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A SUBCLASS OF NOOR-TYPE HARMONIC p-VALENT FUNCTIONS BASED ON HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. In this paper, we introduce a new generalized Noor-type operator of harmonic p-valent functions associated with the Fox-Wright generalized hypergeometric functions (FWGH-functions). Furthermore, we consider a new subclass of complex-valued harmonic multivalent functions based on this new operator. Several geometric properties for this subclass are also discussed.

1. INTRODUCTION

Harmonic function has fruitful applications not only in applied mathematics, but also in physics, engineering. It appears in differential equations, such as harmonic differential equations, wave equations, and heat equations. In geometric function theory (GFT), the famed authors Clunie and Sheil-Small [11] launched the study of harmonic univalent functions in 1984. In their investigates, they provided a class $\mathcal{S}_{\mathcal{H}}$ of harmonic functions $\varphi = \phi + \overline{\psi}$ that are univalent, sense-preserving which is $|\phi'(z)| > |\psi'(z)|$ in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and normalized by the conditions $\varphi(0) = \varphi'(0) - 1 = 0$, where the regular(analytic) part ϕ and the co-regular part ψ are defined as follows:

$$\phi(z) = z + \sum_{\kappa=2}^{\infty} \mu_\kappa z^\kappa, \psi(z) = \sum_{\kappa=1}^{\infty} \nu_\kappa z^\kappa, \quad |\nu_1| < 1.$$

In addition, they studied its geometric properties, which involves coefficient bounds, growth and distortion formulas. Note that, class $\mathcal{S}_{\mathcal{H}}$ reduces to the class $\mathcal{S}$ of regular univalent functions if the co-regular part ψ is zero.

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In 2001, Ahuja and Jahangiri [2] defined a more general class $\mathcal{S}_{\mathcal{H}(p)}$ of harmonic $p-$valent (multivalent) functions, $\varphi = \phi + \overline{\psi}$ that are sense-preserving in $\mathbb{D}$, and ϕ and ψ are of the formula

$$\phi(z) = z^p + \sum_{\kappa=p+1}^{\infty} \mu_\kappa z^\kappa, \psi(z) = \sum_{\kappa=p}^{\infty} \nu_\kappa z^\kappa, \quad |\nu_p| < 1, p \in \mathbb{N} = \{1, 2, \dots\}. \tag{1.1}$$

Note that, class $\mathcal{S}_{\mathcal{H}(p)}$ reduces to the class $\mathcal{M}_p$ of normalized regular $p-$valent functions if the co-regular part ψ is zero. Consequently, the function $\varphi \in \mathcal{M}_p$ are expressed as:

$$\varphi(z) = z^p + \sum_{\kappa=p+1}^{\infty} \mu_\kappa z^\kappa. \tag{1.2}$$

Denoted by $\mathcal{NS}_{\mathcal{H}(p)}$ the subclass of $\mathcal{S}_{\mathcal{H}(p)}$ consisting of functions $\varphi = \phi + \overline{\psi}$ such that the regular functions ϕ and ψ are of the form

$$\phi(z) = z^p - \sum_{\kappa=p+1}^{\infty} |\mu_\kappa| z^\kappa, \psi(z) = -\sum_{\kappa=p}^{\infty} |\nu_\kappa| z^\kappa, \quad |\nu_p| < 1, p \in \mathbb{N} = \{1, 2, \dots\}. \tag{1.3}$$

Convolution (Hadamard) product is a mathematical operation on two regular functions φ_1 and φ_2 to yield a third regular function φ_3. It is used to define various subclasses and linear operators in GFT. This concept owes its origin to Hadamard in 1899 [22]. In the harmonic functions case, Clunie and Sheil-Small [11] studied and defined the following convolution product: for any two functions $\varphi_\imath \in \mathcal{S}_\mathcal{H}$ of the form

$$\varphi_\imath(z) = \phi_\imath(z) + \overline{\psi_\imath(z)} = z + \sum_{\kappa=2}^{\infty} \mu_{\kappa,\imath} z^\kappa + \overline{\sum_{\kappa=1}^{\infty} \nu_{\kappa,\imath} z^\kappa},$$

where $\imath = 1, 2$, $|\nu_{1,1}| < 1$, $|\nu_{1,2}| < 1$, their convolution is denoted by $\varphi_1 * \varphi_2$ and defined as

$$(\varphi_1 * \varphi_2)(z) = z + \sum_{\kappa=2}^{\infty} \mu_{\kappa,1}\, \mu_{\kappa,2}\, z^\kappa + \overline{\sum_{\kappa=1}^{\infty} \nu_{\kappa,1}\, \nu_{\kappa,2}\, z^\kappa}.$$

More generally, the convolution of two functions $\varphi_\imath \in \mathcal{S}_{\mathcal{H}(p)}$ is given by (see, [29]):

$$(\varphi_1 * \varphi_2)(z) = z^p + \sum_{\kappa=p+1}^{\infty} \mu_{\kappa,1}\, \mu_{\kappa,2}\, z^\kappa + \overline{\sum_{\kappa=p}^{\infty} \nu_{\kappa,1}\, \nu_{\kappa,2}\, z^\kappa}, \tag{1.4}$$

where

$$\varphi_\imath(z) = \phi_\imath(z) + \overline{\psi_\imath(z)} = z^p + \sum_{\kappa=p+1}^{\infty} \mu_{\kappa,\imath} z^\kappa + \overline{\sum_{\kappa=p}^{\infty} \nu_{\kappa,\imath} z^\kappa}, \quad \imath = 1, 2, |\nu_{p,1}| < 1, |\nu_{p,2}| < 1.$$

Operators Theory has a significant role in the study GFT. Actually, operators are utilized in defining new subclasses. The technique of convolution has a remarkable part in the evolution of this area. Numerous differential and integral operators (linear operators) can be established in terms of the convolution. In 1915, Alexander [4] introduced the first integral operator on class $\mathcal{A}$ that includes normalized regular functions. Later, several well-known integral operators are investigated by complex analysts, such as Libera [26], Bernardi [9], Miller, Mocanu and Reade [27, 28], Pascu and Pescar [34], Ong et al. [33], Frasin [20], Frasin and Breaz [21], El-Ashwah, Aouf and El-Deeb [16], Deniz [13], Rahrovi [35], Al-Janaby and Ghanim [5], Al-Janaby, Ghanim, Darus [6], Al-Janaby [7] and others. The following are some important linear operators related to results in this study.

In 1975, Ruscheweyh [37] introduced the differential operator $D^\tau\varphi(z)$ so-called the Ruscheweyh differential operator as follows: for $\varphi \in \mathcal{A}$, $\tau > -1$ and $D^\tau : \mathcal{A} \to \mathcal{A}$ is given by

$$D^\tau\varphi(z) = \frac{z}{(1-z)^{\tau+1}} * \varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{(\tau+1)_{\kappa-1}}{(\kappa-1)!}\, \mu_\kappa z^\kappa, \tag{1.5}$$

where $(a)_\kappa = \frac{\Gamma(a+\kappa)}{\Gamma(a)}$ denotes the Pochhammer symbol. Note that $D^0\varphi(z) = \varphi(z)$ and $D^1\varphi(z) = z\varphi'(z)$.

Analogous manner to the Ruscheweyh operator, in 1999, the author Noor [31] presented an integral operator $I_\tau\varphi(z)$, namely Noor Integral of $\tau-$th order, as follows: for a function $\varphi \in \mathcal{A}$ and $\tau \in \mathbb{N}_0$, the Noor integral operator $I_\tau(z)$ is given by $I_\tau : \mathcal{A} \to \mathcal{A}$,

$$I_\tau\omega(z) = \varphi_\tau^{(-1)}(z) * \varphi(z) = \left[\frac{z}{(1-z)^{\tau+1}}\right]^{-1} * \varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{\kappa!}{(\tau+1)_{\kappa-1}}\, \mu_k\, z^\kappa, \tag{1.6}$$

such that $\varphi_\tau(z) * \varphi_\tau^{(-1)}(z) = \frac{z}{(1-z)^2}$. Note that $I_0\varphi(z) = z\varphi'(z)$, $I_1\varphi(z) = \varphi(z)$. This version of integral operator is a considerable gadget in imposing several subclasses of regular functions.

On the other hand, special functions have been applied in GFT. In 1984, de Branges [12] employed hypergeometric function in proving the prominent problem called Bieberbach's conjecture. Since then, the study of hypergeometric function and its generalizations have attracted the attention of many function theorists. The important role played by special functions is defining new operators. The generalized hypergeometric function known as Fox-Wright generalized hypergeometric function (FWGH-function) is defined as: (see for example [19, 40] and [41])

$$\eta\mathcal{W}_\delta[(\rho_l, \mathcal{C}_l)_{1,\eta}; (\sigma_l, \mathcal{D}_l)_{1,\delta}; z] = \eta\mathcal{W}_\delta[(\rho_1, \mathcal{C}_1)\cdots(\rho_\eta, \mathcal{C}_\eta); (\sigma_1, \mathcal{D}_1)\cdots(\sigma_\delta, \mathcal{D}_\delta); z]$$

$$= \sum_{\kappa=0}^{\infty} \frac{\Gamma(\rho_1+\kappa\mathcal{C}_1)\Gamma(\rho_2+\kappa\mathcal{C}_2)\cdots\Gamma(\rho_\eta+\kappa\mathcal{C}_\eta)}{\Gamma(\sigma_1+\kappa\mathcal{D}_1)\Gamma(\sigma_2+\kappa\mathcal{D}_2)\cdots\Gamma(\sigma_\delta+\kappa\mathcal{D}_\delta)}\, \frac{z^\kappa}{\kappa!}$$

(1.7)

$$= \sum_{\kappa=0}^{\infty} \frac{\prod_{j=1}^{\eta}\Gamma(\rho_j+\kappa\mathcal{C}_j)}{\prod_{j=1}^{\delta}\Gamma(\sigma_j+\kappa\mathcal{D}_j)}\, \frac{z^\kappa}{\kappa!},$$

where $\mathcal{C}_\jmath > 0$, $\jmath = 1,2,\dots,\eta$, $\mathcal{B}_\jmath > 0$, $\jmath = 1,2,\dots,\delta$, $1 + \sum\limits_{\jmath=1}^{\eta} \mathcal{C}_\jmath - \sum\limits_{\jmath=1}^{\delta} \mathcal{D}_\jmath \geq 0$, $\rho_\jmath + \kappa\mathcal{C}_\jmath \neq 0,-1,\dots$, $\jmath = 1,2,\dots,\eta, \kappa = 0,1,\dots$, $\sigma_\jmath + \kappa\mathcal{D}_\jmath \neq 0,-1,\dots$, $\jmath = 1,2,\dots$, $\delta, \kappa = 0,1,\dots$ and $z \in \mathbb{C}$. The condition $1 + \sum\limits_{\jmath=1}^{\eta} \mathcal{C}_\jmath - \sum\limits_{\jmath=1}^{\delta} \mathcal{D}_\jmath \geq 0$ is essential so that the series in (1.7) is absolutely convergent for all $z \in \mathbb{C}$, and is an entire function of z (for details, see [25]). Special case of FWGH-function defined in (1.7), given as: if $\mathcal{C}_\jmath = 1$, $\jmath = 1,2,\dots,\eta$, $\mathcal{D}_\jmath = 1$, $\jmath = 1,2,\dots,\delta$, $\eta \leq \delta + 1$ and

$$\Xi = \left(\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath)\right) \left(\prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath)\right)^{-1}, \tag{1.8}$$

then

$$\Xi\ \eta\mathcal{W}_\delta[(\rho_\jmath,1)_{1,\eta};(\sigma_\jmath,1)_{1,\delta};z] = \eta\mathcal{F}_\delta[(\rho_1,...\rho_\eta;\sigma_1,...\sigma_\delta;z],$$

where $\eta\mathcal{F}_\delta[(\rho_1,\dots,\rho_\eta;\sigma_1,\dots,\sigma_\delta;z]$ is a generalized hypergeometric function, [14]. Other special cases of FWGH-function were presented in [25].

