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APPROXIMATE SOLUTION OF BRATU DIFFERENTIAL EQUATIONS USING TRIGONOMETRIC BASIC FUNCTIONS

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Abstract. In this paper, I have proposed a method for finding an approximate function for Bratu differential equations (BDEs), in which trigonometric basic functions are used. First, by defining trigonometric basic functions, I define the values of the transformation function in relation to trigonometric basis functions (TBFs). Following that, the approximate function is defined as a linear combination of trigonometric base functions and values of transform function which is named trigonometric transform method (TTM), and the convergence of the method is also presented. To get an approximate solution function with discrete derivatives of the solution function, we have determined the approximate solution function which satisfies in the Bratu differential equations (BDEs). In the end, the algorithm of the method is elaborated with several examples. In one example, I have presented an absolute error comparison of some approximate methods.

1. Introduction

A problem of the non-linear eigenvalue problem in n dimensions is the Bratu differential equations (BDEs) as follows [13]

$$\sum_{i=1}^{n}\left(\frac{\partial}{\partial t_i}\right)^2 \Phi(t_1,t_2,\dots,t_n)+\lambda\exp(\Phi(t_1,t_2,\dots,t_n))=0, \tag{1.1}$$

in which $|x_i|\leq 1$ for $i=1,2,\dots,n$, with the following boundary conditions as $|x_i|=1$,

$$\Phi(t_1,t_2,\dots,t_n)=0. \tag{1.2}$$

The main objective in this paper is to offer a simple method in which it is possible to apply trigonometric transform method (TTM) to tackle with the one-dimensional

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(1D) BDEs of the following form

$$u''(t) + \lambda \exp(u(t)) = 0, \quad 0 < t \leq T, \tag{1.3}$$

$$u(0) = u_0, \quad u_t(0) = u'_0, \tag{1.4}$$

where $\lambda > 0$ and $t \in \mathbb{R}$ are constant functions (see [12, 23]).

The analytic solution for BDEs is presented as follows:

$$u(t) = \log \left(\frac{\cosh\left(\frac{\phi}{2}\left(t - \frac{1}{2}\right)\right)}{\cosh\left(\frac{\phi}{4}\right)} \right)^{-2},$$

in which ϕ is the solution of $\phi = \sqrt{2\lambda}\cosh\left(\frac{\phi}{4}\right)$ (see [12, 23]). Whereas $\lambda_\epsilon = 3.513830719$, the BDEs has

- one solutions when $\lambda = \lambda_\epsilon$;
- two solutions if $\lambda < \lambda_\epsilon$;
- no solution when $\lambda > \lambda_\epsilon$.

Researchers and scholars are requested to check papers that have been introduced to get a better grasp of thoroughgoing introduction about BDEs and its history in [10, 18].

On the importance and motivation for Bratu differential equation, it should noted that it has a key role in many of the physical phenomena, chemical models and other sciences. Such applications include the model of thermal reaction process, the fuel ignition model of the thermal combustion theory, the Chandrasekhar model of the expansion of the universe, the radiative heat transfer nanotechnology and the chemical reaction theory (see [9, 10, 12, 18]).

As another instance, mathematical modeling in chemistry for the electro-spinning process is related to BDEs via thermo-electro-hydrodynamics balance equations. Colantoni and his co-author in [5] represented a model that is the mono-dimensional Bratu equation as follows:

$$u''(t) - \lambda \exp(u(t)) = 0, \tag{1.5}$$

featuring $\lambda = \frac{18\,E^2(I - r^2 k\,E)^2}{\rho^2\,r^4}$, in which

- r is the radius of the jet at axial coordinate X in the Figure 1;
- I is the electrical current intensity;
- E is the electric area in the axial direction;
- ρ is the material density;
- k is a fixed value which is only dependent on temperature with regard to incompressible polymer.

