SOME IDENTITIES IN RINGS AND NEAR-RINGS WITH DERIVATIONS

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Abstract. In the present paper we investigate commutativity in prime rings and 3-prime near-rings admitting a generalized derivation satisfying certain algebraic identities. Some well-known results characterizing commutativity of prime rings and 3-prime near-rings have been generalized.

1. Introduction

In this paper, $\mathcal{N}$ will denote a right near-ring with center $Z(\mathcal{N})$. A near-ring $\mathcal{N}$ is called zero-symmetric if $x0 = 0$ for all $x \in \mathcal{N}$ (recall that right distributivity yields $0x = 0$). A non empty subset $U$ of $\mathcal{N}$ is said to be a semigroup left (resp. right) ideal of $\mathcal{N}$ if $\mathcal{N}U \subseteq U$ (resp. $UN \subseteq U$) and if $U$ is both a semigroup left ideal and a semigroup right ideal, it is called a semigroup ideal of $\mathcal{N}$. As usual for all $x, y$ in $\mathcal{N}$, the symbol $[x, y]$ stands for Lie product (commutator) $xy - yx$ and $x \circ y$ stands for Jordan product (anticommutator) $xy + yx$. We note that for a near-ring, $-(x + y) = -y - x$. Recall that $\mathcal{N}$ is 3-prime if for $a, b$ in $\mathcal{N}$, $aNb = \{0\}$ implies that $a = 0$ or $b = 0$. $\mathcal{N}$ is said to be 2-torsion free if whenever $2x = 0$, with $x \in \mathcal{N}$, then $x = 0$. An additive mapping $d : \mathcal{N} \rightarrow \mathcal{N}$ is a derivation if $d(xy) = xd(y) + d(x)y$ for all $x, y \in \mathcal{N}$, or equivalently, as noted in [20], that $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{N}$. The concept of derivation in rings has been generalized in several ways by various authors. Generalized derivation has been introduced already in rings by M. Brešar [10]. Also the notions of generalized derivation has been introduced in near-rings by Öznur Gölbasi [14]. An additive mapping $\mathcal{F} : \mathcal{N} \rightarrow \mathcal{N}$ is called a right generalized derivation with associated derivation $d$ if $\mathcal{F}(xy) = \mathcal{F}(x)y + xd(y)$ for all $x, y \in \mathcal{N}$ and $\mathcal{F}$ is called a left generalized derivation with associated derivation $d$ if...
\( \mathcal{F}(xy) = d(x)y + x\mathcal{F}(y) \), for all \( x, y \in \mathbb{N} \). \( \mathcal{F} \) is called a generalized derivation with associated derivation \( d \) if it is both a left as well as a right generalized derivation with associated derivation \( d \). An additive mapping \( \mathcal{F} : \mathbb{N} \to \mathbb{N} \) is said to be a left (resp. right) multiplier (or centralizer) if \( \mathcal{F}(xy) = \mathcal{F}(x)y \) (resp. \( \mathcal{F}(xy) = x\mathcal{F}(y) \)) holds for all \( x, y \in \mathbb{N} \). \( \mathcal{F} \) is said to be a multiplier if it is both left as well as right multiplier. Notice that a right (resp. left) generalized derivation with associated derivation \( d = 0 \) is a left (resp. right) multiplier.

Lemma 2.1. Let \( K \) be a nonzero ideal of \( \mathbb{N} \). In 2002, Rehman [18] established that if a prime ring of a characteristic not 2 admits a generalized derivation \( F \) with a nonzero derivation such that \( F([x, y]) = [x, y] \) (resp. \( F([x, y]) = -[x, y] \)) for all \( x, y \) in a nonzero square closed Lie ideal \( U \), then \( U \subseteq Z(\mathbb{R}) \). Quadri, Khan and Rehman [16], without the characteristic assumption on the ring, proved that a prime ring must be commutative if it admits a generalized derivation \( F \) associated with a nonzero derivation, such that \( F([x, y]) = [x, y] \) (resp. \( F([x, y]) = -[x, y] \)) for all \( x, y \) in a nonzero ideal \( I \). Motivated by the above results, in the following theorem we explore the commutativity of a prime ring, provided with a generalized derivation \( F \) and left multiplier \( G \) satisfying the following conditions: \( F([x, y]_{\alpha, \beta}) = [x, y]_{u, v}, \quad F([x, y]_{\alpha, \beta}) = G([\beta(x), y]) \) for all \( x, y \in \mathbb{R} \), where \( \alpha, \beta, u, v \) automorphisms of \( \mathbb{R} \) and \( [x, y]_{\alpha, \beta} = \alpha(x)y - y\beta(x) \).

2. Some Preliminaries

For the proofs of our main theorems, we need the following lemmas. The first lemmas appear in [7] and [20] in the context of left near-rings, and it is easy to see that they hold for right near-rings as well.

**Lemma 2.1.** Let \( \mathbb{N} \) be a 3-prime near-ring and \( U \) be a nonzero semigroup ideal of \( \mathbb{N} \). Let \( d \) be a nonzero derivation on \( \mathbb{N} \).

