Kragujevac Journal of Mathematics
Volume 45(2) (2021), Pages 275–288.

ON THE NON-NEGATIVE RADIAL SOLUTIONS OF THE TWO DIMENSIONAL BRATU EQUATION

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Abstract. In this paper, we study the boundary value problem on the unit circle for the Bratu's equation depending on the real parameter μ. From the parameter estimate, the existence of non-negative solution is set. A numerical method is suggested to justify the theoretical result. It is a combination of the adaptation of finite difference and Gauss-Seidel method allowing us to obtain a good approximation of μ_c, with respect to the exact theoretical method $\mu_c = \lambda = 5.7831859629467$.

The vast majority of phenomena that occur in nature are described by a non-linear differential equation or by a system of non-linear equations. Among these equations, the Bratu's equation, given by

$$\nabla^2 u + \mu\, e^u = 0\,,$$

is a classical example of equation with a strong nonlinear exponential term and a real parameter μ. This equation arises originally as a simplified model for the description of the combustion of solid fuels. Also it is often appears in science and engineering as a model in various physical applications, from chemical reactions, thermal combustion theory, heat transfer radiation until the Chandrasekhar's model of the universe expansion and even nanotechnology [2, 3, 9, 13]. In [5], the dynamics of the Bratu equation were analyzed and the existence of bifurcations was shown. They are also devoted to describe the Gaussian curvature problem in Riemannian geometry [15], the mean field limit of vortices in Euler flows [8], the Onsager formulation in statistical mechanics [6], the Keller-Siegel system of chemotaxis [19] and the Chern-Simon-Higgs

Key words and phrases. Non-linear eigenvalue problem, finite difference method, Gauss-Seidel method.
2010 *Mathematics Subject Classification.* Primary: 35J25. Secondary: 35J60.
DOI 10.46793/KgJMat2102.275N
Received: May 14, 2018.
Accepted: December 10, 2018.

gauge theory [7, 21].

Recently, most of the research has focused on better and more efficient solution methods for determining solutions, approximate or exact, analytical or numerical to this non-linear Bratu model [1, 4, 11, 12, 17, 18, 20].

In this paper we study the two-dimensional Bratu's equation depending on a real parameter μ on the unit circle with the Dirichlet homogeneous boundary condition. We prove the existence of non-negative radial solutions for a certain range of the real parameter μ. A numerical method is suggested to justify the theoretical result.

1. Theoretical Result: Existence of the Solution

We study the two dimensional Bratu's equation on the unit circle with the homogeneous boundary condition,

$$\begin{cases} -\nabla^2 u(x,y) = \mu e^{u(x,y)}, & x^2+y^2<1, \\ u(x,y) = 0\,, & x^2+y^2=1, \end{cases} \tag{1.1}$$

where μ is a real parameter. The existence of the solution for the problem (1.1) beyond a certain limit of the parameter μ is based on a general theory of the non-linear eigenvalue problem

$$\begin{cases} -\nabla^2 u(x) = \mu f(x,u), & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega, \end{cases} \tag{1.2}$$

where Ω is an open bounded region of $\mathbb{R}^n$ and $f(x,u)$ is a non-negative and continuous function on $(x,u)\in\Omega\times\mathbb{R}$. We have the next result [17]:

Theorem 1.1. *Assume that*

$$f(x,u) \geq h(x) + r(x)u\,, \quad (x,u)\in\Omega\times[0,\infty), \tag{1.3}$$

where h and r are non-negative and continuous functions in Ω. Then the non-linear eigenvalue problem (1.2) *has no non-negative solutions for any $\mu \geq \lambda$, where λ is the principal eigenvalue of the linear eigenvalue problem*

$$\begin{cases} -\nabla^2 u(x) = \lambda r(x) u, & x\in\Omega, \\ u(x) = 0, & x\in\partial\Omega\,. \end{cases} \tag{1.4}$$

In particular, due to the estimate

$$f(x,u) = e^u \geq 1+u,$$

the linear eigenvalue problem (1.4) corresponding to the nonlinear problem (1.1) is given by $h(x) = r(x) = 1$, i.e.,

$$\begin{cases} -\nabla^2 u(x,y) = \lambda\, u(x,y), & x^2+y^2<1, \\ u(x,y) = 0, & x^2+y^2=1. \end{cases} \tag{1.5}$$