In the well-known theory of regular univalent functions, there are numerous investigates on hypergeometric functions associated with classes of regular functions. In 2004, Ahuja and Silverman [1] discovered the corresponding connections between hypergeometric functions and harmonic univalent functions. Recently, the connections between WGHF and harmonic univalent functions were discussed by some authors, such that Murugusundaramoorthy and Raina [30], Sharma [39], Raina and Sharma [36], Ahuja and Sharma [3] and Hussain et al.[23]. In addition, several operators have been extended to harmonic functions by authors. For instance, Chandrashekar et al. [10], El-Ashwah, and Aouf [17] Yaşar and Yalçin [42], Seoudy [38], Al-Janaby [8] and others. Some previous studies that involving hypergeometric and FWGH functions are presented in this paper.

In 2004, Dziok and Raina [15] considered the linear operator $W(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta}]$ by means of FWGH-function on $\mathcal{A}$ as:

$$W(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta}]\varphi(z) = z + \sum_{\kappa=2}^{\infty} \Xi\ \vartheta_\kappa\, \mu_\kappa\, z^\kappa,$$

where

$$\vartheta_\kappa = \frac{\Gamma(\rho_1 + (\kappa-1)\mathcal{C}_1)\Gamma(\rho_2 + (\kappa-1)\mathcal{C}_2)\cdots\Gamma(\rho_\eta + (\kappa-1)\mathcal{C}_\eta)}{\Gamma(\sigma_1 + (\kappa-1)\mathcal{D}_1)\Gamma(\sigma_2 + (\kappa-1)\mathcal{D}_2)\cdots\Gamma(\sigma_\delta + (\kappa-1)\mathcal{D}_\delta)(\kappa-1)!},$$

and Ξ is defined in (1.8). Following that, in 2016, Hussain, Rasheed and Darus [23] introduced a new subclass of harmonic functions by using the extension of the above linear operator to harmonic functions. Also, they investigated various properties such as coefficient bounds, extreme points, and inclusion results and closed under an integral operator for this subclass.

In 2006, the author Noor [32] again imposed the integral operator $I_\tau(\zeta,\xi;\gamma)$ by employing the Gauss hypergeometric function as follows:

$$I_\tau(\zeta,\xi;\gamma)\varphi(z) = [z\mathcal{F}(\zeta,\xi;\zeta;z)]^{(-1)} * \varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{(\gamma)_{\kappa-1}(\tau+1)_{\kappa-1}}{(\zeta)_{\kappa-1}(\xi)_{\kappa-1}} \mu_\kappa z^\kappa, \tag{1.9}$$

where

$$[z\mathcal{F}(\zeta,\xi;\gamma;z)] * [z\mathcal{F}(\zeta,\xi;\gamma;z)]^{(-1)} = \frac{z}{(1-z)^{\tau+1}} = z + \sum_{\kappa=2}^{\infty} \frac{(\tau+1)_{\kappa-1}}{(\kappa-1)!} z^\kappa.$$

In 2008, Ibrahim and Darus [24] studied the following generalized integral operator $I_\tau[(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta}]$ associated with FWGH-function on $\mathcal{A}$, where

$$I_\tau[(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta}]\varphi(z) = z + \sum_{\kappa=2}^{\infty} \frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-1)\mathcal{D}_\jmath)}{\prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-1)\mathcal{C}_\jmath)} (\tau+1)_{\kappa-1} \mu_\kappa z^\kappa \tag{1.10}$$

and

$$\frac{\Gamma(\sigma_1)\cdots\Gamma(\sigma_\delta)}{\Gamma(\rho_1)\cdots\Gamma(\rho_\eta)} = 1.$$

Posterior, in 2016, the authors El-Ashwah and Hassan [18] established the linear operator $\Theta_\kappa[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\tau};(\nu_\jmath,\mathcal{D}_\jmath)_{1,\varsigma}]$ on the class $\mathcal{M}_p$ of regular $p-$valent functions in $\mathbb{D}$ as:

$$\Theta_\kappa[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta}]\varphi(z) = z^p + \sum_{\kappa=p+1}^{\infty} \Xi\, \vartheta_\kappa\, \rho_\kappa\, z^\kappa,$$

where

$$\vartheta_\kappa = \frac{\Gamma(\rho_1+(\kappa-p)\mathcal{C}_1)\Gamma(\rho_2+(\kappa-p)\mathcal{C}_2)\cdots\Gamma(\rho_\eta+(\kappa-p)\mathcal{C}_\eta)}{\Gamma(\sigma_1+(\kappa-p)\mathcal{D}_1)\Gamma(\sigma_2+(\kappa-p)\mathcal{D}_2)\cdots\Gamma(\sigma_\delta+(\kappa-p)\mathcal{D}_\delta)(\kappa-p)!},$$

and Ξ is defined in (1.8).

In this study, we continue our investigates in the theory of operators. Here we'll introduce a new generalized Noor-type operator of harmonic $p-$valent functions associated with FWGH-functions. We then define a new subclass and discuss several of its properties.

2. Imposed Operator $\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$

This section proposes a new generalized Noor-type operator $\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ for harmonic p-valent functions based on FWGH-function in (1.7).

By giving an extension of the FWGH-function in (1.7)

$$\eta\mathcal{M}_\delta[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};z] := \Omega z^p \eta\mathcal{W}_\delta[(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta};(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};z]$$

$$= z^p + \sum_{\kappa=p+1}^{\infty} \frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)} \cdot \frac{z^\kappa}{(\kappa-p)!}, \tag{2.1}$$

where

$$\Omega = \Big(\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath)\Big)\Big(\prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath)\Big)^{-1}. \tag{2.2}$$

We define a new generalization of the extended FWGH-function in (2.1) in terms of $\ell-$th convolution product as:

$$\begin{aligned} &\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z] \\ :=& \underbrace{\eta\mathcal{M}_\delta[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z] * \cdots * \eta\mathcal{M}_\delta[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]}_{\ell-\text{times}} \\ =& z^p + \sum_{\kappa=p+1}^{\infty} \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^\ell z^\kappa. \end{aligned} \tag{2.3}$$

Then we introduce a new function $\Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1}$ as:

$$\begin{aligned} &\Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1} \\ =& z^p + \sum_{\kappa=p+1}^{\infty} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}\right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} z^\kappa, \end{aligned} \tag{2.4}$$

such that for $\tau > -p$

$$\begin{aligned} &\Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big) * \Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1} \\ =& \frac{z^p}{(1-z)^{\tau+p}} = \sum_{\kappa=p}^{\infty} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} z^\kappa. \end{aligned}$$

Next, we consider the following linear operator: $\mathcal{J}_p^\ell[(\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; (\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}] : \mathcal{M}_p \to \mathcal{M}_p$, where

$$\begin{aligned} &\mathcal{J}_p^\ell[(\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; (\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}]\varphi(z) = \Big(\eta\mathcal{M}_\delta^\ell[(\rho_\jmath, \mathcal{C}_\jmath)_{1,\eta}; (\sigma_\jmath, \mathcal{D}_\jmath)_{1,\delta}; z]\Big)^{-1} * \varphi(z) \\ =& z^p + \sum_{\kappa=p+1}^{\infty} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}\right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \mu_\kappa z^\kappa. \end{aligned} \tag{2.5}$$

For brevity,

$$\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z) = \mathcal{J}^{\ell}_{p}[(\sigma_\jmath,\mathcal{D}_\jmath)_{1,\delta};(\rho_\jmath,\mathcal{C}_\jmath)_{1,\eta}]\varphi(z). \tag{2.6}$$

Remark 2.1. For suitably chosen parameters p, ℓ, δ, η, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, ρ_1, ρ_2 and σ_1, the generalized Noor-type operator $\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]$ (2.6) reduces to some of the above linear operators. Thus, we obtain the following special cases.

- For $p=1$, $\ell=1$, $\delta=1$, $\eta=2$, $\mathcal{C}_1=\mathcal{C}_2=\mathcal{D}_1=1$, and $\rho_1=\rho_2=\sigma_1=1$ in (2.6), we gain the Ruscheweyh differential operator given by (1.5).
- For $p=1$, $\ell=1$, $\delta=1$, $\eta=2$, $\mathcal{C}_1=\mathcal{C}_2=\mathcal{D}_1=1$, $\rho_1=\rho_2=1+\tau$ and $\sigma_1=1$, the operator (2.6) provides the Noor integral operator in (1.6).
- By taking $p=1$, $\ell=1$, $\delta=1$, $\eta=2$, $\mathcal{C}_1=\mathcal{C}_2=\mathcal{D}_1=1$, $\rho_1=\zeta$, $\rho_2=\xi$ and $\sigma_1=\gamma$ in (2.6), gives us an integral operator defined by (1.9).
- If $p=1$, $\ell=1$ and $\Omega=1$, we yield the linear operator given by (1.10).

The generalized Noor-type operator $\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ (2.6) when extended to harmonic $p-$valent function $\varphi=\phi+\overline{\psi}$ is defined by

$$\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z) = \mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\phi(z) + \overline{\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\psi(z)}, \tag{2.7}$$

where

$$\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\phi(z) = z^p + \sum_{\kappa=p+1}^{\infty}\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,\mu_\kappa\,z^\kappa$$

and

$$\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\psi(z) = \sum_{\kappa=p}^{\infty}\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,\nu_\kappa\,z^\kappa.$$

3. Geometric Results

This section introduces a certain subclass of harmonic $p-$valent functions which includes the generalized Noor-type operator $\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ extended to harmonic functions. This subclass is denoted by $\mathcal{H}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Further, coefficient bounds, growth formula, extreme points, convolution, convex combinations and class-preserving integral operator are also investigated for harmonic functions satisfying the subclass $\mathcal{H}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.

