0^{k-1}\right) \\
\times \displaystyle\int_0^1 (1-s)^{\alpha_k-\delta_{k-1}-1} s^{-\beta_k} ds, \quad k=2,3,\ldots,n.
\end{cases}
\end{aligned}$$

Therefore,

$$
\begin{aligned}
&\left|x_k(t)-x_k(t_0)\right| \\
&\leq \begin{cases}
\dfrac{M_1\left(t^{\alpha_1-\beta_1}-t_0^{\alpha_1-\beta_1}\right)B\left(\alpha_1,1-\beta_1\right)}{\Gamma\left(\alpha_1\right)}, \quad k=1, \\
\dfrac{M_k\left(t^{\alpha_k-\beta_k}-t_0^{\alpha_k-\beta_k}\right)B\left(\alpha_k,1-\beta_k\right)}{\Gamma\left(\alpha_k\right)}+\displaystyle\sum_{j=1}^{k-2}\frac{\left|a_j^k\right|}{j!}\left(t^j-t_0^j\right) \\
+\dfrac{\Gamma\left(k-\delta_{k-1}\right)M_kB\left(\alpha_k-\delta_{k-1},1-\beta_k\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\left(t^{k-1}-t_0^{k-1}\right), \quad k=2,3,\ldots,n,
\end{cases} \\
&\rightarrow 0, \text{ as } t\rightarrow t_0, \quad k=1,2,\ldots,n.
\end{aligned}
$$

Case 3. For t_0 $(0,1]$ and for all $t\in[0,t_0)$. Similarly, as in Case 2, it can be shown that

$$
\begin{aligned}
&\left|x_k(t)-x_k(t_0)\right| \\
&\leq \begin{cases}
\dfrac{M_1\left(t_0^{\alpha_1-\beta_1}-t^{\alpha_1-\beta_1}\right)B\left(\alpha_1,1-\beta_1\right)}{\Gamma\left(\alpha_1\right)}, \quad k=1, \\
\dfrac{M_k\left(t_0^{\alpha_k-\beta_k}-t^{\alpha_k-\beta_k}\right)B\left(\alpha_k,1-\beta_k\right)}{\Gamma\left(\alpha_k\right)}+\displaystyle\sum_{j=1}^{k-2}\frac{\left|a_j^k\right|}{j!}\left(t_0^j-t^j\right) \\
+\dfrac{\Gamma\left(k-\delta_{k-1}\right)M_kB\left(\alpha_k-\delta_{k-1},1-\beta_k\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\left(t_0^{k-1}-t^{k-1}\right), \quad k=2,3,\ldots,n,
\end{cases} \\
&\rightarrow 0, \text{ as } t\rightarrow t_0, \quad k=1,2,\ldots,n.
\end{aligned}
$$

This ends the proof. □

Lemma 2.2. *Let $k-1<\alpha_k<k$, $k=1,2,\ldots,n$, $n\in\mathbb{N}-\{0,1\}$, $0<\beta_k<1$, $f_k:(0,1]\times\mathbb{R}^k\rightarrow\mathbb{R}$ be continuous, and $\lim_{t\rightarrow0^+}f_k(t,\ldots)=\infty$. Assume that $t^{\beta_k}f_k(t,\ldots)$ is continuous on $[0,1]\times\mathbb{R}^k$. Then, the operator $A:S\rightarrow S$ is completely continuous.*

Proof. For all $(x_1,\ldots,x_n)\in S$, let

$$A(x_1,x_2\ldots,x_n)(t)=\left(A_1(x_1),A_2(x_1,x_2),\ldots,A_n(x_1,\ldots,x_n)\right)(t),$$

where

$$
\begin{aligned}
&A_k(x_1,\ldots,x_k)(t) \\
&:=\begin{cases}
\displaystyle\int_0^t\frac{(t-s)^{\alpha_1-1}}{\Gamma(\alpha_1)}f_1(s,x_1(s))\,ds+a_0^1, \quad k=1, \\
\displaystyle\int_0^t\frac{(t-s)^{\alpha_k-1}}{\Gamma(\alpha_k)}f_k(s,x_1(s),\ldots,x_k(s))\,ds+\sum_{j=0}^{k-2}\frac{a_j^k}{j!}t^j-\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}t^{k-1} \\
\displaystyle\times\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}}{\Gamma(\alpha_k-\delta_{k-1})}f_k(s,x_1(s),\ldots,x_k(s))\,ds, \quad k=2,3,\ldots,n.
\end{cases}
\end{aligned}
$$

By Lemma 2.1, we have $A : S \to S$. Let

$$\left(x_1^0, x_2^0, \ldots, x_n^0\right) \in S : \left\|\left(x_1^0, x_2^0, \ldots, x_n^0\right)\right\|_S = \lambda_0$$

and

$$(x_1, x_2, \ldots, x_n) \in S : \ \left\|(x_1, x_2, \ldots, x_n) - \left(x_1^0, x_2^0, \ldots, x_n^0\right)\right\|_S < 1,$$

then

$$\|(x_1, x_2, \ldots, x_n)\|_S < 1 + \lambda_0 = \lambda.$$

By the continuity of $t^{\beta_k} f_k\left(t, x_1, \ldots, x_k\right)$, we know that $t^{\beta_k} f_k\left(t, x_1, \ldots, x_k\right)$ is uniformly continuous on $[0,1] \times [-\lambda, \lambda]^k$.

Hence, for all $t \in [0,1]$ and for each $\epsilon > 0$, there exists $\rho > 0\,(\rho < 1)$, with

$$\left|t^{\beta_k} f_k\left(t, x_1\left(t\right), \ldots, x_k\left(t\right)\right) - t^{\beta_k} f_k\left(t, x_1^0\left(t\right), \ldots, x_k^0\left(t\right)\right)\right| < \epsilon, \tag{2.1}$$

where $(x_1, x_2, \ldots, x_n) \in S$, and $\|(x_1, x_2, \ldots, x_n) - (x_1^0, x_2^0, \ldots, x_n^0)\|_S < \rho$. Then

$$\begin{aligned} &\left\|A\left(x_1, x_2 \ldots, x_n\right) - A\left(x_1^0, x_2^0, \ldots, x_n^0\right)\right\|_S \\ &= \max_{1 \le k \le n} \left\|A_k\left(x_1, \ldots, x_k\right)(t) - A_k\left(x_1^0, \ldots, x_k^0\right)(t)\right\|_\infty . \end{aligned} \tag{2.2}$$

We have

$$\begin{aligned} &\left\|A_k\left(x_1, \ldots, x_k\right)(t) - A_k\left(x_1^0, \ldots, x_k^0\right)(t)\right\|_\infty \\ &\le \begin{cases} \max\limits_{t\in[0,1]} \displaystyle\int_0^t \frac{(t-s)^{\alpha_1-1} s^{-\beta_1}}{\Gamma\left(\alpha_1\right)} \left|s^{\beta_1} f_1\left(s, x_1\left(s\right)\right) - s^{\beta_1} f_1\left(s, x_1^0\left(s\right)\right)\right| ds, \quad k=1, \\ \max\limits_{t\in[0,1]} \displaystyle\int_0^t \frac{(t-s)^{\alpha_k-1} s^{-\beta_k}}{\Gamma\left(\alpha_k\right)} \left|s^{\beta_k} f_k\left(s, x_1\left(s\right), \ldots, x_k\left(s\right)\right)\right. \\ \left. -s^{\beta_k} f_k\left(s, x_1^0\left(s\right), \ldots, x_k^0\left(s\right)\right)\right| ds + \max\limits_{t\in[0,1]} \dfrac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!} t^{k-1} \\ \times \displaystyle\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1} s^{-\beta_k}}{\Gamma\left(\alpha_k-\delta_{k-1}\right)} \left|s^{\beta_k} f_k\left(s, x_1\left(s\right), \ldots, x_k\left(s\right)\right)\right. \\ \left. -s^{\beta_k} f_k\left(s, x_1^0\left(s\right), \ldots, x_k^0\left(s\right)\right)\right| ds, \quad k = 2, 3, \ldots, n. \end{cases} \end{aligned}$$

Using (2.1), we obtain

$$\begin{aligned} &\left\|A_k\left(x_1, \ldots, x_k\right)(t) - A_k\left(x_1^0, \ldots, x_k^0\right)(t)\right\|_\infty \\ &\le \begin{cases} \dfrac{\epsilon}{\Gamma\left(\alpha_1\right)} \max\limits_{t\in[0,1]} \displaystyle\int_0^t (t-s)^{\alpha_1-1} s^{-\beta_1} ds, \ k=1, \\ \\ \dfrac{\epsilon}{\Gamma\left(\alpha_k\right)} \max\limits_{t\in[0,1]} \displaystyle\int_0^t (t-s)^{\alpha_k-1} s^{-\beta_k} ds \\ + \dfrac{\epsilon\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)} \max\limits_{t\in[0,1]} t^{k-1} \displaystyle\int_0^1 (1-s)^{\alpha_k-\delta_{k-1}-1} s^{-\beta_k} ds, \\ k = 2, 3, \ldots, n, \end{cases} \end{aligned} \tag{2.3}$$

$$\leq \begin{cases} \epsilon \dfrac{B(\alpha_1, 1-\beta_1)}{\Gamma(\alpha_1)} \max\limits_{t\in[0,1]} t^{\alpha_1-\beta_1}, \quad k=1, \\ \epsilon \left(\dfrac{B(\alpha_k, 1-\beta_k)}{\Gamma(\alpha_k)} \max\limits_{t\in[0,1]} t^{\alpha_k-\beta_k} + \dfrac{\Gamma(k-\delta_{k-1}) B(\alpha_k-\delta_{k-1}, 1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})} \right), \\ k=2,3,\ldots,n, \end{cases}$$

$$= \begin{cases} \epsilon \dfrac{\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}, \quad k=1, \\ \\ \epsilon \left(\dfrac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)} + \dfrac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)} \right), \quad k=2,3,\ldots,n. \end{cases}$$

We pose:

$$\begin{aligned} \Lambda_1 &:= \frac{\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}, \\ \Lambda_k &:= \frac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)} + \frac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)}. \end{aligned} \tag{2.4}$$

By (2.3) and (2.4), we have

$$\left\| A_k(x_1,\ldots,x_k)(t) - A_k\left(x_1^0,\ldots,x_k^0\right)(t) \right\|_\infty \leq \begin{cases} \epsilon\Lambda_1, & k=1, \\ \epsilon\Lambda_k, & k=2,3,\ldots,n. \end{cases} \tag{2.5}$$

Thanks to (2.2) and (2.5), we get

$$\left\| A(x_1,x_2\ldots,x_n) - A\left(x_1^0,x_2^0,\ldots,x_n^0\right) \right\|_S \leq \epsilon \max_{1\leq k\leq n} \Lambda_k.$$

Therefore,

$$\left\| A(x_1,x_2\ldots,x_n) - A\left(x_1^0,x_2^0,\ldots,x_n^0\right) \right\|_S \to 0,$$

as

$$\left\| (x_1,x_2\ldots,x_n) - \left(x_1^0,x_2^0,\ldots,x_n^0\right) \right\|_S \to 0.$$

Hence, $A : S \to S$ is continuous.

Let $\theta \subset S$ be bounded. Then, there exists a positive constant ς such that $\|(x_1,x_2\ldots,x_n)\|_S \leq \varsigma$, for all $(x_1,x_2\ldots,x_n) \in \theta$. Since $t^{\beta_k} f_k(t,x_1,\ldots,x_k)$, $k = 1,2,\ldots,n$, are continuous on $[0,1]\times[-\varsigma,\varsigma]^k$, there exist positive constants L_k, $k = 1,2,\ldots,n$, such that

$$\left| t^{\beta_k} f_k(t,x_1(t),\ldots,x_k(t)) \right| \leq L_k, \quad \text{for all } t\in[0,1], \text{ for all } (x_1,x_2\ldots,x_n)\in\theta. \tag{2.6}$$

Then

$$\|A(x_1,x_2\ldots,x_n)\|_S = \max_{1\leq k\leq n} \|A_k(x_1,\ldots,x_k)\|_\infty. \tag{2.7}$$

We have

$$\|A_k(x_1,\dots,x_k)\|_\infty \leq \begin{cases} \max\limits_{t\in[0,1]}\displaystyle\int_0^t \frac{(t-s)^{\alpha_1-1}s^{-\beta_1}}{\Gamma(\alpha_1)}\left|s^{\beta_1}f_1(s,x_1)\right|ds+\left|a_0^1\right|, \quad k=1,\\ \max\limits_{t\in[0,1]}\displaystyle\int_0^t \frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma(\alpha_k)}\left|s^{\beta_k}f_k(s,x_1,\dots,x_k)\right|ds \\ +\displaystyle\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\max_{t\in[0,1]}t^j+\frac{\Gamma(k-\delta_{k-1})}{(k-1)!}\max_{t\in[0,1]}t^{k-1} \\ \times\displaystyle\int_0^1 \frac{(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}}{\Gamma(\alpha_k-\delta_{k-1})}\left|s^{\beta_k}f_k(s,x_1,\dots,x_k)\right|ds, \\ k=2,3,\dots,n. \end{cases}$$

Using (2.6), we get

$$\|A_k(x_1,\dots,x_k)\|_\infty \tag{2.8}$$

$$\leq \begin{cases} \dfrac{L_1}{\Gamma(\alpha_1)}\max\limits_{t\in[0,1]}\displaystyle\int_0^t (t-s)^{\alpha_1-1}s^{-\beta_1}ds+\left|a_0^1\right|, \quad k=1,\\ \dfrac{L_k}{\Gamma(\alpha_k)}\max\limits_{t\in[0,1]}\displaystyle\int_0^t (t-s)^{\alpha_k-1}s^{-\beta_k}ds+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!} \\ +\dfrac{\Gamma(k-\delta_{k-1})L_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}\int_0^1(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds, \quad k=2,3,\dots,n, \end{cases}$$

$$\leq \begin{cases} \dfrac{L_1\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}\max\limits_{t\in[0,1]}t^{\alpha_1-\beta_1}+\left|a_0^1\right|, \quad k=1,\\ L_k\left(\dfrac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)}\max\limits_{t\in[0,1]}t^{\alpha_k-\beta_k}+\dfrac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)}\right) \\ +\displaystyle\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, \quad k=2,3,\dots,n, \end{cases}$$

$$\leq \begin{cases} L_1\Lambda_1+\left|a_0^1\right|, \quad k=1,\\ L_k\Lambda_k+\displaystyle\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, \quad k=2,3,\dots,n. \end{cases}$$

Then by (2.7) and (2.8), we get

$$\|A(x_1,x_2\dots,x_n)\|_S \leq \max_{2\leq k\leq n}\left\{L_1\Lambda_1+\left|a_0^1\right|, L_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\right\}.$$

Thus, $A(\theta)$ is bounded.

For all $(x_1,x_2\dots,x_n)\in\theta$, and for all $t_1,t_2\in[0,1]$, $t_1<t_2$, we have:

$$\|A(x_1,x_2\dots,x_n)(t_2)-A(x_1,x_2\dots,x_n)(t_1)\|_S$$

$$= \max_{1\leq k\leq n} \left\| A_k\left(x_1,\ldots,x_k\right)\left(t_2\right) - A_k\left(x_1,\ldots,x_k\right)\left(t_1\right)\right\|_\infty . \tag{2.9}$$

Then

$$\begin{aligned}
&\left\| A_k\left(x_1,\ldots,x_k\right)\left(t_2\right) - A_k\left(x_1,\ldots,x_k\right)\left(t_1\right)\right\|_\infty \\
&\leq \begin{cases}
\max\limits_{t\in[0,1]} \left| \displaystyle\int_0^{t_2} \frac{\left(t_2-s\right)^{\alpha_1-1} s^{-\beta_1}}{\Gamma\left(\alpha_1\right)} s^{\beta_1} f_1\left(s,x_1\right) ds \right. \\
\left. - \displaystyle\int_0^{t_1} \frac{\left(t_1-s\right)^{\alpha_1-1} s^{-\beta_1}}{\Gamma\left(\alpha_1\right)} s^{\beta_1} f_1\left(s,x_1\right) ds \right|, \quad k=1, \\
\max\limits_{t\in[0,1]} \left| \displaystyle\int_0^{t_2} \frac{\left(t_2-s\right)^{\alpha_k-1} s^{-\beta_k}}{\Gamma\left(\alpha_k\right)} s^{\beta_k} f_k\left(s,x_1,\ldots,x_k\right) ds \right. \\
\left. - \displaystyle\int_0^{t_1} \frac{\left(t_1-s\right)^{\alpha_k-1} s^{-\beta_k}}{\Gamma\left(\alpha_k\right)} s^{\beta_k} f_k\left(s,x_1,\ldots,x_k\right) ds \right| \\
+ \displaystyle\sum_{j=0}^{k-2} \frac{\left|a_j^k\right|}{j!} \left(t_2^j - t_1^j\right) + \frac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!} \left(t_2^{k-1} - t_1^{k-1}\right) \\
\times \displaystyle\int_0^1 \frac{\left(1-s\right)^{\alpha_k-\delta_{k-1}-1} s^{-\beta_k}}{\Gamma\left(\alpha_k-\delta_{k-1}\right)} \left| s^{\beta_k} f_k\left(s,x_1,\ldots,x_k\right)\right| ds, \quad k=2,3,\ldots,n.
\end{cases}
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left\| A_k\left(x_1,\ldots,x_k\right)\left(t_2\right) - A_k\left(x_1,\ldots,x_k\right)\left(t_1\right)\right\|_\infty \\
&\leq \begin{cases}
\dfrac{L_1\Gamma\left(1-\beta_1\right)}{\Gamma\left(\alpha_1+1-\beta_1\right)} \left(t_2^{\alpha_1-\beta_1} - t_1^{\alpha_1-\beta_1}\right), \quad k=1, \\[2ex]
\dfrac{L_k\Gamma\left(1-\beta_k\right)\left(t_2^{\alpha_k-\beta_k} - t_1^{\alpha_k-\beta_k}\right)}{\Gamma\left(\alpha_k+1-\beta_k\right)} + \displaystyle\sum_{j=0}^{k-2} \frac{\left|a_j^k\right|\left(t_2^j - t_1^j\right)}{j!} \\[2ex]
+ \dfrac{\Gamma\left(k-\delta_{k-1}\right) L_k\Gamma\left(1-\beta_k\right)\left(t_2^{k-1} - t_1^{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}+1-\beta_k\right)}, \quad k=2,3,\ldots,n.
\end{cases}
\end{aligned} \tag{2.10}$$

Then, by (2.9) and (2.10), we obtain

$$\begin{aligned}
&\left\| A\left(x_1,x_2\ldots,x_n\right)\left(t_2\right) - A\left(x_1,x_2\ldots,x_n\right)\left(t_1\right)\right\|_S \\
&\leq \max\left\{ \frac{L_1\Gamma\left(1-\beta_1\right)}{\Gamma\left(\alpha_1+1-\beta_1\right)} \left(t_2^{\alpha_1-\beta_1} - t_1^{\alpha_1-\beta_1}\right), \frac{L_k\Gamma\left(1-\beta_k\right)\left(t_2^{\alpha_k-\beta_k} - t_1^{\alpha_k-\beta_k}\right)}{\Gamma\left(\alpha_k+1-\beta_k\right)} \right. \\
&\quad \left. + \sum_{j=0}^{k-2} \frac{\left|a_j^k\right|\left(t_2^j - t_1^j\right)}{j!} + \frac{\Gamma\left(k-\delta_{k-1}\right) L_k\Gamma\left(1-\beta_k\right)\left(t_2^{k-1} - t_1^{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}+1-\beta_k\right)} \right\}.
\end{aligned} \tag{2.11}$$

The right-hand side of (2.11) is independent of $(x_1, x_2 \ldots, x_n)$ and tends to zero as $t_1 \to t_2$. Thus $A(\theta)$ is equicontinuous. By Arzela-Ascoli theorem, A is completely continuous. □

Theorem 2.1. *Assume that there exist nonnegative constants* $\left(\omega_j^k\right)_{j=1,\ldots,k}^{k=1,\ldots,n}$, *satisfying*

$$t^{\beta_k}\left|f_k\left(t,x_1,\ldots,x_k\right)-f_k\left(t,y_1,\ldots,y_k\right)\right|\leq\sum_{j=1}^{k}\omega_j^k\left|x_j-y_j\right|, \tag{2.12}$$

for all $t\in[0,1]$ *and all* $\left(x_1,\ldots,x_k\right),\left(y_1,\ldots,y_k\right)\in\mathbb{R}^k$.

If

$$\Sigma:=\max_{2\leq k\leq n}\left(\omega_1^1\Lambda_1,\sum_{j=1}^{k}\omega_j^k\Lambda_k\right)<1, \tag{2.13}$$

then the system (1.1) *has a unique solution on* $[0,1]$.

Proof. We will prove that A is a contractive operator on S.

Let $\left(x_1,x_2\ldots,x_n\right),\left(y_1,y_2\ldots,y_n\right)\in S$ and $t\in[0,1]$, we have

$$\begin{aligned}&\left\|A\left(x_1,x_2\ldots,x_n\right)-A\left(y_1,y_2\ldots,y_n\right)\right\|_S\\ &=\max_{1\leq k\leq n}\left\|A_k\left(x_1,\ldots,x_k\right)(t)-A_k\left(y_1,\ldots,y_k\right)(t)\right\|_\infty.\end{aligned} \tag{2.14}$$

Then

$$\begin{aligned}&\left\|A_k\left(x_1,\ldots,x_k\right)(t)-A_k\left(y_1,\ldots,y_k\right)(t)\right\|_\infty\\ &\leq\begin{cases}\displaystyle\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_1-1}s^{-\beta_1}}{\Gamma\left(\alpha_1\right)}s^{\beta_1}\left|f_1\left(s,x_1(s)\right)-f_1\left(s,y_1(s)\right)\right|ds, \quad k=1,\\ \displaystyle\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma\left(\alpha_k\right)}s^{\beta_k}\left|f_k\left(s,x_1(s),\ldots,x_k(s)\right)\right.\\ \displaystyle\left.-f_k\left(s,y_1(s),\ldots,y_k(s)\right)\right|ds+\max_{t\in[0,1]}\frac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!}t^{k-1}\\ \displaystyle\times\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}}{\Gamma\left(\alpha_k-\delta_{k-1}\right)}s^{\beta_k}\left|f_k\left(s,x_1(s),\ldots,x_k(s)\right)\right.\\ \left.-f_k\left(s,y_1(s),\ldots,y_k(s)\right)\right|ds, \quad k=2,3,\ldots,n.\end{cases}\end{aligned}$$

Thanks to (2.12), we can write

$$\begin{aligned}&\left\|A_k\left(x_1,\ldots,x_k\right)(t)-A_k\left(y_1,\ldots,y_k\right)(t)\right\|_\infty\\ &\leq\begin{cases}\displaystyle\frac{\omega_1^1}{\Gamma\left(\alpha_1\right)}\left\|x_1-y_1\right\|_\infty\max_{t\in[0,1]}\int_0^t(t-s)^{\alpha_1-1}s^{-\beta_1}ds, \quad k=1,\\ \left(\omega_1^k\left\|x_1-y_1\right\|_\infty+\cdots+\omega_k^k\left\|x_k-y_k\right\|_\infty\right)\\ \displaystyle\times\left(\max_{t\in[0,1]}\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma\left(\alpha_k\right)}ds\right.\\ \displaystyle\left.+\frac{\Gamma\left(k-\delta_{k-1}\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\int_0^1(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds\right), \quad k=2,3,\ldots,n,\end{cases}\end{aligned} \tag{2.15}$$

$$\leq \begin{cases} \dfrac{\omega_1^1 B\left(\alpha_1, 1-\beta_1\right)\left\|x_1-y_1\right\|_\infty}{\Gamma\left(\alpha_1\right)} \max\limits_{t\in[0,1]} t^{\alpha_1-\beta_1}, \quad k=1, \\ \sum\limits_{j=1}^{k} \omega_j^k \max\limits_{1\leq k\leq n}\left\|x_k-y_k\right\|_\infty \left(\dfrac{B\left(\alpha_k, 1-\beta_k\right)}{\Gamma\left(\alpha_k\right)} \max\limits_{t\in[0,1]} t^{\alpha_k-\beta_k}\right. \\ \left. +\dfrac{\Gamma\left(k-\delta_{k-1}\right) B\left(\alpha_k-\delta_{k-1}, 1-\beta_k\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}\right)}\right), \quad k=2,3,\ldots,n, \end{cases}$$

$$\leq \begin{cases} \dfrac{\omega_1^1 \Gamma\left(1-\beta_1\right)}{\Gamma\left(\alpha_1+1-\beta_1\right)}\left\|x_1-y_1\right\|_\infty, \quad k=1, \\ \sum\limits_{j=1}^{k} \omega_j^k \left(\dfrac{\Gamma\left(1-\beta_k\right)}{\Gamma\left(\alpha_k+1-\beta_k\right)}+\dfrac{\Gamma\left(k-\delta_{k-1}\right)\Gamma\left(1-\beta_k\right)}{(k-1)!\Gamma\left(\alpha_k-\delta_{k-1}+1-\beta_k\right)}\right) \\ \times\left\|\left(x_1-y_1,\ldots,x_k-y_k\right)\right\|_S, \quad k=2,3,\ldots,n. \end{cases}$$