Many researchers have used numerical methods for the purpose of solving the BDEs. We can refer to a number of familiar methods, including Homotopic perturbation method [8], Finite difference [19], Optimal homotopy asymptotic method [6], Wavelet method [17], Laplace transform decomposition method [15], B-splines method [4], Variational iteration technique [7], Adomian decomposition method [23], Differential

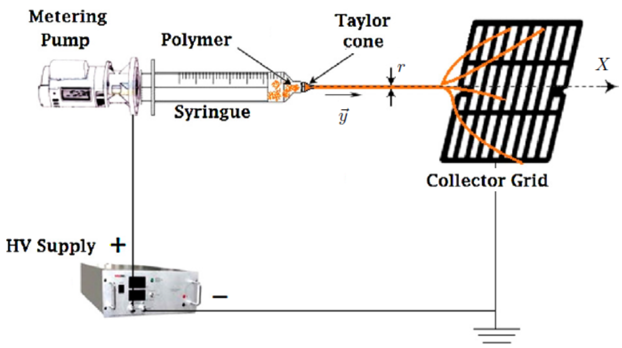

FIGURE 1. Electro-spinning process setup.

quadrature method [21], Lie-group shooting method [1], Reproducing kernel Hilbert space method [2], Pseudo-spectral collocation method [3] and [11, 12, 14, 16, 22].

This paper is organized as what follows: in Section 2, discretization of the derivative is given. In Section 3, we have expressed the trigonometric Basic functions (TBFs). In Section 4, a description of the new approach that is named trigonometric transform method (TTM) is presented. Some numerical examples are offered in Section 5. And conclusions are drawn in Section 6.

2. Discretization of the Derivative

In this section, we introduce discretization of the derivative of a function. The approximation of derivatives by forward differences is one of the most basic tools in finite difference methods for the approximate solution of differential equations, especially initial value problems. The n-th order forward difference is given by

$$u^{(n)}(t) \approx \frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}u\left((n-i)h+t\right), \quad n \in \mathbb{N}.$$

Depending on the application, the spacing h may be a variable or a fixed. In this paper, we consider $\tau = t_{j+1} - t_j$ and $t_j = a + j\,\tau$ for $j = 0, 1, 2, \ldots$. For second order derivative we have:

$$u''(t_{k+1}) \approx \frac{1}{h^2}(u(t_{j+1}) - 2u(t_j) + u(t_{j-1})), \tag{2.1}$$

in which $u(t_0)$ and $u'(t_0)$ are known and $u(t_{-1}) = u(0) - \tau\, u_t(0)$.

3. Trigonometric Basic Functions (TBFs)

In this section, we introduce the trigonometric basis functions and properties that are used in the main sections of the paper to approximate the function of the solution.

Definition 3.1. Presuming that for $n \geq 1$, $a = t_0 < t_1 < \cdots < t_{n-1} < t_n = b$ be specified nodes, we express that basic functions $T_0, T_1, \ldots, T_n$ are defined on $[a,\, b]$

with their trigonometric functions $T_0(t), T_1(t), \dots, T_n(t)$, as follows:

$$T_0(t) = \begin{cases} 0.5\left(1+\cos\frac{\pi}{h_0}(t-t_0)\right), & t_0 \le t \le t_1, \\ 0, & \text{otherwise}, \end{cases}$$

$$T_k(t) = \begin{cases} 0.5\left(1+\cos\frac{\pi}{h_{k-1}}(t-t_k)\right), & t_{k-1} \le t \le t_k, \\ 0.5\left(1+\cos\frac{\pi}{h_k}(t-t_k)\right), & t_k \le t \le t_{k+1}, \quad k=1,2,3,\dots,n-1, \\ 0, & \text{otherwise}, \end{cases} \tag{3.1}$$

$$T_n(t) = \begin{cases} 0.5\left(1+\cos\frac{\pi}{h_{n-1}}(t-t_n)\right), & t_{n-1} \le t \le t_n, \\ 0, & \text{otherwise}, \end{cases}$$

in which $h_k = t_{k+1} - t_k$ for $k = 0, 1, \dots, n-1$.

Remark 3.1. The trigonometric functions introduced in Definition 3.1 are the trigonometric basis functions (TBFs) in which the following properties are satisfied.