Introducing the polar coordinates on the plane

$$x(r,\theta) = r\cos\theta, \quad y(r,\theta) = r\sin\theta, \quad 0\leq r\leq 1, 0\leq\theta\leq 2\pi,$$

we obtain the equivalent, to the (1.5), problem

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \lambda u = 0, \quad u(1,\theta) = 0\,.$$

The standard method of separation of variables

$$u(r,\theta) = R(r)\,\Theta(\theta),$$

with the boundary values

$$R(1) = 0, \quad \Theta(\theta) = \Theta(\theta + 2\pi), \tag{1.6}$$

leads to the ordinary differential equations

$$\Theta''(\theta) + K\,\Theta(\theta) = 0, \tag{1.7}$$

and

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - K)R(r) = 0 \tag{1.8}$$

where K is a constant. In order to obtain a periodic solution, according to the second relation of (1.6) for the equation (1.7), the condition $K = n^2$ is necessarily required, where $n \in \mathbb{N}$. Therefore, the equation (1.8) becomes

$$r^2 R''(r) + rR'(r) + (\lambda r^2 - n^2)R(r) = 0,$$

which is the Bessel's equation with the general solution

$$R(r) = c_1 J_n(\sqrt{\lambda}\,r) + c_2 Y_n(\sqrt{\lambda}\,r)\,,$$

where c_1 and c_2 are arbitrary constants. Requiring a bounded solution, when $r = 0$, we set $c_2 = 0$. Furthermore, using the first relation of (1.6) we obtain

$$J_n(\sqrt{\lambda}) = 0\,,$$

which implies that

$$\lambda_{(m),n} = j^2_{(m),n}\,.$$

Thus, the boundary value problem (1.1) has no non-negative solution for $\mu \geq j^2_{(m),n}$. Since, the first root of the Bessel's function $J_0(x)$ is $j_{1,0} = 2.40482555769577$, then the threshold is $\mu_c = \lambda = 5.7831859629467$.

In the next section we applied this result for the corresponding radial solution.

2. Radial Positive Solution of the Bratu's Equation

The positive solution on the unit disc B_1 has a radial symmetry i.e. depends only on $r = \sqrt{x^2 + y^2}$. In order to prove this we follow the technique developed in [10]. First, we observe that the homogeneous boundary condition implies, that ∂B_1 is a level set of the positive solution $u \in C^2(\bar{B}_1)$ and therefore for the outer unit normal vector to ∂B_1, we have

$$\vec{\nu}(x,y) \equiv (x,y) = \pm\frac{\nabla u}{|\nabla u|} \quad \text{or} \quad (x,y)\cdot\nabla u = \pm|\nabla u|.$$

Furthermore,

$$e^{u(x,y)} = 1 + \int_0^1 \frac{d}{dt} e^{t\,u(x,y)}\,dt = 1 + u(x,y)\int_0^1 e^{t\,u(x,y)}\,dt.$$

Next, we write the Bratu's equation in the equivalent form

$$\nabla^2 u + c(x,y)u = -\mu < 0,$$

where

$$c(x,y) = \mu \int_0^1 e^{tu(x,y)} dt > 0.$$

The Serrin's maximum principle implies, that

$$\frac{\partial u}{\partial \vec{\nu}} = (x,y)\cdot \nabla u = -|\nabla u| < 0, \quad \text{on } \partial B_1. \tag{2.1}$$

Denote by $B_1^+ = B_1 \cap \{(x,y) \in \mathbb{R}^2,\ y > 0\}$ the upper unit half disc and $B_1^- = B_1 \cap \{(x,y) \in \mathbb{R}^2,\ y < 0\}$ the lower unit half disc. Thus, (2.1) implies

$$y\,\frac{\partial u}{\partial y}(x,y) = -|\nabla u(x,y)| - x\,\frac{\partial u}{\partial x}(x,y) < 0, \quad \text{on } \partial B_1^+,$$

which means that $\partial u/\partial y < 0$ on ∂B_1^+. The smoothness of u implies that $\partial u/\partial y < 0$ in B_1^+ close to ∂B_1^+. Thus, the solution u is a decreasing function on the y - direction close to ∂B_1^+. Furthermore, define the sets $l_a = \{(x,a),\ x \in \mathbb{R}\}$, for $0 \le a \le 1$ and $E_a = \{(x,y) \in B_1^+,\ a < y < 1\}$. To any $(x,y) \neq (x,a)$, we assign its reflection with respect to the line l_a, the point $(x, 2a - y)$.