(i) If \( x, y \in \mathbb{N} \) and \( xUy = \{0\} \), then \( x = 0 \) or \( y = 0 \).

(ii) If \( x \in \mathbb{N} \) and \( xU = \{0\} \) or \( Ux = \{0\} \), then \( x = 0 \).

(iii) If \( z \in Z(\mathbb{N}) \), then \( d(z) \in Z(\mathbb{N}) \).

**Lemma 2.2.** Let \( d \) be an arbitrary derivation of a near-ring \( \mathbb{N} \). Then \( \mathbb{N} \) satisfies the following partial distributive laws:

(i) \( z(xd(y) + d(x)y) = zxd(y) + zd(x)y \) for all \( x, y, z \in \mathbb{N} \);

(ii) \( z(d(x)y + xd(y)) = zd(x)y + zxd(y) \) for all \( x, y, z \in \mathbb{N} \).
**Lemma 2.3.** ([5, Theorem 2.1]). Let $N$ be a 3-prime near-ring, $U$ a nonzero semigroup left ideal or semigroup right ideal. If $N$ admits a nonzero derivation $d$ such that $d(U) \subseteq Z(N)$, then $N$ is a commutative ring.

3. SOME RESULTS INVOLVING PRIME RINGS

**Theorem 3.1.** Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $\alpha$, $\beta$, $u$, $v$ automorphisms of $R$ such that $\beta(I) = I$. If $F$ is a generalized derivation of $R$ associated with a derivation $d$ and $G$ is a left multiplier of $R$ which satisfy one of the following conditions:

(i) $F([x, y]_{\alpha, \beta}) = [x, y]_{u, v}$ for all $x, y \in I$;

(ii) $F([x, y]_{\alpha, \beta}) = G([\beta(x), y])$ for all $x, y \in I$;

then $R$ is commutative.

**Proof.** (i) Suppose that

\begin{equation}
F([x, y]_{\alpha, \beta}) = [x, y]_{u, v}, \quad \text{for all } x, y \in I.
\end{equation}

Replacing $y$ by $y\beta(x)$ in (3.1), and using the fact that $[x, y\beta(x)]_{\alpha, \beta} = [x, y]_{\alpha, \beta}\beta(x)$ and $[x, y\beta(x)]_{u, v} = [x, y]_{u, v}\beta(x) + y[v(x), \beta(x)]$ for all $x, y \in I$, we arrive at

\begin{equation}
F([x, y]_{\alpha, \beta})\beta(x) + [x, y]_{\alpha, \beta}d(\beta(x)) = [x, y]_{u, v}\beta(x) + y[v(x), \beta(x)], \quad \text{for all } x, y \in I.
\end{equation}

Using (3.1), (3.2) implies that

\begin{equation}
[x, y]_{\alpha, \beta}d(\beta(x)) = y[v(x), \beta(x)], \quad \text{for all } x, y \in I.
\end{equation}

Substituting $ry$ instead of $y$ in (3.3) where $r \in R$, we arrive at

\[ [\alpha(x), r]Id(\beta(x)) = \{0\}, \quad \text{for all } x \in I, r \in R. \]

By Lemma 2.1 (i), we get $[\alpha(x), r] = 0$ or $d(\beta(x)) = 0$ for all $x \in I, r \in R$ which gives $\alpha(x) \in Z(R)$ or $d(\beta(x)) = 0$ for all $x \in I$. Since $\alpha$ and $\beta$ are automorphisms of $R$, we get $x \in Z(R)$ or $d(\beta(x)) = 0$ for all $x \in I$. Using Lemma 2.1 (iii), we obtain $d(\beta(I)) \subseteq Z(R)$ i.e., $d(I) \subseteq Z(R)$ which forces that $R$ is commutative by Lemma 2.3.

(ii) Assume that

\begin{equation}
F([x, y]_{\alpha, \beta}) = G([\beta(x), y]), \quad \text{for all } x, y \in I.
\end{equation}

Putting $y\beta(x)$ instead of $y$ in (3.4), we get

\[ F([x, y]_{\alpha, \beta})\beta(x) + [x, y]_{\alpha, \beta}d(\beta(x)) = G([\beta(x), y])\beta(x), \quad \text{for all } x, y \in I. \]

Using (3.4), we obtain $[x, y]_{\alpha, \beta}d(\beta(x)) = 0$ for all $x, y \in I$, which implies that

\begin{equation}
\alpha(x)yd(\beta(x)) = y\beta(x)d(\beta(x)), \quad \text{for all } x, y \in I.
\end{equation}

Taking $ry$ in place of $y$ in (3.5) where $r \in R$ and using it again, we conclude that

\[ [\alpha(x), r]Id(\beta(x)) = \{0\}, \quad \text{for all } x \in I, r \in R. \]

By Lemma 2.1 (i), we get $\alpha(x) \in Z(R)$ or $d(\beta(x)) = 0$ for all $x \in R$ and using the same techniques as used above, we conclude that $R$ is commutative.

\[ \square \]
For $\alpha = \beta = u = v = \text{id}_R$, we get the following result.