(1) T_k of $[a, b]$ to $[0, 1]$ is continuous, $\sum\limits_{k=0}^{n} T_k(t) = 1$ for all $t \in [a, b]$ and $T_k(t_k) = 1$, $k = 0, 1, 2, \dots, n$.
(2) $T_k(t) = 0$ if $t \notin (t_{k-1}, t_{k+1})$, for $k = 1, 2, \dots, n-1$, $T_0(t) = 0$ if $t \notin (t_0, t_1)$ and $T_n(t) = 0$ if $t \notin (t_{n-1}, t_n)$.
(3) On subinterval $[t_{k-1}, t_{k+1}]$ for $k = 1, 2, \dots, n-1$, $T_k(t)$, certainly is an increasing function on $[t_{k-1}, t_k]$ and decreasing function on $[t_k, t_{k+1}]$. Basic functions are called uniform as long as $t_{k+1} - t_k = h = \frac{b-a}{n}$ and two additional properties coincide.
(4) $T_k(t_k - t) = T_k(t_k + t)$, for all $t \in [0, h]$ and $k = 1, 2, \dots, n-1$;
(5) $T_k(t) = T_{k-1}(t-h)$ and $T_{k+1}(t) = T_k(t-h)$, for $k = 1, 2, \dots, n-1$ and $t \in [t_k, t_{k+1}]$.

Lemma 3.1 ([20]). *Consider $n \ge 2$, $T_0, T_1, \dots, T_n$, be the* TBFs *which builds on $[a, b]$. Therefore,*

$$\int_{t_0}^{t_1} T_1(t)dt = \int_{t_{n-1}}^{t_n} T_n(t)dt = \frac{h}{2} \tag{3.2}$$

and

$$\int_{t_{k-1}}^{t_{k+1}} T_k(t)dt = h, \tag{3.3}$$

for $k = 1, 2, \dots, n-1$, in which h is the distance between each of the two neighboring nodes.

Definition 3.2. Let f be a function belonging to $C[a, b]$ and T_i, $i = 0, 1, \dots, n$, be the TBFs which buildup on $[a, b]$. We define the F_k that is the transform of function f on $[a, b]$ with respect to basic functions T_k given by

$$F_k = \frac{\int_a^b f(t)T_k(t)dt}{\int_a^b T_k(t)dt}, \quad k = 0, 1, 2, \dots, n. \tag{3.4}$$

Definition 3.3. Let f be a function belonging to $C[a,b]$ and T_i, $i=0,1,\dots,n$, be the TBFs which buildup on $[a,b]$ and F_k be transform of function f on $[a,b]$ with respect to basic functions T_k. Then

$$f_n(t)=\sum_{k=0}^{n} F_k T_k(t)$$

is approximate of function f on $[a,b]$ with respect to TBFs.

Theorem 3.1 (Convergence). *Let f be a uniformly continuous function on $[a,b]$. Thus, for any $\epsilon>0$, there exists n_ϵ such that for all $n \geq n_\epsilon$*

$$|f(t)-f_{n_\epsilon}(t)|<\epsilon. \tag{3.5}$$

Proof. f is a uniformly continuous function on $[a,b]$. Therefore,

$$(\forall \epsilon>0)\ (\exists \delta=\delta(\epsilon))\ (|x-t|<\delta \Rightarrow |f(x)-f(t)|<\epsilon \ \ (0<\delta<\epsilon)).$$

For all $\epsilon>0$, we have

$$|f(t)-f_n(t)|=\left|\sum_{i=0}^{n} T_i(t)f(t)-\sum_{i=0}^{n} F_iT_i(t)\right| \leq \sum_{i=0}^{n} T_i(t)\,|f(t)-F_i|<\epsilon.$$

It is sufficient to show that $|f(t)-F_i|<\epsilon$. Let x, $t\in[x_{i-1},x_{i+1}]$, $i=1,2,\dots,n-1$, so that we can evaluate

$$|f(x)-F_i|=\left|f(x)-\frac{\int_a^b f(t)T_i(t)dt}{\int_a^b T_i(t)dt}\right| \leq \frac{\int_{x_{i-1}}^{x_{i+1}} T_i(t)\,|f(x)-f(t)|\,dt}{\int_{x_{i-1}}^{x_{i+1}} T_i(t)dt}<\epsilon,$$

if and only if

$$\delta<2h<\epsilon \quad \text{or} \quad h<\frac{\epsilon}{2}.$$

Regarding $h=\frac{b-a}{n}$, it is sufficient that $n_\epsilon>\frac{2(b-a)}{\epsilon}$. □

For description of fractional derivative, we have the following proposition.

