Kragujevac Journal of Mathematics
Volume 45(2) (2021), Pages 225–236.

A NOTE ON PAIR OF LEFT CENTRALIZERS IN PRIME RING WITH INVOLUTION

MUZIBUR RAHMAN MOZUMDER[1], ADNAN ABBASI[1], NADEEM AHMAD DAR[2], AND AFTAB HUSSAIN SHAH[3]

Abstract. The purpose of this paper is to study pair of left centralizers in prime rings with involution satisfying certain identities.

1. Introduction

In the present article, R will represent an associative ring with centre $Z(R)$. Q_mr and C represents the maximal ring of quotient and the extended centroid of a prime ring, respectively. For the explanation of Q_mr and C refer to [4]. R is said to be n-torsion free if $na = 0$ (where $a \in R$) implies $a = 0$. R is called prime if $aRb = (0)$ (where $a, b \in R$) implies $a = 0$ or $b = 0$. We write $[x, y]$ for $xy - yx$ and xoy for $xy + yx$, respectively. An additive map $x \mapsto x^*$ of R into itself is called an involution if (i) $(xy)^* = y^*x^*$ and (ii) $(x^*)^* = x$ holds, for all $x, y \in R$. A ring R together with an involution $*$ is said to be a ring with involution or $*$-ring. An element x in a ring with involution $*$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denoted by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the latter case, $S(R) \cap Z(R) \neq (0)$. A description of such rings can be found in [7], where further references can be found.

Following [17], an additive mapping $T : R \to R$ is said to be a left (resp. right) centralizer (multiplier) of R if $T(xy) = T(x)y$ (resp. $T(xy) = xT(y)$) for all $x, y \in R$. An additive mapping T is called a centralizer in case T is a left and a right centralizer of R. Considerable work has been done on left (resp. right) centralizers in prime and semiprime rings during the last few decades (see for example [3, 6, 10, 11, 14–17]) where

Key words and phrases. Prime ring, centralizing mapping, involution.
2010 *Mathematics Subject Classification.* Primary: 16W10, 16N60, 16W25.
DOI 10.46793/KgJMat2102.225M
Received: May 18, 2018.
Accepted: November 26, 2018.

further references can be found. The first result studying the commutativity of prime ring involving a special mapping was due to Divinsky [5], who proved that a simple artinian ring is commutative if it has a commuting non-trivial automorphism. This result was later refined and extended by a number of authors in various directions (see [2,3,8,9,12,13]). Moreover in [3] some related results involving left centralizers have also been discussed. In [10] Oukhtite established similar problems for certain situations involving left centralizers acting on Lie ideals. Recently Ali and Dar [1] proved that if a prime ring with involution of the second kind such that char(R) $\neq 2$ admits a left centralizer $T : R \to R$ satisfying any one of the following conditions:

(i) $T([x, x^*]) = 0$;
(ii) $T(xox^*) = 0$;
(iii) $T([x, x^*]) \pm [x, x^*] = 0$;
(iv) $T(xox^*) \pm (xox^*) = 0$,

for all $x \in R$, then R is commutative. In this paper, we shall consider similar problems involving pair of centralizers. We shall restrict our attention on left centralizers, since all results presented in this article are also true for right centralizers because of left-right symmetry.

Lemma 1.1 ([7], p. 20-23). *Suppose that the elements a_i, b_i in the central closure of a prime ring R satisfy $\sum a_i x b_i = 0$ for all $x \in R$. If $b_i \neq 0$ for some i, then $a_i's$ are C-dependent.*

2. Main Results

Theorem 2.1. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R such that $[T_1(x), T_2(x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$[T_1(x), T_2(x^*)] \in Z(R), \quad \text{for all } x \in R. \tag{2.1}$$

Linearizing (2.1), we get

$$[T_1(x), T_2(y^*)] + [T_1(y), T_2(x^*)] \in Z(R), \quad \text{for all } x, y \in R. \tag{2.2}$$