Definition 3.1. A function $\varphi\in\mathcal{S}_{\mathcal{H}}$ is said to be in subclass $\mathcal{H}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ if it satisfies the following inequality:

$$\operatorname{Re}\left\{(1-\alpha)\,\frac{\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)}{z^p} + \alpha\,\frac{\left[\mathcal{J}^{\eta,\delta}_{p,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)\right]'}{pz^{p-1}}\right\} \geq \frac{\beta}{p}, \tag{3.1}$$

where $\mathcal{J}^{\eta,\,\delta}_{p,\,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)$ is defined by (2.7), $0 \le \alpha \le 1$ and $0 \le \beta < p$.
Also, let $\mathcal{NH}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath]) = \mathcal{H}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath]) \bigcap \mathcal{NS}_{\mathcal{H}(p)}$.

A sufficient coefficient condition for function belonging to the class $\mathcal{H}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ is now derived.

Theorem 3.1. *Let* $\varphi = \phi + \overline{\psi}$ *given by* (1.1). *Then* $\varphi \in \mathcal{H}^{\beta}_{p}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ *if*

(3.2)
$$\sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\,\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\mu_\kappa|$$
$$+\sum_{\kappa=p}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\,\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\nu_\kappa| \le p-\beta,$$

where $0 \le \alpha \le 1$, $0 \le \beta < p$.

Proof. Using the fact that $\mathrm{Re}(\lambda) \ge 0$ if and only if $|1+\lambda| \ge |1-\lambda|$, it suffices to show that

$$|p-\beta+p\theta(z)| \ge |p+\beta-p\theta(z)|, \tag{3.3}$$

where

$$\theta(z) = (1-\alpha)\,\frac{\mathcal{J}^{\eta,\,\delta}_{p,\,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)}{z^p} + \alpha\,\frac{\left[\mathcal{J}^{\eta,\,\delta}_{p,\,\ell}[\sigma_\jmath;\rho_\jmath]\,\varphi(z)\right]'}{pz^{p-1}}.$$

Substituting for ϕ and ψ in θ, we gain

(3.4)
$$|p-\beta+p\theta(z)|$$
$$\ge 2p-\beta-\sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\,\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\mu_\kappa||z|^{\kappa-p}$$
$$-\sum_{\kappa=p}^{\infty} [(\kappa-p)\alpha+p] \left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\,\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\,|\nu_\kappa||z|^{\kappa-p}$$

and

(3.5)

$$\begin{aligned}
&|p+\beta-p\theta(z)|\\
\leq&\beta+\sum_{\kappa=p+1}^{\infty}[(\kappa-p)a+p]\left[\frac{\prod_{j=1}^{\delta}\Gamma(\sigma_j+(\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega\prod_{j=1}^{\eta}\Gamma(\rho_j+(\kappa-p)\mathcal{C}_j)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa||z|^{\kappa-p}\\
&+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{j=1}^{\delta}\Gamma(\sigma_j+(\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega\prod_{j=1}^{\eta}\Gamma(\rho_j+(\kappa-p)\mathcal{C}_j)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa||z|^{\kappa-p}.
\end{aligned}$$

These inequalities (3.4) and (3.5) in conjunction with (3.2) yields

$$\begin{aligned}
&|p-\beta+p\theta(z)|\\
\geq&|p+\beta-p\theta(z)|\\
\geq&2\left[(p-\beta)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{j=1}^{\delta}\Gamma(\sigma_j+(\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega\prod_{j=1}^{\eta}\Gamma(\rho_j+(\kappa-p)\mathcal{C}_j)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|\right.\\
&\left.-\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{j=1}^{\delta}\Gamma(\sigma_j+(\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega\prod_{j=1}^{\eta}\Gamma(\rho_j+(\kappa-p)\mathcal{C}_j)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\right]\geq 0.
\end{aligned}$$

The harmonic function

(3.6)

$$\begin{aligned}
\varphi(z)=&z^p+\sum_{\kappa=p+1}^{\infty}\left[\frac{\Omega\prod_{j=1}^{\eta}\Gamma(\rho_j+(\kappa-p)\mathcal{C}_j)}{\prod_{j=1}^{\delta}\Gamma(\sigma_j+(\kappa-p)\mathcal{D}_j)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!}{(\tau+p)_{\kappa-p}\,[(\kappa-p)\alpha+p]}x_\kappa\,z^\kappa\\
&+\sum_{\kappa=p}^{\infty}\left[\frac{\Omega\prod_{j=1}^{\eta}\Gamma(\rho_j+(\kappa-p)\mathcal{C}_j)}{\prod_{j=1}^{\delta}\Gamma(\sigma_j+(\kappa-p)\mathcal{D}_j)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!}{(\tau+p)_{\kappa-p}\,[(\kappa-p)\alpha+p]}\overline{y}_\kappa\,\overline{z}^\kappa,
\end{aligned}$$

where $\sum_{\kappa=p+1}^{\infty}|x_\kappa|+\sum_{\kappa=p}^{\infty}|y_\kappa|=p-\beta$ shows that the coefficient bound given by (3.2) is sharp.

The functions of the from (3.6) are in subclass $\mathcal{H}_p^{\beta}(\ell,\, \eta,\, \delta)$ because in view of (3.2), we acquire

$$\begin{aligned}&\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|\\&+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\\&\leq\sum_{\kappa=p+1}^{\infty}|x_\kappa|+\sum_{\kappa=p}^{\infty}|y_\kappa|=p-\beta.\end{aligned}$$

This completes the proof. □

Now, we yield the necessary and sufficient condition for the function $\varphi=\phi+\overline{\psi}$ given by (1.3) to be in $\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$.

Theorem 3.2. *Let $\varphi=\phi+\overline{\psi}$ be given by* (1.3). *Then $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$ if and only if the condition* (3.2) *is as follows:*

$$\begin{aligned}&\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|\\&+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\leq p-\beta,\end{aligned}$$

where $0\leq\alpha\leq1$, $0\leq\beta<p$.

Proof. In view of Theorem 3.1 and $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])\subset\mathcal{H}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$, we only need to prove the "only if" part of this theorem. Assume that $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath;\rho_\jmath])$, then by virtue of (3.1), we get

(3.7)

$$\operatorname{Re}\left\{(p-\beta)-\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|\right.$$

$$\times z^{\kappa-p} - \sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\overline{z}^{\kappa-p}\Bigg\}$$

$$\geq 0.$$

This inequality (3.7) must hold for all values of z in $\mathbb{D}$. Upon choosing the values of z on the positive real axis, where $0<|z|=r<1$, (3.7) reduces to

$$(p-\beta) - \sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|r^{\kappa-p}$$
$$-\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|r^{p-k}\geq 0.$$

Letting $r\to -1$ through real values, it follows that

(3.8)
$$(p-\beta) - \sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\mu_\kappa|$$
$$-\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\geq 0.$$

Thus, (3.8) yields (3.2). This completes the proof. □

The following theorem considers the growth bounds for the function φ that belongs to $\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.

Theorem 3.3. *Let $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ and $r=|z|<1$. Then*

$$|\varphi(z)|\leq(1+|\nu_p|)\;r^p+\left[\frac{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^{\ell}\frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\;r^{p+1}$$

and

$$|\varphi(z)| \geq (1+|\nu_p|)\ r^p - \left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^\ell \frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\ r^{p+1}.$$

Proof. Let $\varphi \in \mathcal{NH}_p^\beta(\alpha,\ [\sigma_\jmath;\rho_\jmath])$. By taking the modulus value of φ and using Theorem 3.2, we have

$$\begin{aligned}
|\varphi(z)| \leq& (1+|\nu_p|)\, r^p + \sum_{\kappa=p+1}^{\infty} (|\mu_\kappa|+|\nu_\kappa|)\, r^\kappa \\
\leq& (1+|\nu_p|)\, r^p + r^{p+1} \sum_{\kappa=p+1}^{\infty} (|\mu_\kappa|+|\nu_\kappa|) \\
\leq& (1+|\nu_p|)\, r^p + \frac{r^{p+1}}{[\alpha+p](\tau+p)_1}\left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^\ell \\
&\times \left(\sum_{\kappa=p+1}^{\infty}[\alpha+p](\tau+p)_1\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}\right]^\ell (|\mu_\kappa|+|\nu_\kappa|)\right) \\
\leq& (1+|\nu_p|)\, r^p + \frac{r^{p+1}}{[\alpha+p](\tau+p)_1}\left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^\ell \left(\sum_{\kappa=p+1}^{\infty}[(\kappa-p)\alpha+p]\right. \\
&\times \left.\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\ (|\mu_\kappa|+|\nu_\kappa|)\right) \\
\leq& (1+|\nu_p|)\, r^p + \left[\frac{\Omega\ \prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+\mathcal{D}_\jmath)}\right]^\ell \frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\, r^{p+1}.
\end{aligned}$$

Also,

$$\begin{aligned}
|\varphi(z)| \geq& (1+|\nu_p|)\, r^p - \sum_{\kappa=p+1}^{\infty} (|\mu_\kappa|+|\nu_\kappa|)\, r^\kappa \\
\geq& (1+|\nu_p|)\, r^p - \sum_{\kappa=p+1}^{\infty} (|\mu_\kappa|+|\nu_\kappa|)\, r^{p+1}
\end{aligned}$$

$$
\begin{aligned}
&\geq (1+|\nu_p|)\, r^p - \frac{r^{p+1}}{[\alpha+p](\tau+p)_1} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)} \right]^\ell \left(\sum_{\kappa=p+1}^{\infty} [\alpha+p](\tau+p)_1 \right. \\
&\quad \times \left. \left[\frac{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)}{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)} \right]^\ell (|\mu_\kappa| + |\nu_\kappa|) \right) \\
&\geq (1+|\nu_p|)\, r^p - \frac{r^{p+1}}{[\alpha+p](\tau+p)_1} \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)} \right]^\ell \left(\sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha+p] \right. \\
&\quad \times \left. \left[\frac{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^\ell \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} (|\mu_\kappa| + |\nu_\kappa|) \right) \\
&\geq (1+|\nu_p|) r^p - \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + \mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + \mathcal{D}_\jmath)} \right]^\ell \frac{[p(1-|\nu_p|)-\beta]}{[\alpha+p](\tau+p)_1}\, r^{p+1}.
\end{aligned}
$$

This completes the proof of Theorem 3.3. □

The next theorem determines the extreme points of convex hulls of $\mathcal{NH}_p^\beta(\alpha,\, [\sigma_\jmath; \rho_\jmath])$ denoted by $\overline{\mathrm{co}}\mathcal{NH}_p^\beta(\alpha,\, [\sigma_\jmath; \rho_\jmath])$.

