

## SERIES EXPANSION OF A COTANGENT SUM RELATED TO THE ESTERMANN ZETA FUNCTION

MOULOUD GOUBI<sup>1</sup>

ABSTRACT. In this paper, we study the cotangent sum  $c_0\left(\frac{q}{p}\right)$  related to the Estermann zeta function for the special case when the numerator is equal to 1 and get two useful series expansions of  $c_0\left(\frac{1}{p}\right)$ .

### 1. INTRODUCTION

For a positive integer  $p$  and  $q = 1, 2, \dots, p-1$ , such that  $(p, q) = 1$ , let the cotangent sum (see [10])

$$c_0\left(\frac{q}{p}\right) = -\sum_{k=1}^{p-1} \frac{k}{p} \cot \frac{\pi kq}{p}.$$

$c_0\left(\frac{q}{p}\right)$  is the value at  $s = 0$ ,

$$E_0\left(0, \frac{q}{p}\right) = \frac{1}{4} + \frac{i}{2} c_0\left(\frac{q}{p}\right)$$

of the Estermann zeta function

$$E_0\left(s, \frac{q}{p}\right) = \sum_{k \geq 1} \frac{d(k)}{k^s} \exp\left(\frac{2\pi i k q}{p}\right).$$

It is well-known that the sum  $c_0\left(\frac{q}{p}\right)$  satisfies the reciprocity formula (see [2])

$$c_0\left(\frac{q}{p}\right) + \frac{p}{q} c_0\left(\frac{p}{q}\right) - \frac{1}{\pi q} = \frac{i}{2} \psi_0\left(\frac{q}{p}\right).$$

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The Vasyunin cotangent sum (see [11])

$$V\left(\frac{q}{p}\right) = \sum_{r=1}^{p-1} \left\{ \frac{rq}{p} \right\} \cot\left(\frac{\pi r}{p}\right) = -c_0\left(\frac{q}{p}\right)$$

arises in the study of the Riemann zeta function by virtue of the formula (see [2, 9])

$$\begin{aligned} & \frac{1}{2\pi\sqrt{pq}} \int_{-\infty}^{+\infty} \left| \zeta\left(\frac{1}{2} + it\right) \right|^2 \left(\frac{q}{p}\right)^{it} \frac{dt}{\frac{1}{4} + t^2} \\ &= \frac{\log 2\pi - \gamma}{2} \left(\frac{1}{p} + \frac{1}{q}\right) + \frac{p-q}{2pq} \log \frac{q}{p} - \frac{\pi}{2pq} \left( V\left(\frac{p}{q}\right) + V\left(\frac{q}{p}\right) \right). \end{aligned}$$

This formula is connected to the approach of Nyman, Beurling and Báez-Duarte to the Riemann hypothesis (see [8]), which states that the Riemann hypothesis is true if and only if  $\lim_{n \rightarrow \infty} d_N = 0$ , where

$$d_N^2 = \inf_{A_N} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \left| 1 - \zeta A\left(\frac{1}{2} + it\right) \right|^2 \frac{dt}{\frac{1}{4} + t^2},$$

and the infimum is taken over all Dirichlet polynomials

$$A_N(s) = \sum_{n=1}^N \frac{a_n}{n^s}.$$

In a recent work with A. Bayad [7], we have proved that the sum  $V\left(\frac{q}{p}\right)$  satisfies the reciprocity formula

$$(1.1) \quad V\left(\frac{q}{p}\right) + V\left(\frac{p}{q}\right) = \frac{1}{\pi} \left( G(p, p) + G(q, q) + G(p, q) + (q - p) \log \frac{q}{p} \right),$$

where

$$G(p, q) = \sum_{k \geq 1} \frac{pq}{k(k+1)} \left\{ \frac{k}{p} \right\} \left\{ \frac{k}{q} \right\}.$$

Thereafter the restriction of the relationship (1.1) to  $q = 1$  gives

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi} G(p, p) - (p - 1) \log p.$$

Exactly our interest in this work is the case  $q = 1$  in order to get two series expansions of  $c_0\left(\frac{1}{p}\right)$ . First we recall the different asymptotical writings of  $c_0\left(\frac{1}{p}\right)$  in the literature. In [10, Theorem 1.2, Theorem 1.3] M. Th. Rassias proved that