Proposition 3.1. *With substituting $f_n(t)=\sum\limits_{k=0}^{n} F_k\,T_k(t)$ in (2.1), we will have the next equation for $k=0,1,2,\dots,n-1$:*

$$f_n''(t_{k+1})\approx \frac{1}{h^2}(F_{j+1}-2F_j+F_{j-1}). \tag{3.6}$$

4. Description of the New Approach

Let solution of (1.3) be continuous on $[0,b]$. To gain approximate solution of $u(x)$, we divide $[0,b]$ to n equal partition with step length τ:

$$t_0=0, \quad t_i=t_0+i\tau, \quad i=0,1,\dots,n, \tau=\frac{b}{n}. \tag{4.1}$$

Considering the trigonometric functions with regard to Definition 3.1 on $[0,b]$ and Definition 3.3, we can gain approximate function $u(x)$ by $u_n(x)=\sum\limits_{k=0}^{n} U_k\,T_k(t)$. It is evident that for calculating $u_n(t)$, $t \in [0, b]$, we should calculate U_k, $k = 0, 1, 2, \ldots, n$. In order to gain the approximate solution of the problem (1.3), $u_n(t)$ for points $t_0, t_1, \ldots, t_n$ must be satisfied in (1.3). Due to the boundary conditions (1.4), $u_n(t_0) := u(t_0) = u_0$ and for other points $t_1, t_2, \ldots, t_n$, we have

$$u_n''(t_{k+1}) + \lambda \exp(u_n(t_{k+1})) = 0, \quad k = 0, 1, 2, \ldots, n-1, \tag{4.2}$$

in which $m - 1 < \rho \leq m$ and $m \in \mathbb{Z}^+$.

Using (3.6) and (4.2) converts to the following form for $k = 0, 1, 2 \ldots, n-1$:

$$\frac{1}{h^2}(U_{k+1} - 2\,U_k + U_{k-1}) + \lambda \exp(U_n(t_{k+1})) = 0, \tag{4.3}$$

where $U_0 = u(0)$ and $U_{-1} = u(0) - u'(0)$ are known initial conditions.

Now, using the boundary condition, we can calculate $U_1, U_2, \ldots, U_n$ by the obtained recursive equation (4.3) and then gain the approximate solution $u(t) \approx u_n(t)$ for (1.3).

In order to gain approximation of BDEs, an algorithm by this method is offered in the subsequent algorithm.

Algorithm 1: An algorithm for approximation of BDEs
Step 1: Input n and b.
Step 2: Set $\tau \leftarrow \frac{b}{n}$.
Step 3: Locate $t_k \leftarrow k\,\tau,\ k = 0, 1, 2, \ldots, n$.
Step 4: Choose TBFs $T_k(t)$ toward $k = 0, 1, 2, \ldots, n$.
Step 5: Set recursive equations

$$\frac{1}{h^2}(U_{k+1} - 2\,U_k + U_{k-1}) + \lambda \exp(U_n(t_{k+1})) = 0,$$

where $U_0 = u(0)$ and $U_{-1} = u(0) - \tau u'(0)$.
Step 6: Calculate every U_k, $k = 1, 2, \ldots, n$, of an equation of degree one.
Step 7: The approximate solution is

$$u_n(t) \approx \sum_{k=0}^{n} U_k T_k(t).$$

5. Examples

Now that it is easier to understand trigonometric transform, a number of examples will be given in this section and then will be calculated. These examples include BDEs. In all these examples, software *Mathematica* 11 has been used for calculations and graphs.

Example 5.1. We propose the BDEs for the first example [23]:

$$u''(t) - 2\exp(u(t)) = 0, \quad 0 \leq t \leq 1, \tag{5.1}$$

with the precise solution $u(t) = \log((\cos t)^{-2})$ and the primary conditions:

$$u(0) = 0, \quad u'(0) = 0. \tag{5.2}$$

Following the TTM, according to what was formulated and presented in section 4 for (5.1)–(5.2), we can calculate $U_1, U_2, \ldots, U_n$, and then gain the approximate solution $u_n(t)$ of (5.1).

In Table 1, we can see the estimated solutions for Eq.(5.1), which is derived for various values of n applying TTM. Also, the estimated and approximate solutions are illustrate in Figure 2.

TABLE 1. Approximate result of Example 5.1 with various values of n.