Theorem 2.1. *If $u \in C^2(\bar{B}_1)$ is a positive solution of the Bratu's equation, then u is a function of $r = \sqrt{x^2+y^2}$.*

Proof. It is sufficient to show that $u(x,y) = u(x, 2a-y)$ whenever $a = 0$ i.e. the line l_a coincides with the axis x. To this end, define

$$a_0 = \inf\{a \in [0,1] :\ u(x,y) < u(x, 2\beta - y)\,,\ (x,y) \in E_\beta\,,\ a \le \beta \le 1\}.$$

The above infimum is well defined, since the solution u is a decreasing function on the y - direction close to ∂B_1^+. We will prove that $a_0 = 0$. Suppose that $a_0 > 0$ and define the function

$$v(x,y) = u(x, 2a_0 - y) - u(x,y), \quad (x,y) \in E_{a_0}.$$

Then $v(x,y) > 0$ and

$$\nabla^2 v(x,y) - C(x,y)v(x,y) = 0,$$

where

$$C(x,y) = \Big[u(x,2a_0-y) - u(x,y)\Big]\int_0^1 e^{[tu(x,2a_0-y)+(1-t)u(x,y)]}dt > 0.$$

The Serrin's maximum principle and the above discussion implies that

$$v(x,y) > 0, ,(x,y) \in E_{a_0}, \quad \frac{\partial v}{\partial y}(x,y) < 0, (x,y) \in l_{a_0} \cap B_1^+$$

and, equivalently

$$u(x,y) < u(x, 2a_0 - y), (x,y) \in E_{a_0}, \quad \frac{\partial u}{\partial y}(x,y) < 0, (x,y) \in l_{a_0} \cap B_1^+,$$

with the partial derivatives with respect to y, always taken close to ∂B_1^+. Thus, we have, that the positive solution u is also decreasing function on $l_{a_0} \cap B_1^+$. Choosing any $\varepsilon > 0$, sufficiently small for $0 < \beta = a_0 - \varepsilon < a_0$, we have

$$u(x,y) < u(x, 2a_0 - y) < u(x, 2\beta - y), \quad (x,y) \in l_\beta \cap B_1^+,$$

and by the smoothness of u

$$u(x,y) < u(x, 2\beta - y), \quad (x,y) \in E_\beta, \beta < a_0\,,$$

which contradicts to the definition of a_0. Thus, necessarily $a_0 = 0$, and

$$u(x,y) \leq u(x, -y), \quad (x,y) \in B_1^+.$$

In the same way, we can obtain

$$u(x,y) \geq u(x, -y), \quad (x,y) \in B_1^-,$$

which implies $u(x,y) = u(x,-y)$ in the unit disc. Finally, the axis x can be any diameter of the unit disc, thus we have the radial symmetry of the solution. $\square$

3. Numerical Method

To find the numerical solution of (1.1), we have used an adaptation of the second-order Finite Difference Method (FDM). First, we consider a rectangular region $(\mathcal{R})$ defined by

$$\begin{cases} -1 \leq x \leq 1, \\ -1 \leq y \leq 1, \end{cases}$$

in the cartesian system (OXY), and we insert into $(\mathcal{R})$ the circle $(\mathcal{C})$ defined by $x^2 + y^2 = 1$. Next, the region $(\mathcal{R})$ is subdivided into the grid $n \times n$ equal subregions: $h \times h$ where

$$h = \frac{2}{n}\,,$$

i.e, the axis (OX) and (OY) are partitioned in n equal part each one. So, each point or node (x_i, y_j) of the grid is the intersection of the $x = x_i$ vertical line and the $y = y_j$ horizontal line, where

$$x_i = -1 + ih, \quad i = 0, \ldots, n,$$

and

$$y_j = -1 + jh, \quad j = 0, \ldots, n.$$

Then, it is not difficult to see the following.

(a) For an exterior point or endpoint $P_{i,j} = (x_i, y_j)$ of the circle $(\mathcal{C})$, i.e., $x_i^2 + y_j^2 \geq 1$. See the Figure 1.