Replacing y by ky in (2.2) and using (2.2), we get $2([T_1(y), T_2(x^*)])k \in Z(R)$ for all $x, y \in R$ and $k \in S(R) \cap Z(R)$, since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$. This implies that $[T_1(y), T_2(x^*)] \in Z(R)$ for all $x, y \in R$. Taking $x = x^*$, we have $[T_1(y), T_2(x)] \in Z(R)$ for all $x, y \in R$. This can be further written as

$$[[T_1(y), T_2(x)], r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.3}$$

Replacing y by ym, where $m \in R$ in (2.3) and using (2.3) we get

$$T_1(y)[[m, T_2(x)], r] + [T_1(y), r][m, T_2(x)] + [T_1(y), T_2(x)][m, r] = 0,$$

for all $x, y, m, r \in R$. Replacing m by $T_2(x)$ we get

$$[T_1(y), T_2(x)][T_2(x), r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.4}$$

Replacing r by ru, where $u \in R$ in (2.4) and using (2.4) we get

$$[T_1(y), T_2(x)]r[T_2(x), u] = 0, \quad \text{for all } x, y, r, u \in R.$$

Then by primeness of R, for each fixed $x \in R$, we get either $[T_1(y), T_2(x)] = 0$ for all $y \in R$ or $[T_2(x), u] = 0$ for all $u \in R$. Define $A = \{x \in R \mid [T_2(x), u] = 0 \text{ for all } u \in R\}$ and $B = \{x \in R \mid [T_1(y), T_2(x)] = 0 \text{ for all } y \in R\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence, by Brauer's trick, either $A = R$ or $B = R$. If $A = R$

$$[T_2(x), u] = 0, \quad \text{for all } x, u \in R. \tag{2.5}$$

Then taking $x = xy$ in (2.5), where $y \in R$ and using (2.5) we get $T_2(x)[y, u] = 0$ for all $x, y, u \in R$. Now take $x = xm$, where $m \in R$, then as T_2 is nonzero, applying the primeness of R, we obtain R is commutative. If $B = R$

$$[T_1(y), T_2(x)] = 0, \quad \text{for all } x, y \in R. \tag{2.6}$$

Then replacing y by yv, where $v \in R$ in (2.6) and using (2.6) we get $T_1(y)[v, T_2(x)] = 0$ for all $x, y, v \in R$. Now replace y by yr, where $r \in R$. Then as T_1 is nozero, by primeness of R, we have $[v, T_2(x)] = 0$ for all $v, x \in R$. With similar steps as we did before we get R is commutative. □

Theorem 2.2. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x) \circ T_2(x^*) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x) \circ T_2(x^*) \in Z(R), \quad \text{for all } x \in R. \tag{2.7}$$

Linearizing (2.7), we get

$$T_1(x) \circ T_2(y^*) + T_1(y) \circ T_2(x^*) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.8}$$

Replacing y by ky in (2.8) and using (2.8), we get

$$2(T_1(y) \circ T_2(x^*))k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $T_1(y) \circ T_2(x^*) \in Z(R)$ for all $x, y \in R$. Replacing x by x^* we get $T_1(y) \circ T_2(x) \in Z(R)$ for all $x, y \in R$. This can be further written as

$$[T_1(y) \circ T_2(x), r] = 0,$$
$$[T_1(y)T_2(x), r] + [T_2(x)T_1(y), r] = 0,$$
$$T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) + T_2(x)[T_1(y), r] + [T_2(x), r]T_1(y) = 0, \tag{2.9}$$

for all $x, y, r \in R$. Replacing y by $yT_2(x)$ in (2.9), we get

$$\begin{aligned} (2.10) \qquad & T_1(y)T_2(x)[T_2(x), r] + T_1(y)[T_2(x), r]T_2(x) + [T_1(y), r](T_2(x))^2 \\ & +T_2(x)T_1(y)[T_2(x), r] + T_2(x)[T_1(y), r]T_2(x) + [T_2(x), r]T_1(y)T_2(x) = 0, \end{aligned}$$

for all $x, y, r \in R$. Left multiplying (2.9) by $T_2(x)$ and subtracting it from (2.10), we get

$$(2.11) \qquad (T_1(y)T_2(x) + T_2(x)T_1(y))[T_2(x), r] = 0, \quad \text{for all } x, y, r \in R.$$

