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51

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## Contents

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# INTEGRAL BOUNDARY VALUE PROBLEMS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING HADAMARD AND CAPUTO-HADAMARD FRACTIONAL DERIVATIVES

P. KARTHIKEYAN[1] AND R. ARUL[1]

Abstract. In this paper, we examine the existence and uniqueness of integral boundary value problem for implicit fractional differential equations (IFDE's) involving Hadamard and Caputo-Hadamard fractional derivative. We prove the existence and uniqueness results by utilizing Banach and Schauder's fixed point theorem. Finally, examples are introduced of our results.

## 1. Introduction

FDE's are considered to be a different model to integer differential equations. It has been proved by applying importance in the modeling of various fields of physical sciences, medicine, electronics and wave transformation [8, 16, 21, 23, 26]. The dominant techniques are the method of introducing a parameter for solving an implicit differential equations. In past three years, the most of research paper to developed existence and uniqueness of implicit FDE's involving various derivatives like the Caputo,Riemann-Liouville, Caputo-Hadamard, Hadamard, Hilfer-Hadamard fractional derivatives etc., (see [4–7, 9, 14, 15, 19, 20, 24]).

Caputo Hadamard fractional derivatives were studied in [12] by the authors F. Jarad, T. Abdeljawad and D. Baleanu, where a Caputo-type modification for Hadamard derivatives was introduced and studied. Later, more properties of Hadamard fractional derivatives were investigated in [1, 2, 10, 13].

*Key words and phrases.* Implicit fractional differential equations, Hadamard fractional operators, boundary condition, fixed point theorem, existence and uniqueness.

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The applications of Hadamard fractional differential equations in mathematical physics cuold be found in [11, 17, 18, 22, 25]. In [3] the authors have studied Hilfer-Hadamard FDE's with variable-order fractional integral and fractional derivative. Motivated by the above cited work, we studies the solutions of existence and uniqueness results to the following implicit fractional differential equations with integral boundary conditions of the form

$$ {}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t) = g(t, x(t), {}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t)), \quad t \in \mathcal{J} := (b, \mathcal{T}), \tag{1.1} $$

$$ x(b) = 0,\ x(\mathcal{T}) = \lambda \int_0^{\sigma} x(s)ds, \quad b < \sigma < \mathcal{T}, \lambda \in \mathbb{R}, \tag{1.2} $$

where ${}^{\mathcal{H}}\mathcal{D}^{\vartheta}$ is the Hadamard fractional derivative of order $1 < \vartheta \leq 2$,

$$ {}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t) = g(t, x(t), {}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)), \quad t \in \mathcal{J} := [b, \mathcal{T}], \tag{1.3} $$

$$ x(b) = 0,\ x(\mathcal{T}) = \lambda \int_0^{\sigma} x(s)ds, \quad b < \sigma < \mathcal{T}, \lambda \in \mathbb{R}, \tag{1.4} $$

where ${}^{\mathcal{CH}}\mathcal{D}^{\vartheta}$ is the Caputo-Hadamard fractional derivative of order $1 < \vartheta \leq 2$ and $g : \mathcal{J} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In this paper, Section 2, has definitions and some of the most important basic concepts of the fractional calculus. In Section 3, existence and uniqueness of solutions for integral boundary conditions of implicit fractional differential equations involving Hadamard fractional derivative and Caputo-Hadamard fractional derivatives are proved by utilizing Banach and Schauder's fixed point theorems. In Section 4, an illustrative examples are provided to explain of the results of the problem (1.1)–(1.4).

## 2. Basic Results

In this section, the some most important basic concepts, definitions and some supporting results are used in this paper. By $\mathcal{C}(\mathcal{J}, \mathbb{R})$ we denote the Banach space of all continuous functions form $\mathcal{J}$ into $\mathbb{R}$ with the norm $||x||_\infty = \sup\{|x(t)| : t \in \mathcal{J}\}$.

**Definition 2.1** ([15])**.** The derivative of fractional order $\vartheta > 0$ of a function $g : (0, \infty) \to \mathbb{R}$ is given by

$$ \mathcal{D}^{\vartheta}_{0+}x(t) = \frac{1}{\Gamma(n-\vartheta)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\vartheta-n+1}}ds, $$

where $n = [\vartheta] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

**Definition 2.2** ([15])**.** The **Hadamard fractional integral** of $g$ is defined by

$$ {}_{\mathcal{H}}\mathcal{J}^{\vartheta}x(t) = \frac{1}{\Gamma(\vartheta)}\int_b^t \left(\log\frac{t}{s}\right)^{\vartheta-1}\frac{g(s)}{s}ds, \quad \vartheta > 0. $$

**Definition 2.3** ([15])**.** The **Hadamard fractional derivative** of $g$ is continuous function and further, $\log(\cdot) = \log_e(\cdot)$ is defined as

$$ {}_{\mathcal{H}}\mathcal{D}^{\vartheta}x(t) = \frac{1}{\Gamma(n-\vartheta)}\left(t\frac{d}{dt}\right)^n \int_b^t \left(\log\frac{t}{s}\right)^{n-\vartheta-1}\frac{g(s)}{s}ds, $$

where $n-1<\vartheta<n$, $n=[\vartheta]+1$ and $[\vartheta]$ denotes the integer part of the real number $\vartheta$.

**Definition 2.4** ([12])**.** For at least $n$-times differentiable function $g$, the **Caputo-Hadamard fractional derivative** of order $\vartheta$ is defined as

$$^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)=\frac{1}{\Gamma(n-\vartheta)}\int_a^t\left(\ln\frac{t}{s}\right)^{n-\vartheta-1}\delta^n\frac{g(s)}{s}ds.$$

**Lemma 2.1** (Hadamard fractional derivative)**.** *Let* $v\in\mathcal{C}([b,\mathcal{T}],\mathbb{R})$ *and* $x\in\mathcal{C}^2_\delta([b,\mathcal{T}],\mathbb{R})$. *Then*

$$^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t)=v(t),\quad t\in\mathcal{J}:=[b,\mathcal{T}],$$

$$x(b)=0,x(\mathcal{T})=\lambda\int_0^\sigma x(s)ds,\quad b<\sigma<\mathcal{T},\lambda\in\mathbb{R},\tag{2.1}$$

*is equivalent to the integral equation given by*

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds+\frac{\left(\ln\frac{t}{s}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}-\lambda\left[\int_b^\sigma\left(\ln\frac{s}{b}\right)^{\vartheta-1}ds\right]\right]}\\ &\times\left[\lambda\int_b^\sigma\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{v(r)}{r}drds-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds\right]\end{aligned}\tag{2.2}$$

**Lemma 2.2** (Caputo-Hadamard fractional derivative)**.** *Let* $v\in\mathcal{C}([b,\mathcal{T}],\mathbb{R})$ *and* $x\in\mathcal{C}^2_\delta([b,\mathcal{T}],\mathbb{R})$.

$$^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)=v(t),\quad t\in\mathcal{J}:=[b,\mathcal{T}],$$

$$x(b)=0,\ x(\mathcal{T})=\lambda\int_0^\sigma x(s)ds,\quad b<\sigma<\mathcal{T},\lambda\in\mathbb{R},\tag{2.3}$$

*is equivalent to the integral equation given by*

$$\begin{aligned}x(t)=&\frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds+\frac{\left(\ln\frac{t}{s}\right)}{\Gamma(\vartheta)\left[\left(\ln\frac{\mathcal{T}}{s}\right)-\lambda\left[\sigma\left(\ln\frac{\sigma}{b}-1\right)+b\right]\right]}\\ &\times\left[\lambda\int_b^\sigma\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{v(r)}{r}drds-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{v(s)}{s}ds\right].\end{aligned}\tag{2.4}$$

**Lemma 2.3** (Nonlinear alternative of Lerary-Schauder type, [7])**.** *Let* $\mathcal{B}$ *be a Banach space,* $\mathcal{C}$ *a closed, convex subset of* $\mathcal{B}$, $\mathcal{U}$ *an open subset of* $\mathcal{C}$ *and* $0\in\mathcal{U}$. *Suppose that* $F:\overline{\mathcal{U}}\to\mathcal{C}$ *is a continuous, compact map. Then either* (i) $F$ *has a fixed point in* $\overline{\mathcal{U}}$, *or* (ii) *there is a* $u\in\partial\mathcal{U}$ *and* $\lambda\in(0,1)$, *with* $u=\lambda F(u)$.

## 3. Main Results

To prove the existence and uniqueness results we need the following assumptions.

*Assumption* 3.1. The function $g:\mathcal{J}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

*Assumption* 3.2. There exists constants $K_g > 0$ and $0 < L_g < 1$ such that

$$|g(t,u,v) - g(t,u_1,v_1)| \leq K_g|u-u_1| + L_g|v-v_1|, \quad \text{for any } u,v,u_1,v_1 \in \mathbb{R}.$$

*Assumption* 3.3. There exist a continuous nondecreasing function $\varphi$ on $[0,\infty) \to (0,\infty)$ and a function $p(t) \in \mathcal{C}^1([b,\mathcal{T}],\mathbb{R}^+)$ such that

$$||g(t,u,v)|| \leq p(t)\varphi(||u|| + ||v||).$$

The integral boundary conditions for implicit fractional differential equations with Hadamard fractional derivative (1.1)–(1.2) is equivalent to the integral equation

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s,x(s),{}^{\mathcal{H}}\mathcal{D}^\vartheta x(s))}{s} ds + \frac{\left(\ln \frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right]} \\ &\times \left[\lambda \int_b^\sigma \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{g(r,x(r),{}^{\mathcal{H}}\mathcal{D}^\vartheta x(r))}{r} dr ds \right. \\ &\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{H}}\mathcal{D}^\vartheta x(s))}{s} ds \right], \end{aligned}$$

where $N_1 = \int_b^\sigma \left(\ln \frac{s}{b}\right)^{\vartheta-1} ds$.

The integral boundary conditions for implicit fractional differential equations with Caputo-Hadamard fractional derivative (1.3)–(1.4) is equivalent to the integral equation

$$\begin{aligned} x(t) =& \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^\vartheta x(s))}{s} ds + \frac{\left(\ln \frac{t}{b}\right)}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{b}\right) - \lambda N_2\right]} \\ &\times \left[\lambda \int_b^\sigma \int_r^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{g(r,x(r),{}^{\mathcal{CH}}\mathcal{D}^\vartheta x(r))}{r} dr ds \right. \\ &\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^\vartheta x(s))}{s} ds \right] \end{aligned}$$

or

$$x(t) = I^\vartheta f(s) + \left(\frac{(\ln \frac{t}{s})}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{s}\right) - \lambda N_2\right]}\right) \left[\lambda \int_b^\sigma I^\vartheta f_1(r) ds - I^\vartheta f_2(s)\right],$$

where $N_2 = \sigma\left(\ln \frac{\sigma}{b} - 1\right) + b$ and $f, f_1, f_2 \in \mathcal{C}(\mathcal{J},\mathbb{R})$ satisfies the functional equations

$$\begin{aligned} f(s) &= g(s, I^\vartheta f(s), f(s)), \\ f_1(r) &= g(r, I^\vartheta f_1(r), f_1(r)), \\ f_2(s) &= g(s, I^\vartheta f_2(s), f_2(r)), \\ I^\vartheta f(s) &= \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s,x(s)),{}^{\mathcal{CH}}\mathcal{D}^\vartheta x(s))}{s} ds, \end{aligned}$$

$$I^{\vartheta} f_1(r) = \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{g(r, x(r), {}^{\mathcal{CH}}\mathcal{D}^{\vartheta} x(r))}{r} dr,$$
$$I^{\vartheta} f_2(s) = \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{g(s, x(s)), {}^{\mathcal{CH}}\mathcal{D}^{\vartheta} x(s))}{s} ds.$$

**Theorem 3.1.** *Assume that assumptions* 3.1 *and* 3.2 *hold. If*

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta} + \frac{\left(\ln \frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|}(|\lambda|(\sigma-b)-1)\right]\frac{K_g}{(1-L_g)} < 1,$$

*then there exists a unique solution for* (1.1)–(1.2) *on* $\mathcal{J} := [b, \mathcal{T}]$.

*Proof.* Let $B_r = \{x \in \mathcal{C}([b, \mathcal{T}], \mathbb{R}) : \|x\| \leq r\}$. Consider the operator $\mathcal{H} : \mathcal{C}([b, \mathcal{T}], \mathbb{R}) \to \mathcal{C}([b, \mathcal{T}], \mathbb{R})$ defined by

$$\mathcal{H}(x)(t) = I^{\vartheta} f(s) + \left(\frac{\left(\ln \frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right]}\right)\left(\lambda \int_b^{\sigma} I^{\vartheta} f_1(r) ds - I^{\vartheta} f_2(s)\right), \tag{3.1}$$

where $f, f_1, f_2 \in \mathcal{C}(\mathcal{J}, \mathbb{R})$ satisfies the functional equations

$$f(s) = f(s, I^{\vartheta} f(s), f(s)),$$
$$f_1(r) = f(r, I^{\vartheta} f_1(r), f_1(r)),$$
$$f_2(s) = f(s, I^{\vartheta} f_2(s), f_2(s)),$$

where $N_1 = \int_b^{\sigma} \left(\ln \frac{s}{b}\right)^{\vartheta-1} ds$ and

$$I^{\vartheta} f(s) = \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{g(s, x(s)), {}^{\mathcal{H}}\mathcal{D}^{\vartheta} x(s))}{s} ds,$$
$$I^{\vartheta} f_1(r) = \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{g(r, x(r), {}^{\mathcal{H}}\mathcal{D}^{\vartheta} x(r))}{r} dr,$$
$$I^{\vartheta} f_2(s) = \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{g(s, x(s)), {}^{\mathcal{H}}\mathcal{D}^{\vartheta} x(s))}{s} ds.$$

Clearly, the fixed point of operator $\mathcal{H}$ is solution of problem (1.1)–(1.2). Let $x_1, x_2 \in \mathcal{C}([b, \mathcal{T}], \mathbb{R})$. Then

$$\begin{aligned}(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t) =& \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{f(s) - h(s)}{s} ds \\ &+ \left(\frac{\left(\ln \frac{t}{s}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left[\left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} - \lambda N_1\right]}\right) \\ &\times \left[\lambda \int_b^{\sigma} \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{f(r) - h(r)}{r} drds\right.\end{aligned}$$

$$-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}\frac{f(s)-h(s)}{s}ds\Big],$$

where $f(s), h(s), f(r), h(r) \in \mathcal{C}([b,\mathcal{T}],\mathbb{R})$ are such that

$$\begin{aligned}f(s)&=f(s,x_1(s),f(s)), \quad f(r)=f(r,x_2(r),f(r)),\\ h(s)&=h(s,x_1(s),h(s)), \quad h(r)=h(r,x_2(r),h(r)).\end{aligned}$$

Now,

$$\begin{aligned}|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)| \leq & \frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{|f(s)-h(s)|}{s}ds\\ &+\left(\frac{\left(\ln\frac{t}{s}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}-\lambda N_1\right|}\right)\\ &\times\Big[|\lambda|\int_b^{\sigma}\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{|f(r)-h(r)|}{r}drds\\ &-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}\frac{|f(s)-h(s)|}{s}ds\Big],\end{aligned}\tag{3.2}$$

and, by Assumption 3.2, we have

$$\begin{aligned}&|(f(s)-h(s))| = |g(s,x_1(s),f(s))-g(s,x_2(s),h(s))|,\\ &|(f(s)-h(s))| \leq K_g|x_1(s)-x_2(s)|+L_g|x_1(s)-x_2(s)| \leq \frac{K_g}{1-L_g}|x_1(s)-x_2(s)|,\\ &|(f(s)-h(s))| \leq \frac{K_g}{1-L_g}|x_1(s)-x_2(s)|.\end{aligned}$$

Similary,

$$|(f(r)-h(r))| \leq \frac{K_g}{1-L_g}|x_1(r)-x_2(r)|.$$

The equation (3.2) implies

$$\begin{aligned}|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)| \leq & \frac{1}{\Gamma(\vartheta+1)}\left(\frac{K_g}{1-L_g}\right)||x_1-x_2||\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta}\\ &+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}\\ &\times\left((|\lambda|(\sigma-b)-1)\left(\frac{K_g}{1-L_g}\right)\right)||x_1-x_2||\end{aligned}$$

$$\leq\Bigg(\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}$$
$$\times\left(|\lambda|(\sigma-b)-1\right)\Bigg)\left(\frac{K_g}{1-L_g}\right)||x_1-x_2||_\infty.$$

Thus,

$$|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)|\leq\Bigg(\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}$$
$$\times\left(|\lambda|(\sigma-b)-1\right)\left(\frac{K_g}{1-L_g}\right)||x_1-x_2||_\infty.$$

By (3.1), the operator $\mathcal{H}$ is continuous. Hence, by Banach's contraction principle, $\mathcal{H}$ has a unique fixed point which is a unique solution of the problem (1.1)–(1.2) on $\mathcal{J}:=[b,\mathcal{T}]$. □

**Theorem 3.2.** *Assume that assumptions* 3.1 *and* 3.2 *hold. If*

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta+1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)-\lambda N_2\right|}\left(|\lambda|(\sigma-b)-1\right)\right]\left(\frac{K_g}{1-L_g}\right)<1,$$

*then there exists a unique solution for* (1.3)–(1.4) *on* $\mathcal{J}:=[b,\mathcal{T}]$.

The proof of Theorem 3.2 is similar to the Theorem 3.1.

**Theorem 3.3.** *Assume that assumptions* 3.1 *and* 3.3 *hold. Then there is at least one solution for the problem* (1.1)–(1.2) *on* $\mathcal{J}=:[b,\mathcal{T}]$.

*Proof.* **Step 1.** Show that $\mathcal{H}$ maps bounded sets (balls) into bounded sets in $\mathcal{C}([b,\mathcal{T}],\mathbb{R})$.

For a positive number $r_1$, let $B_{r_1}=\{x\in\mathcal{C}([b,\mathcal{T}],\mathbb{R}):||\mathcal{Z}^*||\leq r_1\}$ be a bounded ball in $\mathcal{C}([b,\mathcal{T}],\mathbb{R})$, where

$$||\mathcal{Z}^*||=\sup_{t\in[b,\mathcal{T}]}(||x||+||g||).$$

Then

$$|\mathcal{H}(x)(t)|\leq\frac{1}{\Gamma(\vartheta)}\int_b^t\left(\ln\frac{t}{s}\right)^{\vartheta-1}\frac{|g(s,x(s)),^c\mathcal{D}^\vartheta x(s))|}{s}ds+\frac{\left(\ln\frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}$$
$$\times\Bigg[|\lambda|\int_b^\sigma\int_b^s\left(\ln\frac{s}{r}\right)^{\vartheta-1}\frac{|g(r,x(r),^c\mathcal{D}^\vartheta x(r))|}{r}drds$$
$$-\int_b^{\mathcal{T}}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta-1}\frac{|g(s,x(s)),^c\mathcal{D}^\vartheta x(s))|}{s}ds\Bigg]$$

$$\leq \frac{1}{\Gamma(\vartheta)} \int_b^t \left(\ln \frac{t}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds + \frac{\left(\ln \frac{t}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|}$$
$$\times \left[|\lambda| \int_b^\sigma \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{r} drds\right.$$
$$\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds\right],$$

i.e.,

$$|\mathcal{H}(x)(t)| \leq \Bigg(\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta} + \frac{\left(\ln \frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|}$$
$$\times (|\lambda|(\sigma-b)-1)\Bigg)\varphi(r)||p||.$$

**Step 2.** Show that $\mathcal{H}$ maps bounded sets (balls) into equicontinuous sets in $\mathcal{C}([b,\mathcal{T}],\mathbb{R})$. Let $\mu_1, \mu_2 \in [b,\mathcal{T}]$, $\mu_1 < \mu_2$. Then, we have

$$||\mathcal{H}(x)(\mu_1) - \mathcal{H}(x)(\mu_2)|| \leq \frac{1}{\Gamma(\vartheta)}\Bigg[\int_b^{\mu_1} \left[\left(\ln \frac{\mu_2}{r}\right)^{\vartheta-1} - \left(\ln \frac{\mu_1}{r}\right)^{\vartheta-1}\right] \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds$$
$$+ \int_{\mu_1}^{\mu_2} \left(\ln \frac{\mu_2}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds\Bigg]$$
$$+ \frac{\left(\ln \frac{\mu_2}{b}\right)^{\vartheta-1} - \left(\ln \frac{\mu_1}{b}\right)^{\vartheta-1}}{\Gamma(\vartheta)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|}$$
$$\times \left[|\lambda| \int_b^\sigma \int_b^s \left(\ln \frac{s}{r}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{r} drds\right.$$
$$\left. - \int_b^{\mathcal{T}} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta-1} \frac{\varphi(||\mathcal{Z}^*||)||p||}{s} ds\right].$$

Obviously, the right-hand side of the above inequality tends to zero independently of $u, v \in B_{r_1}$ as $\mu_2 - \mu_1 \to 0$. As $\mathcal{H}$ satisfies the above assumptions, therefore, by the Arzela-Ascoli theorem, it follows that $\mathcal{H} : \mathcal{C}([b,\mathcal{T}],\mathbb{R}) \to \mathcal{C}([b,\mathcal{T}],\mathbb{R})$ is completely continuous. Let $x$ be a solution. Then, for $t \in [b,\mathcal{T}]$ and following the similar computations as in the first step, we have

$$|x(t)| = \lambda|\mathcal{H}(x)(t)|$$
$$\leq \Bigg(\frac{1}{\Gamma(\vartheta+1)} \left(\ln \frac{\mathcal{T}}{s}\right)^{\vartheta} + \frac{\left(ln \frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln \frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|} (|\lambda|(\sigma-b)-1)\Bigg)$$

$$\times\, \varphi(||x||)||p||.$$

Consequently, we have

$$\frac{||x(t)||}{\left(\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1}-\lambda N_1\right|}\left(|\lambda|(\sigma-b)-1\right)\right)\varphi(||x||)||p||}\leq 1.$$

There exists $\mathcal{M}^*$ such that $||x||\neq\mathcal{M}^*$. Let us set

$$U=\{x\in\mathcal{C}([b,\mathcal{T}],\mathbb{R}):||x||<\mathcal{M}^*\}.$$

Note that the operator $\mathcal{H}:\overline{\mathcal{U}}\to\mathcal{C}([b,\mathcal{T}],\mathbb{R})$ is continuous and completely continuous. From the choice of $\mathcal{U}$, there is no $x\in\partial\mathcal{U}$ such that $x=\lambda\mathcal{H}x$ for some $0\leq\lambda\leq 1$. Consequently, by the nonlinear alternative of Lerary-Schauder type (Lemma 2.3), we deduce that $\mathcal{H}$ has fixed point $x\in\overline{\mathcal{U}}$ which is a solution of the problem (1.1)–(1.2). □

**Theorem 3.4.** *Assume that assumptions* 3.1 *and* 3.3 *hold and there exists a constant* $\mathcal{M}^*>0$, *such that*

$$\mathcal{M}^*>\left(\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{s}\right)^{\vartheta}+\frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta+1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)-\lambda N_1\right|}\left(|\lambda|(\sigma-b)-1\right)\right)||p||\varphi(||x||).$$

*Then, there is at least one solution for the problem* (1.3)–(1.4) *on* $\mathcal{J}=:[b,\mathcal{T}]$.

The proof of Theorem 3.4 is similar to the Theorem 3.3.

## 4. Examples

In this section, some examples are introduced for Hadamard and Caputo-Hadamard fractional derivatives of implicit fractional differential equations with integral boundary conditions.

*Example* 4.1. Consider the implicit Hadamard FDE's with three point integral boundary conditions of the form

$$ {}^{\mathcal{H}}\mathcal{D}^{\frac{10}{7}}x(t)=\frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{H}}\mathcal{D}^{\frac{10}{7}}x(t)|)},\quad 1<\vartheta\leq 2, \tag{4.1}$$

$$x(1)=0,\ x(b)=\lambda\int_b^{\sigma}x(s)ds. \tag{4.2}$$

Here $\vartheta=\frac{10}{7}$,

$$g(t,x(t),{}^{\mathcal{H}}\mathcal{D}^{\vartheta}x(t))=\frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{H}}\mathcal{D}^{\frac{10}{7}}x(t)|)},$$

$\sigma = 3$, $\lambda = 5$. Hence, the Assumption 3.2 holds, with $K_g = L_g = \frac{1}{49}$ and we will check that

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta} + \frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{2\vartheta-1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta-1} - \lambda N_1\right|}\left(|\lambda|(\sigma-b)-1\right)\right]\frac{K_g}{(1-L_g)} < 1.$$

Thus, the Theorem 3.1 is satisfied and shows that the problem (4.1)–(4.2) has a unique solution on $\mathcal{J} =: [b, \mathcal{T}]$.

*Example* 4.2. Consider the implicit Caputo-Hadamard FDE's with three point integral boundary conditions of the form

$$ {}^{\mathcal{CH}}\mathcal{D}^{\frac{10}{7}}x(t) = \frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{CH}}\mathcal{D}^{\frac{10}{7}}x(t)|)}, \quad 1 < \vartheta \leq 2, \tag{4.3}$$

$$ x(1) = 0,\ x(b) = \lambda\int_b^{\sigma} x(s)ds. \tag{4.4}$$

Here $\vartheta = \frac{10}{7}$,

$$g(t, x(t), {}^{\mathcal{CH}}\mathcal{D}^{\vartheta}x(t)) = \frac{|x|}{(t+6)^2(|1+|x|+|{}^{\mathcal{CH}}\mathcal{D}^{\frac{10}{7}}x(t)|)},$$

$\sigma = 3$, $\lambda = 5$. Hence, the Assumption 3.2 holds, with $K_g = L_g = \frac{1}{49}$ and we will check that

$$\left[\frac{1}{\Gamma(\vartheta+1)}\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta} + \frac{\left(\ln\frac{\mathcal{T}}{b}\right)^{\vartheta+1}}{\Gamma(\vartheta+1)\left|\left(\ln\frac{\mathcal{T}}{b}\right) - \lambda N_2\right|}\left(|\lambda|(\sigma-b)-1\right)\right]\frac{K_g}{(1-L_g)} < 1.$$

Thus, the Theorem 3.2 is satisfied and shows that the problem (4.3)–(4.4) has a unique solution on $\mathcal{J} =: [b, \mathcal{T}]$.

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# SERIES EXPANSION OF A COTANGENT SUM RELATED TO THE ESTERMANN ZETA FUNCTION

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Abstract. In this paper, we study the cotangent sum $c_0\left(\frac{q}{p}\right)$ related to the Estermann zeta function for the special case when the numerator is equal to 1 and get two useful series expansions of $c_0\left(\frac{1}{p}\right)$.

## 1. Introduction

For a positive integer $p$ and $q = 1, 2, \ldots, p-1$, such that $(p, q) = 1$, let the cotangent sum (see [10])

$$c_0\left(\frac{q}{p}\right) = -\sum_{k=1}^{p-1} \frac{k}{p} \cot \frac{\pi k q}{p}.$$

$c_0\left(\frac{q}{p}\right)$ is the value at $s = 0$,

$$E_0\left(0, \frac{q}{p}\right) = \frac{1}{4} + \frac{i}{2} c_0\left(\frac{q}{p}\right)$$

of the Estermann zeta function

$$E_0\left(s, \frac{q}{p}\right) = \sum_{k\geq 1} \frac{d(k)}{k^s} \exp\left(\frac{2\pi i k q}{p}\right).$$

It is well-known that the sum $c_0\left(\frac{q}{p}\right)$ satisfies the reciprocity formula (see [2])

$$c_0\left(\frac{q}{p}\right) + \frac{p}{q} c_0\left(\frac{p}{q}\right) - \frac{1}{\pi q} = \frac{i}{2} \psi_0\left(\frac{q}{p}\right).$$

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The Vasyunin cotangent sum (see [11])

$$V\left(\frac{q}{p}\right)=\sum_{r=1}^{p-1}\left\{\frac{rq}{p}\right\}\cot\left(\frac{\pi r}{p}\right)=-c_0\left(\frac{\overline{q}}{p}\right)$$

arises in the study of the Riemann zeta function by virtue of the formula (see [2, 9])

$$\begin{aligned}&\frac{1}{2\pi\sqrt{pq}}\int_{-\infty}^{+\infty}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left(\frac{q}{p}\right)^{it}\frac{dt}{\frac{1}{4}+t^2}\\ =&\frac{\log 2\pi-\gamma}{2}\left(\frac{1}{p}+\frac{1}{q}\right)+\frac{p-q}{2pq}\log\frac{q}{p}-\frac{\pi}{2pq}\left(V\left(\frac{p}{q}\right)+V\left(\frac{q}{p}\right)\right).\end{aligned}$$

This formula is connected to the approach of Nyman, Beurling and Báez-Duarte to the Riemann hypothesis (see [8]), which states that the Riemann hypothesis is true if and only if $\lim\limits_{n\to\infty} d_N=0$, where

$$d_N^2=\inf_{A_N}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\left|1-\zeta A\left(\frac{1}{2}+it\right)\right|^2\frac{dt}{\frac{1}{4}+t^2},$$

and the infimum is taken over all Dirichlet polynomials

$$A_N(s)=\sum_{n=1}^{N}\frac{a_n}{n^s}.$$

In a recent work with A. Bayad [7], we have proved that the sum $V\left(\frac{q}{p}\right)$ satisfies the reciprocity formula

$$V\left(\frac{q}{p}\right)+V\left(\frac{p}{q}\right)=\frac{1}{\pi}\left(G(p,p)+G(q,q)+G(p,q)+(q-p)\log\frac{q}{p}\right), \tag{1.1}$$

where

$$G(p,q)=\sum_{k\geq 1}\frac{pq}{k(k+1)}\left\{\frac{k}{p}\right\}\left\{\frac{k}{q}\right\}.$$

Thereafter the restriction of the relationship (1.1) to $q=1$ gives

$$c_0\left(\frac{1}{p}\right)=-\frac{1}{\pi}G(p,p)-(p-1)\log p.$$

Exactly our interest in this work is the case $q=1$ in order to get two series expansions of $c_0\left(\frac{1}{p}\right)$. First we recall the different asymptotical writings of $c_0\left(\frac{1}{p}\right)$ in the literature. In [10, Theorem 1.2, Theorrem 1.3] M. Th. Rassias proved that

$$c_0\left(\frac{1}{p}\right)=\frac{1}{\pi}p\log p-\frac{p}{\pi}(\log 2\pi-\gamma)+\{\mathcal{O}(\log p)\ \text{ or }\ \mathcal{O}(1)\}.$$

In [9, Theorem 1.7] H. Maier and M. Th. Rassias provide the following improvement. Let $b,n\in\mathbb{N}$, $b\geq 6N$, with $N=\left\lfloor\frac{n}{2}\right\rfloor+1$. There exist absolute real constants $A_1,A_2\geq 1$

and absolute real constants $E_l, l$, with $|E_l| \leq (A_1 l)^{2l}$, such that for each $n \in \mathbb{N}$ we have

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi}(\log 2\pi - \gamma) - \frac{1}{\pi} + \sum_{l=1}^{n} E_l p^{-l} + R_n^{\star}(p),$$

where $|R_n^{\star}(p)| \leq (A_2 n)^{4n} p^{-(n+1)}$.

Only in [9, Theorem 1.9] H. Maier and M. Th. Rassias provide another improvement,

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi}(\log 2\pi - \gamma) + C_1 p + \mathcal{O}(1).$$

We draw attention that S. Bettin finds other reformulations of $c_0\left(\frac{1}{p}\right)$ inspired from continued fraction theory (see [3]).

Finally from another point of view we show in [5] with A. Bayad and M. O. Hernane that

$$\begin{aligned} c_0\left(\frac{1}{p}\right) = &-\frac{1}{\pi}\left(\log \frac{2\pi}{p} - \gamma\right) p + \frac{1}{\pi} + \frac{\pi}{36p} \\ &- \frac{1}{2} \sum_{k=2}^{\left\lfloor \frac{N}{2} \right\rfloor} (-1)^k \frac{4^k \pi^{2k-1} B_{2k}^2}{k(2k)!} \left(\frac{1}{p}\right)^{2k-1} + \mathcal{O}\left(\frac{1}{p^N}\right). \end{aligned}$$

There is a misprint in the formula (1.22) Corollary 1.2 in [5] the correct one is in the formula (1.21) Corollary 1.2.

Otherwise in the same paper [5], an integral representation of $c_0(\frac{1}{p})$ is given by

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} \int_0^1 \frac{(p-2)x^p - px^{p-1} + px - p + 2}{(x-1)^2 (x^p - 1)} dx. \tag{1.2}$$

In this work we prove that

$$(p-2)x^p - px^{p-1} + px - p + 2 = (x-1)^3 \sum_{r=1}^{p-1} (p-r-1) r x^{r-1}$$

and we get another formulation that is

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} \int_0^1 \frac{\sum_{r=1}^{p-1} (p-r-1) r x^{r-1}}{1 + x + \cdots + x^{p-1}} dx.$$

Applying some techniques from the generating function theory [4] to previous integrals; we find two series expansions of $c_0\left(\frac{1}{p}\right)$, as they are well explained in the next section.

## 2. Series Expansion of $c_0\left(\frac{1}{p}\right)$

Let $b_k$ be the integer sequence defined by $b_0 = 1, b_1 = 2$ and the recursive formulae:

$$b_k - 2b_{k-1} + b_{k-2} = 0, \quad 2 \leq k \leq p-1, \ k = p+1,$$

$$b_p - 2b_{p-1} + b_{p-2} = 1$$

and

$$b_k - 2b_{k-1} + b_{k-2} - b_{k-p} + 2b_{k-p-1} - b_{k-p-2} = 0, \quad k \geq p+2.$$

According to the terms $b_k$ we get the first series expansion in the following theorem.

**Theorem 2.1.**

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p\,(p-1)\,(p-2) \sum_{k\geq 0} \frac{b_k}{(k+1)\,(k+p+1)\,(k+2)\,(k+p)}. \tag{2.1}$$

For $p \geq 1$ we define the arithmetic function $a_p$ in the form

$$a_p(k) = \begin{cases} 1, & \text{if } p \mid k, \\ -1, & \text{if } k \equiv 1 \pmod{p}, \\ 0, & \text{otherwise.} \end{cases}$$

This function is not multiplicative. In general the arithmetical functions are defined from the set of natural integers $\mathbb{N}$ into $\mathbb{C}$. We can extend this definition to $\mathcal{F}(\mathbb{C}, \mathbb{C})$; set of functions from $\mathbb{C}$ to $\mathbb{C}$. In that case the corresponding function is $A : \mathbb{N} \to \mathcal{F}(\mathbb{C}, \mathbb{C})$ with $A(p) = a_p$. Furthermore, $A(pq) = \pm A(p)A(q)$ and $|A|$ is multiplicative.

Let the function $M(p,k)$ defined by

$$M(p,0) = \frac{1}{2}p^2 - \frac{3}{2}p + 1$$

and

$$M(p,k) = (p-1)\left(\frac{1}{2}p + k - 1\right) - k\,(p+k-1)\,(H_{p+k-1} - H_k), \quad k \geq 1,$$

where $H_k$ is the Harmonic number

$$H_k = \sum_{j=1}^{k} \frac{1}{j}.$$

Following this function a second series expansion of $c_0\left(\frac{1}{p}\right)$ is given in the following theorem.

**Theorem 2.2.**

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} \sum_{k\geq 0} a_p(k) M(p,k). \tag{2.2}$$

2.1. **Proof of Theorem** 2.1. We take inspiration from the theory of generating functions [4,6], and prove that the sequence $(b_k)$ is generated by the rational function:

$$f(x) = \frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}}.$$

More precisely we get the following lemma.

**Lemma 2.1.**

$$\frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}} = \sum_{k\geq 0} b_k x^k, \quad |x| < 1. \tag{2.3}$$

*Proof.* It is well known that

$$\frac{1}{1-x} = \sum_{k\geq 0} x^k, \quad |x| < 1. \tag{2.4}$$

Since for $0 \leq x < 1$

$$0 < (x-1)^2 (1-x^p) < 1$$

and

$$(x-1)^2 (1-x^p) = 1 - \left(2x - x^2 + x^p - 2x^{p+1} + x^{p+2}\right),$$

then we have

$$0 < 2x - x^2 + x^p - 2x^{p+1} + x^{p+2} < 1.$$

Furthermore, $f(x)$ is developable on entire series to get the result we have to take the quantity $2x - x^2 + x^p - 2x^{p+1} + x^{p+2}$ instead of $x$ in the last formula (2.4). Now, writing

$$\frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}} = \sum_{k\geq 0} d_k x^k$$

and then

$$\left(1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}\right)\left(\sum_{k\geq 0} d_k x^k\right) = 1.$$

To compute this we use the well known Cauchy product of two entire series

$$\left(\sum_{k\geq 0} a_k x^k\right)\left(\sum_{j\geq 0} d_j x^j\right) = \sum_{k\geq 0}\left(\sum_{j=0}^{k} a_j d_{k-j}\right) x^k,$$

which generates the product of a polynomial of degree $n$ with an entire series that also gives an entire series as follows

$$\left(\sum_{k=0}^{n} a_k x^k\right)\left(\sum_{j\geq 0} d_j x^j\right) = \sum_{k\geq 0}\left(\sum_{j=0}^{\min\{n,k\}} a_j d_{k-j}\right) x^k.$$

We return to $f(x)$ in writing

$$1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2} = \sum_{k=0}^{p+2} a_k x^k,$$

with $a_0 = 1,\ a_1 = -2,\ a_2 = 1,\ a_p = -1,\ a_{p+1} = 2,\ a_{p+2} = -1$, and the others are zero. We conclude that $d_0 = 1,\ d_1 = 2.$ The formula

$$\sum_{j=0}^{\min\{p+2,k\}} a_j d_{k-j} = 0$$

states that

$$d_k - 2d_{k-1} + d_{k-2} = 0, \quad 2 \leq k \leq p-1,\ k = p+1,$$
$$d_p - 2d_{p-1} + d_{p-2} = 1$$

and

$$d_k - 2d_{k-1} + d_{k-2} - d_{k-p} + 2d_{k-p-1} - d_{k-p-2} = 0, \quad k \geq p+2.$$

Finally, we see that $d_k$ and $b_k$ are identical for every integer $k \geq 0$. For more information on this approach we refer to [6]. $\square$

To get the result (2.1) of Theorem 2.1 we must substitute the expression (2.3) in the identity (1.2) and one obtains

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\sum_{k\geq 0} b_k \int_0^1 \left((p-2)\,x^{k+p} - px^{k+p-1} + px^{k+1} + (2-p)\,x^k\right) dx.$$

Furthermore,

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\sum_{k\geq 0} b_k \left(\frac{p-2}{k+p+1} - \frac{p}{k+p} + \frac{p}{k+2} - \frac{p-2}{k+1}\right).$$

Finally,

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}p\,(p-1)\,(p-2)\sum_{k\geq 0} \frac{b_k}{(k+1)\,(k+p+1)\,(k+2)\,(k+p)}$$

and $c_0\,(1) = c_0\left(\frac{1}{2}\right) = 0$ is compatible with the definition of $c_0$.

Regarding the identity (2.3) Lemma 2.1 we remark that

$$\frac{1}{(1-x)^2\,(1-x^p)} = \sum_{k\geq 0} b_k x^k, \quad |x| < 1.$$

Furthermore, for $x = \frac{1}{2}$ we deduce that the coefficients $b_k$ satisfy the following statements

$$\sum_{k\geq 0}\frac{b_k}{2^k} = \frac{2^{p+2}}{2^p-1} \quad \text{and} \quad \lim_{k\to\infty}\frac{b_k}{2^k} = 0.$$

2.2. **Proof of Theorem** 2.2. First we began by proving another integral representation of $c_0\left(\frac{1}{p}\right)$.

**Lemma 2.2.**

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi}\int_0^1 \frac{\sum_{r=1}^{p-1}(p-r-1)\,rx^{r-1}}{1+x+\cdots+x^{p-1}}dx. \tag{2.5}$$

*Proof.*

$$\begin{aligned}(x-1)^3\sum_{r=1}^{q-1}(q-r-1)\,rx^{r-1} = &\sum_{r=3}^{q}(q-r+1)\,(r-2)\,x^r - 3\sum_{r=2}^{q-1}(q-r)\,(r-1)\,x^r\\ &+3\sum_{r=1}^{q-2}(q-r-1)\,rx^r - \sum_{r=0}^{q-3}(q-r-2)\,(r+1)\,x^r.\end{aligned}$$

It's obvious to remark that

$$(q-r+1)\,(r-2) - 3\,(q-r)\,(r-1) + 3\,(q-r-1)\,r - (q-r-2)\,(r+1) = 0$$

and the quantity

$$(t-1)^3 \sum_{r=1}^{q-1} (q-r-1)\, r x^{r-1}$$

is reduced to

$$(q-2)\, x^q + 2\,(q-3)\, x^{q-1} + 3(q-4)x^{q-2} - 3(q-2)x^{q-1} - 6(q-3)x^{q-2} - 3(q-2)x^2$$
$$+ 3(q-2)x^{q-2} + 3(q-2)x + 6(q-3)x^2 - q + 2 - 2(q-3)x - 3(q-4)x^2.$$

After simplification we obtain

$$(t-1)^3 \sum_{r=1}^{q-1} (q-r-1)\, r x^{r-1} = (q-2)\, x^q - q x^{q-1} + qx - q + 2. \qquad \square$$

The Theorem 2.2 is immediate from the Lemma 2.2 in the following way. Since

$$\frac{1}{1+x+\cdots+x^{p-1}} = \frac{1-x}{1-x^p}$$

and $|x| < 1$, then

$$\frac{1}{1+x+\cdots+x^{p-1}} = \frac{1-x}{1-x^p} = \sum_{k\geq 0} (1-x)\, x^{pk}.$$

Furthermore,

$$\frac{1}{1+x+\cdots+x^{p-1}} = \sum_{k\geq 0} a_p\,(k)\, x^k$$

and we have

$$\frac{\sum_{r=1}^{p-1} (p-r-1)\, r x^{r-1}}{1+x+\cdots+x^{p-1}} = \sum_{k\geq 0} \sum_{r=1}^{p-1} a_p\,(k)\,(p-r-1)\, r x^{k+r-1}.$$

The passage to the integral inducts

$$c_0\left(\frac{1}{p}\right) = \sum_{k\geq 0} \sum_{r=1}^{p-1} a_p\,(k)\, \frac{(p-r-1)\, r}{k+r}.$$

But

$$\sum_{r=1}^{p-1} \frac{(p-r-1)\, r}{k+r} = (p-1)\left(\frac{1}{2}p + k - 1\right) - k\,(p+k-1) \sum_{r=k+1}^{p+k-1} \frac{1}{r}$$

and the result (2.2) is deduced.

## 3. Connection to Digamma Function

We finish this work by revisiting the proof of the expression of $c_0\left(\frac{1}{p}\right)$ according to the function digamma and Bernoulli polynomials in the work [1] of L. Báez Duarte et al.

$$c_0\left(\frac{1}{p}\right) = \frac{2}{\pi}\sum_{r=1}^{p-1} B_1\left(\frac{r}{p}\right)\psi\left(\frac{r}{p}\right),$$

where $B_1$ is the reduced Bernoulli polynomial

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ \{x\} - \frac{1}{2}, & \text{otherwise,} \end{cases}$$

and $\psi$ the digamma function defined by

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k\geq 1}\left(\frac{1}{k} - \frac{1}{k+z}\right).$$

Starting with the demonstration of a property of $\psi$ that will be used later.

**Proposition 3.1.**

$$\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) = p\int_0^1 \frac{x^{r-1}}{1+x+\cdots+x^{p-1}}dx. \tag{3.1}$$

*Proof.* We quote from [5] the formula

$$\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) = p\sum_{k\geq 0}\frac{1}{(pk+r+1)(pk+r)}.$$

The general term $\frac{1}{(pk+r+1)(pk+r)}$ can be written as following

$$\frac{1}{(pk+r+1)(pk+r)} = \frac{1}{pk+r} - \frac{1}{pk+r+1} = \int_0^1\left(x^{pk+r-1} - x^{pk+r}\right)dx$$

and the passage to the sum states that

$$\sum_{k\geq 0}\frac{1}{(pk+r+1)(pk+r)} = \int_0^1 \frac{x^{r-1}-x^r}{1-x^p}dx.$$

Finally,

$$\sum_{k\geq 0}\frac{1}{(pk+r+1)(pk+r)} = \int_0^1 \frac{x^{r-1}}{1+x+\cdots+x^{p-1}}dx$$

and we have (3.1). Proposition 3.1 follows. □

In [5], it is shown that

$$\log p = \frac{1}{p}\sum_{r=1}^{p-1} r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right).$$

This identity conducts to the following interesting lemma.

**Lemma 3.1.**

$$\sum_{r=1}^{p} \psi\left(\frac{r}{p}\right) = -\gamma p - p\log p. \tag{3.2}$$

*Proof.* Since

$$\sum_{r=1}^{p-1} r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right) = p\log p,$$

then

$$-\sum_{r=1}^{p} \psi\left(\frac{r}{p}\right) + \psi\left(1\right) p = p\log p.$$

Furthermore,

$$\sum_{r=1}^{p} \psi\left(\frac{r}{p}\right) = -\gamma p - p\log p. \qquad \square$$

According to the identity (3.1) Proposition 3.1 and the integral representation (2.5) we conclude that

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi p}\sum_{r=1}^{p-1} (p-r-1)\, r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right).$$

Furthermore combining this result with the identity (3.2) Lemma 3.1 we get

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\log p + \frac{1}{\pi p}\sum_{r=1}^{p-1} (p-r)\, r\left(\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right)\right)$$

and

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi}\log p - \gamma\frac{p-1}{\pi p} + \frac{1}{\pi p}\sum_{r=1}^{p-1} (2r-p-1)\, \psi\left(\frac{r}{p}\right),$$

then

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi p}\sum_{r=1}^{p-1} (2r-p)\, \psi\left(\frac{r}{p}\right).$$

But

$$2r - p = 2p\left(\frac{r}{p} - \frac{1}{2}\right) = 2pB_1\left(\frac{r}{p}\right),$$

which means that

$$c_0\left(\frac{1}{p}\right) = \frac{2}{\pi}\sum_{r=1}^{p} B_1\left(\frac{r}{p}\right)\psi\left(\frac{r}{p}\right).$$

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# FIXED POINT THEOREMS VIA $WF$-CONTRACTIONS

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Abstract. In this paper, we introduce a new class of contractions which remains a mixed type of weak and $F$-contractions but not any of them.

## 1. Introduction and Preliminaries

Investigating fixed point of a mapping continues to be an active topic of research in nonlinear analysis wherein Banach contraction principle remains the main tool as it offers an efficient and plain technique to compute such points. This vital principle has undergone considerable extensions and generalizations in various ways concerning two or three terms in the contraction inequality. One of the noteworthy generalization of this principle involving three terms was due to Alber and Guerre-Delabriere [1] which was refined later by Rhoades [17] and then generalized by Dutta and Choudhury [7].

Let $\Psi$ be the set of all continuous and monotonically nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\psi(t) = 0$ if and only if $t = 0$.

**Theorem 1.1** ([7]). *Let $(X,d)$ be a complete metric space and $f : X \to X$ a weak contractive mapping, i.e.,*

$$\psi(d(fx,fy)) \leq \psi(d(x,y)) - \varphi(d(x,y)),$$

*for all $x,y \in X$, where $\psi,\varphi \in \Psi$. Then $f$ has a unique fixed point.*

Nowadays, there is a tradition of proving unified fixed point results employing an auxiliary function general enough yielding several contractions and henceforth several fixed point results in one go. In 1997, Popa [15] introduced the idea of implicit function which was well followed by [2,3,9,10,16]. Khojasteh et al. [12] introduced the idea of simulation function which is also designed to unify several contractions. For

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further work on simulation functions, one can consult [4, 6, 8, 11, 13, 18] and some other ones. One of the recent widely discussed generalizations of Banach principle (utilizing auxiliary function) is due to Wardowski [19] wherein the author generalized Banach contraction principle by introducing a new type of contractions called $F$-contraction and proved that every such contraction defined on a complete metric space possesses a unique fixed point.

**Definition 1.1** ([19])**.** A self-mapping $f$ on a metric space $(X, d)$ is said to be an $F$*-contraction* if there exists $\tau > 0$ such that

$$d(fx, fy) > 0 \Rightarrow \tau + F(d(fx, fy)) \leq F(d(x, y)), \quad \text{for all } x, y \in X, \tag{1.1}$$

where $F : \mathbb{R}_+ \to \mathbb{R}$ is a mapping satisfying the following conditions:

**F1:** $F$ is strictly increasing;

**F2:** for every sequence $\{s_n\}$ of positive real numbers,

$$\lim_{n\to\infty} s_n = 0 \Leftrightarrow \lim_{n\to\infty} F(s_n) = -\infty;$$

**F3:** there exists $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k F(s) = 0$.

We denote by $\mathcal{F}$ the family of all functions $F$ satisfying conditions (**F1**)-(**F3**). Some natural and known members of $\mathcal{F}$ are $F(s) = \ln s$, $F(s) = s + \ln s$ and $F(s) = \frac{-1}{\sqrt{s}}$.

## 2. $WF$-Contractions

**Definition 2.1.** A self-mapping $f$ on a metric space $(X, d)$ is said to be $WF$*-contraction* if there exist two functions $G, \delta : [0, \infty) \to [0, \infty)$ such that, for all $x, y \in X$ with $d(fx, fy) > 0$, we have

$$\delta(d(x, y)) + G(d(fx, fy)) \leq G(d(x, y)), \tag{2.1}$$

where $G$ and $\delta$ satisfy the following conditions:

**G1:** $G$ is strictly increasing;

**G2:** $\delta(t) > 0$ for all $t > 0$ and for every strictly decreasing sequence $\{s_n\}$ of positive real numbers,

$$\lim_{n\to\infty} \delta(s_n) = 0 \Rightarrow \lim_{n\to\infty} s_n = 0;$$

**G3:** there exists $k \in (0, 1)$ such that $\lim_{s\to 0^+} s^k G(s) = 0$.

In the sequel, $\mathbb{G}$ denotes the family of all functions $G$ meeting the requirements of Definition 2.1 while $\Delta$ stands for the set of all functions $\delta$ enjoying (**G2**). Some members of $\mathbb{G}$ are $G(s) = \ln(s + 1)$, $G(s) = s$, $G(s) = (s + 1) + \frac{1}{(s+1)}$ and $G(s) = \sqrt[n]{s}$, $n \in \mathbb{N}$.

*Example* 2.1. Let $X = [0, \infty)$ and $f$ a self-mapping on $X$ given by

$$f(x) = \begin{cases} \dfrac{x+2}{2}, & \text{for } x \leq 2, \\ 2, & \text{for } x \geq 2. \end{cases}$$

Then $f$ satisfies (2.1) for $G(s) = s + \frac{1}{2(s+1)}$ and $\delta(t) = \frac{t}{8}$. Indeed, the following three cases arise.

**Case 1.** If $2 \le x \le y$, then $d(fx, fy) = 0$. However, inequality (2.1) becomes:

$$\frac{y-x}{8} + \frac{1}{2} \le (y-x) + \frac{1}{2(y-x+1)},$$

which can be written as

$$\frac{1}{2} \le \frac{7}{8}z + \frac{1}{2(z+1)}, \tag{2.2}$$

where $z = y - x \ge 0$. Observe that, the R.H.S of (2.2) is increasing mapping in $z$ for $z \ge 0$ having the value $\frac{1}{2}$ at $z = 0$.

**Case 2.** If $2 \ge y \ge x$, then (2.1) becomes:

$$\frac{y-x}{8} + \frac{y-x}{2} + \frac{1}{(y-x)+2} \le (y-x) + \frac{1}{2(y-x)+2},$$

which can be written as

$$0 \le \frac{3}{8}z + \frac{1}{2z+2} - \frac{1}{z+2}, \tag{2.3}$$

where $z = y - x \ge 0$. Here, also, the R.H.S of (2.3) is increasing mapping in $z$ for $z \ge 0$ with the value 0 at $z = 0$.

**Case 3.** If $x \le 2 \le y$, then (2.1) becomes:

$$\frac{y-x}{8} + \left(1 - \frac{x}{2}\right) + \frac{1}{2\left((1-\frac{x}{2})+1\right)} \le (y-x) + \frac{1}{2((y-x)+1)}$$

or

$$\left(1 - \frac{x}{2}\right) + \frac{1}{4-x} \le \frac{7}{8}(y-x) + \frac{1}{2(y-x)+2}$$

Let $2 - x = a$ and $y - 2 = b$. Then,

$$\frac{a}{2} + \frac{1}{2+a} \le \frac{7}{8}(a+b) + \frac{1}{2(a+b)+2},$$

which is equivalent to

$$\frac{2b+a}{(2+a)(1+a+b)} \le \frac{3a+7b}{4},$$

which is true if we expand it and remember that $a, b \ge 0$.

The following two remarks highlight the relation between $WF$-contractions and the weak and $F$-contractions.

*Remark* 2.1. Observe that $\psi$ in Theorem 1.1 may not belong to $\mathbb{G}$ as it is not required to be strictly increasing. On the other hand, $f$ in Example 2.1 is a $WF$-contraction for $G(s) = s + \frac{1}{2(s+1)}$ but not weak contraction as $G(0) \ne 0$. Consequently, the class of $WF$-contractions and the class of weak contractions are independent.

