# CURVATURE PROPERTIES OF GENERALIZED PP-WAVE METRICS

ABSOS ALI SHAIKH[1], TRAN QUOC BINH[2], AND HARADHAN KUNDU[1]

ABSTRACT. The main objective of the present paper is to investigate the curvature properties of generalized pp-wave metrics. It is shown that a generalized pp-wave spacetime is Ricci generalized pseudosymmetric, 2-quasi-Einstein and generalized quasi-Einstein in the sense of Chaki. As a special case it is shown that pp-wave spacetime is semisymmetric, semisymmetric due to conformal and projective curvature tensors, $R$-space by Venzi and satisfies the pseudosymmetric type condition $P \cdot P = -\frac{1}{3}Q(S, P)$. Again we investigate the sufficient condition for which a generalized pp-wave spacetime turns into pp-wave spacetime, pure radiation spacetime, locally symmetric and recurrent. Finally, it is shown that the energy-momentum tensor of pp-wave spacetime is parallel if and only if it is cyclic parallel. Again the energy momentum tensor is Codazzi type if it is cyclic parallel but the converse is not true as shown by an example. Finally, we make a comparison between the curvature properties of the Robinson-Trautman metric and generalized pp-wave metric.

## 1. Introduction

The class of pp-wave metrics (see [29,71]) arose during the study of exact solutions of Einstein's field equations. The term "pp-wave" is given by Ehlers and Kundt [29], where "pp" abbreviates the term "plane-fronted gravitational waves with parallel rays". The term "plane-fronted gravitational waves" means there admit a geodesic null vector field whose twist, expansion and shear are zero. The term "plane rays" implies that the rotation of the vector field vanishes. For vacuum type N, this ensures

the existence of a covariantly constant vector field which is parallel to the null vector field. There are various forms of generalized pp-wave metrics in different coordinates. The pp-wave belongs to the class of solutions admitting a non-expanding, shear-free and twist-free null congruence and it admits a null Killing vector.

The family of pp-wave space-times was first discussed by Brinkmann [4] and interpreted in terms of gravitational waves by Peres [46]. According to Brinkmann, a pp-wave spacetime is any Lorentzian manifold whose metric tensor can be described, with respect to Brinkmann coordinates, in the form

$$(1.1) \qquad\qquad ds^2 = H(u, x, y)du^2 + 2dudv + dx^2 + dy^2,$$

where $H$ is any nowhere vanishing smooth function. Again it is well known that a Lorentzian manifold with parallel lightlike (null) vector field is called Brinkmann-wave ([4, 30]). A Brinkmann-wave is called pp-wave if its curvature tensor $R$ satisfies the condition $R_{ij}{}^{pq} R_{pqkl} = 0$ ([30, 44, 69]). In 1984, Radhakrishna and Singh [47] presented a class of solutions to Einstein-Maxwell equation for the null electrovac Petrov type $N$ gravitational field. They presented a metric of the form

$$ds^2 = -2U du^2 + 2dudr - \frac{1}{2}P[(dx^3)^2 + (dx^4)^2],$$

where $U = U(u, x^3, x^4)$ and $P = P(x^3, x^4)$ are two nowhere vanishing smooth functions. For the simplicity of notation, we write the variable $u$ as $x$, and the function $U$ as $h$ and $P$ as $f$. Then the aforesaid metric can be written as

$$(1.2) \qquad ds^2 = -2h(x, x^3, x^4)(dx)^2 + 2dxdr - \frac{1}{2}f(x^3, x^4)[(dx^3)^2 + (dx^4)^2].$$

In Section 3 we show that the metric (1.2) admits a covariantly constant null vector field and it satisfies the condition $R_{ij}{}^{pq} R_{pqkl} = 0$ if

$$(1.3) \qquad\qquad f_3^2 + f_4^2 - f(f_{33} + f_{44}) = 0.$$

Thus we can say that the metric (1.2) is a Brinkmann-wave and it becomes a pp-wave if (1.3) holds. Hence we can say the metric (1.2) as "generalized pp-wave metric". We note that for $f \equiv -2$ and $h = -\frac{1}{2}H(u, x^3, x^4)$, the solution (1.2) reduces to the pp-wave metric (1.1).

In the study of differential geometry the notion of manifold of constant curvature has been generalized by many authors in different directions such as locally symmetric manifold by Cartan [5], semisymmetric manifold by Cartan [6] (see also [73–75]), pseudosymmetric manifold by Adamów and Deszcz [1] (see also [15, 22, 24, 34, 35, 78–80]), recurrent manifold by Ruse ([49–51], see also [82]), weakly generalized recurrent manifold by Shaikh and Roy ([52, 66]), hyper generalized recurrent manifold by Shaikh and Patra ([65, 68]), super generalized recurrent manifold by Shaikh et al. [67]. We note that in such geometric structures the first order and second order covariant derivatives of the Riemann curvature tensor and other curvature tensors are involved. We mention that the notion of pseudosymmetry by Deszcz is important in the study of differential geometry due to its application in relativity and cosmology (see [14, 23, 28] and also

references therein). The notion of pseudosymmetry is extended by considering other curvature tensors in its defining condition, such as conformal pseudosymmetry, pseudosymmetric Weyl conformal curvature tensor, Ricci generalized pseudosymmetry etc. and they are called pseudosymmetric type conditions. It may be mentioned that different pseudosymmetric type conditions are admitted by various spacetimes, such as Gödel spacetime ([26, 32]), Som-Raychaudhuri spacetime ([57, 70]), Reissner-Nordström spacetime [37], Vaidya spacetime [63] and Robertson-Walker spacetime ([2, 13, 27]).

The main object of the present paper is to investigate the geometric structures admitted by the generalized pp-wave metric (1.2). It is interesting to note that the metric (1.2) without any other condition admits several geometric structures, such as Ricci generalized pseudosymmetry, 2-quasi Einstein and generalized quasi-Einstein in the sense of Chaki [8]. Again it is shown that the pp-wave metric (i.e., (1.2) with condition (1.3)) is Ricci recurrent but not recurrent, semisymmetric, $R$-space by Venzi, conformal curvature 2-forms are recurrent, Ricci tensor is Riemann compatible and fulfills a pseudosymmetric type condition due to the projective curvature tensor $P$. For the study of pseudosymmetric type conditions with projective curvature tensor we refer the reader to see the recent papers of Shaikh and Kundu ([58, 59]). It is interesting to note that for such a metric $P \cdot R = 0$ but $P \cdot \mathcal{R} \neq 0$.

It is also shown that the metric is weakly Ricci symmetric and weakly cyclic Ricci symmetric for different associated 1-forms, which ensures the existence of infinitely many solutions of associated 1-forms of such structures. Again we investigate the condition for which such a spacetime is locally symmetric and recurrent.

The paper is organized as follows. Section 2 deals with defining conditions of different curvature restricted geometric structures, such as recurrent, semisymmetry, pseudosymmetry, weakly symmetry etc. as preliminaries. Section 3 is devoted to the investigation of curvature restricted geometric structures admitted by the generalized pp-wave metric (1.2). Section 4 is mainly concerned with the geometric structures admitted by pp-wave metric and plane wave metric. Section 5 deals with the investigation of the conditions under which the energy-momentum tensor of such spacetimes are parallel, Codazzi type and cyclic parallel. Finally, the last section is devoted to make a comparison between the curvature properties of the Robinson-Trautman metric and generalized pp-wave metric as well as pp-wave metric.

## 2. Preliminaries

Let $M$ be a connected smooth semi-Riemannian manifold of dimension $n$ ($\geq 3$) equipped with the semi-Riemannian metric $g$. Let $R$, $\mathcal{R}$, $S$, $\mathcal{S}$ and $\kappa$ be respectively the Riemann-Christoffel curvature tensor of type $(0, 4)$, the Riemann-Christoffel curvature tensor of type $(1, 3)$, the Ricci tensor of type $(0, 2)$, the Ricci tensor of type $(1, 1)$ and the scalar curvature of $M$.

For two symmetric $(0, 2)$-tensors $A$ and $E$, their Kulkarni-Nomizu product $A \wedge E$

is defined as (see e.g. [22, 33]):

$$(A \wedge E)(X_1, X_2, X_3, X_4) = A(X_1, X_4)E(X_2, X_3) + A(X_2, X_3)E(X_1, X_4)$$
$$- A(X_1, X_3)E(X_2, X_4) - A(X_2, X_4)E(X_1, X_3),$$

where $X_1, X_2, X_3, X_4 \in \chi(M)$, the Lie algebra of all smooth vector fields on $M$. Throughout the paper we will consider $X, Y, X_1, X_2, \ldots \in \chi(M)$.