We have

$$P_{i,j} : u(x_i, y_j) = w_{i,j} = 0. \tag{3.1}$$

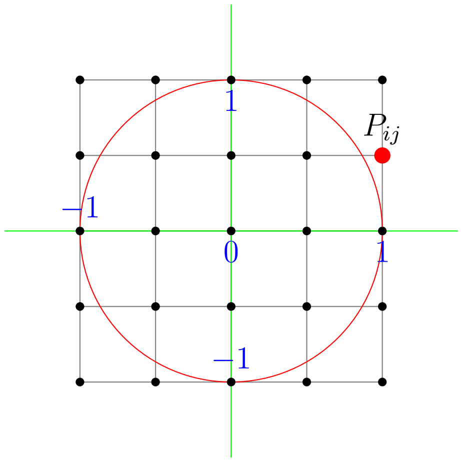

FIGURE 1.

(b) For each interior point $P_{i,j}$ of the circle $(\mathfrak{C})$, i.e., $x_i^2 + y_j^2 < 1$, we apply the Finite Difference Method (FDM) using the Taylor series with the variable x around x_i, and with the variable y around y_j [16], i.e., without loss of generality, around of this point $P_{i,j}$ we suppose the next four points $P_{i+1,j}, P_{i,j+1}, P_{i-1,j}, P_{i,j-1}$ which are known respectively as East(E), North(N), South(S), West(W) point with respect to $P_{i,j}$, see Figure 2.

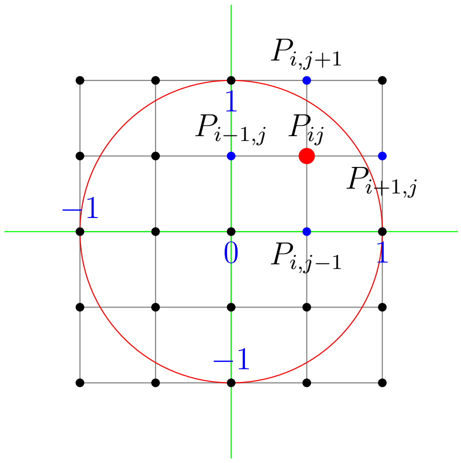

FIGURE 2.

So, we define:

$$P_{i+1,j} : u(x+h, y) = u(x_{i+1}, y_j) = w_{i+1,j},$$
$$P_{i,j+1} : u(x, y+h) = u(x_i, y_{j+1}) = w_{i,j+1},$$
$$P_{i,j-1} : u(x, y-h) = u(x_i, y_{j-1}) = w_{i,j-1},$$
$$P_{i-1,j} : u(x-h, y_j) = u(x_{i-1}, y_j) = w_{i-1,j}.$$

Next, for every interior point (x, y) of the circle, we have

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &\approx \frac{1}{h^2}[u(x+h,y) - 2u(x,y) + u(x-h,y)], \\ \frac{\partial^2 u}{\partial y^2} &\approx \frac{1}{h^2}[u(x,y+h) - 2u(x,y) + u(x,y-h)]. \end{aligned} \tag{3.2}$$

By adding these two equations (3.2), the equation (1.1) for all interior point of the circle can be replaced by the difference equation:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \approx & \frac{1}{h^2}[u(x+h,y) + u(x,y+h) + u(x-h,y) + u(x,y-h) - 4u(x,y)] \\ =& (-\mu)(1 + u(x,y)). \end{aligned}$$

Then, it is easy to see that

$$4w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = (-\mu)(-h^2)(1 + w_{i,j}), \tag{3.3}$$

where $u(x_i, y_j) = w_{i,j}$ and $e^{w_{i,j}} \cong 1 + w_{i,j} + \cdots$.
So, (3.3) implies that for each interior point $P_{i,j}$ of the circle $(\mathcal{C})$, we have

$$(4 - \mu h^2)w_{i,j} - w_{i+1,j} - w_{i-1,j} - w_{i,j+1} - w_{i,j-1} = \mu h^2, \tag{3.4}$$

where $i = 0, \ldots, n$, and $j = 0, \ldots, n$.

The reader may find an illustrative example in the Appendix.

To find the value $u(x_i, y_j) = w_{i,j}$ of the point $P_{i,j}$ on the region $(\mathcal{R})$, the system of linear equations (3.1) and (3.4) is established, moreover, the solution of (1.1) is reduced to the solution $w_{i,j}$ of the system of linear equations (3.1) and (3.4), depending on the parameter μ. Finally, to solve the system of linear equations (3.1) and (3.4), the Gauss-Seidel's Method is used [4].