Replacing r by rm, where $m \in R$ in (2.11) and using (2.11), we get

$$(T_1(y)T_2(x) + T_2(x)T_1(y))r[T_2(x), m] = 0, \quad \text{for all } x, y, r, m \in R.$$

Then by primeness of R, for each fixed $x \in R$, we get either $[T_2(x), m] = 0$ for all $m \in R$ or $T_1(y)T_2(x)+T_2(x)T_1(y) = 0$ for all $y \in R$. Define $A = \{x \in R \mid [T_2(x), m] = 0 \text{ for all } m \in R\}$ and $B = \{x \in R \mid T_1(y)T_2(x) + T_2(x)T_1(y) = 0 \text{ for all } y \in R\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $A = R$ or $B = R$. If $A = R$, then we consider $[T_2(x), r] = 0$ for all $x, r \in R$, proceeding similarly as we did after (2.5), we get R is commutative. Now, consider $B = R$, in this situation

$$(2.12) \qquad T_1(y)T_2(x) + T_2(x)T_1(y) = 0, \quad \text{for all } x, y \in R.$$

Then replacing y by yu, where $u \in R$ in (2.12) and using (2.12), we get $T_1(y)[u, T_2(x)] = 0$ for all $x, y, u \in R$. Replacing y, where $v \in R$ by yv, we get $T_1(y)v[u, T_2(x)] = 0$ for all $x, y, v, u \in R$. Then as T_1 is nonzero, by primeness we get $[u, T_2(x)] = 0$ for all $x, u \in R$. Now, following same line of proof as we did after (2.5), we get R is commutative. □

Theorem 2.3. *Let R be a noncommutative 6-torsion free prime ring with involution $*$ of the second kind. If R admit two nonzero left centralizers T_1 and T_2 from R to R, such that $[T_1(x), T_2(x^*)] \pm [x, x^*] \in Z(R)$ for all $x \in R$, then $T_1 = \lambda T_2$.*

Proof. We have

$$(2.13) \qquad [T_1(x), T_2(x^*)] \pm [x, x^*] \in Z(R), \quad \text{for all } x \in R.$$

Linearizing (2.13), we get

$$(2.14) \quad [T_1(x), T_2(y^*)] + [T_1(y), T_2(x^*)] \pm [x, y^*] \pm [y, x^*] \in Z(R), \quad \text{for all } x, y \in R.$$

Replacing y by ky in (2.14) and using (2.14), we get

$$2([T_1(y), T_2(x^*)] \pm [y, x^*])k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

This further implies that

$$6([T_1(y), T_2(x^*)] \pm [y, x^*])k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since R is 6-torsion free and $S(R) \cap Z(R) \neq (0)$, we have

$$[T_1(y), T_2(x^*)] \pm [y, x^*] \in Z(R), \quad \text{for all } x, y \in R.$$

Replacing x by x^*, we get

$$[T_1(y), T_2(x)] \pm [y, x] \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $y = x$, we have

$$[T_1(x), T_2(x)] \in Z(R), \quad \text{for all } x \in R. \tag{2.15}$$

This further implies that

$$[[T_1(x), T_2(x)], r] = 0, \quad \text{for all } x, r \in R. \tag{2.16}$$

On linearization, we get

$$[[T_1(x), T_2(y)], r] + [[T_1(y), T_2(x)], r] = 0, \quad \text{for all } x, y, r \in R. \tag{2.17}$$

Replacing y by $yT_1(x)$ in (2.17) and using (2.16) and (2.17), we obtain

$$[T_1(x), T_2(y)][T_1(x), r] + [T_1(y), r][T_1(x), T_2(x)] + [T_1(y), T_2(x)][T_1(x), r] = 0, \tag{2.18}$$

for all $x, y, r \in R$. Taking $y = x$ in (2.18) and using (2.15), we arrive at

$$3[T_1(x), T_2(x)][T_1(x), r] = 0, \quad \text{for all } x, r \in R.$$

This further implies that

$$6[T_1(x), T_2(x)][T_1(x), r] = 0, \quad \text{for all } x, r \in R.$$

Since R is 6-torsion free ring, we have

$$[T_1(x), T_2(x)][T_1(x), r] = 0, \quad \text{for all } x, r \in R. \tag{2.19}$$