Theorem 3.4. *A function $\varphi \in \overline{\mathrm{co}}\mathcal{NH}_p^\beta(\alpha,\, [\sigma_\jmath; \rho_\jmath])$ if and only if*

$$
\varphi(z) = \sum_{\kappa=p}^{\infty} \left(X_\kappa h_\kappa(z) + Y_\kappa g_\kappa(z) \right), \tag{3.9}
$$

where

$$
\begin{aligned}
h_p(z) =& z^p, \\
h_\kappa(z) =& z^p - \left[\frac{\Omega \prod\limits_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!} \right]^\ell \frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}} z^\kappa, \\
\kappa =& p+1, p+2, \ldots,
\end{aligned}
$$

$$\begin{aligned} g_\kappa(z) =& z^p - \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa - p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa - p)\mathcal{D}_\jmath)(\kappa - p)!} \right]^\ell \frac{(\kappa - p)!\,(p - \beta)}{[(\kappa - p)\alpha + p](\tau + p)_{\kappa - p}} \overline{z}^\kappa, \\ \kappa =& p, p+1, \ldots, \\ \sum_{\kappa=p}^{\infty} (X_\kappa + Y_\kappa) =& 1,\ X_\kappa \geq 0 \ \mathit{and}\ \ Y_\kappa \geq 0. \end{aligned}$$

Proof. For a function φ of the form (3.9), we acquire

$$\begin{aligned} \varphi(z) =& \sum_{\kappa=p}^{\infty} (X_\kappa h_\kappa(z) + Y_\kappa g_\kappa(z)) \\ =& X_p\, h_p + \sum_{\kappa=p+1}^{\infty} X_\kappa h_\kappa(z) + \sum_{\kappa=p}^{\infty} Y_\kappa g_\kappa(z) \\ =& X_p\, z^p + \sum_{\kappa=p+1}^{\infty} X_\kappa\, z^p \\ & - \sum_{\kappa=p+1}^{\infty} \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa - p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa - p)\mathcal{D}_\jmath)(\kappa - p)!} \right]^\ell \frac{(\kappa - p)!\,(p - \beta)}{[(\kappa - p)\alpha + p](\tau + p)_{\kappa - p}} X_\kappa z^\kappa \\ & + \sum_{\kappa=p}^{\infty} Y_\kappa\, z^p - \sum_{\kappa=p}^{\infty} \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa - p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa - p)\mathcal{D}_\jmath)(\kappa - p)!} \right]^\ell \\ & \times \frac{(\kappa - p)!\,(p - \beta)}{[(\kappa - p)\alpha + p](\tau + p)_{\kappa - p}} Y_\kappa \overline{z}^\kappa \\ =& \sum_{\kappa=p}^{\infty} (X_\kappa + Y_\kappa)\, z^p \\ & - \sum_{\kappa=p+1}^{\infty} \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa - p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa - p)\mathcal{D}_\jmath)(\kappa - p)!} \right]^\ell \frac{(\kappa - p)!\,(p - \beta)}{[(\kappa - p)\alpha + p](\tau + p)_{\kappa - p}} X_\kappa z^\kappa \\ & - \sum_{\kappa=p}^{\infty} \left[\frac{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa - p)\mathcal{C}_\jmath)}{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa - p)\mathcal{D}_\jmath)(\kappa - p)!} \right]^\ell \frac{(\kappa - p)!\,(p - \beta)}{[(\kappa - p)\alpha + p](\tau + p)_{\kappa - p}} Y_\kappa \overline{z}^\kappa \end{aligned}$$

$$
\begin{aligned}
=&z^p-\sum_{\kappa=p+1}^{\infty}\left[\frac{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}X_\kappa z^\kappa\\
&-\sum_{\kappa=p}^{\infty}\left[\frac{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}Y_\kappa \overline{z}^\kappa.
\end{aligned}
$$

Therefore, in view of Theorem 3.2, we gain

$$
\begin{aligned}
&\sum_{k=p+1}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\\
&\left[\left[\frac{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}X_\kappa\right]\\
&+\sum_{k=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\\
&\left[\left[\frac{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}\right]^{\ell}\frac{(\kappa-p)!\,(p-\beta)}{[(\kappa-p)\alpha+p](\tau+p)_{\kappa-p}}Y_\kappa\right]\\
&\le(p-\beta)\left(\sum_{\kappa=p}^{\infty}(X_\kappa+Y_\kappa)-X_p\right)=(p-\beta)\,(1-X_p)\le p-\beta.
\end{aligned}
$$

Therefore, $\varphi\in\overline{\mathrm{co}}\mathcal{N}\mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.

Conversely, suppose that $\varphi\in\overline{\mathrm{co}}\mathcal{N}\mathcal{H}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Set

$$
\begin{aligned}
X_\kappa=&((\kappa-p)\alpha+p)\left[\frac{\prod\limits_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod\limits_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!\,(p-\beta)}\,|\mu_\kappa|,\\
\kappa=&p+1,p+2,\ldots,
\end{aligned}
$$

and

$$Y_\kappa =[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!\,(p-\beta)}\;|\nu_\kappa|,$$
$$\kappa =p,p+1,p+2,\dots$$

On the basis of Theorem 3.2, we note that $0\le X_\kappa\le 1$, $\kappa=p+1,p+2,\dots$ and $0\le Y_\kappa\le 1$, $\kappa=p,p+1,p+2,\dots$ Let $X_p=1-\sum_{\kappa=p+1}^{\infty}X_\kappa+\sum_{\kappa=p}^{\infty}Y_\kappa$ and note that by Theorem 3.2, $X_p\ge 0$. Consequently, $\varphi(z)=\sum_{\kappa=p}^{\infty}(X_\kappa h_\kappa(z)+Y_\kappa g_\kappa(z))$ is obtained as required. □

Using convolution principle, we show the subclass $\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ is closed under convolution.

Theorem 3.5. *For $0\le\lambda\le\beta<p$, let $\varphi\in\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$ and $\mathcal{F}\in\mathcal{NH}_p^{\lambda}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$. Then $\varphi*\mathcal{F}\in\mathcal{NH}_p^{\beta}(\alpha,\,[\sigma_\jmath;\rho_\jmath])\subset\mathcal{NH}_p^{\lambda}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$.*

Proof. Utilizing definition of convolution, let the harmonic function $\varphi(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\mu_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\nu_\kappa|\overline{z}^\kappa$ and $\mathcal{F}(z)=z^p-\sum_{\kappa=p+1}^{\infty}|A_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|B_\kappa|\overline{z}^\kappa$. Then, the convolution of φ and $\mathcal{F}$ is

$$(\varphi*\mathcal{F})(z)=z^p-\sum_{\kappa=p+1}^{\infty}|\mu_\kappa A_\kappa|z^\kappa-\sum_{\kappa=p}^{\infty}|\nu_\kappa B_\kappa|\overline{z}^\kappa.$$

For $\mathcal{F}\in\mathcal{NH}_p^{\lambda}(\alpha,\,[\sigma_\jmath;\rho_\jmath])$, by Theorem 3.2, we conclude that $|A_\kappa|\le 1$ and $|B_\kappa|\le 1$. Now for the convolution $\varphi*\mathcal{F}$, we gain

$$\begin{aligned}
&\sum_{\kappa=p+1}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-c)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\;|\mu_\kappa|\,|A_\kappa|\\
&+\sum_{\kappa=p}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-c)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\;|\nu_\kappa|\,|B_\kappa|\\
\le&\sum_{\kappa=p+1}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-\beta)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\;|\mu_\kappa|\\
&+\sum_{\kappa=p}^{\infty}\frac{[(\kappa-p)\alpha+p]}{(p-\beta)}\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}\;|\nu_\kappa|\le 1,
\end{aligned}$$

since $0 \leq \lambda \leq \beta < p$ and $\varphi \in \mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath; \rho_\jmath])$. Therefore, $\varphi * \mathcal{F} \in \mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath; \rho_\jmath]) \subset \mathcal{NH}_p^{\lambda}(\alpha,\, [\sigma_\jmath; \rho_\jmath])$. □

In this theorem, we show that $\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath; \rho_\jmath])$ is closed under convex combination of its members. Let the functions $\varphi_\imath$ be defined, for $\imath = 1, 2, \ldots$, by

$$\varphi_\imath(z) = z^p + \sum_{\kappa=p+1}^{\infty} |\mu_{\imath,\kappa}| z^\kappa - \sum_{\kappa=p}^{\infty} |\nu_{\imath,\kappa}| \overline{z}^\kappa. \tag{3.10}$$

Theorem 3.6. *Let the functions $\varphi_\imath$ given by* (3.10) *be in $\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath; \rho_\jmath])$ for every $\imath = 1, 2, \ldots$ Then, the function θ defined by*

$$\theta(z) = \sum_{\imath=1}^{\infty} c_\imath \omega_\imath(z), \quad 0 \leq c_\imath < 1, \tag{3.11}$$

is also in the subclass $\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_\jmath; \rho_\jmath])$, where $\sum_{\imath=1}^{\infty} c_\imath = 1$.

Proof. According to the definition of θ, we can write

$$\theta(z) = z^p + \sum_{\kappa=p+1}^{\infty} \left(\sum_{\imath=1}^{\infty} c_\imath |\mu_{\imath,\kappa}| \right) z^\kappa - \sum_{\kappa=p}^{\infty} \left(\sum_{\imath=1}^{\infty} c_\imath |\nu_{\imath,\kappa}| \right) \overline{z}^\kappa.$$

Then, by Theorem 3.2, we have

$$\sum_{\kappa=p+1}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \left(\sum_{\imath=1}^{\infty} c_\imath |\mu_{\imath,\kappa}| \right)$$

$$+ \sum_{\kappa=p}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \left(\sum_{\imath=1}^{\infty} c_\imath |\nu_{\imath,\kappa}| \right)$$

$$= \sum_{\imath=1}^{\infty} c_\imath \left(\sum_{\kappa=p+1}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\mu_{\imath,\kappa}| \right.$$

$$\left. + \sum_{\kappa=p}^{\infty} \frac{[(\kappa-p)\alpha+p]}{(p-\beta)} \left[\frac{\prod_{\jmath=1}^{\delta} \Gamma(\sigma_\jmath + (\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega \prod_{\jmath=1}^{\eta} \Gamma(\rho_\jmath + (\kappa-p)\mathcal{C}_\jmath)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\nu_{\imath,\kappa}| \right)$$

$$\leq \sum_{\imath=1}^{\infty} c_\imath = 1.$$

Hence, the proof is completed. $\square$

Finally, we discuss a closure property of subclass $\mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_j;\rho_j])$ under the generalized Bernardi-Libera-Livingston integral operator $\mathcal{F}$ which is given as (see [9]):

$$\mathcal{F}(z) = \frac{(\lambda+p)}{z^{\lambda}} \int_0^z t^{\lambda-1}\, \varphi(t)dt, \quad \lambda > -p.$$