In terms of Kulkarni-Nomizu product, the conformal curvature tensor $C$, the concircular curvature tensor $W$, the conharmonic curvature tensor $K$ ([36, 83]) and the Gaussian curvature tensor $\mathfrak{G}$ can be expressed as

$$C = R - \frac{1}{n-2}(g \wedge S) + \frac{\kappa}{2(n-2)(n-1)}(g \wedge g),$$
$$W = R - \frac{r}{2n(n-1)}(g \wedge g),$$
$$K = R - \frac{1}{n-2}(g \wedge S),$$
$$\mathfrak{G} = \frac{1}{2}(g \wedge g).$$

Again the projective curvature tensor $P$ of type $(0,4)$ is given by

$$P(X_1, X_2, X_3, X_4) = R(X_1, X_2, X_3, X_4)$$
$$- \frac{1}{n-1}[g(X_1, X_4)S(X_2, X_3) - g(X_2, X_4)S(X_1, X_3)].$$

For a symmetric $(0,2)$-tensor $A$, we get an endomorphism $\mathcal{A}$ defined by $g(\mathcal{A}X, Y) = A(X, Y)$. Then its $k$-th level tensor $A^k$ of type $(0,2)$ is given by

$$A^k(X, Y) = A(\mathcal{A}^{k-1}X, Y),$$

where $\mathcal{A}^{k-1}$ is the endomorphism corresponding to $A^{k-1}$, $k = 2, 3, \ldots$, and $A^1 = A$.

**Definition 2.1** ([3, 60])**.** A semi-Riemannian manifold $M$ is said to be Ein(2), Ein(3) and Ein(4) respectively if

$$S^2 + \lambda_1 S + \lambda_2 g = 0,$$
$$S^3 + \lambda_3 S^2 + \lambda_4 S + \lambda_5 g = 0,$$
$$S^4 + \lambda_6 S^3 + \lambda_7 S^2 + \lambda_8 S + \lambda_9 g = 0,$$

holds for some scalars $\lambda_i$, $1 \le i \le 9$.

**Definition 2.2** ([3, 60])**.** A semi-Riemannian manifold $M$ is said to be generalized Roter type ([56, 60]) if its Riemann-Christoffel curvature tensor $R$ can be expressed as a linear combination of $g \wedge g$, $g \wedge S$, $S \wedge S$, $g \wedge S^2$, $S \wedge S^2$ and $S^2 \wedge S^2$. Again $M$ is said to be Roter type (see [16, 17]) if $R$ can be expressed as a linear combination of $g \wedge g$, $g \wedge S$ and $S \wedge S$.

Again for a $(0,4)$-tensor $D$, an endomorphism $\mathscr{D}(X,Y)$ and the corresponding $(1,3)$-tensor $\mathcal{D}$ can be defined as

$$\mathscr{D}(X,Y)X_1 = \mathcal{D}(X,Y)X_1, \quad D(X_1,X_2,X_3,X_4) = g(\mathcal{D}(X_1,X_2)X_3,X_4).$$

Again for a symmetric $(0,2)$-tensor $A$, another endomorphism $X \wedge_A Y$ (see [13, 22]) can be defined as

$$(X \wedge_A Y)X_1 = A(Y,X_1)X - A(X,X_1)Y.$$

By operating $\mathscr{D}(X,Y)$ and $X \wedge_A Y$ on a $(0,k)$-tensor $B$, $k \geq 1$, we can obtain two $(0,k+2)$-tensors $D \cdot B$ and $Q(A,B)$ respectively given by (see [18, 19, 25, 54, 55] and also references therein):

$$\begin{aligned}
&D \cdot B(X_1,X_2,\ldots,X_k,X,Y)\\
&= -B(\mathcal{D}(X,Y)X_1,X_2,\ldots,X_k) - \cdots - B(X_1,X_2,\ldots,\mathcal{D}(X,Y)X_k).
\end{aligned}$$

and

$$\begin{aligned}
&Q(A,B)(X_1,X_2,\ldots,X_k,X,Y)\\
&=((X \wedge_A Y) \cdot B)(X_1,X_2,\ldots,X_k)\\
&=A(X,X_1)B(Y,X_2,\ldots,X_k) + \cdots + A(X,X_k)B(X_1,X_2,\ldots,Y)\\
&\quad - A(Y,X_1)B(X,X_2,\ldots,X_k) - \cdots - A(Y,X_k)B(X_1,X_2,\ldots,X).
\end{aligned}$$

In terms of local coordinates system, $D \cdot B$ and $Q(A,B)$ can be written as

$$\begin{aligned}
(D \cdot B)_{i_1 i_2 \cdots i_k jl} &= -g^{pq}\left[D_{jli_1q}B_{pi_2\cdots i_k} + \cdots + D_{jli_kq}B_{i_1i_2\cdots p}\right],\\
Q(A,B)_{i_1 i_2 \cdots i_k jl} &= A_{li_1}B_{ji_2\cdots i_k} + \cdots + A_{li_k}B_{i_1i_2\cdots j}\\
&\quad - A_{ji_1}B_{li_2\cdots i_k} - \cdots - A_{ji_k}B_{i_1i_2\cdots l}.
\end{aligned}$$

**Definition 2.3** ([45, 49–51]). A semi-Riemannian manifold $M$ is said to be $B$-recurrent if $\nabla B = \Pi \otimes B$ for an 1-form $\Pi$. In particular for $B = R$ (resp., $S$), the manifold $M$ is called a recurrent (resp., Ricci recurrent) manifold.

**Definition 2.4** ([1, 6, 15, 55, 73]). A semi-Riemannian manifold $M$ is said to be $B$-semisymmetric type if $D \cdot B = 0$ and it is said to be $B$-pseudosymmetric type if $\left(\sum_{i=1}^{k} c_i D_i\right) \cdot B = 0$ for some scalars $c_i$'s, where $D$ and each $D_i$, $i = 1, \ldots, k$, $k \geq 2$, are $(0,4)$ curvature tensors. In particular, if $c_i$'s are all constants, then it is called $B$-pseudosymmetric type manifold of constant type or otherwise non-constant type.

In particular, if $i = 2$, $D_1 = R$, $D_2 = \mathfrak{G}$ and $B = R$, then $M$ is called Ricci generalized pseudosymmetric [12].

**Definition 2.5.** A semi-Riemannian manifold $M$ is said to be quasi-Einstein (resp. 2-quasi-Einstein) if at each point of $M$, $\mathrm{rank}(S - \alpha g) \leq 1$ (resp., $\leq 2$) for a scalar $\alpha$. Also $M$ is said to be generalized quasi-Einstein in the sense of Chaki [8] if

$$S = \alpha g + \beta \Pi \otimes \Pi + \gamma[\Pi \otimes \Omega + \Pi \otimes \Omega],$$

for some 1-forms $\Pi$ and $\Omega$.

Quasi-Einstein, as well as 2-quasi-Einstein manifolds were investigated among others in $[2, 9\text{--}11, 19, 21, 22]$ and $[25]$.

**Definition 2.6.** Let $D$ be a $(0,4)$-tensor and $E$ be a symmetric $(0,2)$-tensor on a semi-Riemannian manifold $M$. Then $E$ is said to be $D$-compatible $([20, 39, 40])$ if

$$D(\mathcal{E}X_1, X, X_2, X_3) + D(\mathcal{E}X_2, X, X_3, X_1) + D(\mathcal{E}X_3, X, X_1, X_2) = 0$$

holds, where $\mathcal{E}$ is the endomorphism corresponding to $E$ defined as $g(\mathcal{E}X_1, X_2) = E(X_1, X_2)$. Again an 1-form $\Pi$ is said to be $D$-compatible if $\Pi \otimes \Pi$ is $D$-compatible.

**Definition 2.7.** A semi-Riemannian manifold $M$ is said to be weakly cyclic Ricci symmetric [64] if its Ricci tensor satisfies the condition

$$(2.1) \qquad (\nabla_X S)(X_1, X_2) + (\nabla_{X_1} S)(X, X_2) + (\nabla_{X_2} S)(X_1, X)$$
$$= \Pi(X) \, S(X_1, X_2) + \Omega(X_1) \, S(X, X_2) + \Theta(X_2) \, S(X_1, X),$$

for three 1-forms $\Pi$, $\Omega$ and $\Theta$ on $M$. Such a manifold is called weakly cyclic Ricci symmetric manifold with solution $(\Pi, \Omega, \Theta)$. Moreover if the first term of left hand side is equal to the right hand side, then it is called weakly Ricci symmetric manifold [77].

**Definition 2.8.** Let $D$ be a $(0,4)$ tensor and $Z$ be a $(0,2)$-tensor on a semi-Riemannian manifold $M$. Then the corresponding curvature 2-forms $\Omega^m_{(D)l}$ (see [3, 38]) are called recurrent if and only if $([41\text{--}43])$

$$(\nabla_{X_1} D)(X_2, X_3, X, Y) + (\nabla_{X_2} D)(X_3, X_1, X, Y) + (\nabla_{X_3} D)(X_1, X_2, X, Y)$$
$$= \Pi(X_1) D(X_2, X_3, X, Y) + \Pi(X_2) D(X_3, X_1, X, Y) + \Pi(X_3) D(X_1, X_2, X, Y),$$

and 1-forms $\Lambda_{(Z)l}$ [72] are called recurrent if and only if

$$(\nabla_{X_1} Z)(X_2, X) - (\nabla_{X_2} Z)(X_1, X) = \Pi(X_1) Z(X_2, X) - \Pi(X_2) Z(X_1, X),$$

for an 1-form $\Pi$.