*Remark* 2.2. Notice that, $G(s) = s,\ s \in [0, \infty)$, is a member of $\mathbb{G}$ which is not in $\mathcal{F}$. On the other hand, $F \in \mathcal{F}$ given by $F(s) = \ln s$ is not in $\mathbb{G}$ (for $\delta \equiv \tau$).

*Remark* 2.3. Every $WF$-contraction mapping is a contractive mapping and hence continuous. This fact follows from (**G1**) and (2.1), i.e.,

$$d(fx, fy) < d(x, y), \quad \text{for all } x, y \in X, x \neq y.$$

**Lemma 2.1.** *Every $WF$-contraction mapping has at most one fixed point.*

*Proof.* If $x, y \in X$ are two distinct fixed points of $f$, then (2.1) gives rise $\delta(d(x, y)) \leq 0$, which is a contradiction as $\delta(t) > 0$ for all $t > 0$. □

**Lemma 2.2.** *Let $(X, d)$ be a metric space space and $\{t_n\}$ a sequences of positive real numbers such that*

$$\delta(t_n) + G(t_{n+1}) \leq G(t_n), \tag{2.4}$$

*for all $n$, where $G \in \mathbb{G}$ and $\delta \in \Delta$. Then the sequence $\{t_n\}$ is decreasing and $\sum_{i=0}^{\infty} \delta(t_i) < \infty$.*

*Proof.* As $\delta(t) > 0$ for all $t > 0$, we have $G(t_{n+1}) < G(t_n)$ for all $n \in \mathbb{N}$. Since $G$ is strictly increasing, we get $t_{n+1} < t_n$, for all $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} t_n = r$ for some $r \geq 0$. Then $G(r) \leq G(t_{n+1})$ for all $n \geq 0$. In view of (2.4), we have

$$\begin{aligned} G(t_{n+1}) &\leq G(t_n) - \delta(t_n) \\ &\leq G(t_{n-1}) - [\delta(t_n) + \delta(t_{n-1})] \\ &\ \vdots \\ &\leq G(t_0) - \sum_{i=0}^{n} \delta(t_i). \end{aligned} \tag{2.5}$$

Therefore, $\sum_{i=0}^{n} \delta(t_i) \leq G(t_0)$ for all $n \geq 0$. □

Now, we are equipped to state and prove our main result.

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space and $f : X \to X$ a $WF$-contraction for some $G \in \mathbb{G}$ and $\delta \in \Delta$. Then $f$ has a unique fixed point.*

*Proof.* Let $x_0 \in X$ be arbitrary and define a sequence $\{x_n\}$ in $X$ by $x_{n+1} := fx_n$ for all $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Notice that, if $x_n = x_{n+1}$ for some $n \in \mathbb{N}_0$, then $x_n$ is the required fixed point and we are done. Henceforth, we assume that such equality does not occur for all $n \in \mathbb{N}_0$. Denote $t_n = d(x_n, x_{n+1})$. On setting $x = x_n$ and $y = x_{n+1}$ in (2.1), we have

$$\delta(t_n) + G(t_{n+1}) \leq G(t_n). \tag{2.6}$$

In view of Lemma 2.2, $\sum_{i=0}^{\infty} \delta(t_i) < \infty$ so that $\lim_{n\to\infty} \delta(t_n) = 0$ and hence, in view of **(G2)**,

$$\lim_{n\to\infty} t_n = 0. \tag{2.7}$$

We assert that $\{x_n\}$ is a Cauchy sequence. From **(G3)**, there is $k \in (0,1)$ such that

$$\lim_{n\to\infty} t_n^k G(t_n) = 0. \tag{2.8}$$

Let $M = \min \delta(t_i),\ 0 \le i \le n$. In view of (2.5), we have

$$\begin{aligned} t_{n+1}^k\Big(G(t_{n+1}) - G(t_0)\Big) &\le t_{n+1}^k\Big(\Big[G(t_0) - \sum_{i=0}^{n} \delta(t_i)\Big] - G(t_0)\Big) \\ &= -t_{n+1}^k \sum_{i=0}^{n} \delta(t_i) \\ &\le -n t_{n+1}^k M \\ &\le 0. \end{aligned}$$

Letting $n \to \infty$ (in view of (2.7) and (2.8)) gives rise

$$\lim_{n\to\infty} n t_n^k = 0.$$

Therefore, there exists $n \in \mathbb{N}$ such that $n t_n^k \le 1$ for all $n \ge N$ so that

$$t_n \le \frac{1}{n^{1/k}}, \quad \text{for all } n \ge N. \tag{2.9}$$

Hence, for $m, n \in \mathbb{N}$ with $m > n \ge N$, we have

$$d(x_m, x_n) \le \sum_{i=n}^{m} t_i < \sum_{i=n}^{\infty} t_i \le \sum_{i=n}^{\infty} \frac{1}{i^{1/k}} < \infty.$$

Therefore, $\{x_n\}$ is a Cauchy sequence. In view of Remark 2.3 and the completeness of $X$, we have

$$x = \lim_{n\to\infty} x_{n+1} = f(\lim_{n\to\infty} x_n) = fx.$$

Now, Lemma 2.1 concludes the proof. ☐

*Remark* 2.4. $f$ in Example 2.1 is a $WF$-contraction. As $X$ is complete, $f$ has a unique fixed point (namely $x = 2$).

## 3. Consequences

**Corollary 3.1** (Banach Contraction Principle)**.** *Every self-mapping $f$ on a complete metric space $(X, d)$ has a unique fixed point if it satisfies the following:*

$$d(fx, fy) \le \beta d(x, y), \quad \textit{for all } x, y \in X,\ \textit{where } \beta \in (0,1). \tag{3.1}$$

*Proof.* The result is a direct consequence of Theorem 2.1 by taking $G(s) = s$ and $\delta(s) = \lambda s$ where $\lambda = 1 - \beta$. ☐

**Corollary 3.2.** *Every self-mapping $f$ on a complete metric space $(X, d)$ has a unique fixed point if it satisfies the following: for all $x, y \in X$ with $d(fx, fy) > 0$, we have*

$$d(fx, fy) \le e^{-\tau}[d(x, y) + 1] - 1, \quad \textit{where } \tau > 0. \tag{3.2}$$

*Proof.* Follows from Theorem 2.1 by taking $G(s) = \ln(s + 1)$ and $\delta(s) \equiv \tau$. ☐

One can list further consequences by varying the functions $G$ and $\delta$ suitably such as in above two corollaries.

## 4. Application

Finally, we discuss the application of fixed point methods to the following two-point boundary value problem of second order differential equation:

$$\begin{cases} x''(t) = u(t, x(t)), & t \in J = [0,1], \\ x(0) = x(1) = 0, \end{cases} \tag{4.1}$$

where $u : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the Green function $G(t,s)$ associated to (4.1) is given by

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t < s \le 1, \\ s(1-t), & 0 \le s < t \le 1. \end{cases}$$

Let $\mathcal{C}(J)$ denotes the space of all continuous functions defined on $J$. We know that $(\mathcal{C}(J), d)$ is a complete metric space (see [5, 14]) where

$$d(x,y) = \|u - v\|_\infty = \max_{t \in J} \left\{ |x(t) - y(t)| e^{-\tau t} \right\}, \quad \tau > 0. \tag{4.2}$$

Now, we prove the following result on the existence and uniqueness solution of the problem described by (4.1).

**Theorem 4.1.** *Problem* (4.1) *has at least one solution* $x^* \in \mathcal{C}^2$ *provided the following condition hold:*

$$\Big| G(t,s)u(s, x(s)) - G(t,s)u(s, y(s)) \Big| \le \tau e^{-2\tau} |x(s) - y(s)| - 1,$$

*for all* $t, s \in J$ *and* $x, y \in \mathcal{C}(J)$ *where* $\tau$ *is a given positive number.*

*Proof.* Observe that $x \in \mathcal{C}^2$ is a solution of the problem described by (4.1) if and only if $x \in \mathcal{C}$ is a solution of the integral equation

$$x(t) = \int_0^1 G(t,s) u(s, x(s)) ds, \quad \text{for all } t \in J. \tag{4.3}$$

Define a function $f : \mathcal{C}(J) \to \mathcal{C}(J)$ by

$$fx(t) = \int_0^1 G(t,s) u(s, x(s)) ds, \quad \text{for all } t \in J. \tag{4.4}$$

Clearly, if $x \in \mathcal{C}(J)$ is a fixed point of $f$, then $x \in \mathcal{C}(J)$ is a solution of (4.3) and hence of (4.1). Let $x, y \in \mathcal{C}(J)$ then, by the hypothesis, we have

$$\begin{aligned}
|fx(t) - fy(t)| =& \Big| \int_0^1 G(t,s)u(s,x(s))ds - \int_0^1 G(t,s)u(s,y(s))ds \Big| \\
\leq& \int_0^1 \Big| G(t,s)u(s,x(s)) - G(t,s)u(s,y(s)) \Big| ds \\
\leq& \int_0^1 \Big[ \tau e^{-2\tau} |y(s) - x(s)| e^{-\tau s} e^{\tau s} - 1 \Big] ds \\
=& \int_0^1 \tau e^{-2\tau} e^{\tau s} |y(s) - x(s)| e^{-\tau s} ds - 1 \\
\leq& \tau e^{-2\tau} d(x,y) \int_0^1 e^{\tau s} ds - 1 \\
\leq& e^{-\tau} d(x,y) - 1 \\
\leq& e^{-\tau} d(x,y) + e^{-\tau} - 1,
\end{aligned}$$

so that

$$|fx(t) - fy(t)| e^{-\tau t} \leq e^{-\tau} d(x,y) + e^{-\tau} - 1.$$

Thus, $d(fx, fy) \leq e^{-\tau} d(x,y) + e^{-\tau} - 1$ so that condition (3.2) is satisfied. Now, Corollary 3.2 ensures the existence of a unique solution of 4.1. □

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# A CATEGORICAL CONNECTION BETWEEN CATEGORIES $(m, n)$-HYPERRINGS AND $(m, n)$-RING VIA THE FUNDAMENTAL RELATION $\Gamma^*$

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Abstract. Let $R$ be an $(m, n)$-hyperring. The $\Gamma^*$-relation on $R$ in the sense of Mirvakili and Davvaz [34] is the smallest strong compatible relation such that the quotient $R/\Gamma^*$ is an $(m, n)$-ring. We use $\Gamma^*$-relation to define a fundamental functor, $F$ from the category of $(m, n)$-hyperrings to the category of $(m, n)$-rings. Also, the concept of a fundamental $(m, n)$-ring is introduced and it is shown that every $(m, n)$-ring is isomorphic to $R/\Gamma^*$ for a nontrivial $(m, n)$-hyperring $R$. Moreover, the notions of partitionable and quotientable are introduced and their mutual relationship is investigated. A functor $G$ from the category of classical $(m, n)$-rings to the category of $(m, n)$-hyperrings is defined and a natural transformation between the functors $F$ and $G$ is given.

## 1. Introduction

The notion of $n$-ary groups (also called $n$-group or multiary group) is a generalization of that of groups. An $n$-ary group $(G, f)$ is a pair of a set $G$ and a map $f : G\times\cdots\times G \to G$, which is called an $n$-ary operation. The earliest work on these structures was done in 1904 by Krasner [24] and in 1928 by Dörnte [22]. Such $n$-ary groups have many applications to computer science, coding theory, topology, combinatorics and quantum physic (see [18–21, 36] and [38]). One of the applications is the entering into algebraic hyperstructures theory defined by Marty in [30]. This work is initiated by Davvaz and Vougiouklis [16] by defining $n$-ary hypergroups. By its generalization, $(m, n)$-hyperrings and $(m, n)$-hypermodules were introduced and studied in different contexts. Some of the studies can be seen in [2, 5, 11, 27–29, 32, 33] and [34].

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On the other hand, fundamental relations are one of important concepts in algebraic hyperstructures theory which classical algebraic structures will be obtained from algebraic hyperstructures by them. The relations have been studied and investigated on hypergroups in [23] and [25], on hyperrings in [1, 13, 15] and [42], and on hypermodules in [3] and [4]. After defining $n$-ary hyperstructures, fundamental relations were extended on them. This extension done on $n$-ary hypergroups in [12] and [16], on $(m, n)$-hyperrings in [34] and $(m, n)$-hypermodules in [5]. The $\Gamma^*$-relation in the sense of Mirvakili and Davvaz [34] is one of relations on an $(m, n)$-hyperring by which an $(m, n)$-ring is induced via the quotient.

In this paper, in Section 2, we give some basic preliminaries about $(m, n)$-rings and $(m, n)$-hyperrings. In Section 3, we define the concept of a fundamental $(m, n)$-ring and prove that every $(m, n)$-ring is isomorphic to $R/\Gamma^*$ for a nontrivial $(m, n)$-hyperring $R$. In Section 4, we define the notion of quotiontable and partitionable $(m, n)$-hyperrings and study a relationship between them. Finally, in Section 5, we introduce the category of $(m, n)$-hyperrings, denoted by $(m, n) - \mathcal{H}_r$ and investigate functorial connections between the categories of $(m, n)$-hyperrings and $(m, n)$-rings via $\Gamma^*$-relation. Moreover, a natural transformation between these functors is characterized.

## 2. $(m, n)$-Rings and $(m, n)$-Hyperrings

In this section we recall some definitions about $(m, n)$-rings and $(m, n)$-hyperrings based on [9, 16] and [34] for development of our paper.

Let $H$ be a nonempty set. A mapping $f : \underbrace{H \times \cdots \times H}_{n} \longrightarrow H$ $(\mathcal{P}^*(H))$, where $\mathcal{P}^*(H)$ is the set of all nonempty subsets of $H$, is called an $n$-ary operation (hyperoperation). A pair $(H, f)$ consisting of a set $H$ and an $n$-ary operation (hyperoperation) $f$ on $H$ is called an $n$-ary groupoid (hypergroupoid). Note that for abbreviation, the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by $x_i^j$ and for $j < i$, $x_i^j$ is the empty set. Also, $f(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, z_{j+1}, \ldots, z_n)$ will be written as $f(x_1^i, y_{i+1}^j, z_{j+1}^n)$. In the case when $y_{i+1} = \cdots = y_j = y$ the last expression will be written as $f(x_1^i, y^{(j-i)}, z_{j+1}^n)$. If $f$ is an $n$-ary operation (hyperoperation) and $t = l(n-1)+1$ for some $l \geq 1$, then $t$-ary operation (hyperoperation) $f_{(l)}$ is defined by

$$f_{(l)}(x_1^{l(n-1)+1}) = \underbrace{f(f(\ldots, f(f}_{l}(x_1^n), x_{n+1}^{2n-1}), \ldots), x_{(l-1)(n-1)+2}^{l(n-1)+1}).$$

An $n$-ary operation (hyperoperation) $f$ is called associative, if

$$f\left(x_1^{i-1}, f\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = f\left(x_1^{j-1}, f\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right),$$

holds, for every $1 \leq i < j \leq n$ and all $x_1^{2n-1} \in H$. An $n$-ary groupoid (hypergroupoid) with the associative $n$-ary operation (hyperoperation) is called an $n$-ary semigroup (semihypergroup). An $n$-ary groupoid (hypergroupoid) $(H, f)$ in which the equation $b = f(a_1^{i-1}, x_i, a_{i+1}^n)$ $(b \in f(a_1^{i-1}, x_i, a_{i+1}^n))$ has a solution $x_i \in H$, for every $a_1^{i-1}, a_{i+1}^n, b \in H$ and $1 \leq i \leq n$, is called an $n$-ary quasigroup (quasihypergroup).

If $(H, f)$ is an $n$-ary semigroup (semihypergroup) and an $n$-ary quasigroup (quasihypergroup), then $(H, f)$ is called an $n$-ary group (hypergroup). An $n$-ary groupoid (hypergroupoid) $(H, f)$ is commutative, if for all $\sigma \in \mathbb{S}_n$ and for every $a_1^n \in H$, we have $f(a_1, \ldots, a_n) = f(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$. If $a_1^n \in H$, then we denote $(a_{\sigma(1)}, \ldots, a_{\sigma(n)})$ by $a_{\sigma(1)}^{\sigma(n)}$.

**Definition 2.1.** Let $(H, f)$ be an $n$-ary group (hypergroup). A non-empty subset $B$ of $H$ is called an $n$-ary subgroup (subhypergroup) of $(H, f)$, if $f(x_1^n) \in B$ ($f(x_1^n) \subseteq B$) for all $x_1^n \in B$, and the equation $b = f(b_1^{i-1}, x_i, b_{i+1}^n)$ ($b \in f(b_1^{i-1}, x_i, b_{i+1}^n)$) has a solution $x_i \in B$, for all $b_1^{i-1}, b_{i+1}^n, b \in B$ and $1 \leq i \leq n$.

**Definition 2.2.** An $(m, n)$-ring (hyperring) is an algebraic structure $(R, f, g)$, which satisfies the following axioms:

(1) $(R, f)$ is an $m$-ary group (hypergroup);
(2) $(R, g)$ is an $n$-ary semigroup (semihypergroup);
(3) the $n$-ary operation (hyperoperation) $g$ is distributive with respect to the $m$-ary operation (hyperoperation) $f$, i.e., for all $a_1^{i-1}, a_{i+1}^n, x_1^m \in R$, and $1 \leq i \leq n$
$$g(a_1^{i-1}, f(x_1^m), a_{i+1}^n) = f(g(a_1^{i-1}, x_1, a_{i+1}^n), \ldots, g(a_1^{i-1}, x_m, a_{i+1}^n)).$$

We say that an $(m, n)$-ring (hyperring) $(R, f, g)$ has an identity element if there exists $1 \in R$ such that $x = g(1^{(i)}, x, 1^{(n-i-1)})$ ($\{x\} = g(1^{(i)}, x, 1^{(n-i-1)})$) for all $0 \leq i \leq n-1$.

*Example* 2.1. Consider the ring $(\mathbb{Z}, +, \cdot)$ where "$+$" and "$\cdot$" are ordinary addition and multiplication on the set of all integers. It is easy to see that $\mathbb{Z}$ with $f(x, y, z) = x+y+z$ and $g(x, y, z) = x \cdot y \cdot z$ for all $x, y, z \in \mathbb{Z}$ will give rise to a $(3, 3)$-ring. Now, consider the following 3-ary hyperoperations on $\mathbb{Z}$ $h(x, y, z) = \{x, y, z, x+y, x+z, y+z, x+y+z\}$ and $k(x, y, z) = \{x \cdot y \cdot z\}$. Then, it can be seen that $(\mathbb{Z}, h, k)$ is a $(3, 3)$-hyperring.

Let $(R_1, f_1, g_1)$ and $(R_2, f_2, g_2)$ be two $(m, n)$-hyperrings. The mapping $\varphi : R_1 \to R_2$ is called a homomorphism from $R_1$ to $R_2$, if for all $x_1^m, y_1^n \in R_1$ we have
$$\varphi(f_1(x_1^m)) = f_2(\varphi(x_1), \ldots, \varphi(x_m)) \quad \text{and} \quad \varphi(g_1(y_1^n)) = g_2(\varphi(y_1), \ldots, \varphi(y_n)).$$

## 3. Fundamental $(m, n)$-Rings

Let $(R, f, g)$ be an $(m, n)$-hyperring and $\rho$ be an equivalence relation on $R$. If $A$ and $B$ are non-empty subsets of $R$, then $A\bar{\rho}B$ means that for every $a \in A$, there exists $b \in B$ such that $a\rho b$ and for every $\nu \in B$, there exists $u \in A$ that $u\rho\nu$. We write $A\bar{\bar{\rho}}B$ if $a\rho b$ for any $a \in A$ and $b \in B$. The equivalence relation $\rho$ is called compatible on $(R, f)$, if $a_i\rho b_i$ for all $1 \leq i \leq m$ implies that $f(a_1^m)\bar{\rho}f(b_1^m)$. Moreover, it is called strongly compatible if $f(a_1^m)\bar{\bar{\rho}}f(b_1^m)$ when $a_i\rho b_i$ for $1 \leq i \leq m$.

Now assume that $\frac{R}{\rho} = \{\rho(r) \mid r \in R\}$, be the set of all equivalence classes of $R$ with respect to $\rho$. Define $m$-ary and $n$-ary hyperoperations $f/\rho$ and $g/\rho$ on $\frac{R}{\rho}$ as follow:
$$f/\rho(\rho(a)_1^m) = \{\rho(c) \mid c \in f(\rho(a)_1^m)\} \quad \text{and} \quad g/\rho(\rho(a)_1^n) = \{\rho(c) \mid c \in g(\rho(a)_1^n)\}.$$

Based on [16], in [34], it was shown that $(R/\rho, f/\rho, g/\rho)$ is an $(m,n)$-hyperring (ring) if and only if $\rho$ is (strongly) compatible relation on $R$. Mirvakili and Davvaz in [34] introduced the strongly compatible relation $\Gamma^*$ on $(m,n)$-hyperrings as follows.

Let $(R,f,g)$ be an $(m,n)$-hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$, where $s = k(m-1)+1$, the relation $\Gamma_{k;l_1^s}$ is defined by

$$x\Gamma_{k;l_1^s}y \;\Leftrightarrow\; \{x,y\} \subseteq f_{(k)}(u_1,\dots,u_s),$$

where $u_i = g_{(l_i)}(x_{i1}^{it_i})$ for some $x_{i1}^{it_i} \in R$ with $t_i = l_i(n-1)+1$ such that $1 \le i \le s$. Now, set $\Gamma_k = \bigcup\limits_{l_1^s \in N} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup\limits_{k \in \mathbb{N}^*} \Gamma_k$. The results [34, Theorem 5.5 and 5.6] yield that the transitive closure of $\Gamma$, $\Gamma^*$, is a strongly compatible relation on $R$ that is the smallest equivalence relation such that $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an $(m,n)$-ring. Hence, $\Gamma^*$ is said to be a *fundamental* relation on $R$.

**Lemma 3.1.** *Let $(R,f,g)$, $(S,f',g')$ be $(m,n)$-hyperrings and $h : R \to S$ be a homomorphism. Then, for all $x,y \in R$,*

- $(i)$ *$x\Gamma^*y$ implies $h(x)\Gamma^*h(y)$;*
- $(ii)$ *if $h$ is an injection, then $h(x)\Gamma^*h(y)$ implies that $x\Gamma^*y$;*
- $(iii)$ *if $h$ is a bijection, then $x\Gamma^*y$ if and only if $h(x)\Gamma^*h(y)$;*
- $(iv)$ *if $h$ is a bijection, then $h(\Gamma^*(x)) = \Gamma^*(h(x))$.*

*Proof.* $(i)$ Let $x\Gamma^*y$. Then there exist $k, l_1^s \in \mathbb{N}$ and $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$ and $1 \le i \le s$ such that $\{x,y\} \subseteq f_{(k)}(u_1,\dots,u_s)$, where $u_i = g_{(l_i)}(x_{i1}^{it_i})$. Since $h$ is homomorphism, we have

$$\begin{aligned}\{h(x),h(y)\} = h\{x,y\} &\subseteq h\Big(f_{(k)}(u_1,\dots,u_s)\Big)\\ &= f'_{(k)}\Big(h(u_1,\dots,u_s)\Big)\\ &= f'_{(k)}\Big(h\Big(g_{(l_1)}(x_{11}^{1t_1}),\dots,g_{(l_s)}(x_{s1}^{st_s})\Big)\Big)\\ &= f'_{(k)}\Big(g'_{(l_1)}\Big(h(x)_{11}^{1t_1}\Big),\dots,g'_{(l_s)}\Big(h(x)_{s1}^{st_s}\Big)\Big).\end{aligned}$$

So, $h(x)\Gamma^*h(y)$.

$(ii)$ For $x,y \in R$, since $h(x)\Gamma^*h(y)$, there exist $k, l_1^s \in \mathbb{N}$ and $z_{i1}^{it_i} \in S$, where $t_i = l_i(n-1)+1$ and $1 \le i \le s$ such that $\{h(x),h(y)\} \subseteq f'_{(k)}(u_1,\dots,u_s)$ for $u_i = g'_{(l_i)}(z_{i1}^{it_i})$. Now, for an injection $h : (R,f,g) \to (S,f',g')$ we have

$$\begin{aligned}\{x,y\} = \Big\{h^{-1}(h(x)),h^{-1}(h(y))\Big\} &= h^{-1}\Big(\{h(x),h(y)\}\Big)\\ &\subseteq h^{-1}\Big(f'_{(k)}(u_1,\dots,u_s)\Big)\\ &= f_{(k)}\Big(g_{(l_1)}\Big(h^{-1}(z)_{11}^{1t_1}\Big),\dots,g_{(l_s)}\Big(h^{-1}(z)_{s1}^{st_s}\Big)\Big).\end{aligned}$$

So, $x\Gamma^* y$.

$(iii)$ It is clear by $(i)$ and $(ii)$.

$(iv)$ Let $x \in R$. By $(iii)$, we have

$$h(\Gamma^*(x)) = \bigcup_{y\in\Gamma^*(x)} h(y) = \bigcup_{x\Gamma^* y} h(y) = \bigcup_{h(x)\Gamma^* h(y)} h(y) = \Gamma^*(h(x)). \qquad \square$$

**Corollary 3.1.** *Let $(R_1, f_1, g_1)$ and $(R_2, f_2, g_2)$ be isomorphic $(m, n)$-hyperrings. Then $R_1/\Gamma^* \cong R_2/\Gamma^*$.*

*Proof.* Let $h : (R_1, f_1, g_1) \to (R_2, f_2, g_2)$ be an isomorphism. Define $\eta : R_1/\Gamma^* \to R_2/\Gamma^*$ by $\eta\Big(\Gamma^*(x)\Big) = \Gamma^*\Big(h(x)\Big)$. By Lemma 3.1, $\eta$ is well-defined, one to one and onto. Hence, $\eta$ is an isomorphism, since $h$ is a homomorphism. $\square$

**Definition 3.1.** An $(m, n)$-ring $(R, f, g)$ is called a *fundamental* $(m, n)$*-ring* if there exists a non-trivial $(m, n)$-hyperring, say $(S, f', g')$, such that $(S/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \cong (R, f, g)$.

*Remark* 3.1. It is needed to explain what a non-trivial $(m, n)$-hyperring is. An $(m, n)$-hyperring $(S, f', g')$ is said to be *trivial* if $|f'(x_1^m)| = |g'(y_1^n)| = 1$ for all $x_1^m, y_1^n \in S$. For example, let $(R, f, g)$ be an $(m, n)$-ring. Define $m$-ary and $n$-ary hyperoperations $f'(x_1^m) = \{f(x_1^m)\}$ and $g'(y_1^n) = \{g(y_1^n)\}$ for all $x_1^m, y_1^n \in R$. Then $(R, f', g')$ is a trivial $(m, n)$-hyperring.

**Lemma 3.2.** *Let $(R, f, g)$ be an $(m, n)$-ring with identity, then for any $(m, n)$-ring $S$ with identity, there exist $m$-ary and $n$-ary hyperoperations "$f'$" and "$g'$" on $R \times S$ such that $(R \times S, f', g')$ is an $(m, n)$-hyperring.*

*Proof.* Let $S$ be a non-zero $(m, n)$-ring with identity 1. Define $m$-ary and $n$-ary hyperoperations "$f'$" and "$g'$" on $R \times S$ as follows:

$$f'\Big((r_1, s_1), \ldots, (r_m, s_m)\Big) = \Big\{(f(r_1^m), s_1), \ldots, (f(r_1^m), s_m)\Big\},$$
$$g'\Big((r_1, s_1), \ldots, (r_n, s_n)\Big) = \Big\{(g(r_1^n), s_1), \ldots, (g(r_1^n), s_n)\Big\}.$$

(For abbreviation, $f'\Big((r_1, s_1), \ldots, (r_m, s_m)\Big)$ denoted by $f'\Big((r, s)_1^m\Big)$, similarly this is for $g'$). Clearly "$f'$" and "$g'$" are associative and "$g'$" is distributive with respect to "$f'$". Also, we have

$$\begin{aligned}
f'\Big((r, s)_1^{i-1}, R \times S, (r, s)_{i+1}^m\Big) &= \bigcup_{(r',s')\in R\times S} f'\Big((r, s)_1^{i-1}, (r', s'), (r, s)_{i+1}^m\Big) \\
&= \bigcup_{(r',s')\in R\times S} \Big\{(f(r_1^{i-1}, r', r_{i+1}^m), s_1), \ldots, (f(r_1^{i-1}, r', r_{i+1}^m), s_{i-1}), \\
&\quad (f(r_1^{i-1}, r', r_{i+1}^m), s'), (f(r_1^{i-1}, r', r_{i+1}^m), s_{i+1}), \\
&\quad \ldots, (f(r_1^{i-1}, r', r_{i+1}^m), s_m)\Big\} \\
&= R \times S.
\end{aligned}$$

Thus, $(R \times S, f', g')$ is an $(m,n)$-hyperring. $\square$

The $(m,n)$-hyperring $(R \times S, f', g')$ is called an *associated* $(m,n)$*-hyperring* to $R$ (via $S$) and denoted by $R_S$.

**Theorem 3.1.** *Let $(R, f, g)$ and $(T, f, g)$ be isomorphic $(m,n)$-rings with identity. Then, for any $(m,n)$-ring $S$ with identity, $R_S$ and $T_S$ are isomorphic $(m,n)$-hyperrings.*

*Proof.* Let $h : R \to T$ be an homomorphism. Define $\omega : (R\times S, f', g') \to (T\times S, f', g')$ by $\omega(r, s) = (h(r), s)$ for all $(r, s) \in R \times S$. Since $h$ is an isomorphism, it is easy to see that $\omega$ is well-defined and a bijection. Now we verify that $\omega$ is a homomorphism.

$$\begin{aligned}
\omega\Big(f'\Big((r,s)_1^m\Big)\Big) &= \omega\Big(\Big\{(f(r_1^m), s_1), \ldots, (f(r_1^m), s_m)\Big\}\Big)\\
&= \Big\{\omega\Big(f(r_1^m), s_1\Big), \ldots, \omega\Big(f(r_1^m), s_m\Big)\Big\}\\
&= \Big\{\Big(h(f(r_1^m)), s_1\Big), \ldots, \Big(h(f(r_1^m)), s_m\Big)\Big\}\\
&= \Big\{\Big(f(h(r)_1^m), s_1\Big), \ldots, \Big(f(h(r)_1^m), s_m\Big)\Big\}\\
&= f'\Big((h(r), s)_1^m\Big)\\
&= f'\Big(\omega\Big((r,s)_1^m\Big)\Big).
\end{aligned}$$

Similarly, $\omega\Big(g'\Big((r,s)_1^n\Big)\Big) = g'\Big(\omega\Big((r,s)_1^n\Big)\Big)$. Thus, $(R \times S, f', g') \cong (T \times S, f', g')$. $\square$

**Theorem 3.2.** *Every $(m,n)$-ring is a fundamental $(m,n)$-ring.*

*Proof.* Let $(R, f, g)$ be an $(m,n)$-ring. By Lemma 3.2, for any $(m,n)$-ring $S$, $(R \times S, f', g')$ is an $(m,n)$-hyperring. For any $r \in R$ and $(s, s') \in S \times S$ we have $\{(r,s),(r,s')\} = g'((r,s),(1,s')_1^{n-1})$, so $(r,s)\Gamma^*(r,s')$. Hence, $(r,s') \in \Gamma^*(r,s)$. Thus, $\Gamma^*(r,s) = \{(r,x) \mid x \in S\}$. Define the mapping $\theta : (R\times S/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \to (R, f, g)$ by $\theta(\Gamma^*(r,s)) = r$. It is clear that $\theta$ is well-defined and one to one, since for any $(r,s),(r',s') \in R \times S$, $\Gamma^*(r,s) = \Gamma^*(r',s')$ if and only if $(r',s') \in \Gamma^*(r,s)$ if and only if $r = r'$ if and only if $\theta(\Gamma^*(r,s)) = \theta(\Gamma^*(r',s'))$. $\theta$ is a homomorphism. Let $(r,s)_1^m$, $(r,s)_1^n \in R \times S$. We have

$$\begin{aligned}
\theta\Big(f'/\Gamma^*(\Gamma^*(r,s)_1^m)\Big) =&\theta\Big(\Gamma^*(f(r_1^m), s_1)\Big) = \cdots = \theta\Big(\Gamma^*(f(r_1^m), s_m)\Big) = f(r_1^m)\\
=&f\Big(\theta(\Gamma^*(r,s))_1^m\Big)\theta\Big(g'/\Gamma^*(\Gamma^*(r,s)_1^n)\Big) = \theta\Big(\Gamma^*(g(r_1^n), s_1)\Big)\\
=&\cdots = \theta\Big(\Gamma^*(g(r_1^n), s_n)\Big) = g(r_1^n) = g\Big(\theta(\Gamma^*(r,s))_1^n\Big).
\end{aligned}$$

Since for any $r \in R$, $\theta(\Gamma^*(r,0)) = r$, then $\theta$ is onto. Thus, $\theta$ is an isomorphism. $\square$

**Theorem 3.3.** *Let $(R, f, g)$ be an $(m, n)$-hyperring. Then there exist an $(m, n)$-ring $S$, $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ on $R \times S$ such that $(R, f, g)$ can be embedded in $(R \times S, f', g')$.*

*Proof.* Let $(R, f, g)$ be an $(m, n)$-hyperring and set $S = (R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$. Define $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ on $R \times R/\Gamma^*$, as following:

$$f'\Big((r, \Gamma^*(v))_1^m\Big) = \Big(f(r_1^m), \Gamma^*(f(v_1^m))\Big),$$
$$g'\Big((r, \Gamma^*(v))_1^n\Big) = \Big(g(r_1^n), \Gamma^*(g(v_1^n))\Big).$$

Let $(r, \Gamma^*(v))_1^m = (r', \Gamma^*(v'))_1^m$, then $r_j = {r'}_j$ and $\Gamma^*(v_j) = \Gamma^*(v'_j)$ for all $1 \leq j \leq m$. Since $\Gamma^*(v_j) = \Gamma^*({v'}_j)$ for all $j = 1, \ldots, m$, there exist $k_j$, $l_{1_j}^{s_j} \in \mathbb{N}$ and $x_{i_j1}^{i_jt_{i_j}} \in R$, where $t_{i_j} = l_{i_j}(n-1)+1$ and $i_j = 1_j, \ldots, s_j$, such that $\{v_j, {v'}_j\} \subseteq f_{(k_j)}(u_{1_j}, \ldots, u_{s_j})$, where $u_{i_j} = g_{(l_j)}(x_{i_j1}^{i_jt_{i_j}})$. Hence,

$$\begin{aligned}\{f(v_1^m), f({v'}_1^m)\} &\subseteq \Big\{f(v_1^m), f(v_1, {v'}_2^m), f(v'_1, v_2, {v'}_3^m), \ldots, f({v'}_1^m)\Big\}\\ &\subseteq f\Big(f_{(k_1)}(u_{1_1}, \ldots, u_{s_1}), \ldots, f_{(k_m)}(u_{1_m}, \ldots, u_{s_m})\Big)\end{aligned}$$

and

$$\begin{aligned}\{g(v_1^n), g({v'}_1^n)\} &\subseteq \Big\{g(v_1^n), g(v_1, {v'}_2^n), g({v'}_1, v_2, {v'}_3^n), \ldots, g({v'}_1^n)\Big\}\\ &\subseteq g\Big(f_{(k_1)}(u_{1_1}, \ldots, u_{s_1}), \ldots, f_{(k_n)}(u_{1_n}, \ldots, u_{s_n})\Big).\end{aligned}$$

Thus, $\Gamma^*(f(v_1^m)) = \Gamma^*(f({v'}_1^m))$ and $\Gamma^*(g(v_1^n)) = \Gamma^*(g({v'}_1^n))$. So, $(f(r_1^m), \Gamma^*(f(v_1^m)) = f({r'}_1^m), \Gamma^*(f({v'}_1^m))$ and $(g(r_1^n), \Gamma^*(g(v_1^n)) = (g({r'}_1^n), \Gamma^*(g({v'}_1^n)))$. Therefore, the $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ are well-defined. Now, we show that $(R \times S, f', g')$ is an $(m, n)$-hyperring. Let $(r, \Gamma^*(v))_1^m \in R \times S$. Then for any $i, j \in \{1, \cdots, m\}$, since "$f$" is associative, it follows that:

$$\begin{aligned}&f'\Big((r, \Gamma^*(v))_1^{i-1}, f'((r, \Gamma^*(v))_i^{m+i-1}), (r, \Gamma^*(v))_{m+i}^{2m-1}\Big)\\ =&\Big(f(r_1^{i-1}, f(r_i^{m+i-1}), r_{m+i}^{2m-1}), \Gamma^*(f(v_1^{i-1}, f(v_i^{m+i-1}), v_{m+i}^{2m-1})\Big)\\ =&\Big(f(r_1^{j-1}, f(r_j^{m+j-1}), r_{m+j}^{2m-1}), \Gamma^*(f(v_1^{j-1}, f(v_j^{m+j-1}), v_{m+j}^{2m-1})\Big)\\ =&f'\Big((r, \Gamma^*(v))_1^{j-1}, f'((r, \Gamma^*(v))_j^{m+j-1}), (r, \Gamma^*(v))_{m+j}^{2m-1}\Big).\end{aligned}$$

So, $f'$ is associative. Similarly, it can be shown that $g'$ is associative on $R \times S$. Now, we verify the reproduction property. Since $f(r_1^{i-1}, R, r_{i+1}^m) = R$ and $R/\Gamma^* = \bigcup_{t \in R} \Gamma^*(t)$, so

$$\begin{aligned}
& f'\Big((r,\Gamma^*(v))_1^i, R\times S, (r,\Gamma^*(v))_{i+1}^m\Big)\\
=& \bigcup_{(r',\Gamma^*(v'))\in R\times S} f'\Big((r,\Gamma^*(v))_1^i, (r',\Gamma^*(v')), (r,\Gamma^*(v))_{i+1}^m\Big)\\
=& \bigcup_{(r',\Gamma^*(v'))\in R\times S} \Big(f(r_1^i, r', r_{i+1}^m), \Gamma^*(f(v_1^i, s', v_{i+1}^m))\Big)\\
&= R\times\Gamma^*(R) = R\times S.
\end{aligned}$$

To investigate distributivity law, let $(r',\Gamma^*(v'))_1^m \in R\times S$, $(r,\Gamma^*(v))_1^n \in R\times S$. Since $g$ is distributive with respect to $f$, then

$$\begin{aligned}
& g'\Big((r,\Gamma^*(v))_1^{i-1}, f'((r',\Gamma^*(v'))_1^m), (r,\Gamma^*(v))_{i+1}^n\Big)\\
=&\Big(g(r_1^{i-1}, f({r'}_1^m), r_{i+1}^n), \Gamma^*(g(v_1^{i-1}, f({v'}_1^m), v_{i+1}^n))\Big)\\
=&\Big(f(g(r_1^{i-1}, {r'}_1, r_{i+1}^n), \ldots, g(r_1^{i-1}, {r'}_m, r_{i+1}^n)),\\
& \Gamma^*(f(g(v_1^{i-1}, {v'}_1, v_{i+1}^n), \cdots, g(v_1^{i-1}, {v'}_m, v_{i+1}^n)))\Big)\\
=&f'\Big(g'((r,\Gamma^*(v))_1^{i-1}, (r',\Gamma^*(v'))_1, (r,\Gamma^*(v))_{i+1}^n), \ldots,\\
& g'((r,\Gamma^*(v))_1^{i-1}, (r',\Gamma^*(v'))_m, (r,\Gamma^*(v))_{i+1}^n)\Big).
\end{aligned}$$

So, $(R\times S, f', g')$ is an $(m,n)$-hyperring. Now, define the mapping $\theta : (R,f,g) \to (R\times S, f', g')$, by $\theta(r) = (r, \Gamma^*(r))$. Let $r, r' \in R$. Then $r = r'$ if and only if $(r,\Gamma^*(r)) = (r', \Gamma^*(r'))$ if and only if $\theta(r) = \theta(r')$. Let $r_1^m, r_1^n \in R$. Then

$$\theta(f(r_1^m)) = (f(r_1^m), \Gamma^*(f(r_1^m)) = f'((r,\Gamma^*(r))_1^m) = f'(\theta(r)_1^m)$$

and

$$\theta(g(r_1^n)) = (g(r_1^n), \Gamma^*(g(r_1^n)) = g'((r,\Gamma^*(r))_1^n) = g'(\theta(r)_1^n),$$

where $\theta(r)_1^k$ means $\theta(r_1), \ldots, \theta(r_k)$ for $k = m$ or $k = n$. Thus, $(R,f,g)$ can be embedded in $(R\times S, f', g')$. □

**Theorem 3.4.** *Let $R$ and $S$ be two sets such that $|R| = |S|$. If $(R,f,g)$ is an $(m,n)$-hyperring, then there exist $m$-ary and $n$-ary hyperoperations "$f'$" and "$g'$" on "$S$", such that $(R,f,g)$ and $(S,f',g')$ are isomorphic $(m,n)$-hyperrings*

*Proof.* Since $|R| = |S|$, then there exists a bijection $\phi : R \to S$. For any $s_1^m, s_1^n \in S$, define the $m$-ary and $n$-ary hyperoperations "$f'$" and "$g'$" as follows:

$$f'(s_1^m) = \phi(f(r_1^m)), \quad g'(s_1^n) = \phi(g(r_1^n)).$$

First we prove that $f'$ and $g'$ are well-defined. Let $s_i = {s'}_i$, where $s_i = \phi(r_i)$, $s'_i = \phi(r'_i)$ and $r_i, r'_i \in R$ for $i = 1, \ldots, m$. So, $s_i = {s'}_i$ implies that $\phi(r_i) = \phi(r'_i)$. Since $\phi$ is bijection, then $r_i = r'_i$ for $i = 1, \ldots, m$ and so $f'(s_1^m) = \phi(f(r_1^m)) = \phi(f({r'}_1^m)) = f'({s'}_1^m)$, similarly $g'(s_1^n) = g'({s'}_1^n)$. Moreover, since

$$\begin{aligned} \phi(f(r_1^m)) &= f'(\phi(r)_1^m), \\ \phi(g(r_1^n)) &= g'(\phi(r)_1^n), \end{aligned} \tag{3.1}$$

$\phi$ is a homomorphism. Now, it is enough to show that $(S, f', g')$ is an $(m, n)$-hyperring. Define the map $\theta : (R, f, g) \to (S, f', g')$ by $\theta(x) = \phi(x)$. Since $\phi$ is bijection then $\theta$ is a bijection. Now we show that $\theta$ is a homomorphism. Let $r_1^m \in R$. Then, by (3.1), $\theta(f(r_1^m)) = \phi(f(r_1^m)) = f'(\phi(r)_1^m) = f'(\theta(r)_1^m)$ and $\theta(g(r_1^n)) = \phi(g(r_1^n)) = g'(\phi(r)_1^n) = g'(\theta(r)_1^n)$. Thus, $\theta$ is an isomorphism and so $(S, f', g')$ is an $(m, n)$-hyperring. □

**Corollary 3.2.** *Let $(R, f, g)$ be an $(m, n)$-ring of infinite order. Then there exist $m$-ary and $n$-ary hyperoperations "$f'$" and "$g'$" on $R$ such that $(R, f, g)$ is a fundamental $(m, n)$-ring of itself, i.e., $(R/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \cong (R, f, g)$.*

*Proof.* For a given $(m, n)$-ring $(R, f, g)$, consider the smallest associated $(m, n)$-hyperring $(R \times \mathbb{Z}_2, f', g')$. By Theorem 3.2, $\left(\frac{(R\times\mathbb{Z}_2, f', g')}{\Gamma^*}, f'/\Gamma^*, g'/\Gamma^*\right) \cong (R, f, g)$. Since $R$ is infinite set, then $|R| = |R \times \mathbb{Z}_2|$ and, by Theorem 3.4, there exist $m$-ary and $n$-ary hyperoperations "$f''$" and "$g''$" on $(R, f, g)$, such that $(R, f'', g'')$ and $(R \times \mathbb{Z}_2, f', g')$, are isomorphic $(m, n)$-hyperrings. Now, we have

$$(R, f, g) \cong \left(\frac{(R \times \mathbb{Z}_2, f', g')}{\Gamma^*}, f'/\Gamma^*, g'/\Gamma^*\right) \cong \left(\frac{(R, f'', g'')}{\Gamma^*}, f'/\Gamma^*, g'/\Gamma^*\right).$$

Hence, $(R, f, g)$ is a fundamental $(m, n)$-ring of itself. □

We recall the relation $\beta_f = \bigcup_{k\geq 1} \beta_k$ on an $n$-ary semihypergroup $(R, f)$ defined by Davvaz and Vougiouklis in [16], where $x\beta_k y$ if and only if there exist $t = k(m-1)+1$ and $z_1^t \in R$ such that $\{x, y\} \subseteq f_{(k)}(z_1^t)$. It is well known that $\beta_f$ is the smallest strongly compatible equivalence relation on $n$-ary semihypergroup $(R, f)$ such that $(R/\beta_f, f/\beta_f)$ is an $n$-ary semigroup. Clearly, $\beta_f \subseteq \Gamma$ and so $\beta_f^* \subseteq \Gamma^*$.

**Theorem 3.5.** *Every finite $(m, n)$-ring is not its fundamental $(m, n)$-ring.*

*Proof.* Let $(R, f, g)$ be a finite $(m, n)$-ring, $|R| = n$. If "$f'$" and "$g'$", are $m$-ary and $n$-ary hyperoperations on $R$, such that $(R, f, g)$ is an $(m, n)$-hyperring, then there exist $x_1^m \in R$ such that $|f'(x_1^m)| \geq 2$. Hence, there are $a, b \in f(x_1^m)$. So $a\beta_f b$ and then $a\Gamma b$. Therefore, $a\Gamma^* b$ and $\Gamma^*(a) = \Gamma^*(b)$. Since $R/\Gamma^* = \{\Gamma^*(t) \mid t \in R\}$, then $|R/\Gamma^*| < n$. Thus, $(R, f, g) \ncong (R/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*)$. □

## 4. Embeddable $(m, n)$-Hyperring

In this section we introduce the concepts of partitionable and quotientable $(m, n)$-hyperrings and investigate the relation between them. Also, we give some results concerning about these concepts.

**Definition 4.1.** An $(m,n)$-hyperring $(R, f_1, g_1)$ is said to be a partitionable $(m,n)$-hyperring if there exists an $(m,n)$-ring $(S, f, g)$, an equivalence relation $\rho$ on $(S, f, g)$, non-trivial $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ such that $(S/\rho, f', g') \cong (R, f_1, g_1)$.

**Theorem 4.1.** *Every $(m,n)$-hyperring is a partitionable $(m,n)$-hyperring.*

*Proof.* Let $(R, f, g)$ be an $(m,n)$-hyperring. Then we consider three cases.

Case 1. Let $R$ be finite and $|R| = n$. Define on $\mathbb{Z}$ the equivalence relation $\rho$ by

$$x\rho y \Leftrightarrow x \equiv y \pmod{n}.$$

Clearly $|R| = |\mathbb{Z}/\rho|$. So, by Theorem 3.4, there exist $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ on $\mathbb{Z}/\rho$, such that $(\mathbb{Z}/\rho, f', g')$ is an $(m,n)$-hyperring and $(R, f, g) \cong (\mathbb{Z}/\rho, f', g')$.

Case 2. Let $R$ be infinite countable. Then $|R| = |\mathbb{Z}|$. Let $\mathcal{A} = \{A_i\}_{i\in\mathbb{Z}}$ be a partition of $\mathbb{Z}$ such that there exists an index $j \in \mathbb{Z}$ such that $|A_j| = 2$ and for any $j \neq i \in \mathbb{Z}$, $|A_i| = 1$. Clearly, the binary relation $\rho$ on $\mathbb{Z}$, by

$$r\rho s \Leftrightarrow (\exists k \in \mathbb{Z}) \text{ s.t } \{r, s\} \subseteq A_k$$

is an equivalence relation on $\mathbb{Z}$ and clearly $|\mathbb{Z}| = |\mathcal{A}| = \left|\frac{\mathbb{Z}}{\rho}\right|$. Thus, by Theorem 3.4, there exist $m$-ary and $n$-ary hyperoperations "$f'$" and "$g'$" on $\mathbb{Z}/\rho$, such that $(\mathbb{Z}/\rho, f', g')$ is an $(m,n)$-hyperring and $(R, f_1, g_1) \cong (\mathbb{Z}/\rho, f', g')$.

Case 3. Let $R$ be uncountable. Then $|R| = |\mathbb{R}|$ and similarly as in case 2 it can be concluded that $R$ is a partitionable $(m,n)$-hyperring. □

Let $(R, f, g)$ be an $(m,n)$-ring. We say that $(N, g)$ is a *normal* subgroup of $n$-semigroup $(R, g)$, if $g(a_1^{i-1}, N, a_{i+1}^n) = g(a_{\sigma(1)}^{\sigma(i-1)}, N, a_{\sigma(i+1)}^{\sigma(n)})$, for all $a_1^n \in R$, $\sigma \in \mathbb{S}_n$ and $1 \leq i \leq n$. Also, for a normal subgroup $N$ of $(S, g)$, we set

$$S/N = \{g(x_1^{i-1}, N, x_{i+1}^n) \mid x_i \in S, 1 \leq i \leq n\}.$$

**Definition 4.2.** An $(m,n)$-hyperring $(R, f, g)$ is called a quotientable $(m,n)$-hyperring if there exist an $(m,n)$-ring $(S, h, k)$, non-trivial $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ such that $(S/N, f', g') \cong (R, f, g)$, where $N$ is a normal subgroup of the $n$-semigroup of $(S, k)$.

**Theorem 4.2.** *Every $(m,n)$-hyperring is a quotientable $(m,n)$-hyperring.*

*Proof.* Let $(R, f, g)$ be an $(m,n)$-hyperring and consider the following cases.

Case 1. Let $R$ be finite and $|R| = n$. Consider $(\mathbb{Z}_n^* = \mathbb{Z}_n \setminus \{\bar{0}\}, \odot)$ and set $g(x_1^n) = \bigodot_{i=1}^{n} x_i$ for $x_1^n \in \mathbb{Z}_n$. Clearly, $N = \{\bar{1}\}$ is a normal subgroup of $(\mathbb{Z}_n^*, g)$ and $|R| = |\mathbb{Z}_n/N|$. Thus, by Theorem 3.4, there exist $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ on $\mathbb{Z}_n/N$ such that $(\mathbb{Z}_n/N, f', g')$ is an $(m,n)$-hyperring and $(R, f, g) \cong (\mathbb{Z}_n/N, f', g')$.

Case 2. Let $R$ be infinite countable and $|R| = |\mathbb{Z} \times \mathbb{Z}|$. Note that $(\mathbb{Z} \times \mathbb{Z}, f, g)$ is an $(m,n)$-ring such that $f((a,b)_1^m) = (a_1 + \cdots + a_m, b_1 + \cdots + b_m)$ and $g((a,b)_1^n) =$

$(a_1 \cdot a_2 \cdots a_n, b_1 \cdot b_2 \cdots b_n)$ for any $a_1^m, a_1^n, b_1^m, b_1^n \in \mathbb{Z}$, where " $+$ " and " $\cdot$ " are ordinary binary operations on $\mathbb{Z}$. Now, let $N = \{(-1,1),(1,1)\}$. Then $N$ is a normal in $((\mathbb{Z}\times\mathbb{Z})^*, g)$. Clearly $|\mathbb{Z}\times\mathbb{Z}| = |(\mathbb{Z}\times\mathbb{Z})/N|$. Hence, by Theorem 3.4, there exist $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ on $(\mathbb{Z}\times\mathbb{Z})/N$ such that $((\mathbb{Z}\times\mathbb{Z})/N, f', g')$ is an $(m,n)$-hyperring and $(R,f,g) \cong ((\mathbb{Z}\times\mathbb{Z})/N, f', g')$.

Case 3. Let $R$ be uncountable. Then $|R| = |\mathbb{R}\times\mathbb{R}|$ and similarly as in case 2 we conclude that $R$ is a quotientable $(m,n)$-hyperring. □

**Theorem 4.3.** *Every quotientable $(m,n)$-hyperring is a partitionable $(m,n)$-hyperring.*

*Proof.* Let $(R, f_1, g_1)$ be a quotientable $(m,n)$-hyperring. Then, there exist an $(m,n)$-ring $(S,f,g)$, non-trivial $m$-ary and $n$-ary hyperoperations $f'$ and $g'$ such that $(S/N, f', g') \cong (R, f_1, g_1)$, where $N$ is a normal subgroup the $n$-semigroup $(S,g)$. Define, the binary relation $\rho$ on $S$ as follows:

$$x\rho y \Leftrightarrow g(x, x_2^{i-1}, N, x_{i+1}^n) = g(y, x_2^{i-1}, N, x_{i+1}^n).$$

Clearly $\rho$ is an equivalence relation on $S$ and for any $s \in S$, $\rho(s) = g(s, x_2^{i-1}, N, x_{i+1}^n)$. Hence, $(R, f_1, g_1)$ is a partitionable $(m,n)$-hyperring. □

*Remark* 4.1. Consider the $(m,n)$-hyperring $(\mathbb{Z}_3, f, g)$ with the $m$-ary and $n$-ary hyperoperations $f(x_1^m) = \mathbb{Z}_3$ and $g(y_1^n) = \mathbb{Z}_3$ for all $x_1^m, y_1^n \in \mathbb{Z}_3$. Define on $\mathbb{Z}$ the relation $\rho$ by $\rho = \{(0,0),(2k,2k'),(2k+1,2k'+1)\}$ Clearly $\rho$ is an equivalence relation and $|\mathbb{Z}_3| = |\frac{\mathbb{Z}}{\rho}|$. Hence, by Theorem 4.1, $(\mathbb{Z}_3, f, g)$ is a partitionable $(m,n)$-hyperring. But $\rho$ is not a multiplicative normal $n$-subgroup of $\mathbb{Z}$. Thus, the converse of Theorem 4.3, is not valid.