Theorem 3.7. *Let $\varphi \in \mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_j;\rho_j])$. Then $\mathcal{F} \in \mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_j;\rho_j])$.*

Proof. Let

$$\varphi(z) = z^p - \sum_{\kappa=p+1}^{\infty} |\mu_\kappa| z^{\kappa} - \sum_{\kappa=p}^{\infty} |\nu_\kappa| \overline{z}^{\kappa}.$$

From the representation of $\mathcal{F}$, it follows that

$$\begin{aligned}
\mathcal{F}(z) &= \frac{\lambda+p}{z^{\lambda}} \int_0^z t^{\lambda-1} \left\{ \phi(z) + \overline{\psi(z)} \right\} dt \\
&= \frac{\lambda+p}{z^{\lambda}} \left\{ \int_0^z t^{\lambda-1} \left(t^p - \sum_{\kappa=p+1}^{\infty} |\mu_\kappa|\, t^{\kappa} \right) dt - \overline{\int_0^z t^{\lambda-1} \left(\sum_{\kappa=p}^{\infty} |\nu_\kappa|\, t^{\kappa} \right) dt} \right\} \\
&= z^p - \sum_{\kappa=p+1}^{\infty} A_\kappa z^{\kappa} - \sum_{\kappa=p}^{\infty} B_\kappa \overline{z}^{\kappa},
\end{aligned}$$

where

$$A_\kappa = \left(\frac{\lambda+p}{\lambda+\kappa} \right) |\mu_\kappa| \quad \text{and} \quad B_\kappa = \left(\frac{\lambda+p}{\lambda+\kappa} \right) |\nu_\kappa|.$$

Therefore, since $\varphi \in \mathcal{NH}_p^{\beta}(\alpha,\, [\sigma_j;\rho_j])$,

$$\begin{aligned}
&\sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha + p] \left[\frac{\prod_{j=1}^{\delta} \Gamma(\sigma_j + (\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega \prod_{j=1}^{\eta} \Gamma(\rho_j + (\kappa-p)\mathcal{C}_j)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \left(\frac{\lambda+p}{\lambda+\kappa} \right) |\mu_\kappa| \\
&+ \sum_{\kappa=p}^{\infty} [(\kappa-p)\alpha + p] \left[\frac{\prod_{j=1}^{\delta} \Gamma(\sigma_j + (\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega \prod_{j=1}^{\eta} \Gamma(\rho_j + (\kappa-p)\mathcal{C}_j)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} \left(\frac{\lambda+p}{\lambda+\kappa} \right) |\nu_\kappa| \\
&\leq \sum_{\kappa=p+1}^{\infty} [(\kappa-p)\alpha + p] \left[\frac{\prod_{j=1}^{\delta} \Gamma(\sigma_j + (\kappa-p)\mathcal{D}_j)(\kappa-p)!}{\Omega \prod_{j=1}^{\eta} \Gamma(\rho_j + (\kappa-p)\mathcal{C}_j)} \right]^{\ell} \frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!} |\mu_\kappa|
\end{aligned}$$

$$+\sum_{\kappa=p}^{\infty}[(\kappa-p)\alpha+p]\left[\frac{\prod_{\jmath=1}^{\delta}\Gamma(\sigma_\jmath+(\kappa-p)\mathcal{D}_\jmath)(\kappa-p)!}{\Omega\prod_{\jmath=1}^{\eta}\Gamma(\rho_\jmath+(\kappa-p)\mathcal{C}_\jmath)}\right]^{\ell}\frac{(\tau+p)_{\kappa-p}}{(\kappa-p)!}|\nu_\kappa|\leq p-\beta.$$

By considering Theorem 3.2, we yield $\mathcal{F}(z)\in\mathcal{NH}_p^{\beta}(\alpha,[\sigma_\jmath;\rho_\jmath])$. □

4. Conclusion

In this paper, we have introduced a new generalized Noor-type integral operator $\mathcal{J}_{p,\ell}^{\eta,\delta}[\sigma_\jmath;\rho_\jmath]$ on the class of harmonic $p-$valent functions Correlating with FWGH-functions in the unit disc $\mathbb{D}$. A certain subclass including this new operator is studied. In addition, some outcomes are obtained by involving coefficient condition and by showing this significance condition for negative coefficient, growth bounds, extreme points, convolution property, convex linear combination and a class-preserving integral operator.

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PERFECT NILPOTENT GRAPHS

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Abstract. Let R be a commutative ring with identity. The nilpotent graph of R, denoted by $\Gamma_N(R)$, is a graph with vertex set $Z_N(R)^*$, and two vertices x and y are adjacent if and only if xy is nilpotent, where $Z_N(R) = \{x \in R \mid xy \text{ is nilpotent, for some } y \in R^*\}$. A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. In this paper, we characterize all rings whose $\Gamma_N(R)$ is perfect. In addition, it is shown that for a ring R, if R is Artinian, then $\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = |\mathrm{Nil}(R)^*| + |\mathrm{Max}(R)|$.

1. Introduction

The theory of graphs associated with rings was started by Beck [4] in 1981 and has grown a lot since then. Anderson and Livingston [2] modified Beck's definition and introduced the notion of zero-divisor graph. Surely, this is the most important graph associated with a ring and not only zero-divisor graphs but also various generalizations of it have attracted many researchers, see for instance [9,11] and [10]. The *zero-divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are joined by an edge if and only if $xy = 0$, where $Z(R)$ is set of zero-divisors of R. In [6], Chen defined a kind of graph structure of rings. He let all the elements of ring R be the vertices of the graph and two vertices x and y are adjacent if and only if xy is nilpotent. However, in 2010, Li and Li [10] modified and studied the *nilpotent graph* $\Gamma_N(R)$ of R is a graph with vertex set $Z_N(R)^*$, and two vertices x and y are adjacent if and only if xy is nilpotent, where $Z_N(R) = \{x \in R \mid xy \text{ is nilpotent, for some } y \in R^*\}$. Note that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of the graph $\Gamma_N(R)$. B. Smith determine all values of n for which zero-divisor graph of $\mathbb{Z}_n$ is perfect [13]. Also, Patil et al. [12] characterize

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various algebraic and order structures whose zero-divisor graphs are perfect graph. Therefore, this paper is devoted to study the perfect of a super graph of zero-divisor graphs. First let us recall some necessary notation and terminology from ring theory and graph theory.

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\mathrm{U}(R)$, $\mathrm{Max}(R)$ and $\mathrm{Nil}(R)$, the set of all zero-divisors, the set of all unit elements of R, the set of all maximal ideals of R and the set of all nilpotent elements of R, respectively. For a subset A of a ring R, we let $A^* = A \setminus \{0\}$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. Some more definitions about commutative rings can be find in [3, 5, 15].

We use the standard terminology of graphs following [7, 14]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\bar{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph* of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{u, v \in E | u, v \in V_0\}$. Also G is called a *null graph* if it has no edge. A complete graph of n vertices is denoted by K_n. An n-partite graph is one whose vertex set can be partitioned into n subsets, so that no edge has both ends in any one subset. A complete n-partite graph is one in which each vertex is jointed to every vertex that is not in the same subset. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect* graph G is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Using the Strong Perfect Graph Theorem, in Section 2 we completely determine all Artinian rings for which the nilpotent graph of R is perfect, leading to our main theorem. In Section 3 for an Artinian ring R, it is shown that the graph $\Gamma_N(R)$ is weakly perfect. Moreover, the exact value of the $\chi(\Gamma_N(R))$ is given.

2. On Perfect Graph

We start with some properties of the nilpotent elements of a ring. The following remark is useful in our proofs.

Remark 2.1. ([10, Remark 2, 3]). Let R be a non-reduced ring. Then the following statements hold.

(1) For every $x \in \mathrm{Nil}(R)^*$, x is adjacent to all non-zero elements of R and so $Z_N(R) = R$.
(2) $\Gamma_N(R)[\mathrm{Nil}(R)^*]$ is a (induced) complete subgraph of $\Gamma_N(R)$.

To prove our main results we need the following celebrate theorem.

Theorem 2.1 (The Strong Perfect Graph Theorem [7])**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

The following result, which is proved in [1, Corollary 2.2], will be helpful in our main results and used frequently in the sequel.

Corollary 2.1. *Let G be a graph and $\{V_1, V_2\}$ be a partition of $V(G)$. If $G[V_i]$ is a complete graph, for every $1 \leq i \leq 2$, then G is a perfect graph.*

The following lemmas have a key role in this paper.

Lemma 2.1. *Let n be a positive integer and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a ring, for every $1 \leq i \leq n$. If $\Gamma_N(R)$ contains no induced odd cycle of length at least 5, then $n \leq 4$.*

Proof. Suppose that $n \geq 5$. Then we can easily get

$$\begin{aligned}&(1,0,0,1,0,0,\dots,0) - (0,1,0,0,1,0,\dots,0) - (1,0,1,0,0,0,\dots,0)\\ &- (0,0,0,1,1,0,\dots,0) - (0,1,1,0,0,0,\dots,0) - (1,0,0,1,0,0,\dots,0)\end{aligned}$$

is a cycle of length 5. Thus, Theorem 2.1 lead to a contradiction. So, $n \leq 4$. □

Before proving first main result of this paper, we bring the following remark, which shows that Artinian rings share the following nice property.

Remark 2.2. Let $R \cong R_1 \times \cdots \times R_n$, $a = (x_1, x_2, \dots, x_n)$ and $b = (y_1, y_2, \dots, y_n)$, where n is a positive integer, every R_i is an Artinian local ring and $x_i, y_i \in R_i$ for every $1 \leq i \leq n$. Then

(1) a is adjacent to b in $\Gamma_N(R)$ if and only if $x_i y_i \in \mathrm{Nil}(R_i)$ for all $1 \leq i \leq n$;
(2) a is not adjacent to b in $\Gamma_N(R)$ if and only if $x_j y_j \in \mathrm{U}(R_j)$ for some $1 \leq j \leq n$;
(3) a is adjacent to b in $\overline{\Gamma_N(R)}$ if and only if $x_i y_i \in \mathrm{U}(R_i)$ for some $1 \leq i \leq n$;
(4) a is not adjacent to b in $\overline{\Gamma_N(R)}$ if and only if $x_j y_j \in \mathrm{Nil}(R_j)$ for all $1 \leq j \leq n$.

By using a similar way as used in the proof of [1, Lemma 2.3], one can prove the following result.

Lemma 2.2. *Let S_1, S_2, S_3, S_4 be rings such that $S_1 \cong R_1$, $S_2 \cong R_1 \times R_2$, $S_3 \cong R_1 \times R_2 \times R_3$ and $S_4 \cong R_1 \times R_2 \times R_3 \times R_4$, where R_i is a ring for every $1 \leq i \leq 4$. Then, if $\Gamma_N(S_4)$ is a perfect graph, then $\Gamma_N(S_3)$, $\Gamma_N(S_2)$ and $\Gamma_N(S_1)$ are perfect graphs.*

We are now in a position to state our first main result in this section.