**Definition 2.9** ([48, 59, 81]). Let $\mathcal{L}(M)$ be the vector space formed by all 1-forms $\Theta$ on a semi-Riemannian manifold $M$ satisfying

$$\Theta(X_1) D(X_2, X_3, X_4, X_5) + \Theta(X_2) D(X_3, X_1, X_4, X_5) + \Theta(X_3) D(X_1, X_2, X_4, X_5) = 0,$$

where $D$ is a $(0,4)$-tensor. Then $M$ is said to be a $D$-space by Venzi if $\dim \mathcal{L}(M) \geq 1$.

## 3. Curvature Properties of Generalized pp-Wave Metric

We can now write the metric tensor $g$ of the generalized pp-wave metric (1.2) as follows:

$$g = \begin{pmatrix} -2h & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}f & 0 \\ 0 & 0 & 0 & -\frac{1}{2}f \end{pmatrix}.$$

Then the non-zero components of its Riemann-Christoffel curvature tensor $R$, Ricci tensor $S$ and scalar curvature $\kappa$ of (1.2) are given by

$$R_{1313} = \frac{-f_3 h_3 + f_4 h_4 + 2f h_{33}}{2f}, \quad R_{1314} = \frac{-f_4 h_3 - f_3 h_4 + 2f h_{34}}{2f},$$

$$R_{1414} = \frac{f_3 h_3 - f_4 h_4 + 2f h_{44}}{2f}, \quad R_{3434} = \frac{-f_3^2 - f_4^2 + f f_{33} + f f_{44}}{4f},$$

$$S_{11} = \frac{2\left(h_{33} + h_{44}\right)}{f}, \quad S_{33} = S_{44} = \frac{-f_3^2 - f_4^2 + f f_{33} + f f_{44}}{2f^2},$$

$$\kappa = \frac{2\left(f_3^2 + f_4^2 - f\left(f_{33} + f_{44}\right)\right)}{f^3},$$

where $f_3 = \frac{\partial f}{\partial x^3}$, $f_4 = \frac{\partial f}{\partial x^4}$, $f_{33} = \frac{\partial f_3}{\partial x^3}$, $f_{34} = \frac{\partial f_3}{\partial x^4}$, $f_{44} = \frac{\partial f_4}{\partial x^4}$ etc.
Again the non-zero components of the conformal curvature tensor $C$ and the projective curvature tensor $P$ are given by

$$C_{1313} = -C_{1414} = \frac{-f_3 h_3 + f_4 h_4 + f h_{33} - f h_{44}}{2f}, \quad C_{1314} = \frac{-f_4 h_3 - f_3 h_4 + 2f h_{34}}{2f},$$

$$P_{1211} = \frac{2\left(h_{33} + h_{44}\right)}{3f}, \quad P_{1313} = \frac{-3f_3 h_3 + 3f_4 h_4 + 4f h_{33} - 2f h_{44}}{6f},$$

$$P_{1314} = -P_{1341} = P_{1413} = -P_{1431} = \frac{-f_4 h_3 - f_3 h_4 + 2f h_{34}}{2f},$$

$$P_{1441} = -\frac{f_3 h_3 - f_4 h_4 + 2f h_{44}}{2f},$$

$$P_{1331} = -\frac{-f_3 h_3 + f_4 h_4 + 2f h_{33}}{2f}, \quad P_{1414} = -\frac{-3f_3 h_3 + 3f_4 h_4 + 2f h_{33} - 4f h_{44}}{6f}.$$

Now the non-zero components (upto symmetry) of the energy momentum tensor

$$T = \frac{c^4}{8\pi G}\left[S - \left(\frac{\kappa}{2} - \Lambda\right)g\right],$$

where $c$ = speed of light in vacuum, $G$ = gravitational constant and $\Lambda$ = cosmological constant, are given by

$$T_{11} = -\frac{c^4\left(f^3 h\Lambda - f^2 h_{33} - f^2 h_{44} + f_{33}fh + f_{44}fh - f_3^2 h - f_4^2 h\right)}{4\pi f^3 G},$$

$$T_{12} = \frac{c^4\left(f^3\Lambda + f_{33}f + f_{44}f - f_3^2 - f_4^2\right)}{8\pi f^3 G}, \quad T_{33} = T_{44} = -\frac{c^4 f\Lambda}{16\pi G}.$$

Then the non-zero components (upto symmetry) of covariant derivative $\nabla T$ of the energy momentum tensor $T$ are given by

$$T_{11,1} = \frac{c^4 (h_{133} + h_{144})}{4\pi f G},$$

$$T_{11,3} = \frac{c^4}{4\pi f^4 G} \left( -f^2 h f_{344} - f^2 h f_{333} + f^3 h_{333} + f^3 h_{344} - f_3 f^2 h_{33} - f_3 f^2 h_{44} \right.$$
$$\left. + 4 f_3 f_{33} f h + 2 f_4 f_{34} f h + 2 f_3 f_{44} f h - 3 f_3^3 h - 3 f_3 f_4^2 h \right),$$

$$T_{11,4} = \frac{c^4}{4\pi f^4 G} \left( -f^2 h f_{444} - f^2 h f_{344} + f^3 h_{334} + f^3 h_{444} - f_4 f^2 h_{33} - f_4 f^2 h_{44} \right.$$
$$\left. + 2 f_4 f_{33} f h + 2 f_3 f_{34} f h + 4 f_4 f_{44} f h - 3 f_4^3 h - 3 f_3^2 f_4 h \right),$$

$$T_{12,3} = \frac{c^4 \left( f^2 f_{344} + f^2 f_{333} + 3 f_3^3 + 3 f_4^2 f_3 - 4 f f_{33} f_3 - 2 f f_{44} f_3 - 2 f f_4 f_{34} \right)}{8\pi f^4 G},$$

$$T_{12,4} = \frac{c^4 \left( f^2 f_{444} + f^2 f_{334} + 3 f_4^3 + 3 f_3^2 f_4 - 2 f f_{33} f_4 - 4 f f_{44} f_4 - 2 f f_3 f_{34} \right)}{8\pi f^4 G},$$

$$T_{13,1} = \frac{c^4 \left( -f_3^2 - f_4^2 + f f_{33} + f f_{44} \right) h_3}{8\pi f^3 G},$$

$$T_{14,1} = \frac{c^4 \left( -f_3^2 - f_4^2 + f f_{33} + f f_{44} \right) h_4}{8\pi f^3 G}.$$

From above we see that the Ricci tensor $S$ of (1.2) is of the form

(3.1) $$S(X,Y) = \alpha g(X,Y) + \beta \eta(X)\eta(Y) + \gamma[\eta(X)\delta(Y) + \eta(Y)\delta(X)],$$

where $\alpha = \frac{f_3^2 + f_4^2 - f f_{33} - f f_{44}}{f^3}$, $\beta = 1$, $\gamma = 1$, $\eta = \{1, 0, 0, 0\}$ and

$$\delta = \left\{ \frac{2 f^2 (h_{33} + h_{44}) - 2 (f_{33} + f_{44}) f h + 2 (f_3^2 + f_4^2) h - f^3}{2 f^3}, \frac{f (f_{33} + f_{44} - f_3^2 - f_4^2)}{f^3}, 0, 0 \right\}.$$

Therefore, the metric (1.2) is generalized quasi-Einstein in the sense of Chaki. Moreover, $||\eta|| = 0$, $||\delta||^2 = \frac{(f_3^2 + f_4^2 - f(f_{33}+f_{44}))(f - 2(h_{33}+h_{44}))}{f^4}$, $g(\eta, \delta) = -\frac{f_3^2 + f_4^2 - f(f_{33}+f_{44})}{f^3}$ and $\nabla \eta = 0$. So there exists a null covariantly constant vector field $\zeta$, where $\zeta$ is the corresponding vector field of $\eta$ (i.e., $g(\zeta, X) = \eta(X)$, for all $X$). Hence we can conclude that the spacetime with the metric (1.2) is a generalized pp-wave metric.

Now from the value of the components of various tensors related to (1.2), we can state the following.

**Theorem 3.1.** *The generalized pp-wave metric* (1.2) *possesses the following curvature restricted geometric structures:*

(i) *Ricci generalized pseudosymmetric such that* $R \cdot R = Q(S, R)$;

(ii) *2-quasi-Einstein, since* $\mathrm{rank} \left( S - \frac{f_3^2 + f_4^2 - f(f_{33}+f_{44})}{f^3} g \right) = 2$;

(iii) *generalized quasi-Einstein in the sense of Chaki such that* (3.1) *holds;*

(iv) *Ricci tensor is Riemann compatible as well as Weyl compatible;*

(v) Ein(3) *manifold such that* $S^3 = -\frac{f_3^2 + f_4^2 - f(f_{33} + f_{44})}{f^3} S^2$.

Now from the components of $R$, we see that the only non-zero component (upto symmetry) of the tensor $D_{ijkl} = R_{ij}^{pq} R_{pqkl}$ is given by $D_{3434} = \frac{\left(f_3^2 + f_4^2 - f(f_{33} + f_{44})\right)^2}{2f^4}$. Hence from definition, we can state the following.

