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CHEBYSHEV POLYNOMIALS FOR CERTAIN SUBCLASS OF BAZILEVIĆ FUNCTIONS ASSOCIATED WITH RUSCHEWEYH DERIVATIVE

ABDUL RAHMAN S. JUMA[1], SABA N. AL-KHAFAJI[2], AND OLGA ENGEL[3]

Abstract. In this paper, through the instrument of the well-known Chebyshev polynomials and subordination, we defined a family of functions, consisting of Bazilević functions of type α, involving the Ruscheweyh derivative operator. Also, we investigate coefficient bounds and Fekete-Szegö inequalities for this class.

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ denote the class of analytic functions $f \in \mathcal{A}$, which are univalent in $\mathbb{U}$ and are normalized with the following conditions:

$$f(0) = 0 \quad \text{and} \quad f'(0) = 1.$$

Let f and g be analytic functions in $\mathbb{U}$. We say that the function f is a subordinate to g in $\mathbb{U}$, written as $f \prec g$, if there exists a Schwarz function w, which is analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$, $(z \in \mathbb{U})$ such that $f(z) = g(w(z))$. Furthermore, if g is univalent in $\mathbb{U}$, then we get

$$f(z) \prec g(z), (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}) \quad \text{(see [6])}.$$

Key words and phrases. Analytic functions, univalent functions, Chebyshev polynomials, Ruscheweyh derivative operator, subordination, Fekete-Szegö inequalities, Bazilević functions.

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The problem of finding the sharp bounds for the non-linear functional $|a_3 - \mu a_2^2|$ for Taylor-Mclaurin series is popularly known as the Fekete-Szegö problem. This problem has a rich history in the geometric functions theory. Its source was in the disproof by Fekete and Szegö of the 1933 guess of Littlewood and Paley that the coefficients of odd univalent functions are limited by unity (see [7], has since received great attention, especially in many subclasses of the family of univalent functions. For that reason Fekete-Szegö functional was studied by many authors and a some assessments were found in a numerous subclasses of normalized univalent functions (see [3, 9, 11, 12, 14]).

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Out of four kinds of Chebyshev polynomials, many researchers dealing with orthogonal polynomials of Chebyshev, contain chiefly results of first and second kinds of Chebyshev polynomials $\mathrm{T}_k(t)$ and $\mathrm{U}_k(t)$ respectively and their numerous uses in different applications. Additionally, one can see those given by the papers in ([1, 2, 4, 5] and [8]). The Chebyshev polynomials of the first and second kinds are well known in the case of a real variable t on $(-1, 1)$, which are defined as follows

$$\begin{aligned}\mathrm{T}_k(t) =& \cos k\theta,\\ \mathrm{U}_k(t) =& \frac{\sin(k+1)\theta}{\sin\theta},\end{aligned}$$

where k is the degree of the polynomial and $t = \cos\theta$.

In [14] (also see [13]) Ruscheweyh introduced the following derivative operator:

$$\begin{aligned}D^0 f(z) =& f(z),\\ D^n f(z) =& \frac{z(z^{n-1}f(z))^{(n)}}{n!},\end{aligned}$$

for $n \in \mathbb{N} = \{1, 2, \dots\}$. The symbol $D^n f$ is called the n^{th} order Ruscheweyh derivative of f.

We observe that

$$\begin{aligned}D^0 f(z) =& f(z),\\ D^1 f(z) =& z f'(z),\end{aligned}$$

and in general

$$D^n f(z) = z + \sum_{k=2}^{\infty} \rho(n,k) a_k z^k,$$

where

$$\rho(n,k) = \begin{pmatrix} n+k-1 \\ n \end{pmatrix}.$$

Definition 1.1. A function $f \in \mathcal{A}$ of the form (1.1) belongs to class $\mathcal{G}(\alpha, t)$, if it satisfies the following subordination

$$\mathcal{G}(\alpha,t) = \left\{ f \in \mathcal{A} : \frac{(D^n f(z))'(D^n f(z))^{\alpha-1}}{z^{\alpha-1}} \prec \frac{1}{1-2tz+z^2} \right\}, \tag{1.2}$$

where $0 \le \alpha \le 1$, $t \in \left(\frac{1}{2}, 1\right]$ and $z \in \mathbb{U}$.

Note that, if $t = \cos\alpha$, $\alpha \in (-\pi/3, \pi/3)$, then

$$\begin{aligned} H(z,t) :=& \frac{1}{1-2\cos\alpha z + z^2} \\ =& 1 + \sum_{k=1}^{\infty} \frac{\sin((k+1)\alpha)}{\sin\alpha} z^k \quad (z \in \mathbb{U}). \end{aligned}$$

Thus

$$H(z,t) = 1 + 2\cos\alpha z + (3\cos^2\alpha - \sin^2\alpha) z^2 + \cdots \quad (z \in \mathbb{U}).$$

Therefore, from [15] we can write

$$H(z,t) = 1 + \mathrm{U}_1(t) z + \mathrm{U}_2(t) z^2 + \cdots \quad (z \in \mathbb{U}, t \in (-1,1)),$$

where

$$\mathrm{U}_{k-1}(t) = \frac{\sin(k \arccos t)}{\sqrt{1-t^2}} \quad (k \in \mathbb{N})$$

denotes the Chebyshev polynomials of the second kind. It is known that

$$\mathrm{U}_k(t) = 2t\mathrm{U}_{k-1}(t) - \mathrm{U}_{k-2}(t)$$

and

$$\begin{aligned} \mathrm{U}_1(t) =& 2t, \\ \mathrm{U}_2(t) =& 4t^2 - 1, \\ \mathrm{U}_3(t) =& 8t^3 - 4t. \\ & \vdots \end{aligned} \tag{1.3}$$

The ordinary generating function for Chebyshev polynomials $\mathrm{T}_k(t)$, $t \in [-1,1]$, of the first kind have the following form

$$\sum_{k=0}^{\infty} \mathrm{T}(t) z^k = \frac{1-tz}{1-2tz+z^2} \quad (z \in \mathbb{U}).$$

The Chebyshev polynomials of the first kinds $\mathrm{T}_k(t)$ and of the second kinds $\mathrm{U}_k(t)$ are connected by the following relations:

$$\begin{aligned} \frac{\mathrm{dT}_k(t)}{\mathrm{d}t} =& k\mathrm{U}_{k-1}(t), \\ \mathrm{T}_k(t) =& \mathrm{U}_k(t) - t\mathrm{U}_{k-1}(t), \\ 2\mathrm{T}_k(t) =& \mathrm{U}_k(t) - \mathrm{U}_{k-2}(t). \end{aligned}$$

By giving specific values to the parameters α and n in this class we obtain the following cases.

(i) If $\alpha = 0$ and $n = 0$, then we get

$$\frac{zf'(z)}{f(z)} \prec H(z,t) := \frac{1}{1-2tz+z^2},$$

it reduces to a special case of the class $\mathcal{B}^{\mu}_{\Sigma}(\lambda, t)$, which was introduced by Bulut, Magesh and Abirami [5].

(ii) If $n = 0$ and $\alpha = 1$, then we get

$$f'(z) \prec H(z,t) := \frac{1}{1-2tz+z^2},$$

it also reduces to a special case of the class $\mathcal{B}^{\mu}_{\Sigma}(\lambda, t)$, which was introduced by Bulut, Magesh and Abirami [5].

(iii) If $n = 0$, then we get

$$f'(z)\left(\frac{z}{f(z)}\right)^{1-\alpha} \prec H(z,t) := \frac{1}{1-2tz+z^2},$$

it reduces to the class $\mathcal{H}(\alpha)$, which was introduced by Bulut, Magesh and Abirami [5], as a special case to the class $\mathcal{B}^{\mu}_{\Sigma}(\lambda, t)$.

The aim of this paper is to provide estimates for initial coefficients of Bazilević functions of type α in the class $\mathcal{G}(\alpha, t)$, involving by the Ruscheweyh derivative operator. Besides that, the problem of Fekete- Szegö in this class is additionally explained.

2. Preliminaries

We need the following lemma to prove our main result.

Lemma 2.1 ([10]). *If $w \in \mathcal{S}$, then for any complex number μ*

$$|w_2 - \mu w_1^2| \leq \max\{1, |\mu|\}.$$

The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

3. Main Results

Theorem 3.1. *Let $f \in \mathcal{A}$ belong to the class $\mathcal{G}(\alpha, t)$. Then*

$$|a_2| \leq \frac{2t}{(\alpha+1)\binom{n+1}{n}}$$

and

$$\begin{aligned}|a_3| \leq & \frac{4t^2+2t-1}{(\alpha+2)\binom{n+2}{n}} + \frac{4t^2}{(\alpha+1)(\alpha+2)\binom{n+2}{n}} - \frac{8\alpha t^2}{(\alpha+1)^2(\alpha+2)\binom{n+2}{n}} \\ & - \frac{4\alpha(\alpha-1)t^2}{2(\alpha+1)^2(\alpha+2)\binom{n+1}{n}\binom{n+2}{n}}.\end{aligned}$$

Proof. If $f \in \mathcal{G}(\alpha, t)$, then from (1.2) we have

$$\frac{(D^n f(z))'(D^n f(z))^{\alpha-1}}{z^{\alpha-1}} = 1 + \mathrm{U}_1(t)w(z) + \mathrm{U}_2(t)w^2(z) + \cdots . \tag{3.1}$$

Replacing the value of $D^n(f(z))$ and $(D^n(f(z)))'$ with their equivalent series expressions in (3.1), it follows that

$$\frac{\left(z + \sum\limits_{k=2}^{\infty} k\rho(n,k)a_k z^k\right)\left(1 + \sum\limits_{k=2}^{\infty} \rho(n,k)a_k(\alpha)z^{k-1}\right)}{z + \sum\limits_{k=2}^{\infty} \rho(n,k)a_k z^k} = 1 + \mathrm{U}_1(t)w(z) + \mathrm{U}_2(t)w^2(z) + \cdots .$$

By using the binomial expansion of $1 + \sum\limits_{k=2}^{\infty} \rho(n,k)a_k(\alpha)z^{k-1}$, upon simplification we obtain

$$\begin{aligned} & \left(z + 2\binom{n+1}{n}a_2z^2 + 3\binom{n+2}{n}a_3z^3 + \cdots\right) \\ & \times \left(1 + \alpha\binom{n+1}{n}a_2 z + \left[\alpha\binom{n+2}{n}a_3 + \frac{\alpha(\alpha-1)}{2!}\binom{n+1}{n}a_2^2\right]z^2 + \cdots\right) \\ = & \left[1 + \mathrm{U}_1(t)w(z) + \mathrm{U}_2(t)w^2(z) + \cdots\right]\left(z + \binom{n+1}{n}a_2z^2 + \binom{n+2}{n}a_3z^3 + \cdots\right),\end{aligned}$$

where w is an analytic function, such that $w(0) = 0$ and

$$|w(z)| = |c_1z + c_2z^2 + c_3z^3 + \cdots| < 1 \quad (z \in \mathbb{U}), \tag{3.2}$$

where

$$|c_j| \leq 1 \quad (j \in \mathbb{N}). \tag{3.3}$$

From (3.2) and (3.3), we have

$$\begin{aligned} & \left(z + 2\binom{n+1}{n}a_2z^2 + 3\binom{n+2}{n}a_3z^3 + \cdots\right) \\ & \times \left(1 + \alpha\binom{n+1}{n}a_2 z + \left[\alpha\binom{n+2}{n}a_3 + \frac{\alpha(\alpha-1)}{2!}\binom{n+1}{n}a_2^2\right]z^2 + \cdots\right) \\ = & \left[1 + \mathrm{U}_1(t)c_1 z + (\mathrm{U}_1(t)c_2 + \mathrm{U}_2(t)c_1^2)z^2 + \cdots\right] \\ & \times \left(z + \binom{n+1}{n}a_2z^2 + \binom{n+2}{n}a_3z^3 + \cdots\right).\end{aligned} \tag{3.4}$$

From (3.4), we obtain

$$a_2 = \frac{\mathrm{U}_1(t)c_1}{(\alpha+1)\binom{n+1}{n}}. \tag{3.5}$$

It is easily seen that from (1.3) and (3.5), we have

$$|a_2| \leq \frac{2t}{(\alpha+1)\binom{n+1}{n}}.$$

Now, in order to find the bound on $|a_3|$, from (9), we get

$$\begin{aligned} a_3 =& \frac{\mathrm{U}_1(t)c_2 + \mathrm{U}_2(t)c_1^2}{(\alpha+2)\binom{n+2}{n}} + \frac{\mathrm{U}_1^2(t)c_1^2}{(\alpha+1)(\alpha+2)\binom{n+2}{n}} - \frac{2\alpha \mathrm{U}_1^2(t)c_1^2}{(\alpha+1)^2(\alpha+2)\binom{n+2}{n}} \\ & - \frac{\alpha(\alpha-1)\mathrm{U}_1^2(t)c_1^2}{2(\alpha+1)^2(\alpha+2)\binom{n+1}{n}\binom{n+2}{n}}. \end{aligned} \tag{3.6}$$

By using (1.3) and (3.5) in (3.6), we get

$$\begin{aligned} |a_3| \leq& \frac{4t^2+2t-1}{(\alpha+2)\binom{n+2}{n}} + \frac{4t^2}{(\alpha+1)(\alpha+2)\binom{n+2}{n}} - \frac{8\alpha t^2}{(\alpha+1)^2(\alpha+2)\binom{n+2}{n}} \\ & - \frac{4\alpha(\alpha-1)t^2}{2(\alpha+1)^2(\alpha+2)\binom{n+1}{n}\binom{n+2}{n}}. \end{aligned}$$

□

Theorem 3.2. *If function f of the form* (1.1) *belongs to the class $\mathcal{G}(\alpha,t)$, then*

$$\begin{aligned} |a_3 - \mu a_2^2| \leq& \frac{2t}{(\alpha^2+2)\binom{n+2}{n}} \max\left\{1, \left|\frac{4t^2-1}{2t} + \frac{2t}{(\alpha^2+1)} - \frac{4\alpha^2 t}{(\alpha^2+1)}\right.\right. \\ & \left.\left. - \frac{\alpha(\alpha-1)t\binom{n+2}{n}}{(\alpha^2+1)\binom{n+1}{n}^2} - \mu\frac{2t\binom{n+2}{n}(\alpha^2+2)}{(\alpha^2+1)^2\binom{n+1}{n}}\right|\right\}. \end{aligned}$$

The result is sharp.

Proof. From (3.5) and (3.6), we get

$$\begin{aligned} a_3 - \mu a_2^2 =& \frac{\mathrm{U}_1(t)c_2 + \mathrm{U}_2(t)c_1^2}{(\alpha+2)\binom{n+2}{n}} + \frac{\mathrm{U}_1^2(t)c_1^2}{(\alpha+1)(\alpha+2)\binom{n+2}{n}} - \frac{2\alpha \mathrm{U}_1^2(t)c_1^2}{(\alpha+1)^2(\alpha+2)\binom{n+2}{n}} \\ & - \frac{\alpha(\alpha-1)\mathrm{U}_1^2(t)c_1^2}{2(\alpha+1)^2(\alpha+2)\binom{n+1}{n}\binom{n+2}{n}} - \mu\frac{\mathrm{U}_1^2(t)c_1^2}{(\alpha+1)^2\binom{n+1}{n}^2}. \end{aligned}$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{\mathrm{U}_1(t)}{(\alpha+2)\binom{n+2}{n}} \Bigg[c_2 + \Bigg(\frac{\mathrm{U}_2(t)}{\mathrm{U}_1(t)} + \frac{\mathrm{U}_1(t)}{(\alpha+1)} - \frac{2\alpha^2 \mathrm{U}_1(t)}{(\alpha+1)^2} - \frac{\alpha(\alpha-1)\mathrm{U}_1(t)\binom{n+2}{n}}{2(\alpha+1)^2\binom{n+1}{n}^2} - \mu \frac{\mathrm{U}_1(t)\binom{n+2}{n}(\alpha+2)}{(\alpha+1)^2\binom{n+1}{n}^2} \Bigg) c_1^2 \Bigg].$$

Then, in view of Lemma 2.1, we conclude that

$$|a_3 - \mu a_2^2| \leq \frac{\mathrm{U}_1(t)}{(\alpha+2)\binom{n+2}{n}} \max \left\{ 1, \left| \frac{\mathrm{U}_2(t)}{\mathrm{U}_1(t)} + \frac{\mathrm{U}_1(t)}{(\alpha+1)} - \frac{2\alpha^2 \mathrm{U}_1(t)}{(\alpha+1)^2} - \frac{\alpha(\alpha-1)\mathrm{U}_1(t)\binom{n+2}{n}}{2(\alpha+1)^2\binom{n+1}{n}^2} - \mu \frac{\mathrm{U}_1(t)\binom{n+2}{n}(\alpha+2)}{(\alpha+1)^2\binom{n+1}{n}^2} \right| \right\},$$

which is equivalent to

$$|a_3 - \mu a_2^2| \leq \frac{2t}{(\alpha+2)\binom{n+2}{n}} \max \left\{ 1, \left| \frac{4t^2-1}{2t} + \frac{2t}{\alpha+1} - \frac{4\alpha^2 t}{(\alpha+1)^2} - \frac{\alpha(\alpha-1)t\binom{n+2}{n}}{(\alpha+1)^2\binom{n+1}{n}^2} - \mu \frac{2t\binom{n+2}{n}(\alpha+2)}{(\alpha+1)^2\binom{n+1}{n}^2} \right| \right\}.$$

This completes the proof. □

Putting $\alpha = 1$ in Theorem 3.2, we obtain the following result.

Corollary 3.1. *If f given by* (1.1) *belongs to the class* $\mathcal{G}(1,t)$, *then*

$$|a_3 - \mu a_2^2| \leq \frac{2t}{3\binom{n+2}{n}} \max \left\{ 1, \left| \frac{4t^2-1}{2t} - t - \mu \frac{3t\binom{n+2}{n}}{2\binom{n+1}{n}} \right| \right\}.$$

The result is sharp.

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A UNITARY TREATMENT OF CERTAIN INEQUALITIES INVOLVING MEANS

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ABSTRACT. The aim of this paper is to state and prove certain inequalities that involve means (e.g., the arithmetic, geometric, logarithmic means) using a particular result. First of all we recall useful properties of a real-valued convex function that will be used in the proof of our inequalities. Further, we present three inequalities, the first involving the logarithmic mean, the second involving the classical arithmetical and geometrical means and in the last we introduce a new mean. Finally, we give alternate proofs to the Schweitzer's inequality and Khanin's inequality.

1. INTRODUCTION

Recall that, for $a, b \in (0, \infty)$, the logarithmic mean is given by the relation

$$L(a,b) := \frac{b-a}{\log b - \log a},$$

and for $k \in \mathbb{N} \setminus \{1\}$ and $x_1, \dots, x_n \in [a, b]$ we introduce the generalized mean as follows

$$\sqrt[k]{\frac{x_1^k + \cdots + x_n^k}{n}}.$$

These means frequently appear in the setting of inequalities.

The subject of inequalities has fascinated a great deal of mathematicians and the proof of this fact lives in the classical and recent results that bear their names. A large number of inequalities have been the subject of well-known books such as [1, 3, 4, 6].

Note that, an important notion that we employ in this paper is that of a convex function, and we are interesed in the property that the maximum of a convex function

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is attained on the boundary of the convex and bounded domain, on which it is defined (for additional details, see, e.g., [2,7,8]).

The aim of our work is to establish certain inequalities using convex functions. In Section 2 we give the important notions that will be used throughout the paper. We introduce here the notion of a convex function and we also give a fundamental result which will be used in the proof of our inequalities, namely Theorem 2.1. Through this particular result we point some properties of convex (see, e.g., [5], [7, pp. 89, 118]) functions of several variables and their applications. In Section 3 we have three important applications. Application 3.1 gives an estimate for the difference between the arithmetic and geometric mean in terms of the logarithmic mean. Application 3.2 gives another estimate for the difference between the arithmetic and geometric mean and in Application 3.3 we give an estimate for the difference between the generalized mean and arithmetic mean using a new mean. We end this section with Schweitzer's inequality (Theorem 3.4) and Khanin's Inequality (Theorem 3.5). We have provided alternative proofs for these inequalities using Theorem 2.1.

2. Prelimiaries

If $\mathcal{C} \subset \mathbb{R}^n$ is a convex set, then, a function $f : \mathcal{C} \to \mathbb{R}$ is said to be **convex** if

$$f((1-\lambda)x + \lambda y) \leq (1-\lambda)f(x) + \lambda f(y), \quad \text{for all } x, y \in \mathcal{C}, \text{ for all } \lambda \in [0,1].$$

We point out the fact that, if $f : [a,b] \to \mathbb{R}$ is convex, then the maximum value of f is attained at the boundary. We shall prove this useful property at Lemma 2.1.

In the latter, for $a, b \in \mathbb{R}$, we have the following notations:

$$\begin{aligned}[a,b]^n =& [a,b] \times \cdots \times [a,b],\\ \{a,b\}^n =& \{a,b\} \times \cdots \times \{a,b\}.\end{aligned}$$

In the latter, we shall prove a theorem that points out that, in certain conditions, the maximum of a convex and continuous function $f : [a,b]^n \to \mathbb{R}$ can be found by taking the maximum of the function on the vertices of the considered hypercube $[a,b]^n$, where $a, b \in \mathbb{R}$.

Before we proceed to the proof of this result, we shall state a useful lemma (see, e.g., [7, Theorem 3.4.6, Theorem 3.4.7]).

Lemma 2.1. *Let $a, b \in \mathbb{R}$ and let $f : [a,b] \to \mathbb{R}$ be a convex and continuous function. Then*

$$\max_{x \in [a,b]} f(x) = \max_{x \in \{a,b\}} f(x).$$

Proof. Since f is continuous on $[a,b]$, we deduce that there is an $\alpha \in [a,b]$ such that

$$f(\alpha) = \max_{x \in [a,b]} f(x).$$

Now, we argue by contradiction. Assume that $\alpha \in (a, b)$. This means that there is some $\lambda \in (0, 1)$, such that

$$\alpha = (1 - \lambda)a + \lambda b.$$

Hence,

$$\begin{aligned} f(\alpha) =& f((1 - \lambda)a + \lambda b) \leq (1 - \lambda)f(a) + \lambda f(b) \\ <& (1 - \lambda)f(\alpha) + \lambda f(\alpha), \end{aligned}$$

i.e., $f(\alpha) < f(\alpha)$, which is absurd and our proof is complete. □

We have the following theorem.

Theorem 2.1. *Let* $f : [a, b]^n \to \mathbb{R}$ *be a* $\mathcal{C}^2$*-class function on* $[a, b]^n$*, such that:*

$$\frac{\partial^2 f}{\partial x_i^2} \geq 0, \quad \text{for all } i = \overline{1, n}. \tag{2.1}$$

Then

$$\max_{x \in [a,b]^n} f(x) = \max_{x \in \{a,b\}^n} f(x).$$

Proof. Define the function $g_1 : [a, b] \to \mathbb{R}$ in the following manner

$$g_1(x_1) = f(x_1, \ldots, x_n),$$

where the variables $x_2, \ldots, x_n$ are arbitrarily fixed. Using condition (2.1) for $i = 1$, we deduce that g_1 is convex, hence:

$$\max_{x_1 \in [a,b]} f(x_1, \ldots, x_n) = \max_{x_1 \in \{a,b\}} f(x_1, \ldots, x_n).$$

Let $x_1 \in \{a, b\}$ be the value for which the maximum is attained. We apply the same steps as above to the function $g_2 : [a, b] \to \mathbb{R}$, $g_2(x_2) = f(x_1, x_2, \ldots, x_n)$ and we deduce that there exists $x_2 \in \{a, b\}$ such that

$$\max_{x_2 \in [a,b]} f(x_1, x_2, \ldots, x_n) = \max_{x_2 \in \{a,b\}} f(x_1, x_2, \ldots, x_n),$$

and we use the same arguments for $x_3, \ldots, x_n$. Consequently, we obtain

$$\max_{x \in [a,b]^n} f(x) = \max_{x \in \{a,b\}^n} f(x),$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

This completes our proof. □

3. Applications of Theorem 2.1

In this section we give the main results of our paper.

Application 3.1. Let $x_1, \ldots, x_n \in [a, b] \subset (0, \infty)$, where $a = \min_{i=\overline{1,n}} x_i$ and $b = \max_{i=\overline{1,n}} x_i$. Then

$$\frac{x_1 + \cdots + x_n}{n} - \sqrt[n]{x_1 \cdots x_n} \leq (L(a,b) - a)\left(\frac{L(a,b)}{L(a, L(a,b))} - 1\right),$$

where

$$L(a,b) = \frac{b-a}{\log b - \log a},$$

denotes the logarithmic mean of a and b.

Proof. Consider the function $f : [a,b]^n \to \mathbb{R}$, given by the relation

$$f(x_1, \ldots, x_n) = \frac{x_1 + \cdots + x_n}{n} - \sqrt[n]{x_1 \cdots x_n}.$$

Take note that

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{n-1}{n^2}(x_1 \cdots x_{i-1} x_{i+1} \cdots x_n)^{\frac{1}{n}} x_i^{-\frac{2n-1}{n}},$$

for $i = \overline{1,n}$.

Since all partial derivatives of second order of f are positive, by applying Theorem 2.1 we obtain

$$\max_{x \in [a,b]^n} f(x) = \max_{x \in \{a,b\}^n} f(x).$$

We shall determine now $\max_{x \in \{a,b\}^n} f(x)$.

Without loss of generality, we may now assume that the maximum of f is obtained at the point $(x_1, \ldots, x_n)$, where

$$x_1 = x_2 = \cdots = x_{n-k} = a,$$
$$x_{n-k+1} = x_{n-k+2} = \cdots = x_n = b.$$

Hence, to determine $\max_{x \in \{a,b\}^n} f(x)$, it is enough to find the maximum of the expression

$$E = \frac{(n-k)a + kb}{n} - \sqrt[n]{a^{n-k} b^k} = a + \frac{k}{n}(b-a) - a\left(\frac{a}{b}\right)^{\frac{k}{n}},$$

for $k = \overline{1, n-1}$.

To this end, let $g : (0,1) \to \mathbb{R}$ be given by

$$g(t) = a + t(b-a) - a\left(\frac{a}{b}\right)^t.$$

The derivative of g is as follows

$$g'(t) = b - a - a\left(\frac{b}{a}\right)^t \log\left(\frac{b}{a}\right).$$

Hence,

$$g'(t) = 0 \Leftrightarrow \left(\frac{b}{a}\right)^t = \frac{L(a,b)}{a} \Leftrightarrow t' := t = \frac{\log(L(a,b)) - \log a}{\log b - \log a} \in (0,1).$$

We deduce that t' is a global maximum point for g. Consequently,

$$\max_{t\in(0,1)} g(t) = g(t').$$

Our claim is that

$$g(t') = (L(a,b) - a)\left[\frac{L(a,b)}{L(a,L(a,b))} - 1\right].$$

Indeed,

$$\begin{aligned}
g(t') &= a + t'(b-a) - a\left(\frac{b}{a}\right)^{t'}\\
&= a + t'(b-a) - a\frac{L(a,b)}{a}\\
&= a + \frac{b-a}{\log b - \log a}\cdot\frac{\log(L(a,b)) - \log a}{L(a,b) - a}(L(a,b) - a) - L(a,b)\\
&= a + L(a,b)\frac{1}{L(a,L(a,b))}(L(a,b) - a) - L(a,b)\\
&= -(L(a,b) - a) + \frac{L(a,b)}{L(a,L(a,b))}(L(a,b) - a)\\
&= (L(a,b) - a)\left[\frac{L(a,b)}{L(a,L(a,b))} - 1\right],
\end{aligned}$$

and our claim is verified. This concludes the proof. □

Application 3.2. Let $x_1, \ldots, x_n \in [a,b] \subset (0,\infty)$, where $a = \min\limits_{i=\overline{1,n}} x_i$ and $b = \max\limits_{i=\overline{1,n}} x_i$. Then

$$\frac{x_1 + \cdots + x_n}{n} - \sqrt[n]{x_1 \cdots x_n} \leq (\sqrt{b} - \sqrt{a})^2.$$

Proof. Using similar arguments to those in the proof of Application 3.1, we consider the expression

$$E = \frac{(n-k)a + kb}{n} - \sqrt[n]{a^{n-k}b^k}.$$

Note that,

$$E \leq \frac{(n-k)a + kb}{n} - \frac{n}{\frac{n-k}{a} + \frac{k}{b}},$$

due to the geometric and harmonic mean inequality, i.e.,

$$\sqrt[n]{a^{n-k}b^k} \geq \frac{n}{\frac{n-k}{a} + \frac{k}{b}}.$$

Thus, one obtains

$$E \leq a + \frac{k}{n}(b-a) - \frac{ab}{b - \frac{k}{n}(b-a)}.$$

On the other hand, we introduce the function $h : (0,1) \to \mathbb{R}$ as follows

$$h(t) = 1 + t(b-a) - \frac{ab}{b - t(b-a)}.$$

The derivative of the function h is

$$h'(t) = b - a - \frac{ab}{(b - t(b-a))^2}(b-a).$$

Then we have

$$h'(t) = 0 \Leftrightarrow t' := t = \frac{\sqrt{b}}{\sqrt{a} + \sqrt{b}} \in (0,1).$$

It follows that t' is a global maximum point for h and therefore,

$$h(t) \leq h(t') = (\sqrt{b} - \sqrt{a})^2.$$

Consequently, we obtain

$$\frac{x_1 + \cdots + x_n}{n} - \sqrt[n]{x_1 \cdots x_n} \leq (\sqrt{b} - \sqrt{a})^2,$$

and our proof is finished. □

Now, we introduce the quantity

$$P_k(a,b) = \sqrt[k]{\frac{b^{k+1} - a^{k+1}}{(k-1)(b-a)}} = \sqrt[k]{\frac{a^k + a^{k-1}b + \cdots + b^k}{k+1}},$$

for $k \in \mathbb{N}$, $k \geq 2$ and $P_1(a,b) = \frac{a+b}{2}$.

One can easily see that $a < P_k(a,b) < b$, and thus $P_k(a,b)$ is a mean.

Application 3.3. Let $x_1, \ldots, x_n \in [a,b] \subset (0,\infty)$, where $a = \min\limits_{i=\overline{1,n}} x_i$ and $b = \max\limits_{i=\overline{1,n}} x_i$. Then

$$\sqrt[k]{\frac{x_1^k + \cdots + x_n^k}{n}} - \frac{x_1 + \cdots + x_n}{n} \leq (P_{k-1}(a,b) - a)\left(1 - \frac{P_{k-1}^{k-1}(a, P_{k-1}(a,b))}{P_{k-1}^{k-1}(a,b)}\right).$$

Proof. Let $f : [a,b]^n \to \mathbb{R}$ be given by

$$f(x_1, \ldots, x_n) = \sqrt[k]{\frac{x_1^k + \cdots + x_n^k}{n}} - \frac{x_1 + \cdots + x_n}{n}.$$

One can easily see that

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{k-1}{n^2} x_i^{k-2} \left(\frac{x_1^k + \cdots + x_n^k}{n}\right)^{\frac{1}{k}-2} \frac{x_1^k + \cdots + x_{i-1}^k + x_{i+1}^k + \cdots + x_n^k}{n} > 0,$$

for all $i = \overline{1,n}$.

We apply Theorem 2.1 and we obtain that

$$\begin{aligned}\max_{x\in[a,b]^n} f(x) &= \max_{p\in\{1,\dots,n-1\}} \sqrt[k]{\frac{(n-p)a^k+pb^k}{n}} - \frac{(n-a)p+pb}{n}\\ &= \max_{p\in\{1,\dots,n-1\}} \left(\sqrt[k]{\frac{p}{n}(b^k-a^k)+a^k} - \frac{p}{n}(b-a)-a\right).\end{aligned}$$

Consider, now, the function $g:(0,1)\to\mathbb{R}$ given by

$$g(t)=\sqrt[k]{t(b^k-a^k)+a^k}-t(b-a)-a.$$

We deduce that

$$g'(t)=\frac{b^k-a^k}{k\left(\sqrt[k]{t(b^k-a^k)+a^k}\right)^{k-1}}-(b-a).$$

Hence,

$$g'(t)=0 \Leftrightarrow \alpha:=t=\frac{P_{k-1}^k(a,b)-a^k}{b^k-a^k}\in(0,1).$$

Consequently, we have

$$\max_{t\in(0,1)} g(t)=g(\alpha).$$

On the other hand, we get

$$\begin{aligned}g(\alpha) &= P_{k-1}(a,b)-\frac{P_{k-1}^k(a,b)-a}{b^k-a^k}(b-a)-a\\ &= P_{k-1}(a,b)-a-\frac{k(b-a)}{b^k-a^k}\frac{P_{k-1}^k(a,b)-a^k}{k(P_{k-1}(a,b)-a)}(P_{k-1}(a,b)-a)\\ &= P_{k-1}(a,b)-a-\frac{1}{P_{k-1}^{k-1}}(a,b)P_{k-1}^{k-1}(a,P(a,b))(P_{k-1}(a,b)-a)\\ &= (P_{k-1}(a,b)-a)\left(1-\frac{P_{k-1}^{k-1}(a,P_{k-1}(a,b))}{P_{k-1}^{k-1}(a,b)}\right).\end{aligned}$$

This concludes our proof. □

Remark 3.1. Setting $n=2$ in Application 3.3 yields the following inequality

$$\sqrt{\frac{x_1^2+\cdots+x_n^2}{n}}-\frac{x_1+\cdots+x_n}{n}\le\frac{(b-a)^2}{4(a+b)}.$$

In the latter, we shall state and prove Schweitzer's inequality.

Theorem 3.4 (Schweitzer)**.** *Let $x_1,\dots,x_n,a,b>0$ such that $x_i\in[a,b]$ for all $i=\overline{1,n}$. Then,*

$$\begin{aligned}(x_1+\cdots+x_n)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)&\le\frac{(a+b)^2}{4ab}n^2, &&\text{for } n \text{ even},\\ (x_1+\cdots+x_n)\left(\frac{1}{x_1}+\cdots+\frac{1}{x_n}\right)&\le\frac{(a+b)^2}{4ab}n^2-\frac{(a-b)^2}{4ab}, &&\text{for } n \text{ odd}.\end{aligned}$$

Proof. Define $f : [a,b]^n \to \mathbb{R}$ as follows

$$f(x_1,\dots,x_n) = (x_1 + \cdots + x_n)\left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right).$$

Then, for all $i = \overline{1,n}$, we have

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{2(x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_n)}{x_i^3} > 0.$$

Apply Theorem 2.1 and deduce that

$$\max_{x\in[a,b]^n} f(x) = \max_{x\in\{a,b\}^n} f(x),$$

where $x = (x_1,\dots,x_n) \in \mathbb{R}^n$. Now, we determine $\max\limits_{x\in\{a,b\}^n} f(x)$. Without loss of generality, we may assume that

$$\begin{aligned} &x_1 = x_2 = \cdots = x_k = a,\\ &x_{k+1} = x_{k+2} = \cdots = x_n = b. \end{aligned}$$

Hence,

$$\max_{x\in\{a,b\}^n} f(x) = \max_{k\in\{1,\dots,n\}} (ka + (n-k)b)\left(\frac{k}{a} + \frac{n-k}{b}\right).$$

We consider the function $g : (0,\infty) \to \mathbb{R}$, defined by

$$g(k) = (ka + (n-k)b)\left(\frac{k}{a} + \frac{n-k}{b}\right).$$

A short computation yields

$$g(k) = -\frac{(b-a)^2}{ab}k^2 + \frac{n(b-a)^2}{ab}k + n^2.$$

One can easily deduce that the maximum of g is obtained when $x = \frac{n}{2}$. Moreover, the restriction of g to the set $\{1,\dots,n\}$ obtains its maximum value at $k = \frac{n}{2}$ if n is even and $k = \frac{n-1}{2}$ or $k = \frac{n+1}{2}$ is n is odd. On the other hand, take note that

$$\begin{aligned} &g\left(\frac{n}{2}\right) = \frac{(a+b)^2}{4ab}n^2, \quad \text{for } n \text{ even},\\ &g\left(\frac{n-1}{2}\right) = g\left(\frac{n+1}{2}\right) = \frac{(a+b)^2}{4ab}n^2 - \frac{(b-a)^2}{4ab}, \quad \text{for } n \text{ odd}. \end{aligned}$$

This concludes our proof. $\square$

Now we focus on the statement and proof of Khanin's inequality.

Theorem 3.5 (Khanin). *Let $x_1, \dots, x_n, a, b \in \mathbb{R}$ such that $x_i \in [a, b]$ for all $i = \overline{1, n}$. Then*

$$\begin{aligned}
(3.1)\qquad &\frac{x_1^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \frac{(b-a)^2}{4}, \quad \textit{for } n \textit{ even},\\
&\frac{x_1^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \left(\frac{b-a}{4}\right)^2 - \frac{(b-a)^2}{4n^2}, \quad \textit{for } n \textit{ odd}.
\end{aligned}$$

Proof. Let $f : [a, b]^n \to \mathbb{R}$, given by

$$f(x_1, \dots, x_n) = \frac{x_1^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + \cdots + x_n}{n}\right)^2.$$

Note that

$$\frac{\partial^2 f}{\partial x_i^2} = \frac{2(n-1)}{n^2} > 0.$$

Now, apply Theorem 2.1 and deduce that

$$\frac{x_1^2 + \cdots + x_n^2}{n} - \left(\frac{x_1 + \cdots + x_n}{n}\right)^2 \leq \frac{ka^2 + (n-k)b^2}{n} - \left(\frac{ka + (n-k)b}{n}\right)^2.$$

Further, we consider the function $g : (0, \infty) \to \mathbb{R}$, given by

$$g(k) = \frac{ka^2 + (n-k)b^2}{n} - \left(\frac{ka + (n-k)b}{n}\right)^2.$$

It can be easily seen that the restriction of g, i.e., $g : \{1, 2, \dots, n\} \to \mathbb{R}$ obtains its maximum value when $k = \frac{n}{2}$ if n is even and when $k = \frac{n-1}{2}$ or $k = \frac{n+1}{2}$ if n is odd. Taking note that

$$\begin{aligned}
g\left(\frac{n}{2}\right) &= \frac{(b-a)^2}{4}, \quad \text{for } n \text{ even},\\
g\left(\frac{n-1}{2}\right) &= g\left(\frac{n+1}{2}\right) = \left(\frac{b-a}{4}\right)^2 - \frac{(b-a)^2}{4n^2}, \quad \text{for } n \text{ odd},
\end{aligned}$$

we obtain relations (3.1).

This completes our proof. □

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SOME REFINEMENTS OF THE NUMERICAL RADIUS INEQUALITIES VIA YOUNG INEQUALITY

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Abstract. In this paper, we get an improvement of the Hölder-McCarthy operator inequality in the case when $r \geq 1$ and refine generalized inequalities involving powers of the numerical radius for sums and products of Hilbert space operators.

1. Introduction

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex Hilbert space and $B(\mathcal{H})$ denote the C^*-algebra of all bounded linear operators on $\mathcal{H}$. Recall that for $A \in B(\mathcal{H})$, $W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \|x\| = 1\}$, $w(A) = \sup\{|\lambda| : \lambda \in W(A)\}$ and $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$, denote the numerical range, the numerical radius and the usual operator norm of A, respectively. Also an operator $A \in B(\mathcal{H})$ is said to be positive if $\langle Ax, x\rangle \geqslant 0$ for each $x \in \mathcal{H}$ and, in this case, is denoted by $A \geqslant 0$.

It is well-known that $\overline{W(A)}$ is a convex subset of the complex plane that contains the convex hull spectrum of A (see [4, p. 7]). It is known that $w(\cdot)$ defines a norm on $B(\mathcal{H})$, which is equivalent to the usual operator norm $\|\cdot\|$ [4, Theorem 1.3-1]. For $A \in B(\mathcal{H})$, we have

$$\frac{1}{2}\|A\| \leq w(A) \leq \|A\|. \tag{1.1}$$

The inequalities in (1.1) have been improved by many mathematicians, (see [2, 7, 10, 13, 17–19]).

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Kittaneh in [7,8] showed that if $A \in B(\mathcal{H})$, then

$$w(A) \leq \frac{1}{2}\||A| + |A^*|\| \leq \frac{1}{2}(\|A\| + \|A^2\|^{\frac{1}{2}}), \tag{1.2}$$

where $|A|^2 = A^*A$, and

$$\frac{1}{4}\|A^*A + AA^*\| \leq w^2(A) \leq \frac{1}{2}\|A^*A + AA^*\|. \tag{1.3}$$

He also obtained the following generalizations of the first inequality in (1.2) and the second inequality in (1.3):

$$w^r(A) \leq \frac{1}{2}\left\||A|^{2\lambda r} + |A^*|^{2(1-\lambda)r}\right\| \tag{1.4}$$

and

$$w^{2r}(A) \leq \left\|\lambda|A|^{2r} + (1-\lambda)|A^*|^{2r}\right\|, \tag{1.5}$$

where $0 < \lambda < 1$, and $r \geq 1$ in [9, Theorem 1, Theorem 2], respectively.

In Section 2 of this paper, we get an improvement of the Hölder-McCarthy operator inequality in the case when $r \geq 1$ and refine inequality (1.4) for $r \geq 1$ and inequality (1.5) for $r \geq 2$, see ([3,12,16]). In addition, we establish some improvements of norm and numerical radius inequalities for sums and powers of operators acting on a Hilbert space in Section 3. For recent work on the numerical radius inequalities, we refer the reader to [13–15,18].

2. Refinements of the Hölder-McCarthy Operator Inequality

In this section, we obtain an improvement of Hölder-McCarthy's operator inequality in the case when $r \geq 1$ and get some improvements of numerical radius inequalities for Hilbert space operators. The following lemmas are essential for our investigation. The first lemma is a simple consequence of the Jensen inequality for convex function $f(t) = t^r$, where $r \geq 1$.

Lemma 2.1. ([13, Lemma 2.1]). *Let $a, b \geq 0$ and $0 \leq \lambda \leq 1$. Then*

$$a^\lambda b^{1-\lambda} \leq \lambda a + (1-\lambda)b \leq (\lambda a^r + (1-\lambda)b^r)^{\frac{1}{r}}, \quad \textit{for } r \geq 1.$$

The second lemma is known as a generalized mixed Schwarz inequality.

Lemma 2.2. ([8, Lemma 5]). *Let $A \in B(\mathcal{H})$ and $x, y \in \mathcal{H}$ be two vectors and $0 \leq \lambda \leq 1$. Then*

$$|\langle Ax, y\rangle|^2 \leq \langle |A|^{2\lambda}x, x\rangle\langle |A^*|^{2(1-\lambda)}y, y\rangle.$$

The third lemma follows from the spectral theorem for positive operators and the Jensen inequality and is known as the Hölder McCarthy inequality.

Lemma 2.3. ([13, Lemma 2.2]). *Suppose that A is a positive operator in $B(\mathcal{H})$ and $x \in \mathcal{H}$ is any unit vector. Then*

(i) *$\langle Ax, x\rangle^r \leq \langle A^r x, x\rangle$ for $r \geq 1$;*

(ii) $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r$ *for* $0 < r \le 1$.

The last lemma is an improvement of Hölder-McCarthy's inequality.

Lemma 2.4. ([6, Corollary 3.1]). *Let A be a positive operator on $\mathcal{H}$. If $x \in \mathcal{H}$ is a unit vector, then*

$$\langle Ax, x\rangle^r \le \langle A^r x, x\rangle - \langle |A - \langle Ax, x\rangle|^r x, x\rangle, \quad \textit{for } r \ge 2.$$

The next theorem is a refinement of inequality (1.5) for $r \ge 2$.