	TTM				
t	$n = 50$	$n = 500$	$n = 1000$	$n = 1500$	Exact
0.0	0.0	0.0	0.0	0.0	0.0
0.2	0.0543317	0.0407728	0.0404703	0.0402949	0.0402695
0.4	0.193714	0.165493	0.164871	0.164416	0.164458
0.6	0.42896	0.385508	0.384559	0.383323	0.38393
0.8	0.799043	0.725417	0.723832	0.722438	0.722781

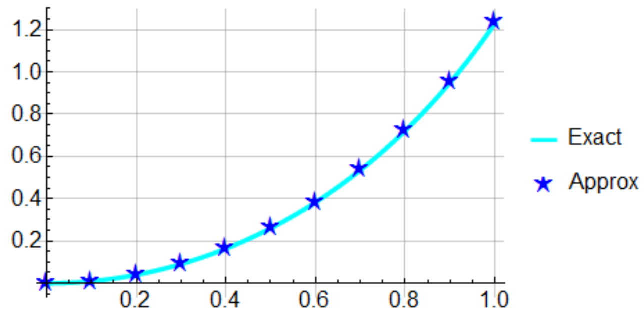

FIGURE 2. Figure for Example 5.1 exact and the approximation solutions.

Noteworthy in the values obtained in the Table 1 is that by increasing the amount n, a more accurate answer for (5.1) can be achieved.

Example 5.2. Consider the BDEs for the second example [23]:

$$u''(t) + \pi^2 \exp(-u(t)) = 0, \quad 0 \le t \le 1, \tag{5.3}$$

given that the primary conditions:

$$u(0) = 0, \quad u'(0) = \pi. \tag{5.4}$$

The unknown coefficient $U_1, U_2, \ldots, U_n$ with due attention to the TTM, according to Section 4 for (5.3)–(5.4) are calculated.

In Table 2 and in Figure 3, we can view the precise and approximate answers for $n = 1500$ through applying TTM.

The approximate solution obtained by the proposed method corresponds to the precise solution $u(t) = \log(1 + \sin(\pi t))$.

In Figure 3, we can see the estimated solutions toward $n = 1500$, which is derived for various value of t applying TTM.

TABLE 2. Approximate result of example 5.2.

t	TTM	Exact	Absolute Error	Relative Error
0.2	0.462127	0.46234	212.789 $\times 10^{-6}$	460.455 $\times 10^{-6}$
0.4	0.66794	0.668371	430.849 $\times 10^{-6}$	645.042 $\times 10^{-6}$
0.6	0.667754	0.668371	616.549 $\times 10^{-6}$	923.317 $\times 10^{-6}$
0.8	0.46142	0.46234	920.306 $\times 10^{-6}$	1.99451 $\times 10^{-3}$

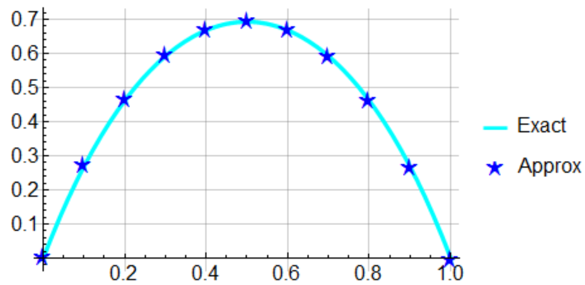

FIGURE 3. Comparison of the approximate solution (5.3) with exact solution for $n = 1500$.

Example 5.3. We offer the BDEs for the third example [23]:

$$u''(t) - \pi^2 \exp(u(t)) = 0, \quad 0 \le t \le 1, \tag{5.5}$$

including the primary conditions:

$$u(0) = 0, \quad u'(0) = -\pi. \tag{5.6}$$

It can be seen in Table 3 and Figure 4 that solving equations with approximate expression is calculated and displayed for $n = 1500$ and various values of t.

TABLE 3. Approximate result of Example 5.3.

t	TTM	Exact	Absolute Error	Relative Error
0.2	0.451242	0.451272	30.7122 $\times 10^{-6}$	68.0615 $\times 10^{-6}$
0.4	-0.227657	-0.226202	1.45505 $\times 10^{-3}$	6.39141 $\times 10^{-3}$
0.6	-0.576992	-0.573173	3.81849 $\times 10^{-3}$	6.61792$\times 10^{-3}$
0.8	-0.699629	-0.69232	7.30951 $\times 10^{-3}$	10.4477 $\times 10^{-3}$

In Table 4, we can see the estimated solutions toward $n = 1500$, which is derived for various values of t applying TTM.