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi} (\log 2\pi - \gamma) + \{ \mathcal{O}(\log p) \text{ or } \mathcal{O}(1) \}.$$

In [9, Theorem 1.7] H. Maier and M. Th. Rassias provide the following improvement. Let  $b, n \in \mathbb{N}$ ,  $b \geq 6N$ , with  $N = \lfloor \frac{n}{2} \rfloor + 1$ . There exist absolute real constants  $A_1, A_2 \geq 1$

and absolute real constants  $E_l, l$ , with  $|E_l| \leq (A_1 l)^{2l}$ , such that for each  $n \in \mathbb{N}$  we have

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi} (\log 2\pi - \gamma) - \frac{1}{\pi} + \sum_{l=1}^n E_l p^{-l} + R_n^*(p),$$

where  $|R_n^*(p)| \leq (A_2 n)^{4n} p^{-(n+1)}$ .

Only in [9, Theorem 1.9] H. Maier and M. Th. Rassias provide another improvement,

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p \log p - \frac{p}{\pi} (\log 2\pi - \gamma) + C_1 p + \mathcal{O}(1).$$

We draw attention that S. Bettin finds other reformulations of  $c_0\left(\frac{1}{p}\right)$  inspired from continued fraction theory (see [3]).

Finally from another point of view we show in [5] with A. Bayad and M. O. Hernane that

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi} \left(\log \frac{2\pi}{p} - \gamma\right) p + \frac{1}{\pi} + \frac{\pi}{36p} - \frac{1}{2} \sum_{k=2}^{\lfloor \frac{N}{2} \rfloor} (-1)^k \frac{4^k \pi^{2k-1} B_{2k}^2}{k(2k)!} \left(\frac{1}{p}\right)^{2k-1} + \mathcal{O}\left(\frac{1}{p^N}\right).$$

There is a misprint in the formula (1.22) Corollary 1.2 in [5] the correct one is in the formula (1.21) Corollary 1.2.

Otherwise in the same paper [5], an integral representation of  $c_0\left(\frac{1}{p}\right)$  is given by

$$(1.2) \quad c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} \int_0^1 \frac{(p-2)x^p - px^{p-1} + px - p + 2}{(x-1)^2(x^p-1)} dx.$$

In this work we prove that

$$(p-2)x^p - px^{p-1} + px - p + 2 = (x-1)^3 \sum_{r=1}^{p-1} (p-r-1)rx^{r-1}$$

and we get another formulation that is

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} \int_0^1 \frac{\sum_{r=1}^{p-1} (p-r-1)rx^{r-1}}{1+x+\dots+x^{p-1}} dx.$$

Applying some techniques from the generating function theory [4] to previous integrals; we find two series expansions of  $c_0\left(\frac{1}{p}\right)$ , as they are well explained in the next section.

## 2. SERIES EXPANSION OF $c_0\left(\frac{1}{p}\right)$

Let  $b_k$  be the integer sequence defined by  $b_0 = 1, b_1 = 2$  and the recursive formulae:

$$b_k - 2b_{k-1} + b_{k-2} = 0, \quad 2 \leq k \leq p-1, \quad k = p+1,$$

$$b_p - 2b_{p-1} + b_{p-2} = 1$$

and

$$b_k - 2b_{k-1} + b_{k-2} - b_{k-p} + 2b_{k-p-1} - b_{k-p-2} = 0, \quad k \geq p + 2.$$

According to the terms  $b_k$  we get the first series expansion in the following theorem.

**Theorem 2.1.**

$$(2.1) \quad c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} p(p-1)(p-2) \sum_{k \geq 0} \frac{b_k}{(k+1)(k+p+1)(k+2)(k+p)}.$$

For  $p \geq 1$  we define the arithmetic function  $a_p$  in the form

$$a_p(k) = \begin{cases} 1, & \text{if } p \mid k, \\ -1, & \text{if } k \equiv 1 \pmod{p}, \\ 0, & \text{otherwise.} \end{cases}$$

This function is not multiplicative. In general the arithmetical functions are defined from the set of natural integers  $\mathbb{N}$  into  $\mathbb{C}$ . We can extend this definition to  $\mathcal{F}(\mathbb{C}, \mathbb{C})$ ; set of functions from  $\mathbb{C}$  to  $\mathbb{C}$ . In that case the corresponding function is  $A : \mathbb{N} \rightarrow \mathcal{F}(\mathbb{C}, \mathbb{C})$  with  $A(p) = a_p$ . Furthermore,  $A(pq) = \pm A(p)A(q)$  and  $|A|$  is multiplicative.