Theorem 2.1. *If $A \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \ge 2$, then*

$$w^{2r}(A) \le \|\lambda |A|^{2r} + (1-\lambda)|A^*|^{2r}\| - \inf_{\|x\|=1} \zeta(x),$$

where

$$\zeta(x) = \left\langle \left(\lambda \left| |A|^2 - \langle |A|^2 x, x\rangle \right|^r + (1-\lambda)\left| |A^*|^2 - \langle |A^*|^2 x, x\rangle \right|^r\right) x, x \right\rangle.$$

Proof. Let $x \in \mathcal{H}$ be a unit vector.

$$\begin{aligned}
|\langle Ax, x\rangle|^2 &\le \langle |A|^{2\lambda} x, x\rangle \langle |A^*|^{2(1-\lambda)} x, x\rangle \quad \text{(by Lemma 2.2)}\\
&\le \langle |A|^2 x, x\rangle^{\lambda} \langle |A^*|^2 x, x\rangle^{1-\lambda} \quad \text{(by Lemma 2.3 (ii))}\\
&\le (\lambda \langle |A|^2 x, x\rangle^r + (1-\lambda)\langle |A^*|^2 x, x\rangle^r)^{\frac{1}{r}} \quad \text{(by Lemma 2.1)}\\
&\le \Bigg(\lambda \left(\langle |A|^{2r} x, x\rangle - \left\langle \left| |A|^2 - \langle |A|^2 x, x\rangle \right|^r x, x\right\rangle\right)\\
&\qquad + (1-\lambda)\left(\langle |A^*|^{2r} x, x\rangle - \left\langle \left| |A^*|^2 - \langle |A^*|^2 x, x\rangle \right|^r x, x\right\rangle\right)\Bigg)^{\frac{1}{r}}\\
&\qquad \text{(by Lemma 2.4)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle Ax, x\rangle|^{2r} &\le \lambda \left(\langle |A|^{2r} x, x\rangle - \left\langle \left| |A|^2 - \langle |A|^2 x, x\rangle \right|^r x, x\right\rangle\right)\\
&\qquad + (1-\lambda)\left(\langle |A^*|^{2r} x, x\rangle - \left\langle \left| |A^*|^2 - \langle |A^*|^2 x, x\rangle \right|^r x, x\right\rangle\right).
\end{aligned}$$

By taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we get the desired relation. □

Recall that the Young inequality says that if $a, b \ge 0$ and $\lambda \in [0,1]$, then

$$(1-\lambda)a + \lambda b \ge a^{1-\lambda} b^{\lambda}.$$

Many mathematicians improved the Young inequality and its reverse. Kober [11], proved that for $a, b > 0$

$$(1-\lambda)a + \lambda b \le a^{1-\lambda} b^{\lambda} + (1-\lambda)(\sqrt{a} - \sqrt{b})^2, \quad \lambda \ge 1. \tag{2.1}$$

By using (2.1), we obtain a refinement of the Hölder-McCarthy inequality.

Lemma 2.5. *Let $A \in B(\mathcal{H})$ be a positive operator. Then*

$$\langle Ax,x\rangle^{\lambda}\left(1+2(\lambda-1)\left(1-\frac{\langle A^{\frac{1}{2}}x,x\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\right)\right)\leq\langle A^{\lambda}x,x\rangle, \tag{2.2}$$

for any $\lambda \geq 1$ and $x \in \mathcal{H}$ with $\|x\| = 1$.

Proof. Applying functional calculus for the positive operator A in (2.1), we get

$$(1-\lambda)aI+\lambda A\leq a^{1-\lambda}A^{\lambda}+(1-\lambda)\left(aI+A-2\sqrt{a}A^{\frac{1}{2}}\right).$$

The above inequality is equivalent to

$$(1-\lambda)a+\lambda\langle Ax,x\rangle\leq a^{1-\lambda}\langle A^{\lambda}x,x\rangle+(1-\lambda)\left(a+\langle Ax,x\rangle-2\sqrt{a}\langle A^{\frac{1}{2}}x,x\rangle\right), \tag{2.3}$$

for any $x \in \mathcal{H}$ with $\|x\| = 1$. By substituting $a = \langle Ax, x\rangle$ in (2.3), we get

$$\langle Ax,x\rangle\leq\langle Ax,x\rangle^{1-\lambda}\langle A^{\lambda}x,x\rangle+2(1-\lambda)\langle Ax,x\rangle\left(1-\frac{\langle A^{\frac{1}{2}}x,x\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\right).$$

By rearranging terms, we get the desired result (2.2). □

Note that by the Hölder-McCarthy inequality, $1 \geq 1-\frac{\langle A^{\frac{1}{2}}x,x\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}} \geq 0$. Hence, the following chain of inequalities are true:

$$\langle Ax,x\rangle^{\lambda}\leq\langle Ax,x\rangle^{\lambda}\left(1+2(\lambda-1)\left(1-\frac{\langle A^{\frac{1}{2}}x,x\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\right)\right)\leq\langle A^{\lambda}x,x\rangle,$$

where A is positive and $\lambda \geq 1$. One can easily see that

$$1-\frac{\left\langle A^{\frac{1}{2}}x,x\right\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\geq\inf\left\{1-\frac{\left\langle A^{\frac{1}{2}}x,x\right\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}:x\in\mathcal{H},\|x\|=1\right\}.$$

So,

$$1+2\left(\lambda-1\right)\left(1-\frac{\left\langle A^{\frac{1}{2}}x,x\right\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\right)\geq1+2\left(\lambda-1\right)\,\inf\left\{1-\frac{\left\langle A^{\frac{1}{2}}x,x\right\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}:x\in\mathcal{H},\|x\|=1\right\}. \tag{2.4}$$

If we denote the right-hand side of inequality (2.4) by $\zeta(x)$, then from inequality (2.2), we get

$$\langle Ax,x\rangle^{\lambda}\leq\frac{1}{\zeta}\langle A^{\lambda}x,x\rangle,\qquad\lambda\geq1. \tag{2.5}$$

The following theorem is an improvement of inequality (1.4).

Theorem 2.2. *Let $A \in B(\mathcal{H})$ be an invertible operator, $0 < \lambda < 1$ and $r > 1$. If for each unit vector $x \in \mathcal{H}$*

$$\zeta(x)=\left(1+2(r-1)\left(1-\frac{\langle|A|^{\lambda}x,x\rangle}{\langle|A|^{2\lambda}x,x\rangle^{\frac{1}{2}}}\right)\right)$$

and

$$\gamma(x)=\left(1+2(r-1)\left(1-\frac{\langle|A^*|^{(1-\lambda)}x,x\rangle}{\langle|A^*|^{2(1-\lambda)}x,x\rangle^{\frac{1}{2}}}\right)\right),$$

then

$$w^r(A)\leq\frac{1}{2\mu}\left\||A|^{2\lambda r}+|A^*|^{2(1-\lambda)r}\right\|,$$

where $\zeta=\inf_{\|x\|=1}\zeta(x)$, $\gamma=\inf_{\|x\|=1}\gamma(x)$ *and* $\mu=\min\{\zeta,\gamma\}$.

Proof. Let $x\in\mathcal{H}$ be a unit vector. Then

$$\begin{aligned}|\langle Ax,x\rangle|&\leq\left\langle|A|^{2\lambda}x,x\right\rangle^{\frac{1}{2}}\left\langle|A^*|^{2(1-\lambda)}x,x\right\rangle^{\frac{1}{2}}\\&\leq\left(\frac{\left\langle|A|^{2\lambda}x,x\right\rangle^r+\left\langle|A^*|^{2(1-\lambda)}x,x\right\rangle^r}{2}\right)^{\frac{1}{r}}\\&\leq\left(\frac{1}{2}\left(\frac{1}{\zeta}\left\langle|A|^{2r\lambda}x,x\right\rangle+\frac{1}{\gamma}\left\langle|A^*|^{2r(1-\lambda)}x,x\right\rangle^r\right)\right)^{\frac{1}{r}}.\end{aligned}$$

Hence,

$$|\langle Ax,x\rangle|^r\leq\frac{1}{2\mu}\left\langle(|A|^{2\lambda r}+|A^*|^{2(1-\lambda)r})x,x\right\rangle.$$

By taking supremum over $x\in\mathcal{H}$ with $\|x\|=1$, we get the desired relation. □

3. Inequalities for Sums and Products of Operators

In this section, we recall that some general result for the product of operators from [5].

If $A,B\in B(\mathcal{H})$, then

$$w(AB)\leq 4w(A)w(B).$$

If A is an isometry and $AB=BA$, or a unitary operator that commutes with another operator B, then

$$w(AB)\leq w(B),$$

(see [4, Corollary 2.5-3]). Dragomir in [1, Theorem 2] showed that for $A,B\in B(\mathcal{H})$, any $\lambda\in(0,1)$ and $r\geq 1$

$$|\langle Ax,By\rangle|^{2r}\leq\lambda\langle(A^*A)^{\frac{r}{\lambda}}x,x\rangle+(1-\lambda)\langle(B^*B)^{\frac{r}{1-\lambda}}y,y\rangle,\tag{3.1}$$

where $x,y\in\mathcal{H}$, with $\|x\|=\|y\|=1$.

Let $A,B\in B(\mathcal{H})$. The Schwarz inequality states that

$$|\langle Ax,By\rangle|^2\leq\langle Ax,Ax\rangle\langle By,By\rangle,\quad\text{for all } x,y\in\mathcal{H}.$$

We get the following refinements of inequality (3.1) for $r\geq 2$.

Lemma 3.1. *For $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$*

$$
\begin{aligned}
|\langle Ax, By\rangle|^{2r} \leq & \lambda\langle (A^*A)^{\frac{r}{\lambda}} x, x\rangle - \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle + (1-\lambda) \\
& \times \langle (B^*B)^{\frac{r}{1-\lambda}} y, y\rangle - (1-\lambda) \left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}} y, y\rangle\right|^r y, y\right\rangle, \qquad (3.2)
\end{aligned}
$$

for any $x, y \in \mathcal{H}$, with $\|x\| = \|y\| = 1$.

Proof. For any unit vectors $x, y \in \mathcal{H}$, we have

$$
\begin{aligned}
|\langle (B^*A)x, y\rangle|^2 \leq & \langle (A^*A)x, x\rangle\langle (B^*B)y, y\rangle \quad \text{(by Schwarz inequality)} \\
= & \langle ((A^*A)^{\frac{1}{\lambda}})^{\lambda} x, x\rangle\langle ((B^*B)^{\frac{1}{1-\lambda}})^{1-\lambda} y, y\rangle \\
\leq & \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle^{\lambda}\langle (B^*B)^{\frac{1}{1-\lambda}} y, y\rangle^{1-\lambda} \quad \text{(by Lemma 2.3)} \\
\leq & (\lambda\langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle^r + (1-\lambda)\langle (B^*B)^{\frac{1}{1-\lambda}} y, y\rangle^r)^{\frac{1}{r}} \quad \text{(by Lemma 2.1)} \\
\leq & \Bigg(\lambda\langle (A^*A)^{\frac{r}{\lambda}} x, x\rangle - \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle \\
& + (1-\lambda)\langle (B^*B)^{\frac{r}{1-\lambda}} y, y\rangle \\
& - (1-\lambda) \left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}} y, y\rangle\right|^r y, y\right\rangle\Bigg)^{\frac{1}{r}} \quad \text{(by Lemma 2.4)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
|\langle Ax, By\rangle|^{2r} \leq & \lambda\langle (A^*A)^{\frac{r}{\lambda}} x, x\rangle - \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle \\
& + (1-\lambda)\langle (B^*B)^{\frac{r}{1-\lambda}} y, y\rangle \\
& - (1-\lambda) \left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}} y, y\rangle\right|^r y, y\right\rangle,
\end{aligned}
$$

for any $x, y \in \mathcal{H}$, with $\|x\| = \|y\| = 1$. $\square$

Theorem 3.1. *Let $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$. Then*

$$\|B^*A\|^{2r} \leq \lambda\|(A^*A)^{\frac{r}{\lambda}}\| + (1-\lambda)\|(B^*B)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1}\zeta(x) - \inf_{\|y\|=1}\zeta(y), \qquad (3.3)$$

where

$$
\begin{aligned}
\zeta(x) = & \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle, \\
\zeta(y) = & (1-\lambda) \left\langle \left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}} y, y\rangle\right|^r y, y\right\rangle.
\end{aligned}
$$

In addition,

$$w^{2r}(B^*A) \leq \|\lambda(A^*A)^{\frac{r}{\lambda}} + (1-\lambda)(B^*B)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1}\gamma(x), \qquad (3.4)$$

where

$$\gamma(x) = \left\langle \left(\lambda \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r + (1-\lambda)\left|(B^*B)^{\frac{1}{1-\lambda}} - \langle (B^*B)^{\frac{1}{1-\lambda}} x, x\rangle\right|^r\right) x, x\right\rangle.$$

Proof. By taking supremum over $x, y \in \mathcal{H}$ with $\|x\| = \|y\| = 1$ in inequality (3.2), we get the required inequality (3.3).

Putting $x = y$ in inequality (3.2), we obtain the numerical radius inequality (3.4). $\square$

Corollary 3.1. *For $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$, the following inequalities hold:*

$$|\langle Ax, y\rangle|^{2r} \leq \lambda\langle (A^*A)^{\frac{r}{\lambda}} x, x\rangle - \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle + (1-\lambda),$$

$$\begin{aligned}|\langle A^2x, y\rangle|^{2r} \leq{}& \lambda\langle (A^*A)^{\frac{r}{\lambda}} x, x\rangle - \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle \\ &+ (1-\lambda)\langle (AA^*)^{\frac{r}{1-\lambda}} y, y\rangle - (1-\lambda) \left\langle \left|(AA^*)^{\frac{1}{1-\lambda}} - \langle (AA^*)^{\frac{1}{1-\lambda}} y, y\rangle\right|^r y, y\right\rangle,\end{aligned}$$

where $x, y \in \mathcal{H}$, $\|x\| = \|y\| = 1$.

Corollary 3.2. *For $A, B \in B(\mathcal{H})$, $0 < \lambda < 1$ and $r \geq 2$, the following norm inequalities and numerical radius inequalities hold:*

(i) $\|A\|^{2r} \leq \lambda\|(A^*A)^{\frac{r}{\lambda}}\| + (1-\lambda) - \inf_{\|x\|=1} \zeta(x)$;

(ii) $\|A^2\|^{2r} \leq \lambda\|(A^*A)^{\frac{r}{\lambda}}\| + (1-\lambda)\|(AA^*)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1} \zeta(x) - \inf_{\|y\|=1} \zeta(y)$;

(iii) $w^{2r}(A) \leq \|\lambda(A^*A)^{\frac{r}{\lambda}} + (1-\lambda)I\| - \inf_{\|x\|=1} \zeta(x)$, *where*

$$\begin{aligned}\zeta(x) ={}& \lambda \left\langle \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r x, x\right\rangle, \\ \zeta(y) ={}& (1-\lambda) \left\langle \left|(AA^*)^{\frac{1}{1-\lambda}} - \langle (AA^*)^{\frac{1}{1-\lambda}} y, y\rangle\right|^r y, y\right\rangle;\end{aligned}$$

(iv) $w^{2r}(A^2) \leq \|\lambda(A^*A)^{\frac{r}{\lambda}} + (1-\lambda)(AA^*)^{\frac{r}{1-\lambda}}\| - \inf_{\|x\|=1} \zeta(x)$, *where*

$$\zeta(x) = \left\langle \left(\lambda \left|(A^*A)^{\frac{1}{\lambda}} - \langle (A^*A)^{\frac{1}{\lambda}} x, x\rangle\right|^r + (1-\lambda) \left|(AA^*)^{\frac{1}{1-\lambda}} - \langle (AA^*)^{\frac{1}{1-\lambda}} x, x\rangle\right|^r\right) x, x\right\rangle.$$

We are going to establish a refinement of a numerical inequality for Hilbert space operators. We need the following lemmas. The first lemma is a generalization of the mixed Schwarz inequality.

Lemma 3.2. ([17, Lemma 2.1]). *Let $A \in B(\mathcal{H})$ and f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\langle Ax, y\rangle| \leq \|f(|A|)x\|\|g(|A^*|)y\|,$$

for all $x, y \in H$.

The next lemma is a consequence of the convexity of the function $f(t) = t^r$, $r \geq 1$.

Lemma 3.3. ([17, Lemma 2.3]). *Let a_i, $i = 1, 2, \ldots, n$, be positive real numbers. Then*

$$\left(\sum_{i=1}^n a_i\right)^r \leq n^{r-1} \sum_{i=1}^n a_i^r, \quad \text{for } r \geq 1.$$

The following theorem is a generalization of the inequalities (1.3) and (1.4).

Theorem 3.2. ([17, Lemma 2.5]). *Let $A_i, X_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2}\left\|\sum_{i=1}^{n}((B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r)\right\|, \quad r \geq 1.$$

We refine the above inequality for $r \geq 1$ by applying a refinement of the Hölder-McCarthy inequality. To achieve our next result, we utilize the strategy of [17, Lemma 2.5].

Theorem 3.3. *Let $A_i, X_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, be invertible operators and let f and g be nonnegative functions on $[0, \infty)$ which are continuous and satisfy in $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then, for all $r > 1$,*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}(B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$, $\zeta = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle (B_i^* f^2(|X_i|)B_i)^{\frac{1}{2}} x, x\right\rangle}{\left\langle (B_i^* f^2(|X_i|)B_i) x, x\right\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}$

and $\gamma = \inf\left\{1 + 2\,(r-1)\left(1 - \frac{\left\langle (A_i^* g^2(|X_i^*|)A_i)^{\frac{1}{2}} x, x\right\rangle}{\left\langle (A_i^* g^2(|X_i^*|)A_i) x, x\right\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}$.

Proof. For every unit vector $x \in H$, we have

$$\begin{aligned}
&\left|\left\langle\left(\sum_{i=1}^{n} A_i^* X_i B_i\right)x, x\right\rangle\right|^r = \left|\sum_{i=1}^{n}\langle (A_i^* X_i B_i)x, x\rangle\right|^r \\
\leq &\left(\sum_{i=1}^{n} |\langle A_i^* X_i B_i x, x\rangle|\right)^r = \left(\sum_{i=1}^{n} |\langle X_i B_i x, A_i x\rangle|\right)^r \\
\leq &\left(\sum_{i=1}^{n}\langle f^2(|X_i|)B_i x, B_i x\rangle^{\frac{1}{2}}\langle g^2(|X_i^*|)A_i x, A_i x\rangle^{\frac{1}{2}}\right)^r \\
&\quad \text{(by Lemma 3.2)} \\
\leq &n^{r-1}\sum_{i=1}^{n}\left\langle B_i^* f^2(|X_i|)B_i x, x\right\rangle^{\frac{r}{2}}\left\langle A_i^* g^2(|X_i^*|)A_i x, x\right\rangle^{\frac{r}{2}} \\
&\quad \text{(by Lemma 3.3)} \\
= &n^{r-1}\sum_{i=1}^{n}\left(\langle B_i^* f^2(|X_i|)B_i x, x\rangle^r\right)^{\frac{1}{2}}\left(\langle A_i^* g^2(|X_i^*|)A_i x, x\rangle^r\right)^{\frac{1}{2}} \\
\leq &\frac{n^{r-1}}{2}\left(\sum_{i=1}^{n}\left(\langle B_i^* f^2(|X_i|)B_i x, x\rangle^r + \langle A_i^* g^2(|X_i^*|)A_i x, x\rangle^r\right)\right) \\
&\quad \text{(by AM} - \text{GM)}
\end{aligned}$$

$$\leq \frac{n^{r-1}}{2}\left(\sum_{i=1}^{n}\left(\frac{1}{\zeta(x)}\langle (B_i^* f^2(|X_i|)B_i)^r x, x\rangle + \frac{1}{\gamma(x)}\langle (A_i^* g^2(|X_i^*|)A_i)^r x, x\rangle\right)\right)$$
(by (2.5))
$$\leq \frac{n^{r-1}}{2\mu}\sum_{i=1}^{n}\left\langle\left((B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r\right)x, x\right\rangle$$
$$= \frac{n^{r-1}}{2\mu}\left\langle\sum_{i=1}^{n}\left((B_i^* f^2(|X_i|)B_i)^r + (A_i^* g^2(|X_i^*|)A_i)^r\right)x, x\right\rangle.$$

Therefore, by taking supremum over $x \in \mathcal{H}$ with $\|x\| = 1$, we have the desired relation. ☐

If we assume that $f(t) = t^\lambda$ and $g(t) = t^{1-\lambda}$, $0 < \lambda < 1$, in Theorem 3.3, then we get the following corollary.

Corollary 3.3. *Let $A_i, X_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, be invertible operators, $r > 1$ and $0 < \lambda < 1$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}(B_i^*|X_i|^{2\lambda}B_i)^r + (A_i^*|X_i^*|^{2(1-\lambda)}A_i)^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle\left(B_i^*|X_i|^{2\lambda}B_i\right)^{\frac{1}{2}}x, x\right\rangle}{\left\langle\left(B_i^*|X_i|^{2\lambda}B_i\right)x : x\right\rangle^{\frac{1}{2}}}\right) : \ \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle\left(A_i^*|X_i|^{2(1-\lambda)}A_i\right)^{\frac{1}{2}}x, x\right\rangle}{\left\langle\left(A_i^*|X_i|^{2(1-\lambda)}A_i\right)x, x\right\rangle^{\frac{1}{2}}}\right) : \ \|x\| = 1\right\}.$$

In particular,

$$w\left(\sum_{i=1}^{n} A_i^* X_i B_i\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}(B_i^*|X_i|B_i + A_i^*|X_i^*|A_i)\right\|.$$

Setting $A_i = B_i = I$, $i = 1, 2, \cdots, n$, in Theorem 3.3, the following inequalities for sums of operators are obtained.

Corollary 3.4. *Let $X_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, be invertible operators and f and g be continuous nonnegative functions on $[0, \infty)$, such that $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then, for $r > 1$,*

$$w^r\left(\sum_{i=1}^{n} X_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n}(f^{2r}(|X_i|) + g^{2r}(|X_i^*|))\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\langle f(|X_i|)x, x\rangle}{\langle f^2(|X_i|)x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\langle g(|X_i^*|)x, x\rangle}{\langle g^2(|X_i^*|)x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

In particular,

$$w^r\left(\sum_{i=1}^{n} X_i\right) \le \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |X_i|^{2\lambda r} + |X_i^*|^{2(1-\lambda)r}\right\|, \quad \lambda \in (0,1),$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\left\langle |X_i|^{\lambda}x, x\right\rangle}{\langle |X_i|^{2\lambda}x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\left\langle |X_i^*|^{(1-\lambda)}x, x\right\rangle}{\langle |X_i^*|^{2(1-\lambda)}x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

If $\lambda = \frac{1}{2}$ in above inequality, we get

$$w^r\left(\sum_{i=1}^{n} X_i\right) \le \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |X_i|^{r} + |X_i^*|^{r}\right\|, \quad r \ge 1,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\left\langle |X_i|^{\frac{1}{2}}x, x\right\rangle}{\langle |X_i|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\left\langle |X_i^*|^{\frac{1}{2}}x, x\right\rangle}{\langle |X_i^*|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

Letting $n = 1$ in inequality (3.3), we obtain

$$w^r(X) \le \frac{1}{2\mu}\left\||X|^r + |X^*|^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\left\langle |X|^{\frac{1}{2}}x, x\right\rangle}{\langle |X|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2\left(r-1\right)\left(1 - \frac{\left\langle |X^*|^{\frac{1}{2}}x, x\right\rangle}{\langle |X^*|x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

Next, we present some numerical radius inequalities for products of operators. Put $X_i = I$, $i = 1, 2, \ldots, n$, in Theorem 3.3, to get the following.

Corollary 3.5. *Let $A_i, B_i \in B(\mathcal{H})$, $i = 1, 2, \ldots, n$, be invertible operators and $r \geq 1$. Then*

$$w^r\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{n^{r-1}}{2\mu}\left\|\sum_{i=1}^{n} |B_i|^{2r} + |A_i|^{2r}\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2(r-1)\left(1 - \frac{\langle |B_i| x, x\rangle}{\langle |B_i| x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2(r-1)\left(1 - \frac{\langle |A_i| x, x\rangle}{\langle |A_i| x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

In particular,

$$w\left(\sum_{i=1}^{n} A_i^* B_i\right) \leq \frac{1}{2}\left\|\sum_{i=1}^{n}(B_i^* B_i + A_i^* A_i)\right\|.$$

Remark 3.1. If we set $n = 1$ in Corollary 3.5, then

$$w^r(A^* B) \leq \frac{1}{2\mu}\left\|(B^* B)^r + (A^* A)^r\right\|,$$

where $\mu = \min\{\zeta, \gamma\}$,

$$\zeta = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle (B^* B)^{\frac{1}{2}} x, x\right\rangle}{\langle (B^* B) x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\},$$

$$\gamma = \inf\left\{1 + 2(r-1)\left(1 - \frac{\left\langle (A^* A)^{\frac{1}{2}} x, x\right\rangle}{\langle (A^* A) x, x\rangle^{\frac{1}{2}}}\right) : \|x\| = 1\right\}.$$

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APPROXIMATE SOLUTION OF BRATU DIFFERENTIAL EQUATIONS USING TRIGONOMETRIC BASIC FUNCTIONS

BAHRAM AGHELI

ABSTRACT. In this paper, I have proposed a method for finding an approximate function for Bratu differential equations (BDEs), in which trigonometric basic functions are used. First, by defining trigonometric basic functions, I define the values of the transformation function in relation to trigonometric basis functions (TBFs). Following that, the approximate function is defined as a linear combination of trigonometric base functions and values of transform function which is named trigonometric transform method (TTM), and the convergence of the method is also presented. To get an approximate solution function with discrete derivatives of the solution function, we have determined the approximate solution function which satisfies in the Bratu differential equations (BDEs). In the end, the algorithm of the method is elaborated with several examples. In one example, I have presented an absolute error comparison of some approximate methods.

1. INTRODUCTION

A problem of the non-linear eigenvalue problem in n dimensions is the Bratu differential equations (BDEs) as follows [13]

$$\sum_{i=1}^{n}\left(\frac{\partial}{\partial t_i}\right)^2 \Phi\left(t_1, t_2, \ldots, t_n\right)+\lambda \exp(\Phi\left(t_1, t_2, \ldots, t_n\right))=0, \tag{1.1}$$

in which $|x_i| \leq 1$ for $i=1,2,\ldots,n$, with the following boundary conditions as $|x_i|=1$,

$$\Phi\left(t_1, t_2, \ldots, t_n\right)=0. \tag{1.2}$$

The main objective in this paper is to offer a simple method in which it is possible to apply trigonometric transform method (TTM) to tackle with the one-dimensional

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(1D) BDEs of the following form

$$u''(t) + \lambda \exp(u(t)) = 0, \quad 0 < t \leq T, \tag{1.3}$$

$$u(0) = u_0, \quad u_t(0) = u'_0, \tag{1.4}$$

where $\lambda > 0$ and $t \in \mathbb{R}$ are constant functions (see [12, 23]).

The analytic solution for BDEs is presented as follows:

$$u(t) = \log \left(\frac{\cosh\left(\frac{\phi}{2}\left(t - \frac{1}{2}\right)\right)}{\cosh\left(\frac{\phi}{4}\right)} \right)^{-2},$$

in which ϕ is the solution of $\phi = \sqrt{2\lambda}\cosh\left(\frac{\phi}{4}\right)$ (see [12, 23]). Whereas $\lambda_\epsilon = 3.513830719$, the BDEs has

- one solutions when $\lambda = \lambda_\epsilon$;
- two solutions if $\lambda < \lambda_\epsilon$;
- no solution when $\lambda > \lambda_\epsilon$.

Researchers and scholars are requested to check papers that have been introduced to get a better grasp of thoroughgoing introduction about BDEs and its history in [10, 18].

On the importance and motivation for Bratu differential equation, it should noted that it has a key role in many of the physical phenomena, chemical models and other sciences. Such applications include the model of thermal reaction process, the fuel ignition model of the thermal combustion theory, the Chandrasekhar model of the expansion of the universe, the radiative heat transfer nanotechnology and the chemical reaction theory (see [9, 10, 12, 18]).

As another instance, mathematical modeling in chemistry for the electro-spinning process is related to BDEs via thermo-electro-hydrodynamics balance equations. Colantoni and his co-author in [5] represented a model that is the mono-dimensional Bratu equation as follows:

$$u''(t) - \lambda \exp(u(t)) = 0, \tag{1.5}$$

featuring $\lambda = \frac{18\, E^2 (I - r^2 k\, E)^2}{\rho^2\, r^4}$, in which

- r is the radius of the jet at axial coordinate X in the Figure 1;
- I is the electrical current intensity;
- E is the electric area in the axial direction;
- ρ is the material density;
- k is a fixed value which is only dependent on temperature with regard to incompressible polymer.

Many researchers have used numerical methods for the purpose of solving the BDEs. We can refer to a number of familiar methods, including Homotopic perturbation method [8], Finite difference [19], Optimal homotopy asymptotic method [6], Wavelet method [17], Laplace transform decomposition method [15], B-splines method [4], Variational iteration technique [7], Adomian decomposition method [23], Differential

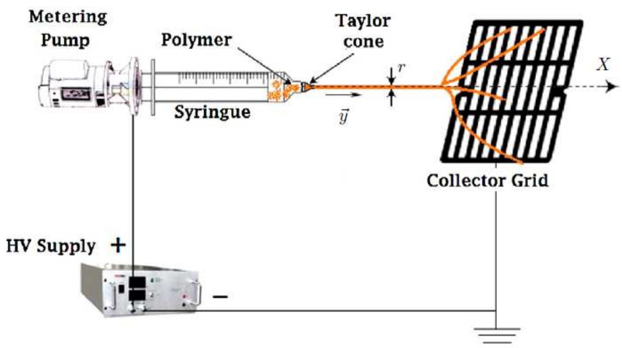

FIGURE 1. Electro-spinning process setup.

quadrature method [21], Lie-group shooting method [1], Reproducing kernel Hilbert space method [2], Pseudo-spectral collocation method [3] and [11, 12, 14, 16, 22].

This paper is organized as what follows: in Section 2, discretization of the derivative is given. In Section 3, we have expressed the trigonometric Basic functions (TBFs). In Section 4, a description of the new approach that is named trigonometric transform method (TTM) is presented. Some numerical examples are offered in Section 5. And conclusions are drawn in Section 6.

2. Discretization of the Derivative

In this section, we introduce discretization of the derivative of a function. The approximation of derivatives by forward differences is one of the most basic tools in finite difference methods for the approximate solution of differential equations, especially initial value problems. The n-th order forward difference is given by

$$u^{(n)}(t) \approx \frac{1}{h^n} \sum_{i=0}^{n} (-1)^i \binom{n}{i} u\left((n-i)h + t\right), \quad n \in \mathbb{N}.$$

Depending on the application, the spacing h may be a variable or a fixed. In this paper, we consider $\tau = t_{j+1} - t_j$ and $t_j = a + j\,\tau$ for $j = 0, 1, 2, \ldots$. For second order derivative we have:

$$u''(t_{k+1}) \approx \frac{1}{h^2}(u(t_{j+1}) - 2u(t_j) + u(t_{j-1})), \tag{2.1}$$

in which $u(t_0)$ and $u'(t_0)$ are known and $u(t_{-1}) = u(0) - \tau\, u_t(0)$.

3. Trigonometric Basic Functions (TBFs)

In this section, we introduce the trigonometric basis functions and properties that are used in the main sections of the paper to approximate the function of the solution.

Definition 3.1. Presuming that for $n \geq 1$, $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be specified nodes, we express that basic functions $T_0, T_1, \ldots, T_n$ are defined on $[a,\, b]$

with their trigonometric functions $T_0(t), T_1(t), \dots, T_n(t)$, as follows:

$$T_0(t) = \begin{cases} 0.5\left(1+\cos\frac{\pi}{h_0}(t-t_0)\right), & t_0 \le t \le t_1, \\ 0, & \text{otherwise}, \end{cases}$$

$$(3.1)\quad T_k(t) = \begin{cases} 0.5\left(1+\cos\frac{\pi}{h_{k-1}}(t-t_k)\right), & t_{k-1} \le t \le t_k, \\ 0.5\left(1+\cos\frac{\pi}{h_k}(t-t_k)\right), & t_k \le t \le t_{k+1}, \quad k = 1,2,3,\dots,n-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$T_n(t) = \begin{cases} 0.5\left(1+\cos\frac{\pi}{h_{n-1}}(t-t_n)\right), & t_{n-1} \le t \le t_n, \\ 0, & \text{otherwise}, \end{cases}$$

in which $h_k = t_{k+1} - t_k$ for $k = 0, 1, \dots, n-1$.

Remark 3.1. The trigonometric functions introduced in Definition 3.1 are the trigonometric basis functions (TBFs) in which the following properties are satisfied.

(1) T_k of $[a,b]$ to $[0,1]$ is continuous, $\sum\limits_{k=0}^{n} T_k(t) = 1$ for all $t \in [a,b]$ and $T_k(t_k) = 1$, $k = 0, 1, 2, \dots, n$.
(2) $T_k(t) = 0$ if $t \notin (t_{k-1}, t_{k+1})$, for $k = 1, 2, \dots, n-1$, $T_0(t) = 0$ if $t \notin (t_0, t_1)$ and $T_n(t) = 0$ if $t \notin (t_{n-1}, t_n)$.
(3) On subinterval $[t_{k-1}, t_{k+1}]$ for $k = 1, 2, \dots, n-1$, $T_k(t)$, certainly is an increasing function on $[t_{k-1}, t_k]$ and decreasing function on $[t_k, t_{k+1}]$. Basic functions are called uniform as long as $t_{k+1} - t_k = h = \frac{b-a}{n}$ and two additional properties coincide.
(4) $T_k(t_k - t) = T_k(t_k + t)$, for all $t \in [0, h]$ and $k = 1, 2, \dots, n-1$;
(5) $T_k(t) = T_{k-1}(t-h)$ and $T_{k+1}(t) = T_k(t-h)$, for $k = 1, 2, \dots, n-1$ and $t \in [t_k, t_{k+1}]$.

Lemma 3.1 ([20])**.** *Consider $n \ge 2$, $T_0, T_1, \dots, T_n$, be the* TBFs *which builds on $[a,b]$. Therefore,*

$$\int_{t_0}^{t_1} T_1(t)dt = \int_{t_{n-1}}^{t_n} T_n(t)dt = \frac{h}{2} \tag{3.2}$$

and

$$\int_{t_{k-1}}^{t_{k+1}} T_k(t)dt = h, \tag{3.3}$$

for $k = 1, 2, \dots, n-1$, in which h is the distance between each of the two neighboring nodes.

Definition 3.2. Let f be a function belonging to $C[a,b]$ and T_i, $i = 0, 1, \dots, n$, be the TBFs which buildup on $[a,b]$. We define the F_k that is the transform of function f on $[a,b]$ with respect to basic functions T_k given by

$$F_k = \frac{\int_a^b f(t)T_k(t)dt}{\int_a^b T_k(t)dt}, \quad k = 0, 1, 2, \dots, n. \tag{3.4}$$

Definition 3.3. Let f be a function belonging to $C[a,b]$ and T_i, $i = 0, 1, \dots, n$, be the TBFs which buildup on $[a,b]$ and F_k be transform of function f on $[a,b]$ with respect to basic functions T_k. Then

$$f_n(t) = \sum_{k=0}^{n} F_k T_k(t)$$

is approximate of function f on $[a,b]$ with respect to TBFs.

Theorem 3.1 (Convergence). *Let f be a uniformly continuous function on $[a,b]$. Thus, for any $\epsilon > 0$, there exists n_ϵ such that for all $n \geq n_\epsilon$*

$$|f(t) - f_{n_\epsilon}(t)| < \epsilon. \tag{3.5}$$

Proof. f is a uniformly continuous function on $[a,b]$. Therefore,

$$(\forall \epsilon > 0)\ (\exists \delta = \delta(\epsilon))\ (|x - t| < \delta \Rightarrow |f(x) - f(t)| < \epsilon \ \ (0 < \delta < \epsilon)).$$

For all $\epsilon > 0$, we have

$$|f(t) - f_n(t)| = \left| \sum_{i=0}^{n} T_i(t) f(t) - \sum_{i=0}^{n} F_i T_i(t) \right| \leq \sum_{i=0}^{n} T_i(t) \left| f(t) - F_i \right| < \epsilon.$$

It is sufficient to show that $|f(t) - F_i| < \epsilon$. Let x, $t \in [x_{i-1}, x_{i+1}]$, $i = 1, 2, \dots, n-1$, so that we can evaluate

$$|f(x) - F_i| = \left| f(x) - \frac{\int_a^b f(t) T_i(t) dt}{\int_a^b T_i(t) dt} \right| \leq \frac{\int_{x_{i-1}}^{x_{i+1}} T_i(t) \left| f(x) - f(t) \right| dt}{\int_{x_{i-1}}^{x_{i+1}} T_i(t) dt} < \epsilon,$$

if and only if

$$\delta < 2h < \epsilon \quad \text{or} \quad h < \frac{\epsilon}{2}.$$

Regarding $h = \frac{b-a}{n}$, it is sufficient that $n_\epsilon > \frac{2(b-a)}{\epsilon}$. □

For description of fractional derivative, we have the following proposition.

Proposition 3.1. *With substituting $f_n(t) = \sum_{k=0}^{n} F_k\, T_k(t)$ in (2.1), we will have the next equation for $k = 0, 1, 2, \dots, n-1$:*

$$f_n''(t_{k+1}) \approx \frac{1}{h^2}(F_{j+1} - 2F_j + F_{j-1}). \tag{3.6}$$

4. Description of the New Approach

Let solution of (1.3) be continuous on $[0,b]$. To gain approximate solution of $u(x)$, we divide $[0,b]$ to n equal partition with step length τ:

$$t_0 = 0, \quad t_i = t_0 + i\tau, \quad i = 0, 1, \dots, n, \tau = \frac{b}{n}. \tag{4.1}$$

Considering the trigonometric functions with regard to Definition 3.1 on $[0,b]$ and Definition 3.3, we can gain approximate function $u(x)$ by $u_n(x) = \sum_{k=0}^{n} U_k\, T_k(t)$. It is evident that for calculating $u_n(t)$, $t \in [0,b]$, we should calculate U_k, $k = 0,1,2,\dots,n$. In order to gain the approximate solution of the problem (1.3), $u_n(t)$ for points $t_0, t_1, \dots, t_n$ must be satisfied in (1.3). Due to the boundary conditions (1.4), $u_n(t_0) := u(t_0) = u_0$ and for other points $t_1, t_2, \dots, t_n$, we have

$$u_n''(t_{k+1}) + \lambda \exp(u_n(t_{k+1})) = 0, \quad k = 0,1,2,\dots,n-1, \tag{4.2}$$

in which $m - 1 < \rho \le m$ and $m \in \mathbb{Z}^+$.

Using (3.6) and (4.2) converts to the following form for $k = 0,1,2\dots,n-1$:

$$\frac{1}{h^2}(U_{k+1} - 2\,U_k + U_{k-1}) + \lambda \exp(U_n(t_{k+1})) = 0, \tag{4.3}$$

where $U_0 = u(0)$ and $U_{-1} = u(0) - u'(0)$ are known initial conditions.

Now, using the boundary condition, we can calculate $U_1, U_2, \dots, U_n$ by the obtained recursive equation (4.3) and then gain the approximate solution $u(t) \approx u_n(t)$ for (1.3).

In order to gain approximation of BDEs, an algorithm by this method is offered in the subsequent algorithm.

Algorithm 1: An algorithm for approximation of BDEs
Step 1: Input n and b.
Step 2: Set $\tau \leftarrow \frac{b}{n}$.
Step 3: Locate $t_k \leftarrow k\,\tau,\ k = 0,1,2,\dots,n$.
Step 4: Choose TBFs $T_k(t)$ toward $k = 0,1,2,\dots,n$.
Step 5: Set recursive equations

$$\frac{1}{h^2}(U_{k+1} - 2\,U_k + U_{k-1}) + \lambda \exp(U_n(t_{k+1})) = 0,$$

where $U_0 = u(0)$ and $U_{-1} = u(0) - \tau u'(0)$.
Step 6: Calculate every U_k, $k = 1,2,\dots,n$, of an equation of degree one.
Step 7: The approximate solution is

$$u_n(t) \approx \sum_{k=0}^{n} U_k T_k(t).$$

5. Examples

Now that it is easier to understand trigonometric transform, a number of examples will be given in this section and then will be calculated. These examples include BDEs. In all these examples, software *Mathematica* 11 has been used for calculations and graphs.

Example 5.1. We propose the BDEs for the first example [23]:

$$u''(t) - 2\exp(u(t)) = 0, \quad 0 \le t \le 1, \tag{5.1}$$

with the precise solution $u(t) = \log((\cos t)^{-2})$ and the primary conditions:

$$u(0) = 0, \quad u'(0) = 0. \tag{5.2}$$

Following the TTM, according to what was formulated and presented in section 4 for (5.1)–(5.2), we can calculate $U_1, U_2, \ldots, U_n$, and then gain the approximate solution $u_n(t)$ of (5.1).

In Table 1, we can see the estimated solutions for Eq.(5.1), which is derived for various values of n applying TTM. Also, the estimated and approximate solutions are illustrate in Figure 2.

TABLE 1. Approximate result of Example 5.1 with various values of n.

t	TTM $n=50$	$n=500$	$n=1000$	$n=1500$	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.0543317	0.0407728	0.0404703	0.0402949	0.0402695
0.4	0.193714	0.165493	0.164871	0.164416	0.164458
0.6	0.42896	0.385508	0.384559	0.383323	0.38393
0.8	0.799043	0.725417	0.723832	0.722438	0.722781

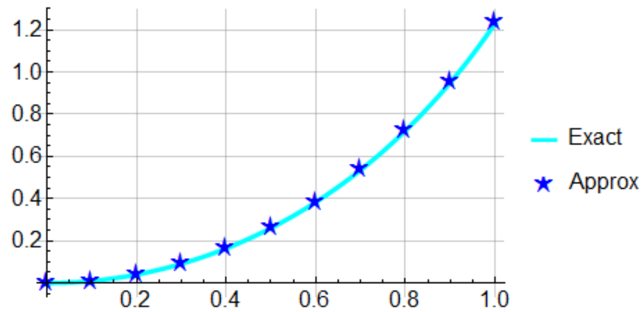

FIGURE 2. Figure for Example 5.1 exact and the approximation solutions.

Noteworthy in the values obtained in the Table 1 is that by increasing the amount n, a more accurate answer for (5.1) can be achieved.

Example 5.2. Consider the BDEs for the second example [23]:

$$u''(t) + \pi^2 \exp(-u(t)) = 0, \quad 0 \le t \le 1, \tag{5.3}$$

given that the primary conditions:

$$u(0) = 0, \quad u'(0) = \pi. \tag{5.4}$$

The unknown coefficient $U_1, U_2, \ldots, U_n$ with due attention to the TTM, according to Section 4 for (5.3)–(5.4) are calculated.

In Table 2 and in Figure 3, we can view the precise and approximate answers for $n = 1500$ through applying TTM.

The approximate solution obtained by the proposed method corresponds to the precise solution $u(t) = \log(1 + \sin(\pi t))$.

In Figure 3, we can see the estimated solutions toward $n = 1500$, which is derived for various value of t applying TTM.

Table 2. Approximate result of example 5.2.

t	TTM	Exact	Absolute Error	Relative Error
0.2	0.462127	0.46234	212.789 $\times 10^{-6}$	460.455 $\times 10^{-6}$
0.4	0.66794	0.668371	430.849 $\times 10^{-6}$	645.042 $\times 10^{-6}$
0.6	0.667754	0.668371	616.549 $\times 10^{-6}$	923.317 $\times 10^{-6}$
0.8	0.46142	0.46234	920.306 $\times 10^{-6}$	1.99451 $\times 10^{-3}$

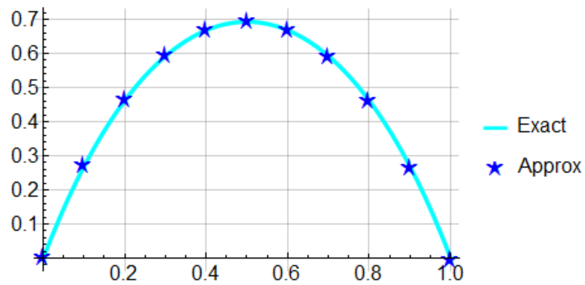

Figure 3. Comparison of the approximate solution (5.3) with exact solution for $n = 1500$.

Example 5.3. We offer the BDEs for the third example [23]:

$$u''(t) - \pi^2 \exp(u(t)) = 0, \quad 0 \le t \le 1, \tag{5.5}$$

including the primary conditions:

$$u(0) = 0, \quad u'(0) = -\pi. \tag{5.6}$$

It can be seen in Table 3 and Figure 4 that solving equations with approximate expression is calculated and displayed for $n = 1500$ and various values of t.

Table 3. Approximate result of Example 5.3.

t	TTM	Exact	Absolute Error	Relative Error
0.2	0.451242	0.451272	30.7122 $\times 10^{-6}$	68.0615 $\times 10^{-6}$
0.4	-0.227657	-0.226202	1.45505 $\times 10^{-3}$	6.39141 $\times 10^{-3}$
0.6	-0.576992	-0.573173	3.81849 $\times 10^{-3}$	6.61792$\times 10^{-3}$
0.8	-0.699629	-0.69232	7.30951 $\times 10^{-3}$	10.4477 $\times 10^{-3}$

In Table 4, we can see the estimated solutions toward $n = 1500$, which is derived for various values of t applying TTM.

Toward $n = 1500$, the solution that we have gained is in accordance with the precise solution $u(t) = \log\left(\frac{1}{1-\sin(1-\pi t)}\right)$.

Example 5.4. Consider the BDEs [1]:

$$u''(t) + 2\exp(u(t)) = 0, \quad 0 \le t \le 1, \tag{5.7}$$

supposing that the primary conditions:

$$u(0) = 0, \quad u'(0) = 0. \tag{5.8}$$

TABLE 4. Absolute error comparison of Example 5.4.