By (2.14) and (2.15), we obtain

$$\begin{aligned} &\left\|A\left(x_1,x_2,\ldots,x_n\right)-A\left(y_1,y_2,\ldots,y_n\right)\right\|_S \\ &\leq \max_{2\leq k\leq n}\left(\omega_1^1\Lambda_1, \sum_{j=1}^{k}\omega_j^k\Lambda_k\right)\left\|\left(x_1-y_1,\ldots,x_k-y_k\right)\right\|_S. \end{aligned}$$

By (2.13), we have $\Sigma := \max_{2\leq k\leq n}\left(\omega_1^1\Lambda_1, \sum_{j=1}^{k}\omega_j^k\Lambda_k\right) < 1$. Hence, A is a contractive operator. Consequently, by Banach fixed point theorem, A has a fixed point which is the unique solution of system (1.1). This completes the proof. □

Example 2.1. Consider the following singular fractional system:

$$\begin{cases} D^{\frac{3}{4}} x_1(t) = \dfrac{\sin x_1(t)}{12\pi\sqrt{t}}, \quad 0<t\leq 1, \\ D^{\frac{3}{2}} x_2(t) = \dfrac{\cos x_1(t)-\cos x_2(t)}{16\pi^3 t^{\frac{5}{7}}}, \quad 0<t\leq 1, \\ D^{\frac{7}{3}} x_3(t) = \dfrac{\left(\sin x_1(t)+\sin x_2(t)+\cos x_3(t)\right)}{24\pi t^{\frac{3}{8}}}, \quad 0<t\leq 1, \\ D^{\frac{7}{2}} x_4(t) = \dfrac{\left|x_1(t)+x_2(t)+x_3(t)+x_4(t)\right|}{32\pi t^{\frac{1}{3}}\left(1+\left|x_1(t)+x_2(t)+x_3(t)+x_4(t)\right|\right)}, \quad 0<t\leq 1, \\ x_1(0)=1, \\ x_2(0)=\sqrt{2},\ D^{\frac{1}{2}}x_2(1)=0, \\ x_3(0)=\dfrac{3}{5},\ x_3'(0)=2\sqrt{3},\ D^{\frac{4}{3}}x_3(1)=0, \\ x_4(0)=\dfrac{1}{2},\ x_4'(0)=\sqrt{5},\ x_4''(0)=1,\ D^{\frac{5}{2}}x_4(1)=0. \end{cases} \tag{2.16}$$

We have:

$$\begin{aligned} &n=4, \quad \alpha_1=\frac{3}{4}, \quad \alpha_2=\frac{3}{2}, \quad \alpha_3=\frac{7}{3}, \quad \alpha_4=\frac{7}{2}, \quad \delta_1=\frac{1}{2}, \quad \delta_2=\frac{4}{3}, \quad \delta_3=\frac{5}{2}, \\ &a_0^1=1, \quad a_0^2=\sqrt{2}, \quad a_0^3=\frac{3}{5}, \quad a_1^3=2\sqrt{3}, \quad a_0^4=\frac{1}{2}, \quad a_1^4=\sqrt{5}, \quad a_2^4=1. \end{aligned}$$

Then, for each $t \in [0,1]$ and (x_1, x_2, x_3, x_4), $(y_1, y_2, y_3, y_4) \in \mathbb{R}^4$, we have:

$$t^{\frac{2}{3}} |f_1(t, x_1) - f_1(t, y_1)| \leq \frac{t^{\frac{1}{6}}}{12\pi} |x_1 - y_1|,$$

$$t^{\frac{6}{7}} |f_2(t, x_1, x_2) - f_2(t, y_1, y_2)| \leq \frac{t^{\frac{1}{7}}}{16\pi^3} |x_1 - y_1| + \frac{t^{\frac{1}{7}}}{16\pi^3} |x_2 - y_2|,$$

$$t^{\frac{7}{8}} |f_3(t, x_1, x_2, x_3) - f_3(t, y_1, y_2, y_3)|$$
$$\leq \left(\frac{\sqrt{t}}{24\pi} |x_1 - y_1| + \frac{\sqrt{t}}{24\pi} |x_2 - y_2| + \frac{\sqrt{t}}{24\pi} |x_3 - y_3| \right),$$

$$t^{\frac{1}{2}} |f_4(t, x_1, x_2, x_3, x_4) - f_4(t, y_1, y_2, y_3, y_4)|$$
$$\leq \left(\frac{t^{\frac{1}{6}}}{32\pi} |x_1 - y_1| + \frac{t^{\frac{1}{6}}}{32\pi} |x_2 - y_2| + \frac{t^{\frac{1}{6}}}{32\pi} |x_3 - y_3| + \frac{t^{\frac{1}{6}}}{32\pi} |x_4 - y_4| \right),$$

where $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{6}{7}$, $\beta_3 = \frac{7}{8}$, $\beta_4 = \frac{1}{2}$.

Moreover, we can take:

$$\omega_1^1 = \frac{1}{12\pi},$$
$$\omega_1^2 = \omega_2^2 = \frac{1}{16\pi^3}, \quad \sum_{j=1}^{2} \omega_j^2 = \frac{1}{8\pi^3},$$
$$\omega_1^3 = \omega_2^3 = \omega_3^3 = \frac{1}{24\pi}, \quad \sum_{j=1}^{3} \omega_j^3 = \frac{1}{8\pi},$$
$$\omega_1^4 = \omega_2^4 = \omega_3^4 = \omega_4^4 = \frac{1}{32\pi}, \quad \sum_{j=1}^{4} \omega_j^4 = \frac{1}{8\pi}.$$

On the other hand, we get

$$\Lambda_1 = 2.7958, \quad \Lambda_2 = 13.4869, \quad \Lambda_3 = 9.4443, \quad \Lambda_4 = 0.5908.$$

Thus,

$$\omega_1^1 \Lambda_1 = 0.0742, \quad \sum_{j=1}^{2} \omega_j^2 \Lambda_2 = 0.0544, \quad \sum_{j=1}^{3} \omega_j^3 \Lambda_3 = 0.3759, \quad \sum_{j=1}^{4} \omega_j^4 \Lambda_4 = 0.0235.$$

Then the singular fractional system (2.16) has a unique solution on $[0,1]$.

Theorem 2.2. *Let $k - 1 < \alpha_k < k$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N} - \{0, 1\}$, $0 < \beta_k < 1$. Assume that $f_k : (0,1] \times \mathbb{R}^k \to \mathbb{R}$ is continuous with $\lim_{t \to 0^+} f_k(t, \ldots) = \infty$ and $t^{\beta_k} f_k(t, \ldots)$ is continuous on $[0,1] \times \mathbb{R}^k$. Then, the system (1.1) has at least one solution on $[0,1]$.*

Proof. Let $P_k = \max_{t \in [0,1]} t^{\beta_k} |f_k(t, x_1(t), \ldots, x_k(t))|$, and define the set $\Delta \subset S$ by

$$\Delta := \{(x_1, x_2, \ldots, x_n) \in S : \|(x_1, x_2, \ldots, x_n)\|_S \leq r\},$$

where

$$r=\max_{2\leq k\leq n}\left(P_1\Lambda_1+\left|a_0^1\right|,P_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\right).$$

We will prove that $A:\Delta\to\Delta$. For $(x_1,x_2,\dots,x_n)\in\Delta$ and $t\in[0,1]$, we have

$$\left\|A\left(x_1,x_2,\dots,x_n\right)\right\|_S=\max_{1\leq k\leq n}\left\|A_k\left(x_1,\dots,x_k\right)(t)\right\|_\infty. \tag{2.17}$$

Then

$$\begin{aligned}
&\left\|A_k\left(x_1,\dots,x_k\right)(t)\right\|_\infty\\
&\leq\begin{cases}\max\limits_{t\in[0,1]}\displaystyle\int_0^t\frac{(t-s)^{\alpha_1-1}s^{-\beta_1}}{\Gamma(\alpha_1)}s^{\beta_1}\left|f_1(s,x_1)\right|ds+\left|a_0^1\right|, \quad k=1,\\
\max\limits_{t\in[0,1]}\displaystyle\int_0^t\frac{(t-s)^{\alpha_k-1}s^{-\beta_k}}{\Gamma(\alpha_k)}s^{\beta_k}\left|f_k(s,x_1,\dots,x_k)\right|ds+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\max_{t\in[0,1]}t^j\\
+\dfrac{\Gamma(k-\delta_{k-1})}{(k-1)!}\max\limits_{t\in[0,1]}t^{k-1}\displaystyle\int_0^1\frac{(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}}{\Gamma(\alpha_k-\delta_{k-1})}s^{\beta_k}\left|f_k(s,x_1,\dots,x_k)\right|ds,\\
k=2,3,\dots,n,\end{cases}\\
&\leq\begin{cases}\dfrac{P_1}{\Gamma(\alpha_1)}\max\limits_{t\in[0,1]}\displaystyle\int_0^t(t-s)^{\alpha_1-1}s^{-\beta_1}ds+\left|a_0^1\right|, \quad k=1,\\
\dfrac{P_k}{\Gamma(\alpha_k)}\max\limits_{t\in[0,1]}\displaystyle\int_0^t(t-s)^{\alpha_k-1}s^{-\beta_k}ds+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\\
+\dfrac{\Gamma(k-\delta_{k-1})P_k}{(k-1)!\Gamma(\alpha_k-\delta_{k-1})}\displaystyle\int_0^1(1-s)^{\alpha_k-\delta_{k-1}-1}s^{-\beta_k}ds, \quad k=2,3,\dots,n.\end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|A_k\left(x_1,\dots,x_k\right)(t)\right\|_\infty \\
&\leq\begin{cases}\dfrac{P_1\Gamma(1-\beta_1)}{\Gamma(\alpha_1+1-\beta_1)}\max\limits_{t\in[0,1]}t^{\alpha_1-\beta_1}+|a_0^1|, \quad k=1,\\
P_k\left(\dfrac{\Gamma(1-\beta_k)}{\Gamma(\alpha_k+1-\beta_k)}\max\limits_{t\in[0,1]}t^{\alpha_k-\beta_k}+\dfrac{\Gamma(k-\delta_{k-1})\Gamma(1-\beta_k)}{(k-1)!\Gamma(\alpha_k-\delta_{k-1}+1-\beta_k)}\right),\\
+\displaystyle\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, \quad k=2,3,\dots,n,\end{cases}\\
&\leq\begin{cases}P_1\Lambda_1+|a_0^1|, \quad k=1,\\
P_k\Lambda_k+\displaystyle\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}, \quad k=2,3,\dots,n.\end{cases}
\end{aligned} \tag{2.18}$$

Using (2.17) and (2.18), we can write

$$\left\|A\left(x_1,x_2,\dots,x_n\right)\right\|_S\leq\max_{2\leq k\leq n}\left(P_1\Lambda_1+\left|a_0^1\right|,P_k\Lambda_k+\sum_{j=0}^{k-2}\frac{\left|a_j^k\right|}{j!}\right). \tag{2.19}$$

Hence, $\|A(x_1, x_2, \ldots, x_n)\|_S \leq r$. By Lemma 2.1, we have $A(x_1, x_2, \ldots, x_n)(t) \in C([0,1])$. Moreover, for $(x_1, x_2, \ldots, x_n) \in \Delta$, we have $A(x_1, x_2, \ldots, x_n) \in \Delta$. So, $A(\Delta) \subset \Delta$, and $A : \Delta \to \Delta$. Then from Lemma 2.2, we get A is completely continuous.

By Lemma 1.5, the system (1.1) has at least one solution on $[0,1]$. Theorem 2.2 is thus proved. □

Example 2.2. Consider the following system:

$$\begin{cases} D^{\frac{1}{2}} x_1(t) = t^{-\frac{1}{3}} e^{-t} \sin x_1(t), \quad 0 < t \leq 1, \\ D^{\frac{4}{3}} x_2(t) = \dfrac{\cos x_1(t)}{\sqrt{t}\,(\pi + \sin x_2(t))}, \quad 0 < t \leq 1, \\ D^{\frac{9}{4}} x_3(t) = \dfrac{e^t \cos x_3}{t^{\frac{1}{5}}\,(4\pi + \sin(x_1 + x_2))}, \quad 0 < t \leq 1, \\ D^{\frac{7}{2}} x_4(t) = t^{-\frac{2}{9}} \dfrac{e^{-2t} \sin(x_1 + x_2)}{16 + \cos(x_3 + x_4)}, \quad 0 < t \leq 1, \\ D^{\frac{14}{3}} x_5(t) = \dfrac{\cos(u_1 + u_2 + u_3 + u_4)}{t^{\frac{1}{4}} e^t}, \quad 0 < t \leq 1, \\ x_1(0) = \sqrt{3}, \\ x_2(0) = \dfrac{2}{3}, \quad D^{\frac{1}{4}} x_2(1) = 0, \\ x_3(0) = -1, \quad x_3'(0) = \dfrac{1}{2}, \quad D^{\frac{3}{2}} x_3(1) = 0, \\ x_4(0) = \dfrac{\sqrt{7}}{2}, \quad x_4'(0) = \dfrac{1}{4}, \quad x_4''(0) = \dfrac{\sqrt{5}}{3}, \quad D^{\frac{11}{5}} x_4(1) = 0, \\ x_5(0) = 1, \quad x_5'(0) = \dfrac{4}{3}, \quad x_5''(0) = \dfrac{3}{7}, \quad x_5'''(0) = \dfrac{2\sqrt{3}}{5}, \quad D^{\frac{10}{3}} x_5(1) = 0. \end{cases} \tag{2.20}$$

We have:

$$n = 5, \quad \alpha_1 = \frac{1}{2}, \quad \alpha_2 = \frac{4}{3}, \quad \alpha_3 = \frac{9}{4}, \quad \alpha_4 = \frac{7}{2}, \quad \alpha_5 = \frac{14}{3}, \quad \delta_1 = \frac{1}{4}, \quad \delta_2 = \frac{3}{2},$$
$$\delta_3 = \frac{11}{5}, \quad \delta_4 = \frac{10}{3}, \quad a_0^1 = \sqrt{3}, \quad a_0^2 = \frac{2}{3}, \quad a_0^3 = -1, \quad a_1^3 = \frac{1}{2}, \quad a_0^4 = \frac{\sqrt{7}}{2},$$
$$a_1^4 = \frac{1}{4}, \quad a_2^4 = \frac{\sqrt{5}}{3}, \quad a_0^5 = 1, \quad a_1^5 = \frac{4}{3}, \quad a_2^5 = \frac{3}{7}, \quad a_3^5 = \frac{2\sqrt{3}}{5}.$$

For $\beta_1 = \frac{2}{3}$, $\beta_2 = \frac{3}{4}$, $\beta_3 = \frac{2}{5}$, $\beta_4 = \frac{4}{9}$, $\beta_5 = \frac{1}{2}$, the system (2.20) has at least one solution on $[0,1]$.

3. Ulam Stability

In this section, we study the Ulam-Hyers stability and the generalized Ulam-Hyers stability of solutions for system (1.1).

Definition 3.1. The singular fractional system (1.1) is Ulam-Hyers stable if there exists a real number $\mu > 0$, such that for all $(\epsilon_1, \epsilon_2, \ldots, \epsilon_n) > 0$, and for all solution

$(x_1, x_2, \ldots, x_n) \in S$ of

$$(3.1)\qquad \begin{cases} \left|D^{\alpha_1} x_1(t) - f_1(t, x_1(t))\right| \leq \epsilon_1, \\ \left|D^{\alpha_2} x_2(t) - f_2(t, x_1(t), x_2(t))\right| \leq \epsilon_2, \\ \vdots \\ \left|D^{\alpha_n} x_n(t) - f_n(t, x_1(t), x_2(t), \ldots, x_n(t))\right| \leq \epsilon_n, \quad 0 < t \leq 1, \end{cases}$$

there exists a solution $(y_1, y_2, \ldots, y_n) \in S$ satisfying

$$(3.2)\qquad \begin{cases} D^{\alpha_1} y_1(t) = f_1(t, y_1(t)), \\ D^{\alpha_2} x_2(t) = f_2(t, y_1(t), y_2(t)), \\ \vdots \\ D^{\alpha_n} x_n(t) = f_n(t, y_1(t), y_2(t), \ldots, y_n(t)), \quad 0 < t \leq 1, k = 1, \\ y_1(0) = a_0^1, \\ y_k^{(j)}(0) = a_j^k, \quad k = 2, 3, \ldots, n, j = 0, 1, \ldots, k-2, \\ D^{\delta_{k-1}} y_k(1) = 0, \quad k = 2, 3, \ldots, n, k-2 < \delta_{k-1} < k-1, \\ k-1 < \alpha_k < k, \quad k = 1, 2, \ldots, n, \end{cases}$$

with

$$\left\|(x_1 - y_1, \ldots, x_n - y_n)\right\|_S \leq \mu\epsilon, \quad \epsilon > 0.$$

Definition 3.2. The singular fractional system (1.1) is generalized Ulam-Hyers stable if there exists $\phi \in C(\mathbb{R}^+, \mathbb{R}^+)$, $\phi(0) = 0$, such that for all $\epsilon > 0$, and for each solution $(x_1, x_2, \ldots, x_n) \in S$ of (3.1), there exists a solution $(y_1, y_2, \ldots, y_n) \in S$ of (3.2) with

$$\left\|(x_1 - y_1, \ldots, x_n - y_n)\right\|_S \leq \phi(\epsilon), \quad \epsilon > 0.$$

Theorem 3.1. *Let $k-1 < \alpha_k < k$, $k = 1, 2, \ldots, n$, $n \in \mathbb{N} - \{0, 1\}$ and $0 < \beta_k < 1$.*

Assume that:

(H_1) $f_k : (0,1] \times \mathbb{R}^k \to \mathbb{R}$ is continuous with $\lim_{t \to 0^+} f_k(t, \ldots) = \infty$ and $t^{\beta_k} f_k(t, \ldots)$ is continuous on $[0,1] \times \mathbb{R}^k$;

(H_2) $\left\|t^{\beta_k} D^{\alpha_k} x_k\right\|_\infty \geq \begin{cases} P_1\Lambda_1 + \left|a_0^1\right|, & k = 1, \\ P_k\Lambda_k + \sum_{j=0}^{k-2} \frac{\left|a_j^k\right|}{j!}, & k = 2, 3, \ldots, n, \end{cases}$

(H_3) all the assumptions of Theorem 2.1 are satisfied;

$(H4)$ $\sum_{j=1}^{k} \omega_j^k < 1$, $k = 1, 2, \ldots, n$.

Then, the singular fractional system (1.1) is generalized Ulam-Hyers stable.

Proof. Using (H_1) we receive (2.19). Thus, for all solution $(x_1, x_2, \ldots, x_n) \in S$ of (3.1), we can write

$$(3.3)\qquad \left\|(x_k)\right\|_\infty \leq \begin{cases} P_1\Lambda_1 + \left|a_0^1\right|, & k = 1, \\ P_k\Lambda_k + \sum_{j=0}^{k-2} \frac{\left|a_j^k\right|}{j!}, & k = 2, 3, \ldots, n. \end{cases}$$

Then, by combining (H_2) with (3.3), we get

$$\|x_k\|_\infty \leq \left\| t^{\beta_k} D^{\alpha_k} x_k \right\|_\infty . \tag{3.4}$$

On the other hand, using (H_3), there exists a solution $(y_1, y_2, \ldots, y_n) \in S$ satisfying (3.2). Therefore, by (3.4) we can write:

$$\begin{aligned}
\|x_k - y_k\|_\infty &\leq \left\| t^{\beta_k} D^{\alpha_k} (x_k - y_k) \right\|_\infty \\
&\leq \Big\| t^{\beta_k} \left(D^{\alpha_k} x_k - f_k (t, x_1, \ldots, x_k) \right) - t^{\beta_k} \left(D^{\alpha_k} y_k - f_k (t, y_1, \ldots, y_k) \right) \\
&\quad + t^{\beta_k} \left(f_k (t, x_1, \ldots, x_k) - f_k (t, y_1, \ldots, y_k) \right) \Big\|_\infty \\
&\leq \left\| t^{\beta_k} \left(D^{\alpha_k} x_k - f_k (t, x_1, \ldots, x_k) \right) \right\|_\infty + \left\| t^{\beta_k} \left(D^{\alpha_k} y_k - f_k (t, y_1, \ldots, y_k) \right) \right\|_\infty \\
&\quad + \left\| t^{\beta_k} \left(f_k (t, x_1, \ldots, x_k) - f_k (t, y_1, \ldots, y_k) \right) \right\|_\infty \\
&\leq \left\| t^{\beta_k} \right\|_\infty \left\| \left(D^{\alpha_k} x_k - f_k (t, x_1, \ldots, x_k) \right) \right\|_\infty \\
&\quad + \left\| t^{\beta_k} \right\|_\infty \left\| \left(D^{\alpha_k} y_k - f_k (t, y_1, \ldots, y_k) \right) \right\|_\infty \\
&\quad + \left\| t^{\beta_k} \left(f_k (t, x_1, \ldots, x_k) - f_k (t, y_1, \ldots, y_k) \right) \right\|_\infty
\end{aligned}$$

From (2.12), (3.1) and (3.2), we obtain

$$\|(x_k - y_k)\|_\infty \leq \epsilon_k + \sum_{j=1}^{k} \omega_j^k \max_{1 \leq k \leq n} \|(x_k - y_k)\|_\infty .$$

Then

$$\max_{1 \leq k \leq n} \|(x_k - y_k)\|_\infty \leq \frac{\epsilon}{1 - \sum_{j=1}^{k} \omega_j^k} := \mu\epsilon, \quad \epsilon = \max_{1 \leq k \leq n} \epsilon_k, \quad \mu = \frac{1}{1 - \sum_{j=1}^{k} \omega_j^k}.$$

Hence,

$$\|(x_1 - y_1, \ldots, x_n - y_n)\|_S \leq \mu\epsilon.$$

Using (H_4), we get $\mu > 0$. Thus, system (1.1) is Ulam-Hyers stable. Taking $\phi(\epsilon) = \mu\epsilon$, we get system (1.1) is generalized Ulam-Hyers stable. This ends the proof. $\square$

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ON λ-PSEUDO BI-STARLIKE FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS ASSOCIATED TO SHELL-LIKE CURVES

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Abstract. In this paper we define a new subclass $\lambda-$pseudo bi-starlike functions with respect to symmetric points of Σ related to shell-like curves connected with Fibonacci numbers and determine the initial Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for $f \in \mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$. Further we determine the Fekete-Szegö result for the function class $\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha, \tilde{p}(z))$ and for special cases, corollaries are stated which some of them are new and have not been studied so far.

1. Introduction

Let $\mathcal{A}$ denote the class of functions f which are *analytic* in the open unit disk $\mathbb{U} = \{z \,:\, z \in \mathbb{C} \,\text{and}\, |z| < 1\}$. Also, let $\mathcal{S}$ denote the class of functions in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$ and are of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The Koebe one quarter theorem [4] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{A}$ contains a disk of radius $\frac{1}{4}$. Thus every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z,\ (z \in \mathbb{U}) \text{ and } f(f^{-1}(w)) = w \ \left(|w| < r_0(f),\ r_0(f) \geq \frac{1}{4}\right).$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions defined in the unit disk $\mathbb{U}$. Since

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$f \in \Sigma$ has the Maclaurian series given by (1.1), a computation shows that its inverse $g = f^{-1}$ has the expansion

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 + \cdots . \tag{1.2}$$

One can see a short history and examples of functions in the class Σ in [17]. Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [2, 3, 11, 17–19]).