Replacing r by rm, where $m \in R$ in (2.19) and making use of (2.19), we get

$$[T_1(x), T_2(x)]r[T_1(x), m] = 0, \quad \text{for all } x, m, r \in R.$$

Using the primeness of R, for each fixed $x \in R$, we have either $[T_1(x), T_2(x)] = 0$ or $[T_1(x), m] = 0$. Define $B = \{x \in R \mid [T_1(x), T_2(x)] = 0\}$ and $A = \{x \in R \mid [T_1(x), m] = 0 \text{ for all } m \in R\}$. Clearly, A and B are additive subgroups of R whose union is R. Hence by Brauer's trick, either $B = R$ or $A = R$. If $B = R$,

$$[T_1(x), m] = 0, \quad \text{for all } x, m \in R. \tag{2.20}$$

Replacing x by xy, where $y \in R$ and using (2.20), we get

$$T_1(x)[y, m] = 0, \quad \text{for all } x, y, m \in R.$$

This further implies that

$$T_1(x)w[y, m] = 0, \quad \text{for all } x, w, y, m \in R.$$

Using the primeness, we get $T_1(x) = 0$ for all $x \in R$ or $[y, m] = 0$ for all $y, m \in R$. Since T_1 is nonzero, therefore we get R is commutative, which is a contradiction to our assumption. Therefore we are left with $B = R$

$$[T_1(x), T_2(x)] = 0, \quad \text{for all } x \in R. \tag{2.21}$$

Linearizing (2.21), we get

$$[T_1(x), T_2(y)] + [T_1(y), T_2(x)] = 0, \quad \text{for all } x, y \in R. \tag{2.22}$$

Replacing x by xz, where $z \in R$ in (2.22) and using (2.22), we get

$$T_1(x)[z, T_2(y)] + T_2(x)[T_1(y), z] = 0, \quad \text{for all } x, y, z \in R. \tag{2.23}$$

Again taking $x = xw$, where $w \in R$ in (2.23), we get

$$T_1(x)w[z, T_2(y)] + T_2(x)w[T_1(y), z] = 0, \quad \text{for all } x, y, z, w \in R. \tag{2.24}$$

In view of Lemma 1.1, we have $[z, T_2(y)] = 0$ for all $y, z \in R$ or $T_1(x) = \lambda(x)T_2(x)$, where $\lambda(x) \in C$. But since $T_2 \neq 0$, $[z, T_2(y)] = 0$ implies R is commutative, a contradiction. Hence we get $T_1(x) = \lambda(x)T_2(x)$, where $\lambda(x) \in C$. Using this in (2.24), we have

$$\lambda(x)T_2(x)w[z, T_2(y)] + T_2(x)w[\lambda(y)T_2(y), z] = 0,$$
$$(\lambda(x)T_2(x) - \lambda(y)T_2(x))w[z, T_2(y)] = 0,$$

for all $x, y, z, w \in R$. Using the primeness of R and Brauer's trick we finally get $T_1 = \lambda T_2$. This completes the proof. □

Theorem 2.4. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x)x^* \pm x^*T_2(x) \in Z(R)$ for all $x \in R$, then either R is commutative or $T_1(y) = \mp T_2(y)$ for all $y \in R$.*

Proof. We have

$$T_1(x)x^* \pm x^*T_2(x) \in Z(R), \quad \text{for all } x \in R. \tag{2.25}$$

Linearaizing (2.25), we get

$$T_1(x)y^* + T_1(y)x^* \pm x^*T_2(y) \pm y^*T_2(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.26}$$

Replacing y by ky in (2.26) and using (2.26), we have

$$2(T_1(y)x^* \pm x^*T_2(y))k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(y)x^* \pm x^*T_2(y) \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $x = x^*$, we get

$$T_1(y)x \pm xT_2(y) \in Z(R), \quad \text{for all } x, y \in R.$$

Replacing x by z, where $z \in Z(R)$ and using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain $T_1(y) \pm T_2(y) \in Z(R)$ for all $y \in R$. This can be further written as

$$[T_1(y), r] \pm [T_2(y), r] = 0, \quad \text{for all } y, r \in R. \tag{2.27}$$