**Theorem 3.2.** *The generalized pp-wave metric* (1.2) *becomes a pp-wave metric if*

$$f_3^2 + f_4^2 - f(f_{33} + f_{44}) = 0.$$

*Example* 3.1. If we consider $f(x^3, x^4) = e^{x^3 + x^4}$, then $f_3^2 + f_4^2 - f(f_{33} + f_{44}) = 0$. Hence by above theorem the metric

$$ds^2 = -2h(x, x^3, x^4)(dx)^2 + 2dxdr - \frac{1}{2}e^{x^3 + x^4}[(dx^3)^2 + (dx^4)^2]$$

is a pp-wave metric.

## 4. CURVATURE PROPERTIES OF PP-WAVE AND PLANE WAVE METRIC

In this section we investigate the curvature restricted geometric structures admitted by the pp-wave metric. Since under the condition (1.3) the generalized pp-wave metric (1.2) becomes a pp-wave metric, putting this condition we get the non-zero components of $R$, $S$, $C$ and $P$ of the pp-wave metric given as follows:

$$R_{1313} = \frac{-f_3 h_3 + f_4 h_4 + 2fh_{33}}{2f}, \quad R_{1314} = \frac{-f_4 h_3 - f_3 h_4 + 2fh_{34}}{2f},$$

$$R_{1414} = \frac{f_3 h_3 - f_4 h_4 + 2fh_{44}}{2f},$$

$$S_{11} = \frac{2(h_{33} + h_{44})}{f},$$

$$C_{1313} = -C_{1414} = \frac{-f_3 h_3 + f_4 h_4 + fh_{33} - fh_{44}}{2f}, \quad C_{1314} = \frac{-f_4 h_3 - f_3 h_4 + 2fh_{34}}{2f},$$

$$P_{1211} = \frac{2(h_{33} + h_{44})}{3f}, \quad P_{1313} = \frac{-3f_3 h_3 + 3f_4 h_4 + 4fh_{33} - 2fh_{44}}{6f},$$

$$P_{1314} = -P_{1341} = P_{1413} = -P_{1431} = \frac{-f_4 h_3 - f_3 h_4 + 2fh_{34}}{2f},$$

$$P_{1441} = -\frac{f_3 h_3 - f_4 h_4 + 2fh_{44}}{2f},$$

$$P_{1331} = -\frac{-f_3 h_3 + f_4 h_4 + 2fh_{33}}{2f}, \quad P_{1414} = -\frac{-3f_3 h_3 + 3f_4 h_4 + 2fh_{33} - 4fh_{44}}{6f}.$$

Using the values of the components of $g$, $R$, $S$ and $C$ we get
(i) $\kappa = 0$, (ii) $R \cdot R = 0$, (iii) $R \cdot S = 0$, (iv) $Q(S, R) = 0$, (v) $R \cdot C = 0$, (vi) $C \cdot R = 0$,

(vii) $C \cdot C = 0$ and (viii) $Q(S, C) = 0$.

The energy momentum tensor $T$ is given by

$$(4.1) \qquad T_{11} = -\frac{c^4 (fh\Lambda - h_{33} - h_{44})}{4\pi fG}, \quad T_{12} = \frac{c^4 \Lambda}{8\pi G}, \quad T_{33} = T_{44} = -\frac{c^4 f\Lambda}{16\pi G}.$$

Then the non-zero components of covariant derivative of $T$ are given by

$$(4.2) \qquad \begin{cases} T_{11,1} = \dfrac{c^4 (h_{144} + h_{133})}{4\pi fG}, \\[2mm] T_{11,3} = \dfrac{c^4 (fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44})}{4\pi f^2 G}, \\[2mm] T_{11,4} = \dfrac{c^4 (fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44})}{4\pi f^2 G}. \end{cases}$$

From the above calculations, we can state the following.

**Theorem 4.1.** *The pp-wave metric (1.2) with the additional condition (1.3) possesses the following curvature properties.*

(i) *$\kappa = 0$ and hence $R = W$ and $C = K$.*

(ii) *$R$-space and $C$-space by Venzi for $\{1, 0, 0, 0\}$. Hence, from second Bianchi identity, the curvature 2-forms $\Omega^m_{(R)l}$ are recurrent for the 1-form $\{1, 0, 0, 0\}$.*

(iii) *It is semisymmetric and hence Ricci semisymmetric, conformally semisymmetric and projectively semisymmetric.*

(iv) *If $\alpha_3 + \alpha_4 \neq 0$, then the conformal 2-forms $\Omega^m_{(C)l}$ are recurrent with 1-form of recurrency*

$$\Pi = \left\{ 1, 0, \frac{\alpha_1 + \alpha_2}{\alpha_3 + \alpha_4}, \frac{\alpha_5 + \alpha_6 + \alpha_7}{\alpha_3 + \alpha_4} \right\},$$

*where*

$$\alpha_1 = (f_3 h_3 - f_4 h_4 - f(h_{33} - h_{44}))(f_3(h_{33} + h_{44}) - f(h_{333} + h_{344})),$$
$$\alpha_2 = (-f_4 h_3 - f_3 h_4 + 2fh_{34})(f(h_{334} + h_{444}) - f_4(h_{33} + h_{44})),$$
$$\alpha_3 = f^2 \left(4h_{34}^2 + (h_{33} - h_{44})^2\right) + f_3^2\left(h_3^2 + h_4^2\right) + f_4^2\left(h_3^2 + h_4^2\right),$$
$$\alpha_4 = 2ff_4\left(h_4(h_{33} - h_{44}) - 2h_3 h_{34}\right) - 2ff_3\left(2h_4 h_{34} + h_3(h_{33} - h_{44})\right),$$
$$\alpha_5 = f^2\left(2h_{34}(h_{333} + h_{344}) + (h_{44} - h_{33})(h_{334} + h_{444})\right)$$
$$\qquad + f_3^2 h_4(h_{33} + h_{44}) + f_4^2 h_4(h_{33} + h_{44}),$$
$$\alpha_6 = ff_3\left(-2h_{34}(h_{33} + h_{44}) - h_4(h_{333} + h_{344}) + h_3(h_{334} + h_{444})\right),$$
$$\alpha_7 = -ff_4\left(-h_{33}^2 + h_{44}^2 + h_3(h_{333} + h_{344}) + h_4(h_{334} + h_{444})\right).$$

(v) *It is not recurrent but if $h_{33} + h_{44} \neq 0$, then it is Ricci recurrent with 1-form of recurrency*

$$\Pi = \left\{ \frac{h_{144} + h_{133}}{h_{33} + h_{44}}, 0, \frac{fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44}}{f(h_{33} + h_{44})}, \frac{fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44}}{f(h_{33} + h_{44})} \right\}.$$

(vi) *If $h_{33}+h_{44} \neq 0$, then it is weakly cyclic Ricci symmetric with solution $(\Pi, \Omega, \Theta)$, given by*

$$\Pi = \left\{\Pi_1, 0, \frac{f(h_{333}+h_{344}) - f_3(h_{33}+h_{44})}{f(h_{33}+h_{44})}, \frac{f(h_{334}+h_{444}) - f_4(h_{33}+h_{44})}{f(h_{33}+h_{44})}\right\},$$

$$\Omega = \left\{\Omega_1, 0, \frac{f(h_{333}+h_{344}) - f_3(h_{33}+h_{44})}{f(h_{33}+h_{44})}, \frac{f(h_{334}+h_{444}) - f_4(h_{33}+h_{44})}{f(h_{33}+h_{44})}\right\},$$

$$\Theta = \left\{\frac{3(h_{133}+h_{144})}{h_{33}+h_{44}} - \Pi_1 - \Omega_1, 0, \frac{f(h_{333}+h_{344}) - f_3(h_{33}+h_{44})}{f(h_{33}+h_{44})},\right.$$
$$\left.\frac{f(h_{334}+h_{444}) - f_4(h_{33}+h_{44})}{f(h_{33}+h_{44})}\right\},$$

*where $\Pi_1$ and $\Omega_1$ are arbitrary scalars.*

(vii) *Ricci simple (i.e., $S = \alpha\eta \otimes \eta$) for*

$$\alpha = \frac{2(h_{33}+h_{44})}{f} \quad and \quad \eta = \{1,0,0,0\}$$

*and hence $S \wedge S = 0$ and $S^2 = 0$. Again, $||\eta|| = 0$ and $\nabla\eta = 0$.*

(viii) $Q(S,R) = Q(S,C) = 0$ *but $R$ or $C$ is not a scalar multiple of $S \wedge S$ as $S$ is of rank* 1.

(ix) $C \cdot R = 0$ *and hence $C \cdot S = 0$, $C \cdot C = 0$ and $C \cdot P = 0$.*

(x) $P \cdot R = 0$ *but $P \cdot \mathcal{R} \neq 0$. Also but $P \cdot S = P \cdot \mathcal{S} = 0$.*

(xi) *Ricci tensor is Riemann compatible as well as Weyl compatible.*

(xii) $P \cdot P = -\frac{1}{3}Q(S,P)$.

*Remark* 4.1. From the value of the local components (presented in Section 4) of various tensors of the pp-wave metric, we can easily conclude that the metric is

(i) not conformally symmetric and hence not locally symmetric or projectively symmetric;

(ii) not conformally recurrent and hence not recurrent or not projectively recurrent;

(iii) not super generalized recurrent [67] and hence not hyper generalized recurrent [65], weakly generalized recurrent [66];

(iv) not weakly symmetric [76] for $R, C, P, W$ and $K$ and hence not Chaki pseudosymmetric [7] for $R, C, P, W$ and $K$;

(v) neither cyclic Ricci parallel [31] nor of Codazzi type Ricci tensor although its scalar curvature is constant;

(vi) not harmonic, i.e., div $R \neq 0$ and moreover div $C \neq 0$, div $P \neq 0$.