An analytic function f is subordinate to an analytic function F in $\mathbb{U}$, written as $f \prec F$ $(z \in \mathbb{U})$, provided there is an analytic function ω defined on $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ satisfying $f(z) = F(\omega(z))$. It follows from Schwarz Lemma that

$$f(z) \prec F(z) \Rightarrow f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}), \quad z \in \mathbb{U}$$

(for details see [4], [10]). We recall important subclasses of $\mathcal{S}$ in geometric function theory such that if $f \in \mathcal{A}$ and $\frac{zf'(z)}{f(z)} \prec p(z)$ and $1 + \frac{zf''(z)}{f'(z)} \prec p(z)$, where $p(z) = \frac{1+z}{1-z}$, then we say that f is starlike and convex, respectively. These functions form known classes denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. An interesting case when the function p is convex but is not univalent was considered in [6]. Ma and Minda [9] unified various subclasses of starlike and convex functions for which either of the quantity $\frac{zf'(z)}{f(z)}$ or $1 + \frac{zf''(z)}{f'(z)}$ is subordinate to a more general superordinate function. Here superordinate functions is an analytic function φ with positive real part in the unit disc $\mathbb{U}$ with $\varphi(0) = 1$, $\varphi'(0) > 0$ and it maps $\mathbb{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis. The class $S^*(\varphi)$ and $K(\varphi)$ denote Ma-Minda starlike and Ma-Minda convex functions, respectively. If we restrict considerations to the absorbing geometric shape of $p(\mathbb{U})$, is parabolic domain or an elliptic domain or in an interior of hyperbola, further the cases, when $p(\mathbb{U})$ is an interior of the right loop of the Lemniscate of Bernoulli or in leaf-like domain in recent past (see [6, 15, 16] and also references cited therein) for the case of functions in $\mathcal{A}$. The behavior of the coefficients are unpredictable when the bi-univalency condition is imposed on the function $f \in \mathcal{A}$ in our present work we attempted to find initial coefficients for $f \in \Sigma$ by considering the geometric shape of $p(\mathbb{U})$ related to a shell-like curve connected with Fibonacci numbers.

Recently, in [14], Sokół introduced the class $\mathcal{SL}$ of shell-like functions as the set of functions $f \in \mathcal{A}$ which is described in the following definition.

Definition 1.1. The function $f \in \mathcal{A}$ belongs to the class $\mathcal{SL}$ if it satisfies the condition that

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z),$$

with $\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$, where $\tau = (1-\sqrt{5})/2 \approx -0.618$.

It should be observed $\mathcal{SL}$ is a subclass of the starlike functions $\mathcal{S}^*$.

The function $\tilde{p}$ is not univalent in $\mathbb{U}$, but it is univalent in the disc $|z| < (3-\sqrt{5})/2 \approx 0.38$. For example, $\tilde{p}(0) = \tilde{p}(-1/2\tau) = 1$ and $\tilde{p}(e^{\mp i \arccos(1/4)}) = \sqrt{5}/5$, and it may also be noticed that

$$\frac{1}{|\tau|} = \frac{|\tau|}{1-|\tau|},$$

which shows that the number $|\tau|$ divides $[0,1]$ such that it fulfils the golden section. The image of the unit circle $|z|=1$ under $\tilde{p}$ is a curve described by the equation given by

$$(10x-\sqrt{5})y^2 = (\sqrt{5}-2x)(\sqrt{5}x-1)^2,$$

which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \le r_0 = (3-\sqrt{5})/2 \approx 0.38$. For $r_0 < r < 1$, it has a loop, and for $r=1$, it has a vertical asymptote. Since τ satisfies the equation $\tau^2 = 1+\tau$, this expression can be used to obtain higher powers τ^n as a linear function of lower powers, which in turn can be decomposed all the way down to a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci numbers u_n:

$$\tau^n = u_n\tau + u_{n-1}.$$

In [13] Raina and Sokół showed that

$$\begin{aligned}
\tilde{p}(z) =& \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} = \left(t+\frac{1}{t}\right)\frac{t}{1-t-t^2} \\
=& \frac{1}{\sqrt{5}}\left(t+\frac{1}{t}\right)\left(\frac{1}{1-(1-\tau)t} - \frac{1}{1-\tau t}\right) \\
=& \left(t+\frac{1}{t}\right)\sum_{n=1}^{\infty} u_n t^n = 1 + \sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n,
\end{aligned}$$

where

$$u_n = \frac{(1-\tau)^n - \tau^n}{\sqrt{5}}, \quad \tau = \frac{1-\sqrt{5}}{2}, \quad n = 1,2,\ldots.$$

This shows that the relevant connection of $\tilde{p}$ with the sequence of Fibonacci numbers u_n, such that $u_0 = 0$, $u_1 = 1$, $u_{n+2} = u_n + u_{n+1}$ for $n = 0,1,2,\ldots$. And they got

$$\begin{aligned}
\tilde{p}(z) =& 1 + \sum_{n=1}^{\infty}\tilde{p}_n z^n \\
=& 1 + (u_0+u_2)\tau z + (u_1+u_3)\tau^2z^2 + \sum_{n=3}^{\infty}(u_{n-3}+u_{n-2}+u_{n-1}+u_n)\tau^n z^n \\
=& 1 + \tau z + 3\tau^2z^2 + 4\tau^3z^3 + 7\tau^4z^4 + 11\tau^5z^5 + \cdots.
\end{aligned}$$

Let $\mathcal{P}(\beta)$, $0 \le \beta < 1$, denote the class of analytic functions p in $\mathbb{U}$ with $p(0) = 1$ and $\operatorname{Re}\{p(z)\} > \beta$. Especially, we will use $\mathcal{P}$ instead of $\mathcal{P}(0)$.

Theorem 1.1 ([7]). *The function $\tilde{p}(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$ belongs to the class $\mathcal{P}(\beta)$ with $\beta = \sqrt{5}/10 \approx 0.2236$.*

Now we give the following lemma which will use in proving.

Lemma 1.1 ([12]). *Let $p \in \mathcal{P}$ with $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$, then $|c_n| \leq 2$, for $n \geq 1$.*

2. Bi-Univalent Function Class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$

In this section, we introduce a new subclass of Σ associated with $\lambda-$pseudo bi-starlike functions with respect to symmetric points related to shell-like curves connected with Fibonacci numbers and obtain the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class by subordination.

The class $\mathcal{L}_\lambda(\gamma)$ of λ-pseudo-starlike functions of order γ $(0 \leq \gamma < 1)$ were introduced and investigated by Babalola [1] whose geometric conditions satisfy

$$\mathrm{Re}\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \gamma.$$

He showed that all pseudo-starlike functions are Bazilevič of type $\left(1 - \frac{1}{\lambda}\right)$ order $\gamma^{\frac{1}{\lambda}}$ and univalent in open unit disk $\mathbb{U}$. If $\lambda = 1$, we have the class of starlike functions of order γ, which in this context, are $1-$pseudo-starlike functions of order γ. Babalola [1] remarked that though for $\lambda > 1$, these classes of $\lambda-$pseudo starlike functions clone the analytic representation of starlike functions, it is not yet known the possibility of any inclusion relations between them.

Motivated by the works of Dziok et al. in [7] on the class of convex and $\alpha-$convex functions related to a shell-like curve connected with Fibonacci numbers, Eker et al. and [5] on bi-pseudo-starlike functions class and obtained the initial coefficients $|a_2|$ and $|a_3|$ with respect to other points in this paper we define the new class named as $\lambda-$pseudo bi-starlike functions with respect to symmetric points related to shell-like curves connected with Fibonacci numbers as follows.

Definition 2.1. Let $0 \leq \alpha \leq 1$ and $\lambda \geq 1$ is real. A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ if the following subordination hold:

$$(1-\alpha)\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} + \alpha \frac{2[(z(f'(z)))']^\lambda}{[f(z) - f(-z)]'} \prec \tilde{p}(z) \tag{2.1}$$

and

$$(1-\alpha)\frac{2w(g'(w))^\lambda}{g(w) - g(-w)} + \alpha \frac{2[(w(g'(w)))']^\lambda}{[g(w) - g(-w)]'} \prec \tilde{p}(w) \tag{2.2}$$

where $\tau = (1 - \sqrt{5})/2 \approx -0.618$ where $z, w \in \mathbb{U}$ and g is given by (1.2).

Specializing the parameter $\lambda = 1$ we have the following definitions, respectively.

Definition 2.2. A function $f \in \Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^{1}_{s,\Sigma}(\alpha, \tilde{p}(z)) \equiv \mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ if the following subordination hold:

$$(1-\alpha)\frac{2zf'(z)}{f(z) - f(-z)} + \alpha \frac{2(z(f'(z)))'}{[f(z) - f(-z)]'} \prec \tilde{p}(z)$$

and

$$(1-\alpha)\frac{2wg'(w)}{g(w)-g(-w)}+\alpha\frac{2(w(g'(w)))'}{[g(w)-g(-w)]'}\prec \tilde{p}(w),$$

where $\tau=(1-\sqrt{5})/2\approx -0.618$ where $z,w\in\mathbb{U}$ and g is given by (1.2).

Definition 2.3. A function $f\in\Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(0,\tilde{p}(z))\equiv\mathcal{SL}^*_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:

$$\frac{2zf'(z)}{f(z)-f(-z)}\prec \tilde{p}(z)$$

and

$$\frac{2wg'(w)}{g(w)-g(-w)}\prec \tilde{p}(w),$$

where $\tau=(1-\sqrt{5})/2\approx -0.618$ where $z,w\in\mathbb{U}$ and g is given by (1.2).

Definition 2.4. A function $f\in\Sigma$ of the form (1.1) is said to be in the class $\mathcal{PSL}^1_{s,\Sigma}(1,\tilde{p}(z))\equiv\mathcal{KL}_{s,\Sigma}(\tilde{p}(z))$ if the following subordination hold:

$$\frac{2(z(f'(z)))'}{[f(z)-f(-z)]'}\prec \tilde{p}(z)$$

and

$$\frac{2(w(g'(w)))'}{[g(w)-g(-w)]'}\prec \tilde{p}(w),$$

where $\tau=(1-\sqrt{5})/2\approx -0.618$, where $z,w\in\mathbb{U}$ and g is given by (1.2).

In the following theorem we determine the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function class $\mathcal{PSL}^\lambda_{s,\Sigma}(\alpha,\tilde{p}(z))$. Later we will reduce these bounds to other classes for special cases as corollaries which are new and have not been studied sofar.

Theorem 2.1. *Let f given by* (1.1) *be in the class $\mathcal{PSL}^\lambda_{s,\Sigma}(\alpha,\tilde{p}(z))$. Then*

$$|a_2|\le\frac{|\tau|}{\sqrt{4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau}} \tag{2.3}$$

and

$$|a_3|\le\frac{2\lambda|\tau|\,[2\lambda(1+\alpha)^2-\{(\lambda-1)(1+3\alpha)-6\lambda(1+\alpha)^2\}\tau]}{(3\lambda-1)(1+2\alpha)\,[4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau]}.$$

Proof. Firstly, let $p(z)=1+p_1z+p_2z^2+\cdots$, and $p\prec\tilde{p}$. Then there exists an analytic function u such that $u(0)=0; |u(z)|<1$ in $\mathbb{U}$ and $p(z)=\tilde{p}(u(z))$. Therefore, the function

$$h(z)=\frac{1+u(z)}{1-u(z)}=1+c_1z+c_2z^2+\cdots$$

is in the class $\mathcal{P}$. It follows that

$$u(z)=\frac{c_1z}{2}+\left(c_2-\frac{c_1^2}{2}\right)\frac{z^2}{2}+\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\frac{z^3}{2}+\cdots$$

and

$$\begin{aligned}\tilde{p}(u(z)) =&1+\frac{\tilde{p}_1 c_1 z}{2}+\left\{\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right\}z^2\\&+\left\{\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right\}z^3+\cdots.\end{aligned}$$

And similarly, there exists an analytic function v such that $v(0)=0; |v(w)|<1$ in $\mathbb{U}$ and $p(w)=\tilde{p}(v(w))$. Therefore, the function

$$k(w)=\frac{1+v(w)}{1-v(w)}=1+d_1w+d_2w^2+\cdots$$

is in the class $\mathcal{P}$. It follows that

$$v(w)=\frac{d_1w}{2}+\left(d_2-\frac{d_1^2}{2}\right)\frac{w^2}{2}+\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\frac{w^3}{2}+\cdots$$

and

$$\begin{aligned}\tilde{p}(v(w)) =&1+\frac{\tilde{p}_1 d_1 w}{2}+\left\{\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right\}w^2\\&+\left\{\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right\}w^3+\cdots.\end{aligned}$$

Let $f\in\mathcal{PSL}_{s,\Sigma}^{\lambda}(\alpha,\tilde{p}(z))$ and $g=f^{-1}$. Considering (2.1) and (2.2), we have

$$(1-\alpha)\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}+\alpha\frac{2[(z(f'(z)))']^{\lambda}}{[f(z)-f(-z)]'}=\tilde{p}(u(z))$$

and

$$(1-\alpha)\frac{2w(g'(w))^{\lambda}}{g(w)-g(-w)}+\alpha\frac{2[(w(g'(w)))']^{\lambda}}{[g(w)-g(-w)]'}=\tilde{p}(v(w)),$$

where $\tau=(1-\sqrt{5})/2\approx-0.618$, $z,w\in\mathbb{U}$ and g is given by (1.2). Since

$$\begin{aligned}&(1-\alpha)\frac{2z(f'(z))^{\lambda}}{f(z)-f(-z)}+\alpha\frac{2[(z(f'(z)))']^{\lambda}}{[f(z)-f(-z)]'}\\=&1+2\lambda(1+\alpha)a_2z+[2\lambda(\lambda-1)(1+3\alpha)a_2^2+(3\lambda-1)(1+2\alpha)a_3]z^2+\cdots\end{aligned}$$

and

$$\begin{aligned}&(1-\alpha)\frac{2w(g'(w))^{\lambda}}{g(w)-g(-w)}+\alpha\frac{2[(w(g'(w)))']^{\lambda}}{[g(w)-g(-w)]'}\\=&1-2\lambda(1+\alpha)a_2w+\{[2(\lambda^2+2\lambda-1)+2\alpha(3\lambda^2+3\lambda-2)]a_2^2\\&-(3\lambda-1)(1+2\alpha)a_3]\}w^2+\cdots.\end{aligned}$$

Thus, we have

$$\begin{aligned}
&1+2\lambda(1+\alpha)a_2z+[2\lambda(\lambda-1)(1+3\alpha)a_2^2+(3\lambda-1)(1+2\alpha)a_3]z^2+\cdots\\
=&1+\frac{\tilde{p}_1c_1z}{2}+\left[\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_1+\frac{c_1^2}{4}\tilde{p}_2\right]z^2\\
&+\left[\frac{1}{2}\left(c_3-c_1c_2+\frac{c_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}c_1\left(c_2-\frac{c_1^2}{2}\right)\tilde{p}_2+\frac{c_1^3}{8}\tilde{p}_3\right]z^3+\cdots
\end{aligned}\tag{2.4}$$

and

$$\begin{aligned}
&1-2\lambda(1+\alpha)a_2w\\
&+\{[2(\lambda^2+2\lambda-1)+2\alpha(3\lambda^2+3\lambda-2)]a_2^2-(3\lambda-1)(1+2\alpha)a_3]\}w^2+\dots\\
=&1+\frac{\tilde{p}_1d_1w}{2}+\left[\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_1+\frac{d_1^2}{4}\tilde{p}_2\right]w^2\\
&+\left[\frac{1}{2}\left(d_3-d_1d_2+\frac{d_1^3}{4}\right)\tilde{p}_1+\frac{1}{2}d_1\left(d_2-\frac{d_1^2}{2}\right)\tilde{p}_2+\frac{d_1^3}{8}\tilde{p}_3\right]w^3+\cdots.
\end{aligned}\tag{2.5}$$

It follows from (2.4) and (2.5) that

$$2\lambda(1+\alpha)a_2=\frac{c_1\tau}{2},\tag{2.6}$$

$$2\lambda(\lambda-1)(1+3\alpha)a_2^2+(3\lambda-1)(1+2\alpha)a_3=\frac{1}{2}\left(c_2-\frac{c_1^2}{2}\right)\tau+\frac{c_1^2}{4}3\tau^2\tag{2.7}$$

and

$$-2\lambda(1+\alpha)a_2=\frac{d_1\tau}{2},\tag{2.8}$$

$$\begin{aligned}
&[2(\lambda^2+2\lambda-1)+2\alpha(3\lambda^2+3\lambda-2)]a_2^2-(3\lambda-1)(1+2\alpha)a_3\\
=&\frac{1}{2}\left(d_2-\frac{d_1^2}{2}\right)\tau+\frac{d_1^2}{4}3\tau^2.
\end{aligned}\tag{2.9}$$

From (2.6) and (2.8), we have

$$c_1=-d_1$$

and

$$a_2^2=\frac{(c_1^2+d_1^2)}{32\lambda^2(1+\alpha)^2}\tau^2.\tag{2.10}$$

Hence,

$$|a_2|\leq\frac{|\tau|}{2\lambda(1+\alpha)}.$$

Now, by summing (2.7) and (2.9), we obtain

$$\left[2(2\lambda^2+\lambda-1)+4\alpha(3\lambda^2-1)\right]a_2^2=\frac{1}{2}(c_2+d_2)\tau-\frac{1}{4}(c_1^2+d_1^2)\tau+\frac{3}{4}(c_1^2+d_1^2)\tau^2.\tag{2.11}$$

Substituting (2.10) in (2.11), we have

$$4\left[4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau\right]a_2^2=(c_2+d_2)\tau^2. \tag{2.12}$$

Therefore, by Lemma (1.1) we obtain

$$|a_2|\leq\frac{|\tau|}{\sqrt{4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau}}. \tag{2.13}$$

It is clear that

$$\min\left\{\frac{|\tau|}{2\lambda(1+\alpha)},\frac{|\tau|}{\sqrt{4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau}}\right\}$$
$$=\frac{|\tau|}{\sqrt{4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau}}.$$

So, we obtain the inequality (2.3).

Now, so as to find the bound on $|a_3|$, let's subtract from (2.7) and (2.9). So, we find

$$2(3\lambda-1)(1+2\alpha)a_3-2(3\lambda-1)(1+2\alpha)a_2^2=\frac{1}{2}(c_2-d_2)\tau. \tag{2.14}$$

Hence, we get

$$2(3\lambda-1)(1+2\alpha)|a_3|\leq 2|\tau|+2(3\lambda-1)(1+2\alpha)|a_2|^2.$$

Then, in view of (2.13), we obtain

$$|a_3|\leq\frac{2\lambda|\tau|\left[2\lambda(1+\alpha)^2-\{(\lambda-1)(1+3\alpha)-6\lambda(1+\alpha)^2\}\tau\right]}{(3\lambda-1)(1+2\alpha)\left[4\lambda^2(1+\alpha)^2-\{2\lambda^2(6\alpha^2+9\alpha+5)-\lambda+2\alpha+1\}\tau\right]}. \qquad \square$$

Taking $\lambda=1$, in the above theorem, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{MSL}_{s,\Sigma}(\alpha,\tilde{p}(z))$.

Corollary 2.1. *Let f given by* (1.1) *be in the class $\mathcal{MSL}_{s,\Sigma}(\alpha,\tilde{p}(z))$. Then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{4(1+\alpha)^2-2(6\alpha^2+10\alpha+5)\tau}}$$

and

$$|a_3|\leq\frac{2(1+\alpha)^2|\tau|\,(1-3\tau)}{(1+2\alpha)\left[4(1+\alpha)^2-2(6\alpha^2+10\alpha+5)\tau\right]}.$$

Further by taking $\alpha=0$ and $\alpha=1$ in Corollary 2.1, we have the following the initial Taylor coefficients $|a_2|$ and $|a_3|$ for the function classes $\mathcal{SL}^*_{s,\Sigma}(\alpha,\tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\alpha,\tilde{p}(z))$, respectively.

Corollary 2.2. *Let f given by* (1.1) *be in the class $\mathcal{SL}^*_{s,\Sigma}(\alpha,\tilde{p}(z))$. Then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{4-10\tau}}$$

and

$$|a_3| \leq \frac{|\tau|(1-3\tau)}{2-5\tau}.$$

Corollary 2.3. *Let f given by* (1.1) *be in the class $\mathcal{KL}_{s,\Sigma}(\alpha, \tilde{p}(z))$. Then*

$$|a_2| \leq \frac{|\tau|}{\sqrt{16-42\tau}}$$

and

$$|a_3| \leq \frac{4|\tau|(1-3\tau)}{3(8-21\tau)}.$$

3. Fekete-Szegö inequality for the Function Class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$

Fekete and Szegö [8] introduced the generalized functional $|a_3 - \mu a_2^2|$, where μ is some real number. Due to Zaprawa [20], in the following theorem we determine the Fekete-Szegö functional for $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$.

Theorem 3.1. *Let f given by* (1.1) *be in the class $\mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $\mu \in \mathbb{R}$. Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{(3\lambda-1)(1+2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)(1+2\alpha)}, \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)(1+2\alpha)}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4\lambda^2(1+\alpha)^2 - \{2\lambda^2(6\alpha^2+9\alpha+5) - \lambda + 2\alpha + 1\}\tau\right]}.$$

Proof. From (2.12) and (2.14) we obtain

$$\begin{aligned} & a_3 - \mu a_2^2 \\ & \quad + \frac{(1-\mu)(c_2+d_2)\tau^2}{4\left[4\lambda^2(1+\alpha)^2 - \{2\lambda^2(6\alpha^2+9\alpha+5) - \lambda + 2\alpha + 1\}\tau\right]} + \frac{\tau(c_2-d_2)}{4(3\lambda-1)(1+2\alpha)} \\ & = \left(\frac{(1-\mu)\tau^2}{4\left[4\lambda^2(1+\alpha)^2 - \{2\lambda^2(6\alpha^2+9\alpha+5) - \lambda + 2\alpha + 1\}\tau\right]} \frac{\tau}{4(3\lambda-1)(1+2\alpha)}\right) c_2 \\ & \quad + \left(\frac{(1-\mu)\tau^2}{4\left[4\lambda^2(1+\alpha)^2 - \{2\lambda^2(6\alpha^2+9\alpha+5) - \lambda + 2\alpha + 1\}\tau\right]} \frac{\tau}{4(3\lambda-1)(1+2\alpha)}\right) d_2. \end{aligned}$$

So, we have

$$a_3 - \mu a_2^2 = \left(h(\mu) + \frac{|\tau|}{4(3\lambda-1)(1+2\alpha)}\right) c_2 + \left(h(\mu) - \frac{|\tau|}{4(3\lambda-1)(1+2\alpha)}\right) d_2, \tag{3.1}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4\lambda^2(1+\alpha)^2 - \{2\lambda^2(6\alpha^2+9\alpha+5) - \lambda + 2\alpha + 1\}\tau\right]}.$$

Then, by taking modulus of (3.1), we conclude that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{(3\lambda-1)(1+2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{4(3\lambda-1)(1+2\alpha)}, \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{4(3\lambda-1)(1+2\alpha)}. \end{cases}$$

□

Taking $\mu = 1$, we have the following corollary.

Corollary 3.1. *If* $f \in \mathcal{PSL}^{\lambda}_{s,\Sigma}(\alpha, \tilde{p}(z))$, *then*

$$|a_3 - a_2^2| \leq \frac{|\tau|}{(3\lambda-1)(1+2\alpha)}.$$

If we can take the parameter $\lambda = 1$ in Theorem 3.1, we can state the following.

Corollary 3.2. *Let f given by* (1.1) *be in the class* $\mathcal{MSL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{2(1+2\alpha)}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{8(1+2\alpha)}, \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{8(1+2\alpha)}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{4\left[4(1+\alpha)^2 - 2\{6\alpha^2 + 10\alpha + 5\}\tau\right]}.$$

Further by taking $\alpha = 0$ and $\alpha = 1$ in the above corollary, we have the following the Fekete-Szegö inequalities for the function classes $\mathcal{SL}^*_{s,\Sigma}(\alpha, \tilde{p}(z))$ and $\mathcal{KL}_{s,\Sigma}(\alpha, \tilde{p}(z))$, respectively.

Corollary 3.3. *Let f given by* (1.1) *be in the class* $\mathcal{SL}^*_{s,\Sigma}(\alpha, \tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{2}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{8}, \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{8}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{8\left[2 - 5\tau\right]}.$$

Corollary 3.4. *Let f given by* (1.1) *be in the class* $\mathcal{KL}_{s,\Sigma}(\alpha, \tilde{p}(z))$ *and* $\mu \in \mathbb{R}$. *Then we have*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \dfrac{|\tau|}{6}, & 0 \leq |h(\mu)| \leq \dfrac{|\tau|}{24}, \\ 4|h(\mu)|, & |h(\mu)| \geq \dfrac{|\tau|}{24}, \end{cases}$$

where

$$h(\mu) = \frac{(1-\mu)\tau^2}{8\left[8 - 21\tau\right]}.$$

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HERMITE-HADAMARD TYPE INEQUALITIES FOR OPERATOR GEOMETRICALLY CONVEX FUNCTIONS II

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Abstract. In this paper, we prove some Hermite-Hadamard type inequalities for operator geometrically convex functions for non-commutative operators.

1. Introduction and Preliminaries

Let $B(H)$ stand for C^*-algebra of all bounded linear operators on a complex Hilbert space H with inner product $\langle \cdot, \cdot \rangle$. An operator $A \in B(H)$ is strictly positive and write $A > 0$ if $\langle Ax, x \rangle > 0$ for all $x \in H$. Let $B(H)^{++}$ stand for all strictly positive operators on $B(H)$.