								TTM
t	BNM	NPS	LTM	DM	BSM	LGSM	SCM	n=1500
0.1	1.91×10^{-14}	9.71×10^{-9}	2.13×10^{-3}	1.52×10^{-2}	1.72×10^{-5}	4.03416×10^{-6}	$6{:}88 \times 10^{-4}$	0
0.3	1.17×10^{-13}	1.98×10^{-8}	6.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}	5.22122×10^{-6}	$8{:}21 \times 10^{-4}$	169.624×10^{-6}
0.5	1.88×10^{-13}	2.60×10^{-8}	9.60×10^{-3}	6.98×10^{-3}	5.56×10^{-5}	1.4554×10^{-8}	$8{:}60 \times 10^{-4}$	341.417×10^{-6}
0.7	1.16×10^{-13}	1.98×10^{-8}	1.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}	5.19455×10^{-6}	$8{:}21 \times 10^{-4}$	424.587×10^{-6}
0.9	1.90×10^{-14}	9.71×10^{-9}	1.09×10^{-3}	1.52×10^{-3}	1.72×10^{-5}	4.01345×10^{-6}	$6{:}88 \times 10^{-4}$	445.899×10^{-6}

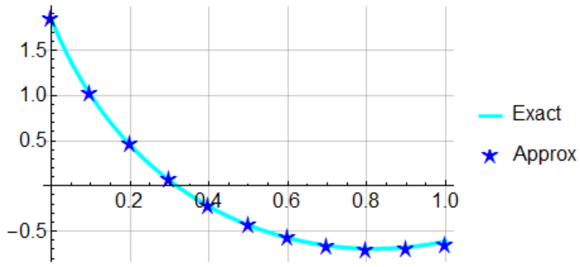

FIGURE 4. Comparison of the approximate solution (5.5) with exact solution for $n = 1500$.

The unknown coefficient $U_1, U_2, \ldots, U_n$, with due attention to the TTM, according to Section 4 for (5.7)–(5.8) are calculated.

Table 4 illustrates an absolute error comparison of the TTM and approximate methods: Block Nyström method (BNM) [12], Non-polynomial spline (NPS) [11], Laplace transform method (LTM) [18], Decomposition method (DM) [16], B-splines method (BSM) [4], Lie-group shooting method (LGSM) [1] and Sinc-collocation method (SCM).

In Figure 5, we can see the estimated solutions toward $n = 1500$, which is derived for various value of t applying TTM.

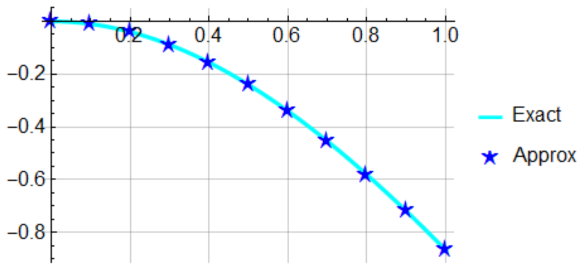

FIGURE 5. Comparison of the approximate solution (5.7) with exact solution for $n = 1500$.

Noteworthy in the values obtained in the last column Table 4 is that by increasing the amount n, a more accurate answer for (5.7) can be achieved.

6. CONCLUSION

I have proposed a method for finding an approximate function of Bratu differential equations (BDEs), in which TTM are used. All examples with absolute and relative errors show that we have favorably applied trigonometric transform method TTM to obtain approximate solution of the BDEs. The obtained solutions that are very close analytical solutions indicate that a little iteration of TTM will result in some useful solutions. As the result seems necessary from the authors' point of view, the suggested technique has the potentials to be practical in solving other similar ordinary differential equations of integer orders and partial differential equations of non integer orders.

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DENSITY PROBLEMS IN SOBOLEV'S SPACES ON TIME SCALES

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Abstract. In this paper, we present a generalization of the density some of the functional spaces on the time scale, for example, spaces of rd-continuous function, spaces of Lebesgue Δ-integral and first-order Sobolev's spaces.

1. Introduction

The theory of time scales, which has recently received a lot of attention, was originally introduced by Stefan Hilger in his Ph.D. Thesis in 1988 in order to unify, extend and generalize continuous and discrete analysis (see Hilger [4]).

Recently, the Lebesgue Δ-integral has been introduced by Bohner and Guseinov in [2, Chapter 5]. For the fundamental relationship between Riemann and Lebesgue Δ-integrals see A. Cabada, D. Vivero [3]. The first study Sobolev's spaces on time scales R. Agarwal et al. (see [7]).

In this paper, we study the density relationship between some of the functional spaces on the time scale, for example, spaces of rd-continuous function, spaces of Lebesgue Δ-integral and first-order Sobolev's spaces.

2. Preliminaries

We will briefly recall some basic definitions and facts from time scale calculus that we will use in the sequel.

Let $\mathbb{T}$ be a closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\},$$

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(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_k := \mathbb{T} - \{m\}$, otherwise, set $\mathbb{T}_k = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^k := \mathbb{T} - \{M\}$, otherwise, set $\mathbb{T}^k = \mathbb{T}$.

Definition 2.1 ([1])**.** The function $\varphi : \mathbb{T} \to \mathbb{R}$ will be called rd-continuous provided it is continuous at each right-dense point and has a left-sided limit at each point, we write $\varphi \in C_{rd}(\mathbb{T}) = C_{rd}(\mathbb{T}, \mathbb{R})$.

Definition 2.2 ([1])**.** Assume $\varphi : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^k$. Then we define φ^{Δ} to be the number (provided it exists), with the property that given any $\varepsilon > 0$, there is a neighbourhood U of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$) for some $\delta > 0$ such that

$$|\varphi(\sigma(t)) - \varphi(s) - \varphi^{\Delta}(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

We call φ^{Δ} the delta (or Hilger) derivative of φ at t.

Lemma 2.1 ([3])**.** *The set of all right-scattered points of* $\mathbb{T}$ *is at most countable, that is, there are* $J \subset N$ *and* $\{t_j\}_{j \in J} \subset \mathbb{T}$ *such that*

$$\mathcal{R} := \{t \in \mathbb{T}, \sigma(t) > t\} = \{t_j\}_{j \in J}.$$

In order to do this, given a function $\varphi : \mathbb{T} \longrightarrow \overline{\mathbb{R}}$, we need an auxiliary function which extends φ to the interval $[a, b]$ defined as

$$\widetilde{\varphi}(t) := \begin{cases} \varphi(t), & \text{if } t \in \mathbb{T}, \\ \varphi(t_j), & \text{if } t \in (t_j, \sigma(t_j)) \text{ for all } j \in J. \end{cases} \tag{2.1}$$

Let $E \subset \mathbb{T}$, we define

$$J_E = \{j \in J : t_j \in E \cap \mathcal{R}\} \quad \text{and} \quad \widetilde{E} = E \cup \bigcup\nolimits_{j \in J_E} (t_j, \sigma(t_j)). \tag{2.2}$$

Proposition 2.1 ([3])**.** *Let* $A \subset \mathbb{T}$. *Then* A *is a* Δ*-measurable if and only if,* A *is Lebesgue measurable.*

In this case the following properties hold for every Δ*-measurable set* A*.*

1. *If* $b \notin A$*, then*

$$\mu_{\Delta}(A) = \mu_L(A) + \sum_{j \in J_A} \mu(t_j); \tag{2.3}$$

2. $\mu_{\Delta}(A) = \mu_L(A)$ *if and only if* $b \notin A$ *and* A *has no right-scattered point.*

Theorem 2.1 ([3])**.** *Let* $E \subset \mathbb{T}$ *be a* Δ*-measurable such that* $b \notin E$*, let* $\widetilde{E}$ *be the set defined in* (2.2)*, let* $\varphi : \mathbb{T} \to \overline{\mathbb{R}}$ *be a* Δ*-measurable function and* $\widetilde{\varphi} : [a, b] \to \overline{\mathbb{R}}$ *be the extension of* φ *to* $[a, b]$*. Then,* φ *is Lebesgue* Δ*-integrable on* E *if and only if* $\widetilde{\varphi}$ *is Lebesgue integrable on* $\widetilde{E}$ *and we have*

$$\int_E \varphi(t)\, \Delta t = \int_{\widetilde{E}} \widetilde{\varphi}(t)\, dt = \int_E \varphi(t)\, dt + \sum_{j \in J_E} \mu(t_j)\, \varphi(t_j). \tag{2.4}$$

We state some of their properties whose proofs can be found in [7, 8].

Definition 2.3 ([7])**.** Let $p \in [1, +\infty)$. Then, the set $L^p_\Delta(\mathbb{T}, \mathbb{R})$ is a Banach space together with the norm defined for every $\varphi \in L^p_\Delta(\mathbb{T}, \mathbb{R})$ as

$$\|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})} = \left(\int_{[a,b)\cap\mathbb{T}} |\varphi(s)|^p \Delta s\right)^{\frac{1}{p}}.$$

We denote by:

$$C^1(\mathbb{T}, \mathbb{R}) := \left\{\varphi : \mathbb{T} \to \mathbb{R} : \varphi \text{ is } \Delta\text{-differentiable on } \mathbb{T}^k \text{ and } \varphi^\Delta \in C\left(\mathbb{T}^k, \mathbb{R}\right)\right\},$$
$$C^1_{rd}(\mathbb{T}, \mathbb{R}) := \left\{\varphi : \mathbb{T} \to \mathbb{R} : \varphi \text{ is } \Delta\text{-differentiable on } \mathbb{T}^k \text{ and } \varphi^\Delta \in C_{rd}\left(\mathbb{T}^k, \mathbb{R}\right)\right\}.$$

Theorem 2.2 ([8])**.** *Let $p \in [1, \infty)$, then, we have the following properties:*

1. $C_{rd}(\mathbb{T}, \mathbb{R})$ *is dense in* $L^p_\Delta(\mathbb{T}, \mathbb{R})$*;*
2. $L^p_\Delta(\mathbb{T}, \mathbb{R})$ *is dense in* $L^1_\Delta(\mathbb{T}, \mathbb{R})$*;*
3. $C^1_{rd}(\mathbb{T}, \mathbb{R})$ *is dense in* $C(\mathbb{T}, \mathbb{R})$*.*

Theorem 2.3 ([7])**.** *Let $p \in [1, +\infty)$. The set $W^{1,p}(\mathbb{T}, \mathbb{R})$ is a Banach space together with the norm defined for every $\varphi \in W^{1,p}(\mathbb{T}, \mathbb{R})$ as*

$$\|\varphi\|_{W^{1,p}(\mathbb{T},\mathbb{R})} = \|\varphi\|_{L^p_\Delta(\mathbb{T},\mathbb{R})} + \left\|\varphi^\Delta\right\|_{L^p_\Delta(\mathbb{T},\mathbb{R})}.$$

3. Main Results

In this section, assume that $\mathbb{T}$ is bounded with $a := \min \mathbb{T}$ and $b := \max \mathbb{T}$ and for simplification, we note

$$[c, d)_\mathbb{T} = [c, d) \cap \mathbb{T} \quad \text{and} \quad [c, d]_\mathbb{T} = [c, d] \cap \mathbb{T}, \quad \text{for all } c, d \in \mathbb{T}.$$

Remark 3.1. $C(\mathbb{T}, \mathbb{R})$ and $C_{rd}(\mathbb{T}, \mathbb{R})$ are Banach spaces together with the norm defined by

$$\|\varphi\|_\infty := \sup_{t\in[a,b]_\mathbb{T}} |\varphi(t)|.$$

Set

$$I := \{j \in J : \rho(t_j) = t_j\}.$$

To derive main results in this section, we need the following lemma.

Lemma 3.1. *Let $p \in [1, +\infty[$, $C(\mathbb{T}, \mathbb{R})$ is dense in $C_{rd}(\mathbb{T}, \mathbb{R})$ provided with the induced topology of $L^p_\Delta(\mathbb{T}, \mathbb{R})$.*

Proof. For all $i \in I$, we defined r_i by $r_i = \{t_j : t_j < t_i\}$. Let $(v^i_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$v^i_n = \frac{t_i - r_i}{(b - a) 2^n} \mu(t_i), \quad \text{for all } i \in I.$$

Then, for all $i \in I$, we have $(v_n^i)_n \in (r_i, t_i)$. Let $(t_n^i)_{n\in\mathbb{N}}$ be a sequence on time scale $\mathbb{T}$ defined by

$$t_n^i = \inf \left[t_i - v_n^i, t_i\right)_{\mathbb{T}}, \quad \text{for all } n \in \mathbb{N},\ i \in I. \tag{3.1}$$

Let $\varphi \in C_{rd}(\mathbb{T}, \mathbb{R})$, we consider the sequence function $(\varphi_n)_{n\in\mathbb{N}}$ given by

$$\varphi_n(t) = \begin{cases} \varphi(t_i) + \dfrac{\varphi(t_i) - \varphi(t_n^i)}{t_i - t_n^i}(t - t_i), & \text{if } t \in [t_n^i, t_i]_{\mathbb{T}} \text{ for all } i \in I, \\ \lim\limits_{t\to b^-} \varphi(t), & \text{if } t = b, \\ \varphi(t), & \text{if not.} \end{cases}$$

Set $t \in [t_n^i, t_i]_{\mathbb{T}}$, for all $i \in I$, which implies that

$$\begin{aligned} |\varphi_n(t) - \varphi(t)| \leq& |\varphi(t_i)| + |\varphi(t)| + \left|\varphi(t_i) - \varphi\left(t_n^i\right)\right| \left|\frac{t - t_i}{t_i - t_n^i}\right| \\ \leq& 2\|\varphi\|_\infty + \left|\varphi(t_i) - \varphi\left(t_n^i\right)\right| \\ \leq& 4\|\varphi\|_\infty . \end{aligned}$$

Finally, we get that$|\varphi_n(t) - \varphi(t)| \leq 4\|\varphi\|_\infty$ for all $t \in [a,b)_{\mathbb{T}}$. It is clear that $(\varphi_n)_n$ is continuous in $\mathbb{T}$. Now, we show that $(\varphi_n)_{n\in\mathbb{N}}$ converges to φ in $L^p_\Delta(\mathbb{T}, \mathbb{R})$. In particular, we have

$$\begin{aligned} \int_{[a,b)_{\mathbb{T}}} |\varphi_n(t) - \varphi(t)|^p \Delta t =& \int_{A_n} |\varphi_n(t) - \varphi(t)|^p \Delta t \leq 4^p \|\varphi\|_\infty^p \int_{A_n} \Delta t \\ =& 4^p \|\varphi\|_\infty^p \mu_\Delta(A_n), \end{aligned}$$

with $A_n = \bigcup\limits_{i\in I} [t_n^i, t_i)_{\mathbb{T}}$, for all $n \in \mathbb{N}$. From (2.3), we have

$$\begin{aligned} \mu_\Delta(A_n) =& \lambda(A_n) + \sum_{i\in I} \sum_{t\in[t_n^i, t_i)_{\mathcal{R}}} \mu(t) \\ \leq& \sum_{i\in I} \lambda\left(\left[t_n^i, t_i\right[\right) + \sum_{i\in I}\left(t_i - t_n^i\right) \\ \leq& 2\sum_{i\in I}\left(t_i - t_n^i\right) \leq \sum_{i\in I} v_n^i \\ \leq& \sum_{i\in I} \frac{t_i - r_i}{(b-a)2^n}\mu(t_i) \leq \frac{b-a}{2^{n-1}}. \end{aligned} \tag{3.2}$$

Therefore, we obtain

$$\|\varphi_n - \varphi\|^p_{L^p_\Delta(\mathbb{T},\mathbb{R})} \leq 4^p \|\varphi\|^p_\infty \frac{b-a}{2^{n-1}}, \quad \text{for all } n \in \mathbb{N}.$$

The proof is complete. □

Remark 3.2. $C^1(\mathbb{T}, \mathbb{R})$ and $C^1_{rd}(\mathbb{T}, \mathbb{R})$ are Banach spaces together with the norm defined by

$$\|\varphi\|_1 := \|\varphi\|_\infty + \|\varphi^\Delta\|_\infty.$$

Let us define a second type of extension for a function φ on $[a,b]$. We introduce the following function

$$\overline{\varphi}(t) := \begin{cases} \varphi(t), & \text{if } t \in \mathbb{T}, \\ \dfrac{\varphi(\sigma(t_j)) - \varphi(t_j)}{\mu(t_j)}(t - t_j) + \varphi(t_j), & \text{if } t \in (t_j, \sigma(t_j)) \text{ for all } j \in J. \end{cases} \tag{3.3}$$

Lemma 3.2. *If $\varphi : [a,b] \to \mathbb{R}$ belongs to $C^1(a,b)$, then $\varphi_{|\mathbb{T}}$ belongs to $C^1_{rd}(\mathbb{T},\mathbb{R})$.*

Proof. We note $\psi = \varphi_{|\mathbb{T}}$, then ψ is $\Delta-$differentiable on $\mathbb{T}^k$, and ψ^Δ is given by

$$\psi^\Delta(t) = \begin{cases} \varphi'(t), & \text{if } t \in \mathbb{T}^k \backslash \mathcal{R}, \\ \dfrac{\varphi(\sigma(t_j)) - \varphi(t_j)}{\mu(t_i)}, & \text{if } t = t_j \in \mathbb{T}^k \text{ for all } j \in J. \end{cases}$$

Now, we show that ψ^Δ is rd-continuous. Let $t \in \mathbb{T}^k$ a left-dense or a right-dense point and prove that

$$\lim_{s\to t} \psi^\Delta(s) = \varphi'(t).$$

Since $\varphi \in C^1(a,b)$, then for all $\varepsilon > 0$, there exists $\alpha > 0$, such that

$$\left|\varphi'(s) - \varphi'(t)\right| \le \varepsilon, \quad \text{for all } s \in (t-\alpha, t+\alpha). \tag{3.4}$$

We define ξ on $(t-\alpha, t+\alpha)$ by $\xi(s) = \varphi(s) - \varphi(t)(t-s)$. By (3.4) we have $\left|\xi'(s)\right| \le \varepsilon$, for all $s \in (t-\alpha, t+\alpha)$. Then ξ is an ε-Lipschitz function on $(t-\alpha, t+\alpha)$, so we get

$$\left|\varphi'(\tau) - \frac{\varphi(\tau) - \varphi(s)}{\tau - s}\right| < \varepsilon, \qquad \text{for all } s, \tau \in (t-\alpha, t+\alpha) \text{ and } \tau \ne s.$$

And we have $\lim_{s\to t} \sigma(s) = t$. There exists $\gamma > 0$, such that $|\sigma(s) - t| \le \varepsilon$, for all $s \in (t-\gamma, t+\gamma) \cap \mathbb{T}$. Put $\delta = \min(\alpha, \gamma)$ for all $s \in (t-\delta, t+\delta) \cap \mathbb{T}$. We consider the following two cases.

If s is right-dense, then

$$\left|\varphi'(\tau) - \psi^\Delta(s)\right| = \left|\varphi'(\tau) - \varphi'(s)\right| \le \varepsilon.$$

If s is right-scattered, one has $\sigma(s), s \in (t-\delta, t+\delta) \cap \mathbb{T}$, then

$$\left|\varphi'(\tau) - \psi^\Delta(s)\right| = \left|\varphi'(\tau) - \frac{\varphi(\sigma(s)) - \varphi(s)}{\sigma(s) - s}\right| \le \varepsilon.$$

Finally, we obtain that ψ^Δ is a continuous function at right-dense points in $\mathbb{T}$, and its left-sided limits exist at left dense points in $\mathbb{T}$. $\square$

Lemma 3.3. *Let $p \in [1, +\infty[$, $C^1(\mathbb{T},\mathbb{R})$ is dense in $C^1_{rd}(\mathbb{T},\mathbb{R})$ provided with the induced topology of $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$.*

Proof. Let $\varphi \in C^1_{rd}(\mathbb{T},\mathbb{R})$, we define $P_{i,n}$ by

$$P_{i,n}(t) = \varphi(t_i) + \varphi^\Delta(t_i)(t - t_i) + \alpha h_2(t, t_i) + \beta h_3(t, t_i), \quad \text{for all } t \in \left[t^i_n, t_i\right]_{\mathbb{T}},$$

where $(t_n^i)_{n\in\mathbb{N}}$ is defined in (3.1) and $(h_k)_k$ are polynomials defined in [1], we choose α and β such that

$$P_{i,n}\left(t_n^i\right)=\varphi\left(t_n^i\right)\quad\text{and}\quad P_{i,n}^{\Delta}\left(t_n^i\right)=\varphi^{\Delta}\left(t_n^i\right),\quad\text{for all } i\in I, n\in\mathbb{N}. \tag{3.5}$$

Then α and β is the solution of the following system

$$\begin{cases}\alpha h_2\left(t_i^n,t_i\right)+\beta h_3\left(t_i^n,t_i\right)=\varphi\left(t_i^n\right)-\varphi\left(t_i\right)-\varphi^{\Delta}\left(t_i\right)h_1\left(t_i^n,t_i\right),\\ \alpha h_1\left(t_i^n,t_i\right)+\beta h_2\left(t_i^n,t_i\right)=\varphi^{\Delta}\left(t_i^n\right)-\varphi^{\Delta}\left(t_i\right).\end{cases}$$

Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\varphi_n\left(t\right)=\begin{cases}P_{i,n}\left(t\right), & \text{if } t\in[t_n^i,t_i]_{\mathbb{T}} \text{ for all } i\in I,\\ \lim\limits_{t\to b^-}\varphi\left(t\right), & \text{if } t=b,\\ \varphi\left(t\right), & \text{if not.}\end{cases}$$

By (3.5), we conclude that φ_n is Δ-differentiable on $\mathbb{T}^k$ and $\left(\varphi_n^{\Delta}\right)$ is continuous in $\mathbb{T}^k$. For all $i\in I$, we get

$$\begin{aligned}\int_{[t_n^i,t_i[\cap\mathbb{T}}\left|\varphi_n\left(t\right)-\varphi\left(t\right)\right|\Delta t\leq&\int_{[t_n^i,t_i[\cap\mathbb{T}}\left(\left|\varphi\left(t\right)\right|+\left|\varphi\left(t_i\right)\right|+\left|\varphi^{\Delta}\left(t_i\right)\right|h_1\left(t,t_i\right)\right)\Delta t\\&+\int_{[t_n^i,t_i[\cap\mathbb{T}}\left|\alpha h_2\left(t,t_i\right)+\beta h_3\left(t,t_i\right)\right|\Delta t\\ \leq&2\left\|\varphi\right\|_{\infty}h_1\left(t_i,t_i^n\right)+\left\|\varphi^{\Delta}\right\|_{\infty}h_1\left(t_i,t_i^n\right)\\&+\left|\alpha h_3\left(t_i^n,t_i\right)+\beta h_4\left(t_i^n,t_i\right)\right|\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}&\int_{[t_n^i,t_i[\cap\mathbb{T}}\left|\varphi_n^{\Delta}\left(t\right)-\varphi^{\Delta}\left(t\right)\right|\Delta t\\ \leq&\int_{[t_n^i,t_i[\cap\mathbb{T}}\left(\left|\varphi^{\Delta}\left(t\right)\right|+\left|\varphi^{\Delta}\left(t_i\right)\right|+\left|\alpha h_1\left(t,t_i\right)+\beta h_2\left(t,t_i\right)\right|\right)\Delta t\\ \leq&2\left\|\varphi^{\Delta}\right\|_{\infty}h_1\left(t_i,t_i^n\right)+\left|\alpha h_2\left(t_i^n,t_i\right)+\beta h_3\left(t_i^n,t_i\right)\right|.\end{aligned}\tag{3.7}$$

For all $i\in I$, we define $\eta_{k,i,n}$ on $[t_n^i,t_i)_{\mathbb{T}}$ by

$$\eta_{k,i,n}\left(s\right)=\alpha h_k\left(s,t_i\right)+\beta h_{k+1}\left(s,t_i\right),\quad\text{for all } k\in\mathbb{N}.$$

Hence, we deduce that

$$\eta_{k,i,n}^{\Delta}\left(s\right)=\alpha h_{k-1}\left(s,t_i\right)+\beta h_k\left(s,t_i\right)=\eta_{k-1,i,n}\left(s\right),\quad\text{for all } s\in\left[t_n^i,t_i\right)_{\mathbb{T}^k},\tag{3.8}$$

by (3.8), we get

$$\left|\eta_{k,i,n}\left(s\right)\right|\leq\int_s^{t_i}\left|\eta_{k-1,i,n}\left(\tau\right)\right|\Delta\tau,\quad\text{for all } k\in\mathbb{N}, s\in\left[t_n^i,t_i\right)_{\mathbb{T}^k}.\tag{3.9}$$

Since, $\left|\eta_{1,i,n}\left(s\right)\right|\leq\left|\eta_{1,i,n}\left(t_i^n\right)\right|$ for all $s\in[t_n^i,t_i)_{\mathbb{T}}$, using the inequality (3.9), we find

$$\left|\eta_{2,i,n}\left(s\right)\right|\leq\left(t_i-t_i^n\right)\left|\eta_{1,i,n}\left(t_i^n\right)\right|,\quad\text{for all } s\in\left[t_n^i,t_i\right)_{\mathbb{T}}\tag{3.10}$$

and

$$|\eta_{3,i,n}(s)| \leq (t_i - t_i^n)^2 |\eta_{1,i,n}(t_i^n)|, \quad \text{for all } s \in \left[t_n^i, t_i\right)_{\mathbb{T}}. \tag{3.11}$$

By (3.10), we obtain

$$\begin{aligned} |\alpha h_2(t_i^n, t_i) + \beta h_3(t_i^n, t_i)| &\leq (t_i - t_i^n) |\eta_{1,i,n}(t_i^n)| \\ &\leq (t_i - t_i^n) \left|\varphi^\Delta(t_i^n) - \varphi^\Delta(t_i)\right| \\ &\leq 2(t_i - t_i^n) \left\|\varphi^\Delta\right\|_\infty, \end{aligned} \tag{3.12}$$

and by (3.11), we have

$$\begin{aligned} |\alpha h_3(t_i^n, t_i) + \beta h_4(t_i^n, t_i)| &\leq (t_i - t_i^n)^2 |\eta_{1,i,n}(t_i^n)| \\ &\leq (t_i - t_i^n)^2 \left|\varphi^\Delta(t_i^n) - \varphi^\Delta(t_i)\right| \\ &\leq 2\left\|\varphi^\Delta\right\|(t_i - t_i^n)^2. \end{aligned} \tag{3.13}$$

Substituting (3.13) in (3.6), we get

$$\begin{aligned} \int_{[a,b)_{\mathbb{T}}} |\varphi_n(t) - \varphi(t)| \Delta t &\leq \left(2\|\varphi\|_\infty + \left\|\varphi^\Delta\right\|_\infty\right) \sum_{i \in I} (t_i - t_i^n) + \left\|\varphi^\Delta\right\|_\infty \sum_{i \in I} (t_i - t_i^n)^2 \\ &\leq \frac{b-a}{2^n}\left(2\|\varphi\|_\infty + (b-a+1)\left\|\varphi^\Delta\right\|_\infty\right). \end{aligned} \tag{3.14}$$

It follows from (3.12) and (3.7), that

$$\int_{[a,b)_{\mathbb{T}}} \left|\varphi_n^\Delta(t) - \varphi^\Delta(t)\right| \Delta t \leq 4\left\|\varphi^\Delta\right\|_\infty \sum_{i \in I} (t_i - t_i^n) \leq \frac{b-a}{2^{n-2}} \left\|\varphi^\Delta\right\|_\infty. \tag{3.15}$$

By inequality (3.14) and (3.15), we obtain that $(\varphi_n)_n$ converges to φ in $W_\Delta^{1,1}(\mathbb{T}, \mathbb{R})$. Finally, by Hölder's inequality, we conclude that $(\varphi_n)_n$ converges to φ in $W_\Delta^{1,p}(\mathbb{T}, \mathbb{R})$. □

Remark 3.3. Let E, F, G be three spaces such that $E \subset F \subset G$ and (G, τ) is a topological space.

1) If F is dense in (G, τ) and E is dense in (F, τ), then E is dense in (G, τ).
2) If E is dense in G, then F is dense in G.

The following theorem is a new generalization of the Theorem 2.2.

Theorem 3.1. *Let $p \in [1, +\infty[$, then $C(\mathbb{T}, \mathbb{R})$ is dense in $L_\Delta^p(\mathbb{T}, \mathbb{R})$.*

Proof. Let $p \in [1, +\infty[$, we have $C(\mathbb{T}, \mathbb{R}) \subset C_{rd}(\mathbb{T}, \mathbb{R}) \subset L_\Delta^p(\mathbb{T}, \mathbb{R})$. By Lemma 3.1 and Theorem 2.2, hence $C_{rd}(\mathbb{T}, \mathbb{R})$ is dense in $L_\Delta^p(\mathbb{T}, \mathbb{R})$ and $C(\mathbb{T}, \mathbb{R})$ is dense in $C_{rd}(\mathbb{T}, \mathbb{R})$ provided with the induced topology of $L_\Delta^p(\mathbb{T}, \mathbb{R})$. Then, by Remark 3.3, we obtain $C(\mathbb{T}, \mathbb{R})$ is dense in $L_\Delta^p(\mathbb{T}, \mathbb{R})$. □

The following results are consequences of Theorem 3.2.

Proposition 3.1. *Let $p \in [1,+\infty[$, then $C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T},\mathbb{R})$.*

Proof. Let $p \in [1,+\infty[$. By Theorem 2.2, we have $C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $C(\mathbb{T},\mathbb{R})$, then $C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $C(\mathbb{T},\mathbb{R})$ provided with the induced topology of $L^p_\Delta(\mathbb{T},\mathbb{R})$, and we have $C(\mathbb{T},\mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T},\mathbb{R})$, by Remark 3.3 , we conclude $C(\mathbb{T},\mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T},\mathbb{R})$. ☐

As a proposition of the previous result, we deduce the following corollary.

Corollary 3.1. *Let $p \in [1,+\infty)$, then $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$ is dense in $C(\mathbb{T},\mathbb{R})$.*

Proof. We have $C^1_{rd}(\mathbb{T},\mathbb{R}) \subset W^{1,p}_\Delta(\mathbb{T},\mathbb{R}) \subset C(\mathbb{T},\mathbb{R})$, by Theorem 2.2, $C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $C(\mathbb{T},\mathbb{R})$. Therefore, Remark 3.3 implies that $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$ is dense in $C(\mathbb{T},\mathbb{R})$. ☐

In the same way, we find the following corollary.

Corollary 3.2. *Let $p \in [1,+\infty)$, then $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$ is dense in $C_{rd}(\mathbb{T},\mathbb{R})$.*

Corollary 3.3. *Let $p \in [1,+\infty)$, then $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$ is dense in $L^p_\Delta(\mathbb{T},\mathbb{R})$.*

The next result show that spaces $C^1_{rd}(\mathbb{T},\mathbb{R})$ and $C^1(\mathbb{T},\mathbb{R})$ are dense in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$.

Theorem 3.2. *Let $p \in [1,+\infty)$, $C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$.*

Proof. Let $\varphi \in W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$, by Corollary 3.9 in [7], we have $\overline{\varphi} \in W^{1,p}(a,b)$. Since $C^1((a,b))$ is dense in $W^{1,p}(a,b)$, then there exists a sequence $(\psi_n)_{n\in\mathbb{N}} \in C^1(a,b)$ that converges to $\overline{\varphi}$ in $W^{1,p}(a,b)$. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence defined by

$$\varphi_n = \psi_{n|\mathbb{T}}, \quad \text{for all } n \in \mathbb{N}.$$

By Lemma 3.2, we get $(\varphi_n)_n \in C^1_{rd}(\mathbb{T},\mathbb{R})$. Now we show that $(\varphi_n)_n$ converges to φ in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$, we have

$$\left\|(\psi_n - \overline{\varphi})_{|\mathbb{T}}\right\|_{W^{1,p}_\Delta(\mathbb{T},\mathbb{R})} = \|\varphi_n - \varphi\|_{W^{1,p}_\Delta(\mathbb{T},\mathbb{R})},$$

by Corollary 3.10 in [7], there exists a constant $C > 0$ which only depends on $(b-a)$ such that

$$\|\varphi_n - \varphi\|_{W^{1,p}_\Delta(\mathbb{T},\mathbb{R})} \leq C\,\|\psi_n - \overline{\varphi}\|_{W^{1,p}((a,b))},$$

we prove that $(\varphi_n)_n$ converges to φ in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$. ☐

Theorem 3.3. *Let $p \in [1,+\infty[$, then $C^1(\mathbb{T},\mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$.*

Proof. Let $p \in [1,+\infty[$. We have $C^1(\mathbb{T},\mathbb{R}) \subset C^1_{rd}(\mathbb{T},\mathbb{R}) \subset W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$. By Lemma 3.3 and Theorem 3.2, hence $C^1_{rd}(\mathbb{T},\mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$ and $C^1(\mathbb{T},\mathbb{R})$ is dense in $C^1_{rd}(\mathbb{T},\mathbb{R})$ provided with the induced topology of $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$. Then, by Remark 3.3, we obtain $C^1(\mathbb{T},\mathbb{R})$ is dense in $W^{1,p}_\Delta(\mathbb{T},\mathbb{R})$. ☐

4. Conclusion

Finally, we give a diagrams that summarizes the main results

$$\begin{array}{ccccc}
C_{rd}^{1}\left(\mathbb{T},\mathbb{R}\right) & & C_{rd}\left(\mathbb{T},\mathbb{R}\right) & & \\
\downarrow & & \downarrow & & \\
W_{\Delta}^{1,p}\left(\mathbb{T},\mathbb{R}\right) & \longrightarrow & L_{\Delta}^{p}\left(\mathbb{T},\mathbb{R}\right) & \longrightarrow & L_{\Delta}^{1}\left(\mathbb{T},\mathbb{R}\right) \\
\uparrow & & \uparrow & & \\
C^{1}\left(\mathbb{T},\mathbb{R}\right) & & C\left(\mathbb{T},\mathbb{R}\right) & &
\end{array}$$

For $\mathbb{T}$ is bounded and $p \in [1,+\infty)$.

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A NOTE ON PAIR OF LEFT CENTRALIZERS IN PRIME RING WITH INVOLUTION

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Abstract. The purpose of this paper is to study pair of left centralizers in prime rings with involution satisfying certain identities.

1. Introduction

In the present article, R will represent an associative ring with centre $Z(R)$. $Q_m r$ and C represents the maximal ring of quotient and the extended centroid of a prime ring, respectively. For the explanation of $Q_m r$ and C refer to [4]. R is said to be n-torsion free if $na = 0$ (where $a \in R$) implies $a = 0$. R is called prime if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y]$ for $xy - yx$ and xoy for $xy + yx$, respectively. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^* x^*$ and (ii) $(x^*)^* = x$ holds, for all $x, y \in R$. A ring R together with an involution $*$ is said to be a ring with involution or $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case, $S(R) \cap Z(R) \neq (0)$. A description of such rings can be found in [7], where further references can be found.

Following [17], an additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer (multiplier) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping T is called a centralizer in case T is a left and a right centralizer of R. Considerable work has been done on left (resp. right) centralizers in prime and semiprime rings during the last few decades (see for example [3, 6, 10, 11, 14–17]) where

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further references can be found. The first result studying the commutativity of prime ring involving a special mapping was due to Divinsky [5], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. This result was later refined and extended by a number of authors in various directions (see [2, 3, 8, 9, 12, 13]). Moreover in [3] some related results involving left centralizers have also been discussed. In [10] Oukhtite established similar problems for certain situations involving left centralizers acting on Lie ideals. Recently Ali and Dar [1] proved that if a prime ring with involution of the second kind such that char(R) $\neq 2$ admits a left centralizer $T : R \to R$ satisfying any one of the following conditions:

(i) $T([x, x^*]) = 0$;
(ii) $T(x o x^*) = 0$;
(iii) $T([x, x^*]) \pm [x, x^*] = 0$;
(iv) $T(x o x^*) \pm (x o x^*) = 0$,

for all $x \in R$, then R is commutative. In this paper, we shall consider similar problems involving pair of centralizers. We shall restrict our attention on left centralizers, since all results presented in this article are also true for right centralizers because of left-right symmetry.

Lemma 1.1 ([7], p. 20-23). *Suppose that the elements a_i, b_i in the central closure of a prime ring R satisfy $\sum a_i x b_i = 0$ for all $x \in R$. If $b_i \neq 0$ for some i, then $a_i^{\prime}s$ are C-dependent.*

2. Main Results

Theorem 2.1. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R such that $[T_1(x), T_2(x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$[T_1(x), T_2(x^*)] \in Z(R), \quad \text{for all } x \in R. \tag{2.1}$$

Linearizing (2.1), we get

$$[T_1(x), T_2(y^*)] + [T_1(y), T_2(x^*)] \in Z(R), \quad \text{for all } x, y \in R. \tag{2.2}$$

Replacing y by ky in (2.2) and using (2.2), we get $2([T_1(y), T_2(x^*)])k \in Z(R)$ for all $x, y \in R$ and $k \in S(R) \cap Z(R)$, since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$. This implies that $[T_1(y), T_2(x^*)] \in Z(R)$ for all $x, y \in R$. Taking $x = x^*$, we have $[T_1(y), T_2(x)] \in Z(R)$ for all $x, y \in R$. This can be further written as

$$[[T_1(y), T_2(x)], r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.3}$$

Replacing y by ym, where $m \in R$ in (2.3) and using (2.3) we get

$$T_1(y)[[m, T_2(x)], r] + [T_1(y), r][m, T_2(x)] + [T_1(y), T_2(x)][m, r] = 0,$$

for all $x, y, m, r \in R$. Replacing m by $T_2(x)$ we get

$$[T_1(y), T_2(x)][T_2(x), r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.4}$$

Replacing r by ru, where $u \in R$ in (2.4) and using (2.4) we get

$$[T_1(y), T_2(x)]r[T_2(x), u] = 0, \quad \text{for all } x, y, r, u \in R.$$

Then by primeness of R, for each fixed $x \in R$, we get either $[T_1(y), T_2(x)] = 0$ for all $y \in R$ or $[T_2(x), u] = 0$ for all $u \in R$. Define $A = \{x \in R \mid [T_2(x), u] = 0 \text{ for all } u \in R\}$ and $B = \{x \in R \mid [T_1(y), T_2(x)] = 0 \text{ for all } y \in R\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence, by Brauer's trick, either $A = R$ or $B = R$. If $A = R$

$$[T_2(x), u] = 0, \quad \text{for all } x, u \in R. \tag{2.5}$$

Then taking $x = xy$ in (2.5), where $y \in R$ and using (2.5) we get $T_2(x)[y, u] = 0$ for all $x, y, u \in R$. Now take $x = xm$, where $m \in R$, then as T_2 is nonzero, applying the primeness of R, we obtain R is commutative. If $B = R$

$$[T_1(y), T_2(x)] = 0, \quad \text{for all } x, y \in R. \tag{2.6}$$

Then replacing y by yv, where $v \in R$ in (2.6) and using (2.6) we get $T_1(y)[v, T_2(x)] = 0$ for all $x, y, v \in R$. Now replace y by yr, where $r \in R$. Then as T_1 is nozero, by primeness of R, we have $[v, T_2(x)] = 0$ for all $v, x \in R$. With similar steps as we did before we get R is commutative. □

Theorem 2.2. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x) \circ T_2(x^*) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x) \circ T_2(x^*) \in Z(R), \quad \text{for all } x \in R. \tag{2.7}$$

Linearizing (2.7), we get

$$T_1(x) \circ T_2(y^*) + T_1(y) \circ T_2(x^*) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.8}$$

Replacing y by ky in (2.8) and using (2.8), we get

$$2(T_1(y) \circ T_2(x^*))k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $T_1(y) \circ T_2(x^*) \in Z(R)$ for all $x, y \in R$. Replacing x by x^* we get $T_1(y) \circ T_2(x) \in Z(R)$ for all $x, y \in R$. This can be further written as

$$[T_1(y) \circ T_2(x), r] = 0,$$
$$[T_1(y)T_2(x), r] + [T_2(x)T_1(y), r] = 0,$$
$$T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) + T_2(x)[T_1(y), r] + [T_2(x), r]T_1(y) = 0, \tag{2.9}$$

for all $x, y, r \in R$. Replacing y by $yT_2(x)$ in (2.9), we get

$$T_1(y)T_2(x)[T_2(x), r] + T_1(y)[T_2(x), r]T_2(x) + [T_1(y), r](T_2(x))^2 \\ +T_2(x)T_1(y)[T_2(x), r] + T_2(x)[T_1(y), r]T_2(x) + [T_2(x), r]T_1(y)T_2(x) = 0, \tag{2.10}$$

for all $x, y, r \in R$. Left multiplying (2.9) by $T_2(x)$ and subtracting it from (2.10), we get

$$(T_1(y)T_2(x) + T_2(x)T_1(y))[T_2(x), r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.11}$$

Replacing r by rm, where $m \in R$ in (2.11) and using (2.11), we get

$$(T_1(y)T_2(x) + T_2(x)T_1(y))r[T_2(x), m] = 0, \quad \text{for all } x, y, r, m \in R.$$

Then by primeness of R, for each fixed $x \in R$, we get either $[T_2(x), m] = 0$ for all $m \in R$ or $T_1(y)T_2(x)+T_2(x)T_1(y) = 0$ for all $y \in R$. Define $A = \{x \in R \mid [T_2(x), m] = 0 \text{ for all } m \in R\}$ and $B = \{x \in R \mid T_1(y)T_2(x) + T_2(x)T_1(y) = 0 \text{ for all } y \in R\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $A = R$ or $B = R$. If $A = R$, then we consider $[T_2(x), r] = 0$ for all $x, r \in R$, proceeding similarly as we did after (2.5), we get R is commutative. Now, consider $B = R$, in this situation

$$T_1(y)T_2(x) + T_2(x)T_1(y) = 0, \quad \text{for all } x, y \in R. \tag{2.12}$$

Then replacing y by yu, where $u \in R$ in (2.12) and using (2.12), we get $T_1(y)[u, T_2(x)] = 0$ for all $x, y, u \in R$. Replacing y, where $v \in R$ by yv, we get $T_1(y)v[u, T_2(x)] = 0$ for all $x, y, v, u \in R$. Then as T_1 is nonzero, by primeness we get $[u, T_2(x)] = 0$ for all $x, u \in R$. Now, following same line of proof as we did after (2.5), we get R is commutative. $\square$

Theorem 2.3. *Let R be a noncommutative 6-torsion free prime ring with involution $*$ of the second kind. If R admit two nonzero left centralizers T_1 and T_2 from R to R, such that $[T_1(x), T_2(x^*)] \pm [x, x^*] \in Z(R)$ for all $x \in R$, then $T_1 = \lambda T_2$.*

Proof. We have

$$[T_1(x), T_2(x^*)] \pm [x, x^*] \in Z(R), \quad \text{for all } x \in R. \tag{2.13}$$

Linearizing (2.13), we get

$$[T_1(x), T_2(y^*)] + [T_1(y), T_2(x^*)] \pm [x, y^*] \pm [y, x^*] \in Z(R), \quad \text{for all } x, y \in R. \tag{2.14}$$

Replacing y by ky in (2.14) and using (2.14), we get

$$2([T_1(y), T_2(x^*)] \pm [y, x^*])k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

This further implies that

$$6([T_1(y), T_2(x^*)] \pm [y, x^*])k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since R is 6-torsion free and $S(R) \cap Z(R) \neq (0)$, we have

$$[T_1(y), T_2(x^*)] \pm [y, x^*] \in Z(R), \quad \text{for all } x, y \in R.$$

Replacing x by x^*, we get

$$[T_1(y), T_2(x)] \pm [y, x] \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $y = x$, we have

$$[T_1(x), T_2(x)] \in Z(R), \quad \text{for all } x \in R. \tag{2.15}$$

This further implies that

$$[[T_1(x), T_2(x)], r] = 0, \quad \text{for all } x, r \in R. \tag{2.16}$$

On linearization, we get

$$[[T_1(x), T_2(y)], r] + [[T_1(y), T_2(x)], r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.17}$$

Replacing y by $yT_1(x)$ in (2.17) and using (2.16) and (2.17), we obtain

$$[T_1(x), T_2(y)][T_1(x), r] + [T_1(y), r][T_1(x), T_2(x)] + [T_1(y), T_2(x)][T_1(x), r] = 0, \tag{2.18}$$

for all $x, y, r \in R$. Taking $y = x$ in (2.18) and using (2.15), we arrive at

$$3[T_1(x), T_2(x)][T_1(x), r] = 0, \quad \text{for all } x, r \in R.$$

This further implies that

$$6[T_1(x), T_2(x)][T_1(x), r] = 0, \quad \text{for all } x, r \in R.$$

Since R is 6-torsion free ring, we have

$$[T_1(x), T_2(x)][T_1(x), r] = 0, \quad \text{for all } x, r \in R. \tag{2.19}$$