*Remark* 4.2. In [12] it was shown that if $Q(S,R) = 0$, then $R = LS \wedge S$ for some scalar $L$ if $S$ is not of rank 1. Recently, in Example 1 of [59] a metric with $Q(S,R) = 0$, on which $S$ is not of rank 1 and $R = e^{x^1} S \wedge S$ is presented. It is interesting to mention that the rank of the Ricci tensor of the pp-wave metric is 1 and here $R \neq 0$ but $S \wedge S = 0$.

*Remark* 4.3. It is well-known that every Ricci recurrent space with $\Pi$ as the 1-form of recurrency, is weakly Ricci symmetric with solution $(\Pi, 0, 0)$. It is interesting to mention that there may infinitely many solutions for a weakly Ricci symmetric manifold. The pp-wave metric ((1.2) with condition (1.3)) is weakly Ricci symmetric with solution

$$\Pi = \left\{ \Pi_1, 0, \frac{f\left(h_{333} + h_{344}\right) - f_3\left(h_{33} + h_{44}\right)}{f\left(h_{33} + h_{44}\right)}, \frac{f\left(h_{334} + h_{444}\right) - f_4\left(h_{33} + h_{44}\right)}{f\left(h_{33} + h_{44}\right)} \right\},$$
$$\Omega = \left\{ \Omega_1, 0, 0, 0 \right\},$$
$$\Theta = \left\{ \frac{h_{133} + h_{144}}{h_{33} + h_{44}} - \Pi_1 - \Omega_1, 0, 0, 0 \right\},$$

where $\Pi_1$ and $\Omega_1$ are arbitrary scalars.

Again it is clear that the pp-wave metric (1.1) in Brinkmann coordinates, is a special case of (1.2) for $f \equiv -2$ and $h = -\frac{1}{2}H$ and hence satisfies (1.3). Therefore from Theorem 4.1, we can state the following about the geometric properties of the metic (1.1).

**Corollary 4.1.** *The metric given in* (1.1) *possesses the following curvature properties.*

(i) $\kappa = 0$ *and hence* $R = W$ *and* $C = K$.

(ii) *$R$-space and $C$-space by Venzi for* $\{1, 0, 0, 0\}$. *Hence, from second Bianchi identity, the curvature 2-forms $\Omega_{(R)l}^m$ are recurrent for the 1-form* $\{1, 0, 0, 0\}$.

(iii) *Semisymmetric and hence Ricci semisymmetric, conformally semisymmetric and projectively semisymmetric.*

(iv) *If* $4H_{34}^2 + (H_{33} - H_{44})^2 \neq 0$, *then its conformal curvature 2-forms $\Omega_{(C)l}^m$ are recurrent with 1-form of recurrency $\Pi$, given by*

$$\Pi_1 = 1, \quad \Pi_2 = 0,$$
$$\Pi_3 = \frac{2H_{34}\left(H_{334} + H_{444}\right) + \left(H_{33} - H_{44}\right)\left(H_{333} + H_{344}\right)}{4H_{34}^2 + \left(H_{33} - H_{44}\right)^2},$$
$$\Pi_4 = \frac{2H_{34}\left(H_{333} + H_{344}\right) - \left(H_{33} - H_{44}\right)\left(H_{334} + H_{444}\right)}{4H_{34}^2 + \left(H_{33} - H_{44}\right)^2}.$$

(v) *If* $H_{33} + H_{44} \neq 0$, *then it is Ricci recurrent with 1-form of recurrency $\Pi$, given by*

$$\Pi = \left\{ 1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}} \right\}.$$

(vi) *If $H_{33} + H_{44} \neq 0$, then it is weakly Ricci symmetric with solution $(\Pi, \Omega, \Theta)$, given by*

$$\Pi = \left\{\Pi_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$

$$\Omega = \left\{\Omega_1, 0, 0, 0\right\},$$

$$\Theta = \left\{\frac{H_{133} + H_{144}}{H_{33} + H_{44}} - \Pi_1 - \Omega_1, 0, 0, 0\right\},$$

*where $\Pi_1$ and $\Omega_1$ are arbitrary scalars.*

(vii) *If $H_{33} + H_{44} \neq 0$, then it is weakly cyclic Ricci symmetric with solution $(\Pi, \Omega, \Theta)$, given by*

$$\Pi = \left\{\Pi_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$

$$\Omega = \left\{\Omega_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$

$$\Theta = \left\{\frac{3\left(H_{133} + H_{144}\right)}{H_{33} + H_{44}} - \Pi_1 - \Omega_1, 0, \frac{H_{333} + H_{344}}{H_{33} + H_{44}}, \frac{H_{334} + H_{444}}{H_{33} + H_{44}}\right\},$$

*where $\Pi_1$ and $\Omega_1$ are arbitrary scalars.*

(viii) *Ricci simple [44] (i.e., $S = \alpha\eta \otimes \eta$) for*

$$\alpha = \frac{1}{2}\left(H_{33} + H_{44}\right) \quad and \quad \eta = \{1, 0, 0, 0\},$$

*and hence $S \wedge S = 0$ and $S^2 = 0$. Here $\|\eta\| = 0$ and $\nabla\eta = 0$.*

(ix) *$Q(S, R) = Q(S, C) = 0$ but $R$ or $C$ is not a scalar multiple of $S \wedge S$ as $S$ is of rank 1.*

(x) *$C \cdot R = 0$ and hence $C \cdot S = 0$, $C \cdot C = 0$ and $C \cdot P = 0$.*

(xi) *$P \cdot R = 0$ but $P \cdot \mathcal{R} \neq 0$. Also but $P \cdot S = P \cdot \mathcal{S} = 0$.*

(xii) *Ricci tensor is Riemann compatible as well as Weyl compatible.*

(xiii) *$P \cdot P = -\frac{1}{3}Q(S, P)$.*

Again the non-vacuum pp-wave solution presented in [69] is a special case of (1.1) for $H(x, x^3, x^4) = 2a_1 e^{a_2 x^3 - a_3 x^4}$. Hence the line element is explicitly given by:

$$(4.3) \qquad ds^2 = 2a_1 e^{a_2 x^3 - a_3 x^4}(dx)^2 + 2dxdr + [(dx^3)^2 + (dx^4)^2].$$

Now the geometric properties of the metric (4.3) can be stated as follows.

**Corollary 4.2.** *The metric given in (4.3) possesses the following curvature properties.*

(i) *It satisfies the curvature conditions (i)-(xiii) of Corollary 4.1 with different associated 1-forms of the corresponding structures.*

(ii) *Moreover it is recurrent for the 1-form of recurrency $\{0, 0, 2a_2, -2a_3\}$. Hence it is Ricci recurrent, conformally recurrent and projectively recurrent. Also it is semisymmetric and hence Ricci semisymmetric, conformally semisymmetric and projectively semisymmetric.*

Again the generalized plane wave metric [71] is given by

(4.4) $$ds^2 = 2H(x, x^3, x^4)(dx)^2 + 2dxdr + (dx^3)^2 + (dx^4)^2,$$

where $H(x, x^3, x^4) = a_1(x^3)^2 + a_2(x^4)^2 + a_3 x^3 x^4 + a_4 x^3 + a_5 x^4 + a_6$, $a_i$'s are scalar. Hence, it is a special case of (1.1) and we can state the following.

**Corollary 4.3.** *The metric given in* (4.4) *possesses the following curvature properties:*

(i) *it satisfies the curvature conditions* (i)-(ix) *of Corollary* 4.1 *with different associated* 1*-forms of the corresponding structures;*

(ii) *moreover it is locally symmetric and hence Ricci symmetric, conformally symmetric and projectively symmetric.*

From Corollary 4.1, Corollary 4.2 and Corollary 4.3, we can state the following about the recurrent structure on a semi-Riemannian manifold.

*Remark* 4.4. From Corollary 4.2 we see that the metric (4.3) is recurrent but not locally symmetric and from Corollary 4.1 we see that the metric (1.1) is Ricci recurrent but not recurrent. These results support the well-known facts that every locally symmetric manifold is recurrent but not conversely, and every recurrent manifold is Ricci recurrent but not conversely.

## 5. Energy-Momentum Tensor of Generalized pp-Wave Metric

In this section we discuss about the energy-momentum tensor of the generalized pp-wave metric (1.2) and also the other special forms, such as (1.1), (4.3) and (4.4). From the values of the energy momentum tensor $T$ of the generalized pp-wave metric (1.2), we can conclude that $T$ is of rank 1 if the cosmological constant is zero and (1.3) holds. In this case

$$T = \frac{c^4 (h_{33} + h_{44})}{4\pi f G} \eta \otimes \eta, \quad \eta = \{1, 0, 0, 0\}.$$

Again it is easy to check that $\eta$ is null. Thus we can state the following.