Let $(R, f_1, g_1)$ be an $(m,n)$-hyperring. Consider the canonical projection $\varphi : (R, f_1, g_1) \to (R/\Gamma^*, f_1/\Gamma^*, g_1/\Gamma^*)$ by $\varphi(r) = \Gamma^*(r)$. Also, by Theorem 4.2, there exist an $(m,n)$-ring $(S,f,g)$, normal $n$-subgroup $N$ such that $\theta : (R, f_1, g_1) \to (S/N, f', g')$ is an isomorphism. Hence, we have the following theorem.

**Theorem 4.4.** *Let $(R, f_1, g_1)$ be a quotientable $(m,n)$-hyperring via an $(m,n)$-ring $(S,f,g)$. Then there exists a unique homomorphism $\psi$, such that $\psi\theta = \varphi$.*

*Proof.* Since $(R, f_1, g_1)$ is a quotientable $(m,n)$-hyperring via an $(m,n)$-ring $(S,f,g)$, there exists a normal subgroup of the $n$-semigroup $(S,g)$ such that $(S/N, f', g') \cong (R, f_1, g_1)$. Define $\psi : S/N \to R/\Gamma^*$ by $\psi(g(s_1^{i-1}, N, s_{i+1}^n)) = \Gamma^*(r)$ such that $\theta(r) = g(s_1^{i-1}, N, s_{i+1}^n)$ for any $s_1^n \in S$. Therefore $\psi = \varphi \circ \theta^{-1}$, so $\psi$ is a homomorphism. Also, $\psi\theta(r) = (\varphi\circ\theta^{-1})(\theta(r)) = \varphi(r)$. Thus, the following diagram is commutative.

$$\begin{tikzcd} R \arrow[r, "\theta"] \arrow[dr, "\varphi"'] & S/N \arrow[d, "\psi"] \\ & R/\Gamma^* \end{tikzcd}$$

Moreover, it is easy to see that $\psi$ is unique. □

**Corollary 4.1.** *Let $(R, f_1, g_1)$ be a quotientable $(m, n)$-hyperring via an $(m, n)$-ring $(S, f, g)$. Then the following diagram is commutative.*

$$\begin{array}{ccc} R & \xrightarrow{\theta} & S/N \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\bar{\varphi}} \\ R/\Gamma^* & \xrightarrow{\bar{\theta}} & (S/N)/\Gamma^* \end{array}$$

*Proof.* Define the maps $\bar{\theta} : R/\Gamma^* \to (S/N)/\Gamma^*$ by $\bar{\theta}(\Gamma^*(r)) = \Gamma^*(\theta(r))$ and $\bar{\varphi} : S/N \to (S/N)/\Gamma^*$ by $\bar{\varphi}(g(s_1^{i-1}, N, s_{i+1}^n)) = \Gamma^*(g(s_1^{i-1}, N, s_{i+1}^n))$. Since $\theta$ and $\varphi$ are homomorphism, $\bar{\theta}$ and $\bar{\varphi}$ are so. Hence, for any $r \in R$

$$\bar{\varphi}\theta(r) = \bar{\varphi}\Big(g(s_1^{i-1}, N, s_{i+1}^n)\Big) = \Gamma^*\Big(g(s_1^{i-1}, N, s_{i+1}^n)\Big) = \Gamma^*(\theta(r)) = \bar{\theta}(\Gamma^*(r)) = \bar{\theta}\varphi(r).$$

□

## 5. Categorical Relations on $(m, n)$-Hyperrings and $(m, n)$-Rings

Now we introduce the category of $(m, n)$-hyperrings, denoted by $(m, n) - \mathcal{H}_r$. This category is defined as follows:

$(i)$ the objects of $(m, n) - \mathcal{H}_r$ are $(m, n)$-hyperrings;

$(ii)$ for the objects $R$ and $R'$ of $(m, n) - \mathcal{H}_r$, the set of all homomorphisms from $R$ to $R'$ are arrows and denoted by $h : R \to R'$.

In this section, we try to investigate the relation between two categories $(m, n) - \mathcal{H}_r$ and $(m, n) - \mathcal{R}_g$ (category of $(m, n)$-rings) and work on natural transformations between them. At first, we define an arrow $F : (m, n) - \mathcal{H}_r \to (m, n) - \mathcal{R}_g$ by $F(R) = R/\Gamma^*$, where $(R, f, g)$ is an object of $(m, n) - \mathcal{H}_r$ and for any arrow $\nu : (R_1, f_1, g_1) \to (R_2, f_2, g_2)$, we define:

$$F(\nu) : R_1/\Gamma^* \to R_2/\Gamma^* \text{ by } F(\nu)(\Gamma^*(x)) = \Gamma^*(\nu(x)), \quad \text{for every } x \in R_1.$$

By Corollary 3.1, $F$ is well-defined. Hence, we have the following.

**Theorem 5.1.** *$F$ is a covariant functor from $(m, n) - \mathcal{H}_r$ to $(m, n) - \mathcal{R}_g$.*

*Proof.* For any object $(R, f, g)$ of $(m, n) - \mathcal{H}_r$, $F(R) = R/\Gamma^*$ is an $(m, n)$-ring and then $F(R)$ is an object in $(m, n) - \mathcal{R}_g$. Now, we show that $F(\nu)$ is an arrow in $(m, n) - \mathcal{R}_g$, for any arrow $\nu : (R_1, f_1, g_1) \to (R_2, f_2, g_2)$. Let $\Gamma^*(x)_1^m,\ \Gamma^*(x)_1^n \in R_1/\Gamma^*$. Thus,

$$\begin{aligned} F(\nu)\Big(f_1/\Gamma^*\Big(\Gamma^*(x)_1^m\Big)\Big) &= F(\nu)\Big(\Gamma^*\Big(f_1(x_1^m)\Big)\Big) = \Gamma^*\Big(\nu\Big(f_1(x_1^m)\Big)\Big) \\ &= \Gamma^*\Big(f_2\Big(\nu(x_1), \dots, \nu(x_m)\Big)\Big) \\ &= f_2/\Gamma^*\Big(\Gamma^*\Big(\nu(x_1)\Big), \dots, \Gamma^*\Big(\nu(x_m)\Big)\Big) \\ &= f_2/\Gamma^*\Big(F(\nu)\Big(\Gamma^*(x_1)\Big), \dots, F(\nu)\Big(\Gamma^*(x_m)\Big)\Big). \end{aligned}$$

Similarly, we have

$$F(\nu)\Big(g_1/\Gamma^*\Big(\Gamma^*(x)_1^n\Big)\Big) = g_2/\Gamma^*\Big(F(\nu)\Big(\Gamma^*(x_1)\Big),\dots,F(\nu)\Big(\Gamma^*(x_n)\Big)\Big).$$

Also for the composition of two arrows $F(\nu)$ and $F(\omega)$ in $(m,n)-\mathcal{R}_g$, where $\nu : (R_1,f_1,g_1) \to (R_2,f_2,g_2)$ and $\omega : (R_2,f_2,g_2) \to (R_3,f_3,g_3)$, we have

$$F(\omega)\circ F(\nu) = F(\omega)(F(\nu)) = F(\omega)(\Gamma^*(\nu)) = \Gamma^*(\omega\circ\nu) = F(\omega\circ\nu).$$

Moreover, for $1_R : R \to R$ and $1_{F(R)} : R/\Gamma^* \to R/\Gamma^*$, we have

$$F(1_R)\Big(\Gamma^*(x)\Big) = \Gamma^*(1_R(x)) = \Gamma^*(x) = 1_{F(R)}(x).$$

Therefore, $F$ is a covariant functor of $(m,n)-\mathcal{H}_r$ to $(m,n)-\mathcal{R}_g$. $\square$

Now, for $(m,n)-\mathcal{H}_r$, $(m,n)-\mathcal{R}_g$, any $(m,n)$-ring $(R,f,g)$ and $S=\mathbb{Z}_2$, define a categorical arrow $U : (m,n)-\mathcal{R}_g \to (m,n)-\mathcal{H}_r$ by $U(R)=R_S$, which for any $(m,n)$-ring homomorphism $\nu : (R_1,f_1,g_1)\to(R_2,f_2,g_2)$ defined by

$$U(\nu)(x,y) = (\nu,1_S)(x,y) = (\nu(x),1_S(y)) = (\nu(x),y).$$

By Theorem 3.1, $U$ is well-defined. Hence, we have the following theorem.

**Theorem 5.2.** *$U$ is a covariant functor from $(m,n)-\mathcal{R}_g$ to $(m,n)-\mathcal{H}_r$.*

*Proof.* For any object $(R,f,g)$ of $(m,n)-\mathcal{R}_g$ by Lemma 3.2, $U(R) = R\times S = R_S$ is an $(m,n)$-hyperring and so $U(R)$ is an object in $(m,n)-\mathcal{H}_r$. Consider any arrow $\nu : (R_1,f_1,g_1)\to(R_2,f_2,g_2)$ in $(m,n)-\mathcal{R}_g$. We show that $U(\nu)$ is an arrow in $(m,n)-\mathcal{H}_r$. Let $(r,s)_1^m,\ (r,s)_1^n \in R_1\times S$. Now, by Lemma 3.2,

$$\begin{aligned}
U(\nu)\Big(f_1'\Big((r,s)_1^m\Big)\Big) &= U(\nu)\Big(\{(f_1(r_1^m),s_1),\dots,(f_1(r_1^m),s_m)\}\Big)\\
&= \Big\{U(\nu)\Big(f_1(r_1^m),s_1\Big),\dots,U(\nu)\Big(f_1(r_1^m),s_m\Big)\Big\}\\
&= \Big\{\Big(\nu(f_1(r_1^m)),s_1\Big),\dots,\Big(\nu(f_1(r_1^m)),s_m\Big)\Big\}\\
&= \Big\{\Big(f_2\Big(\nu(r_1),\dots,\nu(r_m)\Big),s_1\Big),\dots,\Big(f_2\Big(\nu(r_1),\dots,\nu(r_m)\Big),s_m\Big)\Big\}\\
&= f_2'\Big(\Big(\nu(r_1),s_1\Big),\dots,\Big(\nu(r_m),s_m\Big)\Big)\\
&= f_2'\Big(U(\nu)(r_1,s_1),\dots,U(\nu)(r_m,s_m)\Big).
\end{aligned}$$

Similarly, we have $U(\nu)(g_1'\Big((r,s)_1^n\Big)) = g_2'(U(\nu)(r_1,s_1),\dots,U(\nu)(r_n,s_n))$. Thus, $U(\nu) : R_1\times S \to R_2\times S$ is an $(m,n)$-hyperring homomorphism and so is an arrow in $(m,n)-\mathcal{H}_r$. Now, we investigate the composition property. Let $\nu$ and $\omega$ be arrows in $(m,n)-\mathcal{R}_g$. So,

$$U(\nu)\circ U(\omega)(r,s) = U(\nu)\Big(U(\omega)(r,s)\Big) = U(\nu)\Big(\omega(r),s\Big) = \Big(\nu\circ\omega(r),s\Big) = U(\nu\circ\omega)(r,s).$$

Moreover, consider $1_R : R \to R$ and $1_{U(R)} : U(R) \to U(R)$. For any $(r, s) \in R_S$

$$U(1_R)(r, s) = (1_R(r), s) = (r, s) = 1_{U(R)}(r, s).$$

Hence, $U$ is a covariant functor of $(m, n) - \mathcal{R}_g$ to $(m, n) - \mathcal{H}_r$. □

**Theorem 5.3.** *The functor $U : (m, n) - \mathcal{R}_g \to (m, n) - \mathcal{H}_r$ is a faithful functor.*

*Proof.* Let $(R_1, f_1, g_1)$ and $(R_2, f_2, g_2)$ be objects in $(m, n) - \mathcal{R}_g$, $\nu_1, \nu_2 : R_1 \to R_2$ be parallel arrows of $(m, n) - \mathcal{R}_g$ and $U(\nu_1) = U(\nu_2)$. So, for any $(r, s) \in R_{1S}$, $U(\nu_1)(r, s) = U(\nu_2)(r, s)$ and so $\nu_1 = \nu_2$. Thus, $U$ is a faithful functor. □

**Theorem 5.4.** *On objects of $(m, n) - \mathcal{R}_g$, $F \circ U = 1$.*

*Proof.* For any object $(R, f, g)$ in $(m, n) - \mathcal{R}_g$, we have

$$(F \circ U)(R, f, g) = F(R_S, f', g') = (R_S/\Gamma^*, f'/\Gamma^*, g'/\Gamma^*) \cong (R, f, g),$$

by Theorem 3.2. □

**Theorem 5.5.** *For functors $1, F \circ U : (m, n) - \mathcal{R}_g \to (m, n) - \mathcal{R}_g$ there exists a natural transformation $\mu : 1 \to F \circ U$.*

*Proof.* For two functors 1 and $F \circ U$ of $(m, n) - \mathcal{R}_g$ to $(m, n) - \mathcal{R}_g$, define a map $\mu : 1 \to F \circ U$ as follows:

$$\mu : 1(R) \to (F \circ U)(R) \text{ by } \mu(r) = \Gamma^*(r, 0).$$

Now, for any $(m, n)$-ring homomorphism $\nu : (R, f, g) \to (R', f', g')$, consider the following diagram.

$$\begin{array}{ccc} 1(R) & \xrightarrow{\mu_R} & (F \circ U)(R) \\ {\scriptstyle 1(\nu)}\downarrow & & \downarrow{\scriptstyle F\circ U(\nu)} \\ 1(R') & \xrightarrow{\mu_{R'}} & (F \circ U)(R') \end{array}$$

For any $r \in R$, we have

$$\begin{aligned} \Big((F \circ U)(\nu) \circ \mu_R\Big)(r) = F \circ U(\nu)\Big(\mu_R(r)\Big) &= F \circ U(\nu)\Big(\Gamma^*(r, 0)\Big) \\ &= \Gamma^*\Big(\nu(r), 0\Big) \\ &= \mu_{R'}\Big(\nu(r)\Big) \\ &= \mu_{R'}\Big(1(\nu)(r)\Big) = \Big(\mu_{R'} \circ 1(\nu)\Big)(r). \end{aligned}$$

So, $\mu$ is a natural transformation. □

**Theorem 5.6.** *For functors 1 and $U \circ F$ from $(m, n) - \mathcal{H}_r$ to $(m, n) - \mathcal{H}_r$, there exists a transformation $\theta : 1 \to U \circ F$ such that is natural.*

*Proof.* For two functors $1, U\circ F : (m,n){-}\mathcal{H}_r \to (m,n){-}\mathcal{H}_r$, define a map $\theta : 1 \to U\circ F$ as $\theta : 1(R) \to (U\circ F)(R)$ by $\theta(r) = \Big(\Gamma^*(r), 0\Big)$. Now, for any $(m,n)$-hyperring homomorphism $\nu : (R,f,g) \to (R',f',g')$, consider the following diagram.

$$\begin{tikzcd} 1(R) \arrow[r, "\theta_R"] \arrow[d, "1(\nu)"'] & (U\circ F)(R) \arrow[d, "U\circ F(\nu)"] \\ 1(R') \arrow[r, "\theta_{R'}"] & (U\circ F)(R') \end{tikzcd}$$

For any $r \in R$, we have

$$\begin{aligned}\Big((U\circ F)(\nu)\circ\theta_R\Big)(r) = U\circ F(\nu)\Big(\theta_R(r)\Big) &= U\circ F(\nu)\Big(\Gamma^*(r), 0\Big)\\ &= \Big(\Gamma^*\Big(\nu(r)\Big), 0\Big)\\ &= \theta_{R'}\Big(\nu(r)\Big)\\ &= \theta_{R'}\Big(1(\nu)(r)\Big)\\ &= \Big(\theta_{R'}\circ 1(\nu)\Big)(r).\end{aligned}$$

Therefore, $\theta$ is a natural transformation. □

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# APPLICATIONS OF FRACTIONAL DERIVATIVE ON A DIFFERENTIAL SUBORDINATIONS AND SUPERORDINATORS FOR ANALYTIC FUNCTIONS ASSOCIATED WITH DIFFERENTIAL OPERATOR

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Abstract. The purpose of this paper is to derive subordination and superordination results involving fractional derivative of differential operator for analytic functions in the open unit disk. These results are applied to obtain sandwich results. Our results extend corresponding previously known results.

## 1. Introduction and Preliminaries

Let $\mathcal{H} = \mathcal{H}(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in C : |z| < 1\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \quad a \in \mathbb{C}.$$

Also, let $A$ be the subclass of $\mathcal{H}$ consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Let $f, g \in \mathcal{H}$. The function $f$ is said to be subordinate to $g$, or $g$ is said to be superordinate to $f$, if there exists a Schwarz function $w$ analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$, $z \in U$. It is well known that, if the function $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. Let $p, h \in \mathcal{H}$ and

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$\psi(r,s,t;z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If $p$ and $\psi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in $U$ and if $p$ satisfies the second-order differential superordination

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z), \tag{1.2}$$

then $p$ is called a solution of the differential superordination (1.2). An analytic function $q$ is called a subordinate of (1.2), if $q \prec p$ for all $p$ satisfying (1.2). An univalent subordinat $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all the subordinants $q$ of (1.2) is called the best subordinant.

Miller and Mocanu [6] obtained conditions on the functions $h, q$ and $\psi$ for which the following implication holds:

$$h(z) \prec \psi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z).$$

Ali et al. [1] have used the results of Bulboacă [3] to obtain sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where $q_1$ and $q_2$ are given univalent functions in $U$ with $q_1(0) = q_2(0) = 1$.

Also, Tuneski [16] obtain sufficient condition for starlikeness of $f \in A$ in terms of the quantity $\frac{f''(z)f(z)}{(f'(z))^2}$. Shanmugam et al. [14], Goyal et al. [4], Wanas [17, 18] and Attiya and Yassen [2] have obtained sandwich results for certain classes of analytic functions.

**Definition 1.1** ([9])**.** For $f \in A$ the operator $I^{n,m}_{\lambda_1,\lambda_2,\ell,d}$ is defined by $I^{n,m}_{\lambda_1,\lambda_2,\ell,d} : A \to A$,

$$I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z) = \mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z) * R^n f(z), \quad z \in U,$$

where

$$\mathcal{M}^m_{\lambda_1,\lambda_2,\ell,d}(z) = z + \sum_{k=2}^{\infty} \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d}\right]^m z^k,$$

and $R^n f(z)$ denotes the Ruscheweyh derivative operator [10] given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} C(n,k) a_k z^k,$$

where $C(n,k) = \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)}$, $n, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ , $\lambda_2 \geq \lambda_1 \geq 0$, $\ell \geq 0$ and $\ell + d > 0$.

If $f$ given by (1.1), then we easily find that

$$I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+n)}{\Gamma(n+1)\Gamma(k)} \left[\frac{\ell(1 + (\lambda_1 + \lambda_2)(k-1)) + d}{\ell(1 + \lambda_2(k-1)) + d}\right]^m a_k z^k.$$

**Definition 1.2** ([15])**.** The fractional derivative of order $\delta$, $0 \leq \delta < 1$, of a function $f$ is defined by

$$D^\delta_z f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^\delta}\, dt,$$

where the function $f$ is analytic in a simply-connected region of the z-plane containing the origin and the multiplicity of $(z-t)^{-\delta}$ is removed by requiring $\log(z-t)$ to be real, when $\mathrm{Re}(z-t) > 0$.

From Definition 1.1 and Definition 1.2, we have

$$D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z) = \frac{1}{\Gamma(2-\delta)} z^{1-\delta} + \sum_{k=2}^{\infty} \frac{k\Gamma(n+k)}{\Gamma(k-\delta+1)\Gamma(n+1)} \times \left[\frac{\ell(1+(\lambda_1+\lambda_2)(k-1))+d}{\ell(1+\lambda_2(k-1))+d}\right]^m a_k z^{k-\delta}. \tag{1.3}$$

It follows from (1.3) that

$$\ell\lambda_1 z \left(D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)\right)' = [\ell(1+(\lambda_2(k-1))+d]\, D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z) - [\ell(1+(\lambda_2(k-1)-(1-\delta)\lambda_1)+d]\, D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z). \tag{1.4}$$

In order to prove our results, we make use of the following known results.

**Definition 1.3** ([5]). Denote by $Q$ the set of all functions $f$ that are analytic and injective on $\bar{U}\backslash E(f)$, where

$$E(f) = \left\{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\right\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

**Lemma 1.1** ([5]). *Let $q$ be univalent in the unit disk $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set $Q(z) = zq'(z)\phi(q(z))$ and $h(z) = \theta(q(z)) + Q(z)$. Suppose that*

(1) *$Q(z)$ is starlike univalent in $U$;*
(2) $\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} > 0$ *for $z \in U$.*

*If $p$ is analytic in $U$, with $p(0) = q(0)$, $p(U) \subset D$ and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{1.5}$$

*then $p \prec q$ and $q$ is the best dominant of* (1.5).

**Lemma 1.2** ([6]). *Let $q$ be a convex univalent function in $U$ and let $\alpha \in C, \beta \in C\backslash\{0\}$, with*

$$\mathrm{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\mathrm{Re}\left(\frac{\alpha}{\beta}\right)\right\}.$$

*If $p$ is analytic in $U$ and*

$$\alpha p(z) + \beta z p'(z) \prec \alpha q(z) + \beta z q'(z), \tag{1.6}$$

*then $p \prec q$ and $q$ is the best dominant of* (1.6).

**Lemma 1.3** ([6]). *Let $q$ be a convex univalent function in $U$ and let $\beta \in \mathbb{C}$. Further assume that* $\operatorname{Re}(\beta) > 0$. *If $p \in \mathcal{H}[q(0), 1] \cap Q$ and $p(z) + \beta z p'(z)$ is univalent in $U$, then*

$$q(z) + \beta z q'(z) \prec p(z) + \beta z p'(z), \tag{1.7}$$

*which implies that $q \prec p$ and $q$ is the best subordinant of* (1.7).

**Lemma 1.4** ([3]). *Let $q$ be convex univalent in the unit disk $U$ and let $\theta$ and $\phi$ be analytic in a domain $D$ containing $q(U)$. Suppose that*

(1) $\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} > 0$ *for* $z \in U$;
(2) $Q(z) = zq'(z)\phi(q(z))$ *is starlike univalent in* $U$.

*If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(U) \subset D, \theta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $U$, and*

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(p(z)) + zp'(z)\phi(p(z)), \tag{1.8}$$

*then $q \prec p$ and $q$ is the best subordinant of* (1.8).

## 2. Subordination Results

**Theorem 2.1.** *Let $q$ be convex univalent in $U$ with $q(0) = 1$, $\sigma \in \mathbb{C} \setminus \{0\}$, $\gamma > 0$ and suppose that $q$ satisfies*

$$\operatorname{Re}\left\{1 + \frac{zq''(z)}{q'(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{(1-\delta)\gamma}{\sigma}\right)\right\}. \tag{2.1}$$

*If $f \in A$ satisfies the subordination*

$$\begin{aligned}
&\left(1 - \frac{\sigma\left[\ell(1 + (\lambda_2(k-1)) + d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z^{1-\delta}}\right)^{\gamma} \\
&\quad + \frac{\sigma\left[\ell(1 + (\lambda_2(k-1)) + d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta} I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^{\delta} I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}\right) \\
&\prec q(z) + \frac{\sigma}{(1-\delta)\gamma} zq'(z),
\end{aligned} \tag{2.2}$$

*then*

$$\left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z^{1-\delta}}\right)^{\gamma} \prec q(z) \tag{2.3}$$

*and $q$ is the best dominant of* (2.2).

*Proof.* Define the function $p$ by

$$p(z) = \left(\frac{\Gamma(2-\delta)\, D_z^{\delta} I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z^{1-\delta}}\right)^{\gamma}, \quad z \in U. \tag{2.4}$$

Then the function $p$ is analytic in $U$ and $p(0) = 1$. Differentiating (2.4) logarithmically with respect to $z$, we have

$$\frac{zp'(z)}{p(z)} = \gamma \left( \frac{z\left(D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)\right)'}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} - (1-\delta) \right).$$

Now, in view of (1.4), we obtain

$$\frac{zp'(z)}{p(z)} = \frac{\gamma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1} \left( \frac{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} - 1 \right).$$

Therefore,

$$\begin{aligned}\frac{zp'(z)}{(1-\delta)\gamma} =& \frac{\ell(1+(\lambda_2(k-1))+d}{\ell\lambda_1(1-\delta)} \left( \frac{\Gamma(2-\delta)\, D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)}{z^{1-\delta}} \right)^\gamma \\ &\times \left( \frac{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} - 1 \right).\end{aligned}$$

It follows from (2.2) that

$$p(z) + \frac{\sigma}{(1-\delta)\gamma} zp'(z) \prec q(z) + \frac{\sigma}{(1-\delta)\gamma} zq'(z).$$

Thus, an application of Lemma 1.2, with $\alpha = 1$ and $\beta = \frac{\sigma}{(1-\delta)\gamma}$, we obtain (2.3). $\square$

**Theorem 2.2.** *Let $\eta_i \in \mathbb{C}$, $i = 1,2,3,4$, $\gamma > 0$, $t \in \mathbb{C} \setminus \{0\}$ and $q$ be convex univalent in $U$ with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that $q$ satisfies*

$$\operatorname{Re}\left\{ 1 + \frac{\eta_2}{t} q(z) + \frac{2\eta_3}{t} q^2(z) + \frac{3\eta_4}{t} q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0. \tag{2.5}$$

*Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. If $f \in A$ satisfies*

(2.6)

$$\Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right) \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)},$$

*where*

(2.7)

$$\begin{aligned}&\Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right) \\ =&\eta_1 + \eta_2 \left( \frac{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \right)^\gamma + \eta_3 \left( \frac{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \right)^{2\gamma} + \eta_4 \left( \frac{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \right)^{3\gamma} \\ &+ \frac{\gamma t\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1} \left( \frac{D_z^\delta I^{n,m+2}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)} - \frac{D_z^\delta I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d} f(z)}{D_z^\delta I^{n,m}_{\lambda_1,\lambda_2,\ell,d} f(z)} \right),\end{aligned}$$

*then*

$$\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma \prec q(z)$$

*and $q$ is the best dominant of* (2.6).

*Proof.* Define the function $p$ by

$$p(z) = \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma, \quad z \in U. \tag{2.8}$$

Then the function $p$ is analytic in $U$ and $p(0) = 1$.

By a straightforward computation and using (1.4), we have

(2.9)

$$\eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)} = \Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right),$$

where $\Psi\left(\eta_1, \eta_2, \eta_3, \eta_4, \gamma, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.7). From (2.6) and (2.9), we obtain

$$\eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)} \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)}.$$

By setting

$$\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3 \quad \text{and} \quad \phi(w) = \frac{t}{w}, \quad w \neq 0,$$

we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C} \setminus \{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C} \setminus \{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t\frac{zq'(z)}{q(z)}$$

and

$$h(z) = \theta(q(z)) + Q(z) = \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $U$,

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1 + \frac{\eta_2}{t}q(z) + \frac{2\eta_3}{t}q^2(z) + \frac{3\eta_4}{t}q^3(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right\} > 0.$$

Thus, by Lemma 1.1, we get $p(z) \prec q(z)$. By using (2.8), we obtain the desired result. $\square$

**Theorem 2.3.** *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $t \in V \setminus \{0\}$ and $q$ be convex univalent in $U$ with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that $q$ satisfies* (2.5). *Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. If $f \in A$ satisfies*

(2.10)

$$\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right) \prec \eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)},$$

*where*
(2.11)

$$\begin{aligned}
&\Omega\left(\eta_1,\eta_2,\eta_3,\eta_4,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right)\\
=&\eta_1+\eta_2\frac{z^{1-\delta}D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\Gamma(2-\delta)\left(D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^2}+\eta_3\left(\frac{1}{\Gamma(2-\delta)}\right)^2\frac{z^{2(1-\delta)}\left(D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^2}{\left(D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^4}\\
&+\eta_4\left(\frac{1}{\Gamma(2-\delta)}\right)^3\frac{z^{3(1-\delta)}\left(D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^3}{\left(D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^6}+\frac{t\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1}\times\\
&\times\left(1+\frac{D_z^{\delta}I^{n,m+2}_{\lambda_1,\lambda_2,\ell,d}f(z)}{D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}-\frac{2D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right),
\end{aligned}$$

*then*

$$\frac{z^{1-\delta}D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\Gamma(2-\delta)\left(D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^2}\prec q(z)$$

*and $q$ is the best dominant of* (2.10).

*Proof.* Define the function $p$ by

$$p(z)=\frac{z^{1-\delta}D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{\Gamma(2-\delta)\left(D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)^2},\quad z\in U. \tag{2.12}$$

Then the function $p$ is analytic in $U$ and $p(0)=1$.

We note that
(2.13)

$$\eta_1+\eta_2p(z)+\eta_3p^2(z)+\eta_4p^3(z)+t\frac{zp'(z)}{p(z)}=\Omega\left(\eta_1,\eta_2,\eta_3,\eta_4,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right),$$

where $\Omega\left(\eta_1,\eta_2,\eta_3,\eta_4,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right)$ is given by (2.11). From (2.10) and (2.13), we obtain

$$\eta_1+\eta_2p(z)+\eta_3p^2(z)+\eta_4p^3(z)+t\frac{zp'(z)}{p(z)}\prec\eta_1+\eta_2q(z)+\eta_3q^2(z)+\eta_4q^3(z)+t\frac{zq'(z)}{q(z)}.$$

The remaining part of the proof Theorem 2.3 is similar to that of Theorem 2.2 and hence we omit it. □

## 3. Superordination Results

**Theorem 3.1.** *Let $q$ be convex univalent in $U$ with $q(0)=1$, $\gamma>0$ and $\operatorname{Re}\{\sigma\}>0$. Let $f\in A$ satisfies*

$$\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma}\in\mathcal{H}\left[q(0),1\right]\cap Q$$

*and*

$$\left(1-\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma}$$
$$+\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)$$

*be univalent in $U$. If*

$$q(z)+\frac{\sigma}{(1-\delta)\gamma}zq'(z) \tag{3.1}$$
$$\prec\left(1-\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma}$$
$$+\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right),$$

*then*

$$q(z)\prec\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma} \tag{3.2}$$

*and $q$ is the best subordinant of* (3.1).

*Proof.* Define the function $p$ by

$$p(z)=\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma},\quad z\in U. \tag{3.3}$$

Then the function $p$ is analytic in $U$ and $p(0)=1$. Differentiating (3.3) logarithmically with respect to $z$, we get

$$\frac{zp'(z)}{p(z)}=\gamma\left(\frac{z\left(D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)\right)'}{D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}-(1-\delta)\right).$$

After some computations and using (1.4), we find that

$$\left(1-\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma} \tag{3.4}$$
$$+\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\,D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta}I^{n,m+1}_{\lambda_1,\lambda_2,\ell,d}f(z)}{D_z^{\delta}I^{n,m}_{\lambda_1,\lambda_2,\ell,d}f(z)}\right)$$
$$=p(z)+\frac{\sigma}{(1-\delta)\gamma}zp'(z).$$

From (3.1) and (3.4), we have

$$q(z)+\frac{\sigma}{(1-\delta)\gamma}zq'(z)\prec p(z)+\frac{\sigma}{(1-\delta)\gamma}zp'(z).$$

Thus, an application of Lemma 1.3, with $\alpha = 1$ and $\beta = \frac{\sigma}{(1-\delta)\gamma}$, we obtain the results. $\square$

**Theorem 3.2.** *Let $\eta_i \in \mathbb{C}$, $i = 1,2,3,4$, $\gamma > 0$, $t \in \mathbb{C}\setminus\{0\}$ and $q$ be convex univalent in $U$ with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that $q$ satisfies*

$$\operatorname{Re}\left\{\frac{\eta_2}{t}q(z) + \frac{2\eta_3}{t}q^2(z) + \frac{3\eta_4}{t}q^3(z)\right\} > 0. \tag{3.5}$$

*Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. Let $f \in A$ satisfies*

$$\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma \in H\left[q(0),1\right] \cap Q$$

*and $\Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right)$ is univalent in $U$, where $\Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right)$ is given by (2.7). If*

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right), \tag{3.6}$$

*then*

$$q(z) \prec \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma$$

*and $q$ is the best subordinant of* (3.6).

*Proof.* Define the function $p$ by

$$p(z) = \left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^\gamma, \quad z \in U. \tag{3.7}$$

Then the function $p$ is analytic in $U$ and $p(0) = 1$.

By some computation, we have

$$\Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right) = \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)}, \tag{3.8}$$

where $\Psi\left(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z\right)$ is given by (2.7). From (3.6) and (3.8), we obtain

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)}.$$

By setting $\theta(w) = \eta_1 + \eta_2 w + \eta_3 w^2 + \eta_4 w^3$ and $\phi(w) = \frac{t}{w}$, $w \neq 0$, we see that $\theta(w)$ is analytic in $\mathbb{C}$, $\phi(w)$ is analytic in $\mathbb{C}\setminus\{0\}$ and that $\phi(w) \neq 0$, $w \in \mathbb{C}\setminus\{0\}$. Also, we get

$$Q(z) = zq'(z)\phi(q(z)) = t\frac{zq'(z)}{q(z)}.$$

It is clear that $Q(z)$ is starlike univalent in $U$,

$$\operatorname{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))}\right\} = \operatorname{Re}\left\{\frac{\eta_2}{t}q(z) + \frac{2\eta_3}{t}q^2(z) + \frac{3\eta_4}{t}q^3(z)\right\} > 0.$$

Thus, by Lemma 1.4, we get $q(z) \prec p(z)$. By using (3.7), we obtain the desired result. □

**Theorem 3.3.** *Let $\eta_i \in \mathbb{C}$, $i = 1, 2, 3, 4$, $t \in \mathbb{C} \setminus \{0\}$ and $q$ be convex univalent in $U$ with $q(0) = 1$, $q(z) \neq 0$, $z \in U$, and assume that $q$ satisfies* (3.5)*. Suppose that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $U$. Let $f \in A$ satisfies*

$$\frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2} \in H\left[q(0), 1\right] \cap Q$$

*and $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is univalent in $U$, where $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by* (2.11)*. If*

(3.9)

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right),$$

*then*

$$q(z) \prec \frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2}$$

*and $q$ is the best subordinant of* (3.9)*.*

*Proof.* Define the function $p$ by

$$p(z) = \frac{z^{1-\delta} D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)\right)^2}, \quad z \in U. \tag{3.10}$$

Then the function $p$ is analytic in $U$ and $p(0) = 1$.

We note that

(3.11)

$$\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right) = \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)},$$

where $\Omega\left(\eta_1, \eta_2, \eta_3, \eta_4, t, \delta, n, m, \lambda_1, \lambda_2, \ell, d; z\right)$ is given by (2.11). From (3.9) and (3.11), we obtain

$$\eta_1 + \eta_2 q(z) + \eta_3 q^2(z) + \eta_4 q^3(z) + t\frac{zq'(z)}{q(z)} \prec \eta_1 + \eta_2 p(z) + \eta_3 p^2(z) + \eta_4 p^3(z) + t\frac{zp'(z)}{p(z)}.$$

The remaining part of the proof Theorem 3.3 is similar to that of Theorem 3.2 and hence we omit it. □

## 4. Sandwich Results

Combining results of differential subordinations and superordinations, we state the following "sandwich results".

**Theorem 4.1.** *Let $q_1$ and $q_2$ be convex univalent in $U$ with $q_1(0) = q_2(0) = 1$. Suppose $q_2$ satisfies* (2.1), $\gamma > 0$ *and* $\operatorname{Re}\{\sigma\} > 0$. *Let $f \in A$ satisfies*

$$\left(\frac{\Gamma(2-\delta)\; D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma} \in H\left[1,1\right] \cap Q$$

*and*

$$\begin{aligned}&\left(1-\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\; D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}\\&+\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\; D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)\end{aligned}$$

*be univalent in $U$. If*

$$\begin{aligned}&q_1(z)+\frac{\sigma}{(1-\delta)\gamma}zq_1'(z)\\&\prec\left(1-\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\right)\left(\frac{\Gamma(2-\delta)\; D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}\\&\quad+\frac{\sigma\left[\ell(1+(\lambda_2(k-1))+d\right]}{\ell\lambda_1(1-\delta)}\left(\frac{\Gamma(2-\delta)\; D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma}\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)\\&\prec q_2(z)+\frac{\sigma}{(1-\delta)\gamma}zq_2'(z),\end{aligned}$$

*then*

$$q_1(z) \prec \left(\frac{\Gamma(2-\delta)\; D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}{z^{1-\delta}}\right)^{\gamma} \prec q_2(z)$$

*and $q_1$ and $q_2$ are, respectively, the best subordinant and the best dominant.*

**Theorem 4.2.** *Let $q_1$ and $q_2$ be convex univalent in $U$ with $q_1(0) = q_2(0) = 1$. Suppose $q_1$ satisfies* (3.5) *and $q_2$ satisfies (*2.5*). Let $f \in A$ satisfies*

$$\left(\frac{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1} f(z)}{D_z^{\delta} I_{\lambda_1,\lambda_2,\ell,d}^{n,m} f(z)}\right)^{\gamma} \in H\left[1,1\right] \cap Q$$

*and* $\Psi(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z)$ *is univalent in* $U$, *where* $\Psi(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z)$ *is given by* (2.7). *If*

$$\begin{aligned}\eta_1+\eta_2 q_1(z)+\eta_3 q_1^2(z)+\eta_4 q_1^3(z)+t\frac{zq_1'(z)}{q_1(z)} \prec&\Psi(\eta_1,\eta_2,\eta_3,\eta_4,\gamma,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z)\\ \prec&\eta_1+\eta_2 q_2(z)+\eta_3 q_2^2(z)+\eta_4 q_2^3(z)\\ &+t\frac{zq_2'(z)}{q_2(z)},\end{aligned}$$

*then*

$$q_1(z)\prec\left(\frac{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)}\right)^\gamma\prec q_2(z)$$

*and* $q_1$ *and* $q_2$ *are, respectively, the best subordinant and the best dominant.*

**Theorem 4.3.** *Let* $q_1$ *and* $q_2$ *be convex univalent in* $U$ *with* $q_1(0)=q_2(0)=1$. *Suppose* $q_1$ *satisfies* (3.5) *and* $q_2$ *satisfies* (2.5). *Let* $f\in A$ *satisfies*

$$\frac{z^{1-\delta}D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)\right)^2}\in H[1,1]\cap Q$$

*and* $\Omega(\eta_1,\eta_2,\eta_3,\eta_4,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z)$ *is univalent in* $U$, *where* $\Omega(\eta_1,\eta_2,\eta_3,\eta_4,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z)$ *is given by* (2.11). *If*

$$\begin{aligned}\eta_1+\eta_2 q_1(z)+\eta_3 q_1^2(z)+\eta_4 q_1^3(z)+t\frac{zq_1'(z)}{q_1(z)} \prec&\Omega(\eta_1,\eta_2,\eta_3,\eta_4,t,\delta,n,m,\lambda_1,\lambda_2,\ell,d;z)\\ \prec&\eta_1+\eta_2 q_2(z)+\eta_3 q_2^2(z)+\eta_4 q_2^3(z)\\ &+t\frac{zq_2'(z)}{q_2(z)},\end{aligned}$$

*then*

$$q_1(z)\prec\frac{z^{1-\delta}D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m+1}f(z)}{\Gamma(2-\delta)\left(D_z^\delta I_{\lambda_1,\lambda_2,\ell,d}^{n,m}f(z)\right)^2}\prec q_2(z)$$

*and* $q_1$ *and* $q_2$ *are, respectively, the best subordinant and the best dominant.*

*Remark* 4.1. By specifying the function $\phi$ and selecting the particular values of $\eta_1,\eta_2,\eta_3,\eta_4,\gamma,\delta$, $n,m,\lambda_1,\lambda_2,\ell$ and $d$, we can derive a number of known results. Some of them are given below.

(1) Taking $\delta=n=\lambda_2=d=0$ and $\ell=1$ in Theorems 2.1, 3.1, 4.1, we get the results obtained by Răducanu and Nechita [10, Theorem 3.1, Theorem 3.6, Theorem 3.9].
(2) Putting $\delta=n=\lambda_2=\eta_1=\eta_3=\eta_4=d=0$, $\eta_2=\ell=1$ and $\phi(w)=t$ in Theorems 2.3, 3.3, 4.3, we obtain the results obtained by Nechita [8, Theorem 14, Theorem 19, Corollary 21].

(3) For $\delta = n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = d = 0$, $\lambda_1 = \eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we have the results obtained by Shanmugam et al. [13, Theorem 5.4, Theorem 5.5, Theorem 5.6].

(4) By taking $\delta = n = m = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = d = 0$, $\lambda_1 = \eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we get the results obtained by Shanmugam et al. [13, Theorem 3.4, Theorem 3.5, Theorem 3.6].

(5) Putting $\delta = n = \lambda_2 = \eta_1 = \eta_3 = \eta_4 = 0$, $\eta_2 = \ell = 1$ and $\phi(w) = t$ in Theorems 2.3, 3.3, 4.3, we have the results obtained by Shammaky [12, Theorem 3.4, Theorem 3.5, Theorem 3.6].

(6) Taking $\delta = n = m = \lambda_2 = d = 0$ and $\lambda_1 = \ell = 1$ in Theorem 2.1, we obtain the results obtained by Murugusundaramoorthy and Magesh [7, Corollary 3.3].

(7) Putting $\delta = n = m = \lambda_2 = d = 0$ and $\lambda_1 = \ell = 1$ in Theorems 3.1, 4.1, we obtain the results obtained by Răducanu and Nechita [10, Corollary 3.7, Corollary 3.10].

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# NEW GENERALIZED APOSTOL-FROBENIUS-EULER POLYNOMIALS AND THEIR MATRIX APPROACH

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Abstract. In this paper, we introduce a new extension of the generalized Apostol-Frobenius-Euler polynomials $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$. We give some algebraic and differential properties, as well as, relationships between this polynomials class with other polynomials and numbers. We also, introduce the generalized Apostol-Frobenius-Euler polynomials matrix $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$ and the new generalized Apostol-Frobenius-Euler matrix $\mathcal{U}^{[m-1,\alpha]}(c,a;\lambda;u)$, we deduce a product formula for $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$ and provide some factorizations of the Apostol-Frobenius-Euler polynomial matrix $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$, which involving the generalized Pascal matrix.

## 1. Introduction

It is well-known that generalized Frobenius-Euler polynomial $H_n^{(\alpha)}(x;u)$ of order $\alpha$ is defined by means of the following generating function

$$\left(\frac{1-u}{e^z-u}\right)^{\alpha} e^{xz}=\sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u)\frac{z^n}{n!}, \tag{1.1}$$

where $u\in\mathbb{C}$ and $\alpha\in\mathbb{Z}$. Observe that $H_n^{(1)}(x;u)=H_n(x;u)$ denotes the classical Frobenius-Euler polynomials and $H_n^{(\alpha)}(0;u)=H_n^{(\alpha)}(u)$ denotes the Frobenius-Euler numbers of order $\alpha$. $H_n(x;-1)=E_n(x)$ denotes the Euler polynomials (see [2, 7]).

For parameters $\lambda,u\in\mathbb{C}$ and $a,b,c\in\mathbb{R}^+$, the Apostol type Frobenius-Euler polynomials $H_n(x;\lambda;u)$ and the generalized Apostol-type Frobenius-Euler polynomials are

*Key words and phrases.* Generalized Apostol-type polynomials, Apostol-Frobennius-Euler polynomials, Apostol-Bernoulli polynomials of higher order, Apostol-Genocchi polynomials of higher order, Stirling numbers of second kind, generalized Pascal matrix.

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defined by means of the following generating functions (see [8]):

$$\left(\frac{1-u}{\lambda e^z-u}\right)e^{xz}=\sum_{n=0}^{\infty}H_n(x;\lambda;u)\frac{z^n}{n!}, \tag{1.2}$$

$$\left(\frac{a^z-u}{\lambda b^z-u}\right)^{\alpha}c^{xz}=\sum_{n=0}^{\infty}H_n^{(\alpha)}(x;a,b,c;\lambda;u)\frac{z^n}{n!}. \tag{1.3}$$

If we set $x=0$ and $\alpha=1$ in (1.3), we get

$$\frac{a^z-u}{\lambda b^z-u}=\sum_{n=0}^{\infty}H_n(a,b,c;\lambda;u)\frac{z^n}{n!},$$

$H_n(a,b,c;u;\lambda)$ denotes the generalized Apostol-type Frobenius-Euler numbers (see [8]).

In the present paper, we introduce a new class of Frobenius-Euler polynomials considering the work of [8], we give relationships between this polynomials whit other polynomials and numbers, as well as the generalized Apostol-Frobenius-euler polynomials matrix.

The paper is organized as follows. Section 2 contains the definitions of Apostol-type Frobenius-Euler and generalized Apostol-Frobenius-Euler polynomials and some auxiliary results. In Section 3, we define the generalized Apostol-type Frobenius-Euler polynomials and prove some algebraic and differential properties of them, as well as their relation with the Stirling numbers of second kind. Finally, in Section 4 we introduce the generalized Apostol-type Frobenius-Euler polynomial matrix, derive a product formula for it and give some factorizations for such a matrix, which involve summation matrices and the generalized Pascal matrix of first kind in base $c$, respectively.

## 2. Previous Definitions and Notations

Throughout this paper, we use the following standard notions: $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{N}_0=\{0,1,2,\ldots\}$, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers. Furthermore, $(\lambda_0)=1$ and

$$(\lambda)_k=\lambda(\lambda+1)(\lambda+2)\cdots(\lambda+k-1),$$

where $k\in\mathbb{N}$, $\lambda\in\mathbb{C}$. For the complex logarithm, we consider the principal branch. All matrices are in $M_{n+1}(\mathbb{K})$, the set of all $(n+1)\times(n+1)$ matrices over the field $\mathbb{K}$, with $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. Also, for $i,j$ any nonnegative integers we adopt the following convention

$$\binom{i}{j}=0,\qquad \text{whenever } j>i.$$

Now, let us givel some properties of the generalized Apostol-type Frobenius-Euler polynomials and generalized Apostol-type Frobenius-Euler polynomials with parameters $\lambda,a,c$, order $\alpha$ (see [4,8,11]).

**Proposition 2.1.** *For a $m \in \mathbb{N}$, let $\{H_n^{(\alpha)}(x;u)\}_{n\geq 0}$ and $\{H_n(x;\lambda;u)\}_{n\geq 0}$ be the sequences of generalized Apostol-type Frobenius-Euler polynomials, generalized Frobenius-Euler polynomials respectively. Then the following statements hold.*

(a) *Special values: for $n \in \mathbb{N}_0$,*

$$H_n^{(0)}(x;u) = x^n.$$

(b) *Summation formulas:*

$$H_n^{(\alpha)}(x;u;a,b,c;\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(\alpha)}(x;u;a,b,c;\lambda)(x \ln c)^{n-k},$$

$$H_n^{(\alpha+\beta)}(x+y;u;a,b,c;\lambda) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(\alpha)}(x;u;a,b,c;\lambda) H_{n-k}^{(\beta)}(y;u;a,b,c;\lambda),$$

$$((x+y)\ln c)^n = H_{n-k}^{(\alpha)}(y;u;a,b,c;\lambda) H_k^{(-\alpha)}(x;u;a,b,c;\lambda),$$

$$H_n^{(-\alpha)}(x;u^2;a^2,b^2,c^2;\lambda^2) = \sum_{k=0}^{n} \binom{n}{k} H_k^{(-\alpha)}(x;u;a,b,c;\lambda) H_{n-k}^{(-\alpha)}(x;-u;a,b,c;\lambda).$$

**Definition 2.1.** ([5, p. 207]). For $n \in \mathbb{N}_0$ and $x \in \mathbb{C}$, the Stirling numbers of second kind $S(n,k)$ are defined by means of the following expansion

$$x^n = \sum_{k=0}^{n} \binom{x}{k} k! S(n,k).$$

The Jacobi polynomials of the degree $n$ y orde $(\alpha,\beta)$, with $\alpha,\beta > -1$, the $n$-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be defined through Rodrigues' formula

$$P_n^{(\alpha,\beta)}(x) = (1-x)^{-\alpha}(1+x)^{-\beta} \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n} \left\{(1-x)^{n+\alpha}(1+x)^{n+\alpha}\right\}$$

and the values in the end points of the interval $[-1,1]$ is given by

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}, \quad P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}.$$

The relationship between the $n$-th monomial $x^n$ and the $n$-th Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ may be written as

$$x^n = n! \sum_{k=0}^{n} \binom{n+\alpha}{n-k} (-1)^k \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n+1}} P_k^{(\alpha,\beta)}(1-2x). \tag{2.1}$$

**Proposition 2.2.** *For $\lambda \in \mathbb{C}$ and $m \in \mathbb{N}$, let $\{B_n^{[m-1]}(x)\}_{n\geq 0}$, $\{G_n(x)\}_{n\geq 0}$ and $\{\mathcal{E}_n(x;\lambda)\}_{n\geq 0}$ be the sequences of generalized Bernoulli polynomials of level $m$, Genocchi polynomials and Apostol-Euler polynomials, respectively, we have the relationships:*

(a) [12, Equation (4)]

$$x^n = \sum_{k=0}^{n} \binom{n}{k} \frac{k!}{(k+m)!} B_{n-k}^{[m-1]}(x); \tag{2.2}$$

(b) [9, Remark 7]

$$x^n = \frac{1}{2(n+1)}\left[\sum_{k=0}^{n+1}\binom{n+1}{k}G_k(x) + G_{n+1}(x)\right]; \tag{2.3}$$

(c) [10, Equation (32)]

$$x^n = \frac{1}{2}\left[\lambda\sum_{k=0}^{n}\binom{n}{k}\mathcal{E}_k(x;\lambda) + \mathcal{E}_n(x;\lambda)\right]. \tag{2.4}$$

**Definition 2.2.** Let $x$ be any nonzero real number. For $c \in \mathbb{R}^+$, the generalized Pascal matrix of first kind in base $c$ $P_c[x]$ is an $(n+1)\times(n+1)$ matrix whose entries are given by (see [13, 14])

$$p_{i,j,c}(x) := \begin{cases} \binom{i}{j}(x\ln c)^{i-j}, & i \geq j, \\ 0, & \text{otherwise.} \end{cases}$$

When $c = e$, the matrix $P_c[x]$ coincides with the generalized Pascal matrix of first kind $P[x]$. Furthermore, if we adopt the convention $0^0 = 1$, then $P_c[0] = I_{n+1}$, with $I_{n+1} = \operatorname{diag}(1, 1, \ldots, 1)$.

An immediate consequence of the remarks above is the following proposition.

**Proposition 2.3** (Addition Theorem of the argument)**.** *For $x, y \in \mathbb{R}$ is fulfilled*

$$P_c[x+y] = P_c[x]P_c[y].$$

**Proposition 2.4.** *For $c \in \mathbb{R}^+$, let $P_c[x]$ be the generalized Pascal matrix of first kind in base $c$ and order $n+1$. Then the following statements hold.*

(a) *$P_c[x]$ is an invertible matrix and its inverse is given by*

$$P_c^{-1}[x] := (P_c[x])^{-1} = P_c[-x].$$

(e) *The matrix $P_c[x]$ can be factorized as follows*

$$P_c[x] = G_{n,c}[x]G_{n-1,c}[x]\cdots G_{1,c}[x], \tag{2.5}$$

*where $G_{k,c}[x]$ is the $(n+1)\times(n+1)$ summation matrix given by*

$$G_{k,c}[x] = \begin{cases} \begin{bmatrix} I_{n-k} & 0 \\ 0 & S_{k,c}[x] \end{bmatrix}, & k = 1, \ldots, n-1, \\ S_{n,c}[x], & k = n, \end{cases}$$

*being $S_{k,c}[x]$ the $(k+1)\times(k+1)$ matrix whose entries $S_{k,c}(x; i, j)$ are given by*

$$S_{k,c}(x; i, j, c) = \begin{cases} (x\ln c)^{i-j}, & i \geq j, \\ 0, & j > i, \end{cases} \qquad 0 \leq i, j \leq k.$$

## 3. Generalized Apostol-Frobenius-Euler Polynomials $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$

**Definition 3.1.** For $m \in \mathbb{N}$, $\alpha, \lambda, u \in \mathbb{C}$ and $a, c \in \mathbb{R}^+$, the generalized Apostol-type Frobenius-Euler polynomials in the variable $x$, parameters $c, a, \lambda$, order $\alpha$ and level $m$, are defined through the following generating function

$$\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!} - u^m}{\lambda c^z - u^m}\right]^{\alpha} c^{xz} = \sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x;c;a;\lambda;u)\frac{z^n}{n!}, \tag{3.1}$$

where $|z| < \left|\frac{\ln(u^m)}{\ln(c)} - \frac{\ln(\lambda)}{\ln(c)}\right|$.