Replacing r by rm, where $m \in R$ in (2.19) and making use of (2.19), we get

$$[T_1(x), T_2(x)]r[T_1(x), m] = 0, \quad \text{for all } x, m, r \in R.$$

Using the primeness of R, for each fixed $x \in R$, we have either $[T_1(x), T_2(x)] = 0$ or $[T_1(x), m] = 0$. Define $B = \{x \in R \mid [T_1(x), T_2(x)] = 0\}$ and $A = \{x \in R \mid [T_1(x), m] = 0 \text{ for all } m \in R\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $B = R$ or $A = R$. If $B = R$,

$$[T_1(x), m] = 0, \quad \text{for all } x, m \in R. \tag{2.20}$$

Replacing x by xy, where $y \in R$ and using (2.20), we get

$$T_1(x)[y, m] = 0, \quad \text{for all } x, y, m \in R.$$

This further implies that

$$T_1(x)w[y, m] = 0, \quad \text{for all } x, w, y, m \in R.$$

Using the primeness, we get $T_1(x) = 0$ for all $x \in R$ or $[y, m] = 0$ for all $y, m \in R$. Since T_1 is nonzero, therefore we get R is commutative, which is a contradiction to our assumption. Therefore we are left with $B = R$

$$[T_1(x), T_2(x)] = 0, \quad \text{for all } x \in R. \tag{2.21}$$

Linearizing (2.21), we get

$$[T_1(x), T_2(y)] + [T_1(y), T_2(x)] = 0, \quad \text{for all } x, y \in R. \tag{2.22}$$

Replacing x by xz, where $z \in R$ in (2.22) and using (2.22), we get

$$T_1(x)[z, T_2(y)] + T_2(x)[T_1(y), z] = 0, \quad \text{for all } x, y, z \in R. \tag{2.23}$$

Again taking $x = xw$, where $w \in R$ in (2.23), we get

$$T_1(x)w[z, T_2(y)] + T_2(x)w[T_1(y), z] = 0, \quad \text{for all } x, y, z, w \in R. \tag{2.24}$$

In view of Lemma 1.1, we have $[z, T_2(y)] = 0$ for all $y, z \in R$ or $T_1(x) = \lambda(x)T_2(x)$, where $\lambda(x) \in C$. But since $T_2 \neq 0$, $[z, T_2(y)] = 0$ implies R is commutative, a contradiction. Hence we get $T_1(x) = \lambda(x)T_2(x)$, where $\lambda(x) \in C$. Using this in (2.24), we have

$$\begin{aligned}
&\lambda(x)T_2(x)w[z, T_2(y)] + T_2(x)w[\lambda(y)T_2(y), z] = 0,\\
&(\lambda(x)T_2(x) - \lambda(y)T_2(x))w[z, T_2(y)] = 0,
\end{aligned}$$

for all $x, y, z, w \in R$. Using the primeness of R and Brauer's trick we finally get $T_1 = \lambda T_2$. This completes the proof. □

Theorem 2.4. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x)x^* \pm x^*T_2(x) \in Z(R)$ for all $x \in R$, then either R is commutative or $T_1(y) = \mp T_2(y)$ for all $y \in R$.*

Proof. We have

$$T_1(x)x^* \pm x^*T_2(x) \in Z(R), \quad \text{for all } x \in R. \tag{2.25}$$

Linearaizing (2.25), we get

$$T_1(x)y^* + T_1(y)x^* \pm x^*T_2(y) \pm y^*T_2(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.26}$$

Replacing y by ky in (2.26) and using (2.26), we have

$$2(T_1(y)x^* \pm x^*T_2(y))k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(y)x^* \pm x^*T_2(y) \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $x = x^*$, we get

$$T_1(y)x \pm xT_2(y) \in Z(R), \quad \text{for all } x, y \in R.$$

Replacing x by z, where $z \in Z(R)$ and using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain $T_1(y) \pm T_2(y) \in Z(R)$ for all $y \in R$. This can be further written as

$$[T_1(y), r] \pm [T_2(y), r] = 0, \quad \text{for all } y, r \in R. \tag{2.27}$$

Replacing y by yw, where $w \in R$ and using (2.27), we have

$$(T_1(y) \pm T_2(y))[w, r] = 0, \quad \text{for all } y, w, r \in R. \tag{2.28}$$

Replacing w by wm, where $m \in R$ in (2.28) and using (2.28), we obtain

$$(T_1(y) \pm T_2(y))w[m, r] = 0, \quad \text{for all } m, y, w, r \in R.$$

In view of the primeness of R we get either R is commutative or $T_1(y) = \mp T_2(y)$ for all $y \in R$. $\square$

Theorem 2.5. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R, such that $T_1(x)T_2(x^*) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x)T_2(x^*) \in Z(R), \quad \text{for all } x \in R. \tag{2.29}$$

Linearizing (2.29), we get

$$T_1(x)T_2(y^*) + T_1(y)T_2(x^*) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.30}$$

Replacing y by ky in (2.30), where $k \in S(R) \cap Z(R)$ and using (2.30), we have

$$2T_1(y)T_2(x^*)k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(y)T_2(x^*) \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $x = x^*$, we obtain

$$T_1(y)T_2(x) \in Z(R), \quad \text{for all } x, y \in R.$$

This can be further written as

$$T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) = 0, \quad \text{for all } x, y \in R. \tag{2.31}$$

Replacing x by xw, where $w \in R$ in (2.31) and using (2.31), we get

$$T_1(y)T_2(x)[w, r] = 0, \quad \text{for all } x, y, w, r \in R.$$

Replacing y by ym, where $m \in R$, we get

$$T_1(y)RT_2(x)[w, r] = (0), \quad \text{for all } x, y, w, r \in R.$$

This implies in view of the primeness of ring R, either $T_1(y) = 0$ for all $y \in R$ or $T_2(x)[w, r] = 0$ for all $x, w, r \in R$. Since $T_1 \neq 0$, we get $T_2(x)[w, r] = 0$ for all $x, w, r \in R$. This further implies that $T_2(x)y[w, r] = 0$ for all $x, y, w, r \in R$. Since $T_2 \neq 0$, using the primeness of R, we get R is commutative. $\square$

Theorem 2.6. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x)x \pm x^*T_2(x) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x)x \pm x^*T_2(x) \in Z(R), \quad \text{for all } x \in R.$$

Linearizing (2.48), we get

$$T_1(x)y + T_1(y)x \pm x^*T_2(y) \pm y^*T_2(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.32}$$

Replacing y by ky in (2.32) and using (2.32), we arrive at

$$2(T_1(x)y + T_1(y)x \pm x^*T_2(y))k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(x)y + T_1(y)x \pm x^*T_2(y) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.33}$$

Again, replacing x by kx in(2.33) and using (2.33), we get

$$2(T_1(x)y + T_1(y)x)k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, we have

$$T_1(x)y + T_1(y)x \in Z(R), \quad \text{for all } x, y \in R.$$

This can be further written as $T_1(x \circ y) \in Z(R)$ for all $x, y \in R$. Taking $y = z$, where $z \in Z(R)$, we get $T_1(x) \in Z(R)$ for all $x \in R$. This further implies that $[T_1(x), y] = 0$ for all $x, y \in R$. Replacing x by xw, where $w \in R$, we get $T_1(x)[w, y] + [T_1(x), y]w = 0$ for all $x, y, w \in R$. That is, $T_1(x)[w, y] = 0$ for all $x, y, w \in R$. Replacing x by xm, where $m \in R$ and using the facts that $T_1 \neq 0$ and the primeness of R, we obtain $[w, y] = 0$ for all $w, y \in R$. That is, R is commutative. □

Theorem 2.7. *Let R be a noncommutative prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$*. If R admit two nonzero left centralizer T_1 and T_2 from R to R such that $xT_1(x^*) \pm T_2(x)x^* \in Z(R)$ for all $x \in R$, then $T_1 = \mp T_2$.*

Proof. We have

$$xT_1(x^*) \pm T_2(x)x^* \in Z(R), \quad \text{for all } x \in R. \tag{2.34}$$

Linearizing (2.34), we get

$$xT_1(y^*) + yT_1(x^*) \pm T_2(x)y^* \pm T_2(y)x^* \in Z(R), \quad \text{for all } x, y \in R. \tag{2.35}$$

Replacing y by ky in (2.35) and using (2.35), we get

$$2(yT_1(x^*) \pm T_2(y)x^*)k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$yT_1(x^*) \pm T_2(y)x^* \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $x = x^*$, we get

$$yT_1(x) \pm T_2(y)x \in Z(R), \quad \text{for all } x, y \in R.$$

This further implies that

$$[yT_1(x), r] \pm [T_2(y)x, r] = 0, \quad \text{for all } x, y, r \in R.$$

That is,

$$y[T_1(x), r] + [y, r]T_1(x) \pm T_2(y)[x, r] \pm [T_2(y), r]x = 0, \quad \text{for all } x, y, r \in R. \tag{2.36}$$

Replacing x by xw, where $w \in R$ in (2.36) and using (2.36), we obtain

$$(yT_1(x) \pm T_2(y)x)[w, r] = 0, \quad \text{for all } x, y, w, r \in R. \tag{2.37}$$

Replacing w by wm, where $m \in R$ in (2.37) and using (2.37), we get

$$yT_1(x) \pm T_2(y)x = 0, \quad \text{for all } x, y \in R, \tag{2.38}$$

since R is noncommutative. Replacing y by yx in (2.38) and using (2.38), we get

$$y(xT_1(x) - T_1(x)x) = 0, \quad \text{for all } x, y \in R.$$

Using the primeness of R, we get

$$(xT_1(x) - T_1(x)x) = 0, \quad \text{for all } x \in R.$$

Linearizing the above equation, we get

$$xT_1(y) + yT_1(x) - T_1(x)y - T_1(y)x = 0, \quad \text{for all } x, y \in R. \tag{2.39}$$

Replacing y by yu, where $u \in R$ in (2.39) and using (2.39), we arrive at

$$y[T_1(x), u] + T_1(y)[u, x] = 0, \quad \text{for all } x, y, u \in R. \tag{2.40}$$

Replacing x by xm, where $m \in R$ in (2.40) and using (2.40), we get

$$(yT_1(x) - T_1(y)x)[m, u] = 0, \quad \text{for all } x, y, m, u \in R. \tag{2.41}$$

Replacing m by wm, where $m \in R$ in (2.41) and using (2.41), we have T_1 is centralizer, since R is noncommutative. Hence in view of (2.38), we get $(T_1(y) \pm T_2(y))x = 0$ for all $x, y \in R$. Using the primeness of R, we obtain $T_1(y) = \mp T_2(y)$ for all $y \in R$. □

Theorem 2.8. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two left centralizer T_1 and T_2 from R to R such that $T_1(x)T_2(x^*) \pm xx^* \in Z(R)$ for all $x \in R$, then either R is commutative or T_1 and T_2 centralizer.*

Proof. We have

$$T_1(x)T_2(x^*) \pm xx^* \in Z(R), \quad \text{for all } x \in R. \tag{2.42}$$

If either T_1 or T_2 is zero, then we get $\pm xx^* \in Z(R)$ for all $x \in Z(R)$. Replacing x by $x + y$, where $x, y \in R$, we get $xy^* + yx^* \in Z(R)$ for all $x, y \in R$. Taking $y = yk$ where $k \in Z(R) \cap S(R)$ and adding with the previous equation, we get $2yx^*k \in Z(R)$ for all $x, y \in Z(R)$ and $k \in S(R) \cap Z(R)$. Since char(R) $\neq 2$, this implies that $yx^*k \in Z(R)$ for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. Use primeness and the fact that $S(R) \cap Z(R) \neq 0$, we have $yx^* \in Z(R)$ for all $x, y \in R$. This further implies that $yx \in Z(R)$ for all $x, y \in R$. Thus $xz \in Z(R)$ for all $x \in R$ and $z \in Z(R)$. Use primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain R is commutative. Now consider neither T_1 nor T_2 is zero. Linearizing (2.42), we get

$$T_1(x)T_2(y^*) + T_1(y)T_2(x^*) \pm xy^* \pm yx^* \in Z(R), \quad \text{for all } x, y \in R. \tag{2.43}$$

Replacing y by ky in (2.43) and using (2.43), we get $2(T_1(y)T_2(x^*) \pm yx^*)k \in Z(R)$ for all $x, y \in R$. Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $T_1(y)T_2(x^*) \pm$

$yx^* \in Z(R)$ for all $x, y \in R$. This can be written as

$$[T_1(y)T_2(x^*), r] \pm [yx^*, r] = 0,$$
$$T_1(y)[T_2(x^*), r] + [T_1(y), r]T_2(x^*) \pm y[x^*, r] \pm [y, r]x^* = 0,$$

for all $x, y, r \in R$. Taking $x = x^*$, we obtain

$$T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) \pm y[x, r] \pm [y, r]x = 0, \quad \text{for all } x, y, r \in R. \tag{2.44}$$

Replacing x by xw, where $w \in R$ in (2.44) and using (2.44), we get $(T_1(y)T_2(x) \pm yx)[w, r] = 0$ for all $x, y, w, r \in R$. Replacing w by wm, where $m \in R$ and using the previous equation, we get $(T_1(y)T_2(x) \pm yx)w[m, r] = 0$ for all $x, y, w, m, r \in R$. Now using the primeness we get either $T_1(y)T_2(x) \pm yx = 0$ for all $x, y \in R$ or $[m, r] = 0$ for all $m, r \in R$. If $[m, r] = 0$ for $m, r \in R$, this implies that R is commutative. Now suppose

$$T_1(y)T_2(x) \pm yx = 0, \quad \text{for all } x, y \in R. \tag{2.45}$$

Replacing y by $yT_1(w)$ in (2.45), we get

$$T_1(y)T_1(w)T_2(x) \pm yT_1(w)x = 0, \quad \text{for all } x, y, w \in R. \tag{2.46}$$

Taking $y = w$ in (2.45) and left multiplying by $T_1(y)$, we get

$$T_1(y)T_1(w)T_2(x) \pm T_1(y)wx = 0, \quad \text{for all } x, y, w \in R. \tag{2.47}$$

Subtracting (2.47) from (2.46), we have $(\pm yT_1(w) \mp T_1(y)w)x = 0$ for all $x, y, w \in R$. Since $R \neq (0)$ and using primeness of R we get $(\pm yT_1(w) \mp T_1(y)w) = 0$ for all $y, w \in R$. This implies that T_1 is a centralizer. Similarly, we can show that T_2 is a centralizer. □

Theorem 2.9. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x)x^* \pm xT_2(x) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x)x^* \pm xT_2(x) \in Z(R), \quad \text{for all } x \in R. \tag{2.48}$$

Linearizing (2.48), we have

$$T_1(x)y^* + T_2(y)x^* \pm xT_2(y) \pm yT_2(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.49}$$

Replacing x, y by kx, ky in (2.49) where $k \in S(R) \cap Z(R)$ and subtracting it from (2.49), we get

$$-2(T_1(x)y^* + T_2(y)x^*)k^2 \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

This implies that

$$2(T_1(x)y^* + T_2(y)x^*)k^2 \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$T_1(x)y^* + T_2(y)x^* \in Z(R), \quad \text{for all } x, y \in R. \tag{2.50}$$

Replacing y by yk in (2.50), where $k \in S(R) \cap Z(R)$ and using (2.50), we get

$$2T_2(y)x^*k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Taking $x = h$, where $h \in H(R) \cap Z(R)$, we get $2T_2(y)hk \in Z(R)$ for all $y \in R$, $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get $T_2(y) \in Z(R)$ for all $y \in R$. This can be further written as

$$[T_2(y), r] = 0, \quad \text{for all } y, r \in R.$$

Replacing y by ym, where $m \in R$, we get $T_2(y)[m, r] = 0$ for all $y, m, r \in R$. Further, replacing y by yw, where $w \in R$, we get $T_2(y)w[m, r] = 0$ for all $y, w, m, r \in R$. Then by primeness, we get either $T_2 = 0$ or $[m, r] = 0$ for all $m, r \in R$. Since $T_2 \neq 0$, therefore we only have $[m, r] = 0$ for all $m, r \in R$. That is, R is commutative. □

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CURVATURE PROPERTIES OF GENERALIZED PP-WAVE METRICS

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Abstract. The main objective of the present paper is to investigate the curvature properties of generalized pp-wave metrics. It is shown that a generalized pp-wave spacetime is Ricci generalized pseudosymmetric, 2-quasi-Einstein and generalized quasi-Einstein in the sense of Chaki. As a special case it is shown that pp-wave spacetime is semisymmetric, semisymmetric due to conformal and projective curvature tensors, R-space by Venzi and satisfies the pseudosymmetric type condition $P \cdot P = -\frac{1}{3}Q(S, P)$. Again we investigate the sufficient condition for which a generalized pp-wave spacetime turns into pp-wave spacetime, pure radiation spacetime, locally symmetric and recurrent. Finally, it is shown that the energy-momentum tensor of pp-wave spacetime is parallel if and only if it is cyclic parallel. Again the energy momentum tensor is Codazzi type if it is cyclic parallel but the converse is not true as shown by an example. Finally, we make a comparison between the curvature properties of the Robinson-Trautman metric and generalized pp-wave metric.

1. Introduction

The class of pp-wave metrics (see [29, 71]) arose during the study of exact solutions of Einstein's field equations. The term "pp-wave" is given by Ehlers and Kundt [29], where "pp" abbreviates the term "plane-fronted gravitational waves with parallel rays". The term "plane-fronted gravitational waves" means there admit a geodesic null vector field whose twist, expansion and shear are zero. The term "plane rays" implies that the rotation of the vector field vanishes. For vacuum type N, this ensures

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the existence of a covariantly constant vector field which is parallel to the null vector field. There are various forms of generalized pp-wave metrics in different coordinates. The pp-wave belongs to the class of solutions admitting a non-expanding, shear-free and twist-free null congruence and it admits a null Killing vector.

The family of pp-wave space-times was first discussed by Brinkmann [4] and interpreted in terms of gravitational waves by Peres [46]. According to Brinkmann, a pp-wave spacetime is any Lorentzian manifold whose metric tensor can be described, with respect to Brinkmann coordinates, in the form

$$ds^2 = H(u, x, y)du^2 + 2dudv + dx^2 + dy^2, \tag{1.1}$$

where H is any nowhere vanishing smooth function. Again it is well known that a Lorentzian manifold with parallel lightlike (null) vector field is called Brinkmann-wave ([4, 30]). A Brinkmann-wave is called pp-wave if its curvature tensor R satisfies the condition $R_{ij}{}^{pq}R_{pqkl} = 0$ ([30, 44, 69]). In 1984, Radhakrishna and Singh [47] presented a class of solutions to Einstein-Maxwell equation for the null electrovac Petrov type N gravitational field. They presented a metric of the form

$$ds^2 = -2Udu^2 + 2dudr - \frac{1}{2}P[(dx^3)^2 + (dx^4)^2],$$

where $U = U(u, x^3, x^4)$ and $P = P(x^3, x^4)$ are two nowhere vanishing smooth functions. For the simplicity of notation, we write the variable u as x, and the function U as h and P as f. Then the aforesaid metric can be written as

$$ds^2 = -2h(x, x^3, x^4)(dx)^2 + 2dxdr - \frac{1}{2}f(x^3, x^4)[(dx^3)^2 + (dx^4)^2]. \tag{1.2}$$

In Section 3 we show that the metric (1.2) admits a covariantly constant null vector field and it satisfies the condition $R_{ij}{}^{pq}R_{pqkl} = 0$ if

$$f_3^2 + f_4^2 - f\,(f_{33} + f_{44}) = 0. \tag{1.3}$$

Thus we can say that the metric (1.2) is a Brinkmann-wave and it becomes a pp-wave if (1.3) holds. Hence we can say the metric (1.2) as "generalized pp-wave metric". We note that for $f \equiv -2$ and $h = -\frac{1}{2}H(u, x^3, x^4)$, the solution (1.2) reduces to the pp-wave metric (1.1).

In the study of differential geometry the notion of manifold of constant curvature has been generalized by many authors in different directions such as locally symmetric manifold by Cartan [5], semisymmetric manifold by Cartan [6] (see also [73–75]), pseudosymmetric manifold by Adamów and Deszcz [1] (see also [15, 22, 24, 34, 35, 78–80]), recurrent manifold by Ruse ([49–51], see also [82]), weakly generalized recurrent manifold by Shaikh and Roy ([52, 66]), hyper generalized recurrent manifold by Shaikh and Patra ([65, 68]), super generalized recurrent manifold by Shaikh et al. [67]. We note that in such geometric structures the first order and second order covariant derivatives of the Riemann curvature tensor and other curvature tensors are involved. We mention that the notion of pseudosymmetry by Deszcz is important in the study of differential geometry due to its application in relativity and cosmology (see [14, 23, 28] and also references therein). The notion of pseudosymmetry is extended by considering other curvature tensors in its defining condition, such as conformal pseudosymmetry, pseudosymmetric Weyl conformal curvature tensor, Ricci generalized pseudosymmetry etc. and they are called pseudosymmetric type conditions. It may be mentioned that different pseudosymmetric type conditions are admitted by various spacetimes, such as Gödel spacetime ([26, 32]), Som-Raychaudhuri spacetime ([57, 70]), Reissner-Nordström spacetime [37], Vaidya spacetime [63] and Robertson-Walker spacetime ([2, 13, 27]).

The main object of the present paper is to investigate the geometric structures admitted by the generalized pp-wave metric (1.2). It is interesting to note that the metric (1.2) without any other condition admits several geometric structures, such as Ricci generalized pseudosymmetry, 2-quasi Einstein and generalized quasi-Einstein in the sense of Chaki [8]. Again it is shown that the pp-wave metric (i.e., (1.2) with condition (1.3)) is Ricci recurrent but not recurrent, semisymmetric, R-space by Venzi, conformal curvature 2-forms are recurrent, Ricci tensor is Riemann compatible and fulfills a pseudosymmetric type condition due to the projective curvature tensor P. For the study of pseudosymmetric type conditions with projective curvature tensor we refer the reader to see the recent papers of Shaikh and Kundu ([58, 59]). It is interesting to note that for such a metric $P \cdot R = 0$ but $P \cdot \mathcal{R} \neq 0$.

It is also shown that the metric is weakly Ricci symmetric and weakly cyclic Ricci symmetric for different associated 1-forms, which ensures the existence of infinitely many solutions of associated 1-forms of such structures. Again we investigate the condition for which such a spacetime is locally symmetric and recurrent.

The paper is organized as follows. Section 2 deals with defining conditions of different curvature restricted geometric structures, such as recurrent, semisymmetry, pseudosymmetry, weakly symmetry etc. as preliminaries. Section 3 is devoted to the investigation of curvature restricted geometric structures admitted by the generalized pp-wave metric (1.2). Section 4 is mainly concerned with the geometric structures admitted by pp-wave metric and plane wave metric. Section 5 deals with the investigation of the conditions under which the energy-momentum tensor of such spacetimes are parallel, Codazzi type and cyclic parallel. Finally, the last section is devoted to make a comparison between the curvature properties of the Robinson-Trautman metric and generalized pp-wave metric as well as pp-wave metric.

2. Preliminaries

Let M be a connected smooth semi-Riemannian manifold of dimension n (≥ 3) equipped with the semi-Riemannian metric g. Let R, $\mathcal{R}$, S, $\mathcal{S}$ and κ be respectively the Riemann-Christoffel curvature tensor of type $(0,4)$, the Riemann-Christoffel curvature tensor of type $(1,3)$, the Ricci tensor of type $(0,2)$, the Ricci tensor of type $(1,1)$ and the scalar curvature of M.

For two symmetric $(0,2)$-tensors A and E, their Kulkarni-Nomizu product $A \wedge E$

is defined as (see e.g. [22, 33]):

$$\begin{aligned}(A \wedge E)(X_1, X_2, X_3, X_4) =& A(X_1, X_4)E(X_2, X_3) + A(X_2, X_3)E(X_1, X_4) \\ &- A(X_1, X_3)E(X_2, X_4) - A(X_2, X_4)E(X_1, X_3),\end{aligned}$$

where $X_1, X_2, X_3, X_4 \in \chi(M)$, the Lie algebra of all smooth vector fields on M. Throughout the paper we will consider $X, Y, X_1, X_2, \ldots \in \chi(M)$.

In terms of Kulkarni-Nomizu product, the conformal curvature tensor C, the concircular curvature tensor W, the conharmonic curvature tensor K ([36, 83]) and the Gaussian curvature tensor $\mathfrak{G}$ can be expressed as

$$\begin{aligned}C =& R - \frac{1}{n-2}(g \wedge S) + \frac{\kappa}{2(n-2)(n-1)}(g \wedge g), \\ W =& R - \frac{r}{2n(n-1)}(g \wedge g), \\ K =& R - \frac{1}{n-2}(g \wedge S), \\ \mathfrak{G} =& \frac{1}{2}(g \wedge g).\end{aligned}$$

Again the projective curvature tensor P of type $(0, 4)$ is given by

$$\begin{aligned}P(X_1, X_2, X_3, X_4) =& R(X_1, X_2, X_3, X_4) \\ &- \frac{1}{n-1}[g(X_1, X_4)S(X_2, X_3) - g(X_2, X_4)S(X_1, X_3)].\end{aligned}$$

For a symmetric $(0, 2)$-tensor A, we get an endomorphism $\mathcal{A}$ defined by $g(\mathcal{A}X, Y) = A(X, Y)$. Then its k-th level tensor A^k of type $(0, 2)$ is given by

$$A^k(X, Y) = A(\mathcal{A}^{k-1}X, Y),$$

where $\mathcal{A}^{k-1}$ is the endomorphism corresponding to A^{k-1}, $k = 2, 3, \ldots$, and $A^1 = A$.

Definition 2.1 ([3, 60])**.** A semi-Riemannian manifold M is said to be Ein(2), Ein(3) and Ein(4) respectively if

$$\begin{aligned}S^2 + \lambda_1 S + \lambda_2 g =& 0, \\ S^3 + \lambda_3 S^2 + \lambda_4 S + \lambda_5 g =& 0, \\ S^4 + \lambda_6 S^3 + \lambda_7 S^2 + \lambda_8 S + \lambda_9 g =& 0,\end{aligned}$$

holds for some scalars λ_i, $1 \le i \le 9$.

Definition 2.2 ([3, 60])**.** A semi-Riemannian manifold M is said to be generalized Roter type ([56, 60]) if its Riemann-Christoffel curvature tensor R can be expressed as a linear combination of $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. Again M is said to be Roter type (see [16, 17]) if R can be expressed as a linear combination of $g \wedge g$, $g \wedge S$ and $S \wedge S$.

Again for a $(0,4)$-tensor D, an endomorphism $\mathscr{D}(X,Y)$ and the corresponding $(1,3)$-tensor $\mathcal{D}$ can be defined as

$$\mathscr{D}(X,Y)X_1 = \mathcal{D}(X,Y)X_1, \quad D(X_1,X_2,X_3,X_4) = g(\mathcal{D}(X_1,X_2)X_3,X_4).$$

Again for a symmetric $(0,2)$-tensor A, another endomorphism $X \wedge_A Y$ (see [13,22]) can be defined as

$$(X \wedge_A Y)X_1 = A(Y,X_1)X - A(X,X_1)Y.$$

By operating $\mathscr{D}(X,Y)$ and $X \wedge_A Y$ on a $(0,k)$-tensor B, $k \geq 1$, we can obtain two $(0,k+2)$-tensors $D \cdot B$ and $Q(A,B)$ respectively given by (see [18,19,25,54,55] and also references therein):

$$\begin{aligned} &D \cdot B(X_1,X_2,\dots,X_k,X,Y) \\ =& -B(\mathcal{D}(X,Y)X_1,X_2,\dots,X_k) - \cdots - B(X_1,X_2,\dots,\mathcal{D}(X,Y)X_k). \end{aligned}$$

and

$$\begin{aligned} &Q(A,B)(X_1,X_2,\dots,X_k,X,Y) \\ =&((X \wedge_A Y) \cdot B)(X_1,X_2,\dots,X_k) \\ =&A(X,X_1)B(Y,X_2,\dots,X_k) + \cdots + A(X,X_k)B(X_1,X_2,\dots,Y) \\ &- A(Y,X_1)B(X,X_2,\dots,X_k) - \cdots - A(Y,X_k)B(X_1,X_2,\dots,X). \end{aligned}$$

In terms of local coordinates system, $D \cdot B$ and $Q(A,B)$ can be written as

$$\begin{aligned} (D \cdot B)_{i_1 i_2 \cdots i_k jl} =& - g^{pq} \left[D_{jli_1q} B_{pi_2\cdots i_k} + \cdots + D_{jli_kq} B_{i_1 i_2 \cdots p} \right], \\ Q(A,B)_{i_1 i_2 \cdots i_k jl} =& A_{li_1} B_{ji_2\cdots i_k} + \cdots + A_{li_k} B_{i_1 i_2 \cdots j} \\ & - A_{ji_1} B_{li_2 \cdots i_k} - \cdots - A_{ji_k} B_{i_1 i_2 \cdots l}. \end{aligned}$$

Definition 2.3 ([45,49–51])**.** A semi-Riemannian manifold M is said to be B-recurrent if $\nabla B = \Pi \otimes B$ for an 1-form Π. In particular for $B = R$ (resp., S), the manifold M is called a recurrent (resp., Ricci recurrent) manifold.

Definition 2.4 ([1,6,15,55,73])**.** A semi-Riemannian manifold M is said to be B-semisymmetric type if $D \cdot B = 0$ and it is said to be B-pseudosymmetric type if $\left(\sum\limits_{i=1}^{k} c_i D_i\right) \cdot B = 0$ for some scalars c_i's, where D and each D_i, $i = 1,\dots,k$, $k \geq 2$, are (0,4) curvature tensors. In particular, if c_i's are all constants, then it is called B-pseudosymmetric type manifold of constant type or otherwise non-constant type.

In particular, if $i = 2$, $D_1 = R$, $D_2 = \mathfrak{G}$ and $B = R$, then M is called Ricci generalized pseudosymmetric [12].

Definition 2.5. A semi-Riemannian manifold M is said to be quasi-Einstein (resp. 2-quasi-Einstein) if at each point of M, $\operatorname{rank}(S - \alpha g) \leq 1$ (resp., ≤ 2) for a scalar α. Also M is said to be generalized quasi-Einstein in the sense of Chaki [8] if

$$S = \alpha g + \beta \Pi \otimes \Pi + \gamma[\Pi \otimes \Omega + \Pi \otimes \Omega],$$

for some 1-forms Π and Ω.

Quasi-Einstein, as well as 2-quasi-Einstein manifolds were investigated among others in [2, 9–11, 19, 21, 22] and [25].

Definition 2.6. Let D be a $(0,4)$-tensor and E be a symmetric $(0,2)$-tensor on a semi-Riemannian manifold M. Then E is said to be D-compatible ([20, 39, 40]) if

$$D(\mathcal{E}X_1, X, X_2, X_3) + D(\mathcal{E}X_2, X, X_3, X_1) + D(\mathcal{E}X_3, X, X_1, X_2) = 0$$

holds, where $\mathcal{E}$ is the endomorphism corresponding to E defined as $g(\mathcal{E}X_1, X_2) = E(X_1, X_2)$. Again an 1-form Π is said to be D-compatible if $\Pi \otimes \Pi$ is D-compatible.

Definition 2.7. A semi-Riemannian manifold M is said to be weakly cyclic Ricci symmetric [64] if its Ricci tensor satisfies the condition

$$\begin{aligned}(2.1)\qquad &(\nabla_X S)(X_1, X_2) + (\nabla_{X_1} S)(X, X_2) + (\nabla_{X_2} S)(X_1, X)\\ =&\Pi(X)\, S(X_1, X_2) + \Omega(X_1)\, S(X, X_2) + \Theta(X_2)\, S(X_1, X),\end{aligned}$$

for three 1-forms Π, Ω and Θ on M. Such a manifold is called weakly cyclic Ricci symmetric manifold with solution (Π, Ω, Θ). Moreover if the first term of left hand side is equal to the right hand side, then it is called weakly Ricci symmetric manifold [77].

Definition 2.8. Let D be a $(0,4)$ tensor and Z be a $(0,2)$-tensor on a semi-Riemannian manifold M. Then the corresponding curvature 2-forms $\Omega^m_{(D)l}$ (see [3, 38]) are called recurrent if and only if ([41–43])

$$\begin{aligned}&(\nabla_{X_1} D)(X_2, X_3, X, Y) + (\nabla_{X_2} D)(X_3, X_1, X, Y) + (\nabla_{X_3} D)(X_1, X_2, X, Y)\\ =&\Pi(X_1) D(X_2, X_3, X, Y) + \Pi(X_2) D(X_3, X_1, X, Y) + \Pi(X_3) D(X_1, X_2, X, Y),\end{aligned}$$

and 1-forms $\Lambda_{(Z)l}$ [72] are called recurrent if and only if

$$(\nabla_{X_1} Z)(X_2, X) - (\nabla_{X_2} Z)(X_1, X) = \Pi(X_1) Z(X_2, X) - \Pi(X_2) Z(X_1, X),$$

for an 1-form Π.

Definition 2.9 ([48, 59, 81])**.** Let $\mathcal{L}(M)$ be the vector space formed by all 1-forms Θ on a semi-Riemannian manifold M satisfying

$$\Theta(X_1) D(X_2, X_3, X_4, X_5) + \Theta(X_2) D(X_3, X_1, X_4, X_5) + \Theta(X_3) D(X_1, X_2, X_4, X_5) = 0,$$

where D is a $(0,4)$-tensor. Then M is said to be a D-space by Venzi if $\dim \mathcal{L}(M) \geq 1$.

3. Curvature Properties of Generalized pp-Wave Metric

We can now write the metric tensor g of the generalized pp-wave metric (1.2) as follows:

$$g = \begin{pmatrix} -2h & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}f & 0 \\ 0 & 0 & 0 & -\frac{1}{2}f \end{pmatrix}.$$

Then the non-zero components of its Riemann-Christoffel curvature tensor R, Ricci tensor S and scalar curvature κ of (1.2) are given by

$$\begin{aligned}
R_{1313} =& \frac{-f_3h_3 + f_4h_4 + 2fh_{33}}{2f}, \quad R_{1314} = \frac{-f_4h_3 - f_3h_4 + 2fh_{34}}{2f}, \\
R_{1414} =& \frac{f_3h_3 - f_4h_4 + 2fh_{44}}{2f}, \quad R_{3434} = \frac{-f_3^2 - f_4^2 + ff_{33} + ff_{44}}{4f}, \\
S_{11} =& \frac{2\left(h_{33} + h_{44}\right)}{f}, \quad S_{33} = S_{44} = \frac{-f_3^2 - f_4^2 + ff_{33} + ff_{44}}{2f^2}, \\
\kappa =& \frac{2\left(f_3^2 + f_4^2 - f\left(f_{33} + f_{44}\right)\right)}{f^3},
\end{aligned}$$

where $f_3 = \frac{\partial f}{\partial x^3}$, $f_4 = \frac{\partial f}{\partial x^4}$, $f_{33} = \frac{\partial f_3}{\partial x^3}$, $f_{34} = \frac{\partial f_3}{\partial x^4}$, $f_{44} = \frac{\partial f_4}{\partial x^4}$ etc.
Again the non-zero components of the conformal curvature tensor C and the projective curvature tensor P are given by

$$\begin{aligned}
C_{1313} =& - C_{1414} = \frac{-f_3h_3 + f_4h_4 + fh_{33} - fh_{44}}{2f}, \quad C_{1314} = \frac{-f_4h_3 - f_3h_4 + 2fh_{34}}{2f}, \\
P_{1211} =& \frac{2\left(h_{33} + h_{44}\right)}{3f}, \quad P_{1313} = \frac{-3f_3h_3 + 3f_4h_4 + 4fh_{33} - 2fh_{44}}{6f}, \\
P_{1314} =& - P_{1341} = P_{1413} = -P_{1431} = \frac{-f_4h_3 - f_3h_4 + 2fh_{34}}{2f}, \\
P_{1441} =& - \frac{f_3h_3 - f_4h_4 + 2fh_{44}}{2f}, \\
P_{1331} =& - \frac{-f_3h_3 + f_4h_4 + 2fh_{33}}{2f}, \quad P_{1414} = -\frac{-3f_3h_3 + 3f_4h_4 + 2fh_{33} - 4fh_{44}}{6f}.
\end{aligned}$$

Now the non-zero components (upto symmetry) of the energy momentum tensor

$$T = \frac{c^4}{8\pi G}\left[S - \left(\frac{\kappa}{2} - \Lambda\right) g\right],$$

where $c =$ speed of light in vacuum, $G =$ gravitational constant and $\Lambda =$ cosmological constant, are given by

$$\begin{aligned}
T_{11} =& - \frac{c^4\left(f^3h\Lambda - f^2h_{33} - f^2h_{44} + f_{33}fh + f_{44}fh - f_3^2h - f_4^2h\right)}{4\pi f^3G}, \\
T_{12} =& \frac{c^4\left(f^3\Lambda + f_{33}f + f_{44}f - f_3^2 - f_4^2\right)}{8\pi f^3G}, \quad T_{33} = T_{44} = -\frac{c^4f\Lambda}{16\pi G}.
\end{aligned}$$

Then the non-zero components (upto symmetry) of covariant derivative ∇T of the energy momentum tensor T are given by

$$T_{11,1} = \frac{c^4 (h_{133} + h_{144})}{4\pi f G},$$

$$T_{11,3} = \frac{c^4}{4\pi f^4 G}\Big(-f^2 h f_{344} - f^2 h f_{333} + f^3 h_{333} + f^3 h_{344} - f_3 f^2 h_{33} - f_3 f^2 h_{44} + 4 f_3 f_{33} f h + 2 f_4 f_{34} f h + 2 f_3 f_{44} f h - 3 f_3^3 h - 3 f_3 f_4^2 h\Big),$$

$$T_{11,4} = \frac{c^4}{4\pi f^4 G}\Big(-f^2 h f_{444} - f^2 h f_{344} + f^3 h_{334} + f^3 h_{444} - f_4 f^2 h_{33} - f_4 f^2 h_{44} + 2 f_4 f_{33} f h + 2 f_3 f_{34} f h + 4 f_4 f_{44} f h - 3 f_4^3 h - 3 f_3^2 f_4 h\Big),$$

$$T_{12,3} = \frac{c^4 (f^2 f_{344} + f^2 f_{333} + 3 f_3^3 + 3 f_4^2 f_3 - 4 f f_{33} f_3 - 2 f f_{44} f_3 - 2 f f_4 f_{34})}{8\pi f^4 G},$$

$$T_{12,4} = \frac{c^4 (f^2 f_{444} + f^2 f_{334} + 3 f_4^3 + 3 f_3^2 f_4 - 2 f f_{33} f_4 - 4 f f_{44} f_4 - 2 f f_3 f_{34})}{8\pi f^4 G},$$

$$T_{13,1} = \frac{c^4 (-f_3^2 - f_4^2 + f f_{33} + f f_{44}) h_3}{8\pi f^3 G},$$

$$T_{14,1} = \frac{c^4 (-f_3^2 - f_4^2 + f f_{33} + f f_{44}) h_4}{8\pi f^3 G}.$$

From above we see that the Ricci tensor S of (1.2) is of the form

$$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y) + \gamma[\eta(X)\delta(Y) + \eta(Y)\delta(X)], \tag{3.1}$$

where $\alpha = \frac{f_3^2 + f_4^2 - f f_{33} - f f_{44}}{f^3}$, $\beta = 1$, $\gamma = 1$, $\eta = \{1, 0, 0, 0\}$ and

$$\delta = \left\{ \frac{2f^2 (h_{33} + h_{44}) - 2(f_{33} + f_{44}) f h + 2(f_3^2 + f_4^2) h - f^3}{2f^3}, \frac{f(f_{33} + f_{44} - f_3^2 - f_4^2)}{f^3}, 0, 0 \right\}.$$

Therefore, the metric (1.2) is generalized quasi-Einstein in the sense of Chaki. Moreover, $||\eta|| = 0$, $||\delta||^2 = \frac{\left(f_3^2 + f_4^2 - f(f_{33} + f_{44})\right)(f - 2(h_{33} + h_{44}))}{f^4}$, $g(\eta, \delta) = -\frac{f_3^2 + f_4^2 - f(f_{33} + f_{44})}{f^3}$ and $\nabla\eta = 0$. So there exists a null covariantly constant vector field ζ, where ζ is the corresponding vector field of η (i.e., $g(\zeta, X) = \eta(X)$, for all X). Hence we can conclude that the spacetime with the metric (1.2) is a generalized pp-wave metric.

Now from the value of the components of various tensors related to (1.2), we can state the following.

Theorem 3.1. *The generalized pp-wave metric* (1.2) *possesses the following curvature restricted geometric structures:*

(i) *Ricci generalized pseudosymmetric such that* $R \cdot R = Q(S, R)$*;*
(ii) 2*-quasi-Einstein, since* rank $\left(S - \frac{f_3^2 + f_4^2 - f(f_{33} + f_{44})}{f^3} g\right) = 2$*;*
(iii) *generalized quasi-Einstein in the sense of Chaki such that* (3.1) *holds;*
(iv) *Ricci tensor is Riemann compatible as well as Weyl compatible;*

(v) Ein(3) *manifold such that* $S^3 = -\frac{f_3^2+f_4^2-f(f_{33}+f_{44})}{f^3} S^2$.

Now from the components of R, we see that the only non-zero component (upto symmetry) of the tensor $D_{ijkl} = R_{ij}^{pq} R_{pqkl}$ is given by $D_{3434} = \frac{\left(f_3^2+f_4^2-f(f_{33}+f_{44})\right)^2}{2f^4}$. Hence from definition, we can state the following.

Theorem 3.2. *The generalized pp-wave metric* (1.2) *becomes a pp-wave metric if*

$$f_3^2 + f_4^2 - f\left(f_{33} + f_{44}\right) = 0.$$

Example 3.1. If we consider $f(x^3, x^4) = e^{x^3+x^4}$, then $f_3^2 + f_4^2 - f\left(f_{33} + f_{44}\right) = 0$. Hence by above theorem the metric

$$ds^2 = -2h(x, x^3, x^4)(dx)^2 + 2dxdr - \frac{1}{2} e^{x^3+x^4} [(dx^3)^2 + (dx^4)^2]$$

is a pp-wave metric.

4. Curvature Properties of pp-Wave and Plane Wave Metric

In this section we investigate the curvature restricted geometric structures admitted by the pp-wave metric. Since under the condition (1.3) the generalized pp-wave metric (1.2) becomes a pp-wave metric, putting this condition we get the non-zero components of R, S, C and P of the pp-wave metric given as follows:

$$\begin{aligned}
R_{1313} =& \frac{-f_3h_3 + f_4h_4 + 2fh_{33}}{2f}, \quad R_{1314} = \frac{-f_4h_3 - f_3h_4 + 2fh_{34}}{2f}, \\
R_{1414} =& \frac{f_3h_3 - f_4h_4 + 2fh_{44}}{2f}, \\
S_{11} =& \frac{2\left(h_{33} + h_{44}\right)}{f}, \\
C_{1313} =& - C_{1414} = \frac{-f_3h_3 + f_4h_4 + fh_{33} - fh_{44}}{2f}, \quad C_{1314} = \frac{-f_4h_3 - f_3h_4 + 2fh_{34}}{2f}, \\
P_{1211} =& \frac{2\left(h_{33} + h_{44}\right)}{3f}, \quad P_{1313} = \frac{-3f_3h_3 + 3f_4h_4 + 4fh_{33} - 2fh_{44}}{6f}, \\
P_{1314} =& - P_{1341} = P_{1413} = -P_{1431} = \frac{-f_4h_3 - f_3h_4 + 2fh_{34}}{2f}, \\
P_{1441} =& - \frac{f_3h_3 - f_4h_4 + 2fh_{44}}{2f}, \\
P_{1331} =& - \frac{-f_3h_3 + f_4h_4 + 2fh_{33}}{2f}, \quad P_{1414} = -\frac{-3f_3h_3 + 3f_4h_4 + 2fh_{33} - 4fh_{44}}{6f}.
\end{aligned}$$

Using the values of the components of g, R, S and C we get
(i) $\kappa = 0$, (ii) $R \cdot R = 0$, (iii) $R \cdot S = 0$, (iv) $Q(S, R) = 0$, (v) $R \cdot C = 0$, (vi) $C \cdot R = 0$,

(vii) $C \cdot C = 0$ and (viii) $Q(S, C) = 0$.