Let A be a self-adjoint operator in $B(H)$. The Gelfand map establishes a $*$-isometrically isomorphism Φ between the set $C(\mathrm{Sp}(A))$ of all continuous functions defined on the spectrum of A, denoted $\mathrm{Sp}(A)$, and the C^*-algebra $C^*(A)$ generated by A and the identity operator 1_H on H as follows.

For any $f, g \in C(\mathrm{Sp}(A)))$ and any $\alpha, \beta \in \mathbb{C}$ we have:

- $\Phi(\alpha f + \beta g) = \alpha\Phi(f) + \beta\Phi(g)$;
- $\Phi(fg) = \Phi(f)\Phi(g)$ and $\Phi(\bar{f}) = \Phi(f)^*$;
- $\|\Phi(f)\| = \|f\| := \sup_{t \in \mathrm{Sp}(A)} |f(t)|$;
- $\Phi(f_0) = 1_H$ and $\Phi(f_1) = A$, where $f_0(t) = 1$ and $f_1(t) = t$, for $t \in \mathrm{Sp}(A)$.

With this notation we define $f(A) = \Phi(f)$ for all $f \in C(\mathrm{Sp}(A))$, and we call it the continuous functional calculus for a self-adjoint operator A. If A is a self-adjoint operator and both f and g are real valued functions on $\mathrm{Sp}(A)$ then the following important property holds: $f(t) \geq g(t)$ for any $t \in \mathrm{Sp}(A)$ implies that $f(A) \geq g(A)$,

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in the operator order of $B(H)$, see [12]. A real valued continuous function $f:\mathbb{R}\to\mathbb{R}$ is said to be convex (concave) if

$$f(\lambda a+(1-\lambda)b)\leq(\geq)\lambda f(a)+(1-\lambda)f(b),$$

for $a,b\in\mathbb{R}$ and $\lambda\in[0,1]$. The following Hermite-Hadamard inequality holds for any convex function f defined on $\mathbb{R}$

$$\begin{aligned}(b-a)f\left(\frac{a+b}{2}\right)&\leq\int_a^b f(x)dx\\&\leq(b-a)\frac{f(a)+f(b)}{2},\quad\text{for } a,b\in\mathbb{R}.\end{aligned}$$

The author of [8, Remark 1.9.3] gave the following refinement of Hermite-Hadamard inequalities for convex functions

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\leq\frac{1}{2}\left(f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right)\\&\leq\frac{1}{b-a}\int_a^b f(x)dx\\&\leq\frac{1}{2}\left(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right)\\&\leq\frac{f(a)+f(b)}{2}.\end{aligned}$$

A real valued continuous function is operator convex if

$$f(\lambda A+(1-\lambda)B)\leq\lambda f(A)+(1-\lambda)f(B),$$

for self-adjoint operator $A,B\in B(H)$ and $\lambda\in[0,1]$. In [2] Dragomir investigated the operator version of the Hermite-Hadamard inequality for operator convex functions. Let $f:\mathbb{R}\to\mathbb{R}$ be an operator convex function on the interval I then, for any self-adjoint operators A and B with spectra in I, the following inequalities hold

$$\begin{aligned}f\left(\frac{A+B}{2}\right)&\leq 2\int_{\frac{1}{4}}^{\frac{3}{4}} f(tA+(1-t)B)dt\\&\leq\frac{1}{2}\left[f\left(\frac{3A+B}{4}\right)+f\left(\frac{A+3B}{4}\right)\right]\\&\leq\int_0^1 f\left((1-t)A+tB\right)dt\\&\leq\frac{1}{2}\left[f\left(\frac{A+B}{2}\right)+\frac{f(A)+f(B)}{2}\right]\\&\leq\frac{f(A)+f(B)}{2}.\end{aligned}$$

For the first inequality in above, see [10].

A continuous function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ ($\mathbb{R}^+$ denoted positive real numbers) is said to be geometrically convex function (or multiplicatively convex function) if

$$f(a^\lambda b^{1-\lambda}) \leq f(a)^\lambda f(b)^{1-\lambda},$$

for $a, b \in I$ and $\lambda \in [0,1]$.

The author of [7, p. 158] showed that every polynomial $P(x)$ with non-negative coefficients is a geometrically convex function on $[0,\infty)$. More generally, every real analytic function $f(x) = \sum_{n=0}^{\infty} c_n x^n$ with non-negative coefficients is geometrically convex function on $(0, R)$ where R denotes the radius of convergence. Also, see [9,11]. In [10], the following inequalities were obtained for a geometrically convex function

$$\begin{aligned} f(\sqrt{ab}) &\leq \sqrt{\left(f(a^{\frac{3}{4}}b^{\frac{1}{4}})f(a^{\frac{1}{4}}b^{\frac{3}{4}})\right)} \\ &\leq \exp\left(\frac{1}{\log b - \log a}\int_a^b \frac{\log f(t)}{t}dt\right) \\ &\leq \sqrt{f(\sqrt{ab})}.\sqrt[4]{f(a)}.\sqrt[4]{f(b)} \\ &\leq \sqrt{f(a)f(b)}. \end{aligned}$$

In this paper, we prove some Hermite-Hadamard inequalities for operator geometrically convex functions. Moreover, in the final section, we present some examples and remarks.

2. Hermite-Hadamard Inequalities for Geometrically Convex Functions

In this section, we introduce the concept of operator geometrically convex function for positive operators and prove the Hermite-Hadamard type inequalities for this function.

Proposition 2.1. *Let $A, B \in B(H)^{++}$ such that* $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$, *and $t \in [0,1]$. Then* $\mathrm{Sp}(A\sharp_t B) \subseteq I$, *where $A\sharp_t B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^t A^{\frac{1}{2}}$ is t-geometric mean.*

Proof. Let $I = [m, M]$ for some positive real numbers m, M with $m < M$. Since $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$ it is equivalent to $m1_H \leq A \leq M1_H$ and $m1_H \leq B \leq M1_H$. So, by virtue of the fact that if a, b be self-adjoint operators in C^*-algebra $\mathcal{A}$ which $a \leq b$ and $c \in \mathcal{A}$, then $c^*ac \leq c^*bc$, and also by using the operator monotonicity property of the function $f(x) = x^t$ on $(0,\infty)$ for $t \in [0,1]$, we get the result. □

Now, by applying Proposition 2.1, we present the following definition.

Definition 2.1. A continuous function $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is said to be operator geometrically convex if

$$f(A\sharp_t B) \leq f(A)\sharp_t f(B),$$

for $A, B \in B(H)^{++}$ such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$ and $t \in [0,1]$.

We need the following lemmas for proving our theorems.

Lemma 2.1 ([4,5])**.** *Let $A, B \in B(H)^{++}$ and let $t, s, u \in \mathbb{R}$. Then*
$$(A\sharp_t B)\sharp_s(A\sharp_u B) = A\sharp_{(1-s)t+su}B.$$

Lemma 2.2 ([4])**.** *Let A, B, C and D be operators in $B(H)^{++}$ and let $t \in \mathbb{R}$. Then, we have*
$$A\sharp_t B \leq C\sharp_t D,$$
for $A \leq C$ and $B \leq D$.

Lemma 2.3. *Let $A, B \in B(H)^{++}$. If $f : I \subseteq \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function, then*
$$\int_0^1 f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t}B\right)dt \leq \left(\int_0^1 f\left(A\sharp_t B\right)dt\right)\sharp\left(\int_0^1 f\left(A\sharp_{1-t}B,\right)dt\right)$$
such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subseteq I$.

Proof. Since the function $t^{\frac{1}{2}}$ is operator concave, we can write
$$\left(\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}\left(\int_0^1 f(A\sharp_u B)du\right)\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
(by change of variable $v = 1 - u$)
$$= \left(\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}\left(\int_0^1 f(A\sharp_u B)du\right)\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$= \left(\int_0^1\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{1}{2}} f(A\sharp_u B)\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{1}{2}}du\right)^{\frac{1}{2}}$$
$$= \left(\int_0^1\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left((f(A\sharp_{1-u}B))^{\frac{-1}{2}}f(A\sharp_u B)(f(A\sharp_{1-u}B))^{\frac{-1}{2}}\right)\right.$$
$$\left.\times(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}du\right)^{\frac{1}{2}}$$
(by the operator Jensen inequality)
$$\geq \int_0^1\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}f(A\sharp_u B)\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\times(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}du$$
$$= \left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}\int_0^1(f(A\sharp_{1-u}B))^{\frac{1}{2}}\left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}f(A\sharp_u B)\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\times(f(A\sharp_{1-u}B))^{\frac{1}{2}}du\left(\int_0^1 f(A\sharp_v B)dv\right)^{\frac{-1}{2}}$$

(by change of variable $u = 1 - v$)

$$= \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}} \int_0^1 (f(A\sharp_{1-u}B))^{\frac{1}{2}} \left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}} f(A\sharp_u B)(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\times (f(A\sharp_{1-u}B))^{\frac{1}{2}} du \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}.$$

So, we obtain

$$\left(\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}} \left(\int_0^1 f(A\sharp_u B)du\right) \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\geq \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}} \int_0^1 (f(A\sharp_{1-u}B))^{\frac{-1}{2}} \left(\left(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}} f(A\sharp_u B)(f(A\sharp_{1-u}B)\right)^{\frac{-1}{2}}\right)^{\frac{1}{2}}$$
$$\times (f(A\sharp_{1-u}B))^{\frac{1}{2}} du \left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{-1}{2}}.$$

Multiplying both side of the above inequality by $\left(\int_0^1 f(A\sharp_{1-u}B)du\right)^{\frac{1}{2}}$ we obtain

$$\left(\int_0^1 f\left(A\sharp_u B\right) du\right) \sharp \left(\int_0^1 f\left(A\sharp_{1-u}B\right) du\right) \geq \int_0^1 f\left(A\sharp_u B\right) \sharp f\left(A\sharp_{1-u}B\right) du. \qquad \square$$

Before giving our theorems in this section, we mention the following remark.

Remark 2.1. Let $p(x) = x^t$ and $q(x) = x^s$ on $[1, \infty)$, where $0 \leq t \leq s$. If $f(A) \leq f(B)$ then $\operatorname{Sp}\left(f(A)^{\frac{-1}{2}} (f(B)) f(A)^{\frac{-1}{2}}\right) \subseteq [1, \infty)$. By functional calculus, we have

$$p\left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right) \leq q\left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right).$$

So,

$$\left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right)^t \leq \left(f(A)^{\frac{-1}{2}} f(B) f(A)^{\frac{-1}{2}}\right)^s.$$

Now, we are ready to prove Hermite-Hadamard type inequality for operator geometrically convex functions.

Theorem 2.1. *Let f be an operator geometrically convex function. Then, we have*

$$f(A\sharp B) \leq \int_0^1 f(A\sharp_t B)\, dt \leq \int_0^1 f(A)\sharp_t f(B) dt. \tag{2.1}$$

Moreover, if $f(A) \leq f(B)$, then we have

$$\int_0^1 f(A\sharp_t B) dt \leq \int_0^1 f(A)\sharp_t f(B) dt \leq \frac{1}{2}((f(A)\sharp f(B)) + f(B)), \tag{2.2}$$

for $A, B \in B(H)^{++}$.

Proof. Let f be a geometrically convex function. Then we have

$$\begin{aligned} f\left(A\sharp B\right) &= f\left(\left(A\sharp_t B\right)\sharp\left(A\sharp_{1-t}B\right)\right) \qquad \text{(by Lemma 2.1)}\\ &\leq f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t}B\right) \qquad (f \text{ is operator geometrically convex}). \end{aligned}$$

Taking integral of the both sides of the above inequalities on $[0,1]$, we obtain

$$\begin{aligned} f\left(A\sharp B\right) &\leq \int_0^1 f\left(A\sharp_t B\right)\sharp f\left(A\sharp_{1-t}B\right)dt\\ &\leq \left(\int_0^1 f\left(A\sharp_t B\right)dt\right)\sharp\left(\int_0^1 f\left(A\sharp_{1-t}B\right)dt\right) \quad \text{(by Lemma 2.3)}\\ &= \int_0^1 f\left(A\sharp_t B\right)dt\\ &\leq \int_0^1 f\left(A\right)\sharp_t f\left(B\right)dt. \end{aligned}$$

For the case $f(A)\leq f(B)$, by applying Remark 2.1 for $s=\frac{1}{2}$, we have

$$\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t \leq \left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

By integrating the above inequality over $t\in[0,\frac{1}{2}]$, we obtain

$$\int_0^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t dt \leq \frac{1}{2}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Multiplying both sides of the above inequality by $f(A)^{\frac{1}{2}}$, we have

$$\begin{aligned} &\int_0^{\frac{1}{2}} f(A)^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t f(A)^{\frac{1}{2}}dt\\ \leq&\frac{1}{2}\left(f(A)^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^{\frac{1}{2}}f(A)^{\frac{1}{2}}\right). \end{aligned}$$

It follows that

$$\int_0^{\frac{1}{2}} f(A)\sharp_t f(B) \leq \frac{f(A)\sharp f(B)}{2}. \tag{2.3}$$

On the other hand, by considering Remark 2.1 for $s=1$, we have

$$\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t \leq f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}.$$

Integrating the above inequality over $t\in[\frac{1}{2},1]$, we get

$$\int_{\frac{1}{2}}^1\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t dt \leq \frac{1}{2}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right).$$

By multiplying both side of the above inequality by $f(A)^{\frac{1}{2}}$, we have

$$\int_{\frac{1}{2}}^1 f(A)^{\frac{1}{2}}\left(f(A)^{-\frac{1}{2}}f(B)f(A)^{-\frac{1}{2}}\right)^t f(A)^{\frac{1}{2}}dt \leq \frac{f(B)}{2}.$$

It follows that

$$\int_{\frac{1}{2}}^{1} f(A)\sharp_t f(B) \le \frac{f(B)}{2}. \tag{2.4}$$

From inequalities (2.3) and (2.4) we obtain

$$\begin{aligned}\int_0^{\frac{1}{2}} f(A\sharp_t B)dt + \int_{\frac{1}{2}}^{1} f(A\sharp_t B)dt &\le \int_0^{\frac{1}{2}} f(A)\sharp_t f(B)dt + \int_{\frac{1}{2}}^{1} f(A)\sharp_t f(B)dt \\ &\le \frac{f(A)\sharp f(B)}{2} + \frac{f(B)}{2}.\end{aligned}$$

It follows that

$$\int_0^1 f(A\sharp_t B)dt \le \int_0^1 f(A)\sharp_t f(B)dt \le \frac{1}{2}((f(A)\sharp f(B)) + f(B)). \qquad \square$$

By making use of inequalities (2.1) and (2.2), we have the following result.

Corollary 2.1. *Let f be an operator geometrically convex function. Then, if $f(A) \le f(B)$ we have*

$$f(A\sharp B) \le \int_0^1 f\left(A\sharp_t B\right) dt \le \frac{1}{2}\left((f(A)\sharp f(B)) + f(B)\right),$$

for $A, B \in B(H)^{++}$.

Theorem 2.2. *Let f be an operator geometrically convex function. Then, we have*

$$f\left(A\sharp B\right) \le \int_0^1 f\left(A\sharp_t B\right) \sharp f\left(A\sharp_{1-t} B\right) dt \le f(A)\sharp f(B),$$

for $A, B \in B(H)^{++}$.

Proof. We can write

$$\begin{aligned}f(A\sharp B) &= f\left((A\sharp_t B)\sharp(A\sharp_{1-t} B)\right) \quad \text{(by Lemma 2.1)}\\ &\le f(A\sharp_t B)\sharp f(A\sharp_{1-t} B) \quad (f \text{ is operator geometrically convex})\\ &\le \left(f(A)\sharp_t f(B)\right)\sharp\left(f(A)\sharp_{1-t} f(B)\right) \quad \text{(by Lemma 2.2)}\\ &= f(A)\sharp f(B).\end{aligned}$$

So, we obtain

$$f(A\sharp B) \le f(A\sharp_t B)\sharp f(A\sharp_{1-t} B) \le f(A)\sharp f(B).$$

Integrating the above inequality over $t \in [0,1]$ we obtain the desired result. $\square$

We divide the interval $[0,1]$ to the interval $[\nu, 1-\nu]$ when $\nu \in [0, \frac{1}{2})$ and to the interval $[1-\nu, \nu]$ when $\nu \in (\frac{1}{2}, 1]$. The we have the following inequalities.

Theorem 2.3. *Let $A, B \in B(H)^{++}$ such that $f(A) \le f(B)$. Then, we have*

(a) *for $\nu \in [0, \frac{1}{2})$*

$$f(A)\sharp_\nu f(B) \le \frac{1}{1-2\nu}\int_\nu^{1-\nu} f(A)\sharp_t f(B)dt \le f(A)\sharp_{1-\nu} f(B); \tag{2.5}$$

(b) *for* $\nu \in (\frac{1}{2}, 1]$

$$f(A)\sharp_{1-\nu}f(B) \leq \frac{1}{2\nu-1}\int_{1-\nu}^{\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_\nu f(B). \tag{2.6}$$

Proof. Let $\nu \in [0, \frac{1}{2})$, then by Remark 2.1 we have

$$\begin{aligned}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{\nu} &\leq \left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{t}\\ &\leq \left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{1-\nu},\end{aligned}$$

for $\nu \leq t \leq 1-\nu$ and $A, B \in B(H)^{++}$ such that $\operatorname{Sp}(A), \operatorname{Sp}(B) \subseteq I$.
By integrating the above inequality over $t \in [\nu, 1-\nu]$ we obtain

$$\begin{aligned}\int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{\nu}dt &\leq \int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{t}dt\\ &\leq \int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{1-\nu}dt.\end{aligned}$$

It follows that

$$\begin{aligned}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{\nu} &\leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu}\left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{t}dt\\ &\leq \left(f(A)^{\frac{-1}{2}}f(B)f(A)^{\frac{-1}{2}}\right)^{1-\nu}.\end{aligned}$$

Multiplying the both sides of the above inequality by $f(A)^{\frac{1}{2}}$ gives us

$$f(A)\sharp_\nu f(B) \leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_{1-\nu}f(B).$$

Also, we know that

$$\begin{aligned}\lim_{\nu\to\frac{1}{2}} f(A)\sharp_\nu f(B) &= \lim_{\nu\to\frac{1}{2}} \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f(A)\sharp_t f(B)dt\\ &= \lim_{\nu\to\frac{1}{2}} f(A)\sharp_{1-\nu}f(B)\\ &= f(A)\sharp f(B).\end{aligned}$$

Similarily, for $\nu \in (\frac{1}{2}, 1]$, by the same proof as above, we get

$$f(A)\sharp_{1-\nu}f(B) \leq \frac{1}{2\nu-1}\int_{1-\nu}^{\nu} f(A)\sharp_t f(B)dt \leq f(A)\sharp_\nu f(B).$$ □

By definition of geometrically convex function and (2.5) we have

$$\begin{aligned}f\left(A\sharp_\nu B\right) &\leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f\left(A\sharp_t B\right)dt\\ &\leq \frac{1}{1-2\nu}\int_{\nu}^{1-\nu} f(A)\sharp_t f(B)dt\\ &\leq f(A)\sharp_{1-\nu}f(B),\end{aligned}$$

for $\nu \in [0, \frac{1}{2})$. We should mention here that

$$\lim_{\nu \to \frac{1}{2}} \frac{1}{1-2\nu} \int_{\nu}^{1-\nu} f\left(A \sharp_t B\right) dt = \lim_{\nu \to \frac{1}{2}} f(A \sharp_\nu B) = f(A \sharp B).$$

On the other hand, by the definition of geometrically convex function and (2.6) we have

$$\begin{aligned} f\left(A \sharp_{1-\nu} B\right) &\leq \frac{1}{2\nu - 1} \int_{1-\nu}^{\nu} f\left(A \sharp_t B\right) dt \\ &\leq \frac{1}{2\nu - 1} \int_{1-\nu}^{\nu} f(A) \sharp_t f(B) dt \\ &\leq f(A) \sharp_\nu f(B), \end{aligned}$$

for $\nu \in (\frac{1}{2}, 1]$.

3. Examples and Remarks

In this section we give some examples of the results that obtained in the previous section.

Remark 3.1. For positive $A, B \in B(H)$, Ando proved in [1] that if Ψ is a positive linear map, then we have

$$\Psi(A \sharp B) \leq \Psi(A) \sharp \Psi(B).$$

The above inequality shows that we can find some examples for Definition 2.1 when f is linear.

Example 3.1. It is easy to check that the function $f(t) = t^{-1}$ is operator geometrically convex for operators in $B(H)^{++}$.

Definition 3.1. Let ϕ be a map on C^*-algebra $B(H)$. We say that ϕ is 2-positive if the 2×2 operator matrix $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geq 0$, then we have $\begin{bmatrix} \phi(A) & \phi(B) \\ \phi(B^*) & \phi(C) \end{bmatrix} \geq 0$.

In [6], M. Lin gave an example of a 2-positive map over contraction operators (i.e., $\|A\| < 1$). He proved that

$$\phi(t) = (1-t)^{-1} \tag{3.1}$$

is 2-positive.

Example 3.2. Let A and B be two contraction operators in $B(H)^{++}$. Then it is easy to check $A \sharp B$ is also a contraction and positive. Also, we know the 2×2 operator matrix

$$\begin{bmatrix} A & A \sharp B \\ A \sharp B & B \end{bmatrix}$$

is semidefinite positive. Hence, by (3.1) we obtain

$$\begin{bmatrix} (I-A)^{-1} & (I-(A \sharp B))^{-1} \\ (I-(A \sharp B))^{-1} & (I-B)^{-1} \end{bmatrix}$$

is semidefinite positive.

On the other hand, by Ando's characterization of the geometric mean if X is a Hermitian matrix and

$$\begin{bmatrix} A & X \\ X & B \end{bmatrix} \geq 0,$$

then $X \leq A\sharp B$. So we conclude that $(I-(A\sharp B))^{-1} \leq (I-A)^{-1}\sharp(I-B)^{-1}$. Therefore, the function $\phi(t) = (1-t)^{-1}$ is operator geometrically convex.

Also, Lin proved that the function

$$\phi(t) = \frac{1+t}{1-t}$$

is 2-positive over contractions. By the same argument as Example 3.2 we can say the above function is operator geometrically convex too.

Example 3.3. In the proof of [3, Theorem 4.12], by applying Hölder-McCarthy inequality the authors showed the following inequalities

$$\begin{aligned} \langle A\sharp_\alpha Bx, x\rangle &= \left\langle \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right)^{\alpha} A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle \\ &\leq \left\langle \left(A^{\frac{-1}{2}}BA^{\frac{-1}{2}}\right) A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle^{\alpha} \left\langle A^{\frac{1}{2}}x, A^{\frac{1}{2}}x\right\rangle^{1-\alpha} \\ &= \langle Ax, x\rangle^{1-\alpha}\langle Bx, x\rangle^{\alpha} \\ &= \langle Ax, x\rangle\sharp_\alpha\langle Bx, x\rangle, \end{aligned}$$

for $x \in H$ and $\alpha \in [0,1]$. By taking the supremum over unit vector x, we obtain that $f(x) = \|x\|$ is geometrically convex function for usual operator norms.

By the above example and Corollary 2.1, when $\|A\| \leq \|B\|$ we have

$$\|A\sharp B\| \leq \int_0^1 \|A\sharp_t B\| dt \leq \frac{1}{2}(\sqrt{\|A\|\|B\|} + \|B\|),$$

for $A, B \in B(H)^{++}$.

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A NOTE ON PROBABILITY CONVERGENCE DEFINED BY UNBOUNDED MODULUS FUNCTION AND $\alpha\beta$-STATISTICAL CONVERGENCE

SUMIT SOM

Abstract. In this paper we define $f-\alpha\beta$-statistical convergence of order γ in probability and $f-\alpha\beta$-strong p-Cesàro summability of order γ in probability for a sequence of random variables under unbounded modulus function and examine the relation between these two concepts. We show by an example that this notion of $f-\alpha\beta$-statistical convergence of order γ in probability is stronger than $\alpha\beta$-statistical convergence of order γ in probability [9].

1. Introduction

The idea of convergence of a real sequence has been extended to statistical convergence by Fast [10] and Steinhaus [19] and later on reintroduced by Schoenberg [17] independently and is based on the notion of asymptotic density of the subset of natural numbers. However, the first idea of statistical convergence (by different name) was given by Zygmund [20] in the first edition of his monograph published in Warsaw in 1935. Later on it was further investigated from the sequence space point of view and linked with summability theorem by Fridy [11], Connor [5], Šalát [16], Das et. al. [6], Fridy and Orhan [12].

In [3, 4] a different direction was given to the study of statistical convergence where the notion of statistical convergence of order γ $(0 < \gamma < 1)$ was introduced by using the notion of natural density of order γ (where n is replaced by n^γ in the denominator in the definition of natural density). It was observed in [3], that the behavior of this new convergence was not exactly parallel to that of statistical convergence and some

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basic properties were obtained. More results on this convergence can be seen from [18].