Replacing y by yw, where $w \in R$ and using (2.27), we have

$$(T_1(y) \pm T_2(y))[w, r] = 0, \quad \text{for all } y, w, r \in R. \tag{2.28}$$

Replacing w by wm, where $m \in R$ in (2.28) and using (2.28), we obtain

$$(T_1(y) \pm T_2(y))w[m, r] = 0, \quad \text{for all } m, y, w, r \in R.$$

In view of the primeness of R we get either R is commutative or $T_1(y) = \mp T_2(y)$ for all $y \in R$. □

Theorem 2.5. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R, such that $T_1(x)T_2(x^*) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x)T_2(x^*) \in Z(R), \quad \text{for all } x \in R. \tag{2.29}$$

Linearizing (2.29), we get

$$T_1(x)T_2(y^*) + T_1(y)T_2(x^*) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.30}$$

Replacing y by ky in (2.30), where $k \in S(R) \cap Z(R)$ and using (2.30), we have

$$2T_1(y)T_2(x^*)k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(y)T_2(x^*) \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $x = x^*$, we obtain

$$T_1(y)T_2(x) \in Z(R), \quad \text{for all } x, y \in R.$$

This can be further written as

$$T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) = 0, \quad \text{for all } x, y \in R. \tag{2.31}$$

Replacing x by xw, where $w \in R$ in (2.31) and using (2.31), we get

$$T_1(y)T_2(x)[w, r] = 0, \quad \text{for all } x, y, w, r \in R.$$

Replacing y by ym, where $m \in R$, we get

$$T_1(y)RT_2(x)[w, r] = (0), \quad \text{for all } x, y, w, r \in R.$$

This implies in view of the primeness of ring R, either $T_1(y) = 0$ for all $y \in R$ or $T_2(x)[w, r] = 0$ for all $x, w, r \in R$. Since $T_1 \neq 0$, we get $T_2(x)[w, r] = 0$ for all $x, w, r \in R$. This further implies that $T_2(x)y[w, r] = 0$ for all $x, y, w, r \in R$. Since $T_2 \neq 0$, using the primeness of R, we get R is commutative. □

Theorem 2.6. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x)x \pm x^*T_2(x) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x)x \pm x^*T_2(x) \in Z(R), \quad \text{for all } x \in R.$$

Linearizing (2.48), we get

$$T_1(x)y + T_1(y)x \pm x^*T_2(y) \pm y^*T_2(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.32}$$

Replacing y by ky in (2.32) and using (2.32), we arrive at

$$2(T_1(x)y + T_1(y)x \pm x^*T_2(y))k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$T_1(x)y + T_1(y)x \pm x^*T_2(y) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.33}$$

Again, replacing x by kx in(2.33) and using (2.33), we get

$$2(T_1(x)y + T_1(y)x)k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, we have

$$T_1(x)y + T_1(y)x \in Z(R), \quad \text{for all } x, y \in R.$$

This can be further written as $T_1(x \circ y) \in Z(R)$ for all $x, y \in R$. Taking $y = z$, where $z \in Z(R)$, we get $T_1(x) \in Z(R)$ for all $x \in R$. This further implies that $[T_1(x), y] = 0$ for all $x, y \in R$. Replacing x by xw, where $w \in R$, we get $T_1(x)[w, y] + [T_1(x), y]w = 0$ for all $x, y, w \in R$. That is, $T_1(x)[w, y] = 0$ for all $x, y, w \in R$. Replacing x by xm, where $m \in R$ and using the facts that $T_1 \neq 0$ and the primeness of R, we obtain $[w, y] = 0$ for all $w, y \in R$. That is, R is commutative. □

Theorem 2.7. *Let R be a noncommutative prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admit two nonzero left centralizer T_1 and T_2 from R to R such that $xT_1(x^*) \pm T_2(x)x^* \in Z(R)$ for all $x \in R$, then $T_1 = \mp T_2$.*