For $x = 0$ we obtain, the generalized Apostol-Frobennius-Euler numbers of parameters $\lambda \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m \in \mathbb{N}$

$$\mathcal{H}_n^{[m-1,\alpha]}(c,a;\lambda;u) := \mathcal{H}_n^{[m-1,\alpha]}(0;c,a;\lambda;u).$$

According to the Definition 3.1, with $e = \exp(1)$, we have (1.1) and (1.2)

$$\begin{aligned}\mathcal{H}_n^{[0,\alpha]}(x;e,1;1;u) &= H_n^{(\alpha)}(x;\lambda;u),\\ \mathcal{H}_n^{[0,1]}(x;e,1;\lambda;u) &= H_n^{(1)}(x;\lambda;u).\end{aligned}$$

*Example* 3.1. For any $\lambda \in \mathbb{C}$, $m = 2$, $c = 2$, $a = 3$, $\alpha = \frac{1}{2}$ and $u = 2$ the first the generalized Apostol-type Frobenius-Euler polynomials in the variable $x$, parameters $c, a, \lambda$, order $\alpha$ and level $m$ are:

$$\begin{aligned}
\mathcal{H}_0^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2) &= \sqrt{\frac{3}{\lambda-4}},\\
\mathcal{H}_1^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2) &= \sqrt{\frac{-3}{\lambda-4}}x\left[\frac{1}{2}\left(\frac{\ln 3}{\lambda-4}+\frac{3\lambda\ln 2}{(\lambda-4)^2}\right)+x\ln 4\right],\\
\mathcal{H}_2^{[1,(\frac{1}{2})]}(x;2,3;\lambda;2) &= \frac{1}{2}x^2\left[\left(\frac{-3}{4}\sqrt{\frac{-3}{\lambda-4}}\left(\frac{\ln 3}{\lambda-4}+\frac{3\lambda\ln 2}{(\lambda-4)^2}\right)^2\right.\right.\\
&\quad\left.+\frac{1}{2}\sqrt{\frac{-3}{\lambda-4}}\frac{-2\ln 3\ln 2}{(\lambda-4)^2}-\frac{6\lambda^2\ln 4}{(\lambda-4)^3}+\frac{3\lambda\ln 4}{(\lambda-4)^2}\right)\\
&\quad\left.+x\ln 2\sqrt{\frac{-3}{\lambda-4}}\left(\frac{\ln 3}{\lambda-4}+\frac{3\ln 2}{(\lambda-4)^4}\right)+x^2\ln 4\sqrt{\frac{-3}{\lambda-4}}\right].
\end{aligned}$$

*Example* 3.2. For any $\lambda \in \mathbb{C}$, $m = 4$, $c = 2$, $a = 3$, $\alpha = 1$ and $u = 2$ the first the generalized Apostol-type Frobenius-Euler polynomials in the variable $x$, parameters $c, a, \lambda$, order $\alpha$ and level $m$ are:

$$\mathcal{H}_0^{[3,1]}(x;2,3;\lambda;2) = \frac{-15}{\lambda-16},$$

$$\mathcal{H}_1^{[3,1]}(x;2,3;\lambda;2) = x\left[\frac{\ln 3}{\lambda-16} + \frac{\lambda 15\ln 2}{(\lambda-16)^2} - x\frac{15\ln 2}{\lambda-16}\right],$$
$$\mathcal{H}_2^{[3,1]}(x;2,3;\lambda;2) = \frac{1}{2}x^2\left[\frac{\ln 9}{\lambda-16} - \lambda\frac{2\ln 3\ln 2}{(\lambda-16)^2} + x\frac{2\ln 3\ln 2}{\lambda-16} - \lambda^2\frac{30\ln 4}{(\lambda-16)^3}\right.$$
$$\left. + x\frac{30\lambda\ln 4}{(\lambda-16)^2} + \lambda\frac{15\ln 4}{(\lambda-16)^2} - x^2\frac{15\ln 4}{\lambda-16}\right].$$

*Example* 3.3. For any $\lambda \in \mathbb{C}$, $m = 2$, $c = 3$, $a = e$, $\alpha = \frac{1}{3}$, and $u = 5$ the first the generalized Apostol-type Frobenius-Euler polynomials in the variable $x$, parameters $c, a, \lambda$, order $\alpha$ and level $m$ are:

$$\mathcal{H}_0^{[1,(\frac{1}{3})]}(x;3,e;\lambda;5) = \sqrt[3]{\frac{-24}{\lambda-25}},$$
$$\mathcal{H}_1^{[1,(\frac{1}{3})]}(x;3,e;\lambda;5) = x\left[\frac{1}{3}\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^2}\left(\frac{\omega}{\lambda-25} + \lambda\frac{24\ln 3}{(\lambda-25)^2}\right)\right.$$
$$\left. + x\ln 3\sqrt[3]{\frac{-24}{\lambda-25}}\right],$$
$$\mathcal{H}_2^{[1,(\frac{1}{3})]}(x;3,e;\lambda;5) = \frac{1}{2}x^2\left[\left(\frac{2}{9}\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^5}\frac{\omega}{\lambda-25} + \lambda\frac{24\ln 3}{(\lambda-25)^2}\right)^2\right.$$
$$+ \frac{2}{3}x\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^2}\ln 3\left(\frac{\omega}{\lambda-25} + \lambda\frac{24\ln 3}{(\lambda-25)^2}\right)$$
$$+ \frac{1}{3}\sqrt[3]{\left(\frac{\lambda-25}{-24}\right)^2}\left(-2\ln 3\frac{\omega}{(\lambda-25)} - \lambda^2\frac{-48\ln 9}{(\lambda-25)^3}\right.$$
$$\left.\left. + \lambda\frac{24\ln 9}{(\lambda-25)^2}\right) + x^2\ln 9\sqrt[3]{\frac{-24}{\lambda-25}}\right],$$

where $\omega = \ln\left(\frac{3060513257434037}{1125899906842624}\right)$.

**Theorem 3.1.** *For $m \in \mathbb{N}$, let $\{\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)\}_{n\geq 0}$ be the sequence of generalized Apostol-type Frobenius-Euler polynomials, whit parameters $\lambda, u \in \mathbb{C}$ and $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m$. Then the following statements hold.*

(a) *For every $\alpha = 0$ and $n \in \mathbb{N}_0$*
$$\mathcal{H}_n^{[m-1,0]}(x;c;a;\lambda;u) = (x\ln c)^n.$$

(b) *For $\alpha, \lambda \in \mathbb{C}$ and $n, k \in \mathbb{N}_0$, we have the relationship*
$$\mathcal{H}_n^{[m-1,\alpha]}(x;c;a;\lambda;u) = \sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(c;a;\lambda;u)(x\ln c)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha-1]}(c;a;\lambda;u) \mathcal{H}_{k}^{[m-1,1]}(x;c;a;\lambda;u).$$

(c) *Differential relations. For $m \in \mathbb{N}$ and $n, j \in \mathbb{N}_0$ with $0 \le j \le n$, we have*

$$[\mathcal{H}_{n}^{[m-1,\alpha]}(x;c;a;\lambda;u)]^{(j)} = \frac{n!}{(n-j)!}(\ln c)^{j}\, \mathcal{H}_{n-j}^{[m-1,\alpha]}(x;c,a;\lambda;u).$$

(d) *Integral formulas. For $m \in \mathbb{N}$, is fulfilled*

$$\int_{x_0}^{x_1} \mathcal{H}_{n}^{[m-1,\alpha]}(x;c,a;\lambda;u)\, dx = \frac{\ln c}{n+1}\left[\mathcal{H}_{n+1}^{[m-1,\alpha]}(x_1;c,a;\lambda;u) - \mathcal{H}_{n+1}^{[m-1,\alpha]}(x_0;c,a;\lambda;u)\right].$$

(e) *Addition theorem of the argument.*

$$\mathcal{H}_{n}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{k}^{[m-1,\alpha]}(x;c,a;\lambda;u) \mathcal{H}_{n-k}^{[m-1,\beta]}(y;c,a;\lambda;u), \tag{3.2}$$

$$\mathcal{H}_{n}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u)(x\ln c)^{k}, \tag{3.3}$$

$$((x+y)\log c)^{n} = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c;a;\lambda;u) \mathcal{H}_{k}^{[m-1,-\alpha]}(x;c;a;\lambda;u). \tag{3.4}$$

*Proof.* (3.2) From Definition 3.1, we have

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha+\beta]}(x+y,c,a;\lambda;u)\frac{t^{n}}{n!} \\
&= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^{h}}{h!} - u^{m}}{\lambda c^{z} - u^{m}}\right]^{(\alpha+\beta)} c^{(x+y)z} \\
&= \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^{h}}{h!} - u^{m}}{\lambda c^{z} - u^{m}}\right]^{\alpha} c^{xz} \left[\frac{\sum_{h=0}^{m-1}\frac{(z\ln a)^{h}}{h!} - u^{m}}{\lambda c^{z} - u^{m}}\right]^{\beta} c^{yz} \\
&= \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha]}(x;c;a;\lambda;u)\frac{z^{n}}{n!} \sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\beta]}(y;c;a;\lambda;u)\frac{z^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_{k}^{[m-1,\alpha]}(x,c,a;\lambda;u) \mathcal{H}_{n-k}^{[m-1,\beta]}(y,c,a;\lambda;u)\frac{z^{n}}{n!}.
\end{aligned}$$

□

*Proof.* (3.4) Making an adequate modification $\beta = -\alpha$ and aplply (3.2)

$$\sum_{n=0}^{\infty} \mathcal{H}_{n}^{[m-1,\alpha+\beta]}(x+y;c;a;\lambda;u)\frac{z^{n}}{n!}$$

$$=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{(\alpha+\beta)} c^{(x+y)z}$$

$$=\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{\alpha} c^{xz}\left[\frac{\displaystyle\sum_{h=0}^{m-1}\frac{(z\ln a)^h}{h!}-u^m}{\lambda c^z-u^m}\right]^{\beta} c^{yz}$$

$$=\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,\alpha]}(x;c;a;\lambda;u)\frac{z^n}{n!}\sum_{n=0}^{\infty}\mathcal{H}_n^{[m-1,-\alpha]}(y;c;a;\lambda;u)\frac{z^n}{n!}$$

$$=c^{(x+y)z}$$

$$=\sum_{n=0}^{\infty}((x+y)\log c)^n\,\frac{z^n}{n!}.$$

Therefore, (3.4) holds. □

From (2.1) and Proposition 2.2 we deduce some algebraic relations connecting the polynomials $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$ with other families of polynomials.

**Theorem 3.2.** *For $m\in\mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level $m$ $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, by means of the identity.*

(3.5)

$$\begin{aligned}&\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\&=\sum_{k=0}^{n}(-1)^k\sum_{j=k}^{n}j!(\ln c)^j\binom{j+\alpha}{j-k}\binom{n}{j}\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu))P_k^{(\alpha,\beta)}(1-2x).\end{aligned}$$

*Proof.* By substituting (2.1) into the right-hand side of (3.3) and using appropriate binomial coefficient identities (see, for instance [1, 5, 6]), we see that

$$\begin{aligned}&\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\&=\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)(n-j)!(\ln c)^{n-j}\sum_{k=0}^{n-j}(-1)^k\binom{n-j+\alpha}{n-j-k}\\&\times\frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}}P_k^{(\alpha,\beta)}(1-2x)\\&=\sum_{j=0}^{n}\sum_{k=0}^{n-j}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)(n-j)!(\ln c)^{n-j}(-1)^k\binom{n-j+\alpha}{n-j-k}\end{aligned}$$

$$\times \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}} P_k^{(\alpha,\beta)}(1-2x)$$
$$= \sum_{k=0}^{n}(-1)^k \sum_{j=0}^{n-k} \binom{n}{j} \binom{n-j+\alpha}{n-j-k} \mathcal{H}_j^{[m-1,\mu]}(y;c,a;\lambda;u)(n-j)!(\ln c)^{n-j}$$
$$\times \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{n-j+1}} P_k^{(\alpha,\beta)}(1-2x)$$
$$= \sum_{k=0}^{n}(-1)^k \sum_{j=k}^{n} j!(\ln c)^j \binom{j+\alpha}{j-k} \binom{n}{j} \frac{(1+\alpha+\beta+2k)}{(1+\alpha+\beta+k)_{j+1}}$$
$$\times \mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u) P_k^{(\alpha,\beta)}(1-2x).$$

Therefore, (3.5) holds. □

**Theorem 3.3.** *For $m \in \mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level $m$ $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the generalized Bernoulli polynomials of level $m$ $B_n^{[m-1]}(x)$, by means of the following identity*

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \sum_{k=0}^{n}\sum_{j=k}^{n} \frac{k!(\ln c)^j}{(k+m)!} \binom{n}{j} \binom{j}{k} \mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;\mu;\nu) B_{j-k}^{[m-1]}(x).$$

*Proof.* By substituting (2.2) into the right-hand side of (3.3), it suffices to follow the proof given in Theorem 3.2, making the corresponding modifications. □

**Theorem 3.4.** *For $m \in \mathbb{N}$, the generalized Apostol-type Frobenius-Euler polynomials of level $m$ $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the Genocchi polynomials $G_n(x)$, by means of*

$$\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$$
(3.6)
$$= \frac{1}{2}\sum_{k=0}^{n} \frac{(\ln c)^k}{k+1} \left[ \binom{n}{k} \mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u) + \sum_{j=k}^{n} \binom{n}{j} \binom{j}{k} \mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{j-k} \right] G_{k+1}(x).$$

*Proof.* By substituting (2.3) into the right-hand side of (3.3), we see that

$$\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$$
$$= \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u) \frac{(\ln c)^{n-j}}{2(n-j+1)} \left[ \sum_{k=0}^{n-j} \binom{n-j+1}{k+1} G_{k+1}(x) + G_{n-j+1}(x) \right]$$
$$= \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u) \frac{(\ln c)^{n-j}}{2(n-j+1)} \sum_{k=0}^{n-j} \binom{n-j+1}{k+1} G_{k+1}(x)$$
$$+ \sum_{j=0}^{n} \binom{n}{j} \mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u) \frac{(\ln c)^{n-j}}{2(n-j+1)} G_{n-j+1}(x).$$

Then, using appropriate combinational identities and summations (see, for instance [1, 5, 6]), we obtain

$$\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)$$

$$=\frac{1}{2}\sum_{k=0}^{n}\frac{(\ln c)^k}{k+1}\left[\sum_{j=k}^{n}\binom{n}{j}\binom{j}{k}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{j-k}+\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u)\right]G_{k+1}(x).$$

Therefore, (3.6) holds. □

**Theorem 3.5.** *For $m\in\mathbb{N}$, the generalized Apostol-type Frobeniu-Euler polynomials of level $m$ $\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)$, are related with the Apostol-Euler polynomials $\mathcal{E}_n(x;\lambda)$, by means of the following identity*

$$\begin{aligned}(3.7)\quad &\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\ &=\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\left[\lambda\mathcal{H}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;u)+(\ln c)^j\mathcal{H}_n^{[m-1,\alpha]}(y;c,a;\lambda;u)\right]\mathcal{E}_{n-j}(x;\lambda).\end{aligned}$$

*Proof.* By substituting (2.4) into the right-hand side of (3.3), we can see that

(3.8)

$$\begin{aligned}&\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\ &=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{1}{2}\right)\left[\lambda\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)+\mathcal{E}_{n-k}(x;\lambda)\right]\\ &=\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)\\ &\quad+\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{1}{2}\right)\mathcal{E}_{n-k}(x;\lambda).\end{aligned}$$

The first sum in (3.8) becomes

$$\begin{aligned}(3.9)\qquad &\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n-k}\binom{n-k}{j}\mathcal{E}_j(x;\lambda)\\ &=\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n}{k}(\ln c)^{n-k}\left(\frac{\lambda}{2}\right)\binom{n-k}{j}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)\mathcal{E}_j(x;\lambda)\\ &=\sum_{j=0}^{n}\left(\frac{\lambda}{2}\right)\binom{n}{j}\mathcal{E}_j(x;\lambda)\sum_{k=0}^{n-j}\binom{n-j}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\\ &=\sum_{j=0}^{n}\left(\frac{\lambda}{2}\right)\binom{n}{j}\mathcal{E}_j(x;\lambda)\mathcal{H}_{n-j}^{[m-1,\alpha]}(y+1;c,a;\lambda;u).\end{aligned}$$

For the second sum in (3.8), we obtain

$$\begin{aligned}(3.10)\qquad &\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_k^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-k}\left(\frac{1}{2}\right)\mathcal{E}_{n-k}(x;\lambda)\\ &=\frac{1}{2}\sum_{k=0}^{n}\binom{n}{k}\mathcal{H}_{n-k}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{k}\mathcal{E}_k(x;\lambda).\end{aligned}$$

Combining (3.9) and (3.10) we get

$$\begin{aligned}
&\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)\\
=&\left(\frac{\lambda}{2}\right)\sum_{j=0}^{n}\binom{n}{j}\mathcal{E}_j(x;\lambda)\mathcal{H}_{n-j}^{[m-1,\alpha]}(y+1;c,a;\lambda;u)\\
&+\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^j\mathcal{E}_j(x;\lambda)\\
=&\frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}\left[\lambda\mathcal{H}_n^{[m-1,\alpha]}(y+1;c,a;\lambda;u)+(\ln c)^j\mathcal{H}_n^{[m-1,\alpha]}(y;c,a;\lambda;u)\right]\mathcal{E}_{n-j}(x;\lambda).
\end{aligned}$$

Therefore, (3.7) holds. □

**Proposition 3.1.** *For $m\in\mathbb{N}$, $\alpha,\lambda,u,\in\mathbb{C}$, $a,c\in\mathbb{R}^+$ and $n\in\mathbb{N}_0$, we have*

$$\begin{aligned}
\mathcal{H}_n^{[m-1,\alpha]}(x+y;c,a;\lambda;u)&=\sum_{k=0}^{n}k!\binom{x}{k}\sum_{j=0}^{n-k}\binom{n}{j}\mathcal{H}_j^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{n-j}S(n-j,k)\\
&=\sum_{k=0}^{n}k!\binom{x}{k}\sum_{j=k}^{n}\binom{n}{n-j}\mathcal{H}_{n-j}^{[m-1,\alpha]}(y;c,a;\lambda;u)(\ln c)^{j}S(j,k).
\end{aligned}$$

## 4. The Generalized Apostol-Frobenius-Euler Polynomials Matrix

**Definition 4.1.** The generalized $(n+1)\times(n+1)$ Apostol-Frobenius-Euler polynomials matrix $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)$ with $m\in\mathbb{N}$, $\alpha,\lambda,u\in\mathbb{C}$ and $a,c$ positive real numbers is defined by

$$\mathcal{U}_{i,j}^{[m-1,\alpha]}(x;c,a;\lambda;u)=\begin{cases}\binom{i}{j}\mathcal{H}_{i-j}^{[m-1,\alpha]}(x;c,a;\lambda;u), & i\geq j,\\ \\ 0, & \text{otherwise.}\end{cases}$$

While, the matrices

$$\begin{aligned}
\mathcal{U}^{[m-1]}(x;c,a;\lambda;u)&:=\mathcal{U}^{[m-1,1]}(x;c,a;\lambda;u),\\
\mathcal{U}^{[m-1]}(c,a;\lambda;u)&:=\mathcal{U}^{[m-1]}(0;c,a;\lambda;u)
\end{aligned}$$

are called the Apostol-Frobenius-Euler polynomial matrix and the Apostol-Frobenius-Euler matrix, respectively.

Since $\mathcal{H}_n^{[m-1,0]}(x;c,a;\lambda;u)=(x\ln(c))^n$, we have $\mathcal{U}^{[m-1,0]}(x;c,a;\lambda;u)=P_c[x]$. It is clear that (3.3) yields the following matrix identity:

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u)=\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)P_c[x].$$

**Theorem 4.1.** *For a fixed $m\in\mathbb{N}$, let $\{\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)\}_{n\geq0}$ and $\{\mathcal{H}_n^{[m-1,\beta]}(x;c,a;\lambda;u)\}_{n\geq0}$ be the sequences of generalized Apostol-type Frobenius-Euler* *polynomials in the variable $x$, parameters $\lambda, u \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m$. Then satisfies the following product formula:*

$$\begin{aligned}\mathcal{U}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;u) &= \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\beta]}(x;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\beta]}(x;c,a;\lambda;u).\end{aligned} \tag{4.1}$$

*Proof.* Let $B_{i,j,c}^{[m-1,\alpha,\beta]}(a;\lambda;u)(x,y)$ be the $(i,j)$-th entry of the matrix product $\mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)\,\mathcal{U}^{[m-1,\beta]}(y;c,a;\lambda;u)$, then by the addition formula (3.2) we have

$$\begin{aligned}B_{i,j,c}^{[m-1,\alpha,\beta]}(a;\lambda;u)(x,y) &= \sum_{k=0}^{n}\binom{i}{k}\mathcal{H}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\binom{k}{j}\mathcal{H}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \sum_{k=j}^{i}\binom{i}{k}\mathcal{H}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\binom{k}{j}\mathcal{H}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \sum_{k=j}^{i}\binom{i}{j}\binom{i-j}{i-k}\mathcal{H}_{i-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{H}_{k-j}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \binom{i}{j}\sum_{k=0}^{i-j}\binom{i-j}{k}\mathcal{H}_{i-j-k}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{H}_{k}^{[m-1,\beta]}(y;c,a;\lambda;u)\\ &= \binom{i}{j}\mathcal{H}_{i-j}^{[m-1,\alpha+\beta]}(x+y;c,a;\lambda;u),\end{aligned}$$

which implies the first equality of the theorem. The second and third equalities of can be derived in a similar way. ☐

**Corollary 4.1.** *For a fixed $m \in \mathbb{N}$, let $\{\mathcal{H}_n^{[m-1,\alpha]}(x;c,a;\lambda;u)\}_{n\geq 0}$ and $\{\mathcal{H}_n^{[m-1,\beta]}(x;c,a;\lambda;u)\}_{n\geq 0}$ be the sequences of generalized Apostol-type Frobenius-Euler polynomials in the variable $x$, parameters $\lambda, u \in \mathbb{C}$, $a, c \in \mathbb{R}^+$, order $\alpha \in \mathbb{C}$ and level $m$ and $P_c[x]$ the generalized Pascal matrix of first kind in base $c$. Then*

$$\begin{aligned}\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) &= \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)P_c[y]\\ &= P_c[x]\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)P_c[x].\end{aligned}$$

*In particular,*

$$\begin{aligned}\mathcal{U}^{[m-1]}(x+y;c,a;\lambda;u) &= P_c[x]\mathcal{U}^{[m-1]}(y;c,a;\lambda;u)\\ &= P_c[y]\mathcal{U}^{[m-1]}(x;c,a;\lambda;u).\end{aligned}$$

*Proof.* The substitution $\beta = 0$ into (4.1) yields

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)\mathcal{U}^{[m-1,0]}(y;c,a;\lambda;u).$$

Since $\mathcal{U}^{[m-1,0]}(y;c,a;\lambda;u) = P_c[y]$, we obtain

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)P_c[y]. \tag{4.2}$$

A similar argument allows to show that

$$\begin{aligned}\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) &= P_c[x]\mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)\\ &= \mathcal{U}^{[m-1,\alpha]}(y;c,a;\lambda;u)P_c[x].\end{aligned}$$

Finally, the substitution $\alpha=1$ into (4.2) and its combination with the previous equations completes the proof. □

Using the relation (2.5) and Corollary 4.1 we obtain the following factorization for $\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u)$ in terms of summation matrices.

$$\mathcal{U}^{[m-1,\alpha]}(x+y;c,a;\lambda;u) = \mathcal{U}^{[m-1,\alpha]}(x;c,a;\lambda;u)G_{n,c}[y]G_{n-1,c}[y]\cdots G_{1,c}[y].$$

Under the appropriate choice on the parameters, level and order, it is possible to provide some illustrative examples of the generalized Apostol-Frobenius-Euler polynomials matrices.

*Example* 4.1. For $m=1$, $c=a=e=\exp(1)$, $\alpha=1$, $\lambda=-1$, The first four polynomials $\mathcal{H}_k^{[1-1,1]}(x;e,e;1;u)$, $k=0,1,2,3$ are

$$\begin{aligned}\mathcal{H}_0^{[1-1,1]}(x;e,e;1;u) &= 1,\\ \mathcal{H}_1^{[1-1,1]}(x;e,e;1;u) &= x-\frac{1}{1-u},\\ \mathcal{H}_2^{[1-1,1]}(x;e,e;1;u) &= x^2-\frac{2}{1-u}x+\frac{1+u}{(1-u)^2},\\ \mathcal{H}_3^{[1-1,1]}(x;e,e;1;u) &= x^3-\frac{3}{1-u}x^2+\frac{3(1+u)}{(1-u)^2}x-\frac{u^2+4u+1}{(1-u)^3}.\end{aligned}$$

Hence, for $n=3$, we have

$$\mathcal{U}^{[m-1,1]}(x;e,e;1;u) = \begin{bmatrix}1 & 0 & 0 & 0\\ u_{10} & 1 & 0 & 0\\ u_{20} & u_{21} & 1 & 0\\ u_{30} & u_{31} & u_{32} & 1\end{bmatrix},$$

where

$$\begin{aligned}u_{10} = u_{21} = u_{32} &= \mathcal{H}_1^{[1-1,1]}(x;e,e;1;u),\\ u_{20} = u_{31} &= \mathcal{H}_2^{[1-1,1]}(x;e,e;1;u),\\ u_{30} &= \mathcal{H}_3^{[1-1,1]}(x;e,e;1;u).\end{aligned}$$

*Example* 4.2. For $m = 1$, $c = a = e = \exp(1)$, $\lambda = 1$ and $u = -1$, The first four polynomials $\mathcal{H}_k^{[1-1,\alpha]}(x; e, e; 1; -1)$, $k = 0, 1, 2, 3$, are

$$\begin{aligned}
\mathcal{H}_0^{[1-1,\alpha]}(x; e, e; 1; -1) &= 1, \\
\mathcal{H}_1^{[1-1,\alpha]}(x; e, e; 1; -1) &= x - \frac{\alpha}{2}, \\
\mathcal{H}_2^{[1-1,\alpha]}(x; e, e; 1; -1) &= x^2 - \alpha x + \frac{\alpha(\alpha-1)}{4}, \\
\mathcal{H}_3^{[1-1,\alpha]}(x; e, e; 1; -1) &= x^3 - \frac{3\alpha}{2}x^2 + \frac{3\alpha(\alpha-1)}{4}x - \frac{3\alpha^2(\alpha-1)}{8}.
\end{aligned}$$

Then, for $n = 3$, we have

$$\mathcal{U}^{[m-1,\alpha]}(x; e, e; 1; -1) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ u_{10} & 1 & 0 & 0 \\ u_{20} & 2u_{21} & 1 & 0 \\ u_{30} & 3u_{31} & 3u_{32} & 1 \end{bmatrix},$$

where

$$\begin{aligned}
u_{10} = u_{21} = u_{32} &= \mathcal{H}_1^{[1-1,\alpha]}(x; e, e; 1; -1), \\
u_{20} = u_{31} &= \mathcal{H}_2^{[1-1,\alpha]}(x; e, e; 1; -1), \\
u_{30} &= \mathcal{H}_3^{[1-1,\alpha]}(x; e, e; 1; -1).
\end{aligned}$$

*Example* 4.3. For $\lambda \in \mathbb{C}$, $m = c = 2$, $a = 3$, $\alpha = \frac{1}{2}$, $u = 2$, we have the Example 3.1. Therefore,

$$\mathcal{U}^{[1,\frac{1}{2}]}(x; 2, 3; \lambda; 2) \quad = \quad \begin{bmatrix} \sqrt{\frac{3}{\lambda-4}} & 0 & 0 \\ \mathcal{H}_1^{[1,(\frac{1}{2})]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} & 0 \\ \frac{32}{\sqrt{1+\lambda}} & 0 & 0 \\ \mathcal{H}_2^{[1,(\frac{1}{2})]}(x; 2, 3; \lambda; 2) & 2\mathcal{H}_1^{[1,(\frac{1}{2})]}(x; 2, 3; \lambda; 2) & \sqrt{\frac{3}{\lambda-4}} \end{bmatrix}.$$

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# ON A FAMILY OF $(p,q)$-HYBRID POLYNOMIALS

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ABSTRACT. In this paper, the class of $(p,q)$-Bessel-Appell polynomials is introduced. The generating function, series definition and determinant definition of this class are established. Certain members of $(p,q)$-Bessel-Appell polynomials are considered and some properties of these members are also derived. Further, the class of 2D $(p,q)$-Bessel-Appell polynomials is introduced by means of the generating function and series definition. In addition, the graphical representations of some members of $(p,q)$-Bessel-Appell polynomials and 2D $(p,q)$-Bessel-Appell polynomials are plotted with the help of Matlab.

## 1. INTRODUCTION

The quantum calculus (or called $q$-calculus) has been extensively studied and has applications in various fields of mathematics, physics and engineering. Further, motivated and inspired by these applications, many mathematicians and physicist have developed the theory of post quantum calculus (based on $(p,q)$ numbers), an extension of the $q$-calculus and is denoted by $(p,q)$-calculus. The recent interest in the subject is due to the fact that the $(p,q)$-calculus has popped in such diverse areas as quantum algebra, number theory etc. [3–5, 12]. Recently, Duran et al. [5] defined $(p,q)$-analogues of Bernoulli, Euler and Genocchi polynomials and derived the $(p,q)$-analogues of some known earlier formulae. We now review briefly some definitions and notations of $(p,q)$-calculus taken from [3, 4, 12].

The $(p,q)$-numbers are defined as follows:

$$[\alpha]_{p,q} = p^{\alpha-1} + p^{\alpha-2}q + p^{\alpha-3}q^2 + \cdots + pq^{\alpha-2} + q^{\alpha-1} = \frac{p^\alpha - q^\alpha}{p-q}, \quad q < p \leq 1, \alpha \in \mathbb{N}.$$

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We note that $[\alpha]_{p,q} = p^{\alpha-1}[\alpha]_{q/p}$, where $[\alpha]_{q/p}$ is the $q$-number given by $[\alpha]_{q/p} = \frac{(q/p)^{\alpha}-1}{(q/p)-1}$. By appropriately using the relation $[\alpha]_{p,q} = p^{\alpha-1}[\alpha]_{q/p}$, most (if not all) of the $(p,q)$-results can be derived from the corresponding known $q$-results by merely changing the parameters and variables involved. In case of $p = 1$, $(p,q)$-numbers reduce to $q$-numbers [8,9].

The $(p,q)$-factorial $[m]_{p,q}!$ is defined by

$$[m]_{p,q}! = \prod_{s=1}^{m}[s]_{p,q} = [1]_{p,q}[2]_{p,q}[3]_{p,q}\cdots[m]_{p,q}, \quad m \in \mathbb{N},\ [0]_{p,q}! = 1.$$

The $(p,q)$-binomial coefficient $\begin{bmatrix} m \\ s \end{bmatrix}_{p,q}$ is defined by

$$\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} = \frac{[m]_{p,q}!}{[s]_{p,q}!\ [m-s]_{p,q}!}, \quad s = 0,1,2,\ldots,m.$$

The $(p,q)$-analogue of $(x+y)^n$ is given by

$$(x+y)_{p,q}^{m} = \sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q} p^{\binom{m-s}{2}}q^{\binom{s}{2}}x^s y^{m-s}, \quad (p,q)\text{-Gauss Binomial Formula.}$$

The $(p,q)$-analogue of the classical derivative $Df$ of a function $f$ with respect to $t$ is defined by

$$D_{p,q}f(t) = \frac{f(pt)-f(qt)}{pt-qt}, \quad t \neq 0.$$

Also, we note that

(i) $(D_{p,q}f)(0) = f^{'}(0)$, provided that $f$ is differentiable at 0;
(ii) $D_{p,q}(a_1 f(t) + a_2\ g(t)) = a_1 D_{p,q}f(t) + a_2 D_{p,q}g(t)$;
(iii)

$$D_{p,q}(fg)(t) = f(pt)D_{p,q}g(t) + g(qt)D_{p,q}f(t) = g(pt)D_{p,q}f(t) + f(qt)D_{p,q}g(t);$$

(iv)

$$D_{p,q}\left(\frac{f(t)}{g(t)}\right) = \frac{g(pt)D_{p,q}f(t) - f(pt)D_{p,q}g(t)}{g(pt)g(qt)} = \frac{g(qt)D_{p,q}f(t) - f(qt)D_{p,q}g(qt)}{g(pt)g(qt)}.$$

The $(p,q)$-exponential functions are given as:

$$e_{p,q}(t) = \sum_{m=0}^{\infty} p^{\binom{m}{2}}\frac{t^m}{[m]_{p,q}!}, \tag{1.1}$$

$$E_{p,q}(t) = \sum_{m=0}^{\infty} q^{\binom{m}{2}}\frac{t^m}{[m]_{p,q}!}, \tag{1.2}$$

which satisfy the following properties:

$$D_{p,q}e_{p,q}(t) = e_{p,q}(pt), \quad D_{p,q}E_{p,q}(t) = E_{p,q}(qt), \tag{1.3}$$

$$e_{p,q}(t)E_{p,q}(-t) = E_{p,q}(t)e_{p,q}(-t) = 1. \tag{1.4}$$

The class of Appell polynomials was introduced and characterized completely by Appell [2]. Further, Throne [16], Sheffer [15] and Varma [17] studied this class of polynomials from different points of views. Sharma and Chak [14] introduced a $q$-analogue for the class of Appell polynomials and called this sequence of polynomials as $q$-Harmonic. Later, Al-Salam [1] introduced the class of $q$-Appell polynomials $\{\mathcal{A}_{m,q}(x)\}_{m=0}^{\infty}$ and studied some of its properties. These polynomials arise in numerous problems of applied mathematics, theoretical physics, approximation theory and many other branches of mathematics. Recently, many researchers introduced and studied some hybrid special polynomials related to $q$-Appell polynomials (see for example [19]). The polynomials $\mathcal{A}_{m,q}(x)$ (of degree $m$) are called $q$-Appell provided that they satisfy the $q$-differential equation given by:

$$D_{q,x}\{\mathcal{A}_{m,q}(x)\} = [m]_q \mathcal{A}_{m-1,q}(x), \quad m = 0, 1, 2, 3, \ldots, q \in \mathbb{C}, 0 < |q| < 1. \tag{1.5}$$

The $(p, q)$-Appell polynomials (pqAP) $\{\mathcal{A}_{m,p,q}(x)\}_{m=0}^{\infty}$ (see [11]) are defined by means of the followin generating functions

$$\mathcal{A}_{p,q}(t)\ e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}, \tag{1.6}$$

where

$$\mathcal{A}_{p,q}(t) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}\ \frac{t^m}{[m]_{p,q}!}, \quad \mathcal{A}_{p,q}(t) \neq 0, \mathcal{A}_{0,p,q} = 1 \tag{1.7}$$

and $\mathcal{A}_{m,p,q} := \mathcal{A}_{m,p,q}(0)$ denotes the $(p, q)$-Appell numbers.

The explicit form of the pqAP $\mathcal{A}_{m,p,q}(x)$ given as (see [11]):

$$\mathcal{A}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} p^{\binom{m-s}{2}} \mathcal{A}_{s,p,q} x^{m-s}. \tag{1.8}$$

The function $\mathcal{A}_{p,q}(t)$ may be called the determining function for the set $\mathcal{A}_{m,p,q}(x)$. Based on suitable selections for the function $\mathcal{A}_{p,q}(t)$, different members belonging to the family of $(p, q)$-Appell polynomial $\mathcal{A}_{m,p,q}(x)$ can be obtained. These members are mentioned in Table 1.

Table 1. Some known $(p, q)$-Appell polynomials

| **S. No.** | $\mathcal{A}_{\mathbf{p,q}}(\mathbf{t})$ | **Generating Functions** | **Polynomials** |
|---|---|---|---|
| I. | $\mathcal{A}_{p,q}(t) = \frac{t}{(e_{p,q}(t)-1)}$ | $\frac{t}{(e_{p,q}(t)-1)} e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathfrak{B}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}$ | The $(p, q)$-Bernoulli polynomials [6] (see also [11]) |
| II. | $\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q}}{(e_{p,q}(t)+1)}$ | $\frac{[2]_{p,q}}{(e_{p,q}(t)+1)} e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathcal{E}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}$ | The $(p, q)$-Euler polynomials [6] |
| III. | $\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q} t}{(e_{p,q}(t)+1)}$ | $\frac{[2]_{p,q} t}{(e_{p,q}(t)+1)} e_{p,q}(xt) = \sum_{m=0}^{\infty} \mathcal{G}_{m,p,q}(x) \frac{t^m}{[m]_{p,q}!}$, | The $(p, q)$-Genocchi polynomials [6] |

The Bessel polynomials form a set of orthogonal polynomials on the unit circle in the complex plane. They are important in certain problems of mathematical physics, for example, they arise in the study of electrical networks and when the wave equation is considered in spherical coordinates. Several important properties and applications of these polynomials can be found in [7].

The Bessel polynomials $\rho_m(x)$ [18] are defined by means of the following generating function

$$\sum_{m=0}^{\infty} \rho_m(x)\frac{t^m}{m!} = e^{x(1-\sqrt{1-2t})}.$$

This paper is organized as follows. In Section 2, the $(p,q)$-Bessel-Appell polynomials are introduced by means of the generating function and series definition. Also, the determinant definition and some properties for the $(p,q)$-Bessel-Appell polynomials are established. Further, some members of $(p,q)$-Bessel-Appell polynomials are considered. In Section 3, the 2D $(p,q)$-Bessel-Appell polynomials are introduced by means of the generating function and series definition. In Section 4, the graphical representations of some members belonging to $(p,q)$-Bessel-Appell and 2D $(p,q)$-Bessel-Appell families are plotted for suitable values of the indices.

## 2. $(p,q)$-Bessel-Appell Polynomials

In this section, we introduce the $(p,q)$-Bessel-Appell polynomials (pqBeAP) by means of generating function, series definition and determinant definition. First, we introduce the $(p,q)$-analogue of the Bessel polynomials denoted as $(p,q)$-Bessel polynomials $\rho_{m,p,q}(x)$.

**Definition 2.1.** The $(p,q)$-analogue of the Bessel polynomials $p_n(x)$ are defined by the following generating function:

$$\sum_{m=0}^{\infty} \rho_{m,p,q}(x)\frac{t^m}{[m]_{p,q}!} = e_{p,q}(x(1-\sqrt{1-2t})) \tag{2.1}$$

and posses the following series expansion:

$$\rho_{m,p,q}(x) = \sum_{s=0}^{m-1} \frac{[m-1+s]_{p,q}!\ x^{m-s}}{[m-1-s]_{p,q}![s]_{p,q}!\ 2^s}.$$

In order to establish the generating function for the pqBeAP, the following result is proved.

**Theorem 2.1.** *The following generating function for the $(p,q)$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ holds true:*

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t})) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}. \tag{2.2}$$

*Proof.* By expanding the $(p,q)$-exponential function $e_{p,q}(xt)$ in the left hand side of the equation (1.6) and then replacing the powers of $x$, i.e., $x^0, x, x^2, \ldots, x^m$ by the corresponding polynomials $\rho_{0,p,q}(x), \rho_{1,p,q}(x), \rho_{2,p,q}(x), \ldots, \rho_{m,p,q}(x)$ in the left hand side and $x$ by $\rho_{1,p,q}(x)$ in the right hand side of the resultant equation, we have

$$\mathcal{A}_{p,q}(t)\left(1+\rho_{1,p,q}(x)\frac{t}{[1]_{p,q}!}+\rho_{2,p,q}(x)\quad\frac{t^2}{[2]_{p,q}!}+\cdots+\rho_{m,p,q}(x)\frac{t^m}{[m]_{p,q}!}+\cdots\right)$$

$$=\sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x))\ \frac{t^m}{[m]_{p,q}!}. \tag{2.3}$$

Further, summing up the series in left hand side and then using equation (2.1) in the resultant equation, we get

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t}))=\sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x))\ \frac{t^m}{[m]_{p,q}!}.$$

Finally, denoting the resultant pqBeAP in the right hand side of the above equation by ${}_{\rho}\mathcal{A}_{m,p,q}(x)$, that is

$$\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x))={}_{\rho}\mathcal{A}_{m,p,q}(x),$$

the assertion (2.2) is proved. □

*Remark* 2.1. It is remarked that for $p=1$, the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ reduce to the $q$-Bessel-Appell polynomials (qBeAP) ${}_{\rho}\mathcal{A}_{m,q}(x)$ such that

$${}_{\rho}\mathcal{A}_{m,q}(x):={}_{\rho}\mathcal{A}_{m,1,q}(x).$$

Thus, taking $p=1$ in equation (2.2), we get

$$\mathcal{A}_{q}(t)e_{q}(x(1-\sqrt{1-2t}))=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,q}(x)\ \frac{t^m}{[m]_{q}!},$$

which is the generating function for the $q$-Bessel-Appell polynomials.

Next, the series definition for the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ is derived by proving the following result.

**Theorem 2.2.** *The $(p,q)$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ are defined by the following series definition:*

$${}_{\rho}\mathcal{A}_{m,p,q}(x)=\sum_{s=0}^{m}\begin{bmatrix}m\\s\end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\ \rho_{m-s,p,q}(x). \tag{2.4}$$

*Proof.* In view of equations (1.7) and (2.1), equation (2.2) can be written as:

$$\sum_{s=0}^{\infty}\mathcal{A}_{s,p,q}\ \frac{t^s}{[s]_{p,q}!}\sum_{m=0}^{\infty}\rho_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!},$$

which on using the Cauchy product rule gives

$$\sum_{m=0}^{\infty}\sum_{s=0}^{m}\begin{bmatrix}m\\s\end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\ \rho_{m-s,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}.$$

Equating the coefficients of like powers of $t$ in both sides of the above equation, we arrive at our assertion (2.4). □

*Remark* 2.2. For $p=1$, series definition (2.4) becomes

$$_{\rho}\mathcal{A}_{m,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q \mathcal{A}_{s,q}\ \rho_{m-s,q}(x),$$

which is the series definition for the $q$-Bessel-Appell polynomials.

Next, we establish the determinant definition for the pqBeAP $_{\rho}\mathcal{A}_{m,p,q}(x)$.

**Theorem 2.3.** *The $(p,q)$-Bessel-Appell polynomials $_{\rho}\mathcal{A}_{m,p,q}(x)$ of degree $m$ are defined by*

$$_{\rho}\mathcal{A}_{0,p,q}(x) = \frac{1}{\mathcal{B}_{0,p,q}}, \tag{2.5}$$

$$_{\rho}\mathcal{A}_{m,p,q}(x) = \frac{(-1)^m}{(\mathcal{B}_{0,p,q})^{m+1}} \times \begin{vmatrix} 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x) \\ \mathcal{B}_{0,p,q} & \mathcal{B}_{1,p,q} & \mathcal{B}_{2,p,q} & \cdots & \mathcal{B}_{m-1,p,q} & \mathcal{B}_{m,p,q} \\ 0 & \mathcal{B}_{0,p,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \cdots & \begin{bmatrix} m-1 \\ 1 \end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix} m \\ 1 \end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q} \\ 0 & 0 & \mathcal{B}_{0,p,q} & \cdots & \begin{bmatrix} m-1 \\ 2 \end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \begin{bmatrix} m \\ 2 \end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{B}_{0,p,q} & \begin{bmatrix} m \\ m-1 \end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} \end{vmatrix}, \tag{2.6}$$

$$\mathcal{B}_{m,p,q} = -\frac{1}{\mathcal{A}_{0,p,q}}\left(\sum_{s=1}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q}\right), \quad m=1,2,3,\ldots,$$

*where $\mathcal{B}_{0,p,q} \neq 0$, $\mathcal{B}_{0,p,q} = \frac{1}{\mathcal{A}_{0,p,q}}$ and $\rho_{m,p,q}(x)$, $m=0,1,2,\ldots,$ are the $(p,q)$-Bessel polynomials of degree $m$.*

*Proof.* Consider $_{\rho}\mathcal{A}_{m,p,q}(x)$ to be a sequence of the pqBeAP defined by equation (2.2) and $\mathcal{A}_{m,p,q}$, $\mathcal{B}_{m,p,q}$ be two numerical sequences (the coefficients of $q$-Taylor's series expansions of functions) such that

$$\begin{aligned} \mathcal{A}_{p,q}(t) =& \mathcal{A}_{0,p,q} + \mathcal{A}_{1,p,q}\frac{t}{[1]_{p,q}!} + \mathcal{A}_{2,p,q}\frac{t^2}{[2]_{p,q}!} + \cdots + \mathcal{A}_{m,p,q}\frac{t^m}{[m]_{p,q}!} + \cdots, \\ & m=0,1,2,3,\ldots,\ \mathcal{A}_{0,p,q} \neq 0, \end{aligned} \tag{2.7}$$

$$\begin{aligned} \hat{\mathcal{A}}_{p,q}(t) =& \mathcal{B}_{0,p,q} + \mathcal{B}_{1,p,q}\frac{t}{[1]_{p,q}!} + \mathcal{B}_{2,p,q}\frac{t^2}{[2]_{p,q}!} + \cdots + \mathcal{B}_{m,p,q}\frac{t^m}{[m]_{p,q}!} + \cdots, \\ & m=0,1,2,3,\ldots,\ \mathcal{B}_{0,p,q} \neq 0, \end{aligned} \tag{2.8}$$

satisfying

$$\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t) = 1. \tag{2.9}$$

On using Cauchy product rule for the two series production $\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t)$, we get

$$\begin{aligned}\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t) &= \sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}\frac{t^m}{[m]_{p,q}!}\sum_{m=0}^{\infty}\mathcal{B}_{m,p,q}\frac{t^m}{[m]_{p,q}!}\\ &= \sum_{m=0}^{\infty}\sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q}\frac{t^m}{[m]_{p,q}!}.\end{aligned}$$

Consequently,

$$\sum_{s=0}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q} = \begin{cases} 1, & \text{if } m=0,\\ 0, & \text{if } m>0.\end{cases} \tag{2.10}$$

That is

$$\begin{cases} \mathcal{B}_{0,p,q} = \dfrac{1}{\mathcal{A}_{0,p,q}},\\ \mathcal{B}_{m,p,q} = -\dfrac{1}{\mathcal{A}_{0,p,q}}\left(\displaystyle\sum_{s=1}^{m}\begin{bmatrix} m \\ s \end{bmatrix}_{p,q}\mathcal{A}_{s,p,q}\mathcal{B}_{m-s,p,q}\right), \quad m=1,2,\ldots\end{cases} \tag{2.11}$$

Next, multiplying both sides of equation (2.2) by $\hat{\mathcal{A}}_{p,q}(t)$, we get

$$\mathcal{A}_{p,q}(t)\hat{\mathcal{A}}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t})) = \hat{\mathcal{A}}_{p,q}(t)\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}.$$

Further, in view of equations (2.1), (2.8) and (2.9), the above equation becomes

$$\sum_{m=0}^{\infty}\rho_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty}\mathcal{B}_{m,p,q}\frac{t^m}{[m]_{p,q}!}\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}. \tag{2.12}$$

Now, on using Cauchy product rule for the two series in the r.h.s of equation (2.12), we obtain the following infinite system for the unknowns ${}_{\rho}\mathcal{A}_{m,p,q}(x)$:

(2.13)
$$\begin{cases} \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) = 1,\\ \mathcal{B}_{1,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) = \rho_{1,p,q}(x),\\ \mathcal{B}_{2,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]_{p,q}\mathcal{B}_{1,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{2,p,q}(x) = \rho_{2,p,q}(x),\\ \vdots\\ \mathcal{B}_{m-1,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \left[\begin{smallmatrix} m-1 \\ 1 \end{smallmatrix}\right]_{p,q}\mathcal{B}_{m-2,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) + \cdots + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{m-1,p,q}(x)\\ = \rho_{m-1,p,q}(x),\\ \mathcal{B}_{m,p,q}\ {}_{\rho}\mathcal{A}_{0,p,q}(x) + \left[\begin{smallmatrix} m \\ 1 \end{smallmatrix}\right]_{p,q}\mathcal{B}_{m-1,p,q}\ {}_{\rho}\mathcal{A}_{1,p,q}(x) + \cdots + \mathcal{B}_{0,p,q}\ {}_{\rho}\mathcal{A}_{m,p,q}(x) = \rho_{m,p,q}(x),\\ \vdots \end{cases}$$

Obviously the first equation of system (2.13) leads to our first assertion (2.5). The coefficient matrix of system (2.13) is lower triangular, so, this helps us to obtain the unknowns ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ by applying Cramer rule to the first $m+1$ equations of system (2.13). According to this, we can obtain

(2.14)
$$
{}_{\rho}\mathcal{A}_{m,p,q}(x) = \frac{\begin{vmatrix} \mathcal{B}_{0,p,q} & 0 & 0 & \cdots & 0 & 1 \\ \mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & 0 & \cdots & 0 & \rho_{1,p,q}(x) \\ \mathcal{B}_{2,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & \cdots & 0 & \rho_{2,p,q}(x) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{B}_{m-1,p,q} & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \cdots & \mathcal{B}_{0,p,q} & \rho_{m-1,p,q}(x) \\ \mathcal{B}_{m,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \cdots & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \rho_{m,p,q}(x) \end{vmatrix}}{\begin{vmatrix} \mathcal{B}_{0,p,q} & 0 & 0 & \cdots & 0 & 1 \\ \mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & 0 & \cdots & 0 & 0 \\ \mathcal{B}_{2,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathcal{B}_{m-1,p,q} & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \cdots & \mathcal{B}_{0,p,q} & 0 \\ \mathcal{B}_{m,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \cdots & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \mathcal{B}_{0,p,q} \end{vmatrix}},
$$

where $m = 1, 2, 3, \ldots$, which on expanding the determinant in the denominator and taking the transpose of the determinant in the numerator, yields to

$$
\begin{aligned}
{}_{\rho}\mathcal{A}_{m,p,q}(x) = & \frac{1}{(\mathcal{B}_{0,p,q})^{m+1}} \\
& \times \begin{vmatrix} \mathcal{B}_{0,p,q} & \mathcal{B}_{1,p,q} & \mathcal{B}_{2,p,q} & \cdots & \mathcal{B}_{m-1,p,q} & \mathcal{B}_{m,p,q} \\ 0 & \mathcal{B}_{0,p,q} & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} & \cdots & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\mathcal{B}_{m-1,p,q} \\ 0 & 0 & \mathcal{B}_{0,p,q} & \cdots & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-3,p,q} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\mathcal{B}_{m-2,p,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{B}_{0,p,q} & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\mathcal{B}_{1,p,q} \\ 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x) \end{vmatrix}.
\end{aligned} \tag{2.15}
$$

Finally, after $m$ circular row exchanges, that is after moving the $j^{\text{th}}$ row to the $(j+1)^{\text{th}}$ position for $j = 1, 2, 3, \ldots, m-1$, we arrive at our assertion (2.6). $\square$

On taking $p = 1$ in Theorem 2.3, we get the determinant definition for the $q$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,q}(x)$.

**Corollary 2.1.** *The $q$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,q}(x)$ of degree $m$ are defined by*

$$
{}_{\rho}\mathcal{A}_{0,q}(x) = \frac{1}{\mathcal{B}_{0,q}}, \tag{2.16}
$$

$$
(2.17) \qquad {}_{\rho}\mathcal{A}_{m,q}(x) = \frac{(-1)^m}{(\mathcal{B}_{0,q})^{m+1}} \begin{vmatrix} 1 & \rho_{1,q}(x) & \rho_{2,q}(x) & \dots & \rho_{m-1,q}(x) & \rho_{m,q}(x) \\ \mathcal{B}_{0,q} & \mathcal{B}_{1,q} & \mathcal{B}_{2,q} & \dots & \mathcal{B}_{m-1,q} & \mathcal{B}_{m,q} \\ 0 & \mathcal{B}_{0,q} & \left[{2 \atop 1}\right]_q \mathcal{B}_{1,q} & \dots & \left[{m-1 \atop 1}\right]_q \mathcal{B}_{m-2,q} & \left[{m \atop 1}\right]_q \mathcal{B}_{m-1,q} \\ 0 & 0 & \mathcal{B}_{0,q} & \dots & \left[{m-1 \atop 2}\right]_q \mathcal{B}_{m-3,q} & \left[{m \atop 2}\right]_q \mathcal{B}_{m-2,q} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \mathcal{B}_{0,q} & \left[{m \atop m-1}\right]_q \mathcal{B}_{1,q} \end{vmatrix},
$$

$$
\mathcal{B}_{m,q} = -\frac{1}{\mathcal{A}_{0,q}} \left( \sum_{s=1}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_q \mathcal{A}_{s,q} \mathcal{B}_{m-s,q} \right), \quad m = 1, 2, 3, \dots
$$

**Theorem 2.4.** *The following identity for the pqBeAP* ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ *holds true:*

$$
{}_{\rho}\mathcal{A}_{m,p,q}(x) = \frac{1}{\mathcal{B}_{0,p,q}} \left( \rho_{m,p,q}(x) - \sum_{s=0}^{m-1} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{B}_{m-s,p,q} \; {}_{\rho}\mathcal{A}_{s,p,q}(x) \right), \quad m = 1, 2, \dots
$$

*Proof.* Expanding the determinant in equation (2.6) with respect to the $(m+1)^{\text{th}}$ row and using the same technique used in [10], we get the required result. □

On taking $p = 1$ in Theorem 2.4, we get the following result for the $q$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,q}(x)$.