Theorem 2.2. *Let R be a non-reduced Artinian ring. Then $\Gamma_N(R)$ is a perfect graph if and only if $|\mathrm{Max(R)}| \leq 4$.*

Proof. For one direction assume that $|\mathrm{Max(R)}| \leq 4$. This together with [3, Theorem 8.7] implies that there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$ and $n \leq 4$. By Theorem 2.1, it is enough to show that $\Gamma_N(R)$ and $\overline{\Gamma_N(R)}$ contains no induced odd cycle of length at least 5. By Lemma 2.2, we need to prove the case $n = 4$. So let $R \cong R_1 \times R_2 \times R_3 \times R_4$, where R_i is an Artinian local ring. We have the following two claims.

Claim 1. $\Gamma_N(R)$ contains no induced odd cycle of length at least 5. Note that if R is an Artinian non-reduced ring, then $Z_N(R) = R = U(R) \cup Z(R)$, where $U(R) = U(R_1) \times \cdots \times U(R_4)$. We consider the following partition for non-zero zero-divisors of R:

$$A = \{\{(x_1, x_2, x_3, x_4) \mid x_i \in \mathrm{Nil}(R_i) \text{ for all } i\} \backslash \{(0, 0, 0, 0)\}\},$$
$$B = \{(x_1, x_2, x_3, x_4) \mid \text{for some } i, x_i \notin \mathrm{Nil}(R_i)\}.$$

Thus $A \cap B = \varnothing, A \cap U(R) = \varnothing, B \cap U(R) = \varnothing$ and $V(\Gamma_N(R)) = A \cup B \cup U(R)$. Also we consider the following partition for B:

$$\begin{aligned}
B_1 =&\{(x, y, z, w) \in B \mid x \in \mathrm{U}(R_1)\},\\
B_2 =&\{(x, y, z, w) \in B \mid x \in \mathrm{Nil}(R_1) \text{ and } y \in \mathrm{U}(R_2)\},\\
B_3 =&\{(x, y, z, w) \in B \mid x \in \mathrm{Nil}(R_1),\ y \in \mathrm{Nil}(R_2) \text{ and } z \in \mathrm{U}(R_3)\},\\
B_4 =&\{(x, y, z, w) \in B \mid x \in \mathrm{Nil}(R_1),\ y \in \mathrm{Nil}(R_2),\ z \in \mathrm{Nil}(R_3) \text{ and } w \in \mathrm{U}(R_4)\}.
\end{aligned}$$

It is easy to see that $B = \cup_{i=1}^{4} B_i$ and $B_i \cap B_j = \varnothing$ for every $i \neq j$. The elements of $V(\Gamma_N(R))$ have form $a_i = (x_i, y_i, z_i, w_i)$, where $x_i \in R_1, y_i \in R_2, z_i \in R_3$ and $w_i \in R_4$ for each $i \in \mathbb{N}$. Now, assume to the contrary that $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $\Gamma_N(R)$. We have the following cases.

Case 1. $\{a_1, \ldots, a_n\} \cap \mathrm{U}(R) = \varnothing$. Assume to the contrary and with no loss of generality that $a_1 = (x_1, y_1, z_1, w_1) \in \mathrm{U}(R)$. Then a_2 and a_n must be in $\mathrm{Nil}(R)^*$. Therefore, a_n is adjacent to a_2, which is a contradiction.

Case 2. $\{a_1, \ldots, a_n\} \cap A = \varnothing$. Let $a_i \in \{a_1, \ldots, a_n\} \cap A$, for some $1 \leq i \leq n$. Then by Remark 2.1, a_i is adjacent to all other vertices, a contradiction. Thus, $\{a_1, \ldots, a_n\} \cap A = \varnothing$.

Case 3. $\{a_1, \ldots, a_n\} \cap B_4 = \varnothing$. To show this, for a contradiction assume that $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Since a_2 and a_n are adjacent to a_1 and

$$a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4),$$

we see that the fourth components of a_2 and a_n must be in $\mathrm{Nil}(R_4)$. Now since x_3x_1, y_1y_3 and z_1z_3 are nilpotent elements and a_3 is not adjacent to a_1, by Part (2) of Remark 2.2, we conclude that the fourth component of a_3 must be in $\mathrm{U}(R_4)$. This together with the fact that a_4 is adjacent to a_3 imply that the fourth component of a_4 is nilpotent element and so $a_4a_1 \in \mathrm{Nil}(R)$. Therefore, a_4 is adjacent to a_1, which is a contradiction. So the assertion is proved.

Case 4. $\{a_1, \ldots, a_n\} \cap B_1 = \varnothing$. Assume to the contrary and with no loss of generality, $a_1 = (x_1, y_1, z_1, w_1) \in B_1$. It is easy to see that for every $1 \leq i \leq 4$, there is no edge between any two vertices of B_i. This together with the above cases imply that a_n and a_2 are in $B_2 \cup B_3$. We distinguish the following three subcases.

Subcase 4.1. $\{a_n, a_2\} \subset B_3$. In this case, we have

$$\{a_n, a_2\} \subset \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times R_4.$$

Then the third components of a_1 and a_3 must be in $\mathrm{Nil}(R_3)$. Also, since a_n is not adjacent to a_3, by Part (2) of Remark 2.2, the fourth components of a_n and a_3 must be in $\mathrm{U}(R_4)$. This yields

$$\begin{aligned} a_1 \in& \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times R_4, \\ a_3 \in& R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4), \\ a_n \in& \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4). \end{aligned}$$

Then the fourth components of a_1 and a_2 must be in $\mathrm{Nil}(R_4)$. Hence, we find that

$$\begin{aligned} a_1 \in& \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4), \\ a_2 \in& \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4). \end{aligned}$$

Now, since a_2 is not adjacent to a_4, the third components of a_4 must be in $\mathrm{U}(R_3)$. This implies that a_4 is not adjacent to a_n and so $n \geq 7$. It is easy to see that the third component of a_5 must be in $\mathrm{Nil}(R_3)$ and so $a_5a_2 \in \mathrm{Nil}(R)$. This implies that $a_5 - a_2$, a contradiction. So, in this case the assertion is proved.

Subcase 4.2. $\{a_n, a_2\} \subset B_2$. By a similar way as used in Subcase (4.1), we get a contradiction.

Subcase 4.3. $a_n \in B_2$ and $a_2 \in B_3$. By a similar way as used in Subcase (4.1), we get a contradiction. Thus $\{a_1, \ldots, a_n\} \cap B_1 = \varnothing$.

By the above cases, $\{a_1, \ldots, a_n\} \subseteq B_2 \cup B_3$, but this is contradicts the fact $\Gamma_N(R)[B_2 \cup B_3]$ is a bipartite graph, and thus, $\Gamma_N(R)$ contains no induced odd cycle of length at least 5.

In Claim 2, $\mathrm{U}(R)$, A, B and B_i are sets that mentioned in Claim 1.

Claim 2. $\overline{\Gamma_N(R)}$ contains no induced odd cycle of length at least 5. We show that $\overline{\Gamma_N(R)}$ contains no induced odd cycle at least 5. Assume to the contrary that

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\overline{\Gamma_N(R)}$. It is clear that $\overline{\Gamma_N(R)}[A]$ is a null graph and so $\{a_1, \ldots, a_n\} \cap A = \varnothing$. Also, we show that

$$\{a_1, \ldots, a_n\} \cap \mathrm{U}(R) = \varnothing.$$

Assume to the contrary and with no loss of generality that $a_1 \in \mathrm{U}(R)$. Obviously, a_1 is just adjacent to all of vertices of $Z_N(R) \setminus \mathrm{Nil}(R)$. This together with the fact that $\{a_1, \ldots, a_n\} \subset Z_N(R) \setminus \mathrm{Nil}(R)$ imply that a_1 is adjacent to all other vertices, a contradiction. Thus $\{a_1, \ldots, a_n\} \cap \mathrm{U}(R) = \varnothing$. We claim that

$$\{a_1, \ldots, a_n\} \cap B_4 = \varnothing.$$

Indeed, if not, there would exist an $a_i \in B_4$. Without loss of generality, we may assume that $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Then $a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4)$. This together with Part (3) of Remark 2.2 implies that the forth components of a_2 and a_n must be in $\mathrm{U}(R_4)$ and so we have

$$a_n \in R_1 \times R_2 \times R_3 \times \mathrm{U}(R_4),$$
$$a_2 \in R_1 \times R_2 \times R_3 \times \mathrm{U}(R_4).$$

It is easy to see that a_2 is adjacent to a_n, a contradiction, and so,

$$\{a_1, \ldots, a_n\} \cap B_4 = \varnothing.$$

Finally to complete the proof, we prove that $\{a_1, \ldots, a_n\} \cap B_3 = \varnothing$. To get a contradiction, let $a_1 = (x_1, y_1, z_1, w_1) \in B_3$. Then

$$a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times R_4.$$

Since $a_1 - a_n$, $a_1 - a_2$ and a_2 is not adjacent to a_n, we consider the following two cases.

Case 1.

$$a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4),$$
$$a_2 \in R_1 \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4),$$
$$a_n \in R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4).$$

Since a_3 is not adjacent to a_1, the third and the fourth components a_3 must be nilpotent. On the other hand, a_3 is adjacent to a_2. This implies that $x_3x_2 \in \mathrm{U}(R_1)$ or $y_2y_3 \in \mathrm{U}(R_2)$.

First suppose that $x_3x_2 \in \mathrm{U}(R_1)$. Now, we know that

$$a_3 \in \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4),$$
$$a_2 \in \mathrm{U}(R_1) \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4).$$

This together with that a_3 is adjacent to a_4 implies that $x_3x_4 \in \mathrm{U}(R_1)$ or $y_3y_4 \in \mathrm{U}(R_2)$. If $x_3x_4 \in \mathrm{U}(R_1)$, then we have $x_2x_4 \in \mathrm{U}(R_1)$. Therefore, a_4 is adjacent to a_2, which is a contradiction. Thus, we conclude that $y_3y_4 \in \mathrm{U}(R_2)$. This yields

$$a_3 \in \mathrm{U}(R_1) \times \mathrm{U}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4),$$
$$a_4 \in \mathrm{Nil}(R_1) \times \mathrm{U}(R_2) \times R_3 \times R_4.$$

Since a_4 is not adjacent to a_1, we have

$$a_4 \in \mathrm{Nil}(R_1) \times \mathrm{U}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4).$$

Thus a_4 is not adjacent to a_n and so $n \geq 7$. On the other hand, since $a_4 - a_5$, the second components of a_5 must be unit and so a_5 is adjacent to a_2, which is a contradiction.