**Theorem 5.1.** *For zero cosmological constant, the generalized pp-wave metric* (1.2) *is a pure radiation metric if and only if it is a pp-wave metric.*

**Corollary 5.1.** *For zero cosmological constant, the pp-wave metric* ((1.2) *with* (1.3) *or* (1.1)) *is a pure radiation metric.*

We note that recently Shaikh et al. [62] studied the curvature properties of pure radiation metric. Again from the values of the components of $\nabla T$ of the metric (1.2),

we get

$$T_{11,1} + T_{11,1} + T_{11,1}$$
$$= \frac{3c^4 \left(h_{133} + h_{144}\right)}{4\pi f G},$$
$$T_{11,3} + T_{13,1} + T_{31,1}$$
$$= \frac{c^4}{4\pi f^4 G} \left( f^3 h_{333} + f^3 h_{344} - f_{333} f^2 h - f_{344} f^2 h + f_{33} f^2 h_3 + f_{44} f^2 h_3 - f_3 f^2 h_{33} \right.$$
$$\left. - f_3 f^2 h_{44} + 4 f_3 f_{33} f h + 2 f_4 f_{34} f h + 2 f_3 f_{44} f h - f_3^2 f h_3 - f_4^2 f h_3 - 3 f_3^3 h - 3 f_3 f_4^2 h \right),$$
$$T_{11,4} + T_{14,1} + T_{41,1}$$
$$= \frac{c^4}{4\pi f^4 G} \left( f^3 h_{334} + f^3 h_{444} - f_{334} f^2 h - f_{444} f^2 h + f_{44} f^2 h_4 - f_4 f^2 h_{33} - f_4 f^2 h_{44} \right.$$
$$\left. + 2 f_4 f_{33} f h + 2 f_3 f_{34} f h + f_{33} f^2 h_4 + 4 f_4 f_{44} f h - f_3^2 f h_4 - f_4^2 f h_4 - 3 f_4^3 h - 3 f_3^2 f_4 h \right),$$
$$T_{12,3} + T_{23,1} + T_{31,2}$$
$$= \frac{c^4 \left( f^2 f_{333} + f^2 f_{344} + 3 f_3^3 + 3 f_4^2 f_3 - 4 f f_{33} f_3 - 2 f f_{44} f_3 - 2 f f_4 f_{34} \right)}{8\pi f^4 G},$$
$$T_{12,4} + T_{24,1} + T_{41,2}$$
$$= \frac{c^4 \left( f^2 f_{334} + f^2 f_{444} + 3 f_4^3 + 3 f_3^2 f_4 - 2 f f_{33} f_4 - 4 f f_{44} f_4 - 2 f f_3 f_{34} \right)}{8\pi f^4 G}$$

and

$$T_{11,3} - T_{13,1} = \frac{c^4}{8\pi f^4 G} \left( 2 f^3 h_{333} + 2 f^3 h_{344} - 2 f_{333} f^2 h - 2 f_{344} f^2 h \right.$$
$$- f_{33} f^2 h_3 - f_{44} f^2 h_3 - 2 f_3 f^2 h_{33} - 2 f_3 f^2 h_{44} + 8 f_3 f_{33} f h + 4 f_4 f_{34} f h$$
$$\left. + 4 f_3 f_{44} f h + f_3^2 f h_3 + f_4^2 f h_3 - 6 f_3^3 h - 6 f_3 f_4^2 h \right),$$
$$T_{11,4} - T_{14,1} = \frac{c^4}{8\pi f^4 G} \left( 2 f^3 h_{334} + 2 f^3 h_{444} - 2 f_{334} f^2 h - 2 f_{444} f^2 h - f_{33} f^2 h_4 \right.$$
$$- f_{44} f^2 h_4 - 2 f_4 f^2 h_{33} - 2 f_4 f^2 h_{44} + 4 f_4 f_{33} f h + 4 f_3 f_{34} f h + 8 f_4 f_{44} f h$$
$$\left. + f_3^2 f h_4 + f_4^2 f h_4 - 6 f_4^3 h - 6 f_3^2 f_4 h \right)$$

and

$$T_{12,3} - T_{13,2} = T_{23,1} - T_{21,3}$$
$$= - \frac{c^4 \left( f^2 f_{333} + f^2 f_{344} + 3 f_3^3 + 3 f_4^2 f_3 - 4 f f_{33} f_3 - 2 f f_{44} f_3 - 2 f f_4 f_{34} \right)}{8\pi f^4 G},$$
$$T_{12,4} - T_{14,2} = T_{24,1} - T_{21,4}$$
$$= - \frac{c^4 \left( f^2 f_{334} + f^2 f_{444} + 3 f_4^3 + 3 f_3^2 f_4 - 2 f f_{33} f_4 - 4 f f_{44} f_4 - 2 f f_3 f_{34} \right)}{8\pi f^4 G}.$$

Now putting the condition (1.3) to above, we get

$$(5.1) \quad \begin{cases} T_{11,3} - T_{13,1} = \dfrac{c^4 \left( fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44} \right)}{4\pi f^2 G}, \\ T_{11,4} - T_{14,1} = \dfrac{c^4 \left( fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44} \right)}{4\pi f^2 G}, \end{cases}$$

and

$$(5.2) \quad \begin{cases} T_{11,1} + T_{11,1} + T_{11,1} = \dfrac{3c^4 \left( h_{144} + h_{133} \right)}{4\pi f G}, \\ T_{11,3} + T_{13,1} + T_{31,1} = \dfrac{c^4 \left( fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44} \right)}{4\pi f^2 G}, \\ T_{14,1} + T_{41,1} + T_{11,4} = \dfrac{c^4 \left( fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44} \right)}{4\pi f^2 G}. \end{cases}$$

Now, from (4.2), (5.1) and (5.2), we can state the following.

**Proposition 5.1.** *The energy momentum tensor $T$ of the pp-wave metric (1.2) with the condition (1.3) is*

(i) *parallel if*

$$h_{144} + h_{133} = fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44} = fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44} = 0;$$

   *(For example: $f(x^3, x^4) = e^{x^3 + x^4}$ and $h(x^1, x^3, x^4) = e^{x^3 + x^4}$).*

(ii) *Codazzi type if*

$$fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44} = fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44} = 0;$$

   *(For example: $f(x^3, x^4) = e^{x^3 + x^4}$ and $h(x^1, x^3, x^4) = e^{x + x^3 + x^4}$).*

(iii) *cyclic parallel if*

$$h_{144} + h_{133} = fh_{344} + fh_{333} - f_3 h_{33} - f_3 h_{44} = fh_{444} + fh_{334} - f_4 h_{33} - f_4 h_{44} = 0.$$

   *(For example: $f(x^3, x^4) = e^{x^3 + x^4}$ and $h(x^1, x^3, x^4) = e^{x^3 + x^4}$).*

**Proposition 5.2.** *The energy-momentum tensor of the pp-wave metric of the form (1.1) is*

(i) *parallel if $H_{144} + H_{133} = H_{344} + H_{333} = H_{444} + H_{334} = 0$;*
   *(For example: $H(u, x, y) = (x)^2 + (y)^2$).*

(ii) *Codazzi type if $H_{344} + H_{333} = H_{444} + H_{334} = 0$;*
   *(For example: $H(u, x, y) = (ux)^2 + (uy)^2$).*

(iii) *cyclic parallel if $H_{144} + H_{133} = H_{344} + H_{333} = H_{444} + H_{334} = 0$.*
   *(For example: $H(u, x, y) = (x)^2 + (y)^2$).*

Now we can conclude the following.

**Theorem 5.2.** *On a pp-wave spacetime (endowed with the metric (1.2) with (1.3) or (1.1))*

(i) *the energy-momentum tensor is parallel if and only if it is cyclic parallel;*

(ii) *the energy-momentum tensor is Codazzi type if it is cyclic parallel but not conversely (see Example 5.1);*

(iii) *the Ricci tensor is zero, i.e., the space is vacuum if $h$ or $H$ is harmonic, i.e., $h_{33} + h_{44} = 0$ or $H_{33} + H_{44} = 0$ and in this case $\nabla T = 0$.*

*Example* 5.1. We now consider a special form of the generalized pp-wave metric (1.2) as

$$(5.3) \qquad ds^2 = -2e^{x+x^3-x^4}(dx)^2 + 2dxdr - \frac{1}{2}e^{x^3-x^4}[(dx^3)^2 + (dx^4)^2].$$

Then the non-zero components of its $R$, $\nabla R$, $S$ and $\nabla S$ are given by

$$R_{1313} = R_{1414} = e^{x+x^3-x^4}, \quad R_{1313,1} = R_{1414,1} = e^{x+x^3-x^4},$$
$$S_{11} = 4e^x, \quad S_{11,1} = 4e^x.$$

It is easy to check that the scalar curvature of this metric is zero and it is conformally flat. Now the non-zero components of its energy momentum tensor $T$ and its derivative $\nabla T$ are given by

$$T_{11} = \frac{c^4 e^{x-x^4}\left(2e^{x^4} - \Lambda e^{x^3}\right)}{4\pi G}, \quad T_{12} = \frac{c^4 \Lambda}{8\pi G}, \quad T_{33} = T_{44} = -\frac{c^4 \Lambda e^{x^3-x^4}}{16\pi G},$$
$$T_{11,1} = \frac{c^4 e^x}{2\pi G}.$$

Thus we can easily check that the Ricci tensor and the energy momentum tensor of (5.3) are codazzi type but not cyclic parallel.