**Corollary 2.2.** *The following identity for the qBeAP* ${}_{\rho}\mathcal{A}_{m,q}(x)$ *holds true:*

$$
{}_{\rho}\mathcal{A}_{m,q}(x) = \frac{1}{\mathcal{B}_{0,q}} \left( \rho_{m,q}(x) - \sum_{s=0}^{m-1} \begin{bmatrix} m \\ s \end{bmatrix}_{q} \mathcal{B}_{m-s,q} \; {}_{\rho}\mathcal{A}_{s,q}(x) \right), \quad m = 1, 2, \dots
$$

2.1. **Certain Members of the $(p, q)$-Bessel-Appell Polynomials.** Recently, different members of the family of $(p, q)$-Appell polynomials are studied by many researchers (see for example [4, 5]). By making suitable selections for the function $\mathcal{A}_{p,q}(t)$, the members belonging to the family of the $(p, q)$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ can be obtained. The $(p, q)$-Bernoulli polynomials (pqBP) $\mathfrak{B}_{m,p,q}(x)$, $(p, q)$-Euler polynomials (pqEP) $\mathcal{E}_{m,p,q}(x)$ and $(p, q)$-Genocchi polynomials (pqGP) $\mathcal{G}_{m,p,q}(x)$ are important members of the $(p, q)$-Appell family. In this subsection, we introduce the $(p, q)$-Bessel-Bernoulli polynomials (pqBeBP) ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$, $(p, q)$-Bessel-Euler polynomials (pqBeEP) ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ and $(p, q)$-Bessel-Genocchi polynomials (pqBeGP) ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ by means of the generating functions, series definitions and determinant definitions.

2.1.1. $(p, q)$-*Bessel-Bernoulli polynomials.* Since, for $\mathcal{A}_{p,q}(t) = \frac{t}{e_{p,q}(t)-1}$, the pqAP $\mathcal{A}_{m,p,q}(x)$ reduce to the pqBP $\mathfrak{B}_{m,p,q}(x)$ (Table 1 (I)). Therefore, for the same choice of $\mathcal{A}_{p,q}(t)$, the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ reduce to pqBeBP ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$, which are defined by means of following generating function:

$$
(2.18) \qquad \frac{t}{e_{p,q}(t) - 1} \; e_{p,q}(x(1 - \sqrt{1 - 2t})) = \sum_{m=0}^{\infty} {}_{\rho}\mathfrak{B}_{m,p,q}(x) \; \frac{t^m}{[m]_{p,q}!}.
$$

The pqBeBP ${}_\rho\mathfrak{B}_{m,p,q}(x)$ of degree $m$ are defined by the series

$$ {}_\rho\mathfrak{B}_{m,p,q}(x)=\sum_{s=0}^{m}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathfrak{B}_{s,p,q}\rho_{m-s,p,q}(x). $$

The following identity for the pqBeBP ${}_\rho\mathfrak{B}_{m,p,q}(x)$ holds true:

$$ {}_\rho\mathfrak{B}_{m,p,q}(x)=\frac{1}{\mathcal{B}_{0,p,q}}\left(\rho_{m,p,q}(x)-\sum_{s=0}^{m-1}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathcal{B}_{m-s,p,q}\;{}_\rho\mathfrak{B}_{s,p,q}(x)\right),\quad m=1,2,\ldots \tag{2.19} $$

Further, by taking $\mathcal{B}_{0,p,q}=1$ and $\mathcal{B}_{j,p,q}=\frac{1}{[j+1]_{p,q}}$, $j=1,2,3,\ldots$, in equations (2.5) and (2.6), we obtain the determinant definition of the pqBeBP ${}_\rho\mathfrak{B}_{m,p,q}(x)$.

**Definition 2.2.** The $(p,q)$-Bessel-Bernoulli polynomials ${}_\rho\mathfrak{B}_{m,p,q}(x)$ of degree $m$ are defined by

$$ {}_\rho\mathfrak{B}_{0,p,q}(x)=1, \tag{2.20} $$

$$ {}_\rho\mathfrak{B}_{m,p,q}(x)=(-1)^m\begin{vmatrix} 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \cdots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x)\\ 1 & \frac{1}{[2]_{p,q}} & \frac{1}{[3]_{p,q}} & \cdots & \frac{1}{[m]_{p,q}} & \frac{1}{[m+1]_{p,q}}\\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_{p,q}\frac{1}{[2]_{p,q}} & \cdots & \begin{bmatrix}m-1\\1\end{bmatrix}_{p,q}\frac{1}{[m-1]_{p,q}} & \begin{bmatrix}m\\1\end{bmatrix}_{p,q}\frac{1}{[m]_{p,q}}\\ 0 & 0 & 1 & \cdots & \begin{bmatrix}m-1\\2\end{bmatrix}_{p,q}\frac{1}{[m-2]_{p,q}} & \begin{bmatrix}m\\2\end{bmatrix}_{p,q}\frac{1}{[m-1]_{p,q}}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}m\\m-1\end{bmatrix}_{p,q}\frac{1}{[2]_{p,q}} \end{vmatrix}, \tag{2.21} $$

$$ m=1,2,3,\ldots, $$

where $\rho_{m,p,q}(x)$, $m=0,1,2,3,\ldots$, are the $(p,q)$-Bessel polynomials of degree $m$.

2.1.2. $(p,q)$-*Bessel-Euler polynomials.* Since, for $\mathcal{A}_{p,q}(t)=\frac{[2]_{p,q}}{e_{p,q}(t)+1}$, the pqAP $\mathcal{A}_{m,p,q}(x)$ reduce to the pqEP $\mathcal{E}_{m,p,q}(x)$ (Table 1 (II)). Therefore, for the same choice of $\mathcal{A}_{p,q}(t)$, the pqBeAP ${}_\rho\mathcal{A}_{m,p,q}(x)$ reduce to pqBeEP ${}_\rho\mathcal{E}_{m,p,q}(x)$ which are defined by means of following generating function:

$$ \frac{[2]_{p,q}}{e_{p,q}(t)+1}\;e_{p,q}(x(1-\sqrt{1-2t}))=\sum_{m=0}^{\infty}{}_\rho\mathcal{E}_{m,p,q}(x)\;\frac{t^m}{[m]_{p,q}!}. \tag{2.22} $$

The pqBeEP ${}_\rho\mathcal{E}_{m,p,q}(x)$ of degree $m$ are defined by the series:

$$ {}_\rho\mathcal{E}_{m,p,q}(x)=\sum_{s=0}^{m}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathcal{E}_{s,p,q}\rho_{m-s,p,q}(x). $$

The following identity for the pqBeEP ${}_\rho\mathcal{E}_{m,p,q}(x)$ holds true:

$$ {}_\rho\mathcal{E}_{m,p,q}(x)=\frac{1}{\mathcal{B}_{0,p,q}}\left(\rho_{m,p,q}(x)-\sum_{s=0}^{m-1}\begin{bmatrix} m\\ s\end{bmatrix}_{p,q}\mathcal{B}_{m-s,p,q}\;{}_\rho\mathcal{E}_{s,p,q}(x)\right),\quad m=1,2,\ldots $$

Further, by taking $\mathcal{B}_{0,p,q}=1$ and $\mathcal{B}_{j,p,q}=\frac{1}{2}$, $j=1,2,3,\ldots$, in equations (2.5) and (2.6), we obtain the determinant definition of the pqBeEP ${}_\rho\mathcal{E}_{m,p,q}(x)$.

**Definition 2.3.** The $(p,q)$-Bessel-Euler polynomials ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ of degree $m$ are defined by

$$
\begin{aligned}
&(2.23) \qquad {}_{\rho}\mathcal{E}_{0,p,q}(x) = 1,\\
&(2.24) \qquad {}_{\rho}\mathcal{E}_{m,p,q}(x) = (-1)^m \begin{vmatrix} 1 & \rho_{1,p,q}(x) & \rho_{2,p,q}(x) & \dots & \rho_{m-1,p,q}(x) & \rho_{m,p,q}(x) \\ 1 & \frac{1}{2} & \frac{1}{2} & \dots & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_{p,q}\frac{1}{2} & \dots & \begin{bmatrix} m-1 \\ 1 \end{bmatrix}_{p,q}\frac{1}{2} & \begin{bmatrix} m \\ 1 \end{bmatrix}_{p,q}\frac{1}{2} \\ 0 & 0 & 1 & \dots & \begin{bmatrix} m-1 \\ 2 \end{bmatrix}_{p,q}\frac{1}{2} & \begin{bmatrix} m \\ 2 \end{bmatrix}_{p,q}\frac{1}{2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \begin{bmatrix} m \\ m-1 \end{bmatrix}_{p,q}\frac{1}{2} \end{vmatrix},\\
&\qquad m = 1, 2, 3, \dots,
\end{aligned}
$$

where $\rho_{m,p,q}(x)$, $m = 0, 1, 2, 3, \dots$, are the $(p,q)$-Bessel polynomials of degree $m$.

2.1.3. *$(p,q)$-Bessel-Genocchi polynomials.* Since, for $\mathcal{A}_{p,q}(t) = \frac{[2]_{p,q}t}{e_{p,q}(t)+1}$, the pqAP $\mathcal{A}_{m,p,q}(x)$ reduce to the pqGP $\mathcal{G}_{m,p,q}(x)$ (Table 1 (III)). Therefore, for the same choice of $\mathcal{A}_{p,q}(t)$, the pqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x)$ reduce to pqBeGP ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ which are defined by means of following generating functions:

$$
\frac{[2]_{p,q}t}{e_{p,q}(t)+1}\ e_{p,q}(x(1-\sqrt{1-2t})) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{G}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!}. \tag{2.25}
$$

The pqBeGP ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ of degree $m$ are defined by the series:

$$
{}_{\rho}\mathcal{G}_{m,p,q}(x) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{G}_{s,p,q}\rho_{m-s,p,q}(x).
$$

The following identity for the pqBeGP ${}_{\rho}\mathcal{G}_{m,p,q}(x)$ holds true:

$$
{}_{\rho}\mathcal{G}_{m,p,q}(x) = \frac{1}{\mathcal{B}_{0,p,q}}\left(\rho_{m,p,q}(x) - \sum_{s=0}^{m-1} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} \mathcal{B}_{m-s,p,q}\ {}_{\rho}\mathcal{G}_{s,p,q}(x)\right), \quad m = 1, 2, \dots
$$

## 3. 2D $(p,q)$-Bessel-Appell Polynomials

First, we introduce the $(p,q)$-analogue of the 2D Appell polynomials which are the 2-variable generalization of the $(p,q)$-Appell polynomials denoted as 2D $(p,q)$-Appell polynomials $\mathcal{A}_{m,p,q}(x,y)$.

**Definition 3.1.** The $(p,q)$-analogue of the 2D Appell polynomials $\mathcal{A}_{m,p,q}(x,y)$ are defined by the following generating function:

$$
\mathcal{A}_{p,q}(t)\ e_{p,q}(xt)E_{p,q}(yt) = \sum_{m=0}^{\infty} \mathcal{A}_{m,p,q}(x,y)\ \frac{t^m}{[m]_{p,q}!}, \quad \mathcal{A}_{m,p,q} = \mathcal{A}_{m,p,q}(0,0). \tag{3.1}
$$

TABLE 2. Some members of 2D $(p,q)$-Appell polynomials

| S. No. | $\mathcal{A}_{\mathbf{p},\mathbf{q}}(\mathbf{t})$ | Generating Functions | Polynomials |
|---|---|---|---|
| I. | $\mathcal{A}_{p,q}(t)=\frac{t}{(e_{p,q}(t)-1)}$ | $\frac{t}{(e_{p,q}(t)-1)}e_{p,q}(xt)E_{p,q}(yt) = \sum_{m=0}^{\infty}\mathfrak{B}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$ | The 2D $(p,q)$-Bernoulli polynomials |
| II. | $\mathcal{A}_{p,q}(t)=\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}$ | $\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt) = \sum_{m=0}^{\infty}\mathcal{E}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$ | The 2D $(p,q)$-Euler polynomials |
| III. | $\mathcal{A}_{p,q}(t)=\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}$ | $\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}e_{p,q}(xt)E_{p,q}(yt) = \sum_{m=0}^{\infty}\mathcal{G}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$ | The 2D $(p,q)$-Genocchi polynomials |

Some members of the 2D $(p,q)$-Appell polynomials are listed in Table 2.

The approach used in previous section is further exploited to introduce the 2D $(p,q)$-Bessel-Appell polynomials (2DpqBeAP) and focus on deriving its generating functions and series definitions.

In order to establish the generating function for the 2DpqBeAP, the following result is proved.

**Theorem 3.1.** *The following generating function for the 2D $(p,q)$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$ holds true:*

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{A}_{m,p,q}(x,y)\;\frac{t^m}{[m]_{p,q}!}. \tag{3.2}$$

*Proof.* By expanding the first $(p,q)$-exponential function $e_{p,q}(xt)$ in the left hand side of the equation (3.1) and then replacing the powers of $x$, i.e., $x^0,x,x^2,\dots,x^m$ by the corresponding polynomials $\rho_{0,p,q}(x),\rho_{1,p,q}(x),\rho_{2,p,q}(x),\dots,\rho_{m,p,q}(x)$ in the left hand side and $x$ by $\rho_{1,p,q}(x)$ in the right hand side of the resultant equation, we have

$$\mathcal{A}_{p,q}(t)\left(1+\rho_{1,p,q}(x)\frac{t}{[1]_{p,q}!}+\rho_{2,p,q}(x)\frac{t^2}{[2]_{p,q}!}\quad+\cdots+\rho_{m,p,q}(x)\frac{t^m}{[m]_{p,q}!}+\cdots\right)E_{p,q}(yt)$$
$$=\sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x),y)\;\frac{t^m}{[m]_{p,q}!}.$$

Further, summing up the series in left hand side and then using equation (2.1) in the resultant equation, we get

$$\mathcal{A}_{p,q}(t)e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)=\sum_{m=0}^{\infty}\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x),y)\;\frac{t^m}{[m]_{p,q}!}.$$

Finally, denoting the resultant 2DpqBeAP in the right hand side of the above equation by ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$, that is

$$\mathcal{A}_{m,p,q}(\rho_{1,p,q}(x),y)={}_{\rho}\mathcal{A}_{m,p,q}(x,y),$$

the assertion (3.2) is proved. □

*Remark* 3.1. It is remarked that for $p=1$, the 2DpqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$ reduce to the 2D $q$-Bessel-Appell polynomials (2DqBeAP) ${}_{\rho}\mathcal{A}_{m,q}(x,y)$ such that

$$ {}_{\rho}\mathcal{A}_{m,q}(x,y) := {}_{\rho}\mathcal{A}_{m,1,q}(x,y). $$

Thus, taking $p=1$ in equation (3.2), we get

$$ \mathcal{A}_q(t)e_q(x(1-\sqrt{1-2t}))E_q(yt) = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,q}(x,y)\ \frac{t^m}{[m]_q!}, $$

which is the generating function for the 2D $q$-Bessel-Appell polynomials.

Next, we give the series definition for the 2DpqBeAP ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$, by proving the following result.

**Theorem 3.2.** *The 2D $(p,q)$-Bessel-Appell polynomials ${}_{\rho}\mathcal{A}_{m,p,q}(x,y)$ are defined by the following series definition:*

$$ {}_{\rho}\mathcal{A}_{m,p,q}(x,y) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s\, {}_{\rho}\mathcal{A}_{m-s,p,q}(x). \tag{3.3} $$

*Proof.* In view of equations (1.2) and (2.2), equation (3.2) can be written as:

$$ \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} \sum_{s=0}^{\infty} q^{\binom{s}{2}}\ y^s \frac{t^s}{[s]_{p,q}!} = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x,y)\ \frac{t^m}{[m]_{p,q}!}, $$

which on using the Cauchy product rule gives

$$ \sum_{m=0}^{\infty}\sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{p,q} q^{\binom{s}{2}} y^s\, {}_{\rho}\mathcal{A}_{m-s,p,q}(x)\ \frac{t^m}{[m]_{p,q}!} = \sum_{m=0}^{\infty} {}_{\rho}\mathcal{A}_{m,p,q}(x,y)\ \frac{t^m}{[m]_{p,q}!}. \tag{3.4} $$

Equating the coefficients of like powers of $t$ in both sides of the above equation, we arrive at our assertion (3.3). $\square$

*Remark* 3.2. For $p=1$, series definition (3.3) becomes

$$ {}_{\rho}\mathcal{A}_{m,q}(x,y) = \sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix}_{q} q^{\binom{s}{2}} y^s\, {}_{\rho}\mathcal{A}_{m-s,q}(x), $$

which is the series definition for the 2D $q$-Bessel-Appell polynomials.

Certain members belonging to the 2D $(p,q)$-Appell family are given in Table 2. Since, corresponding to each member belonging to the 2D $(p,q)$-Appell family, there exists a new special polynomial belonging to the 2D $(p,q)$-Bessel-Appell family. Thus, by making suitable choices for the functions $\mathcal{A}_{p,q}(t)$ in equations (3.2) and (3.3), the generating functions and series definitions for the corresponding members belonging to the 2D $(p,q)$-Bessel-Appell family can be obtained. The resultant members of the 2D $(p,q)$-Bessel-Appell family along with their generating functions and series definitions are given in Table 3.

Table 3. Certain members belonging to the 2D $(p,q)$-Bessel-Appell polynomials

| S. No. | $\mathcal{A}_{\mathbf{p,q}}(\mathbf{t})$ | Generating Functions | Series Definition | Polynomials |
|---|---|---|---|---|
| I. | $\frac{t}{(e_{p,q}(t)-1)}$ | $\frac{t}{(e_{p,q}(t)-1)}e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)$ $=\sum_{m=0}^{\infty}{}_{\rho}\mathfrak{B}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$ | ${}_{\rho}\mathfrak{B}_{m,p,q}(x,y)$ $=\sum_{s=0}^{m}\begin{bmatrix}m\\s\end{bmatrix}_{p,q}q^{\binom{s}{2}}y^s{}_{\rho}\mathfrak{B}_{m-s,p,q}(x)$ | The 2D $(p,q)$-Bessel-Bernoulli polynomials |
| II. | $\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}$ | $\frac{[2]_{p,q}}{(e_{p,q}(t)+1)}e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)$ $=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{E}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$ | ${}_{\rho}\mathcal{E}_{m,p,q}(x,y)$ $=\sum_{s=0}^{m}\begin{bmatrix}m\\s\end{bmatrix}_{p,q}q^{\binom{s}{2}}y^s{}_{\rho}\mathcal{E}_{m-s,p,q}(x)$ | The 2D $(p,q)$-Bessel-Euler polynomials |
| III. | $\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}$ | $\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)}e_{p,q}(x(1-\sqrt{1-2t}))E_{p,q}(yt)$ $=\sum_{m=0}^{\infty}{}_{\rho}\mathcal{G}_{m,p,q}(x,y)\frac{t^m}{[m]_{p,q}!}$, | ${}_{\rho}\mathcal{G}_{m,p,q}(x,y)$ $=\sum_{s=0}^{m}\begin{bmatrix}m\\s\end{bmatrix}_{p,q}q^{\binom{s}{2}}y^s{}_{\rho}\mathcal{G}_{m-s,p,q}(x)$ | The 2D $(p,q)$-Bessel-Genocchi polynomials |

## 4. Graphical Representation

In this section with the help of Matlab, we plot the graphs of $(p,q)$-Bessel-Bernoulli polynomials ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$, $(p,q)$-Bessel-Euler polynomials ${}_{\rho}\mathcal{E}_{m,p,q}(x)$. To draw the graphs of these polynomials, we consider the values of the first four $(p,q)$-Bessel polynomials $\rho_{m,p,q}(x)$, the expressions of these polynomials are given in Table 4.

Table 4. Expressions of the first four $\rho_{m,p,q}(x)$.

| m | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $\rho_{m,p,q}(x)$ | 1 | $x$ | $x^2+\frac{[2]_{p,q}}{2}x$ | $x^3+\frac{[3]_{p,q}[2]_{p,q}}{2}x^2+\frac{[4]_{p,q}[3]_{p,q}}{4}x$ |

Next, taking $p=\frac{1}{2}$, $q=\frac{1}{4}$ in the determinant definitions (2.21), (2.24) and using the expressions of the $\rho_{m,p,q}(x)$ from Table 4, we get the results mentioned in Table 5 for $m=0,1,2,3$.

Table 5. The first four expressions of ${}_{\rho}\mathfrak{B}_{m,\frac{1}{2},\frac{1}{4}}(x)$ and ${}_{\rho}\mathcal{E}_{m,\frac{1}{2},\frac{1}{4}}(x)$.

| $m$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| ${}_{\rho}\mathfrak{B}_{m,\frac{1}{2},\frac{1}{4}}(x)$ | 1 | $-\frac{4}{3}+x$ | $x^2-\frac{5}{8}x-\frac{20}{21}$ | $x^3-\frac{161}{384}x^2-\frac{7493}{12288}x-\frac{107}{45}$ |
| ${}_{\rho}\mathcal{E}_{m,\frac{1}{2},\frac{1}{4}}(x)$ | 1 | $-\frac{1}{2}+x$ | $x^2-\frac{5}{16}$ | $x^3-\frac{7}{128}x^2-\frac{791}{4096}x-\frac{165}{512}$ |

Now, with the help of Matlab and using equations (2.20), (2.23) and the expressions of ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$ and ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ from Table 5, we get the graphs at Figure 1 and 2.

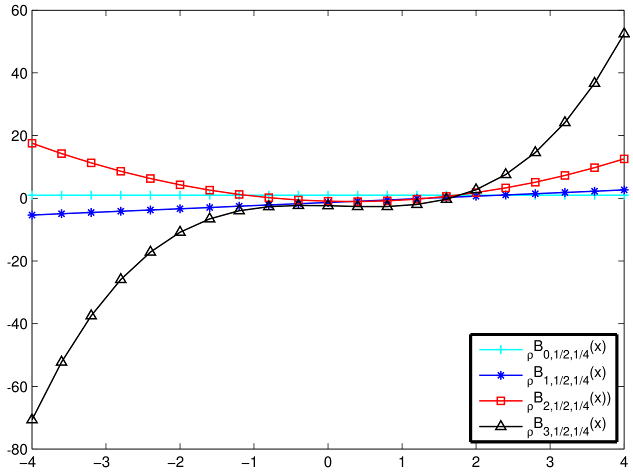


FIGURE 1. Graph of ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$

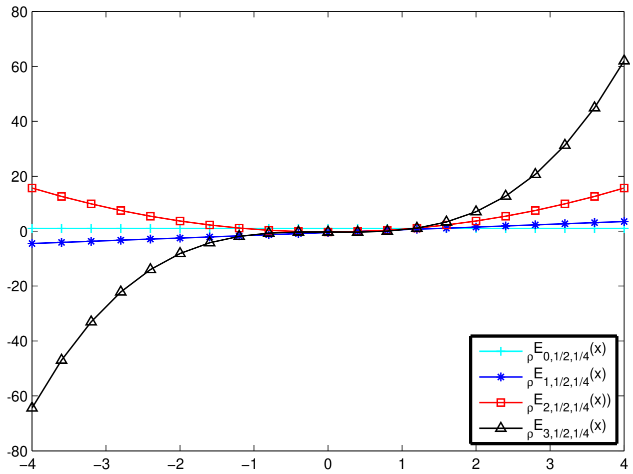


FIGURE 2. Graph of ${}_{\rho}\mathcal{E}_{m,p,q}(x)$

Further, setting $m=3, p=\frac{1}{2}, q=\frac{1}{4}$ in the series definitions of ${}_{\rho}\mathfrak{B}_{m,p,q}(x,y)$, ${}_{\rho}\mathcal{E}_{m,p,q}(x,y)$ given in Table 3 and using the expressions of ${}_{\rho}\mathfrak{B}_{m,p,q}(x)$, ${}_{\rho}\mathcal{E}_{m,p,q}(x)$ from Table 5, we have

(4.1)
$$ {}_{\rho}\mathfrak{B}_{3,\frac{1}{2},\frac{1}{4}}(x,y) = x^3 - \frac{161}{384}x^2 - \frac{7493}{12288}x - \frac{107}{45} + \frac{7}{16}x^2y - \frac{35}{128}xy - \frac{5}{12}y - \frac{7}{48}y^2 + \frac{7}{64}xy^2 + \frac{1}{64}y^3, $$

(4.2)
$$ {}_{\rho}\mathcal{E}_{3,\frac{1}{2},\frac{1}{4}}(x,y) = x^3 - \frac{7}{128}x^2 - \frac{791}{4096}x - \frac{165}{512} + \frac{7}{16}x^2y - \frac{35}{256}y - \frac{7}{128}y^2 + \frac{7}{64}xy^2 + \frac{1}{64}y^3, $$

In view of equations (4.1)–(4.2), we get the surface plots at Figure 3 and 4.

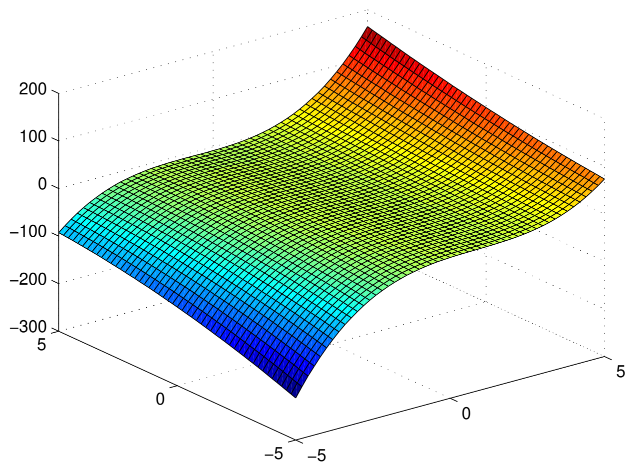


FIGURE 3. Surface plot of ${}_{\rho}\mathfrak{B}_{3,\frac{1}{2},\frac{1}{4}}(x,y)$

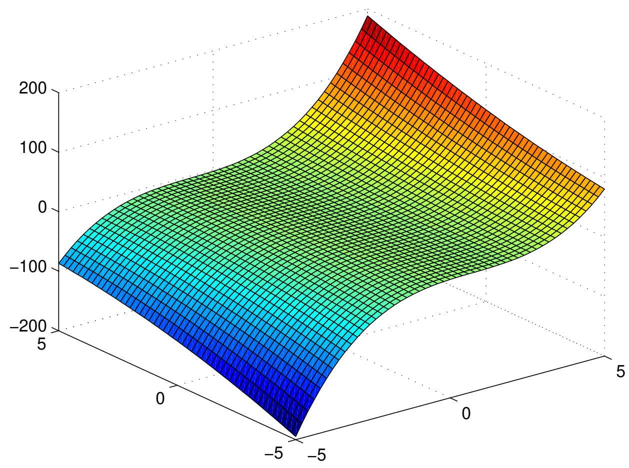


FIGURE 4. Surface plot of ${}_{\rho}\mathcal{E}_{3,\frac{1}{2},\frac{1}{4}}(x,y)$

## 5. Concluding Remarks

The Bernoulli, Euler and Genocchi numbers are among the most interesting and important number sequences in mathematics. These numbers are particularly important in number theory, they have deep connections with calculus of finite differences, combinatorics and other fields. Here, let us recall $(p,q)$-Bernoulli, $(p,q)$-Euler and $(p,q)$-Genocchi numbers.

We note that (see [6])

$$\begin{aligned}\mathfrak{B}_{m,p,q} &:= \mathfrak{B}_{m,p,q}(0), \quad (p,q)\text{-Bernoulli numbers},\\ \mathcal{E}_{m,p,q} &:= \mathcal{E}_{m,p,q}(0), \quad (p,q)\text{-Euler numbers},\\ \mathcal{G}_{m,p,q} &:= \mathcal{G}_{m,p,q}(0), \quad (p,q)\text{-Genocchi numbers}.\end{aligned}$$

Further, we note that

$$\rho_{m,p,q} := \rho_{m,p,q}(0), \quad (p,q)\text{-Bessel numbers}.$$

In this section, we introduce the numbers related to the polynomial families established in Sections 2 and 3.

Taking $x = 0$ in the generating functions of the ${}_\rho\mathfrak{B}_{m,p,q}(x)$, ${}_\rho\mathcal{E}_{m,p,q}(x)$ and ${}_\rho\mathcal{G}_{m,p,q}(x)$ given by equations (2.18), (2.22) and (2.25), the $(p, q)$-Bernoulli, $(p, q)$-Euler and $(p, q)$-Genocchi numbers are obtained. These numbers are listed in Table 6.

TABLE 6. Certain members belonging to $(p, q)$-Bessel-Appell numbers

| S. No. | Notations | Generating Functions | Numbers |
|---|---|---|---|
| I. | ${}_\rho\mathfrak{B}_{m,p,q} := {}_\rho\mathfrak{B}_{m,p,q}(0)$ | $\frac{t}{e_{p,q}(t)-1} = \sum\limits_{m=0}^{\infty} {}_\rho\mathfrak{B}_{m,p,q} \frac{t^m}{[m]_{p,q}!}$ | The $(p, q)$-Bessel-Bernoulli numbers |
| II. | ${}_\rho\mathcal{E}_{m,p,q} := {}_\rho\mathcal{E}_{m,p,q}(0)$ | $\frac{[2]_{p,q}}{(e_{p,q}(t)+1)} = \sum\limits_{m=0}^{\infty} {}_\rho\mathcal{E}_{m,p,q} \frac{t^m}{[m]_{p,q}!}$ | The $(p, q)$-Bessel-Euler numbers |
| III. | ${}_\rho\mathcal{G}_{m,p,q} := {}_\rho\mathcal{G}_{m,p,q}(0)$ | $\frac{[2]_{p,q}t}{(e_{p,q}(t)+1)} = \sum\limits_{m=0}^{\infty} {}_\rho\mathcal{G}_{m,p,q} \frac{t^m}{[m]_{p,q}!},$ | The $(p, q)$-Bessel-Genocchi numbers |

Similarly, on taking $x = y = 0$ in the generating functions of the ${}_\rho\mathfrak{B}_{m,p,q}(x, y)$, ${}_\rho\mathcal{E}_{m,p,q}(x, y)$ and ${}_\rho\mathcal{G}_{m,p,q}(x, y)$ given in Table 3 (I-III), we get the same numbers given in Table 6 (I-III).

We note that the class of numbers introduced in this section are actually the $(p, q)$-Bernoulli, $(p, q)$-Euler and $(p, q)$-Genocchi numbers, respectively.

In this article, the $(p, q)$-analogue of Bessel polynomials and its hybrid form are introduced by means of series expansion and generating function. The determinant form related to these polynomials are derived, which can be helpful for computation purposes and can also be used in finding the solutions of general linear interpolation problems.

Some properties including addition theorem, difference equations and recurrence relations for the $(p, q)$-Appell family have been analyzed and established in [13] (see also [11]). This provides motivation to establish $(p, q)$-difference equations and other properties for $(p, q)$-Bessel-Appell polynomials and their generalized 2D form in future investigation.

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# TWO-DIMENSIONAL DYNAMICS OF CUBIC MAPS

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Abstract. We investigate the global properties of two cubic maps on the plane, we try to explain the basic mechanisms of global bifurcations leading to the creation of nonconnected basins of attraction. It is shown that in some certain conditions the global structure of such systems can be simple. The main results here can be seen as an improvement of the results of stability and bifurcation analysis.

## 1. Introduction

Polynomial diffeomorphisms have been widely studied and they are fundamental to our understanding of dynamical systems. They are of great interest as approximations of more complicated maps with constant Jacobian, and some of them exhibit some of the familiar properties of the quadratic Hénon map.The single Hénon map: $(x', y') = (y + x^2 + a, cx)$ is the simplest polynomial map, and the simplest nontrivial diffeomorphism of the plane containing a single quadratic term as nonlinearity. This map is also known to display chaos for certain parameter values and initial conditions. Due to its simplicity, it has become a benchmark system and has received considerable attention because of its genericity, the complexity and richness of its dynamics, frequently used as an example for demonstrating schemes for analyzing and controlling chaotic behavior.

The set of polynomial maps with polynomial inverse is called the "affine Cremona group", very dynamically interesting maps. The structure of this group is well-known and understood for two-dimensional case; as remarked in Friedland-Milnor's classical work [2], they proved that any map in this group is conjugate to a composite of basic polynomial maps called generalized Hénon maps: $(x', y') = (y + f(x), cx)$, maps with constant and nonzero Jacobian and where $f(x)$ is a polynomial of degree $d \geq 2$. It

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follows that any composition of Hénon maps has an inverse which is a polynomial. Recently, different types of generalization of the standard Hénon map have been studied. Dullin and Meiss in [1] considered polynomial cubic maps. In a recent paper, Sarmah and Paul [7] examined a period doubling route to chaos for a similar model with constant Jacobian. For more details, see the survey of Sibony [8] and the references therein [10, 11], where more light was shed. Silverman [9] studied arithmetic properties of quadratic Hénon maps.

Many of complex behaviors that are observed in dynamical systems are intimately associated with the presence of homoclinic or heteroclinic points of maps [2, 3]. The global bifurcations involving invariant curves have been less investigated, and several open problems are still present. Homoclinic tangencies between stable and unstable invariant manifolds of the same saddle point play a very important role. The existence of transversal homoclinic intersections is considered as the universal criterium of the complexity for maps. At the same time, the presence of non-transversal homoclinic orbits (homoclinic tangencies) indicates an extraordinary richness of bifurcations of such systems and, what is very important, the principal impossibility of providing of a complete description of bifurcations. Therefore, when studying homoclinic bifurcations, the main problems are related to the analysis of their principal bifurcations and characteristic properties of dynamics as a whole.

This work presents a research in the study of cubic polynomial invertible and noninvertible maps of the plane carried out some techniques and numerical simulations. The motivation for studying such maps is, in part, due to the form of these maps which is a generalized version of Hénon map. This set is of fundamental importance in dynamical systems and yields a great deal of interesting characteristics. Our main concerns are the global dynamics characterizing the topological structure of initial conditions which generate interesting path in cubic maps. In addition to the analytical considerations, we also display certain numerical results by using computers to perform rigorous mathematical proofs.

This paper intends to give such a study, particularly to consider two cases of cubic diffeomorphisms. Therefore, it is structured in the following way. In Section 2, division of the parameter plane for the two-dimensional maps into domains of regular and chaotic attractors is studied numerically and analytically. Regularities in the occurrence of different behaviors and transitions are analyzed. The dynamics involves various transitions by bifurcations. In Section 3, we introduce the language mentioned in [5, 6], to analyze these maps, and give some useful definitions. Section 4 focuses on the global dynamics. The impact of invariant manifolds on the structure of basins is investigated. Section 5 gives some results on basin structures of noninvertible maps and their bifurcations, and illustrates properties of homoclinic-heteroclinic bifurcations. We end the paper with a conclusion.

## 2. Division of Parameter Plane

Consider the one-dimensional endomorphism of the $(p+q-2)$ model

$$T_0(x) = ax^{p-1}(1-x)^{q-1}. \tag{2.1}$$

Here, the trivial fixed point $x = 0$ is unstable for $1 < p < 2$ and it is stable for $p > 2$, both cases for any $a > 0$ and $q > 1$. We have a special case for $p = 2$, where $x = 0$ is an unstable fixed point if $r > 1$ and a stable fixed point if $0 < a < 1$, both cases for any $q > 1$. Consequently the set defined by $S = \{(q, a) \in \mathbb{R}^2 : q > 1, a > 0 \text{ for } p = 2\}$ is a bifurcation plane that characterizes the stability of the fixed point $x = 0$ at the parameter space $(p, q, a)$. We consider an imbedding of the model (2.1), which is a one-dimensional noninvertible map into a two-dimensional diffeomorphism rediscovered afresh each time and with a variety of results. We study this diffeomorphism in dependance of at least three parameters and uncover many fascinating dynamical characteristics, using both analytic perturbation theory and numerical methods.

The planar diffeomorphism associated with $T_0$ is the following:

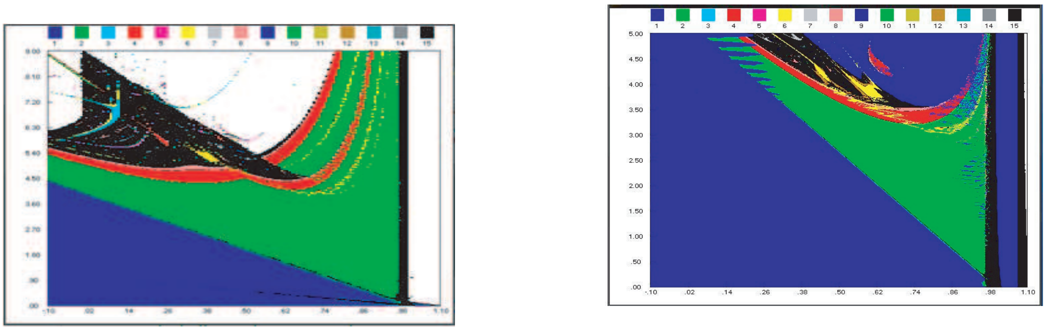
$$T_1 : \begin{cases} x' = T_0(x) + y, \\ y' = cx, \end{cases} \tag{2.2}$$

where $x, y$ are real variables, $a, p, q$ and $c$ are real parameters. $T_1$ has a constant Jacobian determinant $detJ = -c$. We distinguish two types of cubic diffeomorphisms $(p + q - 2 = 3)$, and each type gives different bifurcation diagrams. We only study the most interesting and principal peculiarities of the cubic maps ($p = 3$, $q = 2$ and $p = 2$, $q = 3$).

For $c = 0$, the planar diffeomorphism (2.2) becomes the one-dimensional endomorphism (2.1). The model (2.2) possesses at most three fixed points depending upon the parameter values. To gain preliminary insight into the properties of the dynamical system (2.2) we conducted two-dimensional bifurcation analysis, which provides information on the dependance of the dynamics on parameters. This analysis is expected to reveal the type of attractor to which the dynamics will ultimately settle down after passing an initial transcient phase and within which the trajectory will remain forever. The parameters $(c, a)$ are varied simultaneously to track bifurcations.

We indicate different attractors in different colors in the $(c, a)$-plane for which the mappings were expected to have simple dynamics in the case $p = 3$ and $q = 2$. The Figure 1 give the parameter value for which at least one fixed point is attractive (parameters located in the blue domain will be stabilized at a fixed point). More generally, the Figure $1(a, b)$ gives the regions of the $(c, a)$-plane for which at least a cycle of order $k$ exists $(k = 1, 2, \ldots, 14)$. The black region $(k = 15)$ corresponds to the existence of bounded iterated sequences. Clearly, these figures exhibit the typical period doubling route to chaos obtained by increasing $a$ for fixed $c$. We can recognize, in particular, two typical and well-known structures of the bifurcation diagrams in two-dimensional parameter plane, the so-called “saddle area” in the case $p = 3$ and

$q = 2$, and saddle area with "cross-road area" in the case $p = 2$ and $q = 3$. The saddle area is special because associated with a "degenerate" bifurcation curve for $c = 1$.



(a) Bifurcation structure for $p = 2$ and $q = 3$

(b) Bifurcation structure for $p = 3$ and $q = 2$

FIGURE 1. two-dimensional bifurcation diagrams with colors obtained numerically according to the different orders observed in the plane $(c, a)$.

## 3. Definitions and Fundamental Properties

In this section, we give precise notions in report with invertible polynomial maps, contact and homoclinic bifurcations, and some properties of increasing complexity that try to highlight the important concepts of nonlinear maps (refer to Mira et al. in [6]).

The polynomial map $T$ of the plane has the form

$$(x', \; y') = T(x, y) = (f(x, y; \lambda), g(x, y; \lambda)),$$

where $f$ et $g$ are polynomials in $x$, $y$ and $\lambda$ is a real parameter-vector.The Jacobian determinant is defined as

$$\det J(f, g) = \det T(x, y) = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

3.1. **General properties.** We assume that a closed and invariant set $A$ is called an attracting set if some neighborhood $U$ of $A$ may exist such that $T(U) \subset U$ and $T^n(x, y) \to A$ as $n \to \infty$, for all $(x, y) \in U$. An attracting set $A$ may contain one or several attractors (regular attractors are stable fixed points or cycles) coexisting with sets of repulsive points. The set $D = \cup_{n \geq 0} T^{-n}(U)$ is called the total basin of $A$,it is invariant under backward iteration $T^{-1}$ of $T$, but not necessarily invariant by $T$

$$T^{-1}(\mathrm{D}) = \mathrm{D}, \quad T(\mathrm{D}) \subseteq \mathrm{D}.$$

An attracting set is called of order k if it is made up of $k$ disjoint sets , $A = \cup_{i=1}^{k} A_i$, where each $A_i$ is an attracting set of the map $T^k$.

When $A$ is an attracting set of order $k = 1$, then its total basin is given by $D = D_0$ if it is connected, by $D = \cup_{n\geq 0} T^{-n}(\mathrm{D}_0)$ if it is nonconnected. When $A$ is an attracting set of order $k > 1$, the immediate basin $D_0$ of $A$ is the open set $D_0 = \cup_{i=1}^{n} D_{0,i}$, the $D_{0,i}$ being open disjoints basins of $A_i$. If $A$ is connected attractor, the immediate basin $D_0$ of $A$ is defined as the widest connected component of $D$ containing $A$. When $A$ is the widest attracting set of a map $T$, its basin $D$ is the total basin of bounded iterates. That is, the open set $D$ contains $A$ such that $D$ is the locus of points of the plane having bounded trajectories.

We assume that the existence of an attracting set $A$ is observed through numerical methods.

**Definition 3.1.** Let $S$ be a saddle fixed point of $T$, $W^s(S)$ and $W^u(S)$ denoting its stable and unstable sets. A point $q$ is called homoclinic to $S$, if $q \in W^s(S) \cap W^u(S)$ and $q \neq S$. $q$ is a transversal homoclinic point, so $W^s(S)$ intersects transversely $W^u(S)$.

**Definition 3.2.** One calls homoclinic orbit $O_o(q)$ associated with $q, q$ belonging to a $U(S)$ of $S$, a set constituting of successive iterates of $q$, and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of $T$ in $U(S)$, i.e., $O_o(q) = \left\{T_l^{-n}(q), q, T^n(q) : n > 0\right\} = \{\ldots, q_{-n}, \ldots, q_{-2}, q_{-1}, q, q_1, q_2, \ldots, q_n, \ldots\}$, where $q_n = T^n(q) \to S$ and $q_{-n} = T_l^{-n}(q) \to S$.

**Definition 3.3.** One calls heteroclinic orbit $\varepsilon(q)$ connecting $S$ to $S'$ associated with $q$, the one given by $q$ together with its finite orbit and its infinite sequence of preimages obtained by application of the local inverse map $T_l^{-1}$of $T$ in $U(S)$, i.e., $\varepsilon(q) = \left\{T_l^{-n}(q), q, T^n(q) : n > 0\right\} = \{\ldots, q_{-n}, \ldots, q_{-2}, q_{-1}, q, q_1, q_2, \ldots, q_n, \ldots\}$, where $q_n = T^n(q) \to S'$ and $q_{-n} = T_l^{-n}(q) \to S$.

3.2. **Generalized Hénon map properties.** First we recall the dynamics of the cubic diffeomorphism $T_1$

$$T_1(x, y) = (T_0(x) + y, cx).$$

$T_0(x)$ is a polynomial of degree-3 then $T_1$ is conjugate to Hénon map. We know some results which enable us to detect, predict, determine cycles and fixed points, and locate bifurcation curves in parameter plane. $T_0(x)$ can be equal to $x(1-x)^2$ or to $x^2(1-x)$.

Fixed point $(x_*, y_*)$ of $T_1$ satisfies $y_* = cx_*$, and $(1-c)x_* = T_0(x_*)$, so that $x_*$ is a root of the polynomial $q(x_*) = (c-1)x_* + T_0(x_*)$, thus all fixed points are located on the line $y - cx = 0$ in the plane.

The stability of these fixed points is determined by the Jacobian matrix

$$J_* = \begin{pmatrix} T_0'(x_*) & 1 \\ c & 0 \end{pmatrix},$$

which has trace $TrJ_* = T_0'(x_*)$ and determinant $\det J_* = -c$. The fixed point is stable when the eigenvalues of $J_*$ are less than 1 in magnitude. This is true only when

$J_*$ satisfies the three Jury conditions [4]: $1 - TrJ_* + \det J_* > 0$, $1 + TrJ_* + \det J_* > 0$, $1 - \det J_* > 0$.

It is easy to verify that $T_1$ can have bounded orbits only when there are fixed points of $T_1^n$.

It is sufficient to consider the case $|c| \leq 1$, since the inverse of a generalized Hénon map with $|c| > 1$ is conjugate to a generalized Hénon map with $|c| < 1$ under the reflection $r(x, y) = (y, x)$, and $r \circ T_1^{-1} \circ r = (y - T_0(\frac{x}{c}), \frac{x}{c}), T_1^{-1}(x, y) = (\frac{y}{c}, x - T_0(\frac{y}{c}))$.

*Remark* 3.1. For $c = 1$, the fixed points of $T_1$ are the roots of $T_0$. If $p = 3$ and $q = 2$, the determinant is equal to $-1$ and $T_0'(0) = 0$ with two eigenvalues $-1$,1. There is a fold-flip bifurcation for $O(0, 0)$. For $p = 2$ and $q = 3$, $T_0'(1) = T_1(1) = 0$. These two cases are two nondegenerate codimension-2 bifurcations.

**Theorem 3.1.** *Suppose $T_1$ has no fixed points, then every orbits is unbounded.*

*Proof.* Suppose that $T_1$ has no fixed points, then the fixed point polynomial $q(x) = T_0(x) + cx - x$ is either positive or negative for all $x$. In the first case $q(x)$ is positive, consider $d(x, y) = x + y$, then $d(x', y') = d(x, y) + q(x)$ creases monotonically and must be unbounded. In the other case $q(x)$ is negative, $d(x', y')$ decreases monotonically and then either case there are no bounded orbits.

When there are fixed points, we can find a box that contains all these bounded orbits. □

**Theorem 3.2.** *Every bounded orbit of $T_1$ map is contained in the box*

$$\{(x, y) : |x| \leq M,\ |y| \leq |c|M\},$$

*where $M$ is the largest of the absolute values of the roots of $T_0(x) - (1 + |c|)|x|$.*

*Proof.* See [1], more generally the polynomial determining $M$ is the same as that for the fixed points, up to the absolute value signs. □

**Proposition 3.1.** *Concerning the existence of cycles of order 2, the following holds:*

- *cycles of order 2 occur for $T_1(x, y) = T_1^{-1}(x, y)$;*
- *they have to satisfy $T_0(x) + y = \frac{y}{c}$, $x - T_0\left(\frac{y}{c}\right) = cx$ and $(1 - c)x - T_0\left(\frac{T_0(x)}{1-c)}\right) = 0$.*

*Proof.* Cycles of order-2 are given by the equation $T_1^2(x, y) = (x, y) = T_1^{-1} \circ T_1(x, y)$ and then it is easy to verify that $T_0(x) + y = \frac{y}{c}$, $x - T_0(\frac{y}{c}) = cx$, which is equivalent to $(1 - c)x - T_0(\frac{T_0(x)}{1-c}) = 0$. This equation is divisible by $q(x)$ because fixed points are roots of both equations. Since $T_1$ is cubic then the equation giving 2-cycles is a polynomial of degree-6, there are at most three 2-cycles. □

*Remark* 3.2. By a way analogous to that in the proof of Proposition 3.1, we can determine without any difficulty the equations of cycles of higher order by using $T_1^n(x, y) = (x, y) = T_1^{-1} \circ T_1(x, y)$ which can be reduced to $T_1^{n-1}(x, y) = T_1^{-1}(x, y)$.

Similarly, 3-cycles are solutions of : $T_1^2(x, y) = (T_0(T_0(x) + y) + cx, cy + cT_0(x)) = T_1^{-1}(x, y) = (\frac{y}{c}, x - T_0(\frac{y}{c}))$. They are determined by the system $x_1 - c^2x_0 = T_0(x_0) - cT_0(x_1)$ and $x_0 - cx_1 = T_0(T_0(x_1) + cx_0)$, if we assume that $y = cx_0$.

## 4. Basins and Attractors for the Cubic Diffeomorphism

Now, we examine the behavior of $T_1$ on basin structure and its bifurcations. These bifurcations are characterized by the creation of heteroclinic and homoclinic connections or homoclinic tangles. Especially, we explain basin bifurcations which result from the contact between basin boundaries delimited by stable manifolds of the 2-cycle of saddle type and the nontrivial saddle fixed point (possibly a flip saddle).

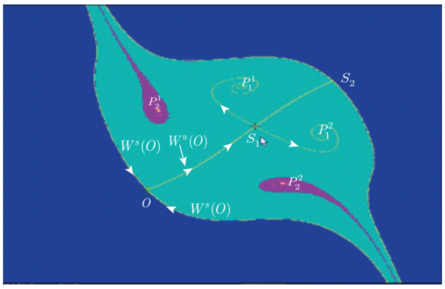
Figures 2 (a), (b), (c), (d), (e), (f) represent the existing attractors (fixed points and 2-cycles), invariant manifolds of saddle points and their basins. The evolution of attractors and their basins is given directly in figures, the parameters $p, q$ have been chosen constant.

We start a qualitative description of bifurcations that are expected to occur as one parameter $a$ or $c$ is varied following a bifurcation path such $c$ close of 1.0, we identify a very fascinating scenario in (a), (b), (c): two nontrivial fixed points are created by a saddle-node bifurcation and one of them ($S_1$) undergoes a period-doubling bifurcation and becomes a flip saddle. A further increase of the parameter $a$ causes a contact between these two boundaries which marks changes in the basins of attraction from connected to nonconnected basins.

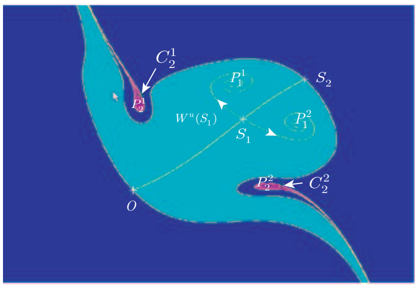
Here, if we consider $T_1 \circ T_1$, instead of $T_1$, points of 2-cycle correspond to fixed points of $T_1^2$ and then a flip bifurcation of $T_1$ corresponds to a pitchfork bifurcation of $T_1^2$. This implies that the same bifurcations are to be expected in the two cases.

The map generates many 2-cycles, we have three 2-cycles of which two are stable. We can see that the bifurcation which is put in evidence can be classified as a global bifurcation, only fixed points and 2-cycles exist and communicate through saddles. This kind of bifurcation involves attracting and repelling invariant curves issuing from saddles. Also, saddles on the boundary of basins play a major role because if they become outside the basins, thus transitions from "connected basin $\leftrightarrow$ nonconnected basin" occur. In particular, we remark that the sequence of bifurcations described in this work, cause the transition of a pair of 2-cycles from inside to outside a stable manifold associated with the saddle $S_2$. This invariant curve, involved in this global structure, exhibits different dynamic behaviors before and after the transition.

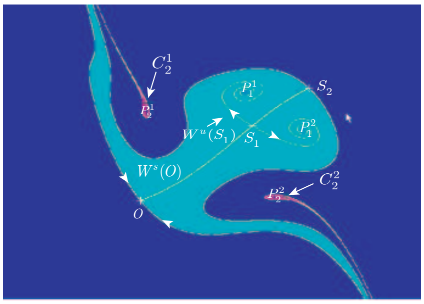
We can also see that in (e), (f) the basin associated with the 2 -cycle $P_2$ is destroyed and the trivial fixed point is outside, still exists and is unstable. In Figure 3 (a), (b), (c), (d), (e) all the fixed points are aligned but the single fixed point that always exists is stable, the other two fixed points are located on the boundary of the big basin and on the boundary of trivial fixed point basin. When the saddle point $S_2$ is outside then the basin becomes nonconnected, each point of 2-cycle has now its own basin. The stable manifold of the saddle point $S_1$ located on the boundary of the trivial fixed point $O(0, 0)$ performs two loops and delimits after the basin of the unique attractor.



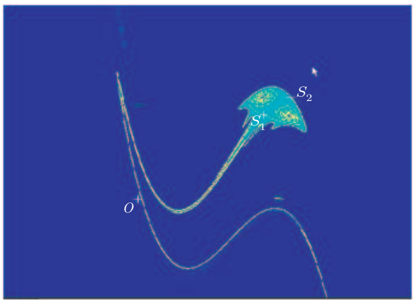
(a) $a = 1,\ c = 0.9300,\ p = 2, q = 3$



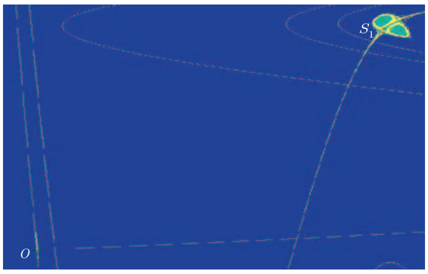
(b) $a = 1.01,\ c = 0.9522,\ p = 2,\ q = 3$



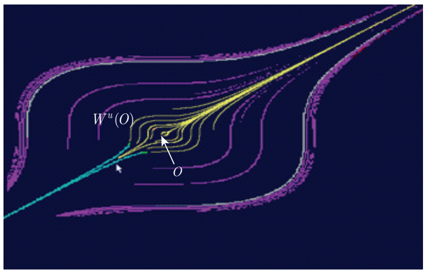
(c) $a = 1.17,\ c = 0.9522,\ p = 2,\ q = 3$



(d) $a = 5,\ c = 0.9522,\ p = 2,\ q = 3$



(e) $a = 16,\ c = 0.9,\ p = 2,\ q = 3$



(f) $a = 21,\ c = 1.0,\ p = 2,\ q = 3$

FIGURE 2. The three fixed points are unstable. The basin of the 2-cycle inside the big basin has a contact with the frontier of the big one, becomes outside and disappears.