So, suppose that $y_2y_3 \in \mathrm{U}(R_2)$. Similarly, we get a contradiction. Thus in this case the assertion is proved.

Case 2.

$$\begin{aligned} a_1 \in& \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4), \\ a_2 \in& R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4), \\ a_n \in& R_1 \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4). \end{aligned}$$

By similar argument that of Case 1, we get a contradiction.

This means that $\{a_1, \ldots, a_n\} \subseteq B_2 \cup B_1$. Clearly, $\overline{\Gamma_N(R)}[B_1]$, $\overline{\Gamma_N(R)}[B_2]$ are complete, and thus by Corollary 2.1, $\overline{\Gamma_N(R)}[B_1 \cup B_2]$ is a perfect graph, a contradiction. Hence $\overline{\Gamma_N(R)}$ contain no induced odd cycle of length at least 5. Therefore, by Claim 1, Claim 2 and Theorem 2.1, $\Gamma_N(R)$ is a perfect graph.

For the other direction, since $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$, then by Theorem 2.1 and Lemma 2.1, $n \leq 4$, as desired. □

3. The Nilpotent Graph of an Artinian Ring is Weakly Perfect

The main goal of this section is to study the coloring of the nilpotent graphs of Artinian rings. For an Artinian ring R, it is shown that the graph $\Gamma_N(R)$ is weakly perfect. Moreover, the exact value of the $\chi(\Gamma_N(R))$ is given.

Theorem 3.1. *Let R be an Artinian ring. Then*

$$\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = |\mathrm{Nil}(R)^*| + |\mathrm{Max}(R)|.$$

Proof. First let R be an Artinian local ring. One may easily check that $V(\Gamma_N(R)) = \mathrm{Nil}(R) \cup \mathrm{U}(R)$ and so $\{\mathrm{Nil}(R), \mathrm{U}(R)\}$ is a partition of $V(\Gamma_N(R))$. By Remark 2.1, we have $\Gamma_N(R)[\mathrm{Nil}(R)^*]$ is a complete subgraph of $\Gamma_N(R)$ and every vertex $x \in \mathrm{Nil}(R)^*$ is adjacent to all other vertices. This together with this fact that there is no adjacency between two vertices of $\mathrm{U}(R)$ imply that $\Gamma_N(R) = \Gamma_N(R)[\mathrm{Nil}(R)^*] \vee \Gamma_N(R)[\mathrm{U}(R)]$ and so

$$\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = \omega(\Gamma_N(R)[\mathrm{Nil}(R)^*]) + \omega(\Gamma_N(R)[\mathrm{U}(R)]) = |\mathrm{Nil}(R)^*| + 1.$$

Now, let R be an Artinian non-local ring. By [3, Theorem 8.7], one can deduce that there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. We put:

$$\begin{aligned} A =& \{\{(x_1, \ldots, x_n) \mid x_i \in \mathrm{Nil}(R_i) \text{ for all } 1 \leq i \leq n\} \setminus \{(0,0,0,0)\}\}, \\ B =& \{(x_1, \ldots, x_n) \mid \text{for some } i, x_i \notin \mathrm{Nil}(R_i)\}, \\ \mathrm{U}(R) =& \{(x_1, \ldots, x_n) \mid x_i \in \mathrm{U}(R_i) \text{ for all } 1 \leq i \leq n\}. \end{aligned}$$

One may easily check that $V(\Gamma_N(R)) = A \cup B \cup \mathrm{U}(R)$, $A \cap B = \varnothing, A \cap \mathrm{U}(R) = \varnothing, B \cap \mathrm{U}(R) = \varnothing$ and so $\{A, B, \mathrm{U}(R)\}$ is a partition of $V(\Gamma_N(R))$. It is clear that $\Gamma_N(R)[U(R)] = \overline{K_{|\mathrm{U}(R)|}}$ and there is no adjacency between two vertices of B and $\mathrm{U}(R)$. To complete the proof, we prove that

$$\begin{aligned} \Gamma_N(R)[A \cup B] =& \Gamma_N(R)[A] \vee \Gamma_N(R)[B], \\ \Gamma_N(R)[A \cup \mathrm{U}(R)] =& \Gamma_N(R)[A] \vee \Gamma_N(R)[\mathrm{U}(R)], \end{aligned}$$

where $\Gamma_N(R)[A]$ is a complete subgraph of $\Gamma_N(R)$ and $\Gamma_N(R)[B]$ is an n-partite subgraph of $\Gamma_N(R)$, which is not an $(n-1)$-partite subgraph of $\Gamma_N(R)$. To see this, by Part (1) of Remark 2.1, we have $\Gamma_N(R)[A]$ is a complete subgraph of $\Gamma_N(R)$ and every vertex $x \in A$ is adjacent to all other vertices.

Now, for every $1 \leq i \leq n$, let $B_i = \{(x_1, \ldots, x_n) \in B \mid x_i \in \mathrm{U}(R_i)$ and $x_j \in \mathrm{Nil}(R_j)$ for every $1 \leq j \leq i\}$. It is easy to see that for every $1 \leq i \leq n$, there is no adjacency between two vertices of B_i. This together with this fact that the set $\{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,0,\ldots,1)\}$ is a clique of $\Gamma_N(R)[B]$ imply that $\Gamma_N(R)[B]$ is an n-partite subgraph of $\Gamma_N(R)$, which is not an $(n-1)$-partite subgraph of $\Gamma_N(R)$. Therefore,

$$\Gamma_N(R)[A \cup B] = \Gamma_N(R)[A] \vee \Gamma_N(R)[B],$$
$$\Gamma_N(R)[A \cup \mathrm{U}(R)] = \Gamma_N(R)[A] \vee \Gamma_N(R)[\mathrm{U}(R)]$$

and so

$$\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = \omega(\Gamma_N(R)[A]) + \omega(\Gamma_N(R)[B]) = |\mathrm{Nil}(R)^*| + |\mathrm{Max}(R)|$$

and the proof is complete. □

We close this paper with the following result.

Theorem 3.2. *Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) $\omega(\Gamma_N(R)) = 2$;
(2) $\chi(\Gamma_N(R)) = 2$;
(3) *either* $\Gamma_N(R) \cong K_{1,2}$ *or* $\Gamma_N(R) \cong K_1 \vee \overline{K_\infty}$.

Proof. (3) $\Rightarrow$ (1), (2) are clear. (2) $\Rightarrow$ (3) is obtained by similar argument to that proof of (1) $\Rightarrow$ (3). (1) $\Rightarrow$ (3) is only thing to prove.

(1) $\Rightarrow$ (3). Suppose that $\omega(\Gamma_N(R)) = 2$. First we show that $|\mathrm{Nil}(R)^*| = 1$. To see this, consider $A = \{a, b, c\}$ where $a, b \in \mathrm{Nil}(R)^*$ and $c \in \mathrm{U}(R)$. Then the subgraph induced by A is isomorphic to K_3, a contradiction. Thus, $|\mathrm{Nil}(R)^*| = 1$.

Now, we have two following cases.

Case 1. $Z(R) = \mathrm{Nil}(R)$. Since $|Z(R)^*| = 1 < \infty$, R is an Artinian (indeed R is finite). By [3, Theorem 8.7] there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is an Artinian local ring. If $n \geq 2$, then $Z(R)^* \geq 2$, a contradiction. So we may assume that R is an Artinian local ring. This, together [8, Example 1.5], implies that $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$ and so $\Gamma_N(R) \cong K_{1,2}$.

Case 2. $Z(R) \neq \mathrm{Nil}(R)$. Since $\omega(\Gamma_N(R)) = 2$ and by Remark 2.1, every $x \in \mathrm{Nil}(R)^*$, x is adjacent to all non-zero elements of R, we have only to show that $|Z(R)| = \infty$. To get a contradiction, let $|Z(R)| < \infty$. Then by [3, Theorem 8.7], we may write $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. Since $Z(R) \neq \mathrm{Nil}(R)$, we have $n \geq 2$. Also, since R is non-reduced, without loss of generality, we can suppose that $a \in \mathrm{Nil}(R_1)^*$. Consider $\phi = \{x, y, z\}$, where $x = (a, 0, \ldots, 0), y = (1, 0, \ldots, 0), z = (0, 1, 0, \ldots, 0)$. Then the subgraph induced by ϕ in $\Gamma_N(R)$ is isomorphic to K_3, a contradiction. Thus, $|Z(R)| = \infty$ and so $\Gamma_N(R) \cong K_1 \vee \overline{K_\infty}$ and the proof is complete. $\square$

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OSCILLATION CRITERIA FOR SECOND ORDER IMPULSIVE DELAY DYNAMIC EQUATIONS ON TIME SCALE

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Abstract. In this work, we study the oscillation of a kind of second order impulsive delay dynamic equations on time scale by using impulsive inequality and Riccati transformation technique. Some examples are given to illustrate our main results.

1. Introduction

Consider a class of second order impulsive nonlinear dynamic equations of the form:

$$(E)\begin{cases}[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta}+q(t)x(\sigma(t)-\delta)=0, & t\in\mathbb{J}_{\mathbb{T}}:=[0,\infty)\cap\mathbb{T},\ t\neq\tau_k,\ t\geq t_0,\\ x(\tau_k^+)=M_k(x(\tau_k)), \quad x^{\Delta}(\tau_k^+)=N_k(x^{\Delta}(\tau_k)), & k\in\mathbb{N},\\ x(t_0^+)=x_0, \quad x^{\Delta}(t_0^+)=x_0^{\Delta}, & t_0-\delta\leq t\leq t_0,\end{cases}$$

under the following hypotheses.

(A_1) $\gamma\geq 1$ is the quotient of odd positive integers, $\mathbb{T}$ is an unbouned above time scale with $0\in\mathbb{T}$ and $\tau_k\in\mathbb{T}$ satisfying the properties $0\leq t_0<\tau_1<\tau_2<\cdots<\tau_k$, $\lim_{k\to\infty}\tau_k=\infty$,

$$x(\tau_k^+)=\lim_{h\to 0^+}x(\tau_k+h), \quad x^{\Delta}(\tau_k^+)=\lim_{h\to 0^+}x^{\Delta}(\tau_k+h),$$

which represent the right limit of $x(t)$ at $t=\tau_k$ in the sense of time scale. If τ_k is right scattered, then $x(\tau_k^+)=x(\tau_k)$, $x^{\Delta}(\tau_k^+)=x^{\Delta}(\tau_k)$. Similarly, we can define $x(\tau_k^-)$, $x^{\Delta}(\tau_k^-)$.

(A_2) $\delta\in\mathbb{R}_+$, $\sigma(t)-\delta\in\mathbb{T}$, $r(t)>0$, $q(t)\in C_{rd}(\mathbb{T},[t_0,\infty)_{\mathbb{T}})$.