Let the function  $M(p, k)$  defined by

$$M(p, 0) = \frac{1}{2}p^2 - \frac{3}{2}p + 1$$

and

$$M(p, k) = (p-1) \left( \frac{1}{2}p + k - 1 \right) - k(p+k-1)(H_{p+k-1} - H_k), \quad k \geq 1,$$

where  $H_k$  is the Harmonic number

$$H_k = \sum_{j=1}^k \frac{1}{j}.$$

Following this function a second series expansion of  $c_0 \left( \frac{1}{p} \right)$  is given in the following theorem.

**Theorem 2.2.**

$$(2.2) \quad c_0 \left( \frac{1}{p} \right) = \frac{1}{\pi} \sum_{k \geq 0} a_p(k) M(p, k).$$

**2.1. Proof of Theorem 2.1.** We take inspiration from the theory of generating functions [4,6], and prove that the sequence  $(b_k)$  is generated by the rational function:

$$f(x) = \frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}}.$$

More precisely we get the following lemma.

**Lemma 2.1.**

$$(2.3) \quad \frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}} = \sum_{k \geq 0} b_k x^k, \quad |x| < 1.$$

*Proof.* It is well known that

$$(2.4) \quad \frac{1}{1-x} = \sum_{k \geq 0} x^k, \quad |x| < 1.$$

Since for  $0 \leq x < 1$

$$0 < (x-1)^2(1-x^p) < 1$$

and

$$(x-1)^2(1-x^p) = 1 - (2x - x^2 + x^p - 2x^{p+1} + x^{p+2}),$$

then we have

$$0 < 2x - x^2 + x^p - 2x^{p+1} + x^{p+2} < 1.$$

Furthermore,  $f(x)$  is developable on entire series to get the result we have to take the quantity  $2x - x^2 + x^p - 2x^{p+1} + x^{p+2}$  instead of  $x$  in the last formula (2.4). Now, writing

$$\frac{1}{1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}} = \sum_{k \geq 0} d_k x^k$$

and then

$$(1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2}) \left( \sum_{k \geq 0} d_k x^k \right) = 1.$$

To compute this we use the well known Cauchy product of two entire series

$$\left( \sum_{k \geq 0} a_k x^k \right) \left( \sum_{j \geq 0} d_j x^j \right) = \sum_{k \geq 0} \left( \sum_{j=0}^k a_j d_{k-j} \right) x^k,$$

which generates the product of a polynomial of degree  $n$  with an entire series that also gives an entire series as follows

$$\left( \sum_{k=0}^n a_k x^k \right) \left( \sum_{j \geq 0} d_j x^j \right) = \sum_{k \geq 0} \left( \sum_{j=0}^{\min\{n,k\}} a_j d_{k-j} \right) x^k.$$

We return to  $f(x)$  in writing

$$1 - 2x + x^2 - x^p + 2x^{p+1} - x^{p+2} = \sum_{k=0}^{p+2} a_k x^k,$$

with  $a_0 = 1, a_1 = -2, a_2 = 1, a_p = -1, a_{p+1} = 2, a_{p+2} = -1$ , and the others are zero. We conclude that  $d_0 = 1, d_1 = 2$ . The formula

$$\sum_{j=0}^{\min\{p+2,k\}} a_j d_{k-j} = 0$$

states that

$$\begin{aligned} d_k - 2d_{k-1} + d_{k-2} &= 0, & 2 \leq k \leq p-1, \quad k = p+1, \\ d_p - 2d_{p-1} + d_{p-2} &= 1 \end{aligned}$$

and

$$d_k - 2d_{k-1} + d_{k-2} - d_{k-p} + 2d_{k-p-1} - d_{k-p-2} = 0, \quad k \geq p+2.$$

Finally, we see that  $d_k$  and  $b_k$  are identical for every integer  $k \geq 0$ . For more information on this approach we refer to [6]. □

To get the result (2.1) of Theorem 2.1 we must substitute the expression (2.3) in the identity (1.2) and one obtains

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi} \sum_{k \geq 0} b_k \int_0^1 \left( (p-2)x^{k+p} - px^{k+p-1} + px^{k+1} + (2-p)x^k \right) dx.$$

Furthermore,

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi} \sum_{k \geq 0} b_k \left( \frac{p-2}{k+p+1} - \frac{p}{k+p} + \frac{p}{k+2} - \frac{p-2}{k+1} \right).$$

Finally,

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} p(p-1)(p-2) \sum_{k \geq 0} \frac{b_k}{(k+1)(k+p+1)(k+2)(k+p)}$$

and  $c_0(1) = c_0\left(\frac{1}{2}\right) = 0$  is compatible with the definition of  $c_0$ .