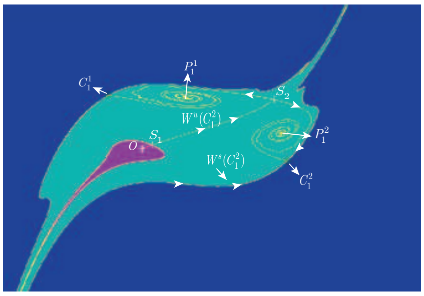
## 5. Bifurcations Basins for the Cubic Endomorphism

Let us consider now the noninvertible map $T_2$ defined by

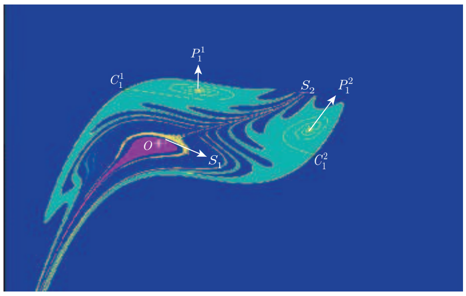
$$T_2 : \begin{cases} x' = T_0(x) + y, \\ y' = cx + dy, \end{cases}$$

where $c, d$ are real parameters.

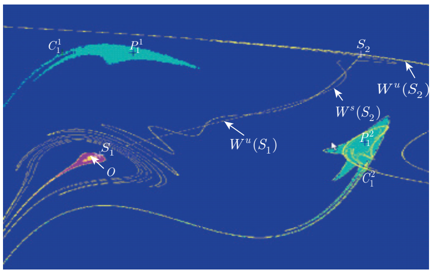
For $d \neq 0$, the system $T_2$ becomes again an endomorphism. We foresee that new phenomena are likely to occur for $T_2$. Figure 4 shows that the dynamics, influenced by the parameter $d$, revolves around fixed points and cycles of order-2 which exist



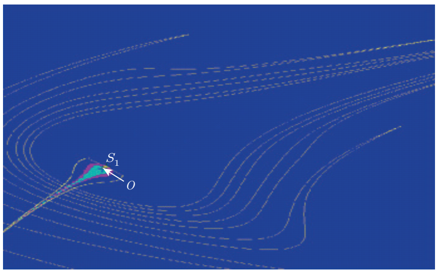
(a) $a = 1.50,\ c = 0.9,\ p = 3,\ q = 2$



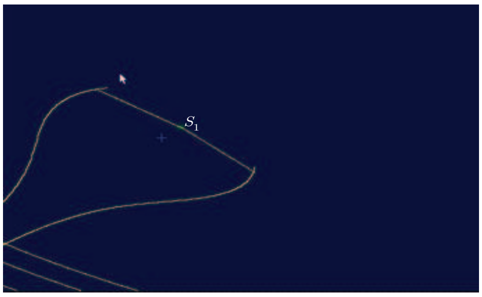
(b) $a = 1.96,\ c = 0.9,\ p = 3,\ q = 2$



(c) $a = 15,\ c = 0.95,\ p = 3,\ q = 2$



(d) $a = 15,\ c = 0.98,\ p = 3,\ q = 2$



(e) $a = 15,\ c = 0.98,\ p = 3,\ q = 2$

FIGURE 3. The red basin is associated with the trivial fixed point. The big attraction basin of a 2-cycle breaks after homoclinic-heteroclinic bifurcations.

respectively in blue and green domains for $p = 2$ and $q = 3$. Close enough to $c = 1$ (in this case $c = 0.952$) only 2-cycles are stable for $a = 1$, here fixed points exist but are unstable after a flip bifurcation.

5.1. **Study of the phase plane.** Our numerical evidence includes the following: for fixed parameter values, we plot attraction basins of attractors. Two types of basins are illustrated in this section. We first choose the parameters so that two attractors coexist. The two attractors do not undergo identical sets of bifurcations in the parameter plane. While one attractor can experience flip bifurcation, the second one undergoes fold bifurcation and we do this by having $c = 0.952$, and negative

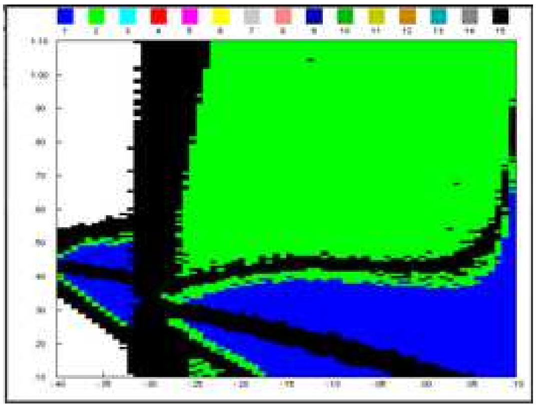


FIGURE 4. Bifurcation diagram in $(d, a)$ parameter plane.

values of $d = -0.07$ which is instructive, with the occurrence of a change of type of bifurcations inside the same basin after heteroclinic bifurcations.

For the value $a = 1$, one has the following situation: two 2-cycles $(P_1^1,\ P_1^2)$ and $(P_2^1,\ P_2^2)$ which interact dynamically with a flip saddle point $S_1$ in the phase plane and their basins are delimited by stable manifolds of the two points of the 2-cycle of saddle type $(C_2^1,\ C_2^2)$ and the unstable manifold of the flip saddle point $S_1$.

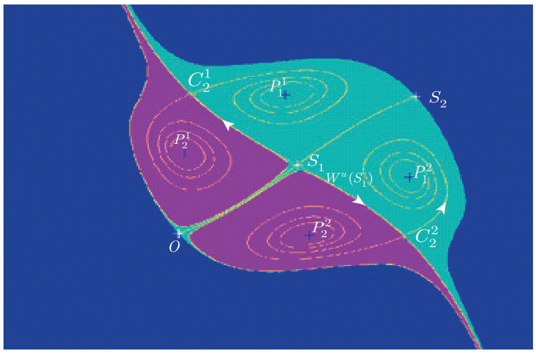



FIGURE 5. For the case $p = 2$, $q = 3$.

We decrease $d$, one has the following situation: the phase portrait of the recurrence $T_2$ at $d = -0.1$ is presented in Figure 6, the two stable 2-cycles exchange their associated saddles. It is in accordance with the bifurcation diagram in Figure 1 (a), the presence of cross-road area allows this change between attractors. For the case $p = 3$ and $q = 2$, we choose $c = 0.9$, $d = -0.32$, and $a = 1.5$, here also we have two 2-cycles which coexist with two flip saddle points and a regular saddle point located on their common frontier.

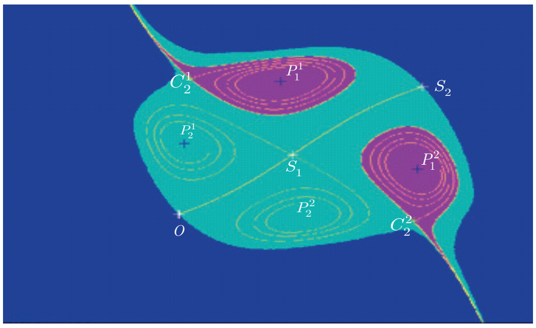



FIGURE 6. For the case $c = 0.952$, $d = -0.1$.

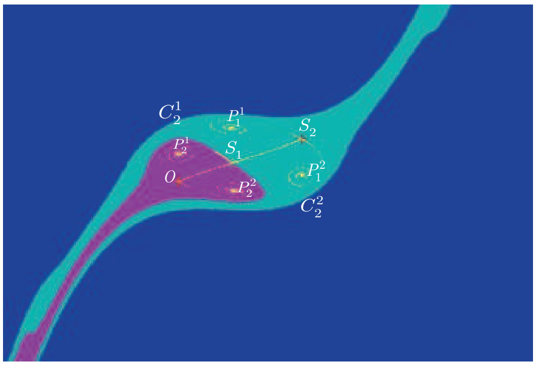



FIGURE 7. For the case $p = 3$, $q = 2$.

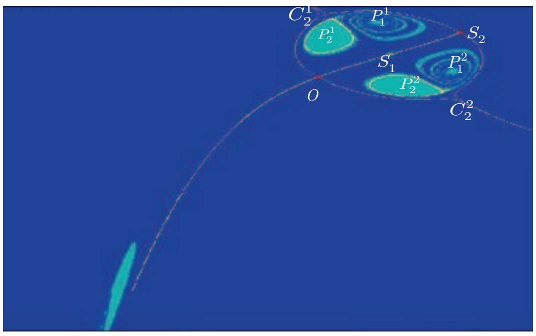



FIGURE 8. For the case $d = -0.355$, $p = 3$ and $q = 2$.

Here $a, c$ are constant but $d = -0.355$, the two basins are now nonconnected and bounded, and a Hopf bifurcation takes place for the 2-cycle $(P_2^1, P_2^2)$. We have a structural stable heteroclinic contour around basins.

## 6. Conclusion

Numerical explorations of cubic maps give interesting results, however, they reveal many intricate phenomena, that can only be understood by means of further specific investigation. A particularly rich bifurcation structure is detected near the limit value $c = 1$. Global bifurcations have important consequences as appearance of saddle connections and basins bifurcations. Heteroclinic bifurcations of saddle points, taking place on and inside the basins of attraction, this phenomenon provides a route for the appearance of nonconnected basins with saddles points located outside.

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# APPLICATION OF JACOBI POLYNOMIAL AND MULTIVARIABLE ALEPH-FUNCTION IN HEAT CONDUCTION IN NON-HOMOGENEOUS MOVING RECTANGULAR PARALLELEPIPED

DINESH KUMAR[1] AND FRÉDÉRIC AYANT[2,3]

Abstract. The present paper deals with an application of Jacobi polynomial and multivariable Aleph-function to solve the differential equation of heat conduction in non-homogeneous moving rectangular parallelepiped. The temperature distribution in the parallelepiped, moving in a direction of the length ($x$-axis) between the limits $x=-1$ and $x=1$ has been considered. The conductivity and the velocity have been assumed to be variables. We shall see two particular cases and the cases concerning Aleph-function of two variables and the $I$-function of two variables.

## 1. Introduction and Preliminaries

We suppose the parallelepiped has heat conductivity $K$, density $\rho$, diffusivity $k$ and specific heat $\sigma$. The partial differential equation satisfied by the temperature $v(x,y,z,t)$ at any time $t$ in a homogeneous parallelepiped bounded by the planes $y=0$ and $y=b$, $z=0$ and $z=c$, moves with a constant velocity $U$ in the direction of its length ($x$-axis) between the limits $x=-1$ and $x=1$, on the lines of Carslaw and Jaeger [4, page 155, (1)] is

$$k\left[\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]-U\frac{\partial v}{\partial x}-\frac{\partial v}{\partial t}=0, \tag{1.1}$$

where $k=\frac{K}{\rho\sigma}$. If we consider a non-homogeneous parallelepiped of variable conductivity $k'\left(1-x^2\right)$ and the velocity $k_0\left[(\alpha-\beta)+(\alpha+\beta)\,x\right]$, where $k', k_0, \alpha$ and $\beta$ are

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constants, the partial differential equation (1.1) reduces to

$$(1.2)\quad \frac{\partial v}{\partial t}=k_0\left[\left(1-x^2\right)\frac{\partial^2 v}{\partial x^2}+\left(\left(\beta-\alpha\right)-\left(\alpha+\beta+2\right)x\right)\frac{\partial v}{\partial x}\right]+k\left[\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right]=0,$$

where $k_0=\frac{k'}{\rho\sigma}$, $\operatorname{Re}(\alpha)>-1$, $\operatorname{Re}(\beta)>-1$.

As physical example we can consider the temperature distribution of the moving mercury parallelepiped between the planes $x=-1$, $x=1$, $y=0$ and $y=b$, $z=0$ and $z=c$ connected by two reservoirs of the mercury at the two ends. The variable flow in the mercury at the end $x=-1$ with a certain speed. The initial temperature distribution in the parallelepiped of mercury can be taken to be $f(x,y,z)$. The surfaces $y=0$ and $y=b$, $z=0$ and $z=c$ of the parallelepiped are supposed to be insulated. The ends $x=-1$ and $x=1$ of the mercury parallelepiped should also be insulated as the conductivity vanishes there.

## 2. Solution of the Problem

By assuming the solution of the partial differential equation (1.2) as $v(x,y,z,t)=X(x)\,Y(y)\,Z(z)\,T(t)$, the solution of the partial differential equation (1.2) reduces to

$$\frac{1}{T}\frac{dT}{dt}=\frac{k_0}{X}\left[\left(1-x^2\right)\frac{d^2X}{dx^2}+\left(-\alpha+\beta-\left(\alpha+\beta+2\right)x\right)\frac{dX}{dx}\right]+\frac{k}{Y}\frac{d^2Y}{dy^2}+\frac{k}{Z}\frac{d^2Z}{dz^2}.$$

Now, taking

$$\frac{k_0}{X}\left[(1-x^2)\frac{d^2X}{dx^2}+\left(-\alpha+\beta-\left(\alpha+\beta+2\right)x\right)\frac{dX}{dx}\right]=-\,k_0 n\left(n+\alpha+\beta+1\right),$$

$$\frac{k}{Y}\frac{d^2Y}{dy^2}=-\,k\lambda^2,$$

and $\frac{k}{Z}\frac{d^2Z}{dz^2}=-k\upsilon^2$, $\lambda$, $\upsilon$ being constants, $n$ being positive integer, we obtain the following equations

$$(2.1)\quad \left(1-x^2\right)\frac{d^2X}{dx^2}+\left(-\alpha+\beta-\left(\alpha+\beta+2\right)x\right)\frac{dX}{dx}+n\left(n+\alpha+\beta+1\right)X=0,$$

$$(2.2)\quad \frac{d^2Y}{dy^2}+\lambda^2 y=0,$$

$$(2.3)\quad \frac{d^2Z}{dz^2}+\upsilon^2 z=0,$$

and

$$(2.4)\quad \frac{dT}{dt}=\left[-n\left(n+\alpha+\beta+1\right)-k\left(\lambda^2+\upsilon^2\right)\right]T.$$

The (2.1) is the differential equation of Jacobi polynomials and its solution is

$$X=P_n^{(\alpha,\beta)}(x).$$

The solution of (2.2), (2.3) and (2.4) are

$$\begin{aligned} Y =& A \cos \lambda y + B \sin \lambda y, \\ Z =& C \cos \upsilon z + D \sin \upsilon z \end{aligned}$$

and

$$T = E \exp \left[ - \left( k_0 \, n \left( n + \alpha + \beta + 1 \right) + k \left( \lambda^2 + \upsilon^2 \right) \right) t \right],$$

where $A$, $B$, $C$, $D$, $E$ are constants.

Hence, the general solution of (1.2), the temperature distribution at any point $M\left(x, y, z\right)$ of the parallelepiped at time $t$ is given by

$$\begin{aligned} v \left(x, y, z, t\right) =& \exp \left[ - \left( k_0 \, n \left( n + \alpha + \beta + 1 \right) + k \left( \lambda^2 + \upsilon^2 \right) \right) t \right] P_n^{(\alpha, \beta)} \left(x\right) \\ & \times \left[ A \cos \lambda y + B \sin \lambda y \right] \left[ C \cos \upsilon z + D \sin \upsilon z \right], \end{aligned} \tag{2.5}$$

if no heats flows from the surfaces $y = 0$ and $y = b$, $z = 0$ and $z = c$, $\left(\frac{\partial v}{\partial y}\right)_{y=b=0} = 0$, $\left(\frac{\partial v}{\partial z}\right)_{z=c=0} = 0$ for all $x$ and $t$. These demand $B = 0$, $D = 0$, $\lambda = \frac{m\pi}{b}$, $\upsilon = \frac{l\pi}{c}$, where $m, l = 0, 1, 2, \ldots$.

Therefore, the solution (2.5) reduces to

$$\begin{aligned} v \left(x, y, z, t\right) = & \sum_{n,m,l=0}^{\infty} A_{nml} \exp \left[ - \left( k_0 \, n \left( n + \alpha + \beta + 1 \right) + k \left( \lambda^2 + \upsilon^2 \right) \right) \, t \right] \\ & \times P_n^{(\alpha, \beta)} \left(x\right) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z. \end{aligned}$$

Here, $\mathrm{Re}\left(\alpha\right) > -1$, $\mathrm{Re}\left(\beta\right) > -1$ and $A_{nml}$ are constants. If the initial temperature distribution in parallelepiped is given by

$$\sum_{n,m,l=0}^{\infty} A_{nml} P_n^{(\alpha, \beta)} \left(x\right) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z. \tag{2.6}$$

Now, multiplying both sides of (2.6) by $\left(1 - x\right)^{\alpha} \left(1 + x\right)^{\beta} \; P_n^{(\alpha, \beta)} \left(x\right) \cos \frac{m\pi}{b} \; y \cos \frac{l\pi}{c} z$, integrating both sides between $x = -1$ and $x = 1$, $y = 0$ and $y = b$, $z = 0$ and $z = c$, and applying the result [5, Vol. II. Page 285, (5)]

$$\int_{-1}^{1} \left(1 - x\right)^{\alpha} \left(1 + x\right)^{\beta} \left[ P_n^{(\alpha, \beta)} \left(x\right) \right]^2 \, \mathrm{d}x = \frac{2^{\alpha+\beta+1} \Gamma \left(\alpha + n + 1\right) \Gamma \left(\beta + n + 1\right)}{n! \, \left(\alpha + \beta + 2n + 1\right) \Gamma \left(\alpha + \beta + n + 1\right)},$$

with $\mathrm{Re}\left(\alpha\right) > -1$, $\mathrm{Re}\left(\beta\right) > -1$, we obtain

$$\begin{aligned} A_{mnl} =& \frac{n! \left(\alpha + \beta + 2n + 1\right) \Gamma \left(\alpha + \beta + n + 1\right)}{2^{\alpha+\beta+1} \Gamma \left(\alpha + n + 1\right) \Gamma \left(\beta + n + 1\right)} \\ & \times \int_0^c \int_0^b \int_{-1}^{1} \left(1 - x\right)^{\alpha} \left(1 + x\right)^{\beta} \; P_n^{(\alpha, \beta)} \left(x\right) \cos \frac{m\pi}{b} y \cos \frac{l\pi}{c} z \, f\left(x, y, z\right) \, dx \, dy \, dz. \end{aligned}$$

Hence, the temperature distribution in the non-homogeneous moving rectangular parallelepiped is given by

(2.7)

$$\begin{aligned}
&v\left(x,y,z,t\right)\\
=&\frac{1}{2^{\alpha+\beta-1}bc}\sum_{n,m,l=0}^{\infty}\frac{n!\,\left(\alpha+\beta+2n+1\right)\Gamma\left(\alpha+\beta+n+1\right)}{2^{\alpha+\beta+1}\Gamma\left(\alpha+n+1\right)\Gamma\left(\beta+n+1\right)}\\
&\times\ \exp\left[-\left(k_0\,n\left(n+\alpha+\beta+1\right)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right]\ P_n^{(\alpha,\beta)}\left(x\right)\,\cos\frac{m\pi}{b}\,y\,\cos\frac{l\pi}{c}z\\
&\times\ \int_0^c\int_0^b\int_{-1}^1\left(1-x\right)^{\alpha}\left(1+x\right)^{\beta}\ P_n^{(\alpha,\beta)}\left(x\right)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}\,z\,f\left(x,y,z\right)\,dx\,dy\,dz,
\end{aligned}$$

with $\mathrm{Re}\left(\alpha\right)>-1$, $\mathrm{Re}\left(\beta\right)>-1$.

*Remark* 2.1. Prasad and Maurya [10] have given application of Jacobi polynomial and multivariable $H$-function in heat conduction in non-homogeneous moving rectangular parallelepiped; Simões et al. [9] have studied Green's functions for heat conduction for unbounded and bounded rectangular spaces.

## 3. Multivariable Aleph-Function

For an illustration, if we take $f\left(x,y,z\right)=f_1\left(x\right)f_2\left(y\right)f_3\left(z\right)$, $f_2\left(y\right)=e^{-\mu y}$, $f_3\left(z\right)=e^{-\delta z}$ and $f_1\left(x\right)$ to be the most general special function in the form of multivariable Aleph-function.

The multivariable Aleph-function is a generalization of the multivariable $H$-function defined by Srivastava and Panda [14,15]. The multivariable Aleph-function is defined by means of the multiple contour integral [3,7]:

$$\begin{aligned}
&\aleph\left(z_1,\ldots,z_r\right)\\
=&\aleph^{0,\,\mathfrak{n}:m_1,n_1,\ldots,m_r,n_r}_{p_i,q_i,\tau_i;R:p_{i^{(1)}},q_{i^{(1)}},\tau_{i^{(1)}};R^{(1)};\ldots;p_{i^{(r)}},q_{i^{(r)}};\tau_{i^{(r)}};R^{(r)}}\left(\begin{array}{c|l} z_1 & \left[\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,\mathfrak{n}}\right],\\ \vdots & \\ z_r & \ldots,\end{array}\right.\\
&\left[\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{\mathfrak{n}+1,p_i}\right]:\left[\left(c_j^{(1)}\right),\left(\gamma_j^{(1)}\right)_{1,n_1}\right],\left[\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)},\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_i^{(1)}}\right];\\
&\left[\tau_i\left(b_{ji};\beta_{ji}^{(1)},\cdots,\beta_{ji}^{(r)}\right)_{m+1,q_i}\right]:\left[\left(d_j^{(1)}\right),\left(\delta_j^{(1)}\right)_{1,m_1}\right],\left[\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)},\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_i^{(1)}}\right];\\
&\quad\ldots;\left[\left(c_j^{(r)}\right),\left(\gamma_j^{(r)}\right)_{1,n_r}\right],\left[\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)},\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_i^{(r)}}\right]\\
&\quad\ldots;\left[\left(d_j^{(r)}\right),\left(\delta_j^{(r)}\right)_{1,m_r}\right],\left[\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)},\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_i^{(r)}}\right]\left.\vphantom{\begin{array}{c}z_1\\ \vdots\\ z_r\end{array}}\right)
\end{aligned}$$

(3.1)

$$=\frac{1}{\left(2\pi\omega\right)^r}\int_{L_1}\cdots\int_{L_r}\psi\left(s_1,\ldots,s_r\right)\prod_{k=1}^{r}\theta_k\left(s_k\right)\,z_k^{s_k}\,ds_1\cdots ds_r,$$

with $\omega = \sqrt{-1}$,

$$\psi(s_1,\ldots,s_r) = \frac{\prod_{j=1}^{\mathfrak{n}} \Gamma\left(1-a_j+\sum_{k=1}^{r}\alpha_j^{(k)}s_k\right)}{\sum_{i=1}^{R}\left[\tau_i \prod_{j=\mathfrak{n}+1}^{p_i}\Gamma\left(a_{ji}-\sum_{k=1}^{r}\alpha_{ji}^{(k)}s_k\right)\prod_{j=1}^{q_i}\Gamma\left(1-b_{ji}+\sum_{k=1}^{r}\beta_{ji}^{(k)}s_k\right)\right]}$$

and

$$\theta_k(s_k) = \frac{\prod_{j=1}^{m_k}\Gamma\left(d_j^{(k)}-\delta_j^{(k)}s_k\right)\prod_{j=1}^{n_k}\Gamma\left(1-c_j^{(k)}+\gamma_j^{(k)}s_k\right)}{\sum_{i^{(k)}=1}^{R^{(k)}}\left[\tau_{i^{(k)}}\prod_{j=m_k+1}^{q_{i^{(k)}}}\Gamma\left(1-d_{ji^{(k)}}^{(k)}+\delta_{ji^{(k)}}^{(k)}s_k\right)\prod_{j=n_k+1}^{p_{i^{(k)}}}\Gamma\left(c_{ji^{(k)}}^{(k)}-\gamma_{ji^{(k)}}^{(k)}s_k\right)\right]}.$$

For more details, see Ayant [1]. The condition for absolute convergence of multiple Mellin-Barnes type contour can be obtained by extension of the corresponding conditions for multivariable $H$-function given by

$$|\arg z_k| < \frac{1}{2}A_i^{(k)}\pi,$$

where

$$A_i^{(k)} = \sum_{j=1}^{\mathfrak{n}}\alpha_j^{(k)} - \tau_i\sum_{j=\mathfrak{n}+1}^{p_i}\alpha_{ji}^{(k)} - \tau_i\sum_{j=1}^{q_i}\beta_{ji}^{(k)} + \sum_{j=1}^{n_k}\gamma_j^{(k)} - \tau_{i^{(k)}}\sum_{j=n_k+1}^{p_{i^{(k)}}}\gamma_{ji^{(k)}}^{(k)}$$

$$+\sum_{j=1}^{m_k}\delta_j^{(k)} - \tau_{i^{(k)}}\sum_{j=m_k+1}^{q_{i^{(k)}}}\delta_{ji^{(k)}}^{(k)} > 0, \tag{3.2}$$

with $k = 1,\ldots,r$, $i = 1,\ldots,R$, $i^{(k)} = 1,\ldots,R^{(k)}$.

The complex numbers $z_i$ are not zero. Throughout this document, we assume the existence and absolute convergence conditions of the multivariable Aleph-function. For convenience, we shall use the following notations in this paper.

$$V = m_1, n_1;\ \ldots;\ m_r, n_r$$
$$W = p_{i^{(1)}}, q_{i^{(1)}}, \tau_{i^{(1)}};\ R^{(1)};\ldots;p_{i^{(r)}}, q_{i^{(r)}}, \tau_{i^{(r)}};\ R^{(r)},$$
$$A = \left\{\left(a_j;\alpha_j^{(1)},\ldots,\alpha_j^{(r)}\right)_{1,n}\right\},\left\{\tau_i\left(a_{ji};\alpha_{ji}^{(1)},\ldots,\alpha_{ji}^{(r)}\right)_{n+1,p_i}\right\}:\left\{\left(c_j^{(1)};\gamma_j^{(1)}\right)_{1,n_1}\right\},$$
$$\left\{\tau_{i^{(1)}}\left(c_{ji^{(1)}}^{(1)};\gamma_{ji^{(1)}}^{(1)}\right)_{n_1+1,p_{i^{(1)}}}\right\};\ldots;\left\{\left(c_j^{(r)};\gamma_j^{(r)}\right)_{1,n_r}\right\},\left\{\tau_{i^{(r)}}\left(c_{ji^{(r)}}^{(r)};\gamma_{ji^{(r)}}^{(r)}\right)_{n_r+1,p_{i^{(r)}}}\right\},$$
$$B = \left\{\tau_i(b_{ji};\beta_{ji}^{(1)},\ldots,\beta_{ji}^{(r)})_{m+1,q_i}\right\}:\left\{\left(d_j^{(1)};\delta_j^{(1)}\right)_{1,m_1}\right\},\left\{\tau_{i^{(1)}}\left(d_{ji^{(1)}}^{(1)};\delta_{ji^{(1)}}^{(1)}\right)_{m_1+1,q_{i^{(1)}}}\right\};$$
$$\ldots;\left\{(d_j^{(r)};\delta_j^{(r)})_{1,m_r}\right\},\left\{\tau_{i^{(r)}}\left(d_{ji^{(r)}}^{(r)};\delta_{ji^{(r)}}^{(r)}\right)_{m_r+1,q_{i^{(r)}}}\right\}.$$

Let

$$f_1(x) = \aleph_{p_i,q_i,\tau_i;R:W}^{0,\mathbf{n}:V}\left(\begin{array}{c|c} p_1(1-x)^{m_1'}(1+x)^{m_1''} & A \\ \vdots & \vdots \\ p_r(1-x)^{m_r'}(1+x)^{m_r''} & B \end{array}\right).$$

Substituting for $f_1(x), f_2(y)$ and $f_3(z)$ in equation (2.7), which is justifiable under the given conditions, we evaluate the $y$ and $z$-integrals, first and the write the multivariable Aleph-function into the Mellin-Barnes contour integral with the help of (3.1), apply the result [5, Vol. II, page 284, (3)],

$$\begin{aligned}(3.3)\quad &\int_{-1}^{1}(1-x)^{l}(1+x)^{\sigma}\; P_n^{(\alpha,\beta)}(x)dx\\ &=\frac{2^{l+\sigma+1}\Gamma(l+1)\Gamma(\sigma+1)}{\Gamma(l+\sigma+2)}\times\,{}_3F_2\left(-n,\alpha+\beta+n+1,l+1;\alpha+1,l+\sigma+2;1\right),\end{aligned}$$

with $\mathrm{Re}(\alpha)>-1$, $\mathrm{Re}(\beta)>-1$, and finally interpret the resulting $\Gamma$-functions with the definition of multivariable Aleph-function. The temperature distribution in a non-homogeneous moving rectangular parallelepiped is then

$$\begin{aligned}v(x,y,z,t)=&\frac{\mu\delta e^{-(\mu b+\delta c)}\Gamma(\alpha+1)}{bc2^{\alpha+\beta-1}}\sum_{n,m,l,N=0}^{\infty}\frac{(1-(-)^{m})\left(1-(-)^{l}\right)}{(\mu^2+m^2\pi^2/b^2)(\delta^2+l^2\pi^2/c^2)}\\ &\times\frac{n!(\alpha+\beta+2n+1)\Gamma(\alpha+\beta+n+N+1)\Gamma(-n+N)}{N!\Gamma(\alpha+N+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)\Gamma(-n)}\\ &\times\exp\left[-\left\{k_0\,n(n+\alpha+\beta+1)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right\}t\right]\\ &\times P_n^{(\alpha,\beta)}(x)\cos\frac{m\pi}{b}y\cos\frac{l\pi}{c}z\,\aleph^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c}p_1 2^{m_1'+m_1''}\\ \vdots\\ p_r 2^{m_r'+m_r''}\end{array}\right|\\ &\left.\begin{array}{c}(-\alpha-N:m_1',\ldots,m_r'),(-\beta:m_1'',\ldots,m_r''),A\\ \vdots\\ (-\alpha-\beta-N-1:m_1'+m_1'',\ldots,m_r'+m_r''),B\end{array}\right).\end{aligned}\tag{3.4}$$

Provide that $\mathrm{Re}(\alpha)>-1$, $\mathrm{Re}(\beta)>-1$, $m_i'$, $m_i''>0$ for $i=1,\ldots,r$, and

$$\mathrm{Re}(\alpha+1)+\sum_{i=1}^{r}m_i'\min_{1\leqslant j\leqslant m_i}\mathrm{Re}\left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,$$

$$\mathrm{Re}(\beta+1)+\sum_{i=1}^{r}m_i''\min_{1\leqslant j\leqslant m_i}\mathrm{Re}\left[\left(\frac{d_j^{(i)}}{\delta_j^{(i)}}\right)\right]>0,$$

$|\arg p_k|<\frac{1}{2}A_i^{(k)}\pi$, where $A_i^{(k)}$ is defined by (3.2).

*Remark* 3.1. For detail and applications of Aleph-function, the reader can refer recent work [2, 8, 16].

## 4. Particular Cases

**(a)** When the rectangular parallelepiped moves with uniform velocity, the partial differential equation (1.2) reduces to the unsteady case of the partial differential equation (1) of Carslaw and Jaeger [4] with no radiation but with variable conductibility. We have $\alpha+\beta=0$ and

$$\frac{\partial v}{\partial t}=k_0\left(\left(1-x^2\right)\frac{\partial^2 v}{\partial x^2}+(\beta-\alpha-2x)\frac{\partial v}{\partial x}\right)+k\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=0,$$

and the temperature distribution in the parallelepiped between $x=-1$ and $x=1$, $y=0$ and $y=b$, $z=0$ and $z=c$ is given by

$$\begin{aligned}
&v\left(x,y,z,t\right)\\
&=\frac{2\mu\delta e^{-(\mu b+\delta c)}\Gamma\left(\alpha+1\right)}{bc}\sum_{n,m,l,N=0}^{\infty}\frac{\left(1-(-)^m\right)\left(1-(-)^l\right)}{\left(\mu^2+m^2\pi^2/b^2\right)\left(\delta^2+l^2\pi^2/c^2\right)}\\
&\quad\times\frac{n!\,\left(2n+1\right)\Gamma\left(n+N+1\right)\Gamma\left(-n+N\right)}{N!\,\Gamma\left(\alpha+N+1\right)\Gamma\left(\alpha+n+1\right)\Gamma\left(\beta+n+1\right)\Gamma\left(-n\right)}\\
&\quad\times\exp\left[-\left(k_0\,n\left(n+1\right)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right]P_n^{(\alpha,\beta)}(x)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}z\\
&\quad\times\aleph^{0,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c}p_1 2^{m'_1+m''_1}\\ \vdots\\ p_r 2^{m'_r+m''_r}\end{array}\middle|\begin{array}{c}\left(-\alpha-N:m'_1,\ldots,m'_r\right),\left(-\beta:m''_1,\ldots,m''_r\right),A\\ \vdots\\ \left(-N-1:m'_1+m''_1,\ldots,m'_r+m''_r\right),\,B\end{array}\right),
\end{aligned}$$

under the same condition that (3.4) with $\alpha+\beta=0$.

**(b)** When the parallelepiped is stationary between $x=-1$ and $x=1$, we have $\alpha=\beta=0$ and the partial differential equation (1.2) reduces to

$$\frac{\partial v}{\partial t}=k_0\left(\left(1-x^2\right)\frac{\partial^2 v}{\partial x^2}-2x\frac{\partial v}{\partial x}\right)+k\left(\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)=0,$$

and the temperature distribution in the parallelepiped between $x=-1$ and $x=1$, $y=0$ and $y=b$, $z=0$ and $z=c$ is given by

$$\begin{aligned}
&v\left(x,y,z,t\right)\\
&=\frac{2\mu\delta e^{-(\mu b+\delta c)}}{bc}\sum_{n,m,l,N=0}^{\infty}\frac{\left(1-(-)^m\right)\left(1-(-)^l\right)}{\left(\mu^2+m^2\pi^2/b^2\right)\left(\delta^2+l^2\pi^2/c^2\right)}\\
&\times\frac{n!\,\left(2n+1\right)\Gamma\left(n+N+1\right)\Gamma\left(-n+N\right)}{N!\,\Gamma\left(N+1\right)\left(\Gamma(n+1)\right)^2\Gamma\left(-n\right)}\,\exp\left[-\left(k_0\,n\left(n+1\right)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right]\\
&\quad\times P_n(x)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}z
\end{aligned}$$

$$\times\ \aleph^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} p_1 2^{m_1'+m_1''} & \left(-N:m_1',\cdots,m_r'\right),\left(0:m_1'',\ldots,m_r''\right),\,A \\ \vdots & \vdots \\ p_r 2^{m_r'+m_r''} & \left(-N-1:m_1'+m_1'',\ldots,m_r'+m_r''\right),\,B \end{array}\right),$$

where $P_n(x)$ is a Legendre's polynomial, under the same condition that (3.4) with $\alpha=\beta=0$.

## 5. Aleph-Function of Two Variables

If $r=2$, the multivariable Aleph-function reduces to Aleph-function of two variables defined by Sharma [13](see also, [6]) and the general solution is

$$\begin{aligned}
&v\left(x,y,z,t\right)\\
=&\frac{\mu\delta e^{-(\mu b+\delta c)}\Gamma\left(\alpha+1\right)}{bc2^{\alpha+\beta-1}}\sum_{n,m,l,N=0}^{\infty}\frac{\left(1-(-)^m\right)\left(1-(-)^l\right)}{\left(\mu^2+m^2\pi^2/b^2\right)\left(\delta^2+l^2\pi^2/c^2\right)}\\
&\times\ \frac{n\,!\left(\alpha+\beta+2n+1\right)\Gamma\left(\alpha+\beta+n+N+1\right)\Gamma\left(-n+N\right)}{N!\,\Gamma\left(\alpha+N+1\right)\Gamma\left(\alpha+n+1\right)\Gamma\left(\beta+n+1\right)\Gamma\left(-n\right)}\\
&\times\ \exp\left[-\left(k_0\,n\left(n+\alpha+\beta+1\right)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right]P_n^{(\alpha,\beta)}(x)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}z\\
&\times\ \aleph^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,\tau_i;R:W}\left(\begin{array}{c|c} p_1 2^{m_1'+m_1''} & \left(-\alpha-N:m_1',m_2'\right),\left(-\beta:m_1'',m_2''\right),\,A \\ \vdots & \vdots \\ p_2 2^{m_2'+m_2''} & \left(-\alpha-\beta-N-1:m_1'+m_1'',m_2'+m_2''\right),\,B \end{array}\right),
\end{aligned}$$

under the same condition that (3.4) with $r=2$.

## 6. $I$-Function of Two Variables

If $r=2$ and $\tau_i,\tau_{i'},\tau_{i''}\to 1$ the multivariable Aleph-function reduces to $I$-function of two variables defined by Sharma and Mishra [12] (see also, [11]) and the general solution is

$$\begin{aligned}
&v\left(x,y,z,t\right)\\
=&\frac{\mu\delta e^{-(\mu b+\delta c)}\Gamma\left(\alpha+1\right)}{bc2^{\alpha+\beta-1}}\sum_{n,m,l,N=0}^{\infty}\frac{\left(1-(-)^m\right)\left(1-(-)^l\right)}{\left(\mu^2+m^2\pi^2/b^2\right)\left(\delta^2+l^2\pi^2/c^2\right)}\\
&\times\ \frac{n!\,\left(\alpha+\beta+2n+1\right)\Gamma\left(\alpha+\beta+n+N+1\right)\Gamma\left(-n+N\right)}{N!\,\Gamma\left(\alpha+N+1\right)\Gamma\left(\alpha+n+1\right)\Gamma\left(\beta+n+1\right)\Gamma\left(-n\right)}\\
&\times\ \exp\left[-\left(k_0\,n\left(n+\alpha+\beta+1\right)+k\pi^2\left(\frac{m^2}{b^2}+\frac{l^2}{c^2}\right)\right)t\right]\,P_n^{(\alpha,\beta)}(x)\,\cos\frac{m\pi}{b}y\,\cos\frac{l\pi}{c}z\\
&\times\ I^{0,\,\mathbf{n}+2:V}_{p_i+2,q_i+1,R:W}\left(\begin{array}{c|c} p_1 2^{m_1'+m_1''} & \left(-\alpha-N:m_1',m_2'\right),\left(-\beta:m_1'',m_2''\right),\,A \\ \vdots & \vdots \\ p_2 2^{m_2'+m_2''} & \left(-\alpha-\beta-N-1:m_1'+m_1'',m_2'+m_2''\right),\,B \end{array}\right),
\end{aligned}$$

under the same condition that (3.4) with $r = 2$ and $\tau_i, \tau_{i'}, \tau_{i''} \to 1$.

## 7. Concluding Remarks

Specializing the parameters of the multivariable Aleph-function, we can obtain a large number of results involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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# OPTIMIZATIONS ON STATISTICAL HYPERSURFACES WITH CASORATI CURVATURES

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Abstract. In the present paper, we study Casorati curvatures for statistical hypersurfaces. We show that the normalized scalar curvature for any real hypersurface (i.e., statistical hypersurface) of a holomorphic statistical manifold of constant holomorphic sectional curvature $k$ is bounded above by the generalized normalized $\delta-$Casorati curvatures and also consider the equality case of the inequality. Some immediate applications are discussed.

## 1. Introduction

In 1985, a notion of statistical manifold has been studied by Amari [1]. The abstract generalizations of statistical models are considered as the statistical manifolds. The geometry of statistical manifolds lies at a junction of several branches of geometry (information geometry, affine differential geometry and Hessian geometry). A statistical structure can be considered as a generalization of a Riemannian structure (a pair of a Riemannian metric and its Levi-Civita connection). It includes the notion of dual connection, also called conjugate connection. The theory of statistical manifold and its statistical submanifold plays a role of central importance in many research fields of differential geometry.

Recently, H. Furuhata investigated the existence of complex structures on statistical manifolds and introduced the concept of holomorphic statistical manifold, as the statistical counterpart of the notion of complex manifold (see [11, 12]). Similarly, by putting a natural affine connection to a Sasakian manifold and a Kenmotsu manifold, Furuhata defined a Sasakian statistical manifold [13] and a Kenmotsu statistical

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manifold [14]. The theory of statistical manifolds and their statistical submanifolds is a very recent geometry. Therefore, it attracts the geometers and several interesting results have been obtained by many of them (for example [3–5, 21, 22, 26, 28]).

The Casorati curvature has been defined by F. Casorati [6] as the normalized square of the length of the second fundamental form of a submanifold of a Riemannian manifold. This notion extends the concept of the principal direction of a hypersurface of a Riemannian manifold. This curvature, which is of interest in computer vision, was preferred by Casorati over the traditional curvatures because it seems to correspond better with the common intuition of curvature. Several geometers have found geometrical interpretation and significance of the (extrinsic) Casorati curvatures. Therefore, it follows that it is of great interest to establish a family of optimal Casorati inequalities for different submanifolds with any codimension of different ambient space forms (for example [9, 10, 15, 16, 18, 19, 24, 25, 27])

In this paper, we obtain a family of optimal inequalities which relate the normalized scalar curvature with the Casorati curvature for statistical hypersurfaces of holomorphic statistical manifolds of constant holomorphic sectional curvature. Equality cases are also verified. Such inequalities were recently obtained for a statistical submanifold, which is obviously a particular class of statistical hypersurfaces. See, for instance [2, 8, 17, 20]. We mention that the ambient spaces in the above mentioned articles are different as compared to the ambient space (that is, a holomorphic statistical manifold of constant holomorphic sectional curvature) in our work, namely a quaternion Kahler-like statistical space form, a Kenmotsu statistical manifold, a statistical manifold, and a Sasakian statistical manifold, respectively.

## 2. Statistical Manifold and its Submanifolds

This section is fully devoted to a brief review of several fundamental formulae and some definitions which are required later.

**Definition 2.1** ([12])**.** Let $\overline{\nabla}$ be an affine connection of Riemannian manifold $(\overline{\mathcal{M}}, \overline{g})$ with Riemannian metric $\overline{g}$ on $\overline{\mathcal{M}}$.

(a) The affine connection $\overline{\nabla}^*$ of $\overline{\mathcal{M}}$ defined by

$$\mathcal{Z}\overline{g}(\mathcal{X}, \mathcal{Y}) = \overline{g}(\overline{\nabla}_{\mathcal{Z}}\mathcal{X}, \mathcal{Y}) + \overline{g}(\mathcal{X}, \overline{\nabla}^*_{\mathcal{Z}}\mathcal{Y}),$$

for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$ is known as the dual connection of $\overline{\nabla}$ with respect to $\overline{g}$.

(b) The triplet $(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g})$ is known as a statistical manifold if the torsion tensor field of $\overline{\nabla}$ vanishes and $\overline{\nabla}\overline{g} \in \Gamma(T\overline{\mathcal{M}}^{(0,3)})$ is symmetric.

*Remark* 2.1. If $(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g})$ is a statistical manifold, so is $(\overline{\mathcal{M}}, \overline{\nabla}^*, \overline{g})$. The dual connections $\overline{\nabla}$ and $\overline{\nabla}^*$ of $\overline{\mathcal{M}}$ satisfy (see [12]) $(\overline{\nabla}^*)^* = \overline{\nabla}$ and $2\overline{\nabla}^0 = \overline{\nabla} + \overline{\nabla}^*$, where $\overline{\nabla}^0$ is Levi-Civita connection for $\overline{\mathcal{M}}$ of $\overline{g}$.

*Example* 2.1. Let $(\overline{\mathcal{M}}, \overline{g})$ be a family of exponential distributions of mean 0

$$\overline{\mathcal{M}} := \{p(u, \Phi) \mid p(u, \Phi) = \Phi e^{-\Phi u}, u \in [0, \infty), \Phi \in (0, \infty)\},$$

a Riemannian metric is given by $\overline{g} := \Phi^{-2}(d\Phi)^2$, and an $\alpha-$connection ($\alpha \in \mathbb{R}$) on $\overline{\mathcal{M}}$ is defined by

$$\overline{\nabla}^{\alpha}_{\frac{\partial}{\partial \Phi}} \frac{\partial}{\partial \Phi} = (\alpha - 1)\Phi^{-1} \frac{\partial}{\partial \Phi}.$$

Then, $(\overline{\mathcal{M}}, \overline{\nabla}^{\alpha}, \overline{g})$ is a 1-dimensional statistical manifold.

We remark that one can also construct examples for higher dimension by defining Fisher information metric and $\alpha-$connection on a family of statistical distribution (for example [12]).

**Definition 2.2** ([12])**.** Let $\left(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g}\right)$ be a statistical manifold and $\mathcal{M}$ be a submanifold of $\overline{\mathcal{M}}$. Then $\left(\mathcal{M}, \nabla, g\right)$ is also a statistical manifold with the induced statistical structure $(\nabla, g)$ on $\mathcal{M}$ from $(\overline{\nabla}, \overline{g})$ and we call $\left(\mathcal{M}, \nabla, g\right)$ as a statistical submanifold in $\left(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g}\right)$.

**Definition 2.3** ([12])**.** Let $\left(\overline{\mathcal{M}}, \overline{g}, \mathcal{J}\right)$ be a Kaehler manifold and $\overline{\nabla}$ be an affine connection on $\overline{\mathcal{M}}$. Then $\left(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g}, \mathcal{J}\right)$ is said to be a holomorphic statistical manifold if

(a) $\left(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g}\right)$ is a statistical manifold, and
(b) a $2-$form $\varpi$ on $\overline{\mathcal{M}}$, given by $\varpi(\mathcal{X}, \mathcal{Y}) = \overline{g}(\mathcal{X}, \mathcal{J}\mathcal{Y})$ for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$, is $\overline{\nabla}-$parallel, that is, $\overline{\nabla}\varpi = 0$.

For a holomorphic statistical manifold $\left(\overline{\mathcal{M}}, \overline{g}, \mathcal{J}\right)$, we have the following relation (see [12]) $\overline{\nabla}_{\mathcal{X}}(\mathcal{J}\mathcal{Y}) = \mathcal{J}\overline{\nabla}^{*}_{\mathcal{X}}\mathcal{Y}$ for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$.

**Lemma 2.1** ([11])**.** *Let $(\overline{\mathcal{M}}, \overline{g}, \mathcal{J})$ be a Kaehler manifold and a connection $\overline{\nabla}$ is defined as $\overline{\nabla} := \nabla^{\overline{g}} + K$, where $K$ is a $(1,2)-$tensor field satisfying the following conditions:*

$$\begin{aligned} K(\mathcal{X}, \mathcal{Y}) =& K(\mathcal{Y}, \mathcal{X}), \\ \overline{g}(K(\mathcal{X}, \mathcal{Y}), \mathcal{Z}) =& \overline{g}(\mathcal{Y}, K(\mathcal{X}, \mathcal{Z})), \end{aligned} \tag{2.1}$$

*and $K(\mathcal{X}, \mathcal{J}\mathcal{Y}) + \mathcal{J}K(\mathcal{X}, \mathcal{Y}) = 0$ for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$. Then, $(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g}, \mathcal{J})$ is a holomorphic statistical manifold.*

By following [26] and Lemma 2.1, we have the following examples.

*Example* 2.2 ([26]). Let $(\overline{g}, \mathcal{J})$ be a Kaehler structure on $\overline{\mathcal{M}}$. We take a vector field $U \in \Gamma(T\overline{\mathcal{M}})$ and set a tensor field $K_1 \in \Gamma(T\overline{\mathcal{M}}^{(1,2)})$ as follows:

$$\begin{aligned} K_1(\mathcal{X}, \mathcal{Y}) =& \Big[\overline{g}(\mathcal{J}U, \mathcal{X})\overline{g}(\mathcal{J}U, \mathcal{Y}) - \overline{g}(U, \mathcal{X})\overline{g}(U, \mathcal{Y})\Big]U \\ &+ \Big[\overline{g}(\mathcal{J}U, \mathcal{X})\overline{g}(U, \mathcal{Y}) + \overline{g}(U, \mathcal{X})\overline{g}(\mathcal{J}U, \mathcal{Y})\Big]\mathcal{J}U, \end{aligned}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$. Then, by simple computation, we see that $K_1$ satisfies three conditions of Lemma 2.1, and hence a holomorphic statistical manifold $(\overline{\mathcal{M}}, \overline{\nabla} := \nabla^{\overline{g}} + K_1, \overline{g}, \mathcal{J})$ is obtained.

*Example* 2.3 ([26]). For a Kaehler manifold $(\overline{\mathcal{M}}, \overline{g}, \mathcal{J})$, we take a vector field $U \in \Gamma(T\overline{\mathcal{M}})$ and set $K_2$ as follows:

$$\begin{aligned}K_2(\mathcal{X}, \mathcal{Y}) =& \Big[\overline{g}(U, \mathcal{J}\mathcal{X})\overline{g}(U, \mathcal{J}\mathcal{Y}) - \overline{g}(U, \mathcal{X})\overline{g}(U, \mathcal{Y}) \\ &- \overline{g}(U, \mathcal{J}\mathcal{X})\overline{g}(U, \mathcal{Y}) - \overline{g}(U, \mathcal{X})\overline{g}(U, \mathcal{J}\mathcal{Y})\Big]U \\ &+ \Big[\overline{g}(U, \mathcal{X})\overline{g}(U, \mathcal{Y}) - \overline{g}(U, \mathcal{J}\mathcal{X})\overline{g}(U, \mathcal{J}\mathcal{Y}) \\ &- \overline{g}(U, \mathcal{J}\mathcal{X})\overline{g}(U, \mathcal{Y}) - \overline{g}(U, \mathcal{X})\overline{g}(U, \mathcal{J}\mathcal{Y})\Big]\mathcal{J}U,\end{aligned}$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(T\overline{\mathcal{M}})$. Then $K_2 \in \Gamma(T\overline{\mathcal{M}}^{(1,2)})$ satisfies three conditions of Lemma 2.1 as in Example 2.2, and hence $(\overline{\mathcal{M}}, \overline{\nabla} := \nabla^{\overline{g}} + K_2, \overline{g}, \mathcal{J})$ becomes a holomorphic statistical manifold.

*Example* 2.4 ([26]). Let us consider a Kaehler manifold

$$(\overline{\mathcal{M}} = \{(u^1, u^2)' \in \mathbb{R}^2 \mid u^1 > 0\}, \overline{g}, \mathcal{J}),$$

where a Riemanian metric $\overline{g}$ and the standard complex structure $\mathcal{J}$ on $\overline{\mathcal{M}}$ are defined by $\overline{g} = u^1\{(du^1)^2 + (du^2)^2\}$ and $\mathcal{J}\partial_1 = \partial_2$, $\mathcal{J}\partial_2 = -\partial_1$, where $\partial_i = \frac{\partial}{\partial u^i}$ for $i = 1, 2$. Now, for any $\kappa \in \mathbb{R}$, we define a $(1,2)$-tensor field $K_3$ on $\mathbb{R}^2$ as follows:

$$K_3 = \sum_{i,j,l=1}^{2} k_{ij}^l \partial_l \otimes du^i \otimes du^j,$$

where $-k_{11}^1 = k_{12}^2 = k_{21}^2 = k_{22}^1 = \kappa$ and $k_{11}^2 = k_{12}^1 = k_{21}^1 = k_{22}^2 = 0$. Then $K_3$ satisfies all three conditions of Lemma 2.1, and hence we get a holomorphic statistical manifold $(\overline{\mathcal{M}}, \overline{\nabla} := \nabla^{\overline{g}} + K_3, \overline{g}, \mathcal{J})$, where an affine connection $\overline{\nabla}$ on $\overline{\mathcal{M}}$ is given by

$$\begin{aligned}\overline{\nabla}_{\partial_1}\partial_1 =& \left(\frac{1}{2}(u^1)^{-1} - \kappa\right)\partial_1, \\ \overline{\nabla}_{\partial_1}\partial_2 =& \overline{\nabla}_{\partial_2}\partial_1 = \left(\frac{1}{2}(u^1)^{-1} + \kappa\right)\partial_2, \\ \overline{\nabla}_{\partial_2}\partial_2 =& -\left(\frac{1}{2}(u^1)^{-1} - \kappa\right)\partial_1.\end{aligned}$$

Now, we pay attention to the concept of statistical hypersurface. Let $(\mathcal{M}, g)$ be a statistical hypersurface of a holomorphic statistical manifold $(\overline{\mathcal{M}}, \overline{g}, \mathcal{J})$. By the Kaehler structure $\mathcal{J}$, one can transfer any tangent vector field $\mathcal{X}$ on $\mathcal{M}$ in $\overline{\mathcal{M}}$ as follows: $\mathcal{J}\mathcal{X} = \mathcal{P}\mathcal{X} + u(\mathcal{X})\mathcal{N}$, where $\mathcal{P}\mathcal{X} = \tan(\mathcal{J}\mathcal{X})$ and $\mathcal{N}$ is a unit normal vector field on $\mathcal{M}$ in $\overline{\mathcal{M}}$.