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(A_3) $M_k, N_k : \mathbb{R} \to \mathbb{R}$ are continuous functions, $M_k(0) = 0 = N_k(0)$ and there exist numbers a_k, a_k^*, b_k, b_k^* such that $a_k^* \leq \frac{M_k(u)}{u} \leq a_k$, $b_k^* \leq \frac{N_k(u)}{u} \leq b_k$, $u \neq 0$, $k \in \mathbb{N}$.

In this work, our objective is to extend the work of [15] to the second order impulsive delay dynamic equations (E). About the time scale concept and fundamentals of time scale calculus we refer the monographs [6] and [7].

Oscillation theory of impulsive differential/difference equation has brought the attention of many researchers, as it provides a more adequate mathematical model for numerous process and phenomena studied in physics, biology, engineering and to mention a few. In the literature, most of the results obtained for difference equations is the continuous analogues of differential equations and vice versa. Hence it was an immediate question to find a way for which one can unify the qualitative properties of both equations. In 1988 Stefen Hilger introduced the concept of time scales calculus, which unify the continuous and discrete calculus in his Ph.D. thesis [12]. The study of impulsive dynamic equations on time scales has been initiated by Benchora et al. [4].

In [15], Huang has considered the second order impulsive dynamic equation of the form

$$\begin{cases} [r(t)(y^\Delta(t))^\gamma]^\Delta + f(t, y^\sigma(t)) = 0, \quad t \in \mathbb{J}_\mathbb{T} := [0,\infty) \cap \mathbb{T},\ t \neq \tau_k,\ t \geq t_0, \\ y(\tau_k^+) = g_k(y(\tau_k)), \quad y^\Delta(\tau_k^+) = h_k(y^\Delta(\tau_k)), \quad k \in \mathbb{N}, \\ y(t_0^+) = y_0, \quad y^\Delta(t_0^+) = y_0^\Delta, \end{cases}$$

and improved the results of [13] and [14].

To the best of the author's knowledge, there is no such results for the impulsive delay dynamic equations on time scales. Hence, in this work an attempt is made to study the impulsive dynamic equations (E) and from which we can find the corresponding results for impulsive differential/difference equation. In this direction, we refer the reader to some works ([2], [13]-[19]) and the references cited there in.

AC^i=$\{x : \mathbb{J}_\mathbb{T} \to \mathbb{R}$ is i-times Δ-differentiable, whose ith delta derivative $x^{\Delta^{(i)}}$ is absolutely continuous$\}$, $PC = \{x : \mathbb{J}_\mathbb{T} \to \mathbb{R}$ is rd-continuous at the points $\tau_k, k \in \mathbb{N}$ for which $x(\tau_k^-)$, $x(\tau_k^+)$, $x^\Delta(\tau_k^-)$ and $x^\Delta(\tau_k^+)$ exist, with $x(\tau_k^-)$=$x(\tau_k)$, $x^\Delta(\tau_k^-)$=$x^\Delta(\tau_k)\}$.

Definition 1.1. A solution of $x(t)$ of (E) is said to be regular if it is defined on some half line $[\tau_x, \infty)_\mathbb{T} \subset [t_0, \infty)_\mathbb{T}$ and $\sup\{|x(t)| : t \geq t_x\} > 0$. A regular solution $x(t)$ of (E) is said to be eventually positive (eventually negative), if there exists $t_1 > 0$ such that $x(t) > 0$ $(x(t) < 0)$ for $t \geq t_1$.

Definition 1.2. A function $x(t) \in PC \cap AC^2(\mathbb{J}_\mathbb{T} \setminus \{\tau_1, \tau_2, \dots\}, \mathbb{R})$ is called a solution of (E) if:

(I) it satisfies (E) a.e. on $\mathbb{J}_\mathbb{T} \setminus \{\tau_k\}, k \in \mathbb{N}$;
(II) for $t = \tau_k, k \in \mathbb{N}$, $x(t)$ satisfies (E);
(III) for any $t \in [t_0 - \delta, t_0]$, $x(t) = \phi(t)$, $x(t_0^+) = x_0$, $x^\Delta(t_0^+) = x_0^\Delta$.

Definition 1.3. A nontrivial solution $x(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0 \geq 0$ such that $x(t)$ has a constant sign for $t \geq t_0$. Otherwise, the solution $x(t)$ is said to be oscillatory.

For completeness in the paper, we give the time scale concept and some fundamentals of time scale calculus in Section 4.

2. Basic Lemmas

We need the time scale version of the following well known results for our use in the sequel.

Lemma 2.1 ([1]). *Let $y, f \in C_{rd}$ and $p \in \mathcal{R}$. Then $y^{\Delta}(t) \leq p(t)y(t) + f(t)$, implies that for all $t \in \mathbb{T}$*

$$y(t) \leq y(t_0)e_p(t,t_0) + \int_{t_0}^{t} e_p(t,\sigma(s))f(s)\Delta s.$$

Lemma 2.2 ([15]). *Assume that*

(*i*) $m \in PC \cap AC^1(\mathbb{J}_{\mathbb{T}} \setminus \{\tau_k\}, \mathbb{R})$;

(*ii*) *for $k \in \mathbb{N}$ and $t \geq t_0$, we have*

$$m^{\Delta}(t) \leq p(t)m(t) + v(t), \quad t \in \mathbb{J}_{\mathbb{T}} = [0,\infty) \cap \mathbb{T}, t \neq \tau_k,$$
$$m(\tau_k^+) \leq d_k m(\tau_k) + e_k.$$

Then the following inequality holds

$$\begin{aligned} m(t) \leq & m(t_0) \prod_{t_0<\tau_k<t} d_k e_p(t_0,t) + \int_{t_0}^{t} \prod_{s<\tau_k<t} d_k e_p(t,\sigma(s))v(s)\Delta s \\ & + \sum_{t_0<\tau_k<t} \Big(\prod_{\tau_k<\tau_j<t} d_j e_p(t,\tau_k) \Big) e_k, t \geq t_0. \end{aligned}$$

Lemma 2.3. *Suppose that (A_1)-(A_3), $a_k, b_k > 0$, $k \in \mathbb{N}$ hold. Furthermore, assume that there exists $T \geq t_0$ such that $x(t) > 0$ for $t \geq T$ and*

(A_4) $\int_T^{\infty} \frac{1}{r^{\frac{1}{\gamma}}(s)} \prod_{T<\tau_k<s} \frac{b_k^*}{a_k} \Delta s = \infty$.

Then $x^{\Delta}(\tau_k^+) \geq 0$ and $x^{\Delta}(t) \geq 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$ and $\tau_k \geq T$.

Proof. Let $x(t)$ be an eventually positive solution of (E) for $t \geq t_0$. Without loss of generality we assume that $x(t) > 0$ and $x(t-\delta) > 0$ for $t \geq t_1 > t_0 + \delta$. From (E), we get $[r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} = -q(t)f(x(t-\delta)) \leq 0$. Therefore, $r(t)(x^{\Delta}(t))^{\gamma}$ is monotonically decreasing on $[t_2,\infty)_{\mathbb{T}}, t_2 > t_1 + \delta$. Assume that $\tau_k > t_2$ for $k \in \mathbb{N}$. Consider the interval $(\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$. We assert that $x^{\Delta}(\tau_k) \geq 0$. If not, there exists $\tau_j \geq t_2$ such that $x^{\Delta}(\tau_j) < 0$ and hence $x^{\Delta}(\tau_j^+) = N_k(x^{\Delta}(\tau_k)) \leq b_k^* x^{\Delta}(\tau_k) < 0$. Let $x^{\Delta}(\tau_j^+) = -\alpha$, $\alpha > 0$. Now for $t \in (\tau_j, \tau_{j+1}]_{\mathbb{T}}$, we have $r(\tau_{j+1})(x^{\Delta}(\tau_{j+1}))^{\gamma} \leq r(\tau_j)(x^{\Delta}(\tau_j^+))^{\gamma}$, that is,

$$x^{\Delta}(\tau_{j+1}) \leq \left(\frac{r(\tau_j)}{r(\tau_{j+1})}\right)^{\frac{1}{\gamma}} x^{\Delta}(t_j^+) = -b_j^*\alpha \left(\frac{r(\tau_j)}{r(\tau_{j+1})}\right)^{\frac{1}{\gamma}} < 0.$$

If $t \in (\tau_{j+1}, \tau_{j+2}]_{\mathbb{T}}$, then

$$
\begin{aligned}
x^{\Delta}(\tau_{j+2}) \leq \left(\frac{r(\tau_{j+1})}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_{j+1}^{+}) &= \left(\frac{r(\tau_{j+1})}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} N_{j+1}(x^{\Delta}(\tau_{j+1})) \\
&\leq b_{j+1}^{*} \left(\frac{r(\tau_{j+1})}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_{j+1}),
\end{aligned}
$$

that is,

$$
x^{\Delta}(\tau_{j+2}) \leq -b_j^{*} b_{j+1}^{*} \alpha \left(\frac{r(\tau_j)}{r(\tau_{j+2})}\right)^{\frac{1}{\gamma}} < 0.
$$

Hence, by the method of induction

$$
\begin{aligned}
x^{\Delta}(\tau_{j+n}) &\leq -b_j^{*} b_{j+1}^{*} b_{j+2}^{*} \cdots b_{j+n-1}^{*} \alpha \left(\frac{r(\tau_j)}{r(\tau_{j+n})}\right)^{\frac{1}{\gamma}} \\
&= -\left(\frac{r(\tau_j)}{r(\tau_{j+n})}\right)^{\frac{1}{\gamma}} \left(\prod_{i=1}^{n-1} b_{j+i}^{*}\right) \alpha < 0,
\end{aligned}
$$

for $t \in (\tau_{j+n-1}, \tau_{j+n}]_{\mathbb{T}}$.

Now, we consider the following impulsive dynamic inequalities

$$
(E_1) \begin{cases} [r(t)(x^{\Delta}(t))^{\gamma}]^{\Delta} \leq 0, & t > \tau_j, t \neq \tau_k, k = j+1, j+2, \ldots, \\ x^{\Delta}(\tau_k^{+}) \leq b_k^{*} x^{\Delta}(\tau_k), & k = j+1, j+2, \ldots \end{cases}
$$

Let $m(t) = r(t)(x^{\Delta}(t))^{\gamma}$, then (E_1) becomes

$$
\begin{cases} m^{\Delta}(t) \leq 0, & t > \tau_j, t \neq \tau_k, k = j+1, j+2, \ldots, \\ m(\tau_k^{+}) \leq (b_k^{*})^{\gamma} m(\tau_k), & k = j+1, j+2, \ldots, \end{cases}
$$

and, by Lemma 2.2, it follows that

$$
m(t) \leq m(\tau_j^{+}) \prod_{\tau_j < \tau_k < t} (b_k^{*})^