Proof. We have

$$xT_1(x^*) \pm T_2(x)x^* \in Z(R), \quad \text{for all } x \in R. \tag{2.34}$$

Linearizing (2.34), we get

$$xT_1(y^*) + yT_1(x^*) \pm T_2(x)y^* \pm T_2(y)x^* \in Z(R), \quad \text{for all } x, y \in R. \tag{2.35}$$

Replacing y by ky in (2.35) and using (2.35), we get

$$2(yT_1(x^*) \pm T_2(y)x^*)k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that

$$yT_1(x^*) \pm T_2(y)x^* \in Z(R), \quad \text{for all } x, y \in R.$$

Taking $x = x^*$, we get

$$yT_1(x) \pm T_2(y)x \in Z(R), \quad \text{for all } x, y \in R.$$

This further implies that

$$[yT_1(x), r] \pm [T_2(y)x, r] = 0, \quad \text{for all } x, y, r \in R.$$

That is,

$$y[T_1(x), r] + [y, r]T_1(x) \pm T_2(y)[x, r] \pm [T_2(y), r]x = 0, \quad \text{for all } x, y, r \in R. \tag{2.36}$$

Replacing x by xw, where $w \in R$ in (2.36) and using (2.36), we obtain

$$(yT_1(x) \pm T_2(y)x)[w, r] = 0, \quad \text{for all } x, y, w, r \in R. \tag{2.37}$$

Replacing w by wm, where $m \in R$ in (2.37) and using (2.37), we get

$$yT_1(x) \pm T_2(y)x = 0, \quad \text{for all } x, y \in R, \tag{2.38}$$

since R is noncommutative. Replacing y by yx in (2.38) and using (2.38), we get

$$y(xT_1(x) - T_1(x)x) = 0, \quad \text{for all } x, y \in R.$$

Using the primeness of R, we get

$$(xT_1(x) - T_1(x)x) = 0, \quad \text{for all } x \in R.$$

Linearizing the above equation, we get

$$xT_1(y) + yT_1(x) - T_1(x)y - T_1(y)x = 0, \quad \text{for all } x, y \in R. \tag{2.39}$$

Replacing y by yu, where $u \in R$ in (2.39) and using (2.39), we arrive at

$$y[T_1(x), u] + T_1(y)[u, x] = 0, \quad \text{for all } x, y, u \in R. \tag{2.40}$$

Replacing x by xm, where $m \in R$ in (2.40) and using (2.40), we get

$$(yT_1(x) - T_1(y)x)[m, u] = 0, \quad \text{for all } x, y, m, u \in R. \tag{2.41}$$

Replacing m by wm, where $m \in R$ in (2.41) and using (2.41), we have T_1 is centralizer, since R is noncommutative. Hence in view of (2.38), we get $(T_1(y) \pm T_2(y))x = 0$ for all $x, y \in R$. Using the primeness of R, we obtain $T_1(y) = \mp T_2(y)$ for all $y \in R$. □

Theorem 2.8. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two left centralizer T_1 and T_2 from R to R such that $T_1(x)T_2(x^*) \pm xx^* \in Z(R)$ for all $x \in R$, then either R is commutative or T_1 and T_2 centralizer.*

Proof. We have

$$T_1(x)T_2(x^*) \pm xx^* \in Z(R), \quad \text{for all } x \in R. \tag{2.42}$$

If either T_1 or T_2 is zero, then we get $\pm xx^* \in Z(R)$ for all $x \in Z(R)$. Replacing x by $x + y$, where $x, y \in R$, we get $xy^* + yx^* \in Z(R)$ for all $x, y \in R$. Taking $y = yk$ where $k \in Z(R) \cap S(R)$ and adding with the previous equation, we get $2yx^*k \in Z(R)$ for all $x, y \in Z(R)$ and $k \in S(R) \cap Z(R)$. Since char(R) $\neq 2$, this implies that $yx^*k \in Z(R)$ for all $x, y \in R$ and $k \in S(R) \cap Z(R)$. Use primeness and the fact that $S(R) \cap Z(R) \neq 0$, we have $yx^* \in Z(R)$ for all $x, y \in R$. This further implies that $yx \in Z(R)$ for all $x, y \in R$. Thus $xz \in Z(R)$ for all $x \in R$ and $z \in Z(R)$. Use primeness and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain R is commutative. Now consider neither T_1 nor T_2 is zero. Linearizing (2.42), we get

$$T_1(x)T_2(y^*) + T_1(y)T_2(x^*) \pm xy^* \pm yx^* \in Z(R), \quad \text{for all } x, y \in R. \tag{2.43}$$