Then, it naturally satisfies the following relations (see [12]):

$$\begin{cases} \mathcal{P}^2\mathcal{X} = -\mathcal{X} + u(\mathcal{X})\xi, \\ u(\xi) = 1, \\ \mathcal{P}\xi = 0. \end{cases}$$

The fundamental equations in the geometry of Riemannian submanifolds are the Gauss and Weingarten formulae and the equations of Gauss, Codazzi and Ricci (see [29]). In the statistical setting, Gauss and Weingarten formulae are, respectively, defined by [12]

$$\begin{cases} \overline{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + \varsigma(\mathcal{X},\mathcal{Y})\mathcal{N}, & \overline{\nabla}^*_{\mathcal{X}}\mathcal{Y} = \nabla^*_{\mathcal{X}}\mathcal{Y} + \varsigma^*(\mathcal{X},\mathcal{Y})\mathcal{N}, \\ \overline{\nabla}_{\mathcal{X}}\mathcal{N} = -\Lambda(\mathcal{X}) + \nu(\mathcal{X})\mathcal{N}, & \overline{\nabla}^*_{\mathcal{X}}\mathcal{N} = -\Lambda^*(\mathcal{X}) + \nu^*(\mathcal{X})\mathcal{N}, \end{cases}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$ and $\mathcal{N} \in \Gamma(T^{\perp}\mathcal{M})$, where $\overline{\nabla}$ and $\overline{\nabla}^*$ (resp. $\nabla$ and $\nabla^*$) are the dual connections on $\overline{\mathcal{M}}$ (resp. on $\mathcal{M}$). Define $\nu$ and $\nu^*$ by $\nu(\mathcal{X}) = g(D_{\mathcal{X}}\mathcal{N},\mathcal{N})$ and $\nu^*(\mathcal{X}) = g(D^*_{\mathcal{X}}\mathcal{N},\mathcal{N})$, respectively. The symmetric and bilinear imbedding curvature tensors of $\mathcal{M}$ in $\overline{\mathcal{M}}$ for $\overline{\nabla}$ and $\overline{\nabla}^*$ are denoted by $\varsigma$ and $\varsigma^*$, respectively. The relation between $\varsigma$ (resp., $\varsigma^*$) and $\Lambda$ (resp. $\Lambda^*$) is defined by [12]

$$\begin{cases} \overline{g}(\varsigma(\mathcal{X},\mathcal{Y}),\mathcal{N}) = g(\Lambda^*\mathcal{X},\mathcal{Y}), \\ \overline{g}(\varsigma^*(\mathcal{X},\mathcal{Y}),\mathcal{N}) = g(\Lambda\mathcal{X},\mathcal{Y}), \end{cases}$$

for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$ and $\mathcal{N} \in \Gamma(T^{\perp}\mathcal{M})$.

**Definition 2.4** ([5])**.** Let $(\mathcal{M},\nabla,g)$ be a submanifold with any codimension of a statistical manifold $(\overline{\mathcal{M}},\overline{\nabla},\overline{g})$. Then $\mathcal{M}$ is said to be

(a) totally geodesic with respect to $\overline{\nabla}$ if $\varsigma = 0$;
(a)* totally geodesic with respect to $\overline{\nabla}^*$ if $\varsigma^* = 0$;
(b) tangentially totally umbilical with respect to $\overline{\nabla}$ if $\varsigma(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathcal{Y})\mathcal{H}$ for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$, (here $\mathcal{H}$ is the mean curvature vector of $\mathcal{M}$ in $\overline{\mathcal{M}}$ for $\overline{\nabla}$);
(b)* tangentially totally umbilical with respect to $\overline{\nabla}^*$ if $\varsigma^*(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathcal{Y})\mathcal{H}^*$ for any $\mathcal{X},\mathcal{Y} \in \Gamma(T\mathcal{M})$, (here $\mathcal{H}^*$ is the mean curvature vector of $\mathcal{M}$ in $\overline{\mathcal{M}}$ for $\overline{\nabla}^*$);
(c) normally totally umbilical with respect to $\overline{\nabla}$ if $\Lambda_{\mathcal{N}}\mathcal{X} = g(\mathcal{H},\mathcal{N})\mathcal{X}$ for any $\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\mathcal{N} \in \Gamma(T^{\perp}\mathcal{M})$;
(c)* normally totally umbilical with respect to $\overline{\nabla}^*$ if $\Lambda^*_{\mathcal{N}}\mathcal{X} = g(\mathcal{H}^*,\mathcal{N})\mathcal{X}$ for any $\mathcal{X} \in \Gamma(T\mathcal{M})$ and $\mathcal{N} \in \Gamma(T^{\perp}\mathcal{M})$.

The curvature tensors with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$ are denoted by $\overline{\mathcal{R}}$ and $\overline{\mathcal{R}}^*$, respectively. Also, $\mathcal{R}$ and $\mathcal{R}^*$ are the curvature tensors with respect to $\nabla$ and $\nabla^*$, respectively. Then the curvature tensor fields of $\overline{\mathcal{M}}$ and $\mathcal{M}$ are respectively defined as (see [12]) $\overline{\mathcal{S}} = \frac{1}{2}(\overline{\mathcal{R}} + \overline{\mathcal{R}}^*)$ and $\mathcal{S} = \frac{1}{2}(\mathcal{R} + \mathcal{R}^*)$.

The sectional curvature $\mathbb{K}$ on $\mathcal{M}$ of $\overline{\mathcal{M}}$ is given by (see [21, 22])

$$\mathbb{K}(\mathcal{X} \wedge \mathcal{Y}) = g(\mathcal{S}(\mathcal{X},\mathcal{Y})\mathcal{Y},\mathcal{X}) = \frac{1}{2}(g(\mathcal{R}(\mathcal{X},\mathcal{Y})\mathcal{Y},\mathcal{X}) + g(\mathcal{R}^*(\mathcal{X},\mathcal{Y})\mathcal{Y},\mathcal{X})),$$

for any orthonormal vectors $\mathcal{X},\mathcal{Y} \in T_{\wp}\mathcal{M}$, $\wp \in \mathcal{M}$.

**Definition 2.5** ([12]). A holomorphic statistical manifold $(\overline{\mathcal{M}}, \overline{\nabla}, \overline{g}, \mathcal{J})$ is said to be of constant holomorphic curvature $k \in \mathbb{R}$ if the following curvature equation holds

$$\overline{\mathcal{S}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} = \frac{k}{4}\Big\{\overline{g}(\mathcal{Y}, \mathcal{Z})\mathcal{X} - \overline{g}(\mathcal{X}, \mathcal{Z})\mathcal{Y} + \overline{g}(\mathcal{J}\mathcal{Y}, \mathcal{Z})\mathcal{J}\mathcal{X} - \overline{g}(\mathcal{J}\mathcal{X}, \mathcal{Z})\mathcal{J}\mathcal{Y} + 2\overline{g}(\mathcal{X}, \mathcal{J}\mathcal{Y})\mathcal{J}\mathcal{Z}\Big\},$$

for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\overline{\mathcal{M}})$. It is denoted by $\overline{\mathcal{M}}(k)$.

The corresponding Gauss equation is given by (see [12])

$$\begin{aligned}
(2.2)\qquad & \frac{k}{2}\Big\{g(\mathcal{Y}, \mathcal{Z})\mathcal{X} - g(\mathcal{X}, \mathcal{Z})\mathcal{Y} + g(\mathcal{P}\mathcal{Y}, \mathcal{Z})\mathcal{P}\mathcal{X} - g(\mathcal{P}\mathcal{X}, \mathcal{Z})\mathcal{P}\mathcal{Y} + 2g(\mathcal{X}, \mathcal{P}\mathcal{Y})\mathcal{P}\mathcal{Z}\Big\} \\
&= 2\overline{\mathcal{S}}(\mathcal{X}, \mathcal{Y})\mathcal{Z} \\
&= 2\mathcal{S}(\mathcal{X}, \mathcal{Y})\mathcal{Z} - g(\Lambda^*\mathcal{Y}, \mathcal{Z})\Lambda\mathcal{X} + g(\Lambda^*\mathcal{X}, \mathcal{Z})\Lambda\mathcal{Y} - g(\Lambda\mathcal{Y}, \mathcal{Z})\Lambda^*\mathcal{X} + g(\Lambda\mathcal{X}, \mathcal{Z})\Lambda^*\mathcal{Y},
\end{aligned}$$

for any $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \Gamma(T\mathcal{M})$.

## 3. Casorati Curvatures for Statistical Hypersurfaces

In this section, we study Casorati curvatures for a statistical hypersurface $\mathcal{M}$ of a holomorphic statistical manifold $\overline{\mathcal{M}}$.

We put $\dim(\mathcal{M}) = m = 2n - 1$ and $\dim(\overline{\mathcal{M}}) = 2n$. Now, we consider a local orthonormal tangent frame $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ of $T\mathcal{M}^m$ and a local orthonormal normal frame $\{\mathcal{E}\}$ of $T^\perp\mathcal{M}^m$ in $\overline{\mathcal{M}}^{2n}$. The scalar curvature $\sigma(\wp)$ of $\mathcal{M}$, $\wp \in \mathcal{M}$, is given by

$$\begin{aligned}
\sigma(\wp) &= \sum_{1 \le i < j \le m} g(\mathcal{S}(\mathcal{E}_i, \mathcal{E}_j)\mathcal{E}_j, \mathcal{E}_i) \\
&= \frac{1}{2}\left\{\sum_{1 \le i < j \le m} g(\mathcal{R}(\mathcal{E}_i, \mathcal{E}_j)\mathcal{E}_j, \mathcal{E}_i) + \sum_{1 \le i < j \le m} g(\mathcal{R}^*(\mathcal{E}_i, \mathcal{E}_j)\mathcal{E}_j, \mathcal{E}_i)\right\},
\end{aligned}$$

and the normalized scalar curvature $\varrho$ of $\mathcal{M}$ is defined as

$$\varrho = \frac{2\sigma(\wp)}{m(m-1)}.$$

The mean curvature vectors $\mathcal{H}$ and $\mathcal{H}^*$ of $\mathcal{M}$ in $\overline{\mathcal{M}}$ are given by

$$\mathcal{H} = \frac{1}{m}\sum_{i=1}^{m} \varsigma(\mathcal{E}_i, \mathcal{E}_i), \quad \left(\text{resp. } \mathcal{H}^* = \frac{1}{m}\sum_{i=1}^{m} \varsigma^*(\mathcal{E}_i, \mathcal{E}_i)\right).$$

Conveniently, let us put

$$\varsigma_{ij} = g(\varsigma(\mathcal{E}_i, \mathcal{E}_j), \mathcal{E}), \quad \left(\text{resp. } \varsigma^*_{ij} = g(\varsigma^*(\mathcal{E}_i, \mathcal{E}_j), \mathcal{E})\right),$$

for $i, j = 1, \ldots, m$.

Then, the squared norm of mean curvature vectors of $\mathcal{M}$ is defined as

$$\|\mathcal{H}\|^2 = \left(\frac{1}{m}\sum_{i=1}^{m} \varsigma_{ii}\right)^2, \quad \left(\text{resp. } \|\mathcal{H}^*\|^2 = \left(\frac{1}{m}\sum_{i=1}^{m} \varsigma^*_{ii}\right)^2\right).$$

The squared norm of second fundamental forms $\varsigma$ and $\varsigma^*$ are denoted by $\mathcal{C}$ and $\mathcal{C}^*$, respectively, called the *Casorati curvatures* of $\mathcal{M}$ in $\overline{\mathcal{M}}$. Therefore, we have

$$\mathcal{C} = \frac{1}{m}\|\varsigma\|^2, \quad \left(\text{resp. } \mathcal{C}^* = \frac{1}{m}\|\varsigma^*\|^2\right),$$

where

$$\|\varsigma\|^2 = \sum_{i,j=1}^{m} \left(\varsigma_{ij}\right)^2, \quad \left(\text{resp. } \|\varsigma^*\|^2 = \sum_{i,j=1}^{m} \left(\varsigma^*_{ij}\right)^2\right).$$

If we consider a $r$-dimensional subspace $\mathcal{W}$ of $T\mathcal{M}$, $r \geq 2$, and an orthonormal basis $\{\mathcal{E}_1, \dots, \mathcal{E}_r\}$ of $\mathcal{W}$. Then the scalar curvature of the $r$-plane section $\mathcal{W}$ is defined as

$$\begin{aligned}\sigma(\mathcal{W}) &= \sum_{1\leq i<j\leq r} \mathcal{S}(\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_i)\\ &=\frac{1}{2}\left\{\sum_{1\leq i<j\leq r} \mathcal{R}(\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_i) + \sum_{1\leq i<j\leq r} \mathcal{R}^*(\mathcal{E}_i, \mathcal{E}_j, \mathcal{E}_j, \mathcal{E}_i)\right\},\end{aligned}$$

and the Casorati curvatures of the subspace $\mathcal{W}$ are the following:

$$\mathcal{C}(\mathcal{W}) = \frac{1}{r}\sum_{i,j=1}^{r} \left(\varsigma_{ij}\right)^2, \quad \left(\text{resp. } \mathcal{C}^*(\mathcal{W}) = \frac{1}{r}\sum_{i,j=1}^{r} \left(\varsigma^*_{ij}\right)^2\right).$$

The normalized Casorati curvatures $\delta_{\mathcal{C}}(m-1)$ and $\widehat{\delta}_{\mathcal{C}}(m-1)$ are defined as

(a)

$$[\delta_{\mathcal{C}}(m-1)]_{\wp} =\frac{1}{2}\mathcal{C}_{\wp} + \left(\frac{m+1}{2m}\right)\inf\{\mathcal{C}(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}$$

$$\left(\text{resp. } [\delta^*_{\mathcal{C}}(m-1)]_{\wp} =\frac{1}{2}\mathcal{C}^*_{\wp} + \left(\frac{m+1}{2m}\right)\inf\{\mathcal{C}^*(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}\right);$$

(b)

$$[\widehat{\delta}_{\mathcal{C}}(m-1)]_{\wp} =2\mathcal{C}_{\wp} - \left(\frac{2m-1}{2m}\right)\sup\{\mathcal{C}(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}$$

$$\left(\text{resp. } [\widehat{\delta}^*_{\mathcal{C}}(m-1)]_{\wp} =2\mathcal{C}^*_{\wp} - \left(\frac{2m-1}{2m}\right)\sup\{\mathcal{C}^*(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}\right).$$

Further, we define the generalized normalized Casorati curvatures $\delta_{\mathcal{C}}(s; m-1)$ and $\widehat{\delta}_{\mathcal{C}}(s; m-1)$ as follows

(a) for $0 < s < m^2 - m$

$$\begin{aligned}[\delta_{\mathcal{C}}(s; m-1)]_{\wp} &=s\mathcal{C}_{\wp} + \zeta(s)\inf\{\mathcal{C}(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}\\ (\text{resp. } [\delta^*_{\mathcal{C}}(s; m-1)]_{\wp} &=s\mathcal{C}^*_{\wp} + \zeta(s)\inf\{\mathcal{C}^*(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\});\end{aligned}$$

(b) for $s > m^2 - m$

$$[\widehat{\delta}_{\mathcal{C}}(s; m-1)]_{\wp} = s\mathcal{C}_{\wp} + \zeta(s)\ \sup\{\mathcal{C}(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}$$
$$(\text{resp. } [\widehat{\delta}^*_{\mathcal{C}}(s; m-1)]_{\wp} = s\mathcal{C}^*_{\wp} + \zeta(s)\ \sup\{\mathcal{C}^*(\mathcal{W}) \mid \mathcal{W} : \text{a hyperplane of } T_{\wp}\mathcal{M}\}),$$

where $\zeta(s) = \frac{1}{sm}(m-1)(m+s)(m^2 - m - s)$, $s \neq m(m-1)$.

Throughout this paper, we work with the above mentioned notations only.

## 4. Bounds of Normalized Scalar Curvature

The most fascinating problem in the theory of Riemannian submanifolds is to find simple relationships between various invariants (intrinsic and extrinsic) of the submanifolds and Riemannian manifolds. Initially, B.-Y. Chen [7] obtained sharp optimal inequalities involving the intrinsic $\delta$-curvatures of Chen and the extrinsic squared mean curvature of submanifolds in a real space form. On the other hand, the study of $\delta-$Casorati [9] curvatures proposed new solutions to the above problem. In this section, we prove such inequalities for a statistical hypersurface $(\mathcal{M}^m, \nabla, g)$ of a holomorphic statistical manifold $(\overline{\mathcal{M}}^{2n}, \overline{\nabla}, \overline{g}, \mathcal{J})$ with constant holomorphic sectional curvature $k$, $\overline{\mathcal{M}}^{2n}(k)$.

**Theorem 4.1.** *Let $\mathcal{M}^m$ $(m = 2n-1)$ be a statistical hypersurface of a $2n$-dimesnional holomorphic statistical manifold with constant holomorphic sectional curvature $k$, $\overline{\mathcal{M}}^{2n}(k)$. Then*

$$\varrho \geq \frac{k(m+3)}{4m} + \frac{m}{m-1}\|\mathcal{H}\|\|\mathcal{H}^*\| - \frac{1}{m(m-1)}\|\varsigma\|\|\varsigma^*\|. \tag{4.1}$$

*Proof.* Let an orthonormal frame of $\mathcal{M}$ be $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ and a unit normal vector to $\mathcal{M}$ be $\{\mathcal{E}\}$. From equation (2.2), we get

$$2\sigma = \frac{k(m+3)(m-1)}{4m} + m^2\|\mathcal{H}\|\|\mathcal{H}^*\| - \sum_{i,j=1}^{m} \varsigma_{ij}\varsigma^*_{ij}.$$

Applying Cauchy-Buniakowski-Schwarz, we have

$$2\sigma \geq \frac{k(m+3)(m-1)}{4m} + m^2\|\mathcal{H}\|\|\mathcal{H}^*\| - \|\varsigma\|\|\varsigma^*\|.$$

From last inequality, we can easily obtain (4.1). This is the required inequality. $\square$

Theorem 4.1 shows that the normalized scalar curvature is bounded below. Now, we switch to our next theorem, which shows that the normalized scalar curvature is bounded above in terms of Casorati curvature. The result is as follows.

**Theorem 4.2.** *Let $\mathcal{M}^m$ $(m = 2n-1)$ be a statistical hypersurface of a $2n-$dimensional holomorphic statistical manifold with constant holomorphic sectional curvature $k$, $\overline{\mathcal{M}}^{2n}(k)$. Then* (*a*) *the generalized normalized Casorati curvatures* $\delta_{\mathcal{C}}(s;m-1)$ *and* $\delta^*_{\mathcal{C}}(s;m-1)$ *satisfy*

$$\varrho \leq \frac{2\delta^0_{\mathcal{C}}(s;m-1)}{m(m-1)} + \left[\frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} - \frac{2m}{m-1}\|\mathcal{H}^0\|^2 + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*)\right], \tag{4.2}$$

*for any* $s \in \mathbb{R}$ *with* $0 < s < m(m-1)$, *where* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ *and* $2\delta^0_{\mathcal{C}}(s;m-1) = \delta_{\mathcal{C}}(s;m-1) + \delta^*_{\mathcal{C}}(s;m-1)$;

(*b*) *the generalized normalized Casorati curvatures* $\widehat{\delta}_{\mathcal{C}}(s;m-1)$ *and* $\widehat{\delta}^*_{\mathcal{C}}(s;m-1)$ *satisfy*

$$\varrho \leq \frac{2\widehat{\delta}^0_{\mathcal{C}}(s;m-1)}{m(m-1)} + \left[\frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} - \frac{2m}{m-1}\|\mathcal{H}^0\|^2 + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*)\right], \tag{4.3}$$

*for any* $s \in \mathbb{R}$, *with* $s > m(m-1)$, *where* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ *and* $2\widehat{\delta}^0_{\mathcal{C}}(s;m-1) = \widehat{\delta}_{\mathcal{C}}(s;m-1) + \widehat{\delta}^*_{\mathcal{C}}(s;m-1)$.

*Proof.* Let an orthonormal frame of $\mathcal{M}$ be $\{\mathcal{E}_1, \ldots, \mathcal{E}_m\}$ and a unit normal vector to $\mathcal{M}$ be $\{\mathcal{E}\}$. From equation (2.2), we get

$$\begin{aligned} 2\sigma(\wp) =& \frac{k}{4}\Big[(m-1)(m+3)\Big] + 2m^2\|\mathcal{H}^0\|^2 - \frac{m^2}{2}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}^*\|^2\Big) \\ &- 2m\,\mathcal{C}^0 + \frac{m}{2}\Big(\mathcal{C} + \mathcal{C}^*\Big). \end{aligned}$$

Let us take a quadratic polynomial $\mathcal{K}$ in the components of the second fundamental form

$$\begin{aligned} \mathcal{K} =& s\mathcal{C}^0 + \zeta(s)\mathcal{C}^0(\mathcal{W}) - 2\sigma(\wp) + \frac{k}{4}\Big[(m-1)(m+3)\Big] \\ &- \frac{m^2}{2}\Big(\|\mathcal{H}\|^2 + \|\mathcal{H}^*\|^2\Big) + \frac{m}{2}\Big(\mathcal{C} + \mathcal{C}^*\Big). \end{aligned} \tag{4.4}$$

Without loss of generality, we assume that $\mathcal{W}$ is spanned by $\mathcal{E}_1, \ldots, \mathcal{E}_m$ and together with (4.4), we find that

$$\mathcal{K} = \frac{m+s}{m}\sum_{i,j=1}^{m}(\varsigma^0_{ij})^2 + \frac{\zeta(s)}{m-1}\sum_{i,j=1}^{m-1}(\varsigma^0_{ij})^2 - \left(\sum_{i=1}^{m}\varsigma^0_{ii}\right)^2$$

or

$$\begin{aligned} \mathcal{K} =& \sum_{i=1}^{m-1}\left[q(\varsigma^0_{ii})^2 + \frac{2(m+s)}{m}(\varsigma^0_{im})^2\right] \\ &+ \left[2q\sum_{1\leq i\neq j\leq m-1}(\varsigma^0_{ij})^2 - 2\sum_{1\leq i\neq j\leq m}(\varsigma^0_{ii}\varsigma^0_{jj}) + \frac{s}{m}(\varsigma^0_{mm})^2\right], \end{aligned} \tag{4.5}$$

where

$$q = \left(\frac{m+s}{m} + \frac{\zeta(s)}{m-1}\right).$$

From (4.5), we observe that the solutions of the following system of linear homogenous equations:

$$\left\{\begin{aligned} \frac{\partial \mathcal{K}}{\partial \varsigma_{ii}^0} &= 2q(\varsigma_{ii}^0) - 2\textstyle\sum_{l=1}^{m} \varsigma_{ll}^0 = 0, \\ \frac{\partial \mathcal{K}}{\partial \varsigma_{mm}^0} &= \frac{2s}{m}\varsigma_{mm}^0 - 2\textstyle\sum_{l=1}^{m-1} \varsigma_{ll}^0 = 0, \\ \frac{\partial \mathcal{K}}{\partial \varsigma_{ij}^0} &= 4q\varsigma_{ij}^0 = 0, \\ \frac{\partial \mathcal{K}}{\partial \varsigma_{im}^0} &= 4(\tfrac{m+s}{m})\varsigma_{im}^0 = 0, \end{aligned}\right. \tag{4.6}$$

are the critical points

$$\varsigma^{0c} = \left(\varsigma_{11}^0, \varsigma_{12}^0, \dots, \varsigma_{mm}^0\right) \tag{4.7}$$

of $\mathcal{K}$, where $i, j = 1, \dots, m-1$, $i \neq j$.

Hence, every solution $\varsigma^{0c}$ has $\varsigma_{ij}^0 = 0$ for $i \neq j$ and the determinant which corresponds to the first two equations of the above system is zero. Furthermore, the Hessian matrix $\mathcal{H}ess_{\mathcal{K}}$ of $\mathcal{K}$ is given by

$$\mathcal{H}ess_{\mathcal{K}} = \begin{pmatrix} I & O & O \\ O & II & O \\ O & O & III \end{pmatrix}, \tag{4.8}$$

where $O$ are the null matrices and the matrices $I, II$ and $III$ are, respectively, given below:

$$I = -2\begin{pmatrix} 1-q & 1 & \dots & 1 & 1 \\ 1 & 1-q & \dots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \dots & 1-q & 1 \\ 1 & 1 & \dots & 1 & \frac{-s}{m} \end{pmatrix},$$

$$II = 4q\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

$$III = \frac{4(m+s)}{m}\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Therefore, the eigenvalues of Hessian matrix $\mathcal{H}ess_{\mathcal{K}}$ are given below:

$$\begin{aligned}\mu_{11} =&0,\ \mu_{22} = 2\left(\frac{2s}{m} + \frac{\zeta(s)}{m-1}\right), \qquad \mu_{33} = \cdots = \mu_{mm} = 2q,\\ \mu_{ij} =&4q,\ \mu_{im} = \frac{4(m+s)}{m}, \quad \text{for all } i,j = 1,2,\ldots,m-1,\ i \neq j.\end{aligned}$$

Thus, we know that $\mathcal{K}$ is parabolic and reaches a minimum $\mathcal{K}(\varsigma^{0c})$ for each solution $\varsigma^{0c}$ of the system (4.6). From the equations (4.5) and (4.6), we arrive at $\mathcal{K}(\varsigma^{0c}) = 0$. Hence $\mathcal{K} \geq 0$, and this further gives following inequality:

$$\begin{aligned}2\sigma(\wp) \leq& s\mathcal{C}^0 + \zeta(s)\mathcal{C}^0(\mathcal{W}) + \frac{k(m-1)(m+3)}{4}\\ &- \frac{m^2}{2}\left(\|\mathcal{H}\|^2 + \|\mathcal{H}^*\|^2\right) + \frac{m}{2}\left(\mathcal{C} + \mathcal{C}^*\right).\end{aligned}$$

Hence, we find that

$$\begin{aligned}\varrho \leq& \frac{s}{m(m-1)}\mathcal{C}^0 + \frac{\zeta(s)}{m(m-1)}\mathcal{C}^0(\mathcal{W}) + \frac{k(m+3)}{4m}\\ &- \frac{2m}{m-1}\|\mathcal{H}^0\|^2 + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*) + \frac{1}{2(m-1)}\left(\mathcal{C} + \mathcal{C}^*\right),\end{aligned}$$

for every tangent hyperplane $\mathcal{W}$ of $\mathcal{M}$. If we take the infimum over all tangent hyperplanes $\mathcal{W}$, our assertion (4.2) follows.

In the same manner, we can establish an inequality (4.3) in the second part of the theorem. □

*Remark* 4.1. The proof of Theorem 4.2 is mainly based on a classical optimization procedure by showing that a quadratic polynomial in the components of the second fundamental form $\varsigma^0$ with respect to Levi-Civita connection is parabolic (see [15, 16, 18, 24, 27]). Since, we have proved that the Hessian matrix (4.8) is positive semi-definite for all points and admits precisely one eigenvalue equal to zero. Therefore, it is easy to say that $\mathcal{K}$ is parabolic and reaches a minimum $\mathcal{K}(\varsigma^{0c})$ for each solution $\varsigma^{0c}$ of the system (4.6). In fact, because of the convexity, the critical point is a global minimum. We note that an alternative proof of Theorem 4.2 can be done by making use of T. Oprea's optimization technique [23], namely analyzing a suitable constrained extremum problem (see also [8, 19, 25]).

The characterisation of equality cases in Theorem 4.2.

**Theorem 4.3.** *Let* $\mathcal{M}^m$ $(m = 2n-1)$ *be a statistical hypersurface of a* $2n-$*dimensional holomorphic statistical manifold with constant holomorphic sectional curvature* $k$*,* $\overline{\mathcal{M}}^{2n}(k)$*. Equalities hold in the relations* (4.2) *and* (4.3) *if and only if*

$$\varsigma_{ij} = -\varsigma^*_{ij}, \quad \textit{for all } i,j = 1,\ldots,m, i \neq j,$$

*and*

$$\varsigma^0_{mm} = \frac{m(m-1)}{s}\varsigma^0_{11} = \cdots = \frac{m(m-1)}{s}\varsigma^0_{m-1\ m-1}.$$

## 5. Some Geometric Applications

In this section, we discuss some immediate applications of the results proved in the previous section. Some immediate consequences of Theorem 4.2 are the following.

**Corollary 5.1.** *Let $\mathcal{M}^m$ $(m = 2n-1)$ be a statistical hypersurface of a $2n-$dimensional holomorphic statistical manifold with constant holomorphic sectional curvature $k$, $\overline{\mathcal{M}}^{2n}(k)$. Then*

(a) *the normalized Casorati curvatures $\delta_{\mathcal{C}}(m-1)$ and $\delta^*_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq 2\delta^0_{\mathcal{C}}(m-1) + \left[\frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} - \frac{2m}{m-1}\|\mathcal{H}^0\|^2 + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*)\right],$$

*where $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ and $2\delta^0_{\mathcal{C}}(m-1) = \delta_{\mathcal{C}}(m-1) + \delta^*_{\mathcal{C}}(m-1)$;*

(b) *the normalized Casorati curvatures $\widehat{\delta}_{\mathcal{C}}(m-1)$ and $\widehat{\delta}^*_{\mathcal{C}}(m-1)$ satisfies*

$$\varrho \leq 2\widehat{\delta}^0_{\mathcal{C}}(m-1) + \left[\frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} - \frac{2m}{m-1}\|\mathcal{H}^0\|^2 + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*)\right],$$

*where $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ and $2\widehat{\delta}^0_{\mathcal{C}}(m-1) = \widehat{\delta}_{\mathcal{C}}(m-1) + \widehat{\delta}^*_{\mathcal{C}}(m-1)$.*

*Remark* 5.1. We remark that one can prove Corollary 5.1 by considering $s = \frac{m(m-1)}{2}$ in $\delta_{\mathcal{C}}(s; m-1)$ (resp. $\delta^*_{\mathcal{C}}(s; m-1)$) and we have the following relation (see [16])

$$\left[\delta_{\mathcal{C}}\left(\frac{m(m-1)}{2}; m-1\right)\right]_{\wp} = m(m-1)\left[\delta_{\mathcal{C}}(m-1)\right]_{\wp}$$

$$\left(\text{resp. } \left[\delta^*_{\mathcal{C}}\left(\frac{m(m-1)}{2}; m-1\right)\right]_{\wp} = m(m-1)\left[\delta^*_{\mathcal{C}}(m-1)\right]_{\wp}\right)$$

at any point $\wp \in \mathcal{M}$.

**Corollary 5.2.** *Let $\mathcal{M}^m$ $(m = 2n-1)$ be a statistical hypersurface of a $2n-$dimensional holomorphic statistical manifold with constant holomorphic sectional curvature $k$, $\overline{\mathcal{M}}^{2n}(k)$. If $\mathcal{M}$ is minimal, i.e., $\mathcal{H}^0 = 0$, then*

(a) *the generalized normalized Casorati curvatures $\delta_{\mathcal{C}}(s; m-1)$ and $\delta^*_{\mathcal{C}}(s; m-1)$ satisfy*

$$\varrho \leq 2\frac{\delta^0_{\mathcal{C}}(s; m-1)}{m(m-1)} + \frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*),$$

*for any $s \in \mathbb{R}$, with $0 < s < m(m-1)$, where $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ and $2\delta^0_{\mathcal{C}}(s; m-1) = \delta_{\mathcal{C}}(s; m-1) + \delta^*_{\mathcal{C}}(s; m-1)$;*

(b) *the generalized normalized Casorati curvatures* $\widehat{\delta}_{\mathcal{C}}(s;m-1)$ *and* $\widehat{\delta}^*_{\mathcal{C}}(s;m-1)$ *satisfy*

$$\varrho \leq 2\frac{\widehat{\delta}^0_{\mathcal{C}}(s;m-1)}{m(m-1)} + \frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*),$$

*for any* $s \in \mathbb{R}$, *with* $s > m(m-1)$, *where* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ *and* $2\widehat{\delta}^0_{\mathcal{C}}(s;m-1) = \widehat{\delta}_{\mathcal{C}}(s;m-1) + \widehat{\delta}^*_{\mathcal{C}}(s;m-1)$.

The following result follows directly from Corollary 5.1.

**Corollary 5.3.** *Let* $\mathcal{M}^m$ $(m = 2n-1)$ *be a statistical hypersurface of a* $2n-$*dimensional holomorphic statistical manifold with constant holomorphic sectional curvature* $k$, $\overline{\mathcal{M}}^{2n}(k)$. *If* $\mathcal{M}$ *is minimal, i.e.,* $\mathcal{H}^0 = 0$, *then*

(a) *the normalized Casorati curvature* $\delta_{\mathcal{C}}(m-1)$ *and* $\delta^*_{\mathcal{C}}(m-1)$ *satisfy*

$$\varrho \leq 2\delta^0_{\mathcal{C}}(m-1) + \frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*),$$

*where* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ *and* $2\delta^0_{\mathcal{C}}(m-1) = \delta_{\mathcal{C}}(m-1) + \delta^*_{\mathcal{C}}(m-1)$;

(b) *the normalized Casorati curvature* $\widehat{\delta}_{\mathcal{C}}(m-1)$ *and* $\widehat{\delta}^*_{\mathcal{C}}(m-1)$ *satisfy*

$$\varrho \leq 2\widehat{\delta}^0_{\mathcal{C}}(m-1) + \frac{k(m+3)}{4m} + \frac{\mathcal{C}^0}{m-1} + \frac{m}{m-1}g(\mathcal{H},\mathcal{H}^*),$$

*where* $2\mathcal{C}^0 = \mathcal{C} + \mathcal{C}^*$ *and* $2\widehat{\delta}^0_{\mathcal{C}}(m-1) = \widehat{\delta}_{\mathcal{C}}(m-1) + \widehat{\delta}^*_{\mathcal{C}}(m-1)$.

Now, we have the following statistical significance of Theorem 4.1.

**Corollary 5.4.** *Let* $\mathcal{M}^m$ $(m = 2n-1)$ *be a statistical hypersurface of a* $2n-$*dimensional holomorphic statistical manifold with constant holomorphic sectional curvature* $k$, $\overline{\mathcal{M}}^{2n}(k)$. *If* $\mathcal{M}$ *is totally umbilical and totally geodesic with respect to* $\overline{\nabla}$ *and* $\overline{\nabla}^*$. *Then*

$$\varrho \geq \frac{k(m+3)}{4m}. \tag{5.1}$$

*Remark* 5.2. In the above Corollary 5.4, we have $\mathcal{M}$ is totally umbilical and totally geodesic with respect to $\overline{\nabla}$ and $\overline{\nabla}^*$, that is, for any $\mathcal{X},\mathcal{Y} \in T_p\mathcal{M}$, $0 = \varsigma(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathcal{Y})\mathcal{H}$, which gives $\mathcal{H} = 0$. Similarly, $0 = \varsigma^*(\mathcal{X},\mathcal{Y}) = g(\mathcal{X},\mathcal{Y})\mathcal{H}^*$ implies $\mathcal{H}^* = 0$. Hence, an inequality (4.1) reduces to (5.1).

Further, we observe the following.

**Corollary 5.5.** *Let* $\mathcal{M}^m$ $(m = 2n-1)$ *be a statistical hypersurface of a* $2n-$*dimensional holomorphic statistical manifold with constant holomorphic sectional curvature* $k$, $\overline{\mathcal{M}}^{2n}(k)$. *Suppose that* $\varrho = \frac{k(m+3)}{4m}$. *Then* $\mathcal{M}$ *is not totally geodesic with respect to* $\overline{\nabla}$ *and* $\overline{\nabla}^*$.

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# QUANTITATIVE UNCERTAINTY PRINCIPLE FOR STURM-LIOUVILLE TRANSFORM

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ABSTRACT. In this paper we consider the Sturm-Liouville transform $\mathcal{F}(f)$ on $\mathbb{R}_+$. We analyze the concentration of this transform on sets of finite measure. In particular, Donoho-Stark and Benedicks-type uncertainty principles are given.

## 1. INTRODUCTION

The uncertainty principle says that a function and its transform cannot concentrate both on small sets. Depending on the precise way to measure "concentration" and "smallness" this principle can assume different forms. This paper focuses on studying different uncertainty principles for the Sturm-Liouville transform, by following the procedures for similar transforms, such as the Fourier transform (the classical setting) we refer to the book [10] and the surveys [4, 7] for further references. The concept of concentration has taken different interpretations in different contexts. For example: Benedicks [2], Slepian and Pollak [18], Landau and Pollak [13], and Donoho and Stark [6] paid attention to the supports of functions and gave quantitative uncertainty principles for the Fourier transforms. Qualitative uncertainty principles are not inequalities, but are theorems that tell us how a function (and its Fourier transform) behave under certain circumstances. For example: Hardy [11], Cowling and Price [5], Beurling [3], Miyachi [15] theorems enter within the framework of the quantitative uncertainty principles. The quantitative and qualitative uncertainty principles have been studied by many authors for various Fourier transforms, for examples (cf. [1, 11, 14, 16]).

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Our aim here is to consider uncertainty principles in which concentration is measured in sense of smallness of the support and when the transform under consideration is the Sturm-Liouville transform.

The first principle that is studied is a Donoho-Stark-type inequality. One can write the classical uncertainty principle in the following way: If a function $f(t)$ is essentially zero outside an interval of length $\Delta t$ and its Fourier transform $f(w)$ is essentially zero outside an interval of length $\Delta w$, then $\Delta t \Delta w \geq 1$. In [6], Donoho and Stark show that it is not necessary to assume that the support and the spectrum are concentrated on intervals and one can replace intervals by measurable sets, and then the length of the interval is naturally replaced by the measure of the set. In Section 2, a version of this inequality for the Sturm-Liouville transform is given, and, as it appears in [6] it is explained how to reconstruct a signal f from a noisy measurement, knowing that the signal is supported on a set $S$.

The second principle, studied in Section 3, is a Benedicks-type result which shows that two measurable sets $(S, \Sigma)$ with finite measure form a strong annihilating pair. This means that a function supported in S cannot have an spectrum in $\Sigma$ giving a quantitative information of the mass of a function whose spectrum is contained in $\Sigma$. The approach is based on the corresponding version of this type of principle for the integral operators transform, studied in [8]. A version of Benedicks type-inequality for integral operators transform with bounded and homogeneous kernel has been proved in [8]. In this paper, we consider a transform of a different nature where in particular the kernel is not homogeneous.

We recall that, Soltani in [19] study what is the relation between the measure and the spectrum of a function $f$ that is $\varepsilon$-concentrated in measurable sets giving. Concentration in support means that the part of the function that is not supported on a set is at least an $\varepsilon$ part of the total mass. The analogous version for spectrum states that the part of the spectrum not supported on a set is an $\varepsilon$ part of the total spectrum. It is shown that if a function is $\varepsilon$-concentrated in space and frequency, then the product of the measures of the support and spectrum is lower bounded by a number close to one.

In order to describe our results, we first need to introduce some facts about harmonic analysis related to Sturm-Liouville transform. We cite here, as briefly as possible, some properties. For more details we refer to [19].

The Sturm-Liouville operator $\Delta$ defined on $\mathbb{R}_+$ by

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\mathcal{A}'(x)}{\mathcal{A}(x)} \frac{\partial}{\partial x} + \rho^2,$$

where $\rho$ is a nonnegative real number and $\mathcal{A}(x) = x^{2\alpha+1} B(x)$, $\alpha > -\frac{1}{2}$, where $B$ is a positive, even, infinitely differentiable function on $\mathbb{R}$ such that $B(0) = 1$. Moreover, we assume that $\mathcal{A}$ and $B$ satisfy the following conditions:

- $\mathcal{A}$ is increasing and $\lim\limits_{x \to \infty} \mathcal{A}(x) = \infty$;

- $\dfrac{\mathcal{A}'}{\mathcal{A}}$ is decreasing and $\lim\limits_{x\to\infty} \frac{\mathcal{A}'(x)}{\mathcal{A}(x)} = 2\rho$;
- there exists a constant $\delta > 0$ such that

$$\frac{\mathcal{A}'(x)}{\mathcal{A}(x)} = \begin{cases} 2\rho + D(x)\exp(-\delta x), & \text{if } \rho > 0, \\ \dfrac{2\alpha+1}{x} + D(x)\exp(-\delta x), & \text{if } \rho = 0, \end{cases}$$

where $D$ is an infinitely differentiable function on $]0,\infty[$, bounded and with bounded derivatives on all intervals $[x_0,\infty[$ for $x_0 > 0$. For all $\lambda \in \mathbb{C}$ the equation

$$\begin{cases} \Delta u = -\lambda^2 u, \\ u(0) = 1, \quad u'(0) = 0, \end{cases}$$

admits a unique solution denoted $\varphi_\lambda$, with the following properties:

- for $x \geq 0$ the function $\lambda \mapsto \varphi_\lambda(x)$ is analytic on $\mathbb{C}$;
- for $\lambda \in \mathbb{C}$ the function $\lambda \mapsto \varphi_\lambda(x)$ is even and infinitely differentiable on $\mathbb{R}$;
- $|\varphi_\lambda(x)| \leq 1$ for all $\lambda, x \in \mathbb{R}$.

For nonzero $\lambda \in \mathbb{C}$ the equation $\Delta u = -\lambda^2 u$ has a solution $\Phi_\lambda$ satisfying

$$\Phi_\lambda(x) = \frac{1}{\sqrt{\mathcal{A}(x)}}\exp(i\lambda x)V(x,\lambda),$$

with $\lim\limits_{x\to\infty} V(x,\lambda) = 1$. Consequently, there exists a function $\lambda \mapsto c(\lambda)$, such that

$$\varphi_\lambda = c(\lambda)\Phi_\lambda + c(-\lambda)\Phi_{-\lambda}, \quad \text{for nonzero } \lambda \in \mathbb{C}.$$

Moreover, there exist positive constants $k_1$, $k_2$ and $k$ such that

$$k_1|\lambda|^{2\alpha+1} \leq |c(\lambda)|^{-2} \leq k_2|\lambda|^{2\alpha+1},$$

for all $\lambda$ such that $\mathrm{Im}\lambda \leq 0$ and $|\lambda| \geq k$.

Let us introduce the dilation operator $D_\rho,\ \rho > 0$, defined by

$$D_\rho f(x) = \frac{1}{\rho^{\alpha+1}} f\left(\frac{x}{\rho}\right).$$

We denote by $L^p(\mathbb{R}_+,\mu),\ 1 \leq p \leq \infty$, the space of measurable functions $f$ on $\mathbb{R}_+$ such that

$$\|f\|_{L^p(\mathbb{R}_+,\mu)} = \left(\int_{\mathbb{R}_+} |f(x)|^p d\mu(x)\right)^{\frac{1}{p}} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$
$$\|f\|_\infty = \mathrm{esssup}_{x\in\mathbb{R}_+}|f(x)| < +\infty, \quad \text{if } p = \infty,$$

where $d\mu(x) = \mathcal{A}(x)dx$.

The Sturm-Liouville transform $\mathcal{F}$ is defined on $L^1(\mathbb{R}_+,\mu)$ by

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}_+} f(x)\varphi_\lambda(x)d\mu(x), \quad \text{for all } \lambda \in \mathbb{R}.$$

Let $\nu$ the measure defined on $[0,\infty[$ by $d\nu(\lambda) = \frac{d\lambda}{2\pi|c(\lambda)|^2}$ and by $L^p(\nu)$, $1 \leq p \leq \infty$, the space of measurable functions $f$ on $[0,\infty[$, such that $\|f\|_{L^p(\mathbb{R}_+,\nu)} < \infty$.

For all $f \in L^1(\mathbb{R}_+, \mu)$, the function $\mathcal{F}(f)$ is continuous on $\mathbb{R}$ and we have

$$\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}_+,\nu)} \leq \|f\|_{L^1(\mathbb{R}_+,\mu)}. \tag{1.1}$$

**Theorem 1.1** (Plancherel theorem)**.** *The Sturm-Liouville transform $\mathcal{F}$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}_+, \mu)$ onto $L^2(\mathbb{R}_+, \nu)$*

$$\int_{\mathbb{R}_+} |f(x)|^2 d\mu(x) = \int_{\mathbb{R}_+} |\mathcal{F}(f)(\lambda)|^2 d\nu(\lambda). \tag{1.2}$$

**Theorem 1.2** (Inversion theorem)**.** *Let $f \in L^1(\mathbb{R}_+, \mu)$ such that $\mathcal{F}(f) \in L^1(\mathbb{R}_+, \nu)$. Then*

$$f(x) = \int_{\mathbb{R}_+} \mathcal{F}(f)(\lambda)\varphi_\lambda(x) d\nu(\lambda) \text{ a.e. } x \in \mathbb{R}^+.$$

**Theorem 1.3** (Riesz's interpolation theorem)**.** *Let $f \in L^p(\mathbb{R}_+ > \mu)$. Then we get the Hausdorff-Young inequality (see* [20]*) $\|\mathcal{F}(f)\|_{L^q(\mathbb{R}_+,\nu)} \leq \|f\|_{L^p(\mathbb{R}_+,\mu)}$.*

**Definition 1.1.** Let $S,\ \Sigma$ be two measurable subsets of $\mathbb{R}^d_+$. Then $(S, \Sigma)$ is called a weak annihilating pair for the Sturm-Liouville transform if $\text{supp}f \subset S$ and $\text{supp}\mathcal{F}_{k,\alpha}(f) \subset \Sigma$, implies that $f = 0$, where $\text{supp}f = \{x : f(x) \neq 0\}$.

**Definition 1.2.** Let $S,\ \Sigma$ be two measurable subsets of $\mathbb{R}_+$. Then $(S, \Sigma)$ is called a strong annihilating pair for the Sturm-Liouville transform if there exists a constant $C(S, \Sigma)$ such that for all function $f \in L^2(\mathbb{R}_+, \mu)$, with $\text{supp}\mathcal{F}(f) \subset \Sigma$,

$$\|f\|_{L^2(\mathbb{R}_+,\mu)} \leq C(S, \Sigma)\|f\|_{L^2(S^c,\mu)}, \tag{1.3}$$

where $S^c = \mathbb{R}_+ \backslash S$ and $\text{supp}f = \{x : f(x) \neq 0\}$.

## 2. The Donoho-Strak's Uncertainty Principle

The classical uncertainty principle says that if a function $f(t)$ is essentially zero outside an interval of light $\Delta t$ and its Fourier transform $\widehat{f}(w)$ is essentially zero outside an interval of length $\Delta w$, then $\Delta t \Delta w \geq 1$. In this section we will prove a quantitative uncertainty inequality about the essential supports of a nonzero function $f \in L^2(\mathbb{R}_+, \mu)$ and its Sturm-Liouville transform.

The first such inequality for the usual Fourier transform was obtained by Donoho-Stark [6].

We consider a pair of orthogonal projections on $L^2(\mathbb{R}_+, \mu)$ defined by $P_S f = \chi_S f$, $Q_\Sigma f = \mathcal{F}^{-1}\left[\chi_\Sigma \mathcal{F}(f)\right]$, where $S$ and $\Sigma$ are measurable subsets of $\mathbb{R}_+$, and $\chi_S$ denote the characteristic function of $S$.

Let $0 < \varepsilon_S, \varepsilon_\Sigma < 1$ and let $f \in L^2(\mathbb{R}_+, \mu)$ be a nonzero function. We say that $f$ is $\varepsilon_S$-time-limited on $S$ if $\|P_{S^c} f\|_{L^2(\mathbb{R}_+,\mu)} \leq \varepsilon_S \|f\|_{L^2(\mathbb{R}_+,\mu)}$. Similarly, we say that $f$ is $\varepsilon_\Sigma$-band-limited on $\Sigma$ for the Sturm-Liouville transform if $\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+,\mu)} \leq \varepsilon_\Sigma \|f\|_{L^2(\mathbb{R}_+,\mu)}$.

We denote by $P_S \cap Q_\Sigma$ for the orthogonal projection onto the intersection of the ranges of $P_S$ and $Q_\Sigma$, we will write $\text{Im}T$ for the range of a linear operator $T$. We denote by $\|T\|_{HS}$ the Hilbert-Schmidt norm of the linear operator $T$. The definition of this norm [21, page 262] implies that for any pair of projections $E$, $F$ one has

$$\dim(\mathrm{Im}P_S \cap \mathrm{Im}Q_\Sigma) = \|P_S \cap Q_\Sigma\|_{HS}^2 \leq \|P_S Q_\Sigma\|_{HS}^2. \tag{2.1}$$

**Theorem 2.1.** *Let $\Sigma$, $S \subset \mathbb{R}_+$ be a pair of measurable subsets and let $\varepsilon_S, \varepsilon_\Sigma > 0$ such that $\varepsilon_S^2 + \varepsilon_\Sigma^2 < 1$. Let $f \in L^2(\mathbb{R}_+, \mu)$ be a non function. If $f$ is $\varepsilon_S$-time-limited on $S$ and $\varepsilon_\Sigma$-band-limited on $\Sigma$ for the Sturm-Liouville transform, then*

$$\mu(S)\nu(\Sigma) \geq \left(1 - \sqrt{\varepsilon_S^2 + \varepsilon_\Sigma^2}\right)^2.$$

We will need the following well-known lemma.

**Lemma 2.1.** *Let $(S, \Sigma)$ be two measurable subsets of $\mathbb{R}_+$. Then the following assertions are equivalent.*

i) $\|P_S Q_\Sigma\| = \|P_S Q_\Sigma\|_{L^2(\mathbb{R}_+,\mu)} < 1$.
ii) *$(S, \Sigma)$ is strongly annihilating pair for the Sturm-Liouville transform. Moreover, we have* $\|f\|_{L^2(\mathbb{R}_+,\mu)}^2 \leq (1 - \|P_S Q_\Sigma\|)^{-2} \left(\|P_{S^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2 + \|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2\right)$.

*Proof.* Firstly we show the following implication i)$\Rightarrow$ii). The identity operator $I$ satisfies $I = P_S + P_{S^c} = P_S Q_\Sigma + P_S Q_{\Sigma^c} + P_{S^c}$, we have from the orthogonality of $P_S$ and $P_{S^c}$

$$\begin{aligned} \|f - P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+,\mu)}^2 =& \|P_S Q_{\Sigma^c} f + P_{S^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2 \\ =& \|P_S Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2 + \|P_{S^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2. \end{aligned}$$

It follows, by $\|P_S\| = 1$, that

$$\|f - P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+,\mu)} \leq \left(\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2 + \|P_{S^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2\right)^{\frac{1}{2}}. \tag{2.2}$$

On the other hand, we have

$$\begin{aligned} \|f - P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+,\mu)} \geq& \|f\|_{L^2(\mathbb{R}_+,\mu)} - \|P_S Q_\Sigma f\|_{L^2(\mathbb{R}_+,\mu)} \\ \geq& \|f\|_{L^2(\mathbb{R}_+,\mu)} - \|P_S Q_\Sigma\| \, \|f\|_{L^2(\mathbb{R}_+,\mu)}. \end{aligned}$$

It follows, from inequality (2.2),

$$(1 - \|P_S Q_\Sigma\|)\|f\|_{L^2(\mathbb{R}_+,\mu)} \leq \left(\|P_{S^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2 + \|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}_+,\mu)}^2\right)^{\frac{1}{2}}. \tag{2.3}$$

As $\|P_S Q_\Sigma\| < 1$, then we obtain the desired result.