Since (1.1) and (3.4) depends on the parameter μ, we determine the threshold μ_c, giving the grid $n \times n$ of $(\mathcal{R})$, and also giving a value the parameter μ for the grid $n \times n$ of $(\mathcal{R})$ the norm $|w_{i,j}| = u_{max}$ is found.

Algorithm

Input: Value n of the subdivision of the region $(\mathcal{R})$; Initial value of $\mu = \mu_0$; k =step of μ.
Output: Value of the threshold μ_c; Approximate solution $u(x_i, y_j) \geq 0$ of the interior point of the circle for certain range of μ; Maximum value u_{max} of the solution $u(x_i, y_j)$.
for $\forall (x_i, y_j) \in (\mathcal{R})$ **do**
  if (x_i, y_j) *satisfy* $x_i^2 + y_j^2 \geq 1$ **then**
    Output: $u(x_i, y_j) = 0$.
  else
    for $\forall (x_i, y_j)$ *that satisfy* $x_i^2 + y_j^2 < 1$ **do**
      (1) Using FDM, establish the system of linear equations $AX = B$ which depends on parameter μ.
      (2) Solve the system of linear equations $AX = B$ by the Gauss-Seidel's Method for given initial value $\mu = \mu_0$.
      if $\exists u(x_i, y_j) < 0$ **then**
        Change Initial value of $\mu = \mu_0$; **goto** (2)
      else
        (3) **while** $\forall u(x_i, y_j) \geq 0$ **do**
          (3.1) $\mu = \mu + k$.
          (3.2) Solve the system of linear equations $AX = B$ by the Gauss-Seidel's Method.
          (3.3) **if** $\exists u(x_i, y_j) < 0$ **then**
            Output: $\mu_c = \mu - k$; $u(x_i, y_j) \geq 0; u_{max} = \max(u(x_i, y_j))$.
            Stop.
          else
            Output: $\mu; u(x_i, y_j) \geq 0$; $u_{max} = \max(u(x_i, y_j))$.
          end
        end
      end
    end
  end
end

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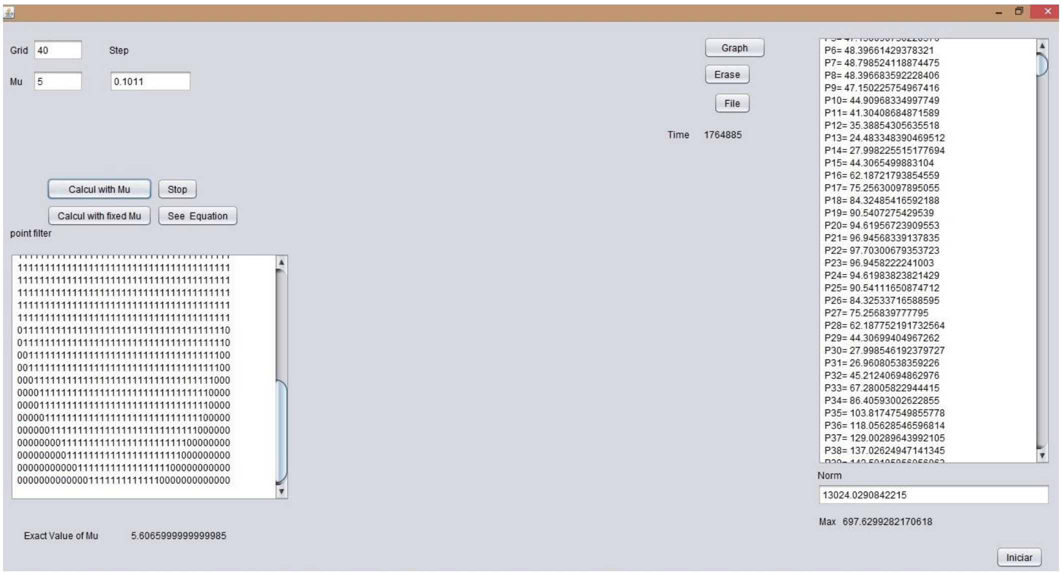
4. Results

We developed a software in high-level programming language (in this case, Java) based on the algorithm mentioned above. The following tables show the result of (1.1) for the respective partition 40×40 and 70×70 (see Figure 3).