## 6. Robinson-Trautman Metric and Generalized pp-Wave Metric

Recently, Shaikh et al. [53] studied the curvature properties of Robinson-Trautman metric. The line element of Robinson-Trautman metric in $\{t, r, x^3, x^4\}$-coordinate is given by

$$(6.1) \qquad ds^2 = -2(a - 2br - qr^{-1})dt^2 + 2dtdr - \frac{r^2}{f^2}[(dx^3)^2 + (dx^4)^2],$$

where $a, b, q$ are constants and $f$ is a function of the real variables $x^3$ and $x^4$. In this section we make a comparison between the curvature properties of the Robinson-Trautman metric (6.1) and generalized pp-wave metric (1.2) as well as the pp-wave metric (1.1).

**Theorem 6.1.** *The Robinson-Trautman metric* (6.1) *and generalized pp-wave metric* (1.2) *have the following similarities and dissimilarities.*
**A. Similarities:**
(i) *both the metrics are 2-quasi-Einstein;*
(ii) *both are generalized quasi-Einstein in the sense of Chaki;*
(iii) *Ricci tensors of both the metrics are Riemann compatible as well as Weyl compatible.*

**B. Dissimilarities:**

(iv) (6.1) *is Deszcz pseudosymmetric whereas* (1.2) *is Ricci generalized pseudosymmetric;*

(v) *the conformal curvature* 2-*forms are recurrent for* (6.1) *but not recurrent for* (1.2);

(vi) *the metric* (6.1) *is Roter type and hence* Ein(2) *but* (1.2) *is not Roter type but* Ein(3).

**Theorem 6.2.** *The Robinson-Trautman metric* (6.1) *and pp-wave metric* (1.1) *have the following similarities and dissimilarities.*

**A. Similarities:**

(i) *for both the metrics, the conformal curvature* 2-*forms are recurrent;*

(ii) *Ricci tensor of both the metrics are Riemann compatible as well as Weyl compatible.*

**B. Dissimilarities:**

(iii) *the metric* (6.1) *is* 2-*quasi-Einstein, where as* (1.1) *is Ricci simple and hence quasi-Einstein;*

(iv) (6.1) *is Deszcz pseudosymmetric whereas* (1.1) *is semisymmetric;*

(v) (6.1) *is pseudosymmetric due to conformal curvature tensor whereas* (1.1) *is semisymmetric due to conformal curvature tensor;*

(vi) *the metric* (6.1) *realizes* $S \wedge S \neq 0$ *but* (1.1) *satisfies* $S \wedge S = 0$;

(vii) *the metric* (6.1) *is Roter type and hence* Ein(2) *but* (1.1) *is not Roter type but* Ein(3).

REFERENCES

[1] A. Adamów and R. Deszcz, *On totally umbilical submanifolds of some class of Riemannian manifolds*, Demonstr. Math. **16** (1983), 39–59.

[2] K. Arslan, R. Deszcz, R. Ezentaş, M. Hotloś and C. Murathan, *On generalized Robertson-Walker spacetimes satisfying some curvature condition*, Turkish J. Math. **38**(2) (2014), 353–373.

[3] A. L. Besse, *Einstein Manifolds*, Springer-Verlag, Berlin, Heidelberg, 1987.

[4] H. W. Brinkmann, *Einstein spaces which are mapped conformally on each other*, Math. Ann. **94** (1925), 119–145.

[5] É. Cartan, *Sur une classe remarquable d'espaces de Riemannian*, Bull. Soc. Math. France **54** (1926), 214–264.

[6] É. Cartan, *Leçons sur la géométrie des espaces de Riemann*, 2nd ed., Gauthier Villars, Paris, 1946.

[7] M. C. Chaki, *On pseudosymmetric manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) **33**(1) (1987), 53–58.

[8] M. C. Chaki, *On generalized quasi-Einstein manifolds*, Publ. Math. Debrecen **58** (2001), 683–691.

[9] B.-Y. Chen, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Scientific, New Jersey, 2017.

[10] B.-Y. Chen, *Classification of torqued vector fields and its applications to Ricci solitons*, Kragujevac J. Math. **41** (2017), 239–250.

[11] J. Chojnacka-Dulas, R. Deszcz, M. Głogowska and M. Prvanović, *On warped products manifolds satisfying some curvature conditions*, J. Geom. Phys. **74** (2013), 328–341.

[12] F. Defever and R. Deszcz, *On semi-Riemannian manifolds satisfying the condition $R \cdot R = Q(S, R)$*, Geometry and Topology of Submanifolds **III**, World Science, River Edge, New Jersey, 1991, 108–130.

[13] F. Defever, R. Deszcz, M. Hotloś, M. Kucharski and Z. Şentürk, *Generalisations of Robertson-Walker spaces*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **43** (2000), 13–24.

[14] F. Defever, R. Deszcz, L. Verstraelen and L. Vrancken, *On pseudosymmetric space-times*, J. Math. Phys. **35**(11) (1994), 5908–5921.

[15] R. Deszcz, *On pseudosymmetric spaces*, Bull. Belg. Math. Soc. Ser. A **44** (1992), 1–34.

[16] R. Deszcz, *On Roter type manifolds*, Book of abstract of V Conference on Geometry and Topology of Manifolds, April 27 - May 3, 2003, Krynica, Poland.

[17] R. Deszcz, *On some Akivis-Goldberg type metrics*, Publ. Inst. Math. (Beograd) (N.S.) **74**(88) (2003), 71–84.

[18] R. Deszcz and M. Głogowska, *Some examples of nonsemisymmetric Ricci-semisymmetric hypersurfaces*, Colloq. Math. **94** (2002), 87–101.

[19] R. Deszcz, M. Głogowska, M. Hotloś and Z. Şentürk, *On certain quasi-Einstein semisymmetric hypersurfaces*, Ann. Univ. Sci. Budapest Eötvös Sect. Math. **41** (1998), 151–164.

[20] R. Deszcz, M. Głogowska, J. Jełowicki, M. Petrović-Torgašev and G. Zafindratafa, *On Riemann and Weyl compatible tensors*, Publ. Inst. Math. (Beograd) (N.S.) **94**(108) (2013), 111–124.

[21] R. Deszcz, M. Głogowska, J. Jełowicki and G. Zafindratafa, *Curvature properties of some class of warped product manifolds*, Int. J. Geom. Methods Mod. Phys. **13** (2016), Paper ID 1550135, 36 pages.

[22] R. Deszcz, M. Głogowska, M. Hotloś and K. Sawicz, *A survey on generalized Einstein metric conditions*, in: M. Plaue, A. D. Rendall, M. Scherfner (Eds.), *Advances in Lorentzian Geometry: Proceedings of the Lorentzian Geometry Conference in Berlin*, AMS/IP Studies in Advanced Mathematics, **49** (2011), 27-46.

[23] R. Deszcz, S. Haesen and L. Verstraelen, *Classification of space-times satisfying some pseudo-symmetry type conditions*, Soochow Journal of Mathematics **30** (2004), 339–349.

[24] R. Deszcz, S. Haesen and L. Verstraelen, *On natural symmetries*, in: A. Mihai, I. Mihai and R. Miron (Eds.), *Topics in Differential Geometry*, Chapter 6, Editura Academiei Române, Bucharest, 2008, 249–308.

[25] R. Deszcz and M. Hotloś, *On hypersurfaces with type number two in spaces of constant curvature*, Ann. Univ. Sci. Budapest Eötvös Sect. Math. **46** (2003), 19–34.

[26] R. Deszcz, M. Hotloś, J. Jełowicki, H. Kundu and A. A. Shaikh, *Curvature properties of Gödel metric*, Int J. Geom. Methods Mod. Phys. **11** (2014), Papaer ID 1450025, 20 pages.

[27] R. Deszcz and M. Kucharski, *On curvature properties of certain generalized Robertson-Walker spacetimes*, Tsukuba J. Math. **23**(1) (1999), 113–130.

[28] R. Deszcz, L. Verstraelen and L. Vrancken, *The symmetry of warped product space-times*, Gen. Relativity Gravitation **23**(6) (1991), 671–681.

[29] J. Ehlers and W. Kundt, *Exact solutions of the gravitational field equations*, in: L. Witten (Ed.), *Gravitation: An Introduction to Current Research*, John Wiley, London, 1962.

[30] A. Galaev and T. Leistner, *Recent developments in pseudo-Riemannian holonomy theory*, in: *Handbook of Pseudo-Riemannian Geometry and Supersymmetry*, Chapter 17, Publishing House, Zurich, 2010, 581–627.

[31] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata **7**(3) (1978), 259–280.

[32] K. Gödel, *An example of a new type of cosmological solutions of Einstein's field equations of gravitation*, Rev. Modern Phys. **21**(3) (1949), 447–450.

[33] M. Głogowska, *Semi-Riemannian manifolds whose Weyl tensor is a Kulkarni-Nomizu square*, Publ. Inst. Math. (Beograd) (N.S.) **72**(86) (2002), 95–106.