The energy momentum tensor T is given by

$$T_{11} = -\frac{c^4 \left(fh\Lambda - h_{33} - h_{44}\right)}{4\pi f G}, \quad T_{12} = \frac{c^4 \Lambda}{8\pi G}, \quad T_{33} = T_{44} = -\frac{c^4 f \Lambda}{16\pi G}. \tag{4.1}$$

Then the non-zero components of covariant derivative of T are given by

$$\begin{cases} T_{11,1} = \dfrac{c^4 \left(h_{144} + h_{133}\right)}{4\pi f G}, \\ T_{11,3} = \dfrac{c^4 \left(f h_{344} + f h_{333} - f_3 h_{33} - f_3 h_{44}\right)}{4\pi f^2 G}, \\ T_{11,4} = \dfrac{c^4 \left(f h_{444} + f h_{334} - f_4 h_{33} - f_4 h_{44}\right)}{4\pi f^2 G}. \end{cases} \tag{4.2}$$

From the above calculations, we can state the following.

Theorem 4.1. *The pp-wave metric* (1.2) *with the additional condition* (1.3) *possesses the following curvature properties.*

(i) $\kappa = 0$ *and hence* $R = W$ *and* $C = K$.

(ii) *R-space and C-space by Venzi for* $\{1, 0, 0, 0\}$. *Hence, from second Bianchi identity, the curvature* 2*-forms* $\Omega^m_{(R)l}$ *are recurrent for the* 1*-form* $\{1, 0, 0, 0\}$.

(iii) *It is semisymmetric and hence Ricci semisymmetric, conformally semisymmetric and projectively semisymmetric.*

(iv) *If* $\alpha_3 + \alpha_4 \neq 0$, *then the conformal* 2*-forms* $\Omega^m_{(C)l}$ *are recurrent with* 1*-form of recurrency*

$$\Pi = \left\{1, 0, \frac{\alpha_1 + \alpha_2}{\alpha_3 + \alpha_4}, \frac{\alpha_5 + \alpha_6 + \alpha_7}{\alpha_3 + \alpha_4}\right\},$$

where

$$\begin{aligned} \alpha_1 &= \left(f_3 h_3 - f_4 h_4 - f\left(h_{33} - h_{44}\right)\right)\left(f_3\left(h_{33} + h_{44}\right) - f\left(h_{333} + h_{344}\right)\right), \\ \alpha_2 &= \left(-f_4 h_3 - f_3 h_4 + 2 f h_{34}\right)\left(f\left(h_{334} + h_{444}\right) - f_4\left(h_{33} + h_{44}\right)\right), \\ \alpha_3 &= f^2\left(4h_{34}^2 + \left(h_{33} - h_{44}\right)^2\right) + f_3^2\left(h_3^2 + h_4^2\right) + f_4^2\left(h_3^2 + h_4^2\right), \\ \alpha_4 &= 2 f f_4\left(h_4\left(h_{33} - h_{44}\right) - 2h_3 h_{34}\right) - 2 f f_3\left(2h_4 h_{34} + h_3\left(h_{33} - h_{44}\right)\right), \\ \alpha_5 &= f^2\left(2h_{34}\left(h_{333} + h_{344}\right) + \left(h_{44} - h_{33}\right)\left(h_{334} + h_{444}\right)\right) \\ &\quad + f_3^2 h_4\left(h_{33} + h_{44}\right) + f_4^2 h_4\left(h_{33} + h_{44}\right), \\ \alpha_6 &= f f_3\left(-2h_{34}\left(h_{33} + h_{44}\right) - h_4\left(h_{333} + h_{344}\right) + h_3\left(h_{334} + h_{444}\right)\right), \\ \alpha_7 &= -f f_4\left(-h_{33}^2 + h_{44}^2 + h_3\left(h_{333} + h_{344}\right) + h_4\left(h_{334} + h_{444}\right)\right). \end{aligned}$$

(v) *It is not recurrent but if* $h_{33} + h_{44} \neq 0$, *then it is Ricci recurrent with* 1*-form of recurrency*

$$\Pi = \left\{\frac{h_{144} + h_{133}}{h_{33} + h_{44}}, 0, \frac{f h_{344} + f h_{333} - f_3 h_{33} - f_3 h_{44}}{f\left(h_{33} + h_{44}\right)}, \frac{f h_{444} + f h_{334} - f_4 h_{33} - f_4 h_{44}}{f\left(h_{33} + h_{44}\right)}\right\}.$$

(vi) *If* $h_{33}+h_{44} \neq 0$*, then it is weakly cyclic Ricci symmetric with solution* (Π, Ω, Θ)*, given by*

$$\Pi = \left\{\Pi_1, 0, \frac{f\,(h_{333}+h_{344}) - f_3\,(h_{33}+h_{44})}{f\,(h_{33}+h_{44})}, \frac{f\,(h_{334}+h_{444}) - f_4\,(h_{33}+h_{44})}{f\,(h_{33}+h_{44})}\right\},$$

$$\Omega = \left\{\Omega_1, 0, \frac{f\,(h_{333}+h_{344}) - f_3\,(h_{33}+h_{44})}{f\,(h_{33}+h_{44})}, \frac{f\,(h_{334}+h_{444}) - f_4\,(h_{33}+h_{44})}{f\,(h_{33}+h_{44})}\right\},$$

$$\Theta = \left\{\frac{3\,(h_{133}+h_{144})}{h_{33}+h_{44}} - \Pi_1 - \Omega_1, 0, \frac{f\,(h_{333}+h_{344}) - f_3\,(h_{33}+h_{44})}{f\,(h_{33}+h_{44})}, \frac{f\,(h_{334}+h_{444}) - f_4\,(h_{33}+h_{44})}{f\,(h_{33}+h_{44})}\right\},$$

where Π_1 *and* Ω_1 *are arbitrary scalars.*

(vii) *Ricci simple* (*i.e.,* $S = \alpha\eta \otimes \eta$) *for*

$$\alpha = \frac{2(h_{33}+h_{44})}{f} \quad \textit{and} \quad \eta = \{1, 0, 0, 0\}$$

and hence $S \wedge S = 0$ *and* $S^2 = 0$*. Again,* $||\eta|| = 0$ *and* $\nabla\eta = 0$*.*

(viii) $Q(S,R) = Q(S,C) = 0$ *but* R *or* C *is not a scalar multiple of* $S \wedge S$ *as* S *is of rank* 1*.*

(ix) $C \cdot R = 0$ *and hence* $C \cdot S = 0$*,* $C \cdot C = 0$ *and* $C \cdot P = 0$*.*

(x) $P \cdot R = 0$ *but* $P \cdot \mathcal{R} \neq 0$*. Also but* $P \cdot S = P \cdot \mathcal{S} = 0$*.*

(xi) *Ricci tensor is Riemann compatible as well as Weyl compatible.*

(xii) $P \cdot P = -\frac{1}{3}Q(S,P)$*.*

Remark 4.1. From the value of the local components (presented in Section 4) of various tensors of the pp-wave metric, we can easily conclude that the metric is

(i) not conformally symmetric and hence not locally symmetric or projectively symmetric;

(ii) not conformally recurrent and hence not recurrent or not projectively recurrent;

(iii) not super generalized recurrent [67] and hence not hyper generalized recurrent [65], weakly generalized recurrent [66];

(iv) not weakly symmetric [76] for R, C, P, W and K and hence not Chaki pseudosymmetric [7] for R, C, P, W and K;

(v) neither cyclic Ricci parallel [31] nor of Codazzi type Ricci tensor although its scalar curvature is constant;

(vi) not harmonic, i.e., $\operatorname{div} \mathrm{R} \neq 0$ and moreover $\operatorname{div} \mathrm{C} \neq 0$, $\operatorname{div} \mathrm{P} \neq 0$.

Remark 4.2. In [12] it was shown that if $Q(S,R) = 0$, then $R = LS \wedge S$ for some scalar L if S is not of rank 1. Recently, in Example 1 of [59] a metric with $Q(S,R) = 0$, on which S is not of rank 1 and $R = e^{x^1} S \wedge S$ is presented. It is interesting to mention that the rank of the Ricci tensor of the pp-wave metric is 1 and here $R \neq 0$ but $S \wedge S = 0$.

Remark 4.3. It is well-known that every Ricci recurrent space with Π as the 1-form of recurrency, is weakly Ricci symmetric with solution $(\Pi, 0, 0)$. It is interesting to mention that there may infinitely many solutions for a weakly Ricci symmetric manifold. The pp-wave metric ((1.2) with condition (1.3)) is weakly Ricci symmetric with solution

$$\begin{aligned}
\Pi =& \left\{\Pi_1, 0, \frac{f\left(h_{333}+h_{344}\right)-f_3\left(h_{33}+h_{44}\right)}{f\left(h_{33}+h_{44}\right)}, \frac{f\left(h_{334}+h_{444}\right)-f_4\left(h_{33}+h_{44}\right)}{f\left(h_{33}+h_{44}\right)}\right\},\\
\Omega =& \left\{\Omega_1, 0, 0, 0\right\},\\
\Theta =& \left\{\frac{h_{133}+h_{144}}{h_{33}+h_{44}} - \Pi_1 - \Omega_1, 0, 0, 0\right\},
\end{aligned}$$

where Π_1 and Ω_1 are arbitrary scalars.

Again it is clear that the pp-wave metric (1.1) in Brinkmann coordinates, is a special case of (1.2) for $f \equiv -2$ and $h = -\frac{1}{2}H$ and hence satisfies (1.3). Therefore from Theorem 4.1, we can state the following about the geometric properties of the metic (1.1).

Corollary 4.1. *The metric given in* (1.1) *possesses the following curvature properties.*

(i) $\kappa = 0$ *and hence* $R = W$ *and* $C = K$.

(ii) *R-space and C-space by Venzi for* $\{1, 0, 0, 0\}$. *Hence, from second Bianchi identity, the curvature* 2*-forms* $\Omega^m_{(R)l}$ *are recurrent for the* 1*-form* $\{1, 0, 0, 0\}$.

(iii) *Semisymmetric and hence Ricci semisymmetric, conformally semisymmetric and projectively semisymmetric.*

(iv) *If* $4H_{34}^2 + (H_{33} - H_{44})^2 \neq 0$, *then its conformal curvature* 2*-forms* $\Omega^m_{(C)l}$ *are recurrent with* 1*-form of recurrency* Π, *given by*

$$\begin{aligned}
\Pi_1 =& 1, \quad \Pi_2 = 0,\\
\Pi_3 =& \frac{2H_{34}\left(H_{334}+H_{444}\right)+\left(H_{33}-H_{44}\right)\left(H_{333}+H_{344}\right)}{4H_{34}^2+\left(H_{33}-H_{44}\right)^2},\\
\Pi_4 =& \frac{2H_{34}\left(H_{333}+H_{344}\right)-\left(H_{33}-H_{44}\right)\left(H_{334}+H_{444}\right)}{4H_{34}^2+\left(H_{33}-H_{44}\right)^2}.
\end{aligned}$$

(v) *If* $H_{33} + H_{44} \neq 0$, *then it is Ricci recurrent with* 1*-form of recurrency* Π, *given by*

$$\Pi = \left\{1, 0, \frac{H_{333}+H_{344}}{H_{33}+H_{44}}, \frac{H_{334}+H_{444}}{H_{33}+H_{44}}\right\}.$$

(vi) *If* $H_{33} + H_{44} \neq 0$*, then it is weakly Ricci symmetric with solution* (Π, Ω, Θ)*, given by*

$$\Pi = \left\{\Pi_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$
$$\Omega = \{\Omega_1, 0, 0, 0\},$$
$$\Theta = \left\{\frac{H_{133} + H_{144}}{H_{33} + H_{44}} - \Pi_1 - \Omega_1, 0, 0, 0\right\},$$

where Π_1 *and* Ω_1 *are arbitrary scalars.*

(vii) *If* $H_{33}+H_{44} \neq 0$*, then it is weakly cyclic Ricci symmetric with solution* (Π, Ω, Θ)*, given by*

$$\Pi = \left\{\Pi_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$
$$\Omega = \left\{\Omega_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$
$$\Theta = \left\{\frac{3\,(H_{133} + H_{144})}{H_{33} + H_{44}} - \Pi_1 - \Omega_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$

where Π_1 *and* Ω_1 *are arbitrary scalars.*

(viii) *Ricci simple* [44] (*i.e.,* $S = \alpha\eta \otimes \eta$) *for*

$$\alpha = \frac{1}{2}(H_{33} + H_{44}) \quad \text{and} \quad \eta = \{1, 0, 0, 0\},$$

and hence $S \wedge S = 0$ *and* $S^2 = 0$*. Here* $||\eta|| = 0$ *and* $\nabla\eta = 0$*.*

(ix) $Q(S, R) = Q(S, C) = 0$ *but* R *or* C *is not a scalar multiple of* $S \wedge S$ *as* S *is of rank* 1*.*

(x) $C \cdot R = 0$ *and hence* $C \cdot S = 0$*,* $C \cdot C = 0$ *and* $C \cdot P = 0$*.*

(xi) $P \cdot R = 0$ *but* $P \cdot \mathcal{R} \neq 0$*. Also but* $P \cdot S = P \cdot \mathcal{S} = 0$*.*

(xii) *Ricci tensor is Riemann compatible as well as Weyl compatible.*

(xiii) $P \cdot P = -\frac{1}{3}Q(S, P)$*.*

Again the non-vacuum pp-wave solution presented in [69] is a special case of (1.1) for $H(x, x^3, x^4) = 2a_1e^{a_2x^3 - a_3x^4}$. Hence the line element is explicitly given by:

$$ds^2 = 2a_1e^{a_2x^3 - a_3x^4}(dx)^2 + 2dxdr + [(dx^3)^2 + (dx^4)^2]. \tag{4.3}$$

Now the geometric properties of the metric (4.3) can be stated as follows.

Corollary 4.2. *The metric given in* (4.3) *possesses the following curvature properties.*

(i) *It satisfies the curvature conditions* (i)-(xiii) *of Corollary* 4.1 *with different associated* 1*-forms of the corresponding structures.*

(ii) *Moreover it is recurrent for the* 1*-form of recurrency* $\{0, 0, 2a_2, -2a_3\}$*. Hence it is Ricci recurrent, conformally recurrent and projectively recurrent. Also it is semisymmetric and hence Ricci semisymmetric, conformally semisymmetric and projectively semisymmetric.*

Again the generalized plane wave metric [71] is given by

$$ds^2 = 2H(x, x^3, x^4)(dx)^2 + 2dxdr + (dx^3)^2 + (dx^4)^2, \tag{4.4}$$

where $H(x, x^3, x^4) = a_1(x^3)^2 + a_2(x^4)^2 + a_3x^3x^4 + a_4x^3 + a_5x^4 + a_6$, a_i's are scalar. Hence, it is a special case of (1.1) and we can state the following.

Corollary 4.3. *The metric given in* (4.4) *possesses the following curvature properties:*

(i) *it satisfies the curvature conditions* (i)-(ix) *of Corollary* 4.1 *with different associated* 1*-forms of the corresponding structures;*
(ii) *moreover it is locally symmetric and hence Ricci symmetric, conformally symmetric and projectively symmetric.*

From Corollary 4.1, Corollary 4.2 and Corollary 4.3, we can state the following about the recurrent structure on a semi-Riemannian manifold.

Remark 4.4. From Corollary 4.2 we see that the metric (4.3) is recurrent but not locally symmetric and from Corollary 4.1 we see that the metric (1.1) is Ricci recurrent but not recurrent. These results support the well-known facts that every locally symmetric manifold is recurrent but not conversely, and every recurrent manifold is Ricci recurrent but not conversely.

5. Energy-Momentum Tensor of Generalized pp-Wave Metric

In this section we discuss about the energy-momentum tensor of the generalized pp-wave metric (1.2) and also the other special forms, such as (1.1), (4.3) and (4.4). From the values of the energy momentum tensor T of the generalized pp-wave metric (1.2), we can conclude that T is of rank 1 if the cosmological constant is zero and (1.3) holds. In this case

$$T = \frac{c^4\,(h_{33} + h_{44})}{4\pi f G}\eta \otimes \eta, \quad \eta = \{1, 0, 0, 0\}.$$

Again it is easy to check that η is null. Thus we can state the following.

Theorem 5.1. *For zero cosmological constant, the generalized pp-wave metric* (1.2) *is a pure radiation metric if and only if it is a pp-wave metric.*

Corollary 5.1. *For zero cosmological constant, the pp-wave metric* ((1.2) *with* (1.3) *or* (1.1)) *is a pure radiation metric.*

We note that recently Shaikh et al. [62] studied the curvature properties of pure radiation metric. Again from the values of the components of ∇T of the metric (1.2), we get

$$
\begin{aligned}
&T_{11,1}+T_{11,1}+T_{11,1}\\
=&\frac{3c^4\left(h_{133}+h_{144}\right)}{4\pi fG},\\
&T_{11,3}+T_{13,1}+T_{31,1}\\
=&\frac{c^4}{4\pi f^4G}\Big(f^3h_{333}+f^3h_{344}-f_{333}f^2h-f_{344}f^2h+f_{33}f^2h_3+f_{44}f^2h_3-f_3f^2h_{33}\\
&-f_3f^2h_{44}+4f_3f_{33}fh+2f_4f_{34}fh+2f_3f_{44}fh-f_3^2fh_3-f_4^2fh_3-3f_3^3h-3f_3f_4^2h\Big),\\
&T_{11,4}+T_{14,1}+T_{41,1}\\
=&\frac{c^4}{4\pi f^4G}\Big(f^3h_{334}+f^3h_{444}-f_{334}f^2h-f_{444}f^2h+f_{44}f^2h_4-f_4f^2h_{33}-f_4f^2h_{44}\\
&+2f_4f_{33}fh+2f_3f_{34}fh+f_{33}f^2h_4+4f_4f_{44}fh-f_3^2fh_4-f_4^2fh_4-3f_4^3h-3f_3^2f_4h\Big),\\
&T_{12,3}+T_{23,1}+T_{31,2}\\
=&\frac{c^4\left(f^2f_{333}+f^2f_{344}+3f_3^3+3f_4^2f_3-4ff_{33}f_3-2ff_{44}f_3-2ff_4f_{34}\right)}{8\pi f^4G},\\
&T_{12,4}+T_{24,1}+T_{41,2}\\
=&\frac{c^4\left(f^2f_{334}+f^2f_{444}+3f_4^3+3f_3^2f_4-2ff_{33}f_4-4ff_{44}f_4-2ff_3f_{34}\right)}{8\pi f^4G}
\end{aligned}
$$

and

$$
\begin{aligned}
T_{11,3}-T_{13,1}=&\frac{c^4}{8\pi f^4G}\Big(2f^3h_{333}+2f^3h_{344}-2f_{333}f^2h-2f_{344}f^2h\\
&-f_{33}f^2h_3-f_{44}f^2h_3-2f_3f^2h_{33}-2f_3f^2h_{44}+8f_3f_{33}fh+4f_4f_{34}fh\\
&+4f_3f_{44}fh+f_3^2fh_3+f_4^2fh_3-6f_3^3h-6f_3f_4^2h\Big),\\
T_{11,4}-T_{14,1}=&\frac{c^4}{8\pi f^4G}\Big(2f^3h_{334}+2f^3h_{444}-2f_{334}f^2h-2f_{444}f^2h-f_{33}f^2h_4\\
&-f_{44}f^2h_4-2f_4f^2h_{33}-2f_4f^2h_{44}+4f_4f_{33}fh+4f_3f_{34}fh+8f_4f_{44}fh\\
&+f_3^2fh_4+f_4^2fh_4-6f_4^3h-6f_3^2f_4h\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&T_{12,3}-T_{13,2}=T_{23,1}-T_{21,3}\\
=&-\frac{c^4\left(f^2f_{333}+f^2f_{344}+3f_3^3+3f_4^2f_3-4ff_{33}f_3-2ff_{44}f_3-2ff_4f_{34}\right)}{8\pi f^4G},\\
&T_{12,4}-T_{14,2}=T_{24,1}-T_{21,4}\\
=&-\frac{c^4\left(f^2f_{334}+f^2f_{444}+3f_4^3+3f_3^2f_4-2ff_{33}f_4-4ff_{44}f_4-2ff_3f_{34}\right)}{8\pi f^4G}.
\end{aligned}
$$

Now putting the condition (1.3) to above, we get

$$\begin{cases} T_{11,3} - T_{13,1} = \dfrac{c^4\left(fh_{344} + fh_{333} - f_3h_{33} - f_3h_{44}\right)}{4\pi f^2 G}, \\ T_{11,4} - T_{14,1} = \dfrac{c^4\left(fh_{444} + fh_{334} - f_4h_{33} - f_4h_{44}\right)}{4\pi f^2 G}, \end{cases} \tag{5.1}$$

and

$$\begin{cases} T_{11,1} + T_{11,1} + T_{11,1} = \dfrac{3c^4\left(h_{144} + h_{133}\right)}{4\pi f G}, \\ T_{11,3} + T_{13,1} + T_{31,1} = \dfrac{c^4\left(fh_{344} + fh_{333} - f_3h_{33} - f_3h_{44}\right)}{4\pi f^2 G}, \\ T_{14,1} + T_{41,1} + T_{11,4} = \dfrac{c^4\left(fh_{444} + fh_{334} - f_4h_{33} - f_4h_{44}\right)}{4\pi f^2 G}. \end{cases} \tag{5.2}$$

Now, from (4.2), (5.1) and (5.2), we can state the following.

Proposition 5.1. *The energy momentum tensor T of the pp-wave metric* (1.2) *with the condition* (1.3) *is*

(i) *parallel if*

$$h_{144} + h_{133} = fh_{344} + fh_{333} - f_3h_{33} - f_3h_{44} = fh_{444} + fh_{334} - f_4h_{33} - f_4h_{44} = 0;$$

(*For example:* $f(x^3, x^4) = e^{x^3+x^4}$ *and* $h(x^1, x^3, x^4) = e^{x^3+x^4}$).

(ii) *Codazzi type if*

$$fh_{344} + fh_{333} - f_3h_{33} - f_3h_{44} = fh_{444} + fh_{334} - f_4h_{33} - f_4h_{44} = 0;$$

(*For example:* $f(x^3, x^4) = e^{x^3+x^4}$ *and* $h(x^1, x^3, x^4) = e^{x+x^3+x^4}$).

(iii) *cyclic parallel if*

$$h_{144} + h_{133} = fh_{344} + fh_{333} - f_3h_{33} - f_3h_{44} = fh_{444} + fh_{334} - f_4h_{33} - f_4h_{44} = 0.$$

(*For example:* $f(x^3, x^4) = e^{x^3+x^4}$ *and* $h(x^1, x^3, x^4) = e^{x^3+x^4}$).

Proposition 5.2. *The energy-momentum tensor of the pp-wave metric of the form* (1.1) *is*

(i) *parallel if* $H_{144} + H_{133} = H_{344} + H_{333} = H_{444} + H_{334} = 0$;
(*For example:* $H(u, x, y) = (x)^2 + (y)^2$).

(ii) *Codazzi type if* $H_{344} + H_{333} = H_{444} + H_{334} = 0$;
(*For example:* $H(u, x, y) = (ux)^2 + (uy)^2$).

(iii) *cyclic parallel if* $H_{144} + H_{133} = H_{344} + H_{333} = H_{444} + H_{334} = 0$.
(*For example:* $H(u, x, y) = (x)^2 + (y)^2$).

Now we can conclude the following.

Theorem 5.2. *On a pp-wave spacetime* (*endowed with the metric* (1.2) *with* (1.3) *or* (1.1))

(i) *the energy-momentum tensor is parallel if and only if it is cyclic parallel;*

(ii) *the energy-momentum tensor is Codazzi type if it is cyclic parallel but not conversely* (*see Example* 5.1)*;*
(iii) *the Ricci tensor is zero, i.e., the space is vacuum if* h *or* H *is harmonic, i.e.,* $h_{33} + h_{44} = 0$ *or* $H_{33} + H_{44} = 0$ *and in this case* $\nabla T = 0$*.*

Example 5.1. We now consider a special form of the generalized pp-wave metric (1.2) as

$$ds^2 = -2e^{x+x^3-x^4}(dx)^2 + 2dxdr - \frac{1}{2}e^{x^3-x^4}[(dx^3)^2 + (dx^4)^2]. \tag{5.3}$$

Then the non-zero components of its R, ∇R, S and ∇S are given by

$$\begin{aligned} R_{1313} =& R_{1414} = e^{x+x^3-x^4}, \quad R_{1313,1} = R_{1414,1} = e^{x+x^3-x^4}, \\ S_{11} =& 4e^x, \quad S_{11,1} = 4e^x. \end{aligned}$$

It is easy to check that the scalar curvature of this metric is zero and it is conformally flat. Now the non-zero components of its energy momentum tensor T and its derivative ∇T are given by

$$\begin{aligned} T_{11} =& \frac{c^4 e^{x-x^4}\left(2e^{x^4} - \Lambda e^{x^3}\right)}{4\pi G}, \quad T_{12} = \frac{c^4\Lambda}{8\pi G}, \quad T_{33} = T_{44} = -\frac{c^4\Lambda e^{x^3-x^4}}{16\pi G}, \\ T_{11,1} =& \frac{c^4 e^x}{2\pi G}. \end{aligned}$$

Thus we can easily check that the Ricci tensor and the energy momentum tensor of (5.3) are codazzi type but not cyclic parallel.

6. Robinson-Trautman Metric and Generalized pp-Wave Metric

Recently, Shaikh et al. [53] studied the curvature properties of Robinson-Trautman metric. The line element of Robinson-Trautman metric in $\{t, r, x^3, x^4\}$-coordinate is given by

$$ds^2 = -2(a - 2br - qr^{-1})dt^2 + 2dtdr - \frac{r^2}{f^2}[(dx^3)^2 + (dx^4)^2], \tag{6.1}$$

where a, b, q are constants and f is a function of the real variables x^3 and x^4. In this section we make a comparison between the curvature properties of the Robinson-Trautman metric (6.1) and generalized pp-wave metric (1.2) as well as the pp-wave metric (1.1).

Theorem 6.1. *The Robinson-Trautman metric* (6.1) *and generalized pp-wave metric* (1.2) *have the following similarities and dissimilarities.*
A. Similarities:
(i) *both the metrics are* 2*-quasi-Einstein;*
(ii) *both are generalized quasi-Einstein in the sense of Chaki;*
(iii) *Ricci tensors of both the metrics are Riemann compatible as well as Weyl compatible.*

B. Dissimilarities:

(iv) (6.1) *is Deszcz pseudosymmetric whereas* (1.2) *is Ricci generalized pseudosymmetric;*

(v) *the conformal curvature* 2*-forms are recurrent for* (6.1) *but not recurrent for* (1.2)*;*

(vi) *the metric* (6.1) *is Roter type and hence* Ein(2) *but* (1.2) *is not Roter type but* Ein(3).

Theorem 6.2. *The Robinson-Trautman metric* (6.1) *and pp-wave metric* (1.1) *have the following similarities and dissimilarities.*

A. Similarities:

(i) *for both the metrics, the conformal curvature* 2*-forms are recurrent;*

(ii) *Ricci tensor of both the metrics are Riemann compatible as well as Weyl compatible.*

B. Dissimilarities:

(iii) *the metric* (6.1) *is* 2*-quasi-Einstein, where as* (1.1) *is Ricci simple and hence quasi-Einstein;*

(iv) (6.1) *is Deszcz pseudosymmetric whereas* (1.1) *is semisymmetric;*

(v) (6.1) *is pseudosymmetric due to conformal curvature tensor whereas* (1.1) *is semisymmetric due to conformal curvature tensor;*

(vi) *the metric* (6.1) *realizes* $S \wedge S \neq 0$ *but* (1.1) *satisfies* $S \wedge S = 0$*;*

(vii) *the metric* (6.1) *is Roter type and hence* Ein(2) *but* (1.1) *is not Roter type but* Ein(3).

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SOME REMARKS ON DIFFERENTIAL IDENTITIES IN RINGS

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Abstract. Let $1 < k$ and $m, k \in \mathbb{Z}^+$. In this manuscript, we analyse the action of (semi)-prime rings satisfying certain differential identities on some suitable subset of rings. To be more specific, we discuss the behaviour of the semiprime ring $\mathcal{R}$ satisfying the differential identities $([d([s,t]_m), [s,t]_m])^k = [d([s,t]_m), [s,t]_m]$ for every $s, t \in \mathcal{R}$.

1. Motivation

The work of this manuscript is motivated by the various results established by many well known algebraists (see [1,2,4,6,7,9–11,14,15,17,19], and references therein). The famous and classical result in this direction is due to Jacobson [12]. The theorem to which we want to mention is the following: *"any ring in which $s^k = s$, $1 < k \in \mathbb{Z}^+$ is necessarily commutative"*. The above mentioned result generalizes Wedderburn theorem, i.e., *"every finite division ring is commutative"*, and also the result that *"any Boolean ring is a commutative ring"*.

In [10], Herstein discussed the commutativity of a ring and he established that *"a ring must be commutative if it satisfies $[s,t]^n = [s,t]$, for every $s, t \in \mathcal{R}$, where $1 < n \in \mathbb{Z}^+$"*. In 2011, Huang [11] proved that *"if a prime ring $\mathcal{R}$ admits a derivation d such that $d([s,t])^k = [s,t]_n$ for all $s, t \in I$, a nonzero ideal of $\mathcal{R}$ (where $1 < k, n \in \mathbb{Z}^+$) R is commutative"*. In 2017, De Filippis et al. [8] proved the following *"let $\mathcal{R}$ be a prime ring of characteristic different from 2, d be a nonzero derivation of $\mathcal{R}$, $f(x_1, \ldots, x_n)$ be a multilinear polynomial over $\mathcal{C}$ and $1 < k \in \mathbb{Z}^+$ such that $\Big([d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)]\Big)^k = [d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)]$ for all*

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$r_1, \dots, r_n \in \mathcal{R}$. *Then* $f(x_1, \dots, x_n)$ *is central valued on* $\mathcal{R}$*"*. In particular, if multilinear polynomial replace by commutator, then we conclude with the commutativity of rings. On the other hand, Giambruno et al. [9] generalizes Herstein result for Engel polynomial and they established that *"a ring must be commutative if it satisfies* $([s,t]_m)^n = [s,t]_m$*"*. In view of Giambruno et al. result Raza and Rehman [18] proved that *"a prime ring* R *is commutative if it satisfies* $d([s,t]_m)^n = [s,t]_m$ *for all* $s, t \in I$*, a nonzero ideal of* $\mathcal{R}$*"*.

In the prospect of above motivation, our intention is to explore the action of prime and semiprime rings satisfying Engel polynomials, which are not multilinear associated with derivations. To be more specific, we discuss the behaviour of the semiprime ring $\mathcal{R}$ satisfying the differential identities $([d([s,t]_m), [s,t]_m])^k = [d([s,t]_m), [s,t]_m]$ for every $s, t \in \mathcal{R}$.

2. The Results

We use the following notations and definitions, unless otherwise mention, $\mathcal{R}$ be a ring, $Z(\mathcal{R})$ be the center of $\mathcal{R}$, $\mathcal{Q}$ be a Martindale quotient ring of $\mathcal{R}$, $\mathcal{U}$ be a Utumi quotient ring of $\mathcal{R}$, and $\mathcal{C}$ be the extended centroid of $\mathcal{R}$ (see [3] for further details). A ring $\mathcal{R}$ is said to be prime if for any $s, t \in \mathcal{R}$, $s\mathcal{R}t = (0)$ implies $s = 0$ or $t = 0$, and $\mathcal{R}$ is semiprime if for any $s \in \mathcal{R}$, $s\mathcal{R}s = (0)$ implies $s = 0$. An additive mapping $d : \mathcal{R} \longrightarrow \mathcal{R}$ is said to be a derivation if it satisfies $d(st) = d(s)t + sd(t)$ for every $s, t \in \mathcal{R}$. If for any fixed $p \in \mathcal{R}$, $d(s) = [p, s]$, for every $s \in \mathcal{R}$, then d is said to be inner derivation. Moreover, d is said to be $\mathcal{Q}$-inner if the extension of d to $\mathcal{Q}$ is inner otherwise $\mathcal{Q}$-outer.

We will proceed by first proving the following auxiliary result.

Lemma 2.1. *Let* $1 < k \in \mathbb{Z}^+$ *and* $1 \leq m \in \mathbb{Z}^+$*. Next,* $\mathcal{R} = \mathcal{M}_m(\mathcal{C})$ *be the ring of* $m \times m$ *matrices over the field* $\mathcal{C}$ *such that* $[[p, [s,t]_m], [s,t]_m]^k = [[p, [s,t]_m], [s,t]_m]$*. Then* $p \in Z(\mathcal{R})$*.*

Proof. Let $p = \sum_{ij} p_{ij} e_{ij}$, where e_{ij} denotes the usual unit matrix with 1 in (i,j)-entry and zero elsewhere and $p_{ij} \in \mathcal{C}$. We show that p is a diagonal matrix. Next, let $s = e_{ij}$ and $t = e_{jj}$ and in view of our hypothesis, we deduce that $([p, e_{ij}]_2)^k = [p, e_{ij}]_2$, i.e, $-2e_{ij}qe_{ij} = 0$ and hence $p_{ji} = 0$ for any $i \neq j$. Therefore, p is a diagonal matrix.

Further, we see that

$$([[\varphi(p), [s,t]_m]]_2)^k = [[\varphi(p), [s,t]_m]]_2$$

is a generalized polynomial identity of $\mathcal{R}$ for $\varphi \in \mathrm{Aut}_{\mathcal{C}}(\mathcal{R})$. This shows that $\varphi(p)$ is diagonal matrix. Precisely, we consider the automorphism $\varphi(p) = (1 + e_{ij})p(1 - e_{ij})$ for any $i \neq j$ and say $\varphi(p) = \sum_{ij} p'_{ij} e_{ij}$, where $p'_{ij} \in \mathcal{C}$. Since $p'_{ij} = 0$, then by easy computation we obtain $0 = p'_{ij} = p_{jj} - p_{ii}$. So, that $p_{jj} = p_{ii}$ holds for any $i \neq j$. This implies that $p \in Z(\mathcal{R})$. This completes the proof. $\square$

Lemma 2.2. *Let $1 < k \in \mathbb{Z}^+$ and $1 \leq m \in \mathbb{Z}^+$. Next, let $\mathcal{R}$ be a non-commutative prime ring of characteristic different from 2. If $p \in \mathcal{Q}$ and $[[p, [s, t]_m], [s, t]_m]^k = [[p, [s, t]_m], [s, t]_m]$ be a generalized polynomial identity for $\mathcal{R}$, then $p \in \mathcal{C}$.*

Proof. We prove this lemma by contradiction, i.e., we assume that $p \notin \mathcal{C}$. Clearly, $\mathcal{Q}$ satisfies $[[p, [s, t]_m], [s, t]_m]^k = [[p, [s, t]_m], [s, t]_m]$ (see [5]). Specifically, as $p \notin \mathcal{C}$ then the above identity is a non-trivial generalized polynomial identity for $\mathcal{Q}$. Thus, $\mathcal{Q}$ is a primitive ring which is isomorphic to a dense ring of linear transformations of a vector space $\mathcal{V}$ over $\mathcal{C}$ (by Martindale's theorem in [16]).

Now firstly we discuss the case when $\dim_{\mathcal{C}}(\mathcal{V}) = m$, where $m > 1$, a finite positive integer. In this case, the contradiction follows by Lemma 2.1.

We now assume that $\dim_{\mathcal{C}} \mathcal{V} = \infty$. Then, we have

$$([p, [s, t]_m]_2)^k = [p, [s, t]_m]_2, \quad \text{for all } s, t \in \mathcal{Q}. \tag{2.1}$$

Moreover, again by Martindale's theorem [16], it follows that $soc(\mathcal{Q}) = \mathcal{H} \neq (0)$ and $e\mathcal{H}e$ is a finite dimensional simple central algebra over $\mathcal{C}$, for any minimal idempotent element $e \in \mathcal{H}$. Moreover, we may assume that $\mathcal{H}$ is non-commutative, otherwise $\mathcal{Q}$ must be commutative. Of course, $\mathcal{H}$ satisfies $([p, [s, t]_m]_2)^k = [p, [s, t]_m]_2$ (see for example proof of [14, Theorem 1]). As $\mathcal{H}$ is a simple ring, either $\mathcal{H}$ does not contain any non-trivial idempotent element or $\mathcal{H}$ is generated by its idempotents. In this last case, suppose that $\mathcal{H}$ contains two minimal orthogonal idempotent elements e and f. By the hypothesis, for $[s, t]_m = [es, f]_m = esf$, we have

$$exf(p)exf = 0, \tag{2.2}$$

in case we get $fpesfpesfpe = 0$, by the primeness of $\mathcal{R}$, we get $fpe = 0$, where e and f are orthogonal idempotent element of rank 1. Specifically, as e of rank 1, we have $ep(1 - e) = 0$ and $(1 - e)pe = 0$, i.e., $ep = epe = pe$. Therefore, $[p, e] = 0$ and $[p, \mathcal{H}] = 0$, where $\mathcal{H}$ is generated by these idempotent elements. This argument gives the conclusion that $p \in \mathcal{C}$ or $\mathcal{R}$ is commutative. In this last case, we conclude with contradiction.

Thus we take the case when $\mathcal{H}$ cannot contain two minimal orthogonal idempotent elements and so, $\mathcal{H} = \mathcal{D}$ for a suitable division ring $\mathcal{D}$ finite dimensional over its center. This implies that $\mathcal{Q} = \mathcal{H}$ and $p \in \mathcal{H}$. By [20, Theorem 2.3.29] ([14, Lemma 2]), $\mathcal{H} \subseteq M_n(K)$, $M_n(K)$ satisfies $([p, [s, t]_m]_2)^k = [p, [s, t]_m]_2$, where K is a field. If $n = 1$, then $\mathcal{H} \subseteq F$, a contradiction. Moreover, if $n \geq 2$, then $p \in Z(\mathcal{M}_n(F))$, as we have just seen.

Finally, consider if $\mathcal{H}$ does not contain any non-trivial idempotent element, then $\mathcal{H}$ is finite dimensional division algebra over $\mathcal{C}$ and $p \in \mathcal{H} = \mathcal{RC} = \mathcal{Q}$. If $\mathcal{C}$ is finite, then $\mathcal{H}$ is finite division ring, that is, $\mathcal{H}$ is a commutative field and so $\mathcal{R}$ is commutative too. If $\mathcal{C}$ is infinite, then $\mathcal{H} \otimes_{\mathcal{C}} K \cong M_n(K)$, where K is a splitting field of $\mathcal{H}$. In this case, we get the conclusion by Lemma 2.1. □

Theorem 2.1. *Let $1 < k \in \mathbb{Z}^+$ and $1 \leq m \in \mathbb{Z}^+$. Next, let $\mathcal{R}$ be a prime ring of characteristic different from 2 and I be a nonzero ideal of $\mathcal{R}$. If I satisfies $[d([s,t]_m),[s,t]_m]^k = [d([s,t]_m),[s,t]_m]$ for all $s,t \in I$, then $\mathcal{R}$ is commutative.*

Proof. We suppose on contrary that $\mathcal{R}$ is non-commutative. Then, by Lemma 2.1 and Lemma 2.2, we discuss the case when d is not $\mathcal{Q}$-inner. Note that

$$\begin{aligned} d([s,t]_m) = &\sum_{m=1}^{k}(-1)^m\binom{k}{m}\sum_{i+j=m-1} t^i d(t)t^j)xy^{k-m} \\ &+\sum_{m=0}^{k}(-1)^m\binom{k}{m}t^m d(s)t^{k-m} \\ &+\sum_{m=0}^{k-1}(-1)^m\binom{k}{m}t^m s\left(\sum_{r+s=k-m-1} t^r d(t)t^s\right). \end{aligned} \tag{2.3}$$

Using the hypothesis and well known results, we can say that $\mathcal{R}$ satisfies

$$[d([s,t]_m),[s,t]_m]^k = [d([s,t]_m),[s,t]_m].$$

Therefore, we obtain

$$\begin{aligned} &\Bigg(\Big[\sum_{m=1}^{k}(-1)^m\binom{k}{m}\sum_{i+j=m-1} t^i d(t)t^j)st^{k-m} + \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^m d(s)t^{k-m} \\ &+\sum_{m=0}^{k-1}(-1)^m\binom{k}{m}t^m s\left(\sum_{r+s=k-m-1} t^r d(t)t^s\right), \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^k xy^{k-m}\Big]\Bigg)^k \\ =&\Big[\sum_{m=1}^{k}(-1)^m\binom{k}{m}\sum_{i+j=m-1} t^i d(t)t^j)st^{k-m} + \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^m d(s)t^{k-m} \\ &+\sum_{m=0}^{k-1}(-1)^m\binom{k}{m}t^m s\left(\sum_{r+s=k-m-1} t^r d(t)t^s\right), \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^k st^{k-m}\Big]. \end{aligned}$$

In view of Kharchenko's theorem [13], we get

$$\begin{aligned} &\Bigg(\Big[\sum_{m=1}^{k}(-1)^m\binom{k}{m}\sum_{i+j=m-1} t^i wt^j)st^{k-m} + \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^m zt^{k-m} \\ &+\sum_{m=0}^{k-1}(-1)^m\binom{k}{m}t^m s\sum_{r+s=k-m-1} t^r wt^s, \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^k st^{k-m}\Big]\Bigg)^k \\ =&\Big[\sum_{m=1}^{k}(-1)^m\binom{k}{m}\sum_{i+j=m-1} t^i wt^j)st^{k-m} + \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^m zt^{k-m} \\ &+\sum_{m=0}^{k-1}(-1)^m\binom{k}{m}t^m s\sum_{r+s=k-m-1} t^r wt^s, \sum_{m=0}^{k}(-1)^m\binom{k}{m}t^k st^{k-m}\Big], \end{aligned}$$

for all $s, t, z, w \in \mathcal{R}$ and hence it satisfied by $\mathcal{Q}$ [3, Theorem 6.4.4]. Thus, $\mathcal{Q}$ is non-commutative as $\mathcal{R}$. Let us take $p' \in \mathcal{Q}$ with $p' \notin \mathcal{C}$. Also, we can see that $d' : \mathcal{Q} \to \mathcal{Q}$ is a nonzero derivation of $\mathcal{Q}$ defined by $d'(s) = [p', s]$ for all $s \in \mathcal{Q}$. Replacing z, w by $d'(s), d'(t)$ in the last identity and using (2.3), we obtain that

$$[d'([s,t]_m), [s,t]_m]^k = [d'([s,t]_m), [s,t]_m],$$

for all $s, t \in \mathcal{Q}$. Thus, we can write

$$[[p', [s,t]_m], [s,t]_m]^k = [[p', [s,t]_m], [s,t]_m],$$

for all $s, t \in \mathcal{R}$. Application of Lemma 2.2 yields the desire conclusion. □

As the immediate consequences of the above theorem, we obtain the following results.

Corollary 2.1. *Let $1 < k \in \mathbb{Z}^+$ and $1 \leq m \in \mathbb{Z}^+$. Next, let $\mathcal{R}$ be a prime ring of characteristic different from* 2 *and d be a derivation of $\mathcal{R}$ such that $[d([s,t]_m), [s,t]_m]^k = [d([s,t]_m), [s,t]_m]$ for all $s, t \in \mathcal{R}$. Then, $\mathcal{R}$ is commutative.*

Corollary 2.2. *Let $1 < k \in \mathbb{Z}^+$. Next, let $\mathcal{R}$ be a prime ring of characteristic different from* 2*, d be a derivation and L be a noncentral Lie ideal of $\mathcal{R}$. If $\mathcal{R}$ satisfies $[d(u), u]^k = [d(u), u]$ for all $u \in L$, then $\mathcal{R}$ is commutative.*

Now, we discuss the our last result for semiprime case. We set out with few preliminary notions which are required for the establishment of the proof of our main theorem. More or less of these notions are classical and we introduce them briefly, let $\mathcal{R}$ be a semiprime ring and $\mathcal{C}$ be the extended centroid $\mathcal{R}$. Also, the orthogonal completion of $\mathcal{R}$ is the intersection of all orthogonally complete subset of $\mathcal{Q}$ containing $\mathcal{R}$ and is denoted by $\mathcal{A} = \mathcal{O}(\mathcal{R})$. In [3, Theorem 3.2.7], Beidar et al. proved that "*if $M \in spec(\mathcal{B})$, then $\mathcal{R}_M = \mathcal{R}/\mathcal{R}M$ is prime, where $\mathcal{B} = \mathcal{B}(\mathcal{C})$ is a Boolean ring of* $\mathcal{C}$ *and spec*$(\mathcal{B})$ *is the set of all maximal ideal of* $\mathcal{B}$". We use the notations Ω-Δ-ring, Horn formulas and Hereditary formulas. For more definitions and related results see ([3], pages 37, 38, 43, 120). Also we use the results obtained by Beidar et el. [3, Proposition 2.5.1 and Theorem 3.2.18] which state that *"any derivation d of a semiprime ring $\mathcal{R}$ can be uniquely extended to a derivation of $\mathcal{U}$ (we shall let d also denote its extension to $\mathcal{U}$)"* and *"let $\mathcal{R}$ be an orthogonally complete Ω-Δ-ring with extended centroid $\mathcal{C}$, $\Theta_i(s_1, s_2, \ldots, s_n)$ Horn formulas of signature of Ω-Δ, $i = 1, 2, \ldots$ and $\Phi(t_1, t_2, \ldots, t_k)$ a hereditary first-order formula such that $\neg\Phi$ is a Horn formula. Further, let $\vec{a} = (a_1, a_2, \ldots, a_n) \in \mathcal{R}^{(n)}$, $\vec{c} = (c_1, c_2, \ldots, c_k) \in \mathcal{R}^{(k)}$. Suppose that $\mathcal{R} \models \Phi(c)$ and for every maximal ideal M of the Boolean ring $\mathcal{B} = \mathcal{B}(\mathcal{C})$, there exists a natural number $i = i(M) > 0$ such that*

$$\mathcal{R}_M \models \Phi(\phi_M(\vec{c})) \Rightarrow \Theta_i(\phi_M(\vec{a})).$$

Then there exist a natural number N and pairwise orthogonal idempotents $e_1, e_2, \ldots, e_N \in \mathcal{B}$ such that $e_1 + e_2 + \cdots + e_N = 1$ and $e_i R \models \Theta_i(e_i\vec{a})$ for all $e_i \neq 0$", respectively.