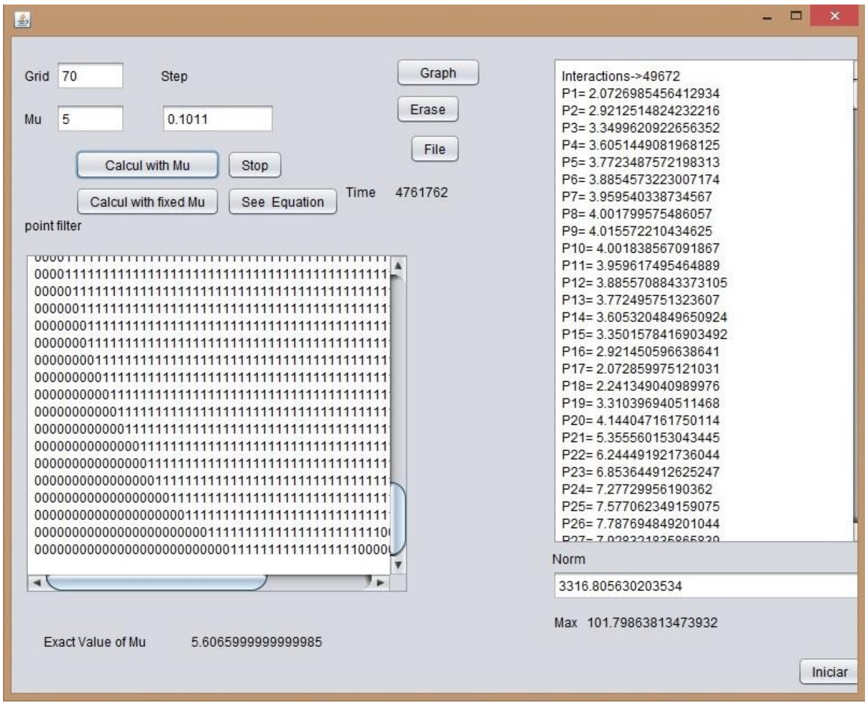


(a) Result for 40×40

4.1. **Finding of μ_c.** For each grid, we find the respective μ_c and the norm u_{max} given in Table 1.

Table 1.

$n \times n$	μ_c	u_{max}
30×30	5.5054999999999999	126.26207775999859
40×40	5.6065999999999985	697.6299282170618
50×50	5.6065999999999985	114.5670432582935
60×60	5.6065999999999985	102.23255929486518
70×70	5.6065999999999985	101.79863813473932
75×75	5.584800000000021	60.012385961525645



(b) Result for 70×70

FIGURE 3. Result for 40×40 and 70×70

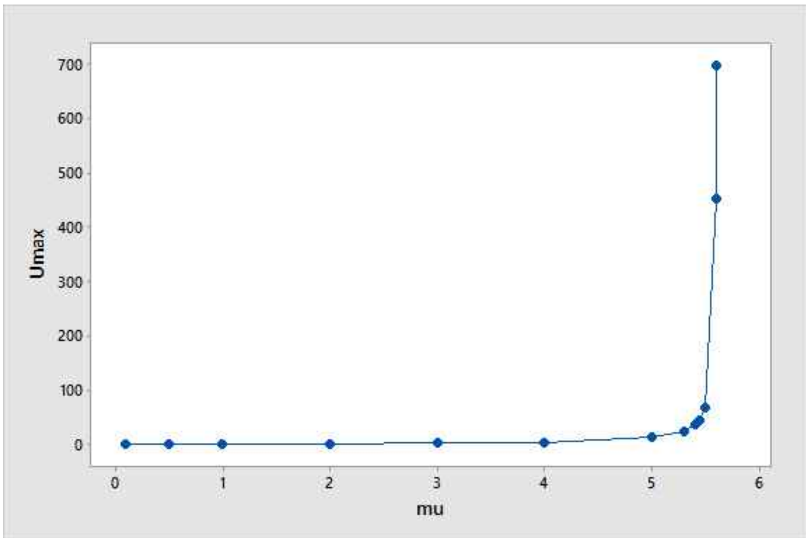


FIGURE 4. Graph of u_{max} vs μ, for 40×40

4.2. **Graph of u_{max} vs μ, for 40×40.** We have drawn the graph of u_{max} vs μ, for 40×40, given in Figure 4.

4.3. **The graph of the solution $u(x,y)$, for 50×50 and 75×75 with $\mu = 4$.** The graphs of the solution $u(x,y)$ of the equation (1.1) are drawn with the parameter $\mu = 4$, for the respective partition 50×50 and 75×75, given in Figure 5.