[34] S. Haesen and L. Verstraelen, *Properties of a scalar curvature invariant depending on two planes*, Manuscripta Math. **122** (2007), 59–72.

[35] S. Haesen and L. Verstraelen, *Natural intrinsic geometrical symmetries*, SIGMA **5** (2009), Paper ID 086, 14 pages.

[36] Y. Ishii, *On conharmonic transformations*, Tensor (N. S.) **11** (1957), 73–80.

[37] D. Kowalczyk, *On the Reissner-Nordström-de Sitter type spacetimes*, Tsukuba J. Math. **30**(2) (2006), 363–381.

[38] D. Lovelock and H. Rund, *Tensors, Differential Forms and Variational Principles*, Courier Dover Publications, New York, 1989.

[39] C. A. Mantica and L. G. Molinari, *Riemann compatible tensors*, Colloq. Math. **128** (2012), 197–210.

[40] C. A. Mantica and L. G. Molinari, *Weyl compatible tensors*, Int. J. Geom. Methods Mod. Phys. **11**(08) (2014), Paper ID 1450070, 20 pages.

[41] C. A. Mantica and Y. J. Suh, *The closedness of some generalized curvature 2-forms on a Riemannian manifold I*, Publ. Math. Debrecen **81**(3-4) (2012), 313–326.

[42] C. A. Mantica and Y. J. Suh, *The closedness of some generalized curvature 2-forms on a Riemannian manifold II*, Publ. Math. Debrecen **82**(1) (2013), 163–182.

[43] C. A. Mantica and Y. J. Suh, *Recurrent conformal 2-forms on pseudo-Riemannian manifolds*, Int. J. Geom. Methods Mod. Phys. **11**(6) (2014), Paper ID 1450056, 29 pages.

[44] C. A. Mantica, and Y. J. Suh, *Pseudo-Z symmetric space-times with divergence-free Weyl tensor and pp-waves*, Int. J. Geom. Methods Mod. Phys. **13**(02) (2016), Paper ID 1650015, 34 pages.

[45] E. M. Patterson, *Some theorems on Ricci recurrent spaces*, J. Lond. Math. Soc. **27** (1952), 287–295.

[46] A. Peres, *Some gravitational waves*, Phys. Rev. Lett. **3** (1959), 571–572.

[47] L. Radhakrishna and N. I. Singh, *Pure radiation fields admitting nontrivial null symmetries*, J. Math. Phys. **25** (1984), 2293–2300.

[48] M. Prvanović, *On weakly symmetric Riemannian manifolds*, Publ. Math. Debrecen **46**(1-2) (1995), 19–25.

[49] H. S. Ruse, *On simply harmonic spaces*, J. Lond. Math. Soc. **21** (1946), 243–247.

[50] H. S. Ruse, *On simply harmonic 'kappa spaces' of four dimensions*, Proc. Lond. Math. Soc. **50** (1949), 317–329.

[51] H. S. Ruse, *Three dimensional spaces of recurrent curvature*, Proc. Lond. Math. Soc. **50** (1949), 438–446.

[52] A. A. Shaikh, F. R. Al-Solamy and I. Roy, *On the existence of a new class of semi-Riemannian manifolds*, Math. Sci. **7**(46) (2013), 1–13.

[53] A. A. Shaikh, M. Ali and Z. Ahsan, *Curvature properties of Robinson-Trautman metric*, J. Geom. **109**(38) (2018), 20 pages, DOI 10.1007/s00022-018-0443-1.

[54] A. A. Shaikh, R. Deszcz, M. Hotloś, J. Jełowicki and H. Kundu, *On pseudosymmetric manifolds*, Publ. Math. Debrecen **86**(3-4) (2015), 433–456.

[55] A. A. Shaikh and H. Kundu, *On equivalency of various geometric structures*, J. Geom. **105** (2014), 139–165.

[56] A. A. Shaikh and H. Kundu, *On warped product generalized Roter type manifolds*, Balkan J. Geom. Appl. **21**(2) (2016), 82–95.

[57] A. A. Shaikh and H. Kundu, *On curvature properties of Som-Raychaudhuri spacetime*, J. Geom. **108**(2) (2016), 501–515.

[58] A. A. Shaikh and H. Kundu, *On warped product manifolds satisfying some pseudosymmetric type conditions*, Differ. Geom. Dyn. Syst. **19** (2017), 119–135.

[59] A. A. Shaikh and H. Kundu, *On some curvature restricted geometric structures for projective curvature tensor*, Int. J. Geom. Methods Mod. Phys. **15**(09) (2018), Paper ID 1850157, 38 pages.

[60] A. A. Shaikh and H. Kundu, *On generlized Roter type manifolds*, Kragujevac J. Math. **43**(3) (2019), 471–493.

[61] A. A. Shaikh, H. Kundu and Md. S. Ali, *On warped product super generalized recurrent manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N. S.) **64**(1) (2018), 85–99.

[62] A. A. Shaikh, H. Kundu, M. Ali and Z. Ahsan, *Curvature properties of a special type of pure radiation metrics*, J. Geom. Phys. **136** (2019), 195–206.

[63] A. A. Shaikh, H. Kundu and J. Sen, *Curvature properties of Vaidya metric*, Indian J. Math. **61**(1) (2019), 41–59.

[64] A. A. Shaikh and S. K. Jana, *On weakly cyclic Ricci symmetric manifolds*, Ann. Polon. Math. **89**(3) (2006), 139–146.

[65] A. A. Shaikh and A. Patra, *On a generalized class of recurrent manifolds*, Arch. Math. (Brno) **46** (2010), 71–78.

[66] A. A. Shaikh and I. Roy, *On weakly generalized recurrent manifolds*, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. **54** (2011), 35–45.

[67] A. A. Shaikh, I. Roy and H. Kundu, *On some generalized recurrent manifolds*, Bull. Iranian Math. Soc. **43**(5) (2017), 1209–1225.

[68] A. A. Shaikh, I. Roy and H. Kundu, *On the existence of a generalized class of recurrent manifolds*, An. Ştiinţ. Univ. Al. I. Cuza Iaşi Mat. (N. S.) **64**(2) (2018), 233–251.

[69] R. Sippel and H. Goenner, *Symmetry classes of pp-waves*, Gen. Relativity Gravitation **18**(12) (1986), 1229–1243.

[70] M. M. Som and A. K. Raychaudhuri, *Cylindrically symmetric charged dust distributions in rigid rotation in general relativity*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **304** (1968), 81–86.

[71] H. Stephani, D. Kramer, M. Maccallum, C. Hoenselaers and E. Herlt, *Exact Solutions of Einstein's Field Equations*, 2nd Ed., Cambridge University Press, UK, Cambridge, 2009.

[72] Y. J. Suh, J-H. Kwon and Y. S. Pyo, *On semi-Riemannian manifolds satisfying the second Bianchi identity*, J. Korean Math. Soc. **40**(1) (2003), 129–167.

[73] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, I. The local version*, J. Differential Geom. **17** (1982), 531–582.

[74] Z. I. Szabó, *Classification and construction of complete hypersurfaces satisfying $R(X, Y) \cdot R = 0$*, Acta Sci. Math. **47** (1984), 321–348.

[75] Z. I. Szabó, *Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$, II. The global version*, Geom. Dedicata **19** (1985), 65–108.

[76] L. Támassy and T. Q. Binh, *On weakly symmetric and weakly projective symmetric Riemannian manifolds*, Coll. Math. Soc. J. Bolyai **50** (1989), 663–670.

[77] L. Tamássy and T. Q. Binh, *On weak symmetries of Einstein and Sasakian manifolds*, Tensor (N. S.) **53** (1993), 140–148.

[78] L. Verstraelen, *Comments on the pseudo-symmetry in the sense of Ryszard Deszcz*, in: Dillen et al. (Eds.), *Geometry and Topology of Submanifolds*, VI, World Science, Singapore, 1994, 119–209.

[79] L. Verstraelen, *Natural extrinsic geometrical symmetries - an introduction*, in: *Recent Advances in the Geometry of Submanifolds Dedicated to the Memory of Franki Dillen (1963-2013)*, Contemporary Mathematics **674**, AMS, New York, 2016, 5–16.

[80] L. Verstraelen, *Foreword*, in: B.-Y. Chen, *Differential Geometry of Warped Product Manifolds and Submanifolds*, World Scientific, Singapore, 2017, vii–xxi.

[81] P. Venzi, *Una generalizzazione degli spazi ricorrenti*, Rev. Roumaine Math. Pures Appl. **30** (1985), 295–305.

[82] A. G. Walker, *On Ruse's spaces of recurrent curvature*, Proc. Lond. Math. Soc. **52** (1950), 36–64.

[83] K. Yano and M. Kon, *Structures on Manifolds*, World Science Publication, Singapore, 1989.

[1]DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF BURDWAN,
GOLAPBAG,
BURDWAN-713104,
WEST BENGAL, INDIA
*Email address*: aask2003@yahoo.co.in, aashaikh@math.buruniv.ac.in
*Email address*: kundu.haradhan@gmail.com

[2]INSTITUTE OF MATHEMATICS AND INFORMATICS,
DEBRECEN UNIVERSITY,
H-4010 DEBRECEN, P.O. BOX 12, HUNGARY
*Email address*: binh@math.klte.hu