**Corollary 3.1.** ([16, Theorem 2.1]). Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ such that $F([x,y]) = [x,y]$ for all $x, y \in I$, then $R$ is commutative.

For $\alpha = \beta = u = \text{id}_R$ and $v = -\text{id}_R$, we get the following result.

**Corollary 3.2.** ([16, Theorem 2.2]). Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a nonzero derivation $d$ such that $F([x,y] + [x,y]) = 0$ for all $x, y \in I$, then $R$ is commutative.

4. SOME RESULTS INVOLVING 3-PRIME NEAR-RINGS

In this section, we will present a very important result that generalizes several theorems that are well known in the literature. More precisely, we will show that a 2-torsion prime near-ring $N$ is a commutative ring if and only if $N$ admits a derivation $d$ and a left multiplier $G$ such that $G([x,y]) = [d(x),y] - [x,d(y)]$ for all $x, y \in U$.

**Theorem 4.1.** Let $N$ be a 2-torsion free prime near-ring and $U$ a nonzero semigroup ideal of $N$. If $N$ admits a derivation $d$ and left multiplier $G$, then the following assertions are equivalents:

1. $G([x,y]) = [d(x),y] - [x,d(y)]$ for all $x, y \in U$;
2. $N$ is a commutative ring.

**Proof.** It is easy to notice that (ii) implies (i).

(i)$\Rightarrow$(ii) Suppose that

$$G([x,y]) = [d(x),y] - [x,d(y)], \quad \text{for all } x, y \in U. \quad (4.1)$$

Replacing $x$ by $xy$ in (4.1) and using the fact that $[xy,y] = [x,y]y$, we obtain

$$[d(xy),y] - [xy,d(y)] = G([x,y])y, \quad \text{for all } x, y \in U.$$ 

Which implies that

$$[d(xy),y] - [xy,d(y)] = ([d(x),y] - [x,d(y)])y, \quad \text{for all } x, y \in U.$$ 

Using Lemma 2.2 and by developing the last expression, we arrive at

$$d(x)y^2 + xd(y)y - yxd(y) - yd(x)y + d(y)xy - xyd(y) = d(x)y^2 - yd(x)y + d(y)xy - xd(y)y.$$ 

For $x = y$, the equation (4.1) and 2-torsion freeness we give easily $d(y)y = yd(y)$ for all $y \in U$. In this case, by a simplification of last equation, we find that

$$xd(y)y = yxd(y), \quad \text{for all } x, y \in U. \quad (4.2)$$

Substituting $tx$ in place of $x$, where $t \in N$ in (4.2) and using it again, we arrive at

$$[y,t]Ud(y) = \{0\}, \quad \text{for all } y \in U, t \in N.$$ 

Using Lemma 2.1 (i), we obtain

$$y \in Z(N) \text{ or } d(y) = 0, \quad \text{for all } y \in U. \quad (4.3)$$
If there exists $y_0 \in Z(\mathcal{N}) \cap U$, then by (4.1), we get $xd(y_0) = d(y_0)x$ for all $x \in U$. In this case, (4.3) gives $xd(y) = d(y)x$ for all $x, y \in U$. Replace $x$ by $tx$, where $t \in \mathbb{N}$, we get $[d(y), t]x = 0$ for all $x, y \in U$, $t \in \mathbb{N}$ which implies that $[d(y), t]U = \{0\}$ for all $y \in U$, $t \in \mathbb{N}$. Since $U \neq \{0\}$, by Lemma 2.1 (ii), we obtain $d(U) \subseteq Z(\mathcal{N})$ and Lemma 2.3 assures that $\mathcal{N}$ is a commutative ring.

If we replace $G$ by the null application or the identical application $id_\mathcal{N}$, we get the following results.

**Corollary 4.1.** ([8, Theorem 2.1]). Let $\mathcal{N}$ be a 2-torsion free prime near-ring. If $\mathcal{N}$ admits a derivation $d$ such that $[d(x), y] = [x, d(y)]$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.

**Corollary 4.2.** Let $\mathcal{N}$ be a 2-torsion free prime near-ring and $U$ a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a derivation $d$, then the following assertions are equivalent:

(i) $[x, y] = [d(x), y] - [x, d(y)]$ for all $x, y \in U$;
(ii) $[d(x), y] = [x, d(y)]$ for all $x, y \in U$;
(iii) $\mathcal{N}$ is a commutative ring.

When $d = 0$, we have the following result.

**Corollary 4.3.** Let $\mathcal{N}$ be a 2-torsion free prime near-ring and $U$ a nonzero semigroup ideal of $\mathcal{N}$. If $\mathcal{N}$ admits a left multiplier $G$, then the following assertions are equivalent:

(i) $G([x, y]) = 0$ for all $x, y \in U$;
(ii) $\mathcal{N}$ is a commutative ring.

**References**

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