Toward $n = 1500$, the solution that we have gained is in accordance with the precise solution $u(t) = \log\left(\frac{1}{1-\sin(1-\pi t)}\right)$.

Example 5.4. Consider the BDEs [1]:

$$u''(t) + 2\exp(u(t)) = 0, \quad 0 \le t \le 1, \tag{5.7}$$

supposing that the primary conditions:

$$u(0) = 0, \quad u'(0) = 0. \tag{5.8}$$

TABLE 4. Absolute error comparison of Example 5.4.

t	BNM	NPS	LTM	DM	BSM	LGSM	SCM	TTM
								n=1500
0.1	1.91×10^{-14}	9.71×10^{-9}	2.13×10^{-3}	1.52×10^{-2}	1.72×10^{-5}	4.03416×10^{-6}	$6{:}88\times10^{-4}$	0
0.3	1.17×10^{-13}	1.98×10^{-8}	6.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}	5.22122×10^{-6}	$8{:}21\times10^{-4}$	169.624×10^{-6}
0.5	1.88×10^{-13}	2.60×10^{-8}	9.60×10^{-3}	6.98×10^{-3}	5.56×10^{-5}	1.4554×10^{-8}	$8{:}60\times10^{-4}$	341.417×10^{-6}
0.7	1.16×10^{-13}	1.98×10^{-8}	1.19×10^{-3}	5.89×10^{-3}	4.49×10^{-5}	5.19455×10^{-6}	$8{:}21\times10^{-4}$	424.587×10^{-6}
0.9	1.90×10^{-14}	9.71×10^{-9}	1.09×10^{-3}	1.52×10^{-3}	1.72×10^{-5}	4.01345×10^{-6}	$6{:}88\times10^{-4}$	445.899×10^{-6}

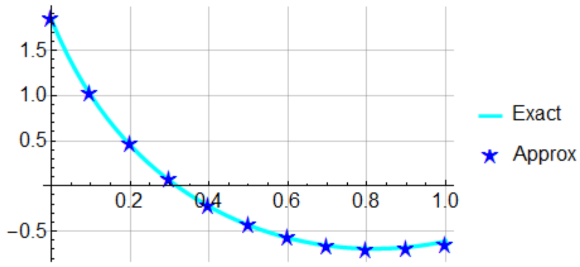

FIGURE 4. Comparison of the approximate solution (5.5) with exact solution for $n = 1500$.

The unknown coefficient $U_1, U_2, \ldots, U_n$, with due attention to the TTM, according to Section 4 for (5.7)–(5.8) are calculated.

Table 4 illustrates an absolute error comparison of the TTM and approximate methods: Block Nyström method (BNM) [12], Non-polynomial spline (NPS) [11], Laplace transform method (LTM) [18], Decomposition method (DM) [16], B-splines method (BSM) [4], Lie-group shooting method (LGSM) [1] and Sinc-collocation method (SCM).

In Figure 5, we can see the estimated solutions toward $n = 1500$, which is derived for various value of t applying TTM.

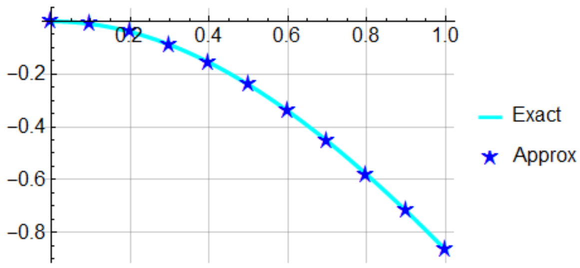

FIGURE 5. Comparison of the approximate solution (5.7) with exact solution for $n = 1500$.

Noteworthy in the values obtained in the last column Table 4 is that by increasing the amount n, a more accurate answer for (5.7) can be achieved.

6. CONCLUSION

I have proposed a method for finding an approximate function of Bratu differential equations (BDEs), in which TTM are used. All examples with absolute and relative errors show that we have favorably applied trigonometric transform method TTM to obtain approximate solution of the BDEs. The obtained solutions that are very close analytical solutions indicate that a little iteration of TTM will result in some useful solutions. As the result seems necessary from the authors' point of view, the suggested technique has the potentials to be practical in solving other similar ordinary differential equations of integer orders and partial differential equations of non integer orders.

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