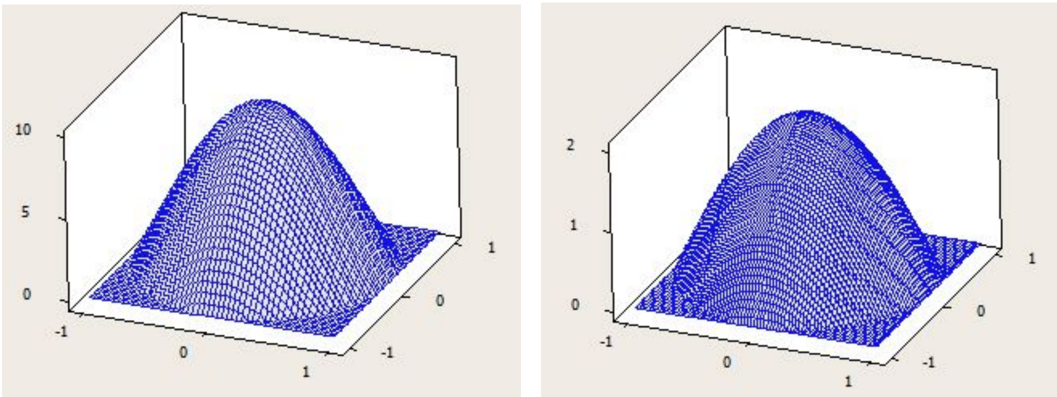


FIGURE 5. Graphs of the solution $u(x,y)$, for 50×50 and 75×75

So, from the Figure 5, we can see that, as $u \in C^2(\bar{B}_1 = \{(x,y) : x^2 + y^2 \leq 1\})$ is a positive solution of the Bratu's equation for $\mu \leq \mu_c$, then u is a function of $r = \sqrt{x^2 + y^2}$.

5. Conclusion

In this paper, we have studied the boundary value problem on the unit circle for the Bratu's equation which depends on a real parameter μ, we show that the boundary equation has the no non-negative solutions when $\mu \geq \mu_c = 5.7831859629467$, where we have implemented the numerical method, that is, the combination of the adaptation of finite difference and Gauss-Seidel method, which allows us to obtain a good approach of μ_c with respect to the exact theoretical method $\mu_c = \lambda = 5.7831859629467$.

A possible application of these results could be to the simplified stationary model for energy functional related to thermo-electro-hydrodynamics description of electrospinning [14].

6. Appendix

Example 6.1. Let $n = 4$, so $h = \frac{2}{n} = \frac{1}{2}$. The grid has $5 \times 5 = 25$ points, in which 9 are interior points of the circle, i.e., $P_{1,1}$, $P_{1,2}$, $P_{1,3}$, $P_{2,1}$, $P_{2,2}$, $P_{2,3}$, $P_{3,1}$, $P_{3,2}$, $P_{3,3}$ (see Figure 6).

The points which satisfy $x_i^2 + y_j^2 \geq 1$, i.e.,

$$P_{0,0} : u(x_0, y_0) = u(-1,-1) = w_{0,0} = 0,$$

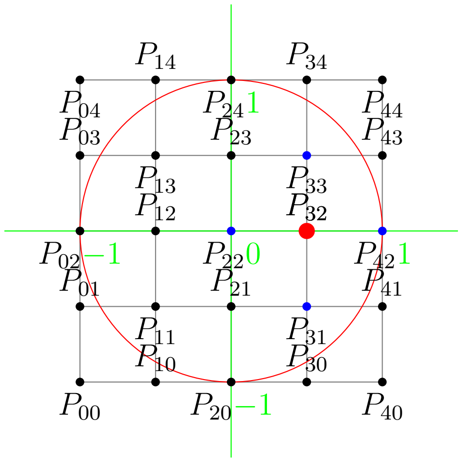


FIGURE 6.