Regarding the identity (2.3) Lemma 2.1 we remark that

$$\frac{1}{(1-x)^2(1-x^p)} = \sum_{k \geq 0} b_k x^k, \quad |x| < 1.$$

Furthermore, for  $x = \frac{1}{2}$  we deduce that the coefficients  $b_k$  satisfy the following statements

$$\sum_{k \geq 0} \frac{b_k}{2^k} = \frac{2^{p+2}}{2^p - 1} \quad \text{and} \quad \lim_{k \rightarrow \infty} \frac{b_k}{2^k} = 0.$$

**2.2. Proof of Theorem 2.2.** First we began by proving another integral representation of  $c_0\left(\frac{1}{p}\right)$ .

**Lemma 2.2.**

$$(2.5) \quad c_0\left(\frac{1}{p}\right) = \frac{1}{\pi} \int_0^1 \frac{\sum_{r=1}^{p-1} (p-r-1)rx^{r-1}}{1+x+\dots+x^{p-1}} dx.$$

*Proof.*

$$\begin{aligned} (x-1)^3 \sum_{r=1}^{q-1} (q-r-1)rx^{r-1} &= \sum_{r=3}^q (q-r+1)(r-2)x^r - 3 \sum_{r=2}^{q-1} (q-r)(r-1)x^r \\ &\quad + 3 \sum_{r=1}^{q-2} (q-r-1)rx^r - \sum_{r=0}^{q-3} (q-r-2)(r+1)x^r. \end{aligned}$$

It's obvious to remark that

$$(q-r+1)(r-2) - 3(q-r)(r-1) + 3(q-r-1)r - (q-r-2)(r+1) = 0$$

and the quantity

$$(t - 1)^3 \sum_{r=1}^{q-1} (q - r - 1) r x^{r-1}$$

is reduced to

$$(q - 2) x^q + 2(q - 3) x^{q-1} + 3(q - 4) x^{q-2} - 3(q - 2) x^{q-1} - 6(q - 3) x^{q-2} - 3(q - 2) x^2 + 3(q - 2) x^{q-2} + 3(q - 2) x + 6(q - 3) x^2 - q + 2 - 2(q - 3) x - 3(q - 4) x^2.$$

After simplification we obtain

$$(t - 1)^3 \sum_{r=1}^{q-1} (q - r - 1) r x^{r-1} = (q - 2) x^q - q x^{q-1} + q x - q + 2. \quad \square$$

The Theorem 2.2 is immediate from the Lemma 2.2 in the following way. Since

$$\frac{1}{1 + x + \dots + x^{p-1}} = \frac{1 - x}{1 - x^p}$$

and  $|x| < 1$ , then

$$\frac{1}{1 + x + \dots + x^{p-1}} = \frac{1 - x}{1 - x^p} = \sum_{k \geq 0} (1 - x) x^{pk}.$$

Furthermore,

$$\frac{1}{1 + x + \dots + x^{p-1}} = \sum_{k \geq 0} a_p(k) x^k$$

and we have

$$\frac{\sum_{r=1}^{p-1} (p - r - 1) r x^{r-1}}{1 + x + \dots + x^{p-1}} = \sum_{k \geq 0} \sum_{r=1}^{p-1} a_p(k) (p - r - 1) r x^{k+r-1}.$$

The passage to the integral inducts

$$c_0 \left( \frac{1}{p} \right) = \sum_{k \geq 0} \sum_{r=1}^{p-1} a_p(k) \frac{(p - r - 1) r}{k + r}.$$

But

$$\sum_{r=1}^{p-1} \frac{(p - r - 1) r}{k + r} = (p - 1) \left( \frac{1}{2} p + k - 1 \right) - k (p + k - 1) \sum_{r=k+1}^{p+k-1} \frac{1}{r}$$

and the result (2.2) is deduced.