Replacing y by ky in (2.43) and using (2.43), we get $2(T_1(y)T_2(x^*) \pm yx^*)k \in Z(R)$ for all $x, y \in R$. Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $T_1(y)T_2(x^*) \pm$

$yx^* \in Z(R)$ for all $x, y \in R$. This can be written as

$$[T_1(y)T_2(x^*), r] \pm [yx^*, r] = 0,$$
$$T_1(y)[T_2(x^*), r] + [T_1(y), r]T_2(x^*) \pm y[x^*, r] \pm [y, r]x^* = 0,$$

for all $x, y, r \in R$. Taking $x = x^*$, we obtain

$$T_1(y)[T_2(x), r] + [T_1(y), r]T_2(x) \pm y[x, r] \pm [y, r]x = 0, \quad \text{for all } x, y, r \in R. \tag{2.44}$$

Replacing x by xw, where $w \in R$ in (2.44) and using (2.44), we get $(T_1(y)T_2(x) \pm yx)[w, r] = 0$ for all $x, y, w, r \in R$. Replacing w by wm, where $m \in R$ and using the previous equation, we get $(T_1(y)T_2(x) \pm yx)w[m, r] = 0$ for all $x, y, w, m, r \in R$. Now using the primeness we get either $T_1(y)T_2(x) \pm yx = 0$ for all $x, y \in R$ or $[m, r] = 0$ for all $m, r \in R$. If $[m, r] = 0$ for $m, r \in R$, this implies that R is commutative. Now suppose

$$T_1(y)T_2(x) \pm yx = 0, \quad \text{for all } x, y \in R. \tag{2.45}$$

Replacing y by $yT_1(w)$ in (2.45), we get

$$T_1(y)T_1(w)T_2(x) \pm yT_1(w)x = 0, \quad \text{for all } x, y, w \in R. \tag{2.46}$$

Taking $y = w$ in (2.45) and left multiplying by $T_1(y)$, we get

$$T_1(y)T_1(w)T_2(x) \pm T_1(y)wx = 0, \quad \text{for all } x, y, w \in R. \tag{2.47}$$

Subtracting (2.47) from (2.46), we have $(\pm yT_1(w) \mp T_1(y)w)x = 0$ for all $x, y, w \in R$. Since $R \neq (0)$ and using primeness of R we get $(\pm yT_1(w) \mp T_1(y)w) = 0$ for all $y, w \in R$. This implies that T_1 is a centralizer. Similarly, we can show that T_2 is a centralizer. □

Theorem 2.9. *Let R be a prime ring with involution $*$ of the second kind such that* char(R) $\neq 2$. *If R admits two nonzero left centralizer T_1 and T_2 from R to R such that $T_1(x)x^* \pm xT_2(x) \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$T_1(x)x^* \pm xT_2(x) \in Z(R), \quad \text{for all } x \in R. \tag{2.48}$$

Linearizing (2.48), we have

$$T_1(x)y^* + T_2(y)x^* \pm xT_2(y) \pm yT_2(x) \in Z(R), \quad \text{for all } x, y \in R. \tag{2.49}$$

Replacing x, y by kx, ky in (2.49) where $k \in S(R) \cap Z(R)$ and subtracting it from (2.49), we get

$$-2(T_1(x)y^* + T_2(y)x^*)k^2 \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

This implies that

$$2(T_1(x)y^* + T_2(y)x^*)k^2 \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get

$$T_1(x)y^* + T_2(y)x^* \in Z(R), \quad \text{for all } x, y \in R. \tag{2.50}$$

Replacing y by yk in (2.50), where $k \in S(R) \cap Z(R)$ and using (2.50), we get

$$2T_2(y)x^*k \in Z(R), \quad \text{for all } x, y \in R \text{ and } k \in S(R) \cap Z(R).$$