Recently the idea of statistical convergence of order γ was further extended to $\alpha\beta$-statistical convergence of order γ in [2] as follows: Let $\alpha = \{\alpha_n\}_{n\in\mathbb{N}}$, $\beta = \{\beta_n\}_{n\in\mathbb{N}}$ be two non-decreasing sequences of positive real numbers satisfying the conditions, $\alpha_n \leq \beta_n$ for all $n \in \mathbb{N}$, and $(\beta_n - \alpha_n) \to \infty$ as $n \to \infty$. This pair of sequence we denoted by (α, β). Then a sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers is said to be $\alpha\beta$-statistically convergent of order γ (where $0 < \gamma \leq 1$) to a real number x if for each $\varepsilon > 0$, the set $K = \{n \in \mathbb{N} : |x_n - x| \geq \varepsilon\}$ has $\alpha\beta$-natural density zero, i.e.,

$$\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^\gamma} |\{k \in [\alpha_n, \beta_n] : |x_k - x| \geq \varepsilon\}| = 0$$

and we write $S^\gamma_{\alpha\beta} - \lim x_n = x$ or $x_n \xrightarrow{S^\gamma_{\alpha\beta}} x$. $\alpha\beta$-statistical convergence of order γ is more general than statistical convergence of order γ, lacunary statistical convergence of order γ and λ statistical convergence of order γ if we take

(i) $\alpha_n = 1$ and $\beta_n = n$, for all $n \in \mathbb{N}$;
(ii) $\alpha_r = (k_{r-1} + 1)$ and $\beta_r = k_r$, for all $r \in \mathbb{N}$, where $\{k_r\}_{r\in\mathbb{N}\cup\{0\}}$ is a lacunary sequence;
(iii) $\alpha_n = (n - \lambda_n + 1)$ and $\beta_n = n$, for all $n \in \mathbb{N}$, respectively.

On the other hand, in probability theory, a new type of convergence called statistical convergence in probability was introduced in [13], as follows: Let $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events $\triangle$ and a given probability function $P : \triangle \to \mathbb{R}$. Then the sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be statistically convergent in probability to a random variable X (where $X : S \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n : P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0.$$

In this case we write $X_n \xrightarrow{PS} X$. The class of all sequences of random variables which are statistically convergent in probability is denoted by PS. One can also see [7,8,14] for related works.

In the year 2014, the concept of f-statistical convergence was introduced by Aizpuru et al. [1] just by replacing $|\{k \leq n : |x_k - c| \geq \varepsilon\}|$ and $\frac{1}{n}$ by $f(|\{k \leq n : |x_k - c| \geq \varepsilon\}|)$ and $\frac{1}{f(n)}$, respectively, where f is an unbounded modulus function. The notion of a modulus function was introduced by Nakano [15]. We recall that a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$;
(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$;
(iii) f is increasing and f is continuous from the right at 0.

The f-density of $K \subset \mathbb{N}$ is denoted by $d_f(K) = \lim_{n\to\infty} \frac{f(|K(n)|)}{f(n)}$. In case of f-density, the relation $d_f(\mathbb{N} \setminus K) = 1 - d_f(K)$ holds only when $d_f(K) = 0$. In other all cases the relation can't hold.

In a natural way in this paper we combine the approches of the above mentioned papers and introduce new and more general methods, namely, $f - \alpha\beta$-statistical convergence of order γ in probability, $f - \alpha\beta$-strong p-Cesàro summability of order γ in probability for a sequence of random variables. We mainly investigate their relationship and also make some observations about these classes. In the way we show that the notion of $f - \alpha\beta$-statistical convergence of order γ in probability is stronger than $\alpha\beta$-statistical convergence of order γ in probability (see [9]). It is important to note that the method of proofs and in particular examples are not analogous to the real case.Throughout the paper f will denote unbounded modulus function.

2. $f - \alpha\beta$-Statistical Convergence of Order γ in Probability

We first introduce the definition of $f - \alpha\beta$-statistical convergence of order γ for a sequence of real numbers as follows.

Definition 2.1. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence of real numbers and f be an unbounded modulus function. The sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be $f - \alpha\beta$-statistically convergent of order γ to a real number x if for any $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} f(|\{k \in [\alpha_n, \beta_n] : |x_k - x| \geq \varepsilon\}|) = 0.$$

The class of all real sequences which are $f - \alpha\beta$-statistically convergent of order γ is denoted by $S^{\gamma,f}_{\alpha\beta}$.

Now we like to introduce the definition of $f - \alpha\beta$-statistical convergence of order γ in probability for a sequence of random variables as follows.

Definition 2.2. Let $(S, \triangle, P)$ be a probability space and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events $\triangle$ and a given probability function $P : \triangle \to \mathbb{R}$. Then the sequence $\{X_n\}_{n\in\mathbb{N}}$ is said to be $f - \alpha\beta$-statistically convergent of order γ (where $0 < \gamma \leq 1$) in probability to a random variable X (where $X : S \to \mathbb{R}$) if for any $\varepsilon, \delta > 0$

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|) = 0$$

or equivalently

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} f(|\{k \in [\alpha_n, \beta_n] : 1 - P(|X_k - X| < \varepsilon) \geq \delta\}|) = 0.$$

In this case we write $S_{\alpha\beta}^{\gamma,f}-\lim P(|X_n-X|\geq\varepsilon)=0$ or $S_{\alpha\beta}^{\gamma,f}-\lim P(|X_n-X|<\varepsilon)=1$ or just $X_n \xrightarrow{PS_{\alpha\beta}^{\gamma,f}} X$. The class of all sequences of random variables which are $f-\alpha\beta$-statistically convergent of order γ in probability is denoted simply by $PS_{\alpha\beta}^{\gamma,f}$.

In Definition 2.2, if we take $f(x)=x$ then $\{X_n\}_{n\in\mathbb{N}}$ is said to be $\alpha\beta$-statistically convergent of order γ in probability to a random variable X. So, $f-\alpha\beta$-statistical convergence of order γ in probability is a generalization of $\alpha\beta$-statistical convergence of order γ in probability for a sequence of random variables.

To show that this is indeed more stronger notion than $\alpha\beta$-statistical convergence of order γ in probability, we will now give an example of a sequence of random variables which is $\alpha\beta$-statistically convergent of order γ in probability but is not $f-\alpha\beta$-statistically convergent of order γ in probability.

Example 2.1. Let a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ be defined by

$$X_n\in\begin{cases}\{-1,1\} \text{ with probability } \frac{1}{2}, \text{ if } n=m^2 \text{ for some } m\in\mathbb{N},\\ \{0,1\} \text{ with probability } P(X_n=0)=\left(1-\frac{1}{n^2}\right) \text{ and } P(X_n=1)=\frac{1}{n^2}, \text{ if } n\neq m^2\\ \text{for any } m\in\mathbb{N}.\end{cases}$$

Let $0<\varepsilon,\delta<1$. Then, we have,

$$P(|X_n-0|\geq\varepsilon)=1, \quad \text{if } n=m^2 \text{ for some } m\in\mathbb{N},$$

and

$$P(|X_n-0|\geq\varepsilon)=\frac{1}{n^2}, \quad \text{if } n\neq m^2 \text{ for any } m\in\mathbb{N}.$$

Let $\frac{1}{2}<\gamma\leq 1$, $\alpha_n=1$, $\beta_n=n^2$, for all $n\in\mathbb{N}$ and $f(x)=\frac{x}{1+x}$ for all $x\geq 0$. Then we have the inequality

$$\frac{1}{n^{2\gamma}}|\{k\in[1,n^2]:P(|X_n-0|\geq\varepsilon)\geq\delta\}|=\left(\frac{n}{n^{2\gamma}}+\frac{d}{n^{2\gamma}}\right)\to 0 \text{ as } n\to\infty,$$

where d is a finite positive integer. So, $X_n \xrightarrow{PS_{\alpha\beta}^{\gamma}} 0$, where $\frac{1}{2}<\gamma\leq 1$. But

$$\lim_{n\to\infty}\frac{1}{f((\beta_n-\alpha_n+1)^\gamma)}f(|\{k\in[\alpha_n,\beta_n]:P(|X_k-0|\geq\varepsilon)\geq\delta\}|)=1.$$

This shows that $\{X_n\}_{n\in\mathbb{N}}$ is not $f-\alpha\beta$-statistically convergent of order γ in probability to 0.

Theorem 2.1. *If a sequence of constants $x_n \xrightarrow{S_{\alpha\beta}^{\gamma,f}} x$, then regarding a constant as a random variable having one point distribution at that point, we may also write $x_n \xrightarrow{PS_{\alpha\beta}^{\gamma,f}} x$.*

Proof. Proof is straight forward, so omitted. □

The following example shows that in general the converse of Theorem 2.1 is not true and also shows that there is a sequence $\{X_n\}_{n\in\mathbb{N}}$ of random variables which is $f-\alpha\beta$-statistically convergent in probability to a random variable X but it is not $f-\alpha\beta$-statistically convergent of order γ in probability for $0<\gamma<1$.

Example 2.2. Let c be a rational number between γ_1 and γ_2. Let the probability density function of X_n be given by

$$f_n(x)=\begin{cases}1, & \text{where } 0<x<1,\\ 0, & \text{otherwise, if } n=[m^{\frac{1}{c}}] \text{ for some } m\in\mathbb{N},\end{cases}$$

$$f_n(x)=\begin{cases}\dfrac{nx^{n-1}}{2^n}, & \text{where } 0<x<2,\\ 0, & \text{otherwise, if } n\neq[m^{\frac{1}{c}}] \text{ for any } m\in\mathbb{N}.\end{cases}$$

Now let $0<\varepsilon,\delta<1$. Then

$$P(|X_n-2|\geq\varepsilon)=1, \quad \text{if } n=[m^{\frac{1}{c}}] \text{ for some } m\in\mathbb{N},$$
$$P(|X_n-2|\geq\varepsilon)=\left(1-\frac{\varepsilon}{2}\right)^n, \quad \text{if } n\neq[m^{\frac{1}{c}}] \text{ for any } m\in\mathbb{N}.$$

Now let $\alpha_n=1,\ \beta_n=n^2,\ f(x)=\sqrt{x}$ for all $x\geq 0$. Consequently, we have the inequality

$$\lim_{n\to\infty}\sqrt{\frac{n^{2c}-1}{n^{2\gamma_1}}}\leq\lim_{n\to\infty}\frac{1}{f(n^{2\gamma_1})}f(|\{k\in[1,n^2]:P(|X_k-2|\geq\varepsilon)\geq\delta\}|)$$

and

$$\lim_{n\to\infty}\frac{1}{f(n^{2\gamma_2})}f(|\{k\in[1,n^2]:P(|X_k-2|\geq\varepsilon)\geq\delta\}|)\leq\lim_{n\to\infty}\sqrt{\frac{n^{2c}+1}{n^{2\gamma_2}}+\frac{d}{n^{2\gamma_2}}},$$

where d is a fixed finite positive integer. This shows that $\{X_n\}_{n\in\mathbb{N}}$ is $f-\alpha\beta$-statistically convergent of order γ_2 in probability to 2 but is not $f-\alpha\beta$-statistically convergent of order γ_1 in probability to 2 whenever $\gamma_1<\gamma_2$ and this is not the usual $f-\alpha\beta$-statistical convergence of order γ of real numbers. So, the converse of Theorem 2.1 is not true. Also by taking $\gamma_2=1$, we see that $X_n\xrightarrow{PS^{1,f}_{\alpha\beta}}2$ but $\{X_n\}_{n\in\mathbb{N}}$ is not $f-\alpha\beta$-statistically convergent of order γ in probability to 2 for $0<\gamma<1$.

Theorem 2.2 (Elementary properties). (i) *If $X_n\xrightarrow{PS^{\gamma,f}_{\alpha\beta}}X$ and $X_n\xrightarrow{PS^{\gamma,g}_{\alpha\beta}}Y$, then $P\{X=Y\}=1$, where f and g are unbounded modulus functions and $0<\gamma\leq 1$.*

(ii) *If $X_n\xrightarrow{PS^{\gamma_1,f}_{\alpha\beta}}X$ and $X_n\xrightarrow{PS^{\gamma_2,f}_{\alpha\beta}}Y$, then $P\{X=Y\}=1$ for any γ_1,γ_2 where $0<\gamma_1,\gamma_2\leq 1$.*

(iii) *Let $0<\gamma_1\leq\gamma_2\leq 1$. Then $PS^{\gamma_1,f}_{\alpha\beta}\subseteq PS^{\gamma_2,f}_{\alpha\beta}$ and this inclusion is strict whenever $\gamma_1<\gamma_2$.*

Proof. (i) If possible let $P\{X = Y\} \neq 1$. Then, there exists two positive real numbers ε and δ such that $P(|X - Y| \geq \varepsilon) = \delta > 0$. Then we have

$$\begin{aligned}&\lim_{n\to\infty} \frac{f(\beta_n - \alpha_n + 1)}{f((\beta_n - \alpha_n + 1)^\gamma)}\\ &\quad - \lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} f\left(\left|\left\{k \in [\alpha_n, \beta_n] : P\left(|X_k - Y| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|\right)\\ \leq &\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} f\left(\left|\left\{k \in [\alpha_n, \beta_n] : P\left(|X_k - X| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right|\right),\end{aligned}$$

which is impossible because the left hand limit is not 0 whereas the right hand limit is 0. So, $P\{X = Y\} = 1$.

(ii) Proof is straightforward and so is omitted.

(iii) The first part is obvious. The inclusion is proper as can be seen from Example 2.2. □

Remark 2.1. In Theorem 2 [3] it was observed that $m_0^{\gamma_1} \subset m_0^{\gamma_2}$ and this inclusion was shown to be strict for at least those γ_1, γ_2 for which there is a $k \in \mathbb{N}$ such that $\gamma_1 < \frac{1}{k} < \gamma_2$. But Example 2.2 shows that the inequality is strict whenever $\gamma_1 < \gamma_2$.

Corollary 2.1. *Let f and g be two unbounded modulus functions and $0 < \gamma \leq 1$. Then $PS_{\alpha\beta}^{\gamma,f} = PS_{\alpha\beta}^{\gamma,g}$.*

Theorem 2.3. *If $X_n \xrightarrow{PS_{\alpha\beta}^{\gamma,f}} X$ (where f is an unbounded modulus function), then $X_n \xrightarrow{PS_{\alpha\beta}^{\gamma}} X$.*

Proof. As $X_n \xrightarrow{PS_{\alpha\beta}^{\gamma,f}} X$ so for any $\varepsilon, \delta > 0$,

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|) = 0.$$

Let $\frac{1}{p_1} > 0$. Then there exists $p \in \mathbb{N}$ such that for all $n \geq p$,

$$\begin{aligned}&f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|) < \frac{1}{p_1} f((\beta_n - \alpha_n + 1)^\gamma)\\ \Rightarrow &f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|)\\ &\quad < \frac{1}{p_1} f\left(\frac{(\beta_n - \alpha_n + 1)^\gamma}{p_1} + \cdots + \frac{(\beta_n - \alpha_n + 1)^\gamma}{p_1}\right)\\ \Rightarrow &f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|) \leq f\left(\frac{(\beta_n - \alpha_n + 1)^\gamma}{p_1}\right)\\ \Rightarrow &|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}| \leq \frac{(\beta_n - \alpha_n + 1)^\gamma}{p_1}\\ \Rightarrow &\lim_{n\to\infty} \frac{1}{(\beta_n - \alpha_n + 1)^\gamma} |\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}| = 0.\end{aligned}$$

This shows that $X_n \xrightarrow{PS^{\gamma}_{\alpha\beta}} X$. $\square$

From Theorem 2.3 and Example 2.1 we see that this notion of $f-\alpha\beta$-statistical convergence of order γ in probability is stronger than $\alpha\beta$-statistical convergence of order γ in probability (see [9]).

Theorem 2.4. *Let f be an unbounded modulus function and $0<\gamma\leq 1$. Let (α,β) and (α',β') are two pairs of sequences of positive real numbers such that $[\alpha'_n,\beta'_n]\subseteq[\alpha_n,\beta_n]$ for all $n\in\mathbb{N}$ and $f((\beta_n-\alpha_n+1)^{\gamma})\leq\varepsilon f((\beta'_n-\alpha'_n+1)^{\gamma})$ for some $\varepsilon>0$. Then we have $PS^{\gamma,f}_{\alpha\beta}\subseteq PS^{\gamma,f}_{\alpha'\beta'}$.*

Proof. Proof is straightforward and so is omitted. $\square$

But if the condition of the Theorem 2.4 is violated, then limit may not be unique for two different (α,β)'s. We now give an example to show this.

Example 2.3. Let $\alpha=\{(2n)!\}$, $\beta=\{(2n+1)!\}$ and $\alpha'=\{(2n+1)!\}$, $\beta'=\{(2n+2)!\}$ and $f(x)=\sqrt{x}$ for all $x\geq 0$.

Let us define a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ by,

$$X_k\in\begin{cases}\{-1,1\} \text{ with probability } P(X_k=-1)=\frac{1}{k},\ P(X_k=1)=(1-\frac{1}{k}),\\ \text{ if } (2n)!<k<(2n+1)!,\\ \{-2,2\} \text{ with probability } P(X_k=-2)=\frac{1}{k},\ P(X_n=2)=(1-\frac{1}{k}),\\ \text{ if } (2n+1)!<k<(2n+2)!,\\ \{-3,3\} \text{ with probability } P(X_k=-3)=P(X_k=3),\\ \text{ if } k=(2n)! \text{ and } k=(2n+1)!.\end{cases}$$

Let $0<\varepsilon,\delta<1$ and $0<\gamma<1$. Then for the sequence (α,β)

$$P(|X_k-1|\geq\varepsilon)=\frac{1}{k},\quad \text{if } (2n)!<k<(2n+1)!$$

and

$$P(|X_k-1|\geq\varepsilon)=1,\quad \text{if } (2n+1)!<k<(2n+2)!$$

and

$$P(|X_k-1|\geq\varepsilon)=1,\quad \text{if } k=(2n)! \text{ and } k=(2n+1)!$$

implies

$$\lim_{n\to\infty}\frac{1}{f(((2n+1)!-(2n)!+1)^{\gamma})}f(|\{k\in[(2n)!,(2n+1)!]:P(|X_k-1|\geq\varepsilon)\geq\delta\}|)=0.$$

So, $X_n \xrightarrow{PS^{\gamma,f}_{\alpha\beta}} 1$.

Similarly, it can be shown that for the sequence $\alpha'=\{(2n+1)!\}$, $\beta'=\{(2n+2)!\}$, $X_n \xrightarrow{PS^{\gamma,f}_{\alpha'\beta'}} 2$.

Definition 2.3. Let $(S, \triangle, P)$ be a probability space and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where each X_n is defined on the same sample space S (for each n) with respect to a given class of events $\triangle$ and a given probability function $P : \triangle \to \mathbb{R}$. A sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ is said to be $f - \alpha\beta$-strong p-Cesàro summable of order γ (where $0 < \gamma \leq 1$ and $p > 0$ is any fixed positive real number) in probability to a random variable X if for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} \sum_{k\in[\alpha_n,\beta_n]} f(\{P(|X_k - X| \geq \varepsilon)\}^p) = 0.$$

In this case we write $X_n \xrightarrow{PW^{\gamma,p,f}_{\alpha\beta}} X$. The class of all sequences of random variables which are $f-\alpha\beta$-strong p-Cesàro summable of order γ in probability is denoted simply by $PW^{\gamma,p,f}_{\alpha\beta}$.

Theorem 2.5. *Let f be an unbounded modulus function such that $f(x) \leq x$ and $f(ax) = af(x)$, for all $x \geq 0$ and $a \in \mathbb{R}$. If $X_n \xrightarrow{PW^{\gamma_1,p,f}_{\alpha\beta}} X$ and $X_n \xrightarrow{PW^{\gamma_2,p,f}_{\alpha\beta}} Y$ (where $p \geq 1$), then $P\{X = Y\} = 1$ for any γ_1, γ_2 where $0 < \gamma_1, \gamma_2 \leq 1$.*

Proof. Proof is straight forward, so omitted. □

Theorem 2.6. *Let f and g be unbounded modulus functions satisfying the conditions stated in Theorem 2.5. If $X_n \xrightarrow{PW^{\gamma,p,f}_{\alpha\beta}} X$ and $X_n \xrightarrow{PW^{\gamma,p,g}_{\alpha\beta}} Y$ (where $p \geq 1$), then $P\{X = Y\} = 1$.*

Proof. If possible let $P\{X = Y\} \neq 1$. Then there exists two positive real numbers ε and δ such that $P(|X - Y| \geq \varepsilon) = \delta > 0$. Then we have

$$\begin{aligned}
&\{P(|X - Y| \geq \varepsilon)\}^p \leq \left\{P\left(|X_k - X| \geq \frac{\varepsilon}{2}\right) + P\left(|X_k - Y| \geq \frac{\varepsilon}{2}\right)\right\}^p \\
\Rightarrow &\sum_{k\in[\alpha_n,\beta_n]} f((P(|X - Y| \geq \varepsilon))^p) \leq 2^p \sum_{k\in[\alpha_n,\beta_n]} f\left(\left(P\left(|X_k - X| \geq \frac{\varepsilon}{2}\right)\right)^p\right) \\
&+ 2^p \sum_{k\in[\alpha_n,\beta_n]} f\left(\left(P\left(|X_k - Y| \geq \frac{\varepsilon}{2}\right)\right)^p\right) \\
\Rightarrow &\frac{f(\delta^p)(\beta_n - \alpha_n + 1)^\gamma}{f((\beta_n - \alpha_n + 1)^\gamma)} - 2^p \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} \sum_{k\in[\alpha_n,\beta_n]} f\left(\left(P\left(|X_k - Y| \geq \frac{\varepsilon}{2}\right)\right)^p\right) \\
&\leq 2^p \frac{1}{f((\beta_n - \alpha_n + 1)^\gamma)} \sum_{k\in[\alpha_n,\beta_n]} f\left(\left(P\left(|X_k - X| \geq \frac{\varepsilon}{2}\right)\right)^p\right),
\end{aligned}$$

which is impossible because the left hand limit is not 0 whereas the right hand limit is 0. So, $P\{X = Y\} = 1$. □

Corollary 2.2. *Let f and g be two unbounded modulus functions satisfying the conditions stated in Theorem 2.5 and $0 < \gamma \leq 1$, $p \geq 1$. Then $PW^{\gamma,p,f}_{\alpha\beta} = PW^{\gamma,p,g}_{\alpha\beta}$.*

Theorem 2.7. (i) *Let* $0 < \gamma_1 \leq \gamma_2 \leq 1$*. Then* $PW_{\alpha\beta}^{\gamma_1,p,f} \subseteq PW_{\alpha\beta}^{\gamma_2,p,f}$*. This inclusion is strict whenever* $\gamma_1 < \gamma_2$*.*

(ii) *Let* $0 < \gamma \leq 1$ *and* $0 < p < q < \infty$*. Then* $PW_{\alpha\beta}^{\gamma,q,f} \subset PW_{\alpha\beta}^{\gamma,p,f}$*.*

Proof. (i) The first part of this theorem is straightforward and so is omitted. For the second part we will give an example to show that there is a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ which is $f-\alpha\beta$-strong p-Cesàro summable of order γ_2 in probability to a random variable X but is not $f-\alpha\beta$-strong p-Cesàro summable of order γ_1 in probability whenever $\gamma_1 < \gamma_2$.

Let $2c$ be a rational number between γ_1 and γ_2. We consider a sequence of random variables :

$$X_n \in \begin{cases} \{-1,1\} \text{ with probability } \frac{1}{2}, \text{ if } n = [m^{\frac{1}{c}}] \text{ for some } m \in \mathbb{N}, \\ \{0,1\} \text{ with probability } P(X_n = 0) = 1 - \frac{1}{\sqrt[p]{n^4}} \text{ and } P(X_n = 1) = \frac{1}{\sqrt[p]{n^4}}, \\ \text{if } n \neq [m^{\frac{1}{c}}] \text{ for any } m \in \mathbb{N}. \end{cases}$$

Then we have, for $0 < \varepsilon < 1$

$$P(|X_n - 0| \geq \varepsilon) = 1, \quad \text{if } n = [m^{\frac{1}{c}}] \text{ for some } m \in \mathbb{N}$$

and

$$P(|X_n - 0| \geq \varepsilon) = \frac{1}{\sqrt[p]{n^4}}, \quad \text{if } n \neq [m^{\frac{1}{c}}] \text{ for any } m \in \mathbb{N}.$$

Let $\alpha_n = 1$ and $\beta_n = n^2$ and $f(x) = \sqrt{x}$ for all $x \geq 0$. So, we have the inequality

$$\lim_{n\to\infty} \frac{n^{2c} - 1}{n^{\gamma_1}} \leq \lim_{n\to\infty} \frac{1}{n^{\gamma_1}} \sum_{k\in[1,n^2]} f(\{P(|X_k - 0| \geq \varepsilon)\}^p)$$

and

$$\lim_{n\to\infty} \frac{1}{n^{\gamma_2}} \sum_{k\in[1,n^2]} f(\{P(|X_k - 0| \geq \varepsilon)\}^p) \leq \lim_{n\to\infty} \left[\frac{n^{2c}+1}{n^{\gamma_2}} + \frac{1}{n^{\gamma_2}} \left(\frac{1}{1^2} + \frac{1}{2^2} + \cdots + \frac{1}{n^4} \right) \right].$$

This shows that $X_n \xrightarrow{PW_{\alpha\beta}^{\gamma_2,p,f}} 0$ but $\{X_n\}_{n\in\mathbb{N}}$ is not $f-\alpha\beta$-strong p-Cesàro summable of order γ_1 in probability to 0.