Now, we are able to discuss our last result.

Theorem 2.2. *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and d be a nonzero derivation of $\mathcal{R}$ such that $([d([s,t]_m),[s,t]_m])^k = d([s,t]_m),[s,t]_m]$ for all $s,t \in \mathcal{R}$. Further, let $\mathcal{A} = \mathcal{O}(\mathcal{R})$ is the orthogonal completion of $\mathcal{R}$ and $\mathcal{B} = \mathcal{B}\mathcal{C}$, where $\mathcal{C}$ is the extended centroid of $\mathcal{R}$. Then there exists a central idempotent element $e \in \mathcal{B}$ such that d vanishes identically on $e\mathcal{A}$ and the ring $(1-e)\mathcal{A}$ is commutative.*

Proof. In view of our hypothesis, $\mathcal{R}$ satisfies

$$([d([s,t]_m),[s,t]_m])^k = [d([s,t]_m),[s,t]_m].$$

Moreover, $\mathcal{U}$ satisfies $([d([s,t]_m),[s,t]_m])^k - [d([s,t]_m),[s,t]_m] = 0$ for every $s,t \in \mathcal{U}$ (see [15]). By Remark 3.1.16 of [3], we conclude that $d(\mathcal{A}) \subseteq \mathcal{A}$ and $d(e) = 0$ for every $e \in \mathcal{B}$. Therefore, $\mathcal{A}$ is an orthogonally complete Ω-Δ- ring, where $\Omega = \{0,+,-,.,d\}$. Consider the formulas

$$\begin{aligned}
\Phi &= \| \ ([d([s,t]_m),[s,t]_m])^k - [d([s,t]_m),[s,t]_m] = 0 \ \|, \quad \text{for all } s,t, \\
\Theta_1 &= \| \ st = ts \ \|, \quad \text{for all } s,t, \\
\Theta_2 &= \| \ d(s) = 0 \ \|, \quad \text{for all } s.
\end{aligned}$$

One can smartly verify that Φ is a hereditary first-order formula and $\neg\Phi$, Θ_1, Θ_2 are Horn formulas. Using Theorem 2.1, we can smartly verify that all the requirements of [3, Theorem 3.2. 18] are satisfied. Therefore, there exist two orthogonal idempotent e_1 and e_2 such that $e_1 + e_2 = 1$ and if $e_i \neq 0$, then $e_i\mathcal{A} \models \Theta_i$, $i = 1,2$. This completes the proof of the theorem. □

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APPLICATION OF THE SUMUDU TRANSFORM TO SOLVE REGULAR FRACTIONAL CONTINUOUS-TIME LINEAR SYSTEMS

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Abstract. In this work, Sumudu transform is used to establish the solution of a regular fractional continuous-time linear system based on Caputo fractional derivative-integral. First results of the proposed method are presented and compared to the existing ones.

1. Introduction

In recent years, fractional calculus and fractional systems appear and play a key role in several applications and domains [7, 9, 12, 13]. The use of the mathematical tools, theories and methods is required to solve such problems. Find the solution of a regular fractional continuous-time linear system with regular pencil is one of the most important problems in systems and control theory [3, 14].

In this paper, instead of Laplace transform [2, 7], the Sumudu transform [1, 16], which is a kind of the Laplace transform but does not require any conditions on the function to be transformable [15], is used to solve such a regular fractional continuous-time linear system based on the Caputo fractional derivative-integral [7]. The Sumudu transform is relatively new but it is as powerful as the Laplace transform and has some good features as for instance, unlike the Laplace transform, the Sumudu transform of the Heaviside step function is also Heaviside step function [6].

More than that, an interesting fact about this transformation is that the original function and its Sumudu transform have the same Taylor coefficients expect $n!$. Hence,

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the Sumudu transform, can be viewed as a power series transformation as shown in [11,16].

Another very interesting property, which makes the Sumudu transform more advantageous then the Laplace transform is the scale and unit preserving properties which could provide convenience when solving differential equations. In other words, the Sumudu transform can be used to solve various mathematical and physical sciences problems without restoring to a new frequency domain [1,16].

Furthermore, the solution of a regular fractional continuous-time linear system, using the Sumudu transform, requires only some boundary conditions and compatibility requirements.

The rest of the present paper is organized as follows. Basic definitions and properties are recalled in Section 2. Then, in Section 3, the solution of the regular fractional continuous-time linear systems is proposed using Sumudu transform followed by some academic and real examples which are presented in Section 4. The obtained results are compared to the state-of-the-art methods [2,7]. Finally, the last Section summarizes and discusses the obtained results.

2. Preliminaries

In the present section, main definitions and properties are recalled.

Definition 2.1 ([4])**.** The function defined by:

$$\mathbf{D}^{\alpha}x(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{x^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\quad x^{(n)}(\tau)=\frac{d^nx(\tau)}{d\tau^n},\tag{2.1}$$

is called the Caputo fractional derivative-integral of the function $x(t)$, where $n-1<\alpha\leq n$, $n\in\mathbb{N}^*$, and Γ refers to the standard Gamma function.

Definition 2.2 ([1,16])**.** Let us consider the set of functions:

$$\mathcal{A}=\left\{x(t)\mid \text{exists } M,\, \tau_1,\, \tau_2>0,\, |x(t)|<Me^{-\frac{|t|}{\tau_j}},\ \text{ if } t\in(-1)^j\times[0,\infty)\right\}.$$

The Sumudu transform $X(v)$ of the function $x(t)$ is defined over the set of functions $\mathcal{A}$ by:

$$X(v)=\mathcal{S}[x(t)](v)=v^{-1}\int_0^{\infty}x(t)e^{-\frac{t}{v}}dv,\quad v\in(-\tau_1,\tau_2).\tag{2.2}$$

Theorem 2.1 ([10])**.** *The Sumudu transform of the fractional derivative-integral* (2.1) *for $n-1<\alpha\leq n$, $n\in\mathbb{N}^*$ has the form:*

$$\mathcal{S}[\mathbf{D}^{\alpha}x(t)](v)=v^{-\alpha}\left(X(v)-\sum_{k=1}^{n}v^{k-1}\Big[x^{(k-1)}(t)\Big]_{t=0}\right),\tag{2.3}$$

where $X(v)$ refers to the Sumudu transform of the function $x(t)$.

Proposition 2.1 ([1])**.** *Let $x_1(t)$ and $x_2(t)$ be in $\mathcal{A}$, having the Sumudu transforms $X_1(v)$ and $X_2(v)$, respectively. Then, the Sumudu transform of the convolution product of x_1 and x_2*

$$(x_1 \star x_2)(t) = \int_0^\infty x_1(t-\tau)\, x_2(\tau)\, d\tau,$$

is given by:

$$\mathcal{S}[(x_1 \star x_2)(t)](v) = v\, X_1(v)\, X_2(v).$$

Proposition 2.2 ([1])**.** *For any $a \in \mathbb{R}_+^*$, the Sumudu transform of $\frac{t^a}{\Gamma(a+1)}$ is:*

$$\mathcal{S}\left[\frac{t^a}{\Gamma(a+1)}\right](v) = v^a.$$

In the following, we denote by $\mathbb{R}^{m\times n}$, the set of real matrices with m rows and n columns and by $\mathbb{R}^m$, the set of real columns vectors.

Proposition 2.3 ([7])**.** *Let $F \in \mathbb{R}^{n\times n}$ be a real matrix. Then, for any $v \in \mathbb{C}$ and $n-1 < \alpha \le n$, $n \in \mathbb{N}^*$, the Laurent series is given by:*

$$(I_n - v^\alpha F)^{-1} = \sum_{k=0}^{\infty} F^k\, v^{k\,\alpha}. \tag{2.4}$$

3. Main Results

Let us consider the following regular fractional continuous-time linear systems:

$$\mathbf{D}^\alpha x(t) = Ax(t) + Bu(t), \tag{3.1}$$

$$y(t) = Cx(t) + Du(t), \tag{3.2}$$

where $\mathbf{D}^\alpha$ is the Caputo fractional derivative-integral, $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, the input and the output vectors of the model respectively, and $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$.

The boundary conditions of the system (3.1) are given by:

$$x(0) = x_0.$$

Furthermore, the solution $x(t)$ is impulse free which is equivalent to the following compatibility conditions:

- $v^{k+i\alpha}\, A^i\, x^{(k)}(0)$ exists for all $i \in \mathbb{N}$, and $0 \le k \le n-1$, $n \in \mathbb{N}^*$ and $v \in (-\tau_1, \tau_2)$;
- $u(t)$ is provided.

It is assumed that the pencil of the pair $(I_n,\, A)$ is regular, i.e.,

$$\det(I_n - v^\alpha A)^{-1} \neq 0, \tag{3.3}$$

for almost $v \in \mathbb{C}$.

By applying the Sumudu transform (formulas (2.2) and (2.3)) to the system (3.1), we obtain:

$$\mathcal{S}[\mathbf{D}^\alpha x(t)](v) = \mathcal{S}[Ax(t) + Bu(t)](v).$$

Let us denote $X(v)$ and $U(v)$ as the Sumudu transforms of $x(t)$ and $u(t)$ respectively. Then, the use of the formula (3.3), yields:

$$X(v) = (I_n - v^\alpha A)^{-1} \left(v^\alpha BU(t) + \sum_{k=1}^{n} v^{k-1} x^{(k-1)}(0) \right),$$

which is equivalent to:

$$X(v) = v^\alpha (I_n - v^\alpha A)^{-1} BU(t) + (I - v^\alpha A)^{-1} \sum_{k=0}^{n-1} v^k x^{(k)}(0).$$

By the Laurent series (2.4), we obtain:

$$X(v) = \sum_{i=0}^{\infty} A^i B\, v \left(v^{(i+1)\alpha - 1} \right) \left(U(v) \right) + \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} A^i v^{i\alpha + k} x^{(k)}(0).$$

Finally, applying the convolution theorem (Proposition 2.1) and the inverse Sumudu transform (Proposition 2.2) give the following theorem.

Theorem 3.1. *The solution of the implicit fractional dynamical system* (3.1) *is given by:*

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} \frac{A^i B}{\Gamma((i+1)\alpha)} \int_0^t (t-\tau)^{(i+1)\alpha - 1} u(\tau) d\tau \\ &+ \sum_{i=0}^{\infty} \sum_{k=0}^{n-1} A^i \frac{t^{i\alpha + k}}{\Gamma(i\alpha + k + 1)} x^{(k)}(0), \end{aligned} \tag{3.4}$$

where α and Γ represent the fractional derivative-integral order and the standard Gamma function, respectively.

If $\alpha = 1$, its remain to the following.

Corollary 3.1. *For $\alpha = 1$, we get:*

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} A^i B \frac{1}{\Gamma(i+1)} \int_0^t (t-\tau)^i\, u(\tau)\, d\tau \\ &+ \sum_{i=0}^{\infty} A^i \frac{t^i}{\Gamma(i+1)} x_0. \end{aligned}$$

4. Experimental Results

This section present academic and real examples. In both cases the obtained results are compared to the existing ones.

Example 4.1. Find the solution of the system (3.1) for $0 < \alpha \leq 1$ and:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad u(t) = \mathbb{I}(t).$$

Using (3.4), it follows that:

$$x(t) = \begin{pmatrix} 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \end{pmatrix},$$

which is the same result obtained in [7].

Example 4.2. Let us consider the following system:

$$\mathbf{D}^{\alpha} Ex(t) = Ax(t) + Bu(t), \tag{4.1}$$

with $0 < \alpha \leq 1$ and the matrices:

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 1 & 1 & 0 & 1 \end{pmatrix}, \quad D = 0 \tag{4.2}$$

and the initial conditions:

$$x_0 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

It is clear that $\det E \neq 0$. Therefore, the system (4.1) becomes:

$$\mathbf{D}^{\alpha} x(t) = \tilde{A}x(t) + \tilde{B}u(t),$$

where

$$\tilde{A} = E^{-1}A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tilde{B} = E^{-1}B = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix}.$$

It follows then

$$x(t) = \begin{pmatrix} 1 \\ 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} \\ -\dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-\tau)^{\alpha-1} u(\tau)\, d\tau - \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} - \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \dfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} \\ 1 + \dfrac{t^{\alpha}}{\Gamma(\alpha+1)} + \dfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \end{pmatrix}.$$

Finally, using the systems (3.2) and (4.2), and the state $x(t)$ the output result is:

$$y(t) = 3 + \frac{2t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

the same result is obtained using Laplace transform [2].

Example 4.3. Let us consider the following regular fractional continuous-time system:

$$\mathbf{D}^{1.5}x(t) = Ax(t) + Bu(t), \tag{4.3}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -w_h^{1.5} \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{and} \quad w_h = 20.$$

The fractional continuous-time linear system (4.3) is derived from the one degree of freedom model of a passive car suspension as shown in Figure 1.

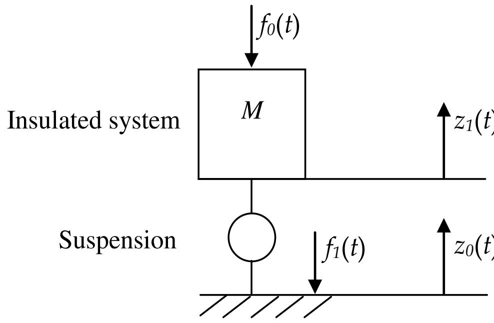

FIGURE 1. One degree of freedom general model of a car suspension [5].

M represents the car quarter mass, z_0 is the profile of the road, f_0 is the efforts applied on the suspension, and z_1, f_1 are the force generated by the suspension and the vertical movement of the mass respectively.

In this example $\alpha = 1.5$. Then, using (3.4), the solution is:

$$\begin{aligned} x(t) = {} & \frac{2\sqrt{\pi}}{\pi} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \int_0^t (t-\tau)^{0.5} u(\tau) d\tau + \frac{1}{2} \begin{pmatrix} 0 \\ 1 \\ -w_h^{1.5} \end{pmatrix} \int_0^t (t-\tau)^2 u(\tau) d\tau \\ & + \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix} + \frac{4\sqrt{\pi}}{3\pi} \begin{pmatrix} x_{0,2} \\ x_{0,3} \\ -x_{0,3}\, w_h^{1.5} \end{pmatrix} t^{1.5} + \begin{pmatrix} x'_{0,1} \\ x'_{0,2} \\ x'_{0,3} \end{pmatrix} t \\ & + \frac{8\sqrt{\pi}}{15\pi} \begin{pmatrix} x'_{0,2} \\ x'_{0,3} \\ -x'_{0,3}\, w_h^{1.5} \end{pmatrix} t^{2.5} \\ & + \sum_{i=2}^{\infty} \begin{pmatrix} (-w_h^{1.5})^{i-2} \\ (-w_h^{1.5})^{i-1} \\ (-w_h^{1.5})^{i} \end{pmatrix} \Bigg[\frac{1}{(1.5i+0.5)\,\Gamma(1.5i+0.5)} \int_0^t (t-\tau)^{1.5i+0.5} u(\tau) d\tau \\ & + \frac{x_{0,3}}{1.5i\Gamma(1.5i)} t^{1.5i} + \frac{x'_{3,0}}{(2.25i^2+1.5i)\Gamma(1.5i)} t^{1.5i+1} \Bigg], \end{aligned}$$

where

$$x(0) = x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \\ x_{0,3} \end{pmatrix} \quad \text{and} \quad x'(0) = \begin{pmatrix} x'_{0,1} \\ x'_{0,2} \\ x'_{0,3} \end{pmatrix}.$$

5. Discussion and Conclusion

In this paper, a new method for solving regular fractional continuous-time linear systems is presented which has been already introduced in [8]. The main idea consist on using the Sumudu transform to solve such a system. Thanks to the interesting properties of the Sumudu transform, the result can be derived easily and the method can be used for several practical applications.

The first results obtained are promising and encourage us to extend the method to singular fractional continuous-time linear systems, and to other type of systems and circuits, and also to other applications as for example to crone suspension which are one of our future research topics and will be discussed in a separate paper.

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ON THE NON-NEGATIVE RADIAL SOLUTIONS OF THE TWO DIMENSIONAL BRATU EQUATION

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Abstract. In this paper, we study the boundary value problem on the unit circle for the Bratu's equation depending on the real parameter μ. From the parameter estimate, the existence of non-negative solution is set. A numerical method is suggested to justify the theoretical result. It is a combination of the adaptation of finite difference and Gauss-Seidel method allowing us to obtain a good approximation of μ_c, with respect to the exact theoretical method $\mu_c = \lambda = 5.7831859629467$.

The vast majority of phenomena that occur in nature are described by a non-linear differential equation or by a system of non-linear equations. Among these equations, the Bratu's equation, given by

$$\nabla^2 u + \mu\, e^u = 0\,,$$

is a classical example of equation with a strong nonlinear exponential term and a real parameter μ. This equation arises originally as a simplified model for the description of the combustion of solid fuels. Also it is often appears in science and engineering as a model in various physical applications, from chemical reactions, thermal combustion theory, heat transfer radiation until the Chandrasekhar's model of the universe expansion and even nanotechnology [2, 3, 9, 13]. In [5], the dynamics of the Bratu equation were analyzed and the existence of bifurcations was shown. They are also devoted to describe the Gaussian curvature problem in Riemannian geometry [15], the mean field limit of vortices in Euler flows [8], the Onsager formulation in statistical mechanics [6], the Keller-Siegel system of chemotaxis [19] and the Chern-Simon-Higgs

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gauge theory [7, 21].

Recently, most of the research has focused on better and more efficient solution methods for determining solutions, approximate or exact, analytical or numerical to this non-linear Bratu model [1, 4, 11, 12, 17, 18, 20].

In this paper we study the two-dimensional Bratu's equation depending on a real parameter μ on the unit circle with the Dirichlet homogeneous boundary condition. We prove the existence of non-negative radial solutions for a certain range of the real parameter μ. A numerical method is suggested to justify the theoretical result.

1. Theoretical Result: Existence of the Solution

We study the two dimensional Bratu's equation on the unit circle with the homogeneous boundary condition,

$$\begin{cases} -\nabla^2 u(x,y) = \mu e^{u(x,y)}, & x^2+y^2<1, \\ u(x,y) = 0\,, & x^2+y^2=1, \end{cases} \tag{1.1}$$

where μ is a real parameter. The existence of the solution for the problem (1.1) beyond a certain limit of the parameter μ is based on a general theory of the non-linear eigenvalue problem

$$\begin{cases} -\nabla^2 u(x) = \mu f(x,u), & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where Ω is an open bounded region of $\mathbb{R}^n$ and $f(x,u)$ is a non-negative and continuous function on $(x,u)\in\Omega\times\mathbb{R}$. We have the next result [17]:

Theorem 1.1. *Assume that*

$$f(x,u)\geq h(x)+r(x)u\,,\quad (x,u)\in\Omega\times[0,\infty), \tag{1.3}$$

where h and r are non-negative and continuous functions in Ω. Then the non-linear eigenvalue problem (1.2) *has no non-negative solutions for any $\mu\geq\lambda$, where λ is the principal eigenvalue of the linear eigenvalue problem*

$$\begin{cases} -\nabla^2 u(x) = \lambda r(x)u, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega\,. \end{cases} \tag{1.4}$$

In particular, due to the estimate

$$f(x,u)=e^u\geq 1+u,$$

the linear eigenvalue problem (1.4) corresponding to the nonlinear problem (1.1) is given by $h(x)=r(x)=1$, i.e.,

$$\begin{cases} -\nabla^2 u(x,y) = \lambda\, u(x,y), & x^2+y^2<1, \\ u(x,y) = 0, & x^2+y^2=1. \end{cases} \tag{1.5}$$

Introducing the polar coordinates on the plane

$$x(r,\theta)=r\cos\theta,\quad y(r,\theta)=r\sin\theta,\quad 0\leq r\leq 1, 0\leq\theta\leq 2\pi,$$

we obtain the equivalent, to the (1.5), problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0, \quad u(1,\theta) = 0\,.$$

The standard method of separation of variables

$$u(r,\theta) = R(r)\,\Theta(\theta),$$

with the boundary values

$$R(1) = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi), \tag{1.6}$$

leads to the ordinary differential equations

$$\Theta''(\theta) + K\,\Theta(\theta) = 0, \tag{1.7}$$

and

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - K)R(r) = 0 \tag{1.8}$$

where K is a constant. In order to obtain a periodic solution, according to the second relation of (1.6) for the equation (1.7), the condition $K = n^2$ is necessarily required, where $n \in \mathbb{N}$. Therefore, the equation (1.8) becomes

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0,$$

which is the Bessel's equation with the general solution

$$R(r) = c_1 J_n(\sqrt{\lambda}\, r) + c_2 Y_n(\sqrt{\lambda}\, r)\,,$$

where c_1 and c_2 are arbitrary constants. Requiring a bounded solution, when $r = 0$, we set $c_2 = 0$. Furthermore, using the first relation of (1.6) we obtain

$$J_n(\sqrt{\lambda}) = 0\,,$$

which implies that

$$\lambda_{(m),n} = j^2_{(m),n}\,.$$

Thus, the boundary value problem (1.1) has no non-negative solution for $\mu \geq j^2_{(m),n}$. Since, the first root of the Bessel's function $J_0(x)$ is $j_{1,0} = 2.40482555769577$, then the threshold is $\mu_c = \lambda = 5.7831859629467$.

In the next section we applied this result for the corresponding radial solution.

2. Radial Positive Solution of the Bratu's Equation

The positive solution on the unit disc B_1 has a radial symmetry i.e. depends only on $r = \sqrt{x^2 + y^2}$. In order to prove this we follow the technique developed in [10]. First, we observe that the homogeneous boundary condition implies, that ∂B_1 is a level set of the positive solution $u \in C^2(\bar{B}_1)$ and therefore for the outer unit normal vector to ∂B_1, we have

$$\vec{\nu}(x,y) \equiv (x,y) = \pm\frac{\nabla u}{|\nabla u|} \quad \text{or} \quad (x,y)\cdot\nabla u = \pm|\nabla u|.$$

Furthermore,

$$e^{u(x,y)} = 1 + \int_0^1 \frac{d}{dt} e^{t\,u(x,y)}\,dt = 1 + u(x,y)\int_0^1 e^{t\,u(x,y)}\,dt.$$

Next, we write the Bratu's equation in the equivalent form

$$\nabla^2 u + c(x,y)u = -\mu < 0,$$

where

$$c(x,y) = \mu \int_0^1 e^{tu(x,y)}dt > 0.$$

The Serrin's maximum principle implies, that

$$\frac{\partial u}{\partial \vec{\nu}} = (x,y)\cdot \nabla u = -|\nabla u| < 0, \quad \text{on } \partial B_1. \tag{2.1}$$

Denote by $B_1^+ = B_1 \cap \{(x,y) \in \mathbb{R}^2,\ y > 0\}$ the upper unit half disc and $B_1^- = B_1 \cap \{(x,y) \in \mathbb{R}^2,\ y < 0\}$ the lower unit half disc. Thus, (2.1) implies

$$y\,\frac{\partial u}{\partial y}(x,y) = -|\nabla u(x,y)| - x\,\frac{\partial u}{\partial x}(x,y) < 0, \quad \text{on } \partial B_1^+,$$

which means that $\partial u/\partial y < 0$ on ∂B_1^+. The smoothness of u implies that $\partial u/\partial y < 0$ in B_1^+ close to ∂B_1^+. Thus, the solution u is a decreasing function on the y - direction close to ∂B_1^+. Furthermore, define the sets $l_a = \{(x,a),\ x \in \mathbb{R}\}$, for $0 \le a \le 1$ and $E_a = \{(x,y) \in B_1^+,\ a < y < 1\}$. To any $(x,y) \neq (x,a)$, we assign its reflection with respect to the line l_a, the point $(x, 2a-y)$.

Theorem 2.1. *If $u \in C^2(\bar{B}_1)$ is a positive solution of the Bratu's equation, then u is a function of $r = \sqrt{x^2+y^2}$.*

Proof. It is sufficient to show that $u(x,y) = u(x, 2a-y)$ whenever $a = 0$ i.e. the line l_a coincides with the axis x. To this end, define

$$a_0 = \inf\{a \in [0,1]:\ u(x,y) < u(x, 2\beta - y)\,,\ (x,y) \in E_\beta\,,\ a \le \beta \le 1\}.$$

The above infimum is well defined, since the solution u is a decreasing function on the y - direction close to ∂B_1^+. We will prove that $a_0 = 0$. Suppose that $a_0 > 0$ and define the function

$$v(x,y) = u(x, 2a_0 - y) - u(x,y), \quad (x,y) \in E_{a_0}.$$

Then $v(x,y) > 0$ and

$$\nabla^2 v(x,y) - C(x,y)v(x,y) = 0,$$

where

$$C(x,y) = \Big[u(x,2a_0-y) - u(x,y)\Big]\int_0^1 e^{[tu(x,2a_0-y)+(1-t)u(x,y)]}dt > 0.$$

The Serrin's maximum principle and the above discussion implies that

$$v(x,y) > 0, , (x,y) \in E_{a_0}, \quad \frac{\partial v}{\partial y}(x,y) < 0, (x,y) \in l_{a_0} \cap B_1^+$$

and, equivalently

$$u(x,y) < u(x, 2a_0 - y), (x,y) \in E_{a_0}, \quad \frac{\partial u}{\partial y}(x,y) < 0, (x,y) \in l_{a_0} \cap B_1^+,$$

with the partial derivatives with respect to y, always taken close to ∂B_1^+. Thus, we have, that the positive solution u is also decreasing function on $l_{a_0} \cap B_1^+$. Choosing any $\varepsilon > 0$, sufficiently small for $0 < \beta = a_0 - \varepsilon < a_0$, we have

$$u(x,y) < u(x, 2a_0 - y) < u(x, 2\beta - y), \quad (x,y) \in l_\beta \cap B_1^+,$$

and by the smoothness of u

$$u(x,y) < u(x, 2\beta - y), \quad (x,y) \in E_\beta, \beta < a_0\,,$$

which contradicts to the definition of a_0. Thus, necessarily $a_0 = 0$, and

$$u(x,y) \le u(x,-y), \quad (x,y) \in B_1^+.$$

In the same way, we can obtain

$$u(x,y) \ge u(x,-y), \quad (x,y) \in B_1^-,$$

which implies $u(x,y) = u(x,-y)$ in the unit disc. Finally, the axis x can be any diameter of the unit disc, thus we have the radial symmetry of the solution. $\square$

3. Numerical Method

To find the numerical solution of (1.1), we have used an adaptation of the second-order Finite Difference Method (FDM). First, we consider a rectangular region $(\mathcal{R})$ defined by

$$\begin{cases} -1 \le x \le 1, \\ -1 \le y \le 1, \end{cases}$$

in the cartesian system (OXY), and we insert into $(\mathcal{R})$ the circle $(\mathcal{C})$ defined by $x^2 + y^2 = 1$. Next, the region $(\mathcal{R})$ is subdivided into the grid $n \times n$ equal subregions: $h \times h$ where

$$h = \frac{2}{n}\,,$$

i.e, the axis (OX) and (OY) are partitioned in n equal part each one. So, each point or node (x_i, y_j) of the grid is the intersection of the $x = x_i$ vertical line and the $y = y_j$ horizontal line, where

$$x_i = -1 + ih, \quad i = 0, \ldots, n,$$

and

$$y_j = -1 + jh, \quad j = 0, \ldots, n.$$

Then, it is not difficult to see the following.

(a) For an exterior point or endpoint $P_{i,j} = (x_i, y_j)$ of the circle $(\mathcal{C})$, i.e., $x_i^2 + y_j^2 \ge 1$. See the Figure 1.

We have

$$P_{i,j} : u(x_i, y_j) = w_{i,j} = 0. \tag{3.1}$$

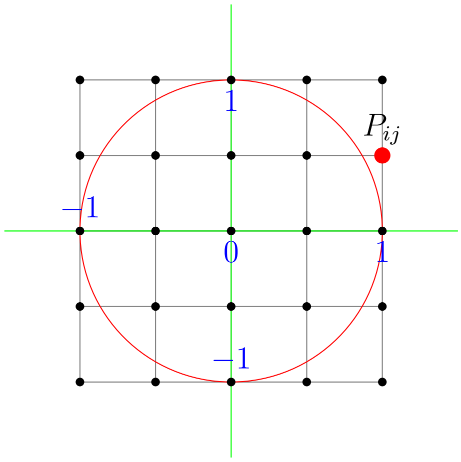


FIGURE 1.

(b) For each interior point $P_{i,j}$ of the circle $(\mathfrak{C})$, i.e., $x_i^2 + y_j^2 < 1$, we apply the Finite Difference Method (FDM) using the Taylor series with the variable x around x_i, and with the variable y around y_j [16], i.e., without loss of generality, around of this point $P_{i,j}$ we suppose the next four points $P_{i+1,j}, P_{i,j+1}, P_{i-1,j}, P_{i,j-1}$ which are known respectively as East(E), North(N), South(S), West(W) point with respect to $P_{i,j}$, see Figure 2.

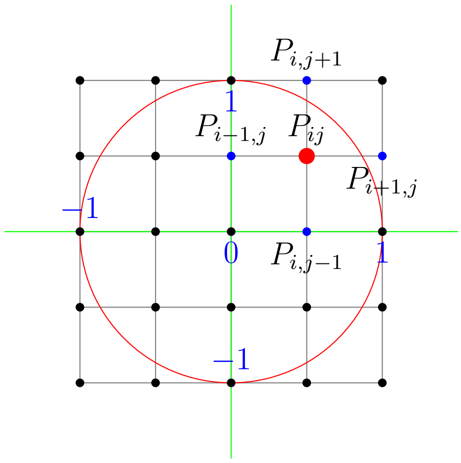


FIGURE 2.

So, we define:

$$\begin{aligned} P_{i+1,j} &: u(x+h,y) = u(x_{i+1},y_j) = w_{i+1,j}, \\ P_{i,j+1} &: u(x,y+h) = u(x_i,y_{j+1}) = w_{i,j+1}, \\ P_{i,j-1} &: u(x,y-h) = u(x_i,y_{j-1}) = w_{i,j-1}, \\ P_{i-1,j} &: u(x-h,y_j) = u(x_{i-1},y_j) = w_{i-1,j}. \end{aligned}$$

Next, for every interior point (x, y) of the circle, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &\approx \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)], \\ \frac{\partial^2 u}{\partial y^2} &\approx \frac{1}{h^2}[u(x,y+h) - 2u(x,y) + u(x,y-h)]. \end{aligned} \tag{3.2}$$

By adding these two equations (3.2), the equation (1.1) for all interior point of the circle can be replaced by the difference equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx & \frac{1}{h^2}[u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)] \\ =& (-\mu)(1 + u(x,y)). \end{aligned}$$

Then, it is easy to see that

$$4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = (-\mu)(-h^2)(1 + w_{i,j}), \tag{3.3}$$

where $u(x_i, y_j) = w_{i,j}$ and $e^{w_{i,j}} \cong 1 + w_{i,j} + \cdots$.

So, (3.3) implies that for each interior point $P_{i,j}$ of the circle $(\mathcal{C})$, we have

$$(4 - \mu h^2)w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = \mu h^2, \tag{3.4}$$

where $i = 0, \ldots, n$, and $j = 0, \ldots, n$.

The reader may find an illustrative example in the Appendix.

To find the value $u(x_i, y_j) = w_{i,j}$ of the point $P_{i,j}$ on the region $(\mathcal{R})$, the system of linear equations (3.1) and (3.4) is established, moreover, the solution of (1.1) is reduced to the solution $w_{i,j}$ of the system of linear equations (3.1) and (3.4), depending on the parameter μ. Finally, to solve the system of linear equations (3.1) and (3.4), the Gauss-Seidel's Method is used [4].

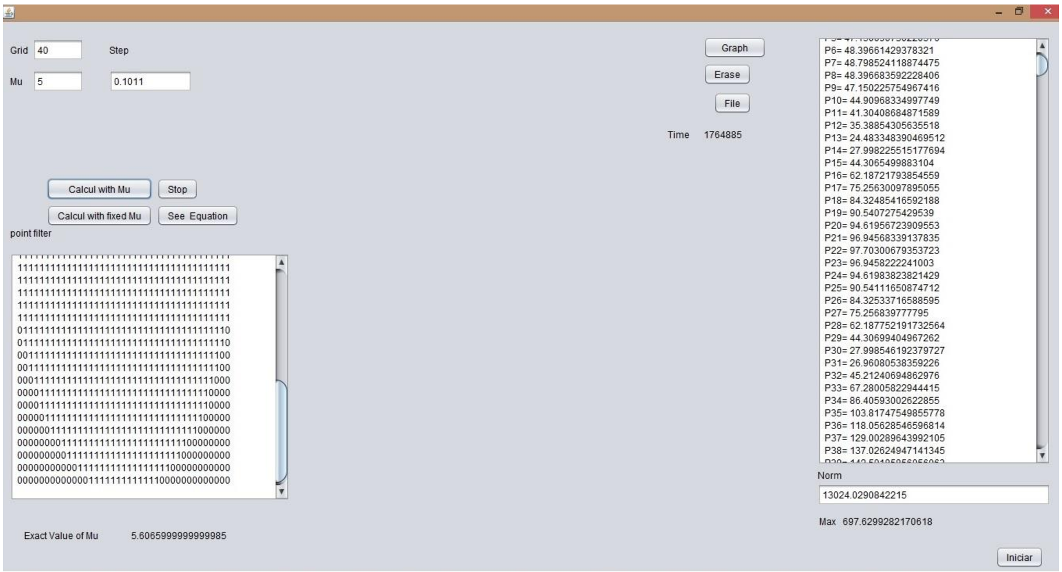
Since (1.1) and (3.4) depends on the parameter μ, we determine the threshold μ_c, giving the grid $n \times n$ of $(\mathcal{R})$, and also giving a value the parameter μ for the grid $n \times n$ of $(\mathcal{R})$ the norm $|w_{i,j}| = u_{max}$ is found.

Algorithm

Input: Value n of the subdivision of the region $(\mathcal{R})$; Initial value of $\mu = \mu_0$; k =step of μ.
Output: Value of the threshold μ_c; Approximate solution $u(x_i, y_j) \geq 0$ of the interior point of the circle for certain range of μ; Maximum value u_{max} of the solution $u(x_i, y_j)$.
for $\forall(x_i, y_j) \in (\mathcal{R})$ **do**
  if (x_i, y_j) *satisfy* $x_i^2 + y_j^2 \geq 1$ **then**
    Output: $u(x_i, y_j) = 0$.
  else
    for $\forall(x_i, y_j)$ *that satisfy* $x_i^2 + y_j^2 < 1$ **do**
      (1) Using FDM, establish the system of linear equations $AX = B$ which depends on parameter μ.
      (2) Solve the system of linear equations $AX = B$ by the Gauss-Seidel's Method for given initial value $\mu = \mu_0$.
      if $\exists u(x_i, y_j) < 0$ **then**
        Change Initial value of $\mu = \mu_0$; **goto** (2)
      else
        (3) **while** $\forall u(x_i, y_j) \geq 0$ **do**
          (3.1) $\mu = \mu + k$.
          (3.2) Solve the system of linear equations $AX = B$ by the Gauss-Seidel's Method.
          (3.3) **if** $\exists u(x_i, y_j) < 0$ **then**
            Output: $\mu_c = \mu - k$; $u(x_i, y_j) \geq 0; u_{max} = \max(u(x_i, y_j))$.
            Stop.
          else
            Output: $\mu; u(x_i, y_j) \geq 0$; $u_{max} = \max(u(x_i, y_j))$.
          end
        end
      end
    end
  end
end

4. Results

We developed a software in high-level programming language (in this case, Java) based on the algorithm mentioned above. The following tables show the result of (1.1) for the respective partition 40×40 and 70×70 (see Figure 3).

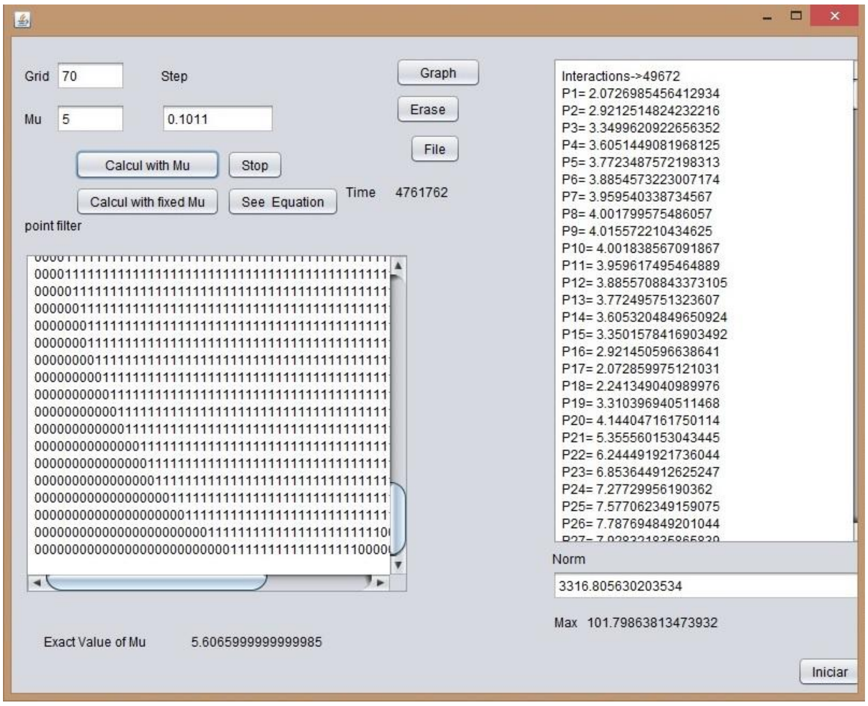


(a) Result for 40×40

4.1. Finding of μ_c. For each grid, we find the respective μ_c and the norm u_{max} given in Table 1.

Table 1.

$n \times n$	μ_c	u_{max}
30×30	5.5054999999999999	126.26207775999859
40×40	5.6065999999999985	697.6299282170618
50×50	5.6065999999999985	114.5670432582935
60×60	5.6065999999999985	102.23255929486518
70×70	5.6065999999999985	101.79863813473932
75×75	5.584800000000021	60.012385961525645



(b) Result for 70×70

FIGURE 3. Result for 40×40 and 70×70

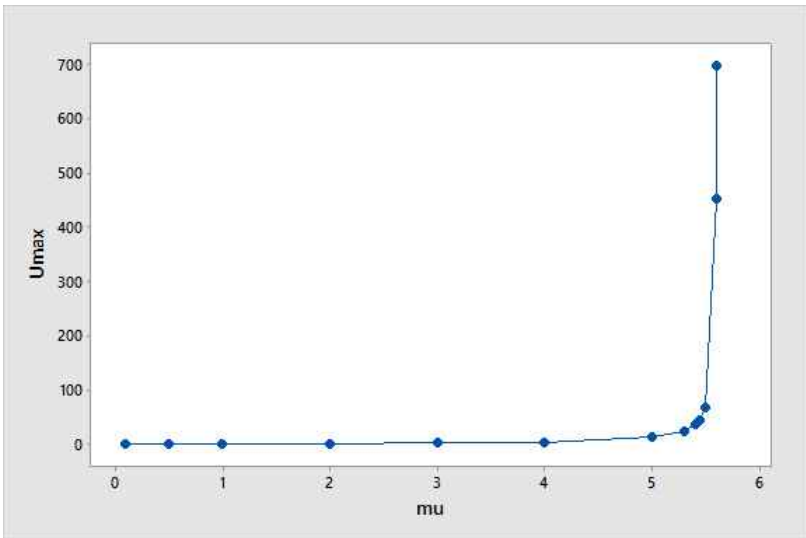


FIGURE 4. Graph of u_{max} vs μ, for 40×40

4.2. **Graph of u_{max} vs μ, for 40×40.** We have drawn the graph of u_{max} vs μ, for 40×40, given in Figure 4.

4.3. **The graph of the solution $u(x, y)$, for 50×50 and 75×75 with $\mu = 4$.** The graphs of the solution $u(x, y)$ of the equation (1.1) are drawn with the parameter $\mu = 4$, for the respective partition 50×50 and 75×75, given in Figure 5.

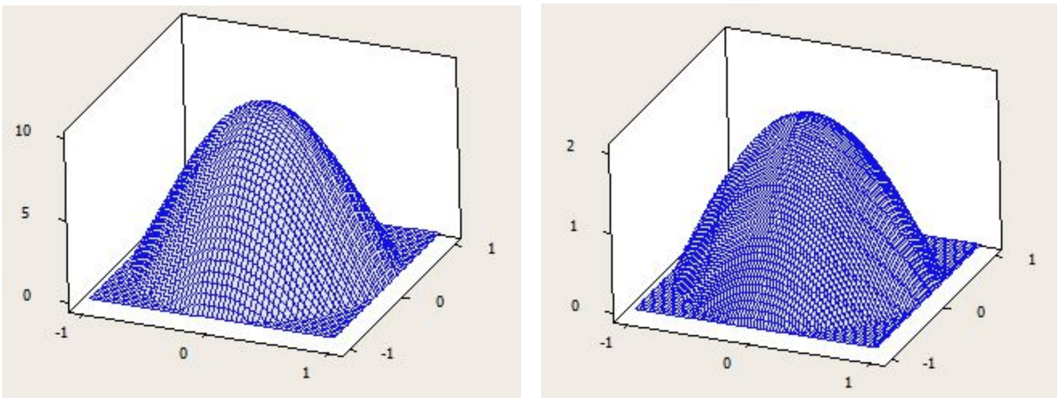


FIGURE 5. Graphs of the solution $u(x, y)$, for 50×50 and 75×75

So, from the Figure 5, we can see that, as $u \in C^2(\bar{B}_1 = \{(x, y) : x^2 + y^2 \leq 1\})$ is a positive solution of the Bratu's equation for $\mu \leq \mu_c$, then u is a function of $r = \sqrt{x^2 + y^2}$.

5. Conclusion

In this paper, we have studied the boundary value problem on the unit circle for the Bratu's equation which depends on a real parameter μ, we show that the boundary equation has the no non-negative solutions when $\mu \geq \mu_c = 5.7831859629467$, where we have implemented the numerical method, that is, the combination of the adaptation of finite difference and Gauss-Seidel method, which allows us to obtain a good approach of μ_c with respect to the exact theoretical method $\mu_c = \lambda = 5.7831859629467$.

A possible application of these results could be to the simplified stationary model for energy functional related to thermo-electro-hydrodynamics description of electrospinning [14].

6. Appendix

Example 6.1. Let $n = 4$, so $h = \frac{2}{n} = \frac{1}{2}$. The grid has $5 \times 5 = 25$ points, in which 9 are interior points of the circle, i.e., $P_{1,1}$, $P_{1,2}$, $P_{1,3}$, $P_{2,1}$, $P_{2,2}$, $P_{2,3}$, $P_{3,1}$, $P_{3,2}$, $P_{3,3}$ (see Figure 6).