Let us now show the second implication ii)⇒i). Recall that

$$\begin{aligned}
\|P_SQ_\Sigma\| = \|Q_\Sigma P_S\| = & \sup_{f\in L^2(\mathbb{R}_+,\mu)} \frac{\|Q_\Sigma P_S f\|_{L^2(\mathbb{R}_+,\mu)}}{\|f\|_{L^2(\mathbb{R}_+,\mu)}} \\
= & \sup_{f:f=P_Sf} \frac{\|Q_\Sigma f\|_{L^2(\mathbb{R}_+,\mu)}}{\|f\|_{L^2(\mathbb{R}_+,\mu)}} \\
= & \sup_{f:f=Q_\Sigma f} \frac{\|P_S f\|_{L^2(\mathbb{R}_+,\mu)}}{\|f\|_{L^2(\mathbb{R}_+,\mu)}} \\
< & 1.
\end{aligned}$$

We suppose that $\|P_SQ_\Sigma\| = 1$. Then we can find a bandlimited sequence $f_n \in L^2(\mathbb{R}_+,\mu)$ on $\Sigma$ of norm 1 (in particular $f_n = Q_\Sigma f_n$) such that

$$\|P_Sf_n\|_{L^2(\mathbb{R}_+,\mu)} \to 1 \text{ as } n \to \infty.$$

By the orthogonality of $S$, we have

$$\|P_{S^c}f_n\|^2_{L^2(\mathbb{R}_+,\mu)} = \|f_n\|^2_{L^2(\mathbb{R}_+,\mu)} - \|P_Sf_n\|_{L^2(\mathbb{R}_+,\mu)} \to 0 \text{ as } n \to \infty,$$

which contradicts (1.3). □

**Lemma 2.2.** *If* $0 < \mu(S)\nu(\Sigma) < 1$*, then for all function* $f \in L^2(\mathbb{R}_+,\mu)$ *such that* $\mathrm{supp}\mathcal{F}(f) \subset \Sigma$ *we have*

$$\|f\|_{L^2(\mathbb{R}_+,\mu)} \le \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1} \|f\|_{L^2(S^c,\mu)}.$$

*Proof.* A straightforward computation shows that $P_SQ_\Sigma$ is an integral operator with kernel $N(t,x) = \chi_S(t)\mathcal{F}^{-1}(\chi_\Sigma\varphi_\lambda(t))(x)$. Indeed, we have

$$\begin{aligned}
P_SQ_\Sigma f(t) = & \chi_S(t)\int_{\mathbb{R}_+} \chi_\Sigma(\xi)\mathcal{F}(f)(\xi)\varphi_\lambda(t)d\nu(\xi) \\
= & \chi_S(t)\int_{\mathbb{R}_+} \chi_\Sigma(\xi)\varphi_\lambda(t)\left(\int_{\mathbb{R}_+} f(x)\varphi_\lambda(x)d\mu(x)\right)d\nu(\xi) \\
= & \int_{\mathbb{R}_+} f(x)N(t,x)d\mu(x),
\end{aligned}$$

where

$$N(t,x) = \chi_S(t)\int_{\mathbb{R}_+} \chi_\Sigma(\xi)\varphi_\lambda(t)\varphi_\lambda(x)d\nu(\xi).$$

Since, $\nu(\Sigma) < \infty$ and $\varphi_\lambda$ is bounded, then for all $t \in \mathbb{R}_+$, $\chi_\Sigma\varphi_\lambda(t) \in L^2(\mathbb{R}_+,\nu)$. Then $P_SQ_\Sigma$ is an integral operator with kernel $N(t,x) = \chi_S(t)\mathcal{F}^{-1}(\chi_\Sigma\varphi_\lambda(t))(x)$. As $\|P_SQ_\Sigma\|_{HS} = \|N\|_{L^2(\mathbb{R}_+\times\mathbb{R}_+,\mu\otimes\mu)}$, it follows from Plancherel's theorem 1.2 that

$$\begin{aligned}
\|P_SQ_\Sigma\|^2_{HS} = & \int_{\mathbb{R}_+} |\chi_S(t)|^2 \left(\int_{\mathbb{R}_+} |\mathcal{F}^{-1}(\chi_\Sigma\varphi_\lambda(t))(x)|^2 d\mu(\xi)\right) d\mu(t) \\
= & \int_{\mathbb{R}_+} \chi_S(t) \int_{\mathbb{R}_+} \chi_\Sigma(\xi)|\varphi_\lambda(t)|^2 d\nu(\xi)d\mu(t).
\end{aligned}$$

We can deduce from $|\varphi_\lambda(t)| < 1$ that

$$\|P_S Q_\Sigma\| \leq \|P_S Q_\Sigma\|_{HS} \leq \sqrt{\mu(S)\nu(\Sigma)}. \tag{2.4}$$

Since $\mu(S)\nu(\Sigma) < 1$, then we have from inequality (2.4) and Lemma 2.1

$$\|f\|^2_{L^2(\mathbb{R}_+,\mu)} \leq \left(1 - \sqrt{\mu(S)\,\nu(\Sigma)}\right)^{-2} \left(\|P_{S^c} f\|^2_{L^2(\mathbb{R}_+,\mu)} + \|Q_{\Sigma^c} f\|^2_{L^2(\mathbb{R}_+,\mu)}\right).$$

Since $\text{supp}\mathcal{F}(f) \subset \Sigma$, it follows from Plancherel's theorem 1.2 that

$$\|Q_{\Sigma^c} f\|^2_{L^2(\mathbb{R}_+,\mu)} = \int_{\Sigma^c} |\mathcal{F}(\xi)|^2 d\nu(\xi) = \|\mathcal{F}(f)\|^2_{L^2(\Sigma^c,\nu)} = 0,$$

which shows the desired result. $\square$

*Proof of Theorem* 2.1. The result follows from inequalities (2.3) and (2.4). Indeed, $f$ is $\varepsilon_S$-time-limited on $S$, then $\|P_{S^c} f\|_{L^2(\mathbb{R}^2,\mu)} \leq \varepsilon_S \|f\|_{L^2(\mathbb{R}^2,\mu)}$. $f$ is $\varepsilon_\Sigma$-band-limited on $\Sigma$ for the Sturm-Liouville transform, then $\|Q_{\Sigma^c} f\|_{L^2(\mathbb{R}^2,\mu)} \leq \varepsilon_\Sigma \|f\|_{L^2(\mathbb{R}^2,\mu)}$. It follows that

$$\|P_{S^c} f\|^2_{L^2(\mathbb{R}^2,\mu)} + \|Q_{\Sigma^c} f\|^2_{L^2(\mathbb{R}^2,\mu)} \leq (\varepsilon_S^2 + \varepsilon_\Sigma^2)\|f\|^2_{L^2(\mathbb{R}^2,\mu)}, \tag{2.5}$$

from (2.3) we deduce that $(1 - \|P_S Q_\Sigma\|)^2 \leq \varepsilon_S^2 + \varepsilon_\Sigma^2$. It follows, from (2.4), that $1 - \sqrt{\varepsilon_S^2 + \varepsilon_\Sigma^2} \leq \|P_S Q_\Sigma\| \leq \sqrt{|S||\Sigma|}$, which proves the desired result. $\square$

*Remark* 2.1. From inequalities (2.1) and (2.4) it follows that

$$\dim(\text{Im}P_S \cap \text{Im}Q_\Sigma) \leq \|P_S Q_\Sigma\|^2_{HS} < \infty. \tag{2.6}$$

The following example is prototypical. A signal $f$ is transmitted to a receiver who know that $f$ is bandlimited on $S$ for the Sturm-Liouville transform, meaning that $f$ is synthesized using only frequency on $S$; equivalently $f = Q_\Sigma f$. Suppose that the observation of $f$ is corrupted by a noise $n \in L^2(\mathbb{R}_+, \mu)$ (which is nonetheless assumed to be small) and an unregistered values on $S$. Thus, the observable function $r$ satisfies

$$r(x) = \begin{cases} f(x) + n(x), & x \in S^c, \\ 0, & x \in S. \end{cases}$$

Here, we have assumed without loss of generality that $n = 0$ on $S$. Equivalently, $r = (I - P_S)f + n$. We say that $f$ can be stably reconstructed from $r$, if there exists a linear operator $K$ and a constant $C$ such that

$$\|f - Kr\|_{L^2(\mathbb{R}_+,\mu)} \leq C\|n\|_{L^2(\mathbb{R}_+,\mu)}. \tag{2.7}$$

The estimate (2.7) shows that the noise $n$ is at most amplified by a factor $C$.

**Corollary 2.1.** *If $S$ and $\Sigma$ are arbitrary measurable sets of $\mathbb{R}_+$ with $0 < \mu(S)\nu(\Sigma) < 1$, then $f$ can be stably reconstructed from $r$. The constant $C$ in equation* (2.7) *is not larger than* $\left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1}$.

*Proof.* If $\mu(S)\nu(\Sigma) < 1$, using (2.4), $\|P_S Q_\Sigma\| < 1$. Hence, $I - P_S Q_\Sigma$ is invertible. Let $K = (I - P_S Q_\Sigma)^{-1}$. Since $f$ is bandlimited on $\Sigma$, then $(I - P_S)f = (I - P_S Q_\Sigma)f$. Therefore,

$$\begin{aligned} f - Kr =& f - K((I - P_S)f + n) \\ =& f - K(I - P_S Q_\Sigma)f - Kn \\ =& f - (I - P_S Q_\Sigma)(I - P_S Q_\Sigma)^{-1} f - Kn \\ =& 0 - Kn. \end{aligned}$$

So, that

$$\begin{aligned} \|f - Kr\|_{L^2(\mathbb{R}_+,\mu)} =& \|Kn\|_{L^2(\mathbb{R}_+,\mu)} \\ \le& \|(I - P_S Q_\Sigma)^{-1}\| \|n\|_{L^2(\mathbb{R}_+,\mu)} \\ \le& \sum_{k=0}^{\infty} \|P_S Q_\Sigma\|^k \|n\|_{L^2(\mathbb{R}_+,\mu)} \\ \le& \sum_{k=0}^{\infty} (\mu(S)\nu(\Sigma))^{\frac{k}{2}} \|n\|_{L^2(\mathbb{R}_+,\mu)} \\ =& \left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1} \|n\|_{L^2(\mathbb{R}_+,\mu)}. \end{aligned}$$

The constant $C$ in equation (2.7) is therefore not larger than $\left(1 - \sqrt{\mu(S)\nu(\Sigma)}\right)^{-1}$. ☐

The identity $K = (I - P_S Q_\Sigma)^{-1} = \sum_{k=0}^{\infty} (P_S Q_\Sigma)^k$ suggests an algorithm for computing $Kr$. Put $f^{(n)} = \sum_{k=0}^{n} (P_S Q_\Sigma)^k r$, then

$$f^{(0)} = r, \quad f^{(n+1)} = r + P_S Q_\Sigma f^{(n)} \quad \text{and} \quad f^{(n)} \to Kr \text{ as } n \to \infty.$$

As $f$ is bandlimited on $\Sigma$ we deduce that

$$f^{(n+1)} - f = P_S Q_\Sigma (f^{(n)} - f). \tag{2.8}$$

Algorithms of this type have applied to a host of problems in signal recovery (see for examples [12, 17]).

## 3. Uncertainty Principles

In this section we will give some remarks about annihilating sets.

**Proposition 3.1.** *Let $f \in L^2(\mathbb{R}_+, \mu)$ has non empty support, then*

$$\nu(\mathrm{supp}\mathcal{F})\mu(\mathrm{supp} f) \ge 1.$$

*In particular, if $\mu(\mathrm{supp} f)\nu(\mathrm{supp}\mathcal{F}) < 1$, then $f = 0$.*

*Proof.* If the function $f \in L^2(\mathbb{R}_+, \mu)$ has non empty support, by the Cauchy-Schwartz inequality and (1.1), we have

$$\begin{aligned}\|\mathcal{F}\|^2_{L^2(\mathbb{R}_+,\nu)} \leq & \nu(\text{supp } \mathcal{F}(f))\|\mathcal{F}(f)\|^2_\infty \\ \leq & \nu(\text{supp } \mathcal{F}(f))\|f\|^2_{L^1(\mathbb{R}_+,\mu)} \\ \leq & \nu(\text{supp } \mathcal{F}(f))\mu(supp\ f)\|f\|^2_{L^2(\mathbb{R}_+,\mu)}.\end{aligned}$$

Using Plancherel's theorem 1.2 we have the following quantitative uncertainty inequality connecting the support of $f$ and the support of its Sturm-Liouville transform $\mathcal{F}$

$$\nu(\text{supp}\mathcal{F})\mu(\text{supp}f) \geq 1. \tag{3.1}$$

It follows that if $\mu(\text{supp}f)\nu(\text{supp}\mathcal{F}) < 1$, then $f = 0$. □

**Proposition 3.2.** *Let $f \in L^1(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then*

$$\|\mathcal{F}\|_{L^q(\mathbb{R}_+,\nu)} \leq \nu(\text{supp}\mathcal{F}(f))^{1/q}\mu(\text{supp}f)^{1/q}\|f\|_{L^p(\mathbb{R}_+,\mu)},$$

*where $q = \frac{p}{p-1}$.*

*Proof.* Let $f \in L^1(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then by Hölder's inequality and (1.1), we get

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^q(\mathbb{R}_+,\nu)} \leq & \nu(\text{supp}\mathcal{F}(f))^{1/q}\|\mathcal{F}(f)\|_\infty \\ \leq & \nu(\text{supp}\mathcal{F}(f))^{1/q}\|f\|_{L^1(\mathbb{R}_+,\mu)} \\ \leq & \nu(\text{supp}\mathcal{F}(f))^{1/q}\mu(\text{supp}f)^{1/q}\|f\|_{L^p(\mathbb{R}_+,\mu)}.\end{aligned}$$

□

**Proposition 3.3.** *Let $f \in L^2(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then*

$$1 < \nu(\text{supp}\mathcal{F}(f))^{\frac{q-2}{2q}}\mu(\text{supp}f)^{\frac{2-p}{2p}},$$

*where $q = \frac{p}{p-1}$.*

*Proof.* Let $f \in L^2(\mathbb{R}_+, \mu) \cap L^p(\mathbb{R}_+, \mu)$, $1 < p \leq 2$, then by (1.1), Hölder's inequality and Riez's interpolation, we get

$$\begin{aligned}\|\mathcal{F}(f)\|_{L^2(\mathbb{R}_+,\nu)} \leq & \nu(\text{supp}\mathcal{F}(f))^{\frac{q-2}{2q}}\|\mathcal{F}(f)\|_{L^q(\mathbb{R}_+,\nu)} \\ \leq & \nu(\text{supp}\mathcal{F}(f))^{\frac{q-2}{2q}}\|f\|_{L^p(\mathbb{R}_+,\mu)} \\ \leq & \nu(\text{supp}\mathcal{F}(f))^{\frac{q-2}{2q}}\mu(\text{supp}f)^{\frac{2-p}{2p}}\|f\|_{L^2(\mathbb{R}_+,\mu)},\end{aligned}$$

by Plancherel's formula we get the desired result. □

**Lemma 3.1.** *Any nonzero function in $\mathcal{C}_0(\mathbb{R}^d_+)$ has linearly independent dilates.*

*Proof.* In the case $d = 1$ this Lemma was proved in [8]. The case $d > 1$ we reduce to the case $d = 1$. Let $f \in \mathcal{C}_0(\mathbb{R}^d_+)$ such that $f \neq 0$, if $x = (x_1, \ldots, x_{d-1}, 0) = r(x', 0) \in \mathbb{R}^d_+$, $\sum_{i=1}^{d-1} |x'_i|^2 = 1$, $r \in \mathbb{R}_+$, we get $g(r) = f(r(x', 0))$.

If $x_d > 0$, $x = r(\theta x', 1)$, where $\sum_{i=1}^{d-1} |x'_i|^2 = 1$, and $r$, $\theta \in \mathbb{R}^+$, we get $g(r) = f(r(\theta x', 1))$. In both cases $g(r) \in \mathcal{C}_0(\mathbb{R}_+)$. □

**Lemma 3.2.** *Let $S_0$ and $\Sigma_0$ be a pair of measurable subsets of $\mathbb{R}_+$ with $0<\mu(S_0),\nu(\Sigma_0)<\infty$, then exist an infinite sequence of distinct numbers $(\rho_j)_{j=0}^{\infty}\subset(0,\infty)$ such that*

$$\mu(\cup_{j=0}^{\infty}\rho_j S_0)<2\mu(S_0)\quad \textit{and}\quad \nu\left(\cup_{j=0}^{\infty}\frac{1}{\rho_j}\Sigma_0\right)<2\nu(\Sigma_0).$$

*Proof.* Let $S_1$ be a measurable subset of $\mathbb{R}_+$ of finite Lebesgue measure such that $S_0\subset S_1$. Define $h:\mathbb{R}_+\to\mathbb{R}_+$ by $h(\rho)=\mu(S_1\cup\rho S_0)$. Since $\chi_{\rho S_0}$ and $\chi_{S_1}$ are in $L^2(\mathbb{R}_+,\mu)$, we may express $h$ in terms of scalar product in $L^2(\mathbb{R}_+,\mu)$

$$h(\rho)=\|\chi_{\rho S_0}-\chi_{S_1}\|^2_{L^2(\mathbb{R}_+,\mu)}+\langle\chi_{\rho S_0},\chi_{S_1}\rangle_{L^2(\mathbb{R}_+,\mu)}.$$

The function $\rho\mapsto h(\rho)$ is a continuous function on $(0,\infty)$. We deduce that there exist an infinite sequence of distinct numbers $(\rho)_{j=0}^{\infty}\subset(0,\infty)$, with $\rho_0=1$ such that $\mu(\cup_{j=0}^{\infty}\rho_j S_0)<2\mu(S_0)$. We can follow the same techniques to prove that

$$\nu\left(\bigcup_{j=0}^{\infty}\frac{1}{\rho_j}\Sigma_0\right)<2\nu(\Sigma_0).\qquad\square$$

We are now in position to prove Benedicks-type theorem for the Sturm-Liouville transform.

**Theorem 3.1.** *Let $S$ and $\Sigma$ be a pair of measurable subsets of $\mathbb{R}_+$, with $0<\mu(S)$, $\nu(\Sigma)<\infty$, then the pair $(S,\Sigma)$ is weakly annihilating pair.*

*Proof.* Suppose that there exist $f_0\neq 0$ such that $S_0=\operatorname{supp}f_0$ and $\Sigma_0=\operatorname{supp}\mathcal{F}(f_0)$ have both finite measure $0<\mu(\operatorname{supp}f_0),\nu(\operatorname{supp}\mathcal{F}(f_0))<\infty$. From Lemma 3.2 we can find an infinite sequence of distinct numbers $(\rho)_{j=0}^{\infty}\subset(0,\infty)$, with $\rho_0=1$, such that, if we denote by $S=\cup_{j=0}^{\infty}\rho_j\operatorname{supp}f_0$ and $\Sigma=\cup_{j=0}^{\infty}\frac{1}{\rho_j}\operatorname{supp}\mathcal{F}(f_0)$ we have $\mu(S)<2\mu(S_0)$, $\nu(\Sigma)<2\nu(\Sigma_0)$.

Put $f_i=D_{\rho_i}f_0$, so that supp $f_i=\rho_i\operatorname{supp}f_0$. As $\mathcal{F}(f_i)=D_{\frac{1}{\rho_i}}\mathcal{F}(f_0)$ we have $\operatorname{supp}\mathcal{F}(f_i)=\frac{1}{\rho_i}\operatorname{supp}\mathcal{F}(f_0)$. Since $\operatorname{supp}\mathcal{F}(f_0)$ has finite measure, $f_0\in\mathcal{C}_0(\mathbb{R}_+)$. It follows from Lemma 3.1 that $(f_i)_{i=0}^{\infty}$ are linearly independent vectors belonging to $\operatorname{Im}P_S\cap\operatorname{Im}Q_\Sigma$ which contradicts (2.6). Then, $(S,\Sigma)$ is weakly annihilating. $\square$

**Theorem 3.2** (Benedicks-type theorem)**.** *Let $S$ and $\Sigma$ be a pair of measurable subsets of $\mathbb{R}_+$ with $0<\mu(S)$, $\nu(\Sigma)<\infty$, then the pair $(S,\Sigma)$ is strong annihilating pair.*

*Proof.* Assume there is no such constant $C(S,\Sigma)$. We can find a sequence $f_n\in L^2(\mathbb{R}_+,\mu)$ of norm 1 weakly convergent in $L^2(\mathbb{R}_+,\mu)$ with some limit $f$ such that

$$\operatorname{supp}f_n\subset S \text{ and } \|\chi_{\Sigma^c}\mathcal{F}(f_n)\|_{L^2(\mathbb{R}_+,\nu)}\to 0 \text{ as } n\to\infty.$$

Since $\mathcal{F}(f_n)$ is the scalar product of $f_n$ and $\chi_S\varphi_\lambda(\cdot)$, it follows that $\mathcal{F}(f_n)$ converge to $\mathcal{F}(f)$. Since $|\mathcal{F}(f_n)|$ as bounded by $\sqrt{\mu(S)}$, it follows from Lebesgue's theorem that $\mathcal{F}(f_n)\chi_\Sigma$ converges to $\mathcal{F}(f)$ in $L^2(\mathbb{R}_+,\nu)$ and the limit $f$ has norm 1. But the function $f$ has support in $S$ and spectrum in $\Sigma$, since $(S,\Sigma)$ is a weak annihilating pair, it follows that $f=0$, which gives a contradiction. $\square$

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# A STUDY OF CONFORMALLY FLAT QUASI-EINSTEIN SPACETIMES WITH APPLICATIONS IN GENERAL RELATIVITY

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Abstract. In this paper we consider conformally flat $(QE)_4$ spacetime and obtained several important results. We study application of conformally flat $(QE)_4$ spacetime in general relativity and Ricci soliton structure in a conformally flat $(QE)_4$ perfect fluid spacetime.

## 1. Introduction

An Einstein manifold is a Riemannian or pseudo-Riemannian manifold whose Ricci tensor $S$ of type $(0,2)$ is non-zero and propotional to the metric tensor. Einstein manifolds form a natural subclass of various classes of Riemannian or pseudo-Riemannian manifolds by a curvature condition imposed on their Ricci tensor [4]. Also in Riemannian geometry as well as in general relativity theory, the Einstein manifold play an very important role.

The quasi-Einstein manifolds are generalization of Einstein manifolds. The notion of quasi-Einstein manifolds was introduced by Chaki and Maity [6] in 2000. According to them, a Riemannian manifold or pseudo-Riemannian manifold is said to be a quasi-Einstein manifold if its Ricci tensor $S$ of type $(0,2)$ is non-zero and satisfies the condition

$$S(X,Y) = \alpha g(X,Y) + \beta A(X)A(Y), \tag{1.1}$$

where $\alpha$ and $\beta$ are real valued non-zero scalar functions and $A$ is a non-zero 1-form equivalent to the vector field $\omega$, i.e., $g(X,\omega) = A(X)$, $g(\omega,\omega) = 1$. Here $A$ is called an associated 1-form and $\omega$ is called a generator. If $\beta = 0$, then the

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manifold reduces to an Einstein manifold. This kind of $n$-dimensional manifold is denoted by $(QE)_n$. Quasi-Einstein manifolds arose during the study of exact solutions of the Einstein field equations as well as during the considerations of quasi-umbilical hypersurfaces of semi-Euclidean spaces. For instance, the Robertson-Walker spacetimes and conformally flat almost pseudo-Ricci symmetric spacetimes are quasi-Einstein manifolds. Also, quasi-Einstein manifolds can be taken as a model of perfect fluid spacetime in general relativity. The importance of quasi-Einstein spacetimes lies in the fact that 4-dimensional pseudo-Riemannian manifolds are related to study of general relativistic fluid spacetime, where the generator vector field $\omega$ is taken as timelike velocity vector field, that is, $g(\omega, \omega) = -1$.

In the paper [5], Chaki and Ray studied spacetimes with covariant constant energy momentum tensor. In recent paper [11,17], they studied the quasi-Einstein spacetime and generalized quasi-Einstein spacetime in general relativity. Additionally, there are many works related with spacetime in general relativity [1,13,14,16,19].

The authors De, Özgür and De showed that conformally flat almost pseudo-Ricci symmetric spacetime can be considered as a model of the perfect fluid spacetime in general relativity and also obeying Einstein equation without cosmological constant and having the vector as velocity vector is infinitesimally spatially isotropic relative to the unit timelike vector field [8]. In [9], they proved that conformally flat perfect fluid spacetime with semisymmetric energy momentum tensor is a spacetime of quasi constant curvature and such spacetime determines an equation of state in quintessence era, where the universe is in an accelerating phase. Therefore it is meaningful to study a conformally flat $(QE)_4$ spacetime in general relativity.

The present paper organized as follows. After preliminaries, in Section 3, we study conformally flat $(QE)_4$ spacetime. In Section 4, we prove that conformally flat Ricci pseudosymmetric $(QE)_4$ spacetime is an $N\left(\frac{2\alpha-5\beta}{6}\right)$ quasi-Einstien spacetime, provided $g(Y,Z)A(X) \neq g(X,Z)A(Y)$. In Section 5, we study conformally flat $(QE)_4$ perfect fluid spacetime and obtained some interesting results on conformally flat $(QE)_4$ spacetime in general reltivity. Finally, we study Ricci soliton structure of conformally flat $(QE)_4$ sapcetime in general relativity.

## 2. Preliminaries

Consider $(QE)_4$ spacetime with associated scalars $\alpha$, $\beta$ and associated 1-form $A$. Then by (1.1), we have

$$r = 4\alpha - \beta, \tag{2.1}$$

where $r$ is a scalar curvature of the spacetime. If $\omega$ is orthogonal unit vector field, then $g(\omega, \omega) = -1$. Again from (1.1), we have

$$\begin{aligned} S(X,\omega) &= (\alpha - \beta)A(X), \\ S(\omega,\omega) &= \beta - \alpha. \end{aligned} \tag{2.2}$$

Let $Q$ be the symmetric endomorphism of the tangent space at each point of the manifold corresponding to the Ricci tensor $S$. Then $g(QX,Y) = S(X,Y)$ for all $X, Y$.

## 3. Conformally Flat $(QE)_4$ Spacetime

A quasi-Einstein spacetime is said to be conformally flat, if the Weyl conformal curvature tensor $C$ vanishes and is defined by [8, 22]

$$\begin{aligned} C(X,Y)Z =& R(X,Y)Z - \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &+ \frac{r}{6}\{g(Y,Z)X - g(X,Z)Y\}, \end{aligned} \tag{3.1}$$

where $Q$ is the Ricci operator defined by $g(QX,Y) = S(X,Y)$ and $r$ is the scalar curvature.

Now, suppose that $(QE)_4$ spacetime is conformally flat. Then by (3.1), we get

$$\begin{aligned} R(X,Y)Z =& \frac{1}{2}\{S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY\} \\ &- \frac{r}{6}\{g(Y,Z)X - g(X,Z)Y\}. \end{aligned} \tag{3.2}$$

From (1.1), we have

$$QX = \alpha X + \beta A(X)\omega. \tag{3.3}$$

Substituting (1.1) and (3.3) in (3.2), we obtain

$$\begin{aligned} R(X,Y,Z,W) =& \left(\frac{2\alpha+\beta}{6}\right)\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ \frac{\beta}{2}\{g(X,W)A(Y)A(Z) - g(Y,W)A(X)A(Z) \\ &+ g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W)\}, \end{aligned} \tag{3.4}$$

which leads to

$$R(X,Y,Z,W) = B(Y,Z)B(X,W) - B(X,Z)B(Y,W), \tag{3.5}$$

where

$$B(X,Y) = \sqrt{\frac{2\alpha+\beta}{6}}g(X,Y) + \frac{\beta\sqrt{3}}{\sqrt{4\alpha+2\beta}}A(X)A(Y). \tag{3.6}$$

It is known that an $n$-dimensional Riemannian or pseudo-Riemannian manifold whose curvature tensor $R$ of type $(0,4)$ satisfies the condition (3.5), where $B$ is a symmetric tensor field of type $(0,2)$, is called a special manifold with the associated symmetric tensor $B$ and is denoted by the symbol $\psi(B)_n$. Recently, these type of manifolds are studied in [15, 18].

By virtue of (3.5) and (3.6), we have the following theorem.

**Theorem 3.1.** *A conformally flat $(QE)_4$ spacetime is $\psi(B)_4$ with associated symmetric tensor $B$ given by* (3.6)*.*

Chen and Yano [7] introduced the concept of manifold of a quasi-constant curvature. A spacetime is said to be of quasi-constant curvature if the curvature tensor $R$ of type $(0,4)$ satisfies

$$\begin{aligned} R(X,Y,Z,W) =& a\{g(Y,Z)g(X,W) - g(X,Z)g(Y,W)\} \\ &+ b\,\{g(X,W)\varpi(Y)\varpi(Z) - g(Y,W)\varpi(X)\varpi(Z) \\ &+ g(Y,Z)\varpi(X)\varpi(W) - g(X,Z)\varpi(Y)\varpi(W)\}, \end{aligned} \tag{3.7}$$

where $a$ and $b$ are scalars and there exists a unit vector field $\nu$ such that $g(X,\nu) = \varpi(X)$. If $b = 0$, then the spacetime is of constant curvature $a$. Comparing the equation (3.4) and (3.7), we have the following.

**Theorem 3.2.** *A conformally flat $(QE)_4$ spacetime is a spacetime of quasi-constant curvature.*

Let $(M^4, g)$ be a conformally flat $(QE)_4$ spacetime. As $C = 0$, we have div$C = 0$, where div denotes the divergence. Hence, from (3.2) we have

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{6}\{g(Y,Z)dr(X) - g(X,Z)dr(Y)\}.$$

In view of (1.1), above relation takes the form

$$\begin{aligned} & d\alpha(X)g(Y,Z) - d\alpha(Y)g(X,Z) + d\beta(X)A(Y)A(Z) - d\beta(Y)A(X)A(Z) \\ & + \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z) \\ & - A(X)(\nabla_Y A)(Z)] \\ =& \frac{1}{6}\left[g(Y,Z)dr(X) - g(X,Z)dr(Y)\right]. \end{aligned} \tag{3.8}$$

Suppose the scalar curvature $r$ is constant, then from (2.1) we have

$$4d\alpha(X) = d\beta(X). \tag{3.9}$$

Using above equation in (3.8), we get

$$\begin{aligned} & d\alpha(X)[g(Y,Z) + 4A(Y)A(Z)] - d\alpha(Y)[g(X,Z) + 4A(X)A(Z)] \\ & + \beta[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - (\nabla_Y A)(X)A(Z) - A(X)(\nabla_Y A)(Z)] = 0. \end{aligned} \tag{3.10}$$

Taking a frame field after contraction over $Y$ and $Z$, we obtain from (3.10) that

$$d\alpha(X) + 4d\alpha(\omega)A(X) + \beta[(\nabla_\omega A)(X) + A(X)\sum_{i=1}^{4}\epsilon_i(\nabla_{e_i}A)(e_i)] = 0, \tag{3.11}$$

where $\epsilon_i = g(e_i, e_i) = \pm 1$. Plugging $\omega$ in place of $Y$ and $Z$ in (3.10), we get

$$3[d\alpha(X) + d\alpha(\omega)A(X)] + \beta(\nabla_\omega A)(X) = 0. \tag{3.12}$$

In view of (3.12) and (3.11), we obtain

$$-2d\alpha(X) + d\alpha(\omega)A(X) + \beta A(X)\sum_{i=1}^{4}\epsilon_i(\nabla_{e_i}A)(e_i) = 0. \tag{3.13}$$

Now, putting $X = \omega$ in the above equation, we get

$$\beta\sum_{i=1}^{4}\epsilon_i(\nabla_{e_i}A)(e_i) = -3d\alpha(\omega). \tag{3.14}$$

From (3.13) and (3.14), it follows that

$$d\alpha(X) = -d\alpha(\omega)A(X). \tag{3.15}$$

Setting $Z = \omega$ in (3.10) and then using (3.13) and $\beta \neq 0$, we get

$$(\nabla_X A)(Y) = (\nabla_Y A)(X). \tag{3.16}$$

Above equation shows that 1-form $A$ is of Codazzi type, this means that generator $\omega$ is irrotational. By virtue of (3.15), (3.12) and $\beta \neq 0$, it follows that

$$(\nabla_\omega A)(X) = 0, \tag{3.17}$$

for all $X$, which implies that $\nabla_\omega \omega = 0$ and hence intergral curves of $\omega$ are geodesic. Again, setting $Y = \omega$ in (3.10) and then using (3.15) and (3.17), we get

$$(\nabla_X A)(Z) = -\frac{d\alpha(\omega)}{4\alpha}[A(X)A(Z) + g(X,Z)]. \tag{3.18}$$

Now, we consider non-vanishing scalar function $f = -\frac{d\alpha(\omega)}{4\alpha}$. Then, we have

$$\nabla_X f = \frac{d\alpha(\omega)}{4\alpha^2}d\alpha(X) - \frac{d^2\alpha(\omega,X)}{4\alpha}. \tag{3.19}$$

By virtue of (3.15), we get $d^2\alpha(X,Y) = -d^2\alpha(\omega,Y)A(X) - d\alpha(\omega)(\nabla_Y A)(X)$. In a Lorentzian manifold, the scalar function $\eta$ satisfies the relation $d^2\eta(X,Y) = d^2\eta(Y,X)$, for all $X, Y$. In view of (3.16), the above relation becomes

$$d^2\alpha(\omega,X)A(Y) = d^2\alpha(\omega,Y)A(X).$$

Taking $Y = \omega$ in the above equation, we get

$$d^2\alpha(\omega,X) = -d^2\alpha(\omega,\omega)A(X) = -\psi A(X), \tag{3.20}$$

where $\psi = d^2\alpha(\omega,\omega)$ is a scalar function. Now in the consequence of (3.20) and (3.15), equation (3.19) takes the form

$$\nabla_X f = -\frac{1}{4\alpha^2}[\{d\alpha(\omega)\}^2 - \alpha\psi]A(X). \tag{3.21}$$

Now consider a 1-form $h$ given by

$$h(X) = -\frac{d\alpha(\omega)}{4\alpha}A(X) = fA(X). \tag{3.22}$$

From (3.16), (3.21) and (3.22) we have $dh(X,Y) = 0$, i.e., the 1-form $h$ is closed. Therefore (3.18) can be written as $(\nabla_X A)(Z) = h(X)A(Z) + fg(X,Z)$. This means that the generator $\omega$ corresponding to the 1-form $A$ is a unit proper concircular vector field [20]. This leads to the following theorem.

**Theorem 3.3.** *In a conformally flat $(QE)_4$ spacetime with constant scalar curvature, the following properties hold:*

i. *the generator vector field $\omega$ is irrotational;*
ii. *the integral curves of $\omega$ are geodesic;*
iii. *the vector field $\omega$ corresponding to the 1-form $A$ is a unit proper concircular vector field.*

**Lemma 3.1.** *In a conformally flat $(QE)_4$ spacetime, the curvature tensor $R$ of type $(1,3)$ satisfies the following properties:*

(i) $R(X,Y)Z = \left(\frac{2\alpha+\beta}{6}\right)\{g(Y,Z)X - g(X,Z)Y\}$;
(ii) $R(X,\omega)Y = \left(\frac{\beta-\alpha}{3}\right)g(X,Y)\omega$;
(iii) $R(X,\omega)\omega = \left(\frac{\beta-\alpha}{3}\right)X$,

*for all $X, Y, Z \in \omega^{\perp}$, the 3-dimensional distribution orthogonal to the generator $\omega$.*

*Proof.* In a conformally flat $(QE)_4$ spacetime, we have the relation (3.4). Since $\omega^{\perp}$ is a 3-dimensional distribution orthogonal to the generator $\omega$, we have $g(X,\omega) = 0$ if and only if $X \in \omega^{\perp}$. Hence (3.4) yields the relation (i)-(iii) for all $X, Y, Z \in \omega^{\perp}$. This proves the lemma. □

Let $X, Y, Z \in \omega^{\perp}$. Let $K_1$ be the sectional curvature of the plane determined by $X$ and $Y$ and $K_2$ be the sectional curvature of the plane determined by $X$ and $\omega$. Then

$$K_1 = \frac{g(R(X,Y)Y,X)}{g(X,X)g(Y,Y) - g(X,Y)^2}, \quad K_2 = \frac{g(R(X,\omega)\omega,X)}{g(X,X)g(\omega,\omega) - g(X,\omega)^2}.$$

By virtue of (i) and (iii) in Lemma 3.1 , we have $K_1 = \frac{2\alpha+\beta}{6}$ and $K_2 = \frac{\alpha-\beta}{3}$. Hence, we state the following.

**Lemma 3.2.** *In a conformally flat $(QE)_4$ spacetime, the sectional curvature of all planes determined by $X, Y \in \omega^{\perp}$ is $\frac{2\alpha+\beta}{6}$ and the sectional curvature of all planes determined by $X$ and $\omega$, where $X \in \omega^{\perp}$ is $\frac{\alpha-\beta}{3}$.*

We note that $K_1$ and $K_2$ are constants if and only if $\alpha$ and $\beta$ are constant. So the following corollary arises.

**Corollary 3.1.** *In a conformally flat $(QE)_4$ spacetime, the sectional curvature $K_1$ of all planes determined by $X$ and $Y$ as well as the sectional curvature $K_2$ of all planes determined by $X$ and $\omega$ are constants if and only if $\alpha$ and $\beta$ are constant.*

*Remark* 3.1. We know that, a pseudo-Riemannian manifold of constant sectional curvature is locally symmetric. Suppose $\alpha$ and $\beta$ are constant, then from Corollary 3.1, we say that conformally flat $(QE)_4$ spacetime is locally symmetric if and only if $\frac{2\alpha+\beta}{6}$ is constant, provided that the vectors are orthogonal to the generator $\omega$.

By virtue of (3.4), we obtain

$$R(X,Y)\omega = \frac{\alpha-\beta}{3}\left\{A(Y)X - A(X)Y\right\},$$
$$R(X,\omega)Y = \frac{\alpha-\beta}{3}\left\{A(Y)X - g(X,Y)\omega\right\}. \tag{3.23}$$

From the Theorem 3.2, we know that conformally flat $(QE)_4$ spacetime is of quasi-constant curvature and is said to be regular if $\alpha - \beta \neq 0$.

Bejan and Crasmareanu [3] proved that a parallel second order symmetric covariant tensor in a regular manifold of quasi-constant curvature is a constant multiple of the metric tensor. Hence we have the following.

**Theorem 3.4.** *A parallel and symmetric second order covariant tensor field in a conformally flat $(QE)_4$ spacetime with $\alpha \neq \beta$, is a constant multiple of the metric tensor, that is $h(X,Y) = h(\omega,\omega)g(X,Y)$, where $h$ is a symmetric tensor field of type $(0,2)$.*

Let us consider a second order symmetric tensor $h = L_\omega g + 2S$, where $L_\omega$ is the Lie derivative with respect to $\omega$. Then

$$h(\omega,\omega) = (L_\omega g)(\omega,\omega) + 2S(\omega,\omega). \tag{3.24}$$

Since $g(\omega,\omega) = -1$, it follows that

$$(\nabla_X A)(\omega) = g(\nabla_X \omega, \omega) = 0.$$

Therefore, $(L_\omega g)(\omega,\omega) = 2g(\nabla_\omega \omega, \omega) = 0$ (because $\nabla_\omega \omega \perp \omega$). In view of (2.2) and (3.24), we obtain

$$h(\omega,\omega) = 2(\beta - \alpha). \tag{3.25}$$

By virtue of Theorem 3.4 and (3.25), we have

$$h(X,Y) = 2(\beta-\alpha)g(X,Y). \tag{3.26}$$

Thus, we have $(L_\omega g)(X,Y) + 2S(X,Y) + 2(\alpha-\beta)g(X,Y) = 0$. This expression defines Ricci soliton on confomally flat $(QE)_4$ spacetime if $(\alpha-\beta)$ is constant. Hence, we conclude the following.

**Theorem 3.5.** *In a conformally flat $(QE)_4$ spacetime, the symmetric tensor field $h = L_\omega g + 2S$ of type $(0,2)$ is parallel with respect to Levi-Civita connection $\nabla$ of $g$, then the relation (3.26) defines a Ricci soliton, provided that $\alpha - \beta$ is constant. In this case, Ricci soliton is called expanding or steady or shrinking according as $\alpha - \beta$ is positive or zero or negative, respectively.*

## 4. Conformally Flat Ricci pseudosymmetric $(QE)_4$ Spacetime

An $n$-dimensional pseudo-Riemannian manifold is said to be Ricci pseudosymmetric if the tensor $R \cdot S$ and Tachibana tensor $Q(g,S)$ are lineraly dependent, i.e.,

$$(R(X,Y) \cdot S(Z,W)) = L_S Q(g,S)(Z,W;X,Y) \tag{4.1}$$

holds on $U_S$, where $U_S = \left\{x \in M : S \neq \frac{r}{n} g \ at \ x\right\}$, $L_S$ is a certain function on $U_S$ and

$$(R(X,Y) \cdot S(Z,W)) = -S(R(X,Y)Z,W) - S(Z,R(X,Y)W), \tag{4.2}$$

$$L_S Q(g,S)(Z,W;X,Y) = -S((X \Lambda_g Y)Z,W) - S(Z,(X \Lambda_g Y)W), \tag{4.3}$$

$$(X \Lambda_g Y)Z = g(Y,Z)X - g(X,Z)Y. \tag{4.4}$$

Suppose a conformally flat $(QE)_4$ spacetime is Ricci pseudosymmetric. Then making use of (4.2)–(4.4) in (4.1), we obtain

$$\begin{aligned} S(R(X,Y)Z,W) + S(Z,R(X,Y)W) =& L_S[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) \\ &+ g(Y,W)S(Z,X) - g(X,W)S(Y,Z)]. \end{aligned} \tag{4.5}$$

Substituting (1.1) in (4.5), we have

$$\begin{aligned} &A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W) \\ =& L_S[g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) + g(Y,W)A(Z)A(X) \\ &- g(X,W)A(Y)A(Z)]. \end{aligned}$$

Plugging $W$ by $\omega$ in previous equation and making use of the property $g(R(X,Y)\omega,\omega) = g(R(\omega,\omega)X,Y) = 0$, we get

$$A(R(X,Y)Z) = L_S[g(Y,Z)A(X) - g(X,Z)A(Y)]. \tag{4.6}$$

In view of (2.1) and (3.4), (4.6) yields

$$\left[L_S - \left(\frac{2\alpha - 5\beta}{6}\right)\right]\{g(Y,Z)A(X) - g(X,Z)A(Y)\} = 0,$$

which yields either $g(Y,Z)A(X) = g(X,Z)A(Y)$ or $\left[L_S - \left(\frac{2\alpha-5\beta}{6}\right)\right] = 0$.

Suppose $g(Y,Z)A(X) \neq g(X,Z)A(Y)$, then we have

$$L_S = \frac{2\alpha - 5\beta}{6}. \tag{4.7}$$

In view of (4.6) and (4.7), we have

$$R(X,Y)Z = \frac{2\alpha - 5\beta}{6}\{g(Y,Z)X - g(X,Z)Y\},$$

which means that the generator vector field $\omega$ belongs to $\frac{2\alpha-5\beta}{6}$-nullity distrbution. This leads to the following.

**Theorem 4.1.** *Every conformally flat Ricci pseudosymmetric $(QE)_4$ spacetime with $g(Y,Z)A(X) \neq g(X,Z)A(Y)$ is an $N\left(\frac{2\alpha-5\beta}{6}\right)$-quasi-Einstein spacetime.*

## 5. Conformally Flat $(QE)_4$ Spacetimes with Applications in General Relativity

Ricci tensor is a part of curvature of spacetime that determines the degree to which matter will tend to converge or diverge in time. It is related to the matter content of universe by means of the Einstein field equation

$$S(X,Y) + \left(\Lambda - \frac{r}{2}\right) g(X,Y) = \kappa T(X,Y), \quad \text{for all } X, Y, \tag{5.1}$$

where $S$ is the Ricci tensor, $r$ is the scalar curvature, $\Lambda$ is the cosmological constant and $\kappa$ is the gravitational constant. Einstein's field equation shows that the energy momentum tensor is symmetric of type $(0,2)$ with divergence zero.

For the perfect fluid matter distribution, the energy momentum tensor is given by

$$T(X,Y) = \rho g(X,Y) + (\sigma + \rho)A(X)A(Y), \tag{5.2}$$

where $\sigma$ is energy density and $\rho$ is the isotropic pressure of the fluid.

Here we consider a conformally flat $(QE)_4$ spacetime obeying Einstein's field equation with cosmological constant whose matter content is perfect fluid. Then, in view of (5.1) and (5.2), Ricci tensor takes the form

$$S(X,Y) = \left(\kappa\rho - \Lambda + \frac{r}{2}\right) g(X,Y) + \kappa(\sigma + \rho)A(X)A(Y). \tag{5.3}$$

Compare (5.3) with (1.1), we have

$$\alpha = \kappa\rho - \Lambda + \frac{r}{2}, \quad \beta = \kappa(\sigma + \rho).$$

Contracting (5.3) and taking into account that $g(\omega, \omega) = -1$, we have

$$r = 4\Lambda + \kappa(\sigma - 3\rho). \tag{5.4}$$

By virtue of (5.4) and (5.3), it follows that

$$S(X,Y) = \left(\Lambda + \frac{\kappa(\sigma - \rho)}{2}\right) g(X,Y) + \kappa(\sigma + \rho)A(X)A(Y). \tag{5.5}$$

Now differentiating (5.5) covariantly, we get

$$\begin{aligned}(\nabla_X S)(Y,Z) =& \frac{\kappa}{2} X(\sigma - \rho) g(Y,Z) + \kappa X(\sigma + \rho)A(Y)A(Z) \\ &+ \kappa(\sigma + \rho)[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)].\end{aligned} \tag{5.6}$$

Let us suppose that conformally flat $(QE)_4$ perfect fluid spacetime is Ricci symmetric, i.e., $\nabla S = 0$, then in view of (3.18) and (5.6), it follows that

$$\begin{aligned}0 =& \frac{\kappa}{2} X(\sigma - \rho) g(Y,Z) + \kappa X(\sigma + \rho)A(Y)A(Z) \\ &+ f\kappa(\sigma + \rho)[2A(X)A(Y)A(Z) + g(X,Y)A(Z) + g(X,Z)A(Y)].\end{aligned} \tag{5.7}$$

Taking contraction on (5.7) over $Y$ and $Z$, we get

$$X(\sigma - 3\rho) = 0. \tag{5.8}$$

This shows that $\sigma - 3\rho$ is constant. Hence, we state the following.

**Theorem 5.1.** *If a conformally flat $(QE)_4$ perfect fluid spacetime obeying Einstein field equation with cosmological constant is Ricci symmetric, then $\sigma - 3\rho$ is constant.*

*Remark* 5.1. Let us take constant as zero in the equation (5.8). Then the isotropic pressure $\rho$ is $\sigma/3$ which means that it characterizes radiation era. Therefore radiation has the equation of state $\upsilon = 1/3$ and it predicts that the resulting universe is isotropic and homogenous [10].

Let us consider the energy momentum tensor which is $\eta$-recurrent, i.e., $(\nabla_X T)(Y,Z) = \eta(X)T(Y,Z)$, where $\eta$ is a nonzero 1-form. By Einstein field equation, this condition becomes

$$(\nabla_X S)(Y,Z) - \frac{dr(X)}{2} g(Y,Z) = \eta(X)S(Y,Z) + \eta(X)\left(\Lambda - \frac{r}{2}\right) g(Y,Z).$$

Recall that the scalar curvature $r$ is constant. Replacing $r$ from (5.4), $S$ from (5.5) and $\nabla S$ from (5.6), we get

$$\begin{aligned}\kappa\rho\eta(X)g(Y,Z) =& \frac{\kappa}{2} X(\sigma-\rho)g(Y,Z) + \kappa X(\sigma+\rho)A(Y)A(Z) \\ &+ \kappa(\sigma+\rho)[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z) - \eta(X)A(Y)A(Z)].\end{aligned}$$

Plugging $Y = Z = \omega$ in the above equation, we have

$$X(\sigma + 3\rho) = 2\eta(X)(2\sigma + \rho). \tag{5.9}$$

Hence, we conclude the following.

**Theorem 5.2.** *If the energy momentum tensor $T$ of conformally flat $(QE)_4$ perfect fluid spacetime is $\eta$-recurrent, then energy density and isotropic pressure satisfies the relation* (5.9).

*Remark* 5.2. For an $\eta$-recurrent energy-momentum tensor, if energy density and isotropic pressure are constants, then $\sigma = -1/2\rho$. For a perfect fluid, $T$ is given in (5.2) which takes the form $T(X,Y) = \rho\left[g(X,Y) + \frac{1}{2}A(X)A(Y)\right]$.

In this case we observe that the equation of state $\upsilon$ is -2 which is less than -1, showimg that the existence of phantom energy. We know that phantom energy is a hypothetical form of dark energy with $\upsilon < -1$ [2]. The existence of phantom energy could cause the expansion of the universe to accelerate so quickly that a scenario known as the **Big Rip**, a possible end to the universe occurs and violates weak energy condition.

## 6. Ricci Soliton Structure in a Conformally Flat $(QE)_4$ Perfect Fluid Spacetime

The present authors recently studied the Ricci soliton structure in perfect fluid spacetime with torse-forming vector field in [19]. In this section, we consder a Ricci soliton structure in a conformally flat $(QE)_4$ perfect fluid spacetime.

The idea of Ricci solitons was introduced by Hamilton[12]. Ricci solitons also correspond to selfsimilar solutions of Hamilton's Ricci flow. They are natural generalizations of Einstein metrics and is defined by

$$(L_V g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0, \tag{6.1}$$

for some constant $\lambda$ and a vector field $V$. The Ricci soliton is said to be shrinking, steady, and expanding according as $\lambda$ is negative, zero, and positive respectively.

In view of (5.5), Ricci soliton equation (6.1) takes the form

$$(L_V g)(Y,Z) = -2\left(\Lambda + \lambda + \frac{\kappa(\sigma-\rho)}{2}\right) g(Y,Z) - 2\kappa(\sigma+\rho)A(Y)A(Z). \tag{6.2}$$

In this case we assume that the energy density $\sigma$ and isotropic pressure $\rho$ are constants. Now differentiating (6.2) covariantly along an arbitrary vector field $X$ provides

$$(\nabla_X L_V g)(Y,Z) = -2\kappa(\sigma+\rho)\left[(\nabla_X A)(Y)A(Z) + A(Y)(\nabla_X A)(Z)\right]. \tag{6.3}$$

Suppose the vector field $\omega$ is concurrent, i.e., $\nabla_X \omega = \xi X$, where $\xi$ is a nonzero constant, then $(\nabla_X A)(Y) = \xi g(X,Y)$. Therefore, (6.3) becomes

$$(\nabla_X L_V g)(Y,Z) = -2\xi\kappa(\sigma+\rho)\left[g(X,Y)A(Z) + g(X,Z)A(Y)\right]. \tag{6.4}$$

The identity

$$(\nabla_X L_V g)(Y,Z) = g((L_V\nabla)(X,Y),Z) + g((L_V\nabla)(X,Z),Y), \tag{6.5}$$

can be found from the commutation formula [21]

$$\left(L_V\nabla_X g - \nabla_X L_V g - \nabla_{[V,X]} g\right)(Y,Z) = -g((L_V\nabla)(X,Y),Z) - g((L_V\nabla)(X,Z),Y).$$

Using (6.4) in (6.5) and a straightforward combinatorial computation shows that

$$(L_V\nabla)(Y,Z) = -2\xi\kappa(\sigma+\rho)A(Z)Y. \tag{6.6}$$

Now, substituting $Y = Z = \omega$ in the well known formula [21], we have

$$(L_V\nabla)(X,Y) = \nabla_X\nabla_Y V - \nabla_{\nabla_X Y} V + R(V,X)Y,$$

and then making use of (6.6) we obtain $\nabla_\omega\nabla_\omega V + R(V,\omega)\omega = 2\xi\kappa(\sigma+\rho)\omega$.

If $\sigma+\rho = 0$, then $\nabla_\omega\nabla_\omega V + R(V,\omega)\omega = 0$, i.e., $V$ is Jacobi along $\omega$.
Next, differentiating the (6.6) along an arbitrary vector field $X$ we have

$$(\nabla_X L_V\nabla)(Y,Z) = -2\xi^2\kappa(\sigma+\rho)g(X,Z)Y. \tag{6.7}$$

According to Yano [21], we have the following commutation formula:

$$(L_V R)(X,Y)Z = (\nabla_X L_V\nabla)(Y,Z) - (\nabla_Y L_V\nabla)(X,Z).$$

In view of (6.7), we obtain

$$(L_V R)(X,Y)Z = 2\xi^2\kappa(\sigma+\rho)[g(Y,Z)X - g(X,Z)Y]. \tag{6.8}$$

Substituting $Y = Z = \omega$ in (6.8), we obtain

$$(L_V R)(X,\omega)\omega = 2\xi^2\kappa(\sigma+\rho)[-X - A(X)\omega]. \tag{6.9}$$

Taking $Y = \omega$ in (3.23), then Lie differentiate along $V$ and making use of (6.2) and (6.9), we find that

$$\begin{aligned}&2\xi^2\kappa(\sigma+\rho)[-X - A(X)\omega] + R(X, L_V\omega)\omega + R(X,\omega)L_V\omega\\ &= \frac{\alpha-\beta}{3}\left[-A(X)L_V\omega + 2\left(\Lambda+\lambda-\frac{\kappa(\sigma+3\rho)}{2}\right)A(X)\omega - g(X, L_V\omega)\omega\right].\end{aligned} \tag{6.10}$$

Plugging $Y = Z = \omega$ in (6.2), we get

$$g(L_V\omega,\omega) = \left[\frac{\kappa(\sigma+3\rho)}{2} - \Lambda - \lambda\right]. \tag{6.11}$$

Contracting (6.10) over $X$, then making use of (5.5) and (6.11) gives

$$\left[\Lambda - \frac{\kappa(\sigma+3\rho)}{2}\right]\cdot\left[\frac{\kappa(\sigma+3\rho)}{2} - \Lambda - \lambda\right] = 3\xi^2(\sigma+\rho). \tag{6.12}$$

If $\sigma + \rho = 0$, then (6.12) gives a relation

$$\lambda = \kappa\rho - \Lambda.$$

This shows that Ricci soliton is expanding if $\kappa\rho > \Lambda$, steady if $\kappa\rho = \Lambda$ and shrinking if $\kappa\rho < \Lambda$. Hence, we can state the following theorem.

**Theorem 6.1.** *Let $M^4$ be a conformally flat $(QE)_4$ perfect fluid spacetime whose energy density and isotropic pressure are constants. If $M^4$ admits a non-trivial (non-Einstein) Ricci soliton with velocity vector of the fluid is concurrent and $\sigma + \rho = 0$, i.e., the spacetime represents inflation, then*

(i) *$V$ is Jacobi along the geodesic determined by $\omega$;*

(ii) *the Ricci soliton is expanding, steady and shrinking according as $\kappa\rho > \Lambda$, $\kappa\rho = \Lambda$ and $\kappa\rho < \Lambda$, respectively.*

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