Taking $x = h$, where $h \in H(R) \cap Z(R)$, we get $2T_2(y)hk \in Z(R)$ for all $y \in R$, $h \in H(R) \cap Z(R)$ and $k \in S(R) \cap Z(R)$. Since char(R) $\neq 2$ and $S(R) \cap Z(R) \neq (0)$, we get $T_2(y) \in Z(R)$ for all $y \in R$. This can be further written as

$$[T_2(y), r] = 0, \quad \text{for all } y, r \in R.$$

Replacing y by ym, where $m \in R$, we get $T_2(y)[m, r] = 0$ for all $y, m, r \in R$. Further, replacing y by yw, where $w \in R$, we get $T_2(y)w[m, r] = 0$ for all $y, w, m, r \in R$. Then by primeness, we get either $T_2 = 0$ or $[m, r] = 0$ for all $m, r \in R$. Since $T_2 \neq 0$, therefore we only have $[m, r] = 0$ for all $m, r \in R$. That is, R is commutative. □

References

[1] S. Ali and N. A. Dar, *On centralizers of prime rings with involution*, Bull. Iranian Math. Soc. **41**(6) (2015), 1465–1475.

[2] A. Ali and M. Yasen, *A note on automorphisms of prime and semiprime rings*, Kyoto J. Math. **45**(2) (2005), 243–246.

[3] M. Ashraf and S. Ali, *On left multipliers and the commutativity of prime rings*, Demonstr. Math. **41**(4) (2008), 764–771.

[4] K. I. Beider, W. S. Martindale and A. V. Mikhalev, *Rings with Generalized Identities*, Monographs and Textbooks in Pure and Applied Mathematics 196(11), Marcel Dekker, New York, 1996.

[5] N. Divinsky, *On commuting automorphisms of rings*, Transactions of the Royal Society of Canada. Section 3 **49**(3) (1955), 19–22.

[6] I. R. Hentzel and T. El-Sayiad, *Left centralizers of rings that are not semiprime*, Rocky Mountain J. Math. **41**(5) (2011), 1471–1482.

[7] I. N. Herstein, *Ring with Involution*, University of Chicago Press, Chicago, 1976.

[8] J. Luh, *A note on automorphisms of rings*, Amer. Math. Monthly **77** (1970) 61–62.

[9] B. Nejjar, A. Kacha, A. Mamouni and L. Oukhtite, *Commutativity theorems in rings with involution*, Comm. Algebra **45** (2017), 698–708.

[10] L. Oukhtite, *Left Multipliers and Lie ideals in rings with involution*, Int.J. Open Problems Compt. Math. **3**(3) (2010), 267–277.

[11] L. Oukhtite, *Left multipliers and Jordan ideals in rings with involution*, Afr. Diaspora J. Math. **11**(1) (2011), 24–28.

[12] E. C. Posner, *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093–1100.

[13] M. F. Smiley, *Remarks on commuting automorphisms*, Amer. Math. Monthly **63** (1956), 466–470.

[14] J. Vukman, *An identity related to centralizer in semiprime rings*, Comment. Math. Univ. Carolin. **40** (1999), 447–456.

[15] J. Vukman, *Centralizers on semiprime rings*, Comment. Math. Univ. Carolin. **42**(2) (2001), 237–245.

[16] J. Vukman and K. U. Irena, *On centralizers of semiprime rings with involution*, Studia Sci. Math. Hungar. **41**(1) (2006), 61–67.

[17] B. Zalar, *On centralizer of semiprime rings*, Comment. Math. Univ. Carolin. **32** (1991), 609–614.

[1]Department of Mathematics,
Aligarh Muslim University,
Aligarh, India
Email address: muzibamu81@gmail.com
Email address: adnan.abbasi001@gmail.com

[2]Department of Computer Science and Engineering,
Islamic University of Science and Technology,
Awantipora, Pulwama, Jammu and Kashmir, India
Email address: ndmdarlajurah@gmail.com

[3]Department of Mathematics,
Central University of Kashmir,
Srinagar, Jammu and Kashmir, India
Email address: aftabshahcuk@gmail.com