The points which satisfy $x_i^2 + y_j^2 \geq 1$, i.e.,

$$P_{0,0} : u(x_0, y_0) = u(-1, -1) = w_{0,0} = 0,$$

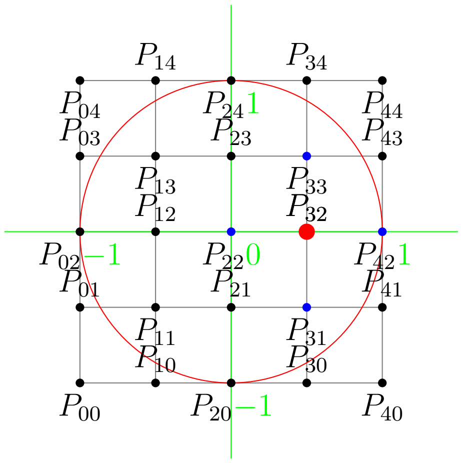


FIGURE 6.

$$
\begin{aligned}
P_{0,1} :& u(x_0, y_1) = u\left(-1, -\frac{1}{2}\right) = w_{0,1} = 0,\\
P_{0,2} :& u(x_0, y_2) = u(-1, 0) = w_{0,2} = 0,\\
P_{0,3} :& u(x_0, y_3) = u\left(-1, \frac{1}{2}\right) = w_{0,3} = 0,\\
P_{0,4} :& u(x_0, y_4) = u(-1, 1) = w_{0,4} = 0,\\
&\vdots\\
P_{4,0} :& u(x_4, y_0) = u(1, -1) = w_{4,0} = 0,\\
P_{4,1} :& u(x_4, y_1) = u\left(1, -\frac{1}{2}\right) = w_{4,1} = 0,\\
P_{4,2} :& u(x_4, y_2) = u(1, 0) = w_{4,2} = 0,\\
P_{4,3} :& u(x_4, y_3) = u(1, \frac{1}{2}) = w_{4,3} = 0,\\
P_{4,4} :& u(x_4, y_4) = u(1, -1) = w_{4,4} = 0.
\end{aligned}
$$

So, for the interior points of the circle which satisfy $x_i^2 + y_j^2 < 1$:

$$
\begin{aligned}
P_{1,1} &: \left(4 - \frac{\mu}{4}\right) w_{1,1} - w_{2,1} - w_{0,1} - w_{1,2} - w_{1,0} = \frac{\mu}{4},\\
P_{1,2} &: \left(4 - \frac{\mu}{4}\right) w_{1,2} - w_{2,2} - w_{0,2} - w_{1,3} - w_{1,1} = \frac{\mu}{4},\\
P_{1,3} &: \left(4 - \frac{\mu}{4}\right) w_{1,3} - w_{2,3} - w_{0,3} - w_{1,4} - w_{1,2} = \frac{\mu}{4},\\
P_{2,1} &: \left(4 - \frac{\mu}{4}\right) w_{2,1} - w_{3,1} - w_{1,1} - w_{2,2} - w_{2,0} = \frac{\mu}{4},\\
P_{2,2} &: \left(4 - \frac{\mu}{4}\right) w_{2,2} - w_{3,2} - w_{1,2} - w_{2,3} - w_{2,1} = \frac{\mu}{4},
\end{aligned}
$$

$$\begin{aligned}
P_{2,3}: \left(4-\frac{\mu}{4}\right) w_{2,3} - w_{3,3} - w_{1,3} - w_{2,4} - w_{2,2} &= \frac{\mu}{4},\\
P_{3,1}: \left(4-\frac{\mu}{4}\right) w_{3,1} - w_{4,1} - w_{2,1} - w_{3,2} - w_{3,0} &= \frac{\mu}{4},\\
P_{3,2}: \left(4-\frac{\mu}{4}\right) w_{3,2} - w_{4,2} - w_{2,2} - w_{3,3} - w_{3,1} &= \frac{\mu}{4},\\
P_{3,3}: \left(4-\frac{\mu}{4}\right) w_{3,3} - w_{4,3} - w_{2,3} - w_{3,4} - w_{3,2} &= \frac{\mu}{4}.
\end{aligned}$$

It is not defficult to establish the system of linear equations $AX = B$, where

$$A=\begin{bmatrix}
(4-\frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
-1 & (4-\frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & -1 & (4-\frac{\mu}{4}) & 0 & 0 & -1 & 0 & 0 & 0\\
-1 & 0 & 0 & (4-\frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0\\
0 & -1 & 0 & 0 & (4-\frac{\mu}{4}) & -1 & 0 & -1 & 0\\
0 & 0 & -1 & 0 & -1 & (4-\frac{\mu}{4}) & 0 & 0 & -1\\
0 & 0 & 0 & -1 & 0 & 0 & (4-\frac{\mu}{4}) & -1 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & -1 & (4-\frac{\mu}{4}) & -1\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & (4-\frac{\mu}{4})
\end{bmatrix},$$

$$X = \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{2,1} & w_{2,2} & w_{2,3} & w_{3,1} & w_{3,2} & w_{3,3}\end{bmatrix}^T,$$

$$B = \begin{bmatrix} \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4}\end{bmatrix}^T.$$

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LYAPUNOV-TYPE INEQUALITY FOR AN ANTI-PERIODIC CONFORMABLE BOUNDARY VALUE PROBLEM

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AND DIPAK KUMAR SATPATHI[3]

Abstract. In this article, we present a Lyapunov-type inequality for a conformable boundary value problem associated with anti-periodic boundary conditions. To demonstrate the applicability of established result, we obtain a lower bound on the eigenvalue of the corresponding eigenvalue problem.

1. Introduction

The subject of fractional calculus deals with the theory and applications of integral and differential operators of arbitrary order. The combined efforts of a number of scientists for many years resulted a strong basic theory of fractional calculus [13, 19]. In this process, several types of fractional differential operators were proposed so far. Unfortunately, each type obeys only some of the properties of the classical derivative.

In 2015, Ortigueira et al. [15] formulated two criteria required by an operator capable of being interpreted as fractional derivative. Recently, Tarasov [20] proposed a principle of nonlocality for fractional derivatives. As a result of these two articles, neither of the conformable differential operators proposed by Khalil et al. [12] are interpreted as fractional derivatives. Further, differential equations with conformable derivatives can be represented as differential equations of integer order for the space of differentiable functions. Subsequently, the conformable derivative was generalized in many ways [1, 10, 11]. Several authors have explored properties [3–7] and physical applications of the conformable derivative [5, 6, 24]. Recently, Anderson et al. [5]

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argued that there is a significant value in exploring the mathematics and physical applications of these derivatives.

The Lyapunov inequality is a necessary condition for the existence of a nontrivial solution of Hill's equation associated with Dirichlet boundary conditions.

Theorem 1.1 ([14]). *If the boundary value problem*

$$\begin{cases} y''(t) + p(t)y(t) = 0, & a < t < b, \\ y(a) = 0, \ y(b) = 0, \end{cases} \tag{1.1}$$

has a nontrivial solution, where $p : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |p(s)|ds > \frac{4}{(b-a)}. \tag{1.2}$$

The Lyapunov inequality (1.2) finds its applications in various problems related to the theory of differential equations and allied fields. Due to its importance, the Lyapunov inequality has been generalized in many forms. For a detailed discussion on Lyapunov-type inequalities and their applications, one can refer [8, 16, 18, 21–23] and the references therein.

On the other hand, Abdeljawad [2] and Gholami et al. [9] independently generalized Theorem 1.1 to the case where the classical second-order derivative in (1.1) is replaced by an α^{th}-order, $1 < \alpha \leq 2$, conformable derivative.

Theorem 1.2 ([2]). *If the boundary value problem*

$$\begin{cases} \left(T^{\alpha}_{a+}y\right)(t) + p(t)y(t) = 0, & a < t < b, \\ y(a) = 0, \quad y(b) = 0, \end{cases}$$

has a nontrivial solution, where $p : [a, b] \to \mathbb{R}$ is a continuous function, then

$$\int_a^b |p(s)|ds > \frac{\alpha^{\alpha}}{(\alpha-1)^{\alpha-1}(b-a)^{\alpha-1}}.$$

Here T^{α}_{a+} denotes the α^{th}-order conformable differential operator. Motivated by these works, in this article, we derive a Lyapunov-type inequality for the following two-point anti-periodic conformable boundary value problem:

$$\begin{cases} \left(T^{\alpha}_{0+}y\right)(t) + p(t)y(t) = 0, & 1 < \alpha \leq 2, 0 < t < T, \\ y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0. \end{cases} \tag{1.3}$$

2. Preliminaries

Throughout, we shall use the following notations, definitions and known results of conformable calculus [1, 12].

Definition 2.1 ([1])**.** Let $y : [a, \infty) \to \mathbb{R}$ and $0 < \alpha \le 1$. The α^{th}-order conformable derivative of y starting from a is defined by

$$\left(T_{a+}^{\alpha} y\right)(t) = \lim_{\varepsilon \to 0} \left[\frac{y\left(t + \varepsilon (t-a)^{1-\alpha}\right) - y(t)}{\varepsilon} \right], \quad t \in (a, \infty).$$

If $\left(T_{a+}^{\alpha} y\right)$ exists on (a, b), then

$$\left(T_{a+}^{\alpha} y\right)(a) = \lim_{t \to a^+} \left(T_{a+}^{\alpha} y\right)(t).$$

Definition 2.2 ([1])**.** Let $y : [a, \infty) \to \mathbb{R}$, $\alpha > 0$ and choose $n \in \mathbb{N}_1$ such that $n - 1 < \alpha \le n$. Assume that $y^{(n-1)}$ exists on (a, ∞). The α^{th}-order conformable derivative of y starting from a is defined by

$$\begin{aligned} \left(T_{a+}^{\alpha} y\right)(t) =& \left(T_{a+}^{\alpha-n+1} y^{(n-1)}\right)(t) \\ =& \lim_{\varepsilon \to 0} \left[\frac{y^{(n-1)}\left(t + \varepsilon (t-a)^{n-\alpha}\right) - y^{(n-1)}(t)}{\varepsilon} \right], \quad t \in (a, \infty). \end{aligned}$$

If $y^{(n)}$ exists on (a, ∞), we have

$$\left(T_{a+}^{\alpha} y\right)(t) = (t-a)^{n-\alpha} y^{(n)}(t), \quad t \in (a, \infty).$$

Also, we define

$$\left(T_{a+}^{0} y\right)(t) = y(t), \quad t \in (a, \infty).$$

Definition 2.3 ([1])**.** Let $y : [a, b] \to \mathbb{R}$, $\alpha > 0$ and choose $n \in \mathbb{N}_1$ such that $n - 1 < \alpha \le n$. The α^{th}-order conformable integral of y starting from a is defined by

$$\left(I_{a+}^{\alpha} y\right)(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} (s-a)^{\alpha-n} y(s) ds, \quad t \in [a, b].$$

Theorem 2.1 ([1])**.** *Let $y : [a, b] \to \mathbb{R}$, $\alpha > 0$ and choose $n \in \mathbb{N}_1$ such that $n - 1 < \alpha \le n$. If $y^{(n-1)}$ exists on (a, b), then*

$$\left(I_{a+}^{\alpha} T_{a+}^{\alpha} y\right)(t) = y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)(t-a)^k}{k!}, \quad t \in (a, b).$$

3. Anti-Periodic Boundary Value Problem

In this section, we derive a few properties of the Green's function for the boundary value problem (1.3) and obtain the corresponding Lyapunov-type inequality.

Theorem 3.1. *Let $1 < \alpha \le 2$ and $h : [0, T] \to \mathbb{R}$ is a continuous function. The conformal boundary value problem*

$$\begin{cases} \left(T_{0+}^{\alpha} y\right)(t) + h(t) = 0, \quad 0 < t < T, \\ y(0) + y(T) = 0, \quad y'(0) + y'(T) = 0, \end{cases} \tag{3.1}$$

has the unique solution

$$y(t) = \int_0^T G(t,s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} \left(\dfrac{T}{4} + \dfrac{t-s}{2}\right) s^{\alpha-2}, & 0 < t \le s \le T, \\ \left(\dfrac{T}{4} + \dfrac{s-t}{2}\right) s^{\alpha-2}, & 0 < s \le t \le T. \end{cases} \tag{3.2}$$

Proof. Applying I_{0+}^{α} on both sides of (3.1) and using Theorem 2.1, we have

$$y(t) = C_1 + C_2 t - \int_0^t (t-s)s^{\alpha-2}h(s)ds. \tag{3.3}$$

Differentiating (3.3) with respect to t, we get

$$y'(t) = C_2 - \int_0^t s^{\alpha-2}h(s)ds. \tag{3.4}$$

Using $y(0) + y(T) = 0$ in (3.3) we get

$$2C_1 + C_2 T = \int_0^T (T-s)s^{\alpha-2}h(s)ds. \tag{3.5}$$

Using $y'(0) + y'(T) = 0$ in (3.4) we get

$$C_2 = \frac{1}{2}\int_0^T s^{\alpha-2}h(s)ds. \tag{3.6}$$

Then, from (3.5) and (3.6), we have

$$2C_1 = -\frac{T}{2}\int_0^T s^{\alpha-2}h(s)ds + \int_0^T (T-s)s^{\alpha-2}h(s)ds,$$

which implies

$$C_1 = \frac{1}{2}\int_0^T \left(\frac{T}{2} - s\right) s^{\alpha-2}h(s)ds. \tag{3.7}$$

Then, from (3.3), (3.6) and (3.7) we have

$$\begin{aligned} y(t) &= \frac{1}{2}\int_0^T \left(\frac{T}{2} - s\right) s^{\alpha-2}h(s)ds + \frac{t}{2}\int_0^T s^{\alpha-2}h(s)ds - \int_0^t (t-s)s^{\alpha-2}h(s)ds \\ &= \frac{1}{2}\int_0^t \left(\frac{T}{2} + t - s\right) s^{\alpha-2}h(s)ds + \frac{1}{2}\int_t^T \left(\frac{T}{2} + t - s\right) s^{\alpha-2}h(s)ds \\ &\quad - \int_0^t (t-s)s^{\alpha-2}h(s)ds \\ &= \int_0^t \left(\frac{T}{4} + \frac{s-t}{2}\right) s^{\alpha-2}h(s)ds + \int_t^T \left(\frac{T}{4} + \frac{t-s}{2}\right) s^{\alpha-2}h(s)ds \\ &= \int_0^T G(t,s)h(s)ds. \end{aligned}$$

The proof is complete. □

Lemma 3.1. *The Green's function $G(t,s)$ defined in* (3.2) *satisfies the following properties:*

(a) $G(t,s) \le G(s,s)$, $(t,s) \in (0,T] \times (0,T]$;
(b) $s^{2-\alpha}G(s,s) = \frac{T}{4}$, $s \in [0,T]$;
(c) $\left|s^{2-\alpha}G(t,s)\right| \le \frac{T}{4}$, $(t,s) \in [0,T] \times [0,T]$;
(d) $\max_{t\in[0,T]} \int_0^T G(t,s)ds = \frac{T^\alpha\left(2-\alpha+2^{\frac{\alpha-2}{\alpha-1}}(\alpha-1)\right)}{4\alpha(\alpha-1)}$;
(e) $\max_{t\in[0,T]} \int_0^T s^{2-\alpha}G(t,s)ds = \frac{T^2}{8}$;
(f) $\max_{t\in[0,T]} \int_0^T G'(t,s)ds = \frac{T^{\alpha-1}}{2(\alpha-1)}$;
(g) $\max_{t\in[0,T]} \int_0^T s^{2-\alpha}G'(t,s)ds = \frac{T}{2}$;
(h) $\max_{t\in[0,T]} \int_0^T \left|G(t,s)\right|ds = \frac{T^\alpha(7\alpha-2)}{4\alpha(\alpha-1)}$.

Proof. Define the functions

$$G_1(t,s) = \left(\frac{T}{4} + \frac{t-s}{2}\right) s^{\alpha-2} \quad \text{and} \quad G_2(t,s) = \left(\frac{T}{4} + \frac{s-t}{2}\right) s^{\alpha-2}.$$

We can easily check that $G_1(t,s)$ is an increasing function of t. Differentiating $G_2(t,s)$ with respect to t for every fixed s, we observe that, $G_2(t,s)$ is a decreasing function of t. Thus, we have (a). The proof of (b) follows from (3.2). Clearly, from (a) and (b), we have

$$s^{2-\alpha}G(t,s) \le \frac{T}{4}, \quad (t,s) \in [0,T] \times [0,T]. \tag{3.8}$$

Consider

$$s^{2-\alpha}G_1(t,s) = \frac{T}{4} + \frac{s-t}{2} \ge \frac{T}{4} + \frac{0-T}{2} \ge -\frac{T}{4},$$

which implies

$$-s^{2-\alpha}G_1(t,s) \le \frac{T}{4}. \tag{3.9}$$

Similarly

$$s^{2-\alpha}G_2(t,s) = \frac{T}{4} + \frac{t-s}{2} \ge \frac{T}{4} + \frac{0-T}{2} \ge -\frac{T}{4},$$

implies

$$-s^{2-\alpha}G_2(t,s) \le \frac{T}{4}. \tag{3.10}$$

So, from (3.9) and (3.10), we get

$$-s^{2-\alpha}G(t,s) \le \frac{T}{4}, \quad (t,s) \in [0,T] \times [0,T]. \tag{3.11}$$

Then, from (3.8) and (3.11), (c) follows. For (d), consider

$$\int_0^T G(t,s)ds = \int_0^t \left(\frac{T}{4} + \frac{s-t}{2}\right) s^{\alpha-2}ds + \int_t^T \left(\frac{T}{4} + \frac{t-s}{2}\right) s^{\alpha-2}ds$$

$$
\begin{aligned}
&= \left(\frac{T}{4} - \frac{t}{2}\right)\left(\frac{t^{\alpha-1}}{\alpha-1}\right) + \frac{t^{\alpha}}{2\alpha} + \left(\frac{T}{4} + \frac{t}{2}\right)\left[\frac{T^{\alpha-1}}{\alpha-1} - \frac{t^{\alpha-1}}{\alpha-1}\right] \\
&\quad - \frac{1}{2}\left[\frac{T^{\alpha}}{\alpha} - \frac{t^{\alpha}}{\alpha}\right].
\end{aligned} \tag{3.12}
$$

Define $H_1(t)$ as the right hand side of (3.12). Now, differentiating $H_1(t)$ with respect to t and equating it to 0, we obtain $t = \frac{T}{2^{\frac{1}{\alpha-1}}}$. Again, differentiating $H_1{}'(t)$ with respect to t, we observe that $H_1{}''(t) \leq 0$ at $t = \frac{T}{2^{\frac{1}{\alpha-1}}}$. So, $H_1(t)$ attains its maximum at $t = \frac{T}{2^{\frac{1}{\alpha-1}}}$. Thus, we have (d). Consider

$$
\begin{aligned}
\int_0^T s^{2-\alpha} G(t,s)ds &= \int_0^t \left(\frac{T}{4} + \frac{s-t}{2}\right) ds + \int_t^T \left(\frac{T}{4} + \frac{t-s}{2}\right) ds \\
&= \left(\frac{T}{4} - \frac{t}{2}\right) t + \frac{t^2}{4} + \left(\frac{T}{4} + \frac{t}{2}\right)(T-t) - \left(\frac{T^2 - t^2}{4}\right).
\end{aligned} \tag{3.13}
$$

Define $H_2(t)$ as the right hand side of (3.13). Now, differentiating $H_2(t)$ with respect to t and equating it to 0, we obtain $t = \frac{T}{2}$. Again, differentiating $H_2{}'(t)$ with respect to t, we observe that $H_2{}''(t) < 0$ at $t = \frac{T}{2}$. So, $H_2(t)$ attains its maximum at $t = \frac{T}{2}$. Thus, we have (e). Consider

$$
\begin{aligned}
\int_0^T G'(t,s)ds &= -\frac{1}{2}\int_0^t s^{\alpha-2} ds + \frac{1}{2}\int_t^T s^{\alpha-2} ds \\
&= -\frac{1}{2}\left[\frac{t^{\alpha-1}}{\alpha-1}\right] + \frac{1}{2}\left[\frac{T^{\alpha-1}}{\alpha-1} - \frac{t^{\alpha-1}}{\alpha-1}\right] \\
&\leq \frac{T^{\alpha-1}}{2(\alpha-1)}.
\end{aligned}
$$

This completes the proof of (f). For (g), consider

$$
\int_0^T s^{2-\alpha} G'(t,s)ds = -\frac{1}{2}\int_0^t ds + \frac{1}{2}\int_t^T ds = -\frac{t}{2} + \frac{T}{2} - \frac{t}{2} = \frac{T}{2} - t \leq \frac{T}{2}.
$$

Consider

$$
\begin{aligned}
\int_0^T |G(t,s)|ds &= \int_0^t |G_1(t,s)|ds + \int_t^T |G_2(t,s)|ds \\
&\leq \int_0^t \left(\frac{T}{4} + \left|\frac{s-t}{2}\right|\right) s^{\alpha-2} ds + \int_t^T \left(\frac{T}{4} + \left|\frac{t-s}{2}\right|\right) s^{\alpha-2} ds \\
&= \frac{Tt^{\alpha-1}}{4(\alpha-1)} - \int_0^t \left(\frac{s-t}{2}\right) s^{\alpha-2} ds + \frac{T}{4}\left(\frac{T^{\alpha-1}}{\alpha-1} - \frac{t^{\alpha-1}}{\alpha-1}\right) \\
&\quad - \int_t^T \left(\frac{t-s}{2}\right) s^{\alpha-2} ds \\
&= \frac{Tt^{\alpha-1}}{4(\alpha-1)} - \frac{t^{\alpha}}{2\alpha} + \frac{t^{\alpha}}{2(\alpha-1)} - \frac{t\left(T^{\alpha-1} - t^{\alpha-1}\right)}{2(\alpha-1)} + \frac{T^{\alpha-1} - t^{\alpha-1}}{2\alpha}
\end{aligned}
$$

$$\leq \frac{T^\alpha}{4(\alpha-1)} + \frac{T^\alpha}{\alpha-1} + \frac{T^\alpha}{2\alpha} = \frac{T^\alpha(7\alpha-2)}{4\alpha(\alpha-1)}. \tag{3.14}$$

Thus, we have (h). The proof is complete. ☐

We are now able to formulate a Lyapunov-type inequality for the anti-periodic boundary value problem.

Theorem 3.2. *If* (1.3) *has a nontrivial solution, then*

$$\int_0^T s^{\alpha-2}|p(s)|ds \geq \frac{4}{T}. \tag{3.15}$$

Proof. Let $C[0,T]$ be the Banach space of continuous functions y on $[0,T]$ with the norm

$$\|y\|_C = \max_{t\in[0,T]} |y(t)|.$$

It follows from Theorem 3.1 that a solution to (1.3) satisfies the equation

$$y(t) = \int_0^T G(t,s)p(s)y(s)ds.$$

Consider

$$\begin{aligned}
|y(t)| &= \Big| \int_0^T G(t,s)p(s)y(s)ds \Big| \\
&\leq \int_0^T \Big|G(t,s)\Big||p(s)||y(s)|ds \\
&\leq \|y\| \int_0^T \Big|G(t,s)\Big||p(s)|ds \\
&= \|y\| \int_0^T \Big[s^{2-\alpha}\Big|G(t,s)\Big|\Big]\Big|s^{\alpha-2}p(s)\Big|ds,
\end{aligned}$$

which implies

$$\|y\| \leq \|y\| \max_{s\in[0,T]} \Big[s^{2-\alpha}\Big|G(t,s)\Big|\Big]\Big[\int_0^T \Big|s^{\alpha-2}p(s)\Big|ds\Big].$$

An application of Lemma 3.1 yields the result. The proof is complete. ☐

4. Application

In this section, we estimate a lower bound for the eigenvalue of the conformable eigenvalue problem corresponding to the conformable boundary value problem (1.3) using three different methods.

Definition 4.1 ([17])**.** A Lyapunov Inequality Lower Bound (LILB) is defined as a lower estimate for the smallest eigenvalue obtained from Lyapunov-type inequality given in (3.15) by setting $p(s) = \lambda$, that is,

$$\lambda \geq \frac{1}{TG_{max}},$$

where $G_{max} = \max_{0\leq t\leq T} |G(t,s)|$.

Definition 4.2 ([17])**.** A Cauchy-Schwartz Inequality Lower Bound (CSILB) is defined as a lower bound for the smallest eigenvalue obtained from Cauchy-Schwartz inequality of type given in (3.15) by setting $p(s) = \lambda$, that is,

$$\lambda \geq \left[\int_0^T \int_0^T G^2(t,s) ds dt \right]^{-\frac{1}{2}}.$$

Definition 4.3 ([17])**.** A Semi Maximum Norm Lower Bound (SMNLB) is defined as a lower bound for the smallest eigenvalue obtained from Semi Maximum Norm inequality of type given in given in (3.15) by setting $p(s) = \lambda$, that is,

$$\lambda \geq \frac{1}{\max_{0 \leq t \leq T} \int_0^T |G(t,s)| ds}. \tag{4.1}$$

Theorem 4.1. *Assume that y is a nontrivial solution of the conformable eigenvalue problem*

$$\begin{cases} \left(T_{0+}^{\alpha} y \right)(t) + \lambda y(t) = 0, & 0 < t < T, \\ y(0) + y(T) = 0, & y'(0) + y'(T) = 0, \end{cases}$$

where $y(t) \neq 0$ for each $t \in (0,T)$. Then

$$\lambda_{(LILB)} \geq \frac{4(\alpha - 1)}{T^{\alpha}}, \quad 1 < \alpha \leq 2, \tag{4.2}$$

$$\lambda_{(CSILB)} \geq \frac{4\sqrt{(2\alpha - 3)}}{T^{\alpha}}, \quad \frac{3}{2} \leq \alpha \leq 2, \tag{4.3}$$

$$\lambda_{(SMNLB)} \geq \frac{4\alpha(\alpha - 1)}{T^{\alpha}(7\alpha - 2)}, \quad 1 < \alpha \leq 2. \tag{4.4}$$

Proof. We choose $p(s) = \lambda$ in (3.15). Then, we obtain,

$$\lambda \int_0^T s^{\alpha - 2} ds \geq \frac{4}{T},$$

implies

$$\lambda \left(\frac{T^{\alpha - 1}}{\alpha - 1} \right) \geq \frac{4}{T}.$$

This proves the result (4.2). Consider,

$$\begin{aligned} \lambda \geq & \left[\int_0^T \int_0^T G^2(t,s) ds dt \right]^{-\frac{1}{2}} \\ = & \left(\int_0^T \int_0^T \left| s^{2-\alpha} G(t,s) \right|^2 s^{2\alpha - 4} ds dt \right)^{-\frac{1}{2}} \\ \geq & \left(\frac{T^2}{16} \int_0^T \int_0^T s^{2\alpha - 4} ds dt \right)^{-\frac{1}{2}} \end{aligned}$$

$$=\frac{4}{T}\left(\frac{T^{2\alpha-2}}{2\alpha-3}\right)^{-\frac{1}{2}}=\frac{4\sqrt{(2\alpha-3)}}{T^{\alpha}}.$$

So, (4.3) is proved. The result (4.4) follows from (4.1) and (3.14). The proof is complete. □

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ON THE HARMONIC INDEX AND THE SIGNLESS LAPLACIAN SPECTRAL RADIUS OF GRAPHS

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Abstract. The harmonic index of a conected graph G is defined as $H(G) = \sum_{uv\in E(G)} \frac{2}{d(u)+d(v)}$, where $E(G)$ is the edge set of G, $d(u)$ and $d(v)$ are the degrees of vertices u and v, respectively. The spectral radius of a square matrix M is the maximum among the absolute values of the eigenvalues of M. Let $q(G)$ be the spectral radius of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix having degrees of the vertices on the main diagonal and $A(G)$ is the $(0, 1)$ adjacency matrix of G. The harmonic index of a graph G and the spectral radius of the matrix $Q(G)$ have been extensively studied. We investigate the relationship between the harmonic index of a graph G and the spectral radius of the matrix $Q(G)$. We prove that for a connected graph G with n vertices, we have

$$\frac{q(G)}{H(G)} \leq \begin{cases} \dfrac{n^2}{2(n-1)}, & \text{if } n \geq 6, \\ \dfrac{16}{5}, & \text{if } n = 5, \\ 3, & \text{if } n = 4, \end{cases}$$

and the bounds are best possible.

1. Introduction

A lot of research has been done on topological indices due to their chemical importance. Chemical-based experiments show that there is a strong relationship between the properties of chemical compounds and their molecular structures. Topological indices are used for modelling properties of chemical compounds and biological activities in chemistry, biochemistry and nanotechnology. We study the harmonic index which is one of the most known topological indices.

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Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. The degree of a vertex $v \in V(G)$, $d(v)$, is the number of edges incident with v. A tree is a connected graph containing no cycles and a unicyclic graph is a connected graph containing exactly one cycle. A bicyclic graph is a connected graph G having $n+1$ edges where n is the number of vertices of G. Let us denote the complete graph, the star and the path having n vertices by K_n, S_n and P_n, respectively.

Let $e_1, e_2, \ldots, e_k \in E(G)$. We denote by $G - \{e_1, e_2, \ldots, e_k\}$ the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{e_1, e_2, \ldots, e_k\}$. An edge-induced subgraph $G[e_1, e_2, \ldots, e_k]$ is a subgraph of G which consists of the edges $e_1, e_2, \ldots, e_k$ and vertices incident with $e_1, e_2, \ldots, e_k$.

The spectral radius of a square matrix M is the maximum among the absolute values of the eigenvalues of M. Let $q(G)$ be the spectral radius of the signless Laplacian matrix $Q(G) = D(G) + A(G)$, where $D(G)$ is the diagonal matrix having degrees of the vertices on the main diagonal and $A(G)$ is the $(0,1)$ adjacency matrix of G. We denote the spectral radius (of the adjacency matrix $A(G)$) of a graph G by $\lambda(G)$.

The Randić index of a graph G is defined as

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d(u)d(v)}}.$$

This topological index has been successfully related to chemical and physical properties of organic molecules, and become one of the most important molecular descriptors. The Randić index was introduced by Randić [16] and generalized by Bollobás and Erdős [3]. Using the AutoGraphiX2 system, Aouchiche, Hansen and Zheng [1, 2] studied lower and upper bounds on $R(G) \oplus i(G)$ in terms of the number of vertices of G, where $i(G)$ is one of the following invariants: the maximum, minimum and average degree, diameter, girth, algebraic and vertex connectivity, matching number and the spectral radius of G, and $\oplus$ denotes one of the four operations $+, -, \times, /$.

The harmonic index

$$H(G) = \sum_{uv \in E(G)} \frac{2}{d(u) + d(v)}$$

of a graph G was introduced by Fajtlowicz [8]. Hansen and Vukicević [12] studied the connection between the Randić index and the chromatic number of graphs. Deng et al. [6] considered the relation connecting the harmonic index and the chromatic number and strengthened the result relating the Randić index and the chromatic number conjectured by the system AutoGraphiX and proved in [12]. Favaron, Mahéo and Saclé [9] considered the relationship between the harmonic index and eigenvalues of a graph.

Using the AutoGraphiX system, Hansen and Lucas [11] gave a conjecture saying that if G is a connected graph having $n \geq 4$ vertices, then

$$\frac{q(G)}{R(G)} \leq \begin{cases} \dfrac{4n-4}{n}, & \text{if } 4 \leq n \leq 12, \\ \dfrac{n}{\sqrt{n-1}}, & \text{if } n \geq 13, \end{cases}$$

with equality if and only if G is K_n for $4 \leq n \leq 12$ and G is S_n for $n \geq 13$. Recently, Ning and Peng [15] solved this conjecture.

Motivated by the work of [15] we study the relationship between the harmonic index $H(G)$ of a graph G and the spectral radius of the signless Laplacian matrix $Q(G)$. In particular, we prove the following theorem.

Theorem 1.1. *Let G be a connected graph having n vertices. Then*

$$\frac{q(G)}{H(G)} \leq \begin{cases} \dfrac{n^2}{2(n-1)}, & \text{if } n \geq 6, \\ \dfrac{16}{5}, & \text{if } n = 5, \\ 3 & \text{if } n = 4, \end{cases}$$

with equality if and only if G is S_n for $n \geq 6$ and G is K_n for $4 \leq n \leq 5$.

2. Preliminaries

In this section, we present known results, which will be used in the proofs of our theorems. Upper bounds on the spectral radius of the signless Laplacian matrix and the adjacency matrix of a graph were given in [10] and [13], respectively.

Lemma 2.1 ([10]). *Let G be a connected graph with n vertices, m edges and let $q(G)$ be the spectral radius of the signless Laplacian matrix of G. Then*

$$q(G) \leq \frac{2m}{n-1} + n - 2,$$

with equality if and only if G is K_n or S_n.

Lemma 2.2 ([13]). *Let G be a connected graph G with n vertices, m edges and let $\lambda(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\lambda(G) \leq \sqrt{2m - n + 1},$$

with equality if and only if G is K_n or S_n.

Let us present three lower bounds on the harmonic index $H(G)$ of a graph G for general graphs, unicyclic graphs and bicyclic graphs.

Lemma 2.3 ([5]). *Let G be a connected graph with n vertices and m edges. Then*

$$H(G) \geq \frac{2m^2}{n\lambda(G)^2}.$$

Lemma 2.4 ([7, 14, 17]). *Let G be a unicyclic graph with $n \geq 3$ vertices. Then*

$$H(G) \geq \frac{5n^2+n-12}{2n(n+1)}.$$

Lemma 2.5 ([7, 14]). *Let G be a bicyclic graph with $n \geq 4$ vertices. Then*

$$H(G) \geq \frac{14}{5} - \frac{2n^2+14n+16}{n(n+1)(n+2)}.$$

3. Results

First we consider graphs with $n \geq 6$ vertices, m edges and the harmonic index at least $\frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$.

Theorem 3.1. *Let G be a connected graph with $n \geq 6$ vertices and m edges. If $H(G) \geq \frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$, then*

$$\frac{q(G)}{H(G)} \leq \frac{n^2}{2(n-1)},$$

with equality if and only if G is S_n.

Proof. Let $H(G) \geq \frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$. By Lemma 2.1, we have $q(G) \leq \frac{2m}{n-1} + n - 2$, with equality if and only if G is either K_n or S_n. Thus,

$$\frac{q(G)}{H(G)} \leq \frac{\frac{2m}{n-1}+n-2}{\frac{2(n-1)}{n}+\frac{4(m-n+1)}{n^2}} = \frac{\left(\frac{2(n-1)}{n}+\frac{4(m-n+1)}{n^2}\right)\frac{n^2}{2(n-1)}}{\frac{2(n-1)}{n}+\frac{4(m-n+1)}{n^2}} = \frac{n^2}{2(n-1)}.$$

Note that for K_n we have $m = \frac{n(n-1)}{2}$ and for this value we get $\frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2} = 4(\frac{n-1}{n})^2$. Since $H(K_n) = \frac{n}{2} > 4(\frac{n-1}{n})^2$ for every $n \geq 6$, we obtain $\frac{q(K_n)}{H(K_n)} < \frac{n^2}{2(n-1)}$.

For the graph S_n we have $m = n-1$. Since $H(S_n) = \frac{2(n-1)}{n} = \frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$ for $m = n-1$, we get $\frac{q(S_n)}{H(S_n)} = \frac{n^2}{2(n-1)}$, which implies that $\frac{q(G)}{H(G)} \leq \frac{n^2}{2(n-1)}$, with equality if and only if G is S_n. □

Let us show that the main result holds for trees and graphs satisfying the inequality $m \geq n + 1 + \frac{6}{n-4}$.

Theorem 3.2. *Let G be a connected graph with $n \geq 6$ vertices and m edges.*

(i) *If $m = n-1$, then $\frac{q(G)}{H(G)} \leq \frac{n^2}{2(n-1)}$, with equality if and only if G is S_n.*

(ii) *If $m \geq n+1+\frac{6}{n-4}$, then $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$.*

(iii) *If $n = 6, 7$ and $m = 10$, then $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$.*

Proof. By Lemmas 2.2 and 2.3 we have $H(G) \geq \frac{2m^2}{n\lambda^2} \geq \frac{2m^2}{n(2m-n+1)}$. Let $f(m) = H(G) - \frac{2(n-1)}{n} - \frac{4(m-n+1)}{n^2}$. Then

$$f(m) \geq \frac{2m^2}{n(2m-n+1)} - \frac{2(n-1)}{n} - \frac{4(m-n+1)}{n^2} = \frac{2 \times g(m)}{n^2(2m-n+1)},$$

where $g(m) = (m - n + 1)[(n - 4)m - (n^2 - 3n + 2)]$.

(i) If $m = n - 1$, then $g(m) = 0$ which implies that $f(m) \geq 0$ and $H(G) \geq \frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$. From Theorem 3.1 we get $\frac{q(G)}{H(G)} \leq \frac{n^2}{2(n-1)}$, with equality if and only if G is S_n.

(ii) If $m \geq n + 1 + \frac{6}{n-4} = \frac{n^2-3n+2}{n-4}$, then $(n-4)m - (n^2 - 3n + 2) \geq 0$. This implies that $g(m) \geq 0$ and $f(m) \geq 0$. Thus, $H(G) \geq \frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$. Since G is not S_n, from Theorem 3.1, we get $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$.

(iii) If $n = 6$ and $m = 10$, or $n = 7$ and $m = 10$, or $n = 10$ and $m = 12$, then $g(m) = 0$. Thus $f(m) \geq 0$ and $H(G) \geq \frac{2(n-1)}{n} + \frac{4(m-n+1)}{n^2}$. Since G is not S_n, from Theorem 3.1 we get $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$. ☐

The following two results solve our problem for every $m = n$ and $m = n + 1$, respectively.

Theorem 3.3. *Let G be a connected graph with $n \geq 6$ vertices and $m = n$ edges. Then*

$$\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}.$$

Proof. If $m = n$, then G is a unicyclic graph and by Lemma 2.4 we have $H(G) \geq \frac{5n^2+n-12}{2n(n+1)}$. It can be checked that $\frac{5n^2+n-12}{2n(n+1)} > \frac{2(n-1)}{n} + \frac{4}{n^2}$. Then from Theorem 3.1 we obtain $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$. ☐

Theorem 3.4. *Let G be a connected graph with $n \geq 6$ and $m = n + 1$ edges. Then*

$$\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}.$$

Proof. If $m = n + 1$, then G is a bicyclic graph and by Lemma 2.5 we have $H(G) \geq \frac{14}{5} - \frac{2n^2+14n+16}{n(n+1)(n+2)}$. It can be checked that $\frac{14}{5} - \frac{2n^2+14n+16}{n(n+1)(n+2)} > \frac{2(n-1)}{n} + \frac{8}{n^2}$. Then, from Theorem 3.1, we obtain $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$. ☐

From Theorems 3.2, 3.3 and 3.4 we obtain the best possible bound on $\frac{q(G)}{H(G)}$ for graphs G having $n \geq 10$ vertices.

Corollary 3.1. *Let G be a connected graph with $n \geq 10$ vertices. Then*

$$\frac{q(G)}{H(G)} \leq \frac{n^2}{2(n-1)},$$

with equality if and only if G is S_n.

Proof. Since $n + 2 \geq n + 1 + \frac{6}{n-4}$ for $n \geq 10$, by Theorem 3.2 (ii), $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$ for every $n \geq 10$ and $m \geq n + 2$. By Theorems 3.3 and 3.4, $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$ for graphs G such that $m = n$ and $m = n + 1$, and by Theorem 3.2 (i), $\frac{q(G)}{H(G)} \leq \frac{n^2}{2(n-1)}$ for graphs G such that $m = n - 1$ with equality if and only if G is S_n. ☐

From Theorems 3.2, 3.3 and 3.4 we also know that if $6 \le n \le 9$, then the only cases which remain unsolved are:

(i) $n = 6$ and $m = 8, 9$;
(ii) $n = 7$ and $m = 9$;
(iii) $n = 8$ and $m = 10$;
(iv) $n = 9$ and $m = 11$.

For this purpose we present results on the spectral radius of connected graphs with n vertices and $n+2 \le m \le n+3$ edges. From [4, Theorems 3.2 and 3.3] and their proofs we obtain the following lemma.

Lemma 3.1 ([4])**.** *The maximum spectral radius $\lambda(G)$ of a connected graph G with $n \ge 4$ vertices and m edges is the maximum root of*

(i) $\varphi_1(\lambda) = \lambda^3 - 2\lambda^2 - (n-1)\lambda + 2(n-4)$ *if* $m = n+2$*;*
(ii) $\varphi_2(\lambda) = \lambda^4 - (n+3)\lambda^2 - 8\lambda + 4(n-6)$ *and*
$\varphi_3(\lambda) = \lambda^6 - (n+3)\lambda^4 - 10\lambda^3 + (4n-21)\lambda^2 + (2n-8)\lambda - (n-5)$ *if* $m = n+3$.

We use Lemma 3.1 in the proof of Theorem 3.5.

Theorem 3.5. *Let G be a connected graph with n vertices and m edges. If*

(i) $n = 6$ *and* $m = 8, 9$*;*
(ii) $n = 7$ *and* $m = 9$*;*
(iii) $n = 8$ *and* $m = 10$*;*
(iv) $n = 9$ *and* $m = 11$*,*

then $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$.

Proof. By Lemma 3.1, we can calculate the upper bounds on the maximum spectral radius λ. We have $\lambda < 3.1775$ if $n = 6$ and $m = 8$, $\lambda < 3.274$ if $n = 7$ and $m = 9$, $\lambda < 3.373$ if $n = 8$ and $m = 10$, $\lambda < 3.475$ if $n = 9$ and $m = 11$, $\lambda < 3.404$ if $n = 6$ and $m = 9$.

Thus, from Lemma 2.3 we obtain the lower bounds on the harmonic index of G. $H(G) > 2.11294$ if $n = 6$ and $m = 8$, $H(G) > 2.15903$ if $n = 7$ and $m = 9$, $H(G) > 2.19739$ if $n = 8$ and $m = 10$, $H(G) > 2.22671$ if $n = 9$ and $m = 11$, $H(G) > 2.33015$ if $n = 6$ and $m = 9$.

By Lemma 2.1, we obtain upper bounds on $q(G)$. We have $q(G) \le \frac{36}{5}$ if $n = 6$ and $m = 8$, $q(G) \le 8$ if $n = 7$ and $m = 9$, $q(G) \le \frac{62}{7}$ if $n = 8$ and $m = 10$, $q(G) \le \frac{39}{4}$ if $n = 9$ and $m = 11$, $q(G) \le \frac{38}{5}$ if $n = 6$ and $m = 9$.

It is easy to verify that $\frac{q(G)}{H(G)} < \frac{n^2}{2(n-1)}$ for all these cases. □

From Theorems 3.2, 3.3, 3.4 and 3.5 we get Corollary 3.2.

Corollary 3.2. *Let G be a connected graph with n vertices where $6 \le n \le 9$. Then*

$$\frac{q(G)}{H(G)} \le \frac{n^2}{2(n-1)},$$

with equality if and only if G is S_n.

It remains to find upper bounds on $\frac{q(G)}{H(G)}$ for graphs G having $n \leq 5$ vertices. For $n = 3$ there are only two non-isomorphic graphs: K_3 and $K_3 - \{e\}$, where $e \in E(K_3)$. We have $H(K_3) = \frac{3}{2}$ and by Lemma 2.1, $q(K_3) = 4$, thus $\frac{q(K_3)}{H(K_3)} = \frac{8}{3}$. For $K_3 - \{e\}$ we obtain $H(K_3 - \{e\}) = \frac{4}{3}$ and by Lemma 2.1, $q(K_3 - \{e\}) \leq 3$, so $\frac{q(K_3 - \{e\})}{H(K_3 - \{e\})} \leq \frac{9}{4}$. Hence $\frac{q(G)}{H(G)} \leq \frac{8}{3}$ for any graph G having 3 vertices with equality if and only if G is K_3.

Let us present bounds for graphs having 4 and 5 vertices.

Theorem 3.6. *Let G be a connected graph with 4 vertices. Then*

$$\frac{q(G)}{H(G)} \leq 3,$$

with equality if and only if G is K_4.

Proof. The only graph with 4 vertices and 6 edges is K_4, and the only graph with 4 vertices and 5 edges is $K_4 - \{e\}$. Since $H(K_4) = 2$ and $q(K_4) = 6$ (by Lemma 2.1), we get $\frac{q(K_4)}{H(K_4)} = 3$.

For $K_4 - \{e\}$ where $e \in E(K_4)$, we obtain $H(K_4 - \{e\}) = \frac{29}{15}$, and from Lemma 2.1 we have $q(K_4 - \{e\}) \leq \frac{16}{3}$, which gives $\frac{q(K_4 - \{e\})}{H(K_4 - \{e\})} \leq \frac{80}{29} < 3$.

We have two non-isomorphic graphs for $m = 4$, namely C_4 and $S_4 + \{e\}$. We get $H(C_4) = 2$ and $q(C_4) \leq \frac{14}{3}$ (by Lemma 2.1), so $\frac{q(C_4)}{H(C_4)} \leq \frac{7}{3} < 3$. Similarly, $H(S_4 + \{e\}) = \frac{9}{5}$ and $q(S_4 + \{e\}) \leq \frac{14}{3}$, thus $\frac{q(S_4 + \{e\})}{H(S_4 + \{e\})} \leq \frac{70}{27} < 3$.

There are two non-isomorphic graphs for $m = 3$, namely S_4 and P_4. We have $H(S_4) = \frac{3}{2}$ and $q(S_4) = 4$ (by Lemma 2.1), thus $\frac{q(S_4)}{H(S_4)} = \frac{8}{3}$. Similarly, $H(P_4) = \frac{11}{6}$ and $q(P_4) \leq 4$, hence $\frac{q(P_4)}{H(P_4)} \leq \frac{24}{11} < 3$. $\square$

Theorem 3.7. *Let G be a connected graph with 5 vertices. Then*

$$\frac{q(G)}{H(G)} \leq \frac{16}{5},$$

with equality if and only if G is K_5.