(ii) Proof is straightforward and so is omitted. □

Theorem 2.8. *Let f be an unbounded modulus function such that $f(x) \leq x$ for all $x \geq 0$ and $0 < \gamma_1 \leq \gamma_2 \leq 1$. Then* $PW_{\alpha\beta}^{\gamma_1,p,f} \subset PS_{\alpha\beta}^{\gamma_2,f}$*.*

Proof. Let $X_n \xrightarrow{PW_{\alpha\beta}^{\gamma_1,p,f}} X$. Then for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^{\gamma_1})} \sum_{k\in[\alpha_n,\beta_n]} f(\{P(|X_k - X| \geq \varepsilon)\}^p) = 0.$$

Then

$$\sum_{k\in[\alpha_n,\beta_n]} f((P(|X_k - X| \geq \varepsilon))^p) \geq \sum_{\substack{k\in[\alpha_n,\beta_n]\\ P(|X_k-X|\geq\varepsilon)\geq\delta}} f((P(|X_k - X| \geq \varepsilon))^p)$$

$$\Rightarrow \sum_{k\in[\alpha_n,\beta_n]} f((P(|X_k - X| \geq \varepsilon))^p) \geq f(\delta^p)\ f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|)$$

$$\Rightarrow \frac{1}{f((\beta_n - \alpha_n + 1)^{\gamma_1})} \sum_{k\in[\alpha_n,\beta_n]} f((P(|X_k - X| \geq \varepsilon))^p)$$

$$\geq f(\delta^p)\ \frac{1}{f((\beta_n - \alpha_n + 1)^{\gamma_2})} f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|).$$

This shows that

$$\lim_{n\to\infty} \frac{1}{f((\beta_n - \alpha_n + 1)^{\gamma_2})} f(|\{k \in [\alpha_n, \beta_n] : P(|X_k - X| \geq \varepsilon) \geq \delta\}|) = 0.$$

This completes the proof. □

But the converse of Theorem 2.8 is not generally true as can be seen from the following example.

Example 2.4. Let a sequence of random variables $\{X_n\}_{n\in\mathbb{N}}$ be defined by,

$$X_n \in \begin{cases} \{-1, 1\} \text{ with probability } \frac{1}{2}, \text{ if } n = m^m \text{ for some } m \in \mathbb{N}, \\ \{0, 1\} \text{ with probability } P(X_n = 0) = 1 - \frac{1}{\sqrt[p]{n}},\ P(X_n = 1) = \frac{1}{\sqrt[p]{n}}, \\ \text{ if } n \neq m^m \text{ for any } m \in \mathbb{N}. \end{cases}$$

Let $0 < \varepsilon < 1$ and $f(x) = \sqrt{x}$ for all $x \geq 0$. Then

$$P(|X_n - 0| \geq \varepsilon) = 1, \quad \text{if } n = m^m \text{ for some } m \in \mathbb{N},$$

and

$$P(|X_n - 0| \geq \varepsilon) = \frac{1}{\sqrt[p]{n}}, \quad \text{if } n \neq m^m \text{ for any } m \in \mathbb{N}.$$

Let $\alpha_n = 1$ and $\beta_n = n^2$. It can be easily seen that $X_n \xrightarrow{PS^{\gamma,f}_{\alpha\beta}} 0$ for each $0 < \gamma \leq 1$.

Let $H = \{n \in \mathbb{N} :\ n \neq m^m \text{ for any } m \in \mathbb{N}\}$. Then

$$\begin{aligned}\frac{1}{n^\gamma} \sum_{k\in[1,n^2]} f(\{P(|X_k - 0| \geq \varepsilon)\}^p) =& \frac{1}{n^\gamma} \sum_{\substack{k\in[1,n^2]\\ k\in H}} f(\{P(|X_k - 0| \geq \varepsilon)\}^p) \\ &+ \frac{1}{n^\gamma} \sum_{\substack{k\in[1,n^2]\\ k\notin H}} f(\{P(|X_k - 0| \geq \varepsilon)\}^p) \\ &= \frac{1}{n^\gamma} \sum_{\substack{k\in[1,n^2]\\ k\in H}} \frac{1}{\sqrt{k}} + \frac{1}{n^\gamma} \sum_{\substack{k\in[1,n^2]\\ k\notin H}} 1 > \frac{1}{n^\gamma} \sum_{k=1}^{n^2} \frac{1}{\sqrt{k}} > \frac{1}{n^{\gamma-1}},\end{aligned}$$

(since $\sum_{k=1}^{n} \frac{1}{\sqrt{k}} > \sqrt{n}$ for $n \geq 2$). So, X_n is not $f - \alpha\beta$-strong p-Cesàro summable of order γ in probability to 0 for $0 < \gamma \leq 1$.

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THE RECIPROCAL COMPLEMENTARY WIENER NUMBER OF GRAPH OPERATIONS

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Abstract. The reciprocal complementary Wiener number of a connected graph G is defined as $\sum_{\{x,y\}\subseteq V(G)} \frac{1}{D+1-d_G(x,y)}$, where D is the diameter of G and $d_G(x,y)$ is the distance between vertices x and y. In this work, we study the reciprocal complementary Wiener number of various graph operations such as join, Cartesian product, composition, strong product, disjunction, symmetric difference, corona product, splice and link of graphs.

1. Introduction

Throughout this work, all graphs considered are simple, connected and finite. Let $G = (V(G), E(G))$ be a connected graph. For $x, y \in V(G)$, the *distance* $d_G(x,y)$ between the vertices x and y is equal to the length of a shortest path that connects x and y. For a vertex x in a connected nontrivial graph G, the *eccentricity* $\varepsilon_G(x)$ of x is the greatest geodesic distance between x and any other vertex of G. Also, the *diameter* $D = D(G)$ of the graph G is defined as the maximum eccentricity of any vertex in G. In other words,

$$\varepsilon_G(x) = \max\{d_G(x,y)|y \in V(G)\}, \quad D = D(G) = \max\{\varepsilon_G(x)|x \in V(G)\}.$$

In mathematical chemistry, a *molecular graph* (or *chemical graph*) is a labeled graph whose vertices correspond to the atoms of the compound and edges correspond to chemical bonds. It is natural to study mathematical properties of these graph models to find chemico-physical properties of the molecule under consideration.

Let G be a n-vertex graph with the vertex-set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and diameter D. The *reciprocal complementary distance matrix* $RCD = [rc_{ij}]$ of G is an $n \times n$

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matrix such that $rc_{ij} = \frac{1}{D+1-d_G(v_i,v_j)}$ if $i \neq j$, and 0 otherwise (see [7]). Ivanciuc et al. [5, 6] introduced the *reciprocal complementary Wiener number* of the graph G as:

$$RCW(G) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} rc_{ij} = \sum_{\{v_i,v_j\} \subseteq V(G)} \frac{1}{D+1-d_G(v_i,v_j)}. \tag{1.1}$$

This invariant has been successfully applied in the structure-property modeling of the molar hear capacity, standard Gibbs energy of formation and vaporization enthalpy of 134 alkanes C_6-C_{10} (see [5]).

Zhou et al. [14] gave various bounds for this quantity and Nordhaus-Gaddum-type result. Moreover, the trees with the smallest, the second smallest and the third smallest RCW, and the unicyclic and bicyclic graphs with the smallest and the second smallest RCW are characterized (see [2]). Zhu et al. [15] obtained the unique tree with $4 \leq D \leq n-3$ and minimum reciprocal complementary Wiener number. They also specified the non-caterpillars with the smallest, the second smallest and the third smallest RCW-value. In [10], some bounds for the reciprocal complementary Wiener index of line graphs are presented.

Up to now, various topological indices have been introduced and used in the QSAR/QSPR studies. The *Wiener index* (or *Wiener number*) is the oldest and is one of the most studied topological quantities, both from a theoretical point of view and applications. This concept is defined as the sum of distances over all unordered vertex pairs in a graph G (see [12]). This invariant obtained wide attention and numerous results have been worked out, see the survey [13]. In special classes of graphs, such as trees, unicyclic and bicyclic graphs, this index has been studied in [3, 9, 11]. After it, a large number of other distance-based topological indices have been proposed and considered in the chemical and mathematico-chemical literature.

Brückler et al. [1] introduced a general distance-based topological index, called Q-index. The Q-index is defined as

$$Q(G) = \sum_{k \geq 0} f(k) D(G,k)\,, \tag{1.2}$$

where f is a function such that $f(0) = 0$, and $D(G,k)$ is the number of vertex pairs at distance k. Q is an additive function of increments associated with pairs of vertices of G. The Wiener, hyper-Wiener, Harary, and reciprocal complementary Wiener indices are all special cases of the Q-index. More precisely, by choosing $f(k) = k, \frac{k^2}{2} + \frac{k}{2}, \frac{1}{k}$ and $\frac{k^3}{6} + \frac{k^2}{2} + \frac{k}{3}$, the Q-index is equal to the Wiener, hyper-Wiener, Harary, and Tratch-Stankevich-Zefirov indices, respectively. In other special case, if consider $f(k) = \frac{1}{D+1-k}$, then the Q-index will be equal to the reciprocal complementary Wiener number. In other words, it holds

$$RCW(G) = \sum_{k=1}^{D} \frac{D(G,k)}{D+1-k}. \tag{1.3}$$

In this research, we study the reciprocal complementary Wiener number of various graph operations like join, Cartesian product, composition, strong product, disjunction, symmetric difference, corona product, splice and link of graphs.

2. Main Results

Throughout this paper, we consider graphs G_i with n_i vertices, m_i edges and the diameter D_i, $i = 1, 2$. Also, note that whenever we say $xy \notin E$, it is assumed that $x \neq y$. Moreover, we use standard notations of graph theory. The path, cycle, star, wheel and complete graphs with n vertices are denoted by P_n, C_n, S_n, W_n and K_n, respectively.

By applying relation (1.3), we compute RCW of some special graphs in the following example.

Example 2.1. Let P_n, K_n, S_n and W_n denote a path graph, complete graph, star graph and wheel graph with n vertices, respectively. Then

$$\begin{aligned} RCW(P_n) &= n-1, \\ RCW(K_n) &= \frac{1}{2}n(n-1), \\ RCW(S_n) &= \frac{1}{2}(n-1)^2, \\ RCW(W_n) &= \begin{cases} 6, & n=4, \\ \frac{1}{2}(n-1)(n-2), & n \geq 5. \end{cases} \end{aligned}$$

We begin by computing the reciprocal complementary Wiener number of join of graphs.

2.1. Join.

The *join* $G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. In the following lemma, we determine the reciprocal complementary Wiener number of join of graphs with respect to their numbers of vertices and edges.

Theorem 2.1. *Let G_1 and G_2 be two n_1- and n_2-vertex graphs, respectively.*

(i) *If G_1 and G_2 are complete graphs, then*

$$RCW(G_1 + G_2) = \frac{1}{2}(n_1 + n_2)(n_1 + n_2 - 1).$$

(ii) *If $\{G_1, G_2\} \neq \{K_{n_1}, K_{n_2}\}$, then*

$$RCW(G_1 + G_2) = \frac{1}{2}\Big(n_1(n_1 + n_2 - 1) + n_2(n_2 - 1) - m_1 - m_2\Big).$$

Proof. Suppose x and y are two vertices of $G_1 + G_2$. By definition of the join of two graphs, one can easily see that

$$d_{G_1+G_2}(x,y) = \begin{cases} 0, & x = y, \\ 1, & xy \in E_1 \text{ or } xy \in E_2 \text{ or } (x \in V_1 \text{ and } y \in V_2), \\ 2, & \text{otherwise.} \end{cases}$$

Assume that G_1 and G_2 are complete graphs, then $G_1 + G_2 = K_{n_1+n_2}$. Therefore, $RCW(G_1 + G_2) = RCW(K_{n_1+n_2}) = \binom{n_1+n_2}{2}$ (see Example 2.1). This completes the proof of part (i). To prove the second part, suppose that at least one of graphs G_1 or G_2 is not complete. So, we have $D = D(G_1 + G_2) = 2$, and

$$\begin{aligned} RCW(G_1+G_2) &= \sum_{\{x,y\}\subseteq V(G_1+G_2)} \frac{1}{3 - d_{G_1+G_2}(x,y)} \\ &= \sum_{\{x,y\}\subseteq V_1} \frac{1}{3 - d_{G_1+G_2}(x,y)} + \sum_{\{x,y\}\subseteq V_2} \frac{1}{3 - d_{G_1+G_2}(x,y)} \\ &\quad + \sum_{\substack{x\in V_1\\ y\in V_2}} \frac{1}{3 - d_{G_1+G_2}(x,y)} \\ &= \sum_{xy\in E_1} \frac{1}{3 - d_{G_1+G_2}(x,y)} + \sum_{xy\notin E_1} \frac{1}{3 - d_{G_1+G_2}(x,y)} \\ &\quad + \sum_{xy\in E_2} \frac{1}{3 - d_{G_1+G_2}(x,y)} + \sum_{xy\notin E_2} \frac{1}{3 - d_{G_1+G_2}(x,y)} \\ &\quad + \sum_{\substack{x\in V_1\\ y\in V_2}} \frac{1}{3 - d_{G_1+G_2}(x,y)} \\ &= \frac{1}{2}\Big(n_1(n_1+n_2-1) + n_2(n_2-1) - m_1 - m_2\Big). \end{aligned}$$

□

Example 2.2. We know that $\overline{K_r} + \overline{K_s} = K_{r,s}$ (in particular, $K_1 + \overline{K_{n-1}} = K_{1,n-1} = S_n$) is the complete bipartite graph. From Theorem 2.1 we obtain explicit formulas for the reciprocal complementary Wiener number of the these graphs

$$RCW(K_{r,s}) = \frac{1}{2}\Big(r(r+s-1) + s(s-1)\Big), \quad RCW(S_n) = \frac{1}{2}(n-1)^2.$$

2.2. **Cartesian product.** The *Cartesian product* $G_1 \square G_2$ of graphs G_1 and G_2 has the vertex set $V_1 \times V_2$ and $(x,y)(u,v)$ is an edge of $G_1 \square G_2$ if ($x = u$ and $yv \in E_2$), or ($xu \in E_1$ and $y = v$). For example, the *ladder graph* $L_{2,n}$ can be obtained as the Cartesian product of two path graphs P_2 and P_n.

Now, we study the reciprocal complementary Wiener number of the Cartesian product of graphs. To do this, we need the following well-known relation related to distance properties of the Cartesian product of two graphs (see [4])

$$d_{G_1\square G_2}\Big((x,y),(u,v)\Big) = d_{G_1}(x,u) + d_{G_2}(y,v). \tag{2.1}$$

Theorem 2.2. *Let G_1 and G_2 be two non-complete graphs. Then*

$$\begin{aligned} RCW(G_1\Box G_2) <& \frac{n_1m_2+n_2m_1}{D_1+D_2}+\frac{2m_1m_2}{D_1+D_2-1}+(n_1+2m_1)\left(RCW(G_2)-\frac{m_2}{D_2}\right) \\ &+(n_2+4m_2)\left(RCW(G_1)-\frac{m_1}{D_1}\right). \end{aligned}$$

Proof. Applying (2.1), we have $D = D(G_1\Box G_2) = D_1 + D_2$. Therefore,

$$\begin{aligned} RCW(G_1\Box G_2) =& \sum_{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)} \frac{1}{D+1-d_{G_1\Box G_2}\big((x,y),(u,v)\big)} \\ =& \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\Box G_2)\\ yv\in E_2}} \frac{1}{D_1+D_2} \\ &+ \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\Box G_2)\\ yv\notin E_2}} \frac{1}{D_1+D_2+1-d_{G_2}(y,v)} \\ &+ \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\Box G_2)\\ xu\in E_1}} \frac{1}{D_1+D_2} \\ &+ \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\Box G_2)\\ xu\notin E_1}} \frac{1}{D_1+D_2+1-d_{G_1}(x,u)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\in E_1, yv\in E_2}} \frac{1}{D_1+D_2-1} \\ &+ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\in E_1, yv\notin E_2}} \frac{1}{D_1+D_2-d_{G_2}(y,v)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\notin E_1, yv\in E_2}} \frac{1}{D_1+D_2-d_{G_1}(x,u)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\Box G_2)\\ xu\notin E_1, yv\notin E_2}} \frac{1}{D_1+D_2+1-d_{G_1}(x,u)-d_{G_2}(y,v)} \\ <& \frac{n_1m_2}{D_1+D_2}+n_1\left(RCW(G_2)-\frac{m_2}{D_2}\right)+\frac{n_2m_1}{D_1+D_2} \\ &+n_2\left(RCW(G_1)-\frac{m_1}{D_1}\right)+\frac{2m_1m_2}{D_1+D_2-1} \\ &+2m_1\left(RCW(G_2)-\frac{m_2}{D_2}\right)+2m_2\left(RCW(G_1)-\frac{m_1}{D_1}\right) \\ &+2m_2\left(RCW(G_1)-\frac{m_1}{D_1}\right) \end{aligned}$$

$$
\begin{aligned}
=&\frac{n_1m_2+n_2m_1}{D_1+D_2}+\frac{2m_1m_2}{D_1+D_2-1}+(n_1+2m_1)\Big(RCW(G_2)-\frac{m_2}{D_2}\Big)\\
&+(n_2+4m_2)\Big(RCW(G_1)-\frac{m_1}{D_1}\Big). \qquad \square
\end{aligned}
$$

Corollary 2.1. *Let $G_1 \cong K_{n_1}$ and $G_2 \ncong K_{n_2}$ be two graphs. Then*

$$
\begin{aligned}
RCW(G_1\square G_2) =&\frac{n_1m_2+n_2\binom{n_1}{2}}{1+D_2}+\frac{2\binom{n_1}{2}m_2}{D_2}+2\binom{n_1}{2}\Big(RCW(G_2)-\frac{m_2}{D_2}\Big)\\
&+\sum_{yv\notin E_2}\frac{n_1}{D_2+2-d_{G_2}(y,v)}\\
<&\frac{n_1m_2+n_2\binom{n_1}{2}}{1+D_2}+\frac{2\binom{n_1}{2}m_2}{D_2}+\Big(n_1+2\binom{n_1}{2}\Big)\Big(RCW(G_2)-\frac{m_2}{D_2}\Big).
\end{aligned}
$$

Example 2.3. Consider the graph whose vertices are the n-tuples $b_1, b_2, \ldots, b_n$ with $b_i \in \{0,1\}$, let two vertices be adjacent if the corresponding tuples differ in precisely one place. Such a graph is called a hypercube of dimension n and denoted by Q_n. It is well-known fact that the hypercube Q_n can be written in the form $Q_n = \underbrace{K_2\square K_2\square\cdots\square K_2}_{n\,\text{times}}$.

For $n = 3$, by Corollary 2.1 we have

$$RCW(Q_3) = K_2\square K_2\square K_2 = K_2\square C_4 = 14.$$

Corollary 2.2. *Let $L_{2,n}$ be the ladder graph, then*

$$RCW(L_{2,n}) = 4n-1-2\sum_{k=1}^{n}\frac{1}{k}.$$

Proof.

$$
\begin{aligned}
RCW(L_{2,n}) =&RCW(P_2\square P_n)\\
=&\frac{2(n-1)+n}{n}+\frac{2(n-1)}{n-1}+2\big(n-2)+2\sum_{yv\notin E(P_n)}\frac{1}{n+1-d_{P_n}(y,v)}\\
=&-\frac{2}{n}+2n+1+2\sum_{k=2}^{n-1}\frac{D(P_n,k)}{n+1-k}.
\end{aligned}
$$

On the other hand, it is clear that $D(P_n, k) = n-k$, for $k = 1, \ldots, n-1$. Therefore,

$$
\begin{aligned}
RCW(L_{2,n}) =& -\frac{2}{n}+2n+1+2\sum_{k=2}^{n-1}\frac{n-k}{n+1-k}\\
=& 4n-1-2\sum_{k=1}^{n}\frac{1}{k}. \qquad \square
\end{aligned}
$$

Corollary 2.3. *Let $G_1 \cong K_{n_1}$ and $G_2 \cong K_{n_2}$ be two complete graphs. Then*

$$RCW(G_1\square G_2) = \frac{n_1n_2}{4}(2n_1n_2-n_1-n_2).$$

2.3. **Composition.** The *composition* $G_1[G_2]$ (also known as the graph *lexicographic product*) of simple undirected graphs G_1 and G_2 is the graph with the vertex set $V(G_1[G_2]) = V_1 \times V_2$ and any two vertices (x, y) and (u, v) are adjacent if and only if $xu \in E_1$ or ($x = u$ and $yv \in E_2$).

Let G_1 and G_2 be graphs on $n_1 > 1$ and n_2 vertices, respectively. It follows from the definition that the distance between two distinct vertices (x, y) and (u, v) of $G_1[G_2]$ is given by

$$d_{G_1[G_2]}\big((x,y),(u,v)\big) = \begin{cases} 0, & x = u \text{ and } y = v, \\ 1, & x = u \text{ and } yv \in E_2, \\ 2, & x = u \text{ and } yv \notin E_2, \\ d_{G_1}(x,u), & x \neq u. \end{cases}$$

Note that if $G_1 \cong K_1$ then $G_1[G_2] \cong G_2$. So, in the following lemma we study the reciprocal complementary Wiener number of composition $G_1[G_2]$ for case $n_1 > 1$.

Theorem 2.3. *Let G_1 and G_2 be two graphs on $n_1 > 1$ and n_2 vertices, respectively.*

(i) *If G_1 is a non-complete graph, then*

$$RCW(G_1[G_2]) = \frac{n_1 m_2}{D_1} + \frac{n_1\left(\binom{n_2}{2} - m_2\right)}{D_1 - 1} + n_2^2 RCW(G_1).$$

(ii) *If $G_1 \cong K_{n_1}$ and G_2 is a non-complete graph, then*

$$RCW(G_1[G_2]) = \frac{n_1 m_2}{2} + n_1\left(\binom{n_2}{2} - m_2\right) + \frac{n_2^2}{2}\binom{n_1}{2}.$$

(iii) *If G_1 and G_2 are complete graphs, then*

$$RCW(G_1[G_2]) = (n_1 + n_2^2)\binom{n_1}{2}.$$

Proof. By the definition of the composition of two graphs one can see that,

$$D = D(G_1[G_2]) = \begin{cases} 1, & G_1 \cong K_{n_1} \text{ and } G_2 \cong K_{n_2}, \\ 2, & G_1 \cong K_{n_1} \text{ and } G_2 \ncong K_{n_2}, \\ D_1 = D(G_1), & G_1 \ncong K_{n_1}. \end{cases}$$

Suppose G_1 and G_2 are non-complete graphs, then

$$\begin{aligned} RCW(G_1[G_2]) &= \sum_{\{(x,y),(u,v)\} \subseteq V(G_1[G_2])} \frac{1}{D + 1 - d_{G_1[G_2]}\big((x,y),(u,v)\big)} \\ &= \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1[G_2]) \\ yv \in E_2}} \frac{1}{D_1} + \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1[G_2]) \\ yv \notin E_2}} \frac{1}{D_1 - 1} \\ &\quad + \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1[G_2]) \\ x \neq u}} \frac{1}{D_1 + 1 - d_{G_1}(x,u)} \end{aligned}$$

$$=\frac{n_1 m_2}{D_1}+\frac{n_1\left(\binom{n_2}{2}-m_2\right)}{D_1-1}+n_2^2 RCW(G_1),$$

which completes part (i).

The proof is completed by a similar argument as proof of the first part. □

2.4. **Disjunction.** The *disjunction* $G_1 \wedge G_2$ of graphs G_1 and G_2 is the graph with vertex set $V_1 \times V_2$ and (x,y) is adjacent with (u,v) whenever $xu \in E_1$ or $yv \in E_2$.