### 3. CONNECTION TO DIGAMMA FUNCTION

We finish this work by revisiting the proof of the expression of  $c_0\left(\frac{1}{p}\right)$  according to the function digamma and Bernoulli polynomials in the work [1] of L. Báez Duarte et al.

$$c_0\left(\frac{1}{p}\right) = \frac{2}{\pi} \sum_{r=1}^{p-1} B_1\left(\frac{r}{p}\right) \psi\left(\frac{r}{p}\right),$$

where  $B_1$  is the reduced Bernoulli polynomial

$$B_1(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Z}, \\ \{x\} - \frac{1}{2}, & \text{otherwise,} \end{cases}$$

and  $\psi$  the digamma function defined by

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{k \geq 1} \left( \frac{1}{k} - \frac{1}{k+z} \right).$$

Starting with the demonstration of a property of  $\psi$  that will be used later.

**Proposition 3.1.**

$$(3.1) \quad \psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) = p \int_0^1 \frac{x^{r-1}}{1+x+\dots+x^{p-1}} dx.$$

*Proof.* We quote from [5] the formula

$$\psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) = p \sum_{k \geq 0} \frac{1}{(pk+r+1)(pk+r)}.$$

The general term  $\frac{1}{(pk+r+1)(pk+r)}$  can be written as following

$$\frac{1}{(pk+r+1)(pk+r)} = \frac{1}{pk+r} - \frac{1}{pk+r+1} = \int_0^1 (x^{pk+r-1} - x^{pk+r}) dx$$

and the passage to the sum states that

$$\sum_{k \geq 0} \frac{1}{(pk+r+1)(pk+r)} = \int_0^1 \frac{x^{r-1} - x^r}{1-x^p} dx.$$

Finally,

$$\sum_{k \geq 0} \frac{1}{(pk+r+1)(pk+r)} = \int_0^1 \frac{x^{r-1}}{1+x+\dots+x^{p-1}} dx$$

and we have (3.1). Proposition 3.1 follows. □

In [5], it is shown that

$$\log p = \frac{1}{p} \sum_{r=1}^{p-1} r \left( \psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) \right).$$

This identity conducts to the following interesting lemma.



**Lemma 3.1.**

$$(3.2) \quad \sum_{r=1}^p \psi\left(\frac{r}{p}\right) = -\gamma p - p \log p.$$

*Proof.* Since

$$\sum_{r=1}^{p-1} r \left( \psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) \right) = p \log p,$$

then

$$-\sum_{r=1}^p \psi\left(\frac{r}{p}\right) + \psi(1)p = p \log p.$$

Furthermore,

$$\sum_{r=1}^p \psi\left(\frac{r}{p}\right) = -\gamma p - p \log p. \quad \square$$

According to the identity (3.1) Proposition 3.1 and the integral representation (2.5) we conclude that

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi p} \sum_{r=1}^{p-1} (p-r-1)r \left( \psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) \right).$$

Furthermore combining this result with the identity (3.2) Lemma 3.1 we get

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi} \log p + \frac{1}{\pi p} \sum_{r=1}^{p-1} (p-r)r \left( \psi\left(\frac{r+1}{p}\right) - \psi\left(\frac{r}{p}\right) \right)$$

and

$$c_0\left(\frac{1}{p}\right) = -\frac{1}{\pi} \log p - \gamma \frac{p-1}{\pi p} + \frac{1}{\pi p} \sum_{r=1}^{p-1} (2r-p-1) \psi\left(\frac{r}{p}\right),$$

then

$$c_0\left(\frac{1}{p}\right) = \frac{1}{\pi p} \sum_{r=1}^{p-1} (2r-p) \psi\left(\frac{r}{p}\right).$$

But

$$2r-p = 2p \left( \frac{r}{p} - \frac{1}{2} \right) = 2p B_1\left(\frac{r}{p}\right),$$

which means that

$$c_0\left(\frac{1}{p}\right) = \frac{2}{\pi} \sum_{r=1}^p B_1\left(\frac{r}{p}\right) \psi\left(\frac{r}{p}\right).$$

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<sup>1</sup>DEPARTMENT OF MATHEMATICS,  
UNIVERSITY OF MOULoud MAMMERI,  
HASNAOUA II TIZI-OUZOU 15000 ALGERIA  
Email address: mouloud.ummto@hotmail.fr