Proof. We consider the cases $m = 7, 8, 9, 10$. The only graph with 5 vertices and 10 edges is K_5. Since $H(K_5) = \frac{5}{2}$ and $q(K_5) = 8$ (by Lemma 2.1), we get $\frac{q(K_5)}{H(K_5)} = \frac{16}{5}$.

The only graph with 5 vertices and 9 edges is $K_5 - \{e\}$ where $e \in E(K_5)$. We have $H(K_5 - \{e\}) = \frac{69}{28}$ and from Lemma 2.1 we obtain $q(K_5 - e) \leq \frac{15}{2}$, which gives $\frac{q(K_5 - e)}{H(K_5 - e)} \leq \frac{210}{69} < \frac{16}{5}$.

For $m = 8$ we have $G = K_5 - \{e_1, e_2\}$ where $e_1, e_2 \in E(K_5)$. There are two non-isomorphic graphs having 8 edges. If e_1 and e_2 are adjacent, then $H(K_5 - \{e_1, e_2\}) = \frac{67}{28}$, and if e_1 and e_2 are not adjacent, then $H(K_5 - \{e_1, e_2\}) = \frac{52}{21}$. So, $H(K_5 - \{e_1, e_2\}) \geq \frac{67}{28}$

and from Lemma 2.1 we obtain $q(K_5 - \{e_1, e_2\}) \leq 7$, which gives

$$\frac{q(K_5 - \{e_1, e_2\})}{H(K_5 - \{e_1, e_2\})} \leq \frac{196}{67} < \frac{16}{5}.$$

For $m = 7$ we have $G = K_5 - \{e_1, e_2, e_3\}$ where $e_1, e_2, e_3 \in E(K_5)$. There are four non-isomorphic graphs having 7 edges. If $G[e_1, e_2, e_3]$ is K_3, then $H(K_5-\{e_1, e_2, e_3\}) = \frac{9}{4}$. If $G[e_1, e_2, e_3]$ is S_4, then $H(K_5 - \{e_1, e_2, e_3\}) = \frac{79}{35}$. If $G[e_1, e_2, e_3]$ is a path, then $H(K_5 - \{e_1, e_2, e_3\}) = \frac{83}{35}$. If $G[e_1, e_2, e_3]$ is not connected ($G[e_1, e_2, e_3]$ is $K_2 \cup P_3$), then $H(K_5 - \{e_1, e_2, e_3\}) = \frac{37}{15}$. Thus $H(K_5 - \{e_1, e_2, e_3\}) \geq \frac{9}{4}$ and from Lemma 2.1 we obtain $q(K_5 - \{e_1, e_2, e_3\}) \leq \frac{13}{2}$, which gives

$$\frac{q(K_5 - \{e_1, e_2, e_3\})}{H(K_5 - \{e_1, e_2, e_3\})} \leq \frac{26}{9} < \frac{16}{5}.$$

If $m = 4, 5$ or 6, it can be proved similarly that $\frac{q(G)}{H(G)} < \frac{16}{5}$. □

From Theorems 3.6 and 3.7, and Corollaries 3.1 and 3.2 we obtain our main result (Theorem 1.1).

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SUMMATION-INTEGRAL TYPE OPERATORS BASED ON LUPAŞ-JAIN FUNCTIONS

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Abstract. We introduce a genuine summation-integral type operators based on Lupaş-Jain type base functions related to the unbounded sequences. We investigated their degree of approximation in terms of modulus of continuity and $\mathcal{K}$-functional for the functions from bounded and continuous functions space. Furthermore, we give some theorems for the local approximation properties of functions belonging to Lipschitz class. Also, we give Voronovskaja theorem for these operators.

1. Introduction

Inspiring by Lupaş's paper [12], Agratini studied the following operators

$$L_n(f,x)=2^{-nx}\sum_{k=0}^{\infty}\frac{(nx)_k}{2^k k!}f\left(\frac{k}{n}\right), \tag{1.1}$$

where $f\in C[0,\infty)$, $C[0,\infty)$ is the space of all real valued continuous functions on $[0,\infty)$ in [1]. Agratini gave some estimations for rates of convergence, an asymptotic formula and a reobtained version by using probabilistic methods at the same study. Also, he introduced a Kantorovich and a Durrmeyer modifications of the operators (1.1). Agratini, in [2], gave some estimations on the Kantorovich variant of the operators (1.1) by using modulus of smoothness. Moreover, he investigated rate of convergence by the step weight function of Lupaş operators, for local Lipschitz class functions. Also, he gave some approximation properties of the operators given by (1.1) using probabilistic methods.

Key words and phrases. Lupaş-Jain functions, summation-integral type operators, moduli of continuity, $\mathcal{K}$-functional, Voronovskaja theorem.

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In [3], Erençin and Taşdelen introduced a generalization of the operators (1.1) with the help of increasing and unbounded sequences of positive numbers (a_n), (b_n). They studied weighted approximation properties of these generalized operators. Later, in [4], they studied convergence properties of the Kantorovich type version of these operators. By using the modulus of continuity and Peetre's $\mathcal{K}$-functional, they gave the rate of convergence of these operators. Also, they investigated convergence properties for the functions from local Lipschitz class.

In [9], we generalized the operators (1.1) based on Lupaş base function by using the sequences (a_n), (b_n) as follows

$$L_{a_n,b_n}\left(f;x\right)=2^{-\frac{a_n}{b_n}x}\sum_{k=0}^{\infty}\frac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!}f\left(\frac{b_n}{a_n}k\right), \tag{1.2}$$

where $(a_n),(b_n)$ are unbounded and increasing sequences of positive real numbers such that

$$\lim_{n\to\infty}\frac{b_n}{a_n}=0 \quad \text{and} \quad \frac{b_n}{a_n}\leq 1. \tag{1.3}$$

We gave and investigated some basic results for these operators. Also, using Lupaş and Szász basis functions we defined summation-integral type operators

$$D_{a_n,b_n}\left(f;x\right)=\frac{a_n}{b_n}\sum_{k=0}^{\infty}l_{n,k}\left(x\right)\int\limits_0^{\infty}P_{n,k}\left(u\right)f\left(u\right)du,$$

where $P_{n,k}\left(x\right)=e^{-\frac{a_n}{b_n}x}\dfrac{\left(\frac{a_n}{b_n}x\right)^k}{k!}$, $l_{n,k}\left(x\right)=\dfrac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!}2^{-\frac{a_n}{b_n}x}$. Then, we gave the degree of approximation of these operators in terms of Ditzian-Totik modulus of smoothness and corresponding $\mathcal{K}$-functional. Also, we examined the convergence by using the Lipschitz class functions and we gave some results in weighted spaces.

Govil et al. in [5], introduced a modification of Lupaş operators with weight of Szász basis functions. They investigated the rate of convergence for the functions which have bounded derivatives. In addition, they gave a new modification of the Lupaş operators as follows

$$D_n\left(f;x\right)=n\sum_{k=1}^{\infty}l_{n,k}(x)\int\limits_0^{\infty}p_{n,k-1}\left(u\right)f\left(u\right)du+l_{n,0}(x)f(0), \quad x\geq 0,$$

where

$$p_{n,k-1}\left(x\right)=e^{-nt}\frac{\left(nx\right)^{k-1}}{(k-1)!}$$

and

$$l_{n,k}\left(x\right)=\frac{\left(nx\right)_k}{2^k k!}2^{-nx}.$$

Jain in [10], introduced the modified form of the Szász-Mirakjan operator as follows

$$P_n^{\beta}\left(f;x\right)=\sum_{k=0}^{\infty}p_{n,k}^{\beta}(x)f\left(\frac{k}{n}\right),\quad x\geq 0, \tag{1.4}$$

where the operators based on certain parameter $0\leq\beta<1$ and the base function

$$p_{n,k}^{\beta}(x)=e^{-nx+k\beta}\frac{nx\left(nx+k\beta\right)^{k-1}}{k!}.$$

The rate of approximation of operators given by (1.4), for some values of n, is better than the rate of approximation of operators Szász-Mirakjan.

Gupta and Greubel established the Durrmeyer variant of the operators (1.4) as follow

$$D_n^{\beta}\left(f;x\right)=\sum_{k=1}^{\infty}\left(\int_0^{\infty}p_{n,k-1}^{\beta}(u)du\right)^{-1}p_{n,k}^{\beta}(x)\int_0^{\infty}p_{n,k-1}^{\beta}\left(u\right)f\left(u\right)du+e^{-nx}f(0)$$

and investigated some approximation properties in [6].

Inspiring by the previous studies, we define a genuine summation-integral type operators by using Jain and Lupaş base functions for integrable functions as follows

$$\begin{aligned} D_{a_n,b_n}^{[\beta]}(f;x) &\\ = \sum_{k=1}^{\infty}&\left(\int_0^{\infty}\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right)du\right)^{-1}l_{n,k}^{*}(x)\int_0^{\infty}\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right)f(u)du+2^{-\frac{a_n}{b_n}x}f(0), \end{aligned} \tag{1.5}$$

where $f\in C[0,\infty)$ is integrable function, $l_{n,k}^{*}(x)=\frac{\left(\frac{a_n}{b_n}x\right)_k}{2^k k!}2^{-\frac{a_n}{b_n}x}$, (a_n) and (b_n) are unbounded and increasing sequences of positive real numbers satisfying the condition (1.3) and the Jain-type base function is

$$\theta_\beta\left(k,\frac{a_n}{b_n}x\right)=\frac{a_n}{b_n}x\left(\frac{a_n}{b_n}x+k\beta\right)^{k-1}\frac{e^{-\left(\frac{a_n}{b_n}x+k\beta\right)}}{k!},$$

for $x\in[0,\infty)$, $n\in\mathbb{N}$ and $\beta\in[0,1)$. Here, by considering the definition of θ_β we see that $\sum_{k=0}^{\infty}\theta_\beta\left(k,\frac{a_n}{b_n}x\right)=1$.

In addition, if we take $\langle f,g\rangle=\int_0^{\infty}f(t)g(t)dt$ at the definition of the operators $D_{a_n,b_n}^{[\beta]}$, we can write these operators as follow (see [6, 7])

$$D_{a_n,b_n}^{[\beta]}(f;x)=\sum_{k=1}^{\infty}\frac{\left\langle\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right),f(t)\right\rangle}{\left\langle\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right),1\right\rangle}l_{n,k}^{*}(x)+2^{-\frac{a_n}{b_n}x}f(0). \tag{1.6}$$

In this paper, we give the degree of approximation of these operators using the modulus of continuity and Peetre's $\mathcal{K}$-functional. Also, we give some theorems about local approximation and Voronovskaja theorem.

1.1. **Direct results for $D^{[\beta]}_{a_n,b_n}$ operators.** In this section, we give some basic properties of the operators $D^{[\beta]}_{a_n,b_n}$ operators.

Lemma 1.1. *For* $0\leq\beta<1$ *we define*

$$P_r^*(k-1,\beta)=\frac{\left\langle\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right),t^r\right\rangle}{\left\langle\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right),1\right\rangle},$$

and we get the following

$$\begin{aligned}
&P_0^*(k-1,\beta)=1,\\
&P_1^*(k-1,\beta)=\frac{b_n}{a_n}\left[(1-\beta)k+\frac{\beta(2-\beta)}{1-\beta}\right],\\
&P_2^*(k-1,\beta)=\left(\frac{b_n}{a_n}\right)^2\left[(1-\beta)^2k^2+(1+4\beta-2\beta^2)k+\frac{\beta^2(3-\beta)}{1-\beta}\right].
\end{aligned}$$

The proof is obtained by method of Lemma 2 at [6].

Lemma 1.2. ([9, Lemma 1]). *For* $f\in C[0,\infty)$ *and* $x\in[0,\infty)$*, the operators* L_{a_n,b_n} *given by* (1.2) *satisfy the following conditions*

$$\begin{aligned}
&L_{a_n,b_n}(e_0;x)=1,\\
&L_{a_n,b_n}(e_1;x)=x,\\
&L_{a_n,b_n}(e_2;x)=x^2+2\frac{b_n}{a_n}x,
\end{aligned}$$

where $e_i(x)=x^i$*,* $i=0,1,2$*, and* (a_n)*,* (b_n) *are sequences of positive real numbers satisfying the condition* (1.3)*.*

Now, we give the following equalities for the test functions of the operators defined with (1.6).

Lemma 1.3. *Let* $e_k(x)=x^k$*,* $k=0,1,2$*, and* (a_n)*,* (b_n) *are unbounded and increasing sequences of positive real numbers satisfying the condition* (1.3)*. For each* $x\in[0,\infty)$*, the operators* $D^{[\beta]}_{a_n,b_n}$ *satisfy the following equalities*

$$\begin{aligned}
&D^{[\beta]}_{a_n,b_n}(e_0;x)=1,\\
&D^{[\beta]}_{a_n,b_n}(e_1;x)=x(1-\beta)+\frac{b_n}{a_n}\frac{\beta(2-\beta)}{(1-\beta)},\\
&D^{[\beta]}_{a_n,b_n}(e_2;x)=(1-\beta)^2x^2+3\frac{b_n}{a_n}x+\left(\frac{b_n}{a_n}\right)^2\frac{\beta^2(3-\beta)}{(1-\beta)},\\
&D^{[\beta]}_{a_n,b_n}(e_3;x)=(1-\beta)^3x^3+3\frac{b_n}{a_n}\left(9-15\beta+9\beta^2-3\beta^3\right)x^2\\
&\qquad+\left(\frac{b_n}{a_n}\right)^2\left(6+14\beta+16\beta^2-2\beta^3+\frac{3\beta^4}{1-\beta}\right)x+\left(\frac{b_n}{a_n}\right)^3\frac{\beta^2(3-\beta)}{1-\beta},
\end{aligned}$$

$$\begin{aligned}D^{[\beta]}_{a_n,b_n}(e_4;x)=&(1-\beta)^4x^4+\frac{b_n}{a_n}\Big(18-28\beta+22\beta^2-8\beta^3+2\beta^4\Big)x^3\\&+\left(\frac{b_n}{a_n}\right)^2\Big(47-40\beta+30\beta^2-20\beta^3+5\beta^4\Big)x^2\\&+\left(\frac{b_n}{a_n}\right)^3\left(42-+30\beta^2-20\beta^3+20\beta^4-\frac{10\beta^5}{1-\beta}\right)x+\left(\frac{b_n}{a_n}\right)^4\frac{\beta^4(5-\beta)}{1-\beta}.\end{aligned}$$

Proof. Considering the definition of the operator (1.6), the properties of Pochammer symbol, and using the equality

$$2^{\frac{a_n}{b_n}x}=\sum_{k=0}^{\infty}\frac{\left(\frac{a_n}{b_n}x\right)_k}{2^kk!},\quad x\in[0,\infty),$$

and by considering Lemma 1.1 and Lemma 1.2, we get

$$D^{[\beta]}_{a_n,b_n}(e_0;x)=\sum_{k=1}^{\infty}P^*_0(k-1,\beta)l^*_{n,k}(x)+2^{-\frac{a_n}{b_n}x}e_0(0)=\sum_{k=1}^{\infty}l^*_{n,k}(x)=L_{a_n,b_n}(e_0;x)=1.$$

For e_1 and e_2 we have following results,

$$\begin{aligned}D^{[\beta]}_{a_n,b_n}(e_1;x)=&\sum_{k=1}^{\infty}P^*_1(k-1,\beta)l^*_{n,k}(x)+2^{-\frac{a_n}{b_n}x}e_1(0)\\=&\sum_{k=1}^{\infty}l^*_{n,k}(x)\left[\frac{b_n}{a_n}\left((1-\beta)k+\frac{\beta}{1-\beta}+1\right)\right]\\=&\frac{b_n}{a_n}(1-\beta)\sum_{k=1}^{\infty}l^*_{n,k}(x)(k)+\frac{b_n}{a_n}\frac{\beta(2-\beta)}{1-\beta}\\=&(1-\beta)L_{a_n,b_n}(e_1;x)+\frac{b_n}{a_n}\frac{\beta(2-\beta)}{1-\beta}L_{a_n,b_n}(e_0;x)\\=&(1-\beta)x+\frac{b_n}{a_n}\frac{\beta(2-\beta)}{1-\beta},\\D^{[\beta]}_{a_n,b_n}(e_2;x)=&\sum_{k=1}^{\infty}P^*_2(k-1,\beta)l^*_{n,k}(x)+2^{-\frac{a_n}{b_n}x}e_2(0)\\=&\left(\frac{b_n}{a_n}\right)^2(1-\beta)^2\sum_{k=1}^{\infty}l^*_{n,k}(x)k^2+\left(\frac{b_n}{a_n}\right)^2(1+4\beta-2\beta^2)\sum_{k=1}^{\infty}l^*_{n,k}(x)k\\&+\left(\frac{b_n}{a_n}\right)^2\frac{\beta^2(3-\beta)}{1-\beta}\sum_{k=1}^{\infty}l^*_{n,k}(x)\\=&(1-\beta)^2L_{a_n,b_n}(e_2;x)+\frac{b_n}{a_n}\Big((1-\beta)^2+\Big(2+4\beta-2\beta^2\Big)\Big)L_{a_n,b_n}(e_1;x)\\&+\left(\frac{b_n}{a_n}\right)^2\frac{\beta^2(3-\beta)}{(1-\beta)}L_{a_n,b_n}(e_0;x)\end{aligned}$$

$$=(1-\beta)^2x^2+3\frac{b_n}{a_n}x+\left(\frac{b_n}{a_n}\right)^2\frac{\beta^2(3-\beta)}{(1-\beta)}.$$

By using same method, if we keep to continue the calculations we can get the values of other test functions. □

As a result of this lemma, we can give the central moments.

Lemma 1.4. *For each* $x\in[0,\infty)$ *and the operators* $D^{[\beta]}_{a_n,b_n}$*, we have*

$$\mu_{n,1}(x):=D^{[\beta]}_{a_n,b_n}((t-x);x)=x(-\beta)+\frac{b_n}{a_n}\frac{\beta(2-\beta)}{(1-\beta)}, \tag{1.7}$$

$$\mu_{n,2}(x):=D^{[\beta]}_{a_n,b_n}\left((t-x)^2;x\right)=\beta^2x^2+2\frac{2\beta^2-7\beta+3}{1-\beta}\frac{b_n}{a_n}x+\left(\frac{b_n}{a_n}\right)^2\frac{\beta^2(3-\beta)}{(1-\beta)}. \tag{1.8}$$

and

$$\begin{aligned}\mu_{n,4}(x):=&D^{[\beta]}_{a_n,b_n}\left((t-x)^4;x\right)\\ =&\beta^4x^4+\frac{b_n}{a_n}\left(32\beta+22\beta^2+4\beta^3-34\beta^4-\frac{4\beta(2-\beta)}{1-\beta}\right)x^3\\ &+\left(\frac{b_n}{a_n}\right)^2\left(23-96\beta-2\beta^2-12\beta^3+5\beta^4-\frac{12\beta^4}{1-\beta}+\frac{6\beta^2(3-\beta)}{1-\beta}\right)x^2\\ &+\left(\frac{b_n}{a_n}\right)^3\left(42+30\beta^2+30\beta^3+20\beta^4-\frac{10\beta^5}{1-\beta}-4\frac{\beta^2(3-\beta)}{(1-\beta)}\right)x+\left(\frac{b_n}{a_n}\right)^2\frac{\beta^2(3-\beta)}{(1-\beta)}.\end{aligned}$$

Proof of the last lemma is obvious from Lemma 1.3.

Remark 1.1. To obtain the Korovkin-type theorem we change β with the sequence (β_n) with the following property $\beta_n\in[0,1)$ for every $n\in\mathbb{N}$ and

$$\lim_{n\to\infty}\beta_n=0. \tag{1.9}$$

At the rest of this paper, we will use the notation $D^{[\beta_n]}_{a_n,b_n}$ instead of $D^{[\beta]}_{a_n,b_n}$. □

Remark 1.2. Using the conditions (1.3) and (1.9) for each $0\leq x<\infty$, we get

$$\mu_{n,1}(x)=D^{[\beta_n]}_{a_n,b_n}((t-x);x)\to0,\quad \textit{as } n\to\infty,$$
$$\mu_{n,2}(x)=D^{[\beta_n]}_{a_n,b_n}\left((t-x)^2;x\right)\to0,\quad \textit{as } n\to\infty,$$

and

$$\mu_{n,4}(x)=D^{[\beta_n]}_{a_n,b_n}\left((t-x)^4;x\right)\to0,\quad \textit{as } n\to\infty.$$

Lemma 1.5. *For all* $n\in\mathbb{N}$ *and* $x\in[0,\infty)$*, if* $\lim_{n\to\infty}\frac{a_n}{b_n}\beta_n=\zeta\in\mathbb{R}$*, we have*

$$\lim_{n\to\infty}\frac{a_n}{b_n}\mu_{n,1}(x)=-\zeta x, \tag{1.10}$$

$$\lim_{n\to\infty}\frac{a_n}{b_n}\mu_{n,2}(x)=3x, \tag{1.11}$$

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)^2\mu_{n,4}(x)=32\zeta x^3+23x^2. \tag{1.12}$$

To satisfy the Korovkin-type theorem we consider the lattice homomorphism T_a: $C[0,\infty)\to C[0,a]$ defined as $T_a(f)=f|_{[0,a]}$ with a fixed $a\geq 0$. It is clear that, from Lemma 1.3 and by using the condition (1.9), we have $T_a\left(D^{[\beta_n]}_{a_n,b_n}(e_k)\right)\to T_a(e_k)=x^k$ uniformly on $[0,a]$, where $k=0,1,2$. Then, by the well-known Korovkin theorem, the following result is proven on any compact subset of $[0,\infty)$ as $n\to\infty$.

Theorem 1.1. *Let* $\left(D^{[\beta_n]}_{a_n,b_n}\right)$ *be the sequence of linear positive operators given by* (1.6), $f\in C[0,a]$ *and* (β_n) *be the sequence satisfying the condition* (1.9). *The sequence* $\left(D^{[\beta_n]}_{a_n,b_n}\right)$ *converges uniformly to* $f(x)$ *on* $[0,a]$.

2. Degree of Approximation

In this section, we give an estimate the degree of approximation for the operators $D^{[\beta]}_{a_n,b_n}(f;x)$ in terms of the modulus of continuity, Ditzian-Totik moduli of smoothness, and the Peetre's $\mathcal{K}$-functional. Also, we give Voronovskaja theorem for $D^{[\beta]}_{a_n,b_n}(f;x)$ operators.

We begin by recalling some definitions and notations. By $C_B[0,\infty)$, we denote the class on real valued continuous and bounded functions f defined on the interval $[0,\infty)$ with the norm $\|f\|=\sup_{x\in[0,\infty)}|f(x)|$. For $f\in C_B[0,\infty)$, $\delta>0$, the mth order modulus of continuity is defined as

$$\omega_m(f,\delta)=\sup_{0<h\leq\delta}\sup_{x\in[0,\infty)}|\Delta^m_h f(x)|,$$

with Δ is the forward difference.

The Petree's $\mathcal{K}$-functional is defined by

$$\mathcal{K}_2(f,\delta)=\inf_{g\in C^2_B[0,\infty)}\{||f-g||+\delta\|g''\|\},\quad \delta>0,$$

where $C^2_B[0,\infty)=\{g\in C_B[0,\infty)\ :\ g',g''\in C_B[0,\infty)\}$ and $||\cdot||$ is the uniform norm on $C_B[0,\infty)$. By ([8], page 10), we have the following inequality

$$\mathcal{K}_2(f,\delta)\leq M\omega_2(f,\sqrt{\delta}), \tag{2.1}$$

where M is a positive constant and ω_2 is the second order modulus of smoothness for $f\in C_B[0,\infty)$ defined as

$$\omega_2(f,\sqrt{\delta})=\sup_{0<h\leq\delta}\sup_{x,x+2h\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|.$$

Now, we can give the following result.

Theorem 2.1. $\left(D^{[\beta_n]}_{a_n,b_n}\right)$ *denotes a sequence of linear positive operators defined by* (1.6) *and let* (β_n) *be the sequence satisfying the condition* (1.9). *Then, for all* $f\in C_B[0,\infty)$

and for each $x\in[0,\infty)$, *the following inequality*

$$\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|\leq 2\omega\left(f,\delta^{[\beta_n]}_{a_n,b_n}\right) \tag{2.2}$$

holds, where $\delta^{[\beta_n]}_{a_n,b_n}(x)=\{\mu_{n,2}(x)\}^{1/2}$ *and* ω *is usual first moduli of continuity.*

Proof. For every $u,t\in[0,\infty)$ and $\delta>0$, considering the definition of modulus of continuity we can write

$$|f(u)-f(x)|\leq(1+\delta^{-1}|u-x|)\omega(f,\delta).$$

Using the definition of (1.5) with the above inequality we have,

$$\begin{aligned}&\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|\\ &\leq\sum_{k=1}^{\infty}\left(\int_0^{\infty}\theta_{\beta_n}\left(k-1,\frac{a_n}{b_n}u\right)du\right)^{-1}l^*_{n,k}(x)\int_0^{\infty}\theta_{\beta_n}\left(k-1,\frac{a_n}{b_n}u\right)\left(1+\delta^{-1}|u-x|\right)du\\ &=\omega(f,\delta)D^{[\beta_n]}_{a_n,b_n}(e_0)(x)+\delta^{-1}\omega(f,\delta)D^{[\beta_n]}_{a_n,b_n}(|u-x|)(x). \qquad (2.3)\end{aligned}$$

Applying Cauchy-Schwartz inequality, we have

$$\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|\leq\omega(f,\delta)+\delta^{-1}\omega(f,\delta)\left\{D^{[\beta_n]}_{a_n,b_n}\left((u-x)^2\right)(x)\right\}^{1/2}.$$

By considering (1.8), if we choose $\delta:=\delta^{[\beta_n]}_{a_n,b_n}(x)$ as follows

$$\delta^{[\beta_n]}_{a_n,b_n}(x)=\{\mu_{n,2}(x)\}^{1/2},$$

we obtain (2.2) for each $x\in[0,\infty)$. □

Now, we give the rate of convergence by means of Peetre's $\mathcal{K}$-functional.

Theorem 2.2. *For each* $x\in[0,\infty)$ *and* $f\in C_B[0,\infty)$, *the following inequalities*

$$\begin{aligned}\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|&\leq 4\mathcal{K}_2\left(f,d^{[\beta_n]}_{a_n,b_n}(x)\right)+\omega_1\left(f,\alpha^{[\beta_n]}_{a_n,b_n}\right)\\ &\leq M\omega_2\left(f;\sqrt{d^{[\beta_n]}_{a_n,b_n}(x)}\right)+\omega_1\left(f,\alpha^{[\beta_n]}_{a_n,b_n}(x)\right)\end{aligned} \tag{2.4}$$

hold, where $d^{[\beta_n]}_{a_n,b_n}(x)=\left(\delta^{[\beta_n]}_{a_n,b_n}(x)\right)^2+\left(\alpha^{[\beta_n]}_{a_n,b_n}(x)\right)^2$ *with* $\delta^{[\beta_n]}_{a_n,b_n}(x)$ *is given in Theorem* 2.1, $\alpha^{[\beta_n]}_{a_n,b_n}(x)=\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}-\beta_n x$ *and* M *is a constant independently of* n *and* x.

Proof. Lets take auxiliary operators for $D^{[\beta_n]}_{a_n,b_n}$ operators as below

$$\overline{D}^{[\beta_n]}_{a_n,b_n}(f;x)=D^{[\beta_n]}_{a_n,b_n}(f;x)+f(x)-f\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}\right). \tag{2.5}$$

So, it is obvious to see that for all $f\in C_B[0,\infty)$

$$\begin{aligned}\left|\overline{D}^{[\beta_n]}_{a_n,b_n}(f;x)\right|&\leq\left|D^{[\beta_n]}_{a_n,b_n}(f;x)\right|+|f(x)|+\left|f\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}\right)\right|\\ &\leq 3\|f\|.\end{aligned} \tag{2.6}$$

From the Lemma 1.3 we see that the new operators $\overline{D}_{a_n,b_n}^{[\beta_n]}(f;x)$ obtain $\overline{D}_{a_n,b_n}^{[\beta_n]}(t;x)=x$ and as a direct result of this they obtain $\overline{D}_{a_n,b_n}^{[\beta_n]}(t-x;x)=0$. For $g\in C_B^2[0,\infty)$, $t\in[0,\infty)$ by using the Taylor formula, we know

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du.$$

Now, we apply the operators $\overline{D}_{a_n,b_n}^{[\beta_n]}$ to both sides of this equality and using the equality (2.5), we get

$$\begin{aligned}\overline{D}_{a_n,b_n}^{[\beta_n]}(g;x)-g(x)=&D_{a_n,b_n}^{[\beta_n]}\left(\int_x^t (t-u)g''(u)du;x\right)\\ &-\int_x^{(1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}}\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}-u\right)g''(u)du.\end{aligned}$$

Now, passing absolute value and later considering the feature of norm give us following inequalities

$$\begin{aligned}&\left|\overline{D}_{a_n,b_n}^{[\beta_n]}(g;x)-g(x)\right|\\ &\leq D_{a_n,b_n}^{[\beta_n]}\left(\int_x^t |t-u||g''(u)|du;x\right)\\ &+\int_x^{(1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}}\left|(1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}-u\right||g''(u)|du\\ &\leq\|g''\|\left(D_{a_n,b_n}^{[\beta_n]}\left((t-x)^2;x\right)+\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}-x\right)^2\right).\end{aligned}\tag{2.7}$$

From Lemma 1.4, by using the (1.8) and taking $(-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}:=\alpha_{a_n,b_n}^{[\beta_n]}(x)$, we get

$$\begin{aligned}D_{a_n,b_n}^{[\beta_n]}\left((t-x)^2;x\right)+\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}-x\right)^2&=\left(\delta_{a_n,b_n}^{[\beta_n]}(x)\right)^2+\left(\alpha_{a_n,b_n}^{[\beta_n]}(x)\right)^2\\ &=d_{a_n,b_n}^{[\beta_n]}(x).\end{aligned}\tag{2.8}$$

Using the inequality (2.7) and equality (2.8) we can write

$$\left|\overline{D}_{a_n,b_n}^{[\beta_n]}(g;x)-g(x)\right|\leq\left(d_{a_n,b_n}^{[\beta_n]}(x)\right)\|g''\|.\tag{2.9}$$

For $f\in C_B[0,\infty)$ and considering (1.5), we can write

$$\left|D_{a_n,b_n}^{[\beta_n]}(f;x)\right|$$

$$\leq \sum_{k=1}^{\infty}\left(\int_0^{\infty}\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right)du\right)^{-1} l^*_{n,k}(x)\int_0^{\infty}\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right)|f(u)|du+2^{-\frac{a_n}{b_n}x}|f(0)|$$

$$\leq \|f\|\sum_{k=1}^{\infty}\left(\int_0^{\infty}\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right)du\right)^{-1} l^*_{n,k}(x)\int_0^{\infty}\theta_\beta\left(k-1,\frac{a_n}{b_n}u\right)du+2^{-\frac{a_n}{b_n}x}$$

$$= \|f\| D^{[\beta_n]}_{a_n,b_n}(e_0;x)=\|f\|. \tag{2.10}$$

Combining (2.9) and (2.6), for $f\in C_B[0,\infty)$ and for $g\in C^{(2)}_B[0,\infty)$, we have

$$\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right| \leq \left|\overline{D}^{[\beta_n]}_{a_n,b_n}((f-g);x)\right|+|(f-g)(x)|+\left|\overline{D}^{[\beta_n]}_{a_n,b_n}(g;x)-g(x)\right|$$
$$+\left|f\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}\right)-f(x)\right|$$
$$\leq 4\|f-g\|+\left(d^{[\beta_n]}_{a_n,b_n}(x)\right)\|g''\|+\left|f\left((1-\beta_n)x+\frac{b_n}{a_n}\frac{\beta_n(2-\beta_n)}{1-\beta_n}\right)-f(x)\right|.$$

Taking the infimum over all $g\in C^2_B[0,\infty)$, we reach the result (2.4) and by using the inequality (2.1) we find, for each $x\in[0,\infty)$

$$\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right| \leq 4M\omega_2\left(f,d^{[\beta_n]}_{a_n,b_n}(x)\right)+\omega_1\left(f,\alpha^{[\beta_n]}_{a_n,b_n}(x)\right),$$

which implies the proof. □

Now we give the result by using Ditzian-Totik moduli of smoothness. Let start with reminding the some definitions which will be used.

Let function $f\in C[0,\infty)$ and if we take step weight function $\phi:[0,\infty)\to\mathbb{R}$. The first order Ditzian-Totik modulus of smoothness and corresponding K- functional are given by, respectively,

$$\omega^{\phi}_1(f,\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\left\{\left|f\left(x+\frac{h\phi(x)}{2}\right)+f\left(x-\frac{h\phi(x)}{2}\right)\right| : x\pm\frac{h\phi(x)}{2}\in[0,\infty)\right\},$$
$$\mathcal{K}_{1,\phi}(f,\delta)=\inf\{\|f-g\|_\infty+\delta\|\phi g'\|_\infty : g\in C'(\phi)\},\quad \delta>0,$$

where $C(\phi)=\{g\in AC_{loc}[0,\infty) : \|\phi g'\|_\infty<\infty\}$. $g\in AC_{loc}[0,\infty)$ shows that the function g is differentiable and g is absolutely continuous on every closed interval $[a,b]\subset[0,\infty)$. It is known that there exists a positive constant $M>0$, such that (see [8], p.68)

$$\frac{1}{M}\omega^{\phi}_1(f,\sqrt{\delta})\leq\mathcal{K}_{1,\phi}(f,\delta)\leq M\omega^{\phi}_1(f,\sqrt{\delta}). \tag{2.11}$$

Theorem 2.3. *Let $f\in C_B[0,\infty)$. For $x\in(0,\infty)$, we have*

$$\left\|D^{[\beta_n]}_{a_n,b_n}(f)-f\right\|\leq 2\mathcal{K}_{1,\phi}\left(f,\delta^{[\beta_n]}_{a_n,b_n}\right)\leq 2M\omega^{\phi}_1\left(f,\sqrt{\delta^{[\beta_n]}_{a_n,b_n}}\right), \tag{2.12}$$

where $\phi(x)=\sqrt{x}$ is a step function.

Proof. For $g\in C(\phi)$, by using Taylor's formula of g, we have

$$g(t)=g(x)+\int_x^t g'(u)du=g(x)+\int_x^t \frac{g'(u)}{\phi(u)}\phi(u)du.$$

Then, for the step function $\phi(x)=\sqrt{x}$, we get

$$|g(t)-g(x)|\leq\|\phi g'\|_\infty\left|\int_x^t \frac{1}{\phi(u)}du\right|=\|\phi g'\|_\infty 2\left|\sqrt{t}-\sqrt{x}\right|=2\|\phi g'\|_\infty\frac{|t-x|}{\sqrt{t}+\sqrt{x}}.$$

From the inequality $\sqrt{t}+\sqrt{x}\geq\sqrt{x}$, we get

$$|g(t)-g(x)|\leq 2\|\phi g'\|_\infty\frac{|t-x|}{\sqrt{x}}=2\|\phi g'\|_\infty\frac{|t-x|}{\phi(x)}. \tag{2.13}$$

Using (2.13) and (2.10), for $f\in C_B[0,\infty)$ and $g\in C(\phi)$, we have

$$\begin{aligned}\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|&\leq\left|D^{[\beta_n]}_{a_n,b_n}((f-g);x)\right|+\left|D^{[\beta_n]}_{a_n,b_n}(g;x)-g(x)\right|+|g(x)-f(x)|\\&\leq 2\|f-g\|+2\frac{\|\phi g'\|_\infty}{\phi(x)}D^{[\beta_n]}_{a_n,b_n}(|t-x|;x).\end{aligned}$$

By applying Cauchy-Schwartz inequality, we can write

$$\begin{aligned}\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|&\leq 2\|f-g\|+2\frac{\|\phi g'\|_\infty}{\phi(x)}\left(D^{[\beta_n]}_{a_n,b_n}\left((t-x)^2;x\right)\right)^{1/2}\\&\leq 2\|f-g\|+2\frac{\|\phi g'\|_\infty}{\phi(x)}\delta^{[\beta_n]}_{a_n,b_n}(x).\end{aligned}$$

Taking the infimum on the right hand side over all $g\in C^2(\phi)$ we obtain

$$|D_{a_n,b_n}(f;x)-f(x)|\leq 2\mathcal{K}_{1,\phi}\left(f,\delta^{[\beta_n]}_{a_n,b_n}\right).$$

Considering (2.11) we get (2.12) which is desired result. □

In this section, we obtain some pointwise estimates of rate of convergence of the operators (1.6). The Lipschitz-type space is given as follow, in [13];

$$Lip^*_M(\eta)=\left\{f\in C[0,\infty):|f(t)-f(x)|\leq M_f\frac{|t-x|^\eta}{(t+x)^{\frac{\eta}{2}}},\ x,t\in(0,\infty)\right\},$$

where M_f is a positive constant and $\eta\in(0,1]$.

Theorem 2.4. *Let $f\in Lip^*_M(\eta)$. Then, for all $x\in(0,\infty)$, we get*

$$\left|D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\right|\leq M\left(\frac{\mu_{n,2}(x)}{x}\right)^{\eta/2}, \tag{2.14}$$

where $\mu_{n,2}(x)$ is the same as in Lemma 1.4.

Proof. For a function $f \in Lip_M^*(\eta)$, by using the definition, we get

$$|f(t)-f(x)| \leq M_f \frac{|t-x|^\eta}{(t+x)^{\frac{\eta}{2}}}.$$

Applying the operators $D_{a_n,b_n}^{[\beta_n]}$ on both sides of the above inequality, we have

$$\left|D_{a_n,b_n}^{[\beta_n]}(f;x)-f(x)\right| \leq M_f D_{a_n,b_n}^{[\beta_n]}\left(\frac{|t-x|^\eta}{(t+x)^{\frac{\eta}{2}}};x\right).$$

By using the Hölder's inequality, with $p=\frac{2}{\eta}$, $q=\frac{2}{2-\eta}$ and, using the Lemma 1.4, we can write

$$\begin{aligned}\left|D_{a_n,b_n}^{[\beta_n]}(f;x)-f(x)\right| &\leq M_f \left(D_{a_n,b_n}^{[\beta_n]}\left(\frac{(t-x)^2}{(t+x)};x\right)\right)^{\frac{\eta}{2}} \left(D_{a_n,b_n}^{[\beta_n]}(1;x)\right)^{\frac{2-\eta}{2}} \\ &\leq M_f \frac{1}{x^{\eta/2}} \left(D_{a_n,b_n}^{[\beta_n]}\left((t-x)^2;x\right)\right)^{\frac{\eta}{2}}.\end{aligned}$$

So, we obtain (2.14). □

Now we give an estimate for the rate of convergence by the means of the general space of the Lipschitz-type maximal functions. Let $\widetilde{C}_B[0,\infty)$ be the space of bounded, uniformly continuous real valued functions on $[0,\infty)$. The Lipschitz-type maximal function of order η of $f \in \widetilde{C}_B[0,\infty)$ is introduced by Lenze [11] as

$$\widetilde{f_\eta}(x) = \sup_{t \neq x, t \in [0,\infty)} \frac{|f(t)-f(x)|}{|t-x|^\eta}, \quad x \in [0,\infty),$$

and $\eta \in (0,1]$.

Theorem 2.5. *Let $\left(D_{a_n,b_n}^{[\beta_n]}\right)$ be a sequence of linear positive operators defined by* (1.6). *Then, for all $f \in \widetilde{C}_B[0,\infty)$, we get*

$$\left|D_{a_n,b_n}^{[\beta_n]}(f;x)-f(x)\right| \leq \widetilde{f_\eta}(x)\left(\delta_{a_n,b_n}^{[\beta_n]}(x)\right)^\eta,$$

where $\delta_{a_n,b_n}^{[\beta_n]}$ is the same as in Theorem 2.1.

Proof. Using the definition of maximal function, we get

$$|f(t)-f(x)| \leq \widetilde{f_\eta}(x)|t-x|^\eta,$$

and applying the operators $D_{a_n,b_n}^{[\beta_n]}$ on both sides of this equation, we get

$$\left|D_{a_n,b_n}^{[\beta_n]}(f;x)-f(x)\right| \ \leq \ \widetilde{f_\eta}(x) D_{a_n,b_n}^{[\beta_n]}(|t-x|^\eta;x).$$

Applying the Hölder's inequality with $p=\frac{2}{\eta}$ and $q=\frac{2}{2-\eta}$, using Lemma 1.3, it follows that

$$\begin{aligned}D_{a_n,b_n}^{[\beta_n]}(|t-x|^\eta;x) &\leq \widetilde{f_\eta}(x)\left(D_{a_n,b_n}^{[\beta_n]}(|t-x|^2;x)\right)^{\frac{\eta}{2}}\left(D_{a_n,b_n}^{[\beta_n]}(1^{\frac{2-\eta}{2}};x)\right)^{\frac{2-\eta}{2}} \\ &\leq \widetilde{f_\eta}(x)\left(D_{a_n,b_n}^{[\beta_n]}((t-x)^2;x)\right)^{\frac{\eta}{2}}\end{aligned}$$

$$\leq\widetilde{f_\eta}(x)\Big(\delta^{[\beta_n]}_{a_n,b_n}(x)\Big)^\eta.$$

Hence, the proof is completed. □

3. Voronovskaja Theorem

We prove a Voronoskaja type theorem for the operators $D^{[\beta_n]}_{a_n,b_n}(f;x)$.

Theorem 3.1. *For every $f\in C_B[0,\infty)$ such that $f',f''\in C_B[0,\infty)$, and for every fixed $x\in[0,\infty)$, we have*

$$\lim_{n\to\infty}\frac{a_n}{b_n}\Big(D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\Big)=-\zeta xf'(x)+\frac{3}{2}xf''(x),$$

where ζ is the same with the Lemma 1.5.

Proof. Let $x\in[0,\infty)$ be a fixed point. For all $t\in[0,\infty)$, by using Taylor expansion we have

$$f(t)=f(x)+(t-x)f'(x)+\frac{1}{2}(t-x)^2f''(x)+R(t,x)(t-x)^2,$$

where $R(t,x)$ is the remainder term, $R(t,x)\in C_B[0,\infty)$, and $R(t,x)\to 0$ as $t\to x$. Applying the operator $D^{[\beta_n]}_{a_n,b_n}$ to both sides of Taylor expansion and considering $D^{[\beta_n]}_{a_n,b_n}(e_0;x)=1$, we get

$$\begin{aligned}\frac{a_n}{b_n}\Big(D^{[\beta_n]}_{a_n,b_n}(f;x)-f(x)\Big)&=\frac{a_n}{b_n}D^{[\beta_n]}_{a_n,b_n}((t-x);x)f'(x)+\frac{1}{2}\frac{a_n}{b_n}D^{[\beta_n]}_{a_n,b_n}\Big((t-x)^2;x\Big)f''(x)\\&+\frac{a_n}{b_n}D^{[\beta_n]}_{a_n,b_n}\Big(R(t,x)(t-x)^2;x\Big)\\&=A_1+A_2+A_3.\end{aligned}$$

Thus, we immediately have

$$\begin{aligned}A_1&=\frac{a_n}{b_n}\mu_{n,1}(x)f'(x),\\A_2&=\frac{1}{2}\frac{a_n}{b_n}\mu_{n,2}(x)f''(x).\end{aligned}$$

Now, we estimate A_3. From Cauchy-Schwartz inequality, we have

$$\begin{aligned}A_3&=\frac{a_n}{b_n}D^{[\beta_n]}_{a_n,b_n}\Big(R(t,x)(t-x)^2;x\Big)\\&\leq\Big\{D^{[\beta_n]}_{a_n,b_n}\Big((R(t,x))^2;x\Big)\Big\}^{1/2}\left\{\left(\frac{a_n}{b_n}\right)^2D^{[\beta_n]}_{a_n,b_n}\Big((t-x)^4;x\Big)\right\}^{1/2}.\end{aligned}$$

The properties of the function $R(t,x)$ implies that $R^2(x,x)=0$ and $R^2(x,x)\in C_B[0,\infty)$. Hence, we obtain

$$\lim_{n\to\infty}D^{[\beta_n]}_{a_n,b_n}\Big((R(t,x))^2;x\Big)=R^2(x,x)=0,\quad x\in[0,\infty).$$

Furthermore, by applying equation (1.12) from Lemma 1.5, we get

$$\lim_{n\to\infty}D^{[\beta_n]}_{a_n,b_n}\Big(R(t,x)(t-x)^2;x\Big)=0.$$

Moreover, if we take limits as $n\to\infty$ over A_1 and A_2, from the equalities (1.10) and (1.11), this implies the desired result. □

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