Let G_1 and G_2 be graphs on $n_1 > 1$ and $n_2 > 1$ vertices, respectively. Clearly, the distance between two vertices (x,y) and (u,v) of $G_1 \wedge G_2$ is given by

$$d_{G_1\wedge G_2}\Big((x,y),(u,v)\Big)=\begin{cases}0, & x=u \text{ and } y=v,\\ 1, & xu\in E_1 \text{ or } yv\in E_2,\\ 2, & \text{otherwise.}\end{cases}$$

Note that if $n_i = 1$ for some $i \in \{1,2\}$, then $G_1 \wedge G_2 \cong G_{n_i'}$, where $n_i' = 3-i$. So, we determine the reciprocal complementary Wiener number of disjunction $G_1 \wedge G_2$ for cases $n_1 > 1$ and $n_2 > 1$.

Theorem 2.4. *Let G_1 and G_2 be graphs on $n_1 > 1$ and $n_2 > 1$ vertices, respectively.*

(i) *If G_1 and G_2 are complete graphs, then*

$$RCW(G_1\wedge G_2)=\frac{1}{2}\left[n_1\binom{n_2}{2}+n_2\binom{n_1}{2}+\binom{n_1}{2}\binom{n_2}{2}\right].$$

(ii) *If $\{G_1, G_2\} \neq \{K_{n_1}, K_{n_2}\}$, then*

$$RCW(G_1\wedge G_2)=\frac{1}{2}\Big(n_1^2n_2^2+2m_1m_2-m_2n_1^2-m_1n_2^2-n_1n_2\Big).$$

Proof. From definition of disjunction it is clear that if at least one of graphs G_1 and G_2 is not complete, then $D = D(G_1 \wedge G_2) = 2$, otherwise $D = 1$. To prove part (ii), assume that $\{G_1, G_2\} \neq \{K_{n_1}, K_{n_2}\}$. Hence, we can write

$$\begin{aligned}RCW(G_1\wedge G_2)=&\sum_{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)}\frac{1}{D+1-d_{G_1\wedge G_2}\Big((x,y),(u,v)\Big)}\\
=&\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ xu\in E_1}}\frac{1}{2}+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ yv\in E_2}}\frac{1}{2}\\
&-\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ xu\in E_1, yv\in E_2}}\frac{1}{2}+\sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\wedge G_2)\\ yv\notin E_2}}1\\
&+\sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\wedge G_2)\\ xu\notin E_1}}1+\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\wedge G_2)\\ xu\notin E_1, yv\notin E_2}}1\\
=&\frac{1}{2}\Big(n_1^2n_2^2+2m_1m_2-m_2n_1^2-m_1n_2^2-n_1n_2\Big).\end{aligned}$$

A similar argument as part (ii), shows that

$$RCW(K_{n_1} \wedge K_{n_2}) = \frac{1}{2}\left[n_1\binom{n_2}{2} + n_2\binom{n_1}{2} + \binom{n_1}{2}\binom{n_2}{2}\right].$$

This completes the proof. $\square$

2.5. **Strong product.** The *strong product* of graphs G_1 and G_2, denoted by $G_1 \boxtimes G_2$, is the graph with vertex set $V_1 \times V_2$ and $(x, y)(u, v)$ is an edge whenever ($x = u$ and $yv \in E_2$), or ($y = v$ and $xu \in E_1$), or ($xu \in E_1$ and $yv \in E_2$).

In the following result, we give a basic property about the strong product of graphs.

Lemma 2.1 ([4])**.** *Let G_1 and G_2 be two connected graphs, $x, u \in V(G_1)$ and $y, v \in V(G_2)$. Then $d_{G_1 \boxtimes G_2}\big((x, y), (u, v)\big) = \max\{d_{G_1}(x, u), d_{G_2}(y, v)\}$.*

Corollary 2.4. *Let $G_1 \boxtimes G_2$ be the strong product of connected graphs G_1 and G_2. Then $D = \max\{D_1, D_2\}$, where D, D_1 and D_2 are the diameter of $G_1 \boxtimes G_2$, G_1 and G_2, respectively.*

Theorem 2.5. *Let $G_1 \boxtimes G_2$ be the strong product of connected graphs G_1 and G_2. Then*

$$\begin{aligned} RCW(G_1 \boxtimes G_2) \leq{}& \frac{1}{D}(2m_1m_2 + n_1m_2 + n_2m_1) + (n_1 + 2m_1)\left(RCW(G_2) - \frac{m_2}{D_2}\right) \\ &+ (n_2 + 2m_2)\left(RCW(G_1) - \frac{m_1}{D_1}\right) + 2\left[\binom{n_1}{2} - m_1\right]\left[\binom{n_2}{2} - m_2\right]. \end{aligned}$$

The equality is satisfied if and only if G_1 or G_2 is a complete graph.

Proof.

$$\begin{aligned} RCW(G_1 \boxtimes G_2) ={}& \sum_{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2)} \frac{1}{D + 1 - d_{G_1 \boxtimes G_2}\big((x, y), (u, v)\big)} \\ ={}& \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1 \boxtimes G_2) \\ yv \in E_2}} \frac{1}{D} + \sum_{\substack{\{(x,y),(x,v)\} \subseteq V(G_1 \boxtimes G_2) \\ yv \notin E_2}} \frac{1}{D + 1 - d_{G_2}(y, v)} \\ &+ \sum_{\substack{\{(x,y),(u,y)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \in E_1}} \frac{1}{D} + \sum_{\substack{\{(x,y),(u,y)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \notin E_1}} \frac{1}{D + 1 - d_{G_1}(x, u)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \in E_1, yv \in E_2}} \frac{1}{D} + \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \in E_1, yv \notin E_2}} \frac{1}{D + 1 - d_{G_2}(y, v)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \notin E_1, yv \in E_2}} \frac{1}{D + 1 - d_{G_1}(x, u)} \\ &+ \sum_{\substack{\{(x,y),(u,v)\} \subseteq V(G_1 \boxtimes G_2) \\ xu \notin E_1, yv \notin E_2}} \frac{1}{D + 1 - \max\{d_{G_1}(x, u), d_{G_2}(y, v)\}}. \end{aligned}$$

By (1.1), we have

$$
\begin{aligned}
n_1 RCW(G_2) &= \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D_2+1-d_{G_2}(y,v)} \\
&\quad + \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\in E_2}} \frac{1}{D_2+1-\underbrace{d_{G_2}(y,v)}_{1}} \\
&= \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D_2+1-d_{G_2}(y,v)} + \frac{n_1 m_2}{D_2},
\end{aligned}
$$

hence,

$$
\begin{aligned}
\sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D+1-d_{G_2}(y,v)} &\leq \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\notin E_2}} \frac{1}{D_2+1-d_{G_2}(y,v)} \\
&= n_1\left(RCW(G_2)-\frac{m_2}{D_2}\right).
\end{aligned}
$$

Similarly, we can check that

$$
\begin{aligned}
\sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1}} \frac{1}{D+1-d_{G_1}(x,u)} &\leq n_2\left(RCW(G_1)-\frac{m_1}{D_1}\right), \\
\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\in E_1, yv\notin E_2}} \frac{1}{D+1-d_{G_2}(y,v)} &\leq 2m_1\left(RCW(G_2)-\frac{m_2}{D_2}\right), \\
\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1, yv\in E_2}} \frac{1}{D+1-d_{G_1}(x,u)} &\leq 2m_2\left(RCW(G_1)-\frac{m_1}{D_1}\right).
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1, yv\notin E_2}} \frac{1}{D+1-\underbrace{\max\{d_{G_1}(x,u), d_{G_2}(y,v)\}}_{\leq D}} &\leq \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\notin E_1, yv\notin E_2}} 1 \\
&= 2\left[\binom{n_1}{2}-m_1\right]\left[\binom{n_2}{2}-m_2\right].
\end{aligned}
$$

On the other hand, it is easy to see that

$$
\sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\boxtimes G_2)\\ yv\in E_2}} \frac{1}{D} = \frac{n_1 m_2}{D}, \qquad \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\boxtimes G_2)\\ xu\in E_1}} \frac{1}{D} = \frac{n_2 m_1}{D},
$$

$$
\sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\boxtimes G_2)\\ xu\in E_1, yv\in E_2}} \frac{1}{D} = \frac{2m_1 m_2}{D}.
$$

Therefore,

$$\begin{aligned} RCW(G_1 \boxtimes G_2) \leq{} & \frac{1}{D}(2m_1m_2 + n_1m_2 + n_2m_1) + (n_1 + 2m_1)\left(RCW(G_2) - \frac{m_2}{D_2}\right) \\ & + (n_2 + 2m_2)\left(RCW(G_1) - \frac{m_1}{D_1}\right) + 2\left[\binom{n_1}{2} - m_1\right]\left[\binom{n_2}{2} - m_2\right]. \end{aligned}$$

□

Using similar arguments as in the proof of Theorem 2.5, one can prove the following result.

Lemma 2.2. *Let K_r be a complete graph on r vertices and G be a graph with n vertices, m edges and diameter d. Then*

$$RCW(G \boxtimes K_r) = \frac{1}{d}\left[rm + dr^2\left(RCW(G) - \frac{m}{d}\right) + (n + 2m)\binom{r}{2}\right].$$

Example 2.4. By the definition of the composition and strong product of two graphs one can see that, $G[K_n] = G \boxtimes K_n$. The open fence graph is the composition (or strong product) of path P_n and K_2. So, from Theorem 2.3 (i) (or Lemma 2.2), we have

$$RCW(P_n[K_2]) = RCW(P_n \boxtimes K_2) = \frac{n}{n-1} + 4n - 4, \quad n \geq 3.$$

As an application, in the following result, we obtain the reciprocal complementary Wiener number of the closed fence graph $C_n \boxtimes K_2$.

Lemma 2.3. *Let C_n be a cycle graph on n vertices. Then*

$$RCW(C_n \boxtimes K_2) = \begin{cases} 4n \displaystyle\sum_{k=1}^{\frac{n}{2}} \frac{1}{k} - 2n + 8, & 2 \mid n, \\ 4n \displaystyle\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{k} + \frac{2n}{n-1}, & 2 \nmid n. \end{cases}$$

Proof. We first obtain the reciprocal complementary Wiener number of a cycle graph C_n on n vertices. Regarding the structure of the cycle C_n, it can easily be concluded that if n is even then $D(C_n, k) = n$, $k = 1, 2, \ldots, \frac{n}{2} - 1$ and $D(C_n, \frac{n}{2}) = \frac{n}{2}$. On the other hand, if n is odd then $D(C_n, k) = n$, $k = 1, 2, \ldots, \frac{n-1}{2}$. Hence, by applying relation (1.3), we have

$$RCW(C_n) = \begin{cases} -\dfrac{n}{2} + n \displaystyle\sum_{k=1}^{\frac{n}{2}} \frac{1}{k}, & 2 \mid n, \\ n \displaystyle\sum_{k=1}^{\frac{n-1}{2}} \frac{1}{k}, & 2 \nmid n. \end{cases}$$

Finally, the proof is completed using Lemma 2.2. □

2.6. **Symmetric difference.** The *symmetric difference* $G_1 \oplus G_2$ of graphs G_1 and G_2 is the graph with vertex set $V_1 \times V_2$ and (x, y) is adjacent with (u, v) whenever $xu \in E_1$ or $yv \in E_2$ but not both. Note that if $n_i = 1$ for some $i \in \{1, 2\}$, then $G_1 \oplus G_2 \cong G_{n'_i}$, where $n'_i = 3 - i$.

In the following lemma, we compute the symmetric difference of two graphs with respect to their numbers of vertices and edges.

Theorem 2.6. *Let G_1 and G_2 be two graphs on $n_1 > 1$ and $n_2 > 1$ vertices, respectively. Then*

$$RCW(G_1 \oplus G_2) = \frac{1}{2}\Big(4m_1m_2 + n_1^2n_2^2 - m_1n_2^2 - m_2n_1^2 - n_1n_2\Big).$$

Proof. By [8, Lemma 4], we have

$$d_{G_1\oplus G_2}\Big((x,y),(u,v)\Big) = \begin{cases} 0, & x = u \text{ and } y = v, \\ 1, & xu \in E_1 \text{ or } yv \in E_2, \text{ but not both}, \\ 2, & \text{otherwise}. \end{cases}$$

Hence, by applying these relations, we get $D = D(G_1 \oplus G_2) = 2$. So,

$$\begin{aligned} RCW(G_1 \oplus G_2) = & \sum_{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2)} \frac{1}{D + 1 - d_{G_1\oplus G_2}\Big((x,y),(u,v)\Big)} \\ = & \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\oplus G_2) \\ yv\in E_2}} \frac{1}{2} + \sum_{\substack{\{(x,y),(x,v)\}\subseteq V(G_1\oplus G_2) \\ yv\notin E_2}} 1 \\ & + \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\oplus G_2) \\ xu\in E_1}} \frac{1}{2} + \sum_{\substack{\{(x,y),(u,y)\}\subseteq V(G_1\oplus G_2) \\ xu\notin E_1}} 1 \\ & + \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2) \\ xu\in E_1, yv\notin E_2}} \frac{1}{2} + \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2) \\ xu\in E_1, yv\in E_2}} 1 \\ & + \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2) \\ xu\notin E_1, yv\in E_2}} \frac{1}{2} + \sum_{\substack{\{(x,y),(u,v)\}\subseteq V(G_1\oplus G_2) \\ xu\notin E_1, yv\notin E_2}} 1 \\ = & \frac{1}{2}\Big(4m_1m_2 + n_1^2n_2^2 - m_1n_2^2 - m_2n_1^2 - n_1n_2\Big). \end{aligned}$$

□

2.7. **Corona product.** Let $V_1 = \{u_1, u_2, \dots, u_{n_1}\}$ and $V_2 = \{v_1, v_2, \dots, v_{n_2}\}$ be the vertex sets of given graphs G_1 and G_2, respectively. The *corona product* of G_1 and G_2 is denoted by $G_1 \circ G_2$ and defined as the graph obtained by taking n_1 copies of G_2 and joining each vertex of the i^{th} copy with vertex u_i of V_1, $i = 1, 2, \dots, n_1$. Denote by G_2^i the i^{th} copy of G_2 joined to the vertex u_i of G_1, and let $V_2^i = \{v_{i1}, v_{i2}, \dots, v_{in_2}\}$, $i = 1, 2, \dots, n_1$.

Theorem 2.7. *Let G_1 and G_2 be two graphs on $n_1 > 1$ and $n_2 \geq 1$ vertices, respectively. Then*

$$RCW(G_1 \circ G_2) < (n_2+1)^2 RCW(G_1) + \frac{n_1 n_2}{D_1+2} + \frac{n_1\left[\binom{n_2}{2} - m_2\right]}{D_1+1}.$$

Proof. From definition of the corona product of graphs, it is easy to check that

$$\begin{aligned}
d_{G_1\circ G_2}(u_i, u_p) &= d_{G_1}(u_i, u_p),\\
d_{G_1\circ G_2}(u_i, v_{pq}) &= d_{G_1}(u_i, u_p) + 1,\\
d_{G_1\circ G_2}(v_{ij}, v_{pq}) &= \begin{cases} 0, & i = p \text{ and } j = q,\\ 1, & i = p \text{ and } v_j v_q \in E_2,\\ 2, & i = p \text{ and } v_j v_q \notin E_2,\\ d_{G_1}(u_i, u_p) + 2, & i \neq p. \end{cases}
\end{aligned}$$

So, we can see that $D = D(G_1 \circ G_2) = D_1 + 2$. Hence,

$$\begin{aligned}
RCW(G_1 \circ G_2) &= \sum_{\{x,y\}\subseteq V(G_1\circ G_2)} \frac{1}{D+1-d_{G_1\circ G_2}(x,y)}\\
&= \sum_{\{x,y\}\subseteq V_1} \frac{1}{D+1-d_{G_1\circ G_2}(x,y)}\\
&\quad + \sum_{i=1}^{n_1} \sum_{\{v_{ij},v_{iq}\}\subseteq V_2^i} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{iq})}\\
&\quad + \sum_{i=1}^{n_1}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(u_i,v_{pq})}\\
&\quad + \sum_{i=1}^{n_1-1}\sum_{p=i+1}^{n_1}\sum_{j=1}^{n_2}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{pq})}.
\end{aligned}$$

Consider now for convenience:

$$\begin{aligned}
S_1 &= \sum_{\{x,y\}\subseteq V_1} \frac{1}{D+1-d_{G_1\circ G_2}(x,y)},\\
S_2 &= \sum_{i=1}^{n_1} \sum_{\{v_{ij},v_{iq}\}\subseteq V_2^i} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{iq})},\\
S_3 &= \sum_{i=1}^{n_1}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(u_i,v_{pq})},\\
S_4 &= \sum_{i=1}^{n_1-1}\sum_{p=i+1}^{n_1}\sum_{j=1}^{n_2}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{pq})}.
\end{aligned}$$

So, we have

$$\begin{aligned}
S_1 &= \sum_{\{x,y\}\subseteq V_1} \frac{1}{D+1-d_{G_1\circ G_2}(x,y)}\\
&= \sum_{\{x,y\}\subseteq V_1} \frac{1}{D_1+3-d_{G_1}(x,y)}\\
&< RCW(G_1),\\
S_2 &= \sum_{i=1}^{n_1} \sum_{\{v_{ij},v_{iq}\}\subseteq V_2^i} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{iq})}\\
&= \sum_{i=1}^{n_1} \sum_{\substack{\{v_{ij},v_{iq}\}\subseteq V_2^i\\ v_jv_q\in E_2}} \frac{1}{D_1+2} + \sum_{i=1}^{n_1} \sum_{\substack{\{v_{ij},v_{iq}\}\subseteq V_2^i\\ v_jv_q\notin E_2}} \frac{1}{D_1+1}\\
&= \frac{n_1m_2}{D_1+2} + \frac{n_1\left[\binom{n_2}{2}-m_2\right]}{D_1+1},\\
S_3 &= \sum_{i=1}^{n_1}\sum_{p=1}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(u_i,v_{pq})}\\
&= \sum_{i=1}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D_1+2} + \sum_{i=1}^{n_1}\sum_{\substack{p=1\\p\neq i}}^{n_1}\sum_{q=1}^{n_2} \frac{1}{D_1+2-d_{G_1}(u_i,u_p)}\\
&< \frac{n_1n_2}{D_1+2} + 2n_2RCW(G_1),\\
S_4 &= \sum_{i=1}^{n_1-1}\sum_{p=i+1}^{n_1}\sum_{j=1}^{n_2}\sum_{q=1}^{n_2} \frac{1}{D+1-d_{G_1\circ G_2}(v_{ij},v_{pq})}\\
&= \sum_{i=1}^{n_1-1}\sum_{p=i+1}^{n_1}\sum_{j=1}^{n_2}\sum_{q=1}^{n_2} \frac{1}{D_1+1-d_{G_1}(u_i,u_p)}\\
&= n_2^2RCW(G_1).
\end{aligned}$$

Therefore,

$$RCW(G_1\circ G_2) < (n_2+1)^2RCW(G_1) + \frac{n_1n_2}{D_1+2} + \frac{n_1\left[\binom{n_2}{2}-m_2\right]}{D_1+1}. \qquad \square$$

2.8. **Splice and link.** Let G_1 and G_2 be two connected graphs with disjoint vertex sets V_1 and V_2, respectively. For given vertices $u\in V_1$ and $v\in V_2$, a *splice* of G_1 and G_2 by vertices u and v is denoted by $(G_1.G_2)(u,v)$ and defined by identifying the vertices u and v in the union of G_1 and G_2. Also, a *link* of G_1 and G_2 by vertices u and v is denoted by $(G_1\sim G_2)(u,v)$ and obtained by joining u and v by an edge in the union of these graphs.

Theorem 2.8. *Let G_1 and G_2 be two graphs on n_1 and n_2 vertices, respectively. Then*

(i) $RCW\Big((G_1.G_2)(u,v)\Big) \leq (n_1-1)(n_2-1) + RCW(G_1) + RCW(G_2)$;
(ii) $RCW\Big((G_1 \sim G_2)(u,v)\Big) \leq n_1n_2 + RCW(G_1) + RCW(G_2)$.

Equality in (i) *holds if and only if one of the following cases occurs:*

(i_1) $n_i = 1$, *for some* $i \in \{1,2\}$;
(i_2) G_1 *and* G_2 *are non-complete graphs and* $\varepsilon_{G_1}(u) = \varepsilon_{G_2}(v) = 1$.

Moreover, equality in (ii) *holds if and only if* $n_1 = n_2 = 1$.

Proof. Suppose $\dot{D}$ and $\tilde{D}$ are the diameter of the splice and link of graphs G_1 and G_2 by vertices u and v, respectively. By above definitions of the splice and link of graphs, one can easily see that

$$d_{(G_1.G_2)(u,v)}(x,y) = \begin{cases} d_{G_1}(x,y), & x,y \in V_1, \\ d_{G_2}(x,y), & x,y \in V_2, \\ d_{G_1}(x,u) + d_{G_2}(y,v), & x \in V_1 \text{ and } y \in V_2, \end{cases}$$

and also,

$$d_{(G_1 \sim G_2)(u,v)}(x,y) = \begin{cases} d_{G_1}(x,y), & x,y \in V_1, \\ d_{G_2}(x,y), & x,y \in V_2, \\ d_{G_1}(x,u) + d_{G_2}(y,v) + 1, & x \in V_1 \text{ and } y \in V_2. \end{cases}$$

Hence, in graph $(G_1.G_2)(u,v)$, if the endpoints of a diametral path (i.e. a shortest path between two vertices whose distance is equal to the diameter of the graph) are in the graph G_1 (or G_2) then $\dot{D}$=D_1 (or D_2), otherwise if one of these endpoints belongs to V_1 and the other endpoint belongs to V_2, then $\dot{D}$=$\varepsilon_{G_1}(u) + \varepsilon_{G_2}(v)$. Thus, $\dot{D}$=max$\{D_1, D_2, \varepsilon_{G_1}(u) + \varepsilon_{G_2}(v)\}$. Similarly, $\tilde{D}$=max$\{D_1, D_2, \varepsilon_{G_1}(u) + \varepsilon_{G_2}(v) + 1\}$. By applying the above obtained relationships and also definitions of the splice and link of graphs, it is obvious that if $n_1 = 1$ or $n_2 = 1$, then the equality in (i) holds. Assume that $n_1, n_2 \geq 2$, then

$$\begin{aligned} RCW\Big((G_1.G_2)(u,v)\Big) &= \sum_{\{x,y\} \subseteq V\left((G_1.G_2)(u,v)\right)} \frac{1}{\dot{D} + 1 - d_{(G_1.G_2)(u,v)}(x,y)} \\ &= \sum_{\{x,y\} \subseteq V_1} \frac{1}{\dot{D} + 1 - d_{G_1}(x,y)} + \sum_{\{x,y\} \subseteq V_2} \frac{1}{\dot{D} + 1 - d_{G_2}(x,y)} \\ &\quad + \sum_{\substack{x \in V_1 \setminus \{u\} \\ y \in V_2 \setminus \{v\}}} \frac{1}{\dot{D} + 1 - d_{G_1}(x,u) - d_{G_2}(y,v)} \\ &\leq (n_1-1)(n_2-1) + RCW(G_1) + RCW(G_2), \end{aligned}$$

and equality holds when $\dot{D} = D_1 = D_2 = d_{G_1}(x,u) + d_{G_2}(y,v)$, for all $x \in V_1 \setminus \{u\}$ and $y \in V_2 \setminus \{v\}$. On the other hand, since G_1 and G_2 are connected graphs, we conclude that equality holds if and only if $d_{G_1}(x,u) = d_{G_2}(y,v) = 1$ and $D_1 = D_2 = 2$, for all $x \in V_1 \setminus \{u\}$ and $y \in V_2 \setminus \{v\}$. This means that G_1 and G_2 are non-complete graphs and $\varepsilon_{G_1}(u) = \varepsilon_{G_2}(v) = 1$, which completes the proof of part (i). The proof of part (ii) can be completed by using the similar arguments as in the proof of part (i). $\square$

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EXTENDED CONVERGENCE OF A TWO-STEP-SECANT-TYPE METHOD UNDER A RESTRICTED CONVERGENCE DOMAIN

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Abstract. We present a local as well as a semi-local convergence analysis of a two-step secant-type method for solving nonlinear equations involving Banach space valued operators. By using weakened Lipschitz and center Lipschitz conditions in combination with a more precise domain containing the iterates, we obtain tighter Lipschitz constants than in earlier studies. This technique lead to an extended convergence domain, more precise information on the location of the solution and tighter error bounds on the distances involved. These advantages are obtained under the same computational effort, since the new constants are special cases of the old ones used in earlier studies. The new technique can be used on other iterative methods. The numerical examples further illustrate the theoretical results.

1. Introduction

Let $F : D \subseteq \mathcal{B}_1 \to \mathcal{B}_2$ be a Fréchet-differentiable operator, $\mathcal{B}_1$ and $\mathcal{B}_2$ be Banach spaces and D be a nonempty convex subset of $\mathcal{B}_1$. One of the most important problems in mathematics and computational sciences is finding a locally unique solution x^* of the equation

$$F(x) = 0. \tag{1.1}$$

Many problems in the aforementioned disciplines can be written in a form like (1.1) using mathematical modeling. The solution x^* is sought in closed form but this can be achieved only in special cases. This is the reason why most solution methods for equation (1.1) are iterative. The most popular methods for generating a sequence

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approximating x^* are one-step Newton or Secant-type or two step Newton or Secant-type methods [1–18].

The study of convergence of iterative algorithms is usually centered into two categories: semi-local and local convergence analysis. The semi-local convergence is based on the information around an initial point, to obtain conditions ensuring the convergence of these algorithms, while the local convergence is based on the information around a solution to find estimates of the computed radii of the convergence balls. Local results are important since they provide the degree of difficulty in choosing initial points.

In the present paper we study the local as well as the semi-local convergence of two-step secant-type method defined for each $n = 0, 1, 2, \ldots$, $A_n = [x_n, y_n; F]$ by

$$\begin{aligned} x_{n+1} &= x_n - A_n^{-1} F(x_n), \\ y_{n+1} &= x_{n+1} - A_n^{-1} F(x_{n+1}), \end{aligned} \tag{1.2}$$

where $x_0, y_0 \in D$ are initial points and $[\cdot, \cdot; F] : D^2 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is a divided difference of order one for F on D satisfying

$$[x, y; F](x - y) = F(x) - F(y), \quad \text{for each } x, y \in D \text{ with } x \neq y,$$

and

$$[x, x; F] = F'(x), \quad \text{for each } x \in D$$

(if F is Fréchet differentiable on D). Notice that in the case of the secant method

$$x_{n+1} = x_n - [x_{n-1}, x_n; F]^{-1} F(x_n)$$

or

$$x_{n+1} = x_n - [x_n, x_{n-1}; F]^{-1} F(x_n),$$

we presented in [13] a convergence analysis under center Lipschitz and weak Lipschitz conditions (see (a_4) and (a_5)) leading to the following advantages (A) over other approaches (using only Lipschitz conditions), (see (a_4) and (c_4)).

(a) Extended convergence domain.
(b) Tighter error bounds on the distances $\|x_{n+1} - x_n\|, \|x_n - x^*\|, \|y_n - x^*\|$.
(c) At least as precise information on the location of the solution.

Our semi-local convergence analysis also improves the corresponding one in [11], since in our article we use the center-Lipschitz condition to locate a subset D_0 of D containing the iterates. This way the Lipschitz constants are tighter than in [11], resulting to the advantages (a)-(c). It is worth noticing that these advantages are obtained under the same computational effort, since the new constants are tighter and special cases of the constants in [11]. Hence, we have extended the applicability of method (1.2). Moreover, we have provided the local convergence analysis of method (1.2) not given in [11].

Notice that extending the semi-local convergence domain is important, especially since the convergence domain of such methods is small in general. Tighter error bounds implies that fewer iterates must be computed to obtain prespecified error tolerance.

The local, semi-local convergence analysis for method (1.2) is given in Section 2, Section 3, respectively, whereas Section 4 contains the numerical examples.

2. Local Convergence

We shall define some scalar functions and parameters to be used in the local convergence analysis of method (1.2). Let $\ell_0, \ell, \ell_1, \ell_2, \ell_3$ and ℓ_4 be nonnegative parameters. Let $r_0 = \frac{1}{\ell_0+\ell}$ and $r_1 = \frac{1}{\ell_0+\ell+\ell_1+\ell_2}$. Define functions g_1, g_2, h_1 and h_2 on the interval $[0, r_0)$ by

$$\begin{aligned} g_1(t) =& \frac{(\ell_1+\ell_2)t}{1-(\ell_0+\ell)t}, \\ g_2(t) =& \frac{\ell_3(g_1(t)t+t)+\ell_4 t}{1-(\ell_0+\ell)t}, \\ h_1(t) =& g_1(t)-1 \end{aligned}$$

and

$$h_2(t) = g_2(t) - 1.$$

We have $h_1(r_1) = 0$ and for each $t \in [0, r_1),\ 0 \le g_1(t) < 1$. Moreover, $h_1(0) = -1$ and $h_2(t) \to +\infty$ as $t \to r_0^-$. Hence, function h_2 has zeros in the interval $(0, r_0)$. Denote by r_2 the smallest such zero. Define functions g_0 and h_0 on the interval $[0, r_0)$ by

$$g_0(t) = \ell_0 g_1(t) + \ell g_2(t)$$

and

$$h_0(t) = g_0(t) - 1.$$

We get that $h_0(t) = -1$ and $h_0(t) \to +\infty$ as $t \to r_0^-$. Denote by ρ the smallest zero of function h_0 on the interval $(0, r_0)$. Then, define functions g_3 and h_3 on the interval $[0, \rho)$ by

$$g_3(t) = \frac{\ell_1 g_1(t) + \ell_2 g_2(t)}{1-(\ell_0 g_1(t) + \ell g_2(t))}$$

and

$$h_3(t) = g_3(t) - 1.$$

We obtain that $h_3(0) = -1$ and $h_3(t) \to +\infty$ as $t \to \rho^-$. Denote by r_3 the smallest zero of function h_3. Define the radius of convergence r by

$$r = \min\{r_i : i = 1, 2, 3\}. \tag{2.1}$$

Then, we have that for each $t \in [0, r)$

$$0 \le g_i(t) < 1. \tag{2.2}$$

Let $B(x, \lambda) = \{y \in X : \|x - y\| < \lambda\}$ and $\bar{B}(x, \lambda)$ be the closure of $B(x, \lambda)$.

Definition 2.1. Set $D_0 = D \cap B(x^*, \frac{1}{\ell})$. The set $T^* = (F, x_0, y_0, x^*)$ belongs to the class $K^* = K^*(\ell_0, \ell, \ell_1, \ell_2, \ell_3, \ell_4)$, if

(a_1) $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a Fréchet differentiable operator and $[.,.;F] : D^2 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is a divided difference for F of order one on D^2;
(a_2) there exists $x^* \in D$ such that $F(x^*) = 0$ and $F'(x^*)^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$;
(a_3) there exist $\ell_0 \geq 0$, $\ell \geq 0$ with ℓ_0, ℓ not both zero such that for each $x, y \in D$

$$\|F'(x^*)^{-1}([x, y; F] - F'(x^*))\| \leq \ell_0\|x - x^*\| + \ell\|y - x^*\|;$$

(a_4) there exist $\ell_i \geq 0$, $i = 1, 2, 3, 4$, such that for each $x, y, z \in D_0$

$$\begin{aligned}\|F'(x^*)^{-1}([x, y; F] - [x, x^*; F])\| \leq& \ell_1\|x - x^*\| + \ell_2\|y - x^*\|,\\ \|F'(x^*)^{-1}([x, y; F] - [z, x^*; F])\| \leq& \ell_3\|x - x^*\| + \ell_4\|y - x^*\|;\end{aligned}$$

(a_5) $\bar{B}(x^*, r) \subseteq D$, where r is defined in (2.1).

The local convergence analysis of method (1.2) follows in the class K^*.

Theorem 2.1. *Suppose that $T^* \subseteq K^*$ holds. Then, sequence $\{x_n\}$ generated for $x_0, y_0 \in B(x^*, r) - \{x^*\}$ is well defined in $B(x^*, r)$, remains in $B(x^*, r)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold*

$$\|x_{n+1} - x^*\| \leq g_1(r)\|x_n - x^*\| \leq \|x_n - x^*\| < r, \tag{2.3}$$
$$\|y_{n+1} - x^*\| \leq g_2(r)\|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\| < r \tag{2.4}$$

and

$$\|x_{n+2} - x^*\| \leq g_3(r)\|x_{n+1} - x^*\| \leq \|x_{n+1} - x^*\|, \tag{2.5}$$

where the functions g_i, $i = 1, 2, 3$, are defined previously. Furthermore, the solution x^* of equation $F(x) = 0$ is unique in $D_1 = D \cap \bar{B}(x^*, R)$ for $R \in [r, \frac{1}{\ell_0 + \ell})$.

Proof. We shall use mathematical induction to show estimates (2.3)-(2.5). By hypothesis $x_0, y_0 \in B(x^*, r) - \{x^*\}$, (2.1), ($a_2$) and ($a_3$), we have in turn that

$$\|F'(x^*)^{-1}(A_0 - F'(x^*))\| \leq \ell_0\|x_0 - x^*\| + \ell\|y_0 - x^*\| \leq (\ell_0 + \ell)r < 1. \tag{2.6}$$

By (2.6) and the Banach lemma on invertible operators [1,4,5,10,15], we deduce that $A_0^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|A_0^{-1}F'(x^*)\| \leq \frac{1}{1 - (\ell_0\|x_0 - x^*\| + \ell\|y_0 - x^*\|)}.$$

Hence, x_1, y_1 are well defined by method (1.2) for $n = 0$. Then, using (a_2), (2.1), (2.2) and (a_4) we get in turn that

$$\begin{aligned}
\|x_1 - x^*\| =& \|x_0 - x^* - A_0^{-1}F(x_0)\| \\
\leq& \|A_0^{-1}F'(x^*)\| \|F'(x^*)^{-1}(A_0 - [x_0, x^*; F])\| \|x_0 - x^*\| \\
\leq& \frac{\ell_1\|x_0 - x^*\| + \ell_2\|y_0 - x^*\|}{1 - (\ell_0\|x_0 - x^*\| + \ell_1\|y_0 - x^*\|)}\|x_0 - x^*\| \\
\leq& g_1(r)\|x_0 - x^*\| \leq \|x_0 - x^*\| < r, \qquad (2.7)\\
\|y_1 - x^*\| \leq& \|A_0^{-1}F'(x^*)\| \|F'(x^*)^{-1}(A_0 - [x_1, x^*; F])\| \|x_1 - x^*\| \\
\leq& \frac{\ell_3\|x_1 - x_0\| + \ell_4\|y_0 - x^*\|}{1 - (\ell_0\|x_0 - x^*\| + \ell_1\|y_0 - x^*\|)}\|x_0 - x^*\| \\
\leq& \frac{\ell_3(\|x_1 - x^*\| + \|x_0 - x^*\|) + \ell_4\|y_0 - x^*\|}{1 - (\ell_0\|x_0 - x^*\| + \ell_1\|y_0 - x^*\|)}\|x_0 - x^*\| \\
\leq& g_2(r)\|x_1 - x^*\| \leq \|x_1 - x^*\| < r, \qquad (2.8)
\end{aligned}$$

and similarly to (2.7)

$$\begin{aligned}
\|x_2 - x^*\| \leq& \frac{\ell_1\|x_1 - x^*\| + \ell_2\|y_0 - x^*\|}{1 - (\ell_0\|x_1 - x^*\| + \ell\|y_1 - x^*\|)}\|x_1 - x^*\| \qquad (2.9)\\
\leq& g_3(r)\|x_1 - x^*\| \leq \|x_1 - x^*\|.
\end{aligned}$$

That is estimates (2.7)–(2.9) show (2.3)–(2.5), respectively for $k = 0$. By simply replacing $x_0, y_0, x_1, y_1, x_2, y_2$ by $x_k, y_k, x_{k+1}, y_{k+1}, x_{k+2}, y_{k+2}$ in the preceding estimates, we complete the induction for (2.3)–(2.5). Then, it follows from the estimate

$$\|x_{k+2} - x^*\| \leq c\|x_{k+1} - x^*\| < r,$$

where $c = g_3(r) \in [0, 1)$ that $\lim_{k\to\infty} x_k = x^*$. Finally, to show the uniqueness part, let $y^* \in D_0$ with $F(y^*) = 0$. Set $E = [x^*, y^*; F]$. Then, by (a_3), we get

$$\|F'(x^*)^{-1}(E - F'(x^*))\| \leq \ell\|y^* - x^*\| \leq \ell R < 1,$$

so $E^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$. Using the identity

$$0 = F(x^*) - F(y^*) = [x^*, y^*; F](x^* - y^*),$$

we conclude that $x^* = y^*$. $\square$

Let $\rho = \min\left\{\frac{1}{\ell_0+\ell+\ell_1+\ell_2}, \frac{1}{\ell_0+\ell+2\ell_3+\ell_4}\right\}$. Define parameters $a_1 = \frac{\ell_1}{1-(\ell_0+\ell)\rho}$, $a_2 = \frac{\ell_2}{1-(\ell_0+\ell)\rho}$, $a_3 = a_4 = \frac{\ell_3}{1-(\ell_0+\ell)\rho}$ and $a_5 = \frac{\ell_4}{1-(\ell_0+\ell)\rho}$. Then, for $x_0, y_0 \in B(x^*, \rho)$, we have by the proof of Theorem 2.1, that

$$\|x_{n+1} - x^*\| \leq (a_1\|x_n - x^*\| + a_2\|y_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho,$$

$$
\begin{aligned}
\|y_{n+1}-x^*\| \leq & (a_3\|x_{n+1}-x^*\|+a_4\|x_n-x^*\|+a_5\|y_n-x^*\|)\|x_{n+1}-x^*\| \\
\leq & \|x_{n+1}-x^*\| < \rho, \\
\|x_{n+2}-x^*\| \leq & (a_1\|x_{n+1}-x^*\|+a_2\|y_{n+1}-x^*\|)\|x_{n+1}-x^*\| \\
\leq & \|x_{n+1}-x^*\|, \\
\|y_{n+2}-x^*\| \leq & (a_3\|x_{n+1}-x^*\|+a_4\|x_{n+1}-x^*\|+a_5\|y_{n+1}-x^*\|)\|x_{n+2}-x^*\| \\
\leq & \|x_{n+2}-x^*\|
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_{n+3}-x^*\| \leq & (a_1\|x_{n+2}-x^*\|+a_2\|y_{n+2}-x^*\|)\|x_{n+2}-x^*\| \\
\leq & (a_1+a_2)\|x_{n+2}-x^*\|^2.
\end{aligned}
$$

Hence, we arrive at following proposition.

Proposition 2.1. *Let $T^* \subset K^*$ with r replaced by ρ. Then, sequence $\{x_n\}$ converges quadratically to x^* provided that $x_0, y_0 \in B(x^*, \rho) - \{x^*\}$. Moreover, the solution x^* of equation $F(x) = 0$ is unique in D_1 for $R \in \left[\rho, \frac{1}{\ell_0+\ell}\right)$.*

3. Semi-Local Convergence Analysis

Let $L_0, L, L_1, L_2 > 0$, $\eta \geq 0$ and $\eta_0 \geq 0$ be given parameters. As in Section 2, we define a set.

Definition 3.1. Set $D_0 = D \cap B(x^*, \frac{1}{L_0+L})$. The set $T = T(F, x_0, y_0)$ belongs to class $K = K(L_0, L, L_1, L_2, \eta_0, \eta)$, if

- (c_1) $F : D \subset \mathcal{B}_1 \to \mathcal{B}_2$ is a Fréchet differentiable operator and $[\cdot,\cdot;F] : D^2 \to \mathcal{L}(\mathcal{B}_1, \mathcal{B}_2)$ is a divided difference for F of order one on D^2;
- (c_2) there exists $x_0, y_0 \in D$ and $\eta \geq 0$, $\eta \geq 0$ such that $A_0^{-1} \in \mathcal{L}(\mathcal{B}_2, \mathcal{B}_1)$, $\|x_0 - y_0\| \leq \eta_0$ and $\|A_0^{-1}F(x_0)\| \leq \eta$;
- (c_3) there exist $L_0 \geq 0$, $L \geq 0$ such that for each $x, y \in D$
$$\|A_0^{-1}([x,y;F]-A_0)\| \leq L_0\|x-x_0\|+L\|y-y_0\|;$$
- (c_4) there exist $L_i \geq 0$, $i = 1, 2$, such that for each $x, y, z \in D_0$
$$\|A_0^{-1}([x,y;F]-[y,z;F])\| \leq L_1\|x-y\|+L_2\|y-z\|;$$
- (c_5) $\bar{B}(x^*, t^*) \subseteq D$, where t^* is given in Lemma 3.1 that follows.

We need to define majorizing sequence $\{t_n\}, \{u_n\}$ by

$$
\begin{aligned}
t_0 =& 0, \quad u_0 = \eta_0, \quad t_1 = \eta, \quad u_1 = L_1(1+L_0t_1+Lu_0), \\
t_2 =& t_1\left(1+\frac{L_0t_1+Lu_0}{1-(L_0t_1+L(u_1+u_0))}\right), \\
u_{n+1} =& t_{n+1}+\frac{L_1(t_{n+1}-t_n)+L_2(u_n-t_n)}{1-(L_0t_n+L(u_n+u_0))}(t_{n+1}-t_n)
\end{aligned}
$$

and

$$t_{n+2} = t_{n+1} + \frac{L_1(t_{n+1} - t_n) + L_2(u_n - t_n)}{1 - (L_0 t_{n+1} + L(u_{n+1} + u_0))}(t_{n+1} - t_n).$$

We also need the convergence result for the aforementioned majorizing sequences.

Lemma 3.1. ([12, Lemma 1, Page 734]). *Let $\alpha \in (0,1)$ be the unique solution of equation $q(t) = 0$, where*

$$q(t) = Lt^3 + L_0 t^2 + (L_1 + L_2)t - (L_1 + L_2).$$

Suppose that

$$0 < \frac{L_0(t_1 - t_0) + Lu_0}{1 - (L_0(t_1 - t_0) + L(u_1 + u_0))} \leq \alpha < 1 - \frac{(L_0 + L)t_1}{1 - Lu_0}.$$

Then, sequences $\{t_n\}, \{u_n\}$ are non-decreasing, bounded from above by $t^{**} = \frac{t_1}{1-\alpha}$ and converge to their unique least upper bound t^* such that $t^* \in [t_1, t^{**}]$. Moreover, for each $n = 1, 2, \ldots$ the following estimates hold:

$$\begin{aligned} 0 \leq & u_{n+1} - t_{n+1} \leq \alpha(t_{n+1} - t_n), \\ 0 \leq & t_{n+2} - t_{n+1} \leq \alpha(t_{n+1} - t_n) \end{aligned}$$

and

$$t_n \leq u_n.$$

Based on Definition 3.1 and Lemma 3.1, we obtain the following semi-local convergence result for method (1.2).

Theorem 3.1. *Suppose $T \subseteq K$ and conditions of Lemma 3.1 hold. Then, sequences $\{x_n\}$ and $\{y_n\}$ generated by method (1.2), starting at $x_0, y_0 \in D$ are well defined in $B(x_0, t^*)$, remain in $B(x_0, t^*)$ for each $n = 0, 1, 2, \ldots$ and converges to the unique solution x^* of equation $F(x) = 0$ in $D_1 = D \cap \bar{B}\left(t^*, \frac{1}{L_0+L}\right)$.*

Proof. It follows from the corresponding proof in [12, Theorem 1, Page 735] but see also the remark that follows. □

Remark 3.1. The semi-local convergence of method (1.2) was also established in [12] but there is a major difference effecting the convergence domain, error bounds on the distances $\|x_{n+1} - x_n\|, \|y_n - x_n\|$ and the uniqueness domain. Indeed, the condition used in [12] instead of (c_4) is

(c_4') $\|A_0^{-1}([x, y; F] - [u, v; F])\| \leq M_1\|x - u\| + M_2\|y - v\|$ for each $x, y, u, v \in D$ and some $M_1 \geq 0$ and $M_2 \geq 0$.

But (c_4) is weaker than $(\bar{c}_4)$ even, if $D_0 = D$. Therefore, $L_1 \leq M_1$ and $L_2 \leq M_2$, hold in general (see [1, 4, 5]). The iterates remain in D_0 which is a more accurate location than D, since $D_0 \subseteq D$ leading to tighter Lipschitz constants and the advantages (A). Define sequences $\{\bar{t}_n\}, \{\bar{u}_n\}$ as $\{t_n\}, \{u_n\}$, respectively but with M_1 replacing L_1 and

M_2 replacing L_2. Then, assuming that the rest of the hypotheses of Theorem 3.1 hold with these changes, a simple inductive argument shows that

$$\begin{aligned}0 \leq & u_{n+1} - t_{n+1} \leq \bar{u}_{n+1} - \bar{t}_{n+1} \leq \bar{\alpha}(\bar{t}_{n+1} - \bar{t}_n),\\ 0 \leq & t_{n+2} - t_{n+1} \leq \bar{t}_{n+2} - \bar{t}_{n+1} \leq \bar{\alpha}(\bar{t}_{n+1} - \bar{t}_n),\\ t_n \leq & \bar{t}_n,\\ u_n \leq & \bar{u}_n\end{aligned}$$

and

$$t^* \leq \bar{t}^* = \lim_{n\to\infty} \bar{t}_n,$$

where $\bar{\alpha} \in (0,1)$ is the unique solution of equation $\bar{q}(t) = 0$, with

$$\bar{q}(t) = Lt^3 + L_0t^2 + (M_1 + M_2)t - (M_1 + M_2).$$

Notice that

$$\begin{aligned}\bar{q}(\alpha) =& L\alpha^3 + L_0\alpha^2 + (M_1 + M_2)\alpha - (M_1 + M_2)\\ =& q(\alpha) + [(M_1 - L_1) + (M_2 - L_2)](\alpha - 1) < 0,\end{aligned}$$

since $q(\alpha) = 0$, $\alpha \in (0,1)$, $L_1 \leq M_1$ and $L_2 \leq M_2$. Therefore, we have $\alpha \leq \bar{\alpha}$. Hence, we justified the claim made in the introduction (see also the numerical examples).

4. Numerical Examples

We present the following examples to test the convergence criteria. Define the divided difference by

$$[x, y; F] = \int_0^1 F'(\tau x + (1-\tau)y)d\tau.$$

Example 4.1. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$ be the space of continuous functions defined in $[0,1]$ equipped with the max norm. Let $D = \{z \in C[0,1] : \|z\| \leq 1\}$. Define F on D by [1, 13]:

$$F(x)(s) = x(s) - f(s) - \frac{1}{8}\int_0^1 G(s,t)x(t)^3\, dt, \quad x \in C[0,1],\ s \in [0,1],$$

where $f \in C[0,1]$ is a given function and the kernel G is the Green's function

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s,\\ s(1-t), & s \leq t.\end{cases}$$

Notice that nonlinear integral equation $F(x)(s) = 0$ is of Chandrasekhar type [1, 4, 5, 10]. Then $F'(x)$ is a linear operator given for each $x \in D$, by

$$[F'(x)(v)](s) = v(s) - \frac{3}{8}\int_0^1 G(s,t)x(t)^2 v(t)\, dt, \quad v \in C[0,1],\ s \in [0,1].$$

If we choose $x_0(s) = f(s) = s$, then we obtain $\|F'(x_0)\| \leq \frac{1}{64}$.

Choose $x_{-1} = 2s$, we see [13] that $L_1 = 0.08125\ldots$, $L_2 = 0.040625\ldots$, $L = 0.0359375\ldots$, $L_0 = 0.071875\ldots$, $t_1 = \eta = 0.0298507$ and $u_1 = \eta_1 = 1$. Notice that hypothesis $0 < \frac{L_0(t_1-t_0)+Lu_0}{1-(L_0(t_1-t_0)+L(u_1+u_0))} \leq \alpha < 1 - \frac{(L_0+L)t_1}{1-Lu_0}$ is satisfied. So, we can guarantee the convergence of the Secant method (1.2) from Theorem 2.1.

Example 4.2. Let $\mathbb{B}_1 = \mathbb{B}_2 = \mathbb{R}^3$, $D = B(0,1)$, $x^* = (0,0,0)^T$ and define F on D by

$$F(x) = F(x_1, x_2, x_3) = \left(e^{x_1} - 1, \frac{e-1}{2}{x_2}^2 + x_2, x_3\right)^T.$$

For the points $u = (u_1, u_2, u_3)^T$, the Fréchet derivative is given by

$$F'(u) = \begin{pmatrix} e^{u_1} & 0 & 0 \\ 0 & (e-1)u_2 + 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Using the norm of the maximum of the rows and (a_3)-(a_4) and since $F'(x^*) = \mathrm{diag}(1,1,1)$, we can define parameters for method (1.2) by $\ell_1 = 0$, $\ell_0 = \ell = \ell_2 = \frac{e-1}{2}$, $\ell_3 = \frac{e^{\frac{1}{e-1}}}{2}$, $\ell_4 = \frac{e-1}{2}$. Then, the radius of convergence using (2.1) is given by $r = 0.2607$. Local results were not given in [12] but if they were, $\bar{\ell}_0 = \bar{\ell} = \frac{e-1}{2}$, $\bar{\ell}_1 = 0$, $\bar{\ell}_2 = \frac{e}{2}$, then $\bar{\ell}_3 = \bar{\ell}_4 = \frac{e}{2}$. Therefore, by (2.1) with ℓ_4 replacing $\bar{\ell}_4$, we get $\bar{r} = 0.2340$.

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