$$
\begin{aligned}
P_{0,1} &: u(x_0, y_1) = u\left(-1, -\frac{1}{2}\right) = w_{0,1} = 0,\\
P_{0,2} &: u(x_0, y_2) = u(-1, 0) = w_{0,2} = 0,\\
P_{0,3} &: u(x_0, y_3) = u\left(-1, \frac{1}{2}\right) = w_{0,3} = 0,\\
P_{0,4} &: u(x_0, y_4) = u(-1, 1) = w_{0,4} = 0,\\
&\vdots\\
P_{4,0} &: u(x_4, y_0) = u(1, -1) = w_{4,0} = 0,\\
P_{4,1} &: u(x_4, y_1) = u\left(1, -\frac{1}{2}\right) = w_{4,1} = 0,\\
P_{4,2} &: u(x_4, y_2) = u(1, 0) = w_{4,2} = 0,\\
P_{4,3} &: u(x_4, y_3) = u(1, \frac{1}{2}) = w_{4,3} = 0,\\
P_{4,4} &: u(x_4, y_4) = u(1, -1) = w_{4,4} = 0.
\end{aligned}
$$

So, for the interior points of the circle which satisfy $x_i^2 + y_j^2 < 1$:

$$
\begin{aligned}
P_{1,1} &: \left(4 - \frac{\mu}{4}\right) w_{1,1} - w_{2,1} - w_{0,1} - w_{1,2} - w_{1,0} = \frac{\mu}{4},\\
P_{1,2} &: \left(4 - \frac{\mu}{4}\right) w_{1,2} - w_{2,2} - w_{0,2} - w_{1,3} - w_{1,1} = \frac{\mu}{4},\\
P_{1,3} &: \left(4 - \frac{\mu}{4}\right) w_{1,3} - w_{2,3} - w_{0,3} - w_{1,4} - w_{1,2} = \frac{\mu}{4},\\
P_{2,1} &: \left(4 - \frac{\mu}{4}\right) w_{2,1} - w_{3,1} - w_{1,1} - w_{2,2} - w_{2,0} = \frac{\mu}{4},\\
P_{2,2} &: \left(4 - \frac{\mu}{4}\right) w_{2,2} - w_{3,2} - w_{1,2} - w_{2,3} - w_{2,1} = \frac{\mu}{4},
\end{aligned}
$$

$$\begin{aligned}
P_{2,3} &: \left(4-\frac{\mu}{4}\right) w_{2,3} - w_{3,3} - w_{1,3} - w_{2,4} - w_{2,2} = \frac{\mu}{4},\\
P_{3,1} &: \left(4-\frac{\mu}{4}\right) w_{3,1} - w_{4,1} - w_{2,1} - w_{3,2} - w_{3,0} = \frac{\mu}{4},\\
P_{3,2} &: \left(4-\frac{\mu}{4}\right) w_{3,2} - w_{4,2} - w_{2,2} - w_{3,3} - w_{3,1} = \frac{\mu}{4},\\
P_{3,3} &: \left(4-\frac{\mu}{4}\right) w_{3,3} - w_{4,3} - w_{2,3} - w_{3,4} - w_{3,2} = \frac{\mu}{4}.
\end{aligned}$$

It is not defficult to establish the system of linear equations $AX = B$, where

$$A = \begin{bmatrix}
(4-\frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0\\
-1 & (4-\frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0 & 0 & 0\\
0 & -1 & (4-\frac{\mu}{4}) & 0 & 0 & -1 & 0 & 0 & 0\\
-1 & 0 & 0 & (4-\frac{\mu}{4}) & -1 & 0 & -1 & 0 & 0\\
0 & -1 & 0 & 0 & (4-\frac{\mu}{4}) & -1 & 0 & -1 & 0\\
0 & 0 & -1 & 0 & -1 & (4-\frac{\mu}{4}) & 0 & 0 & -1\\
0 & 0 & 0 & -1 & 0 & 0 & (4-\frac{\mu}{4}) & -1 & 0\\
0 & 0 & 0 & 0 & -1 & 0 & -1 & (4-\frac{\mu}{4}) & -1\\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & (4-\frac{\mu}{4})
\end{bmatrix},$$

$$X = \begin{bmatrix} w_{1,1} & w_{1,2} & w_{1,3} & w_{2,1} & w_{2,2} & w_{2,3} & w_{3,1} & w_{3,2} & w_{3,3} \end{bmatrix}^T,$$

$$B = \begin{bmatrix} \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} & \frac{\mu}{4} \end{bmatrix}^T.$$

Acknowledgements. This work was partially supported by Instituto Politécnico Nacional, UPIIZ, and by DICIS-University of Guanajuato.

We would like to thank the reviewers for their useful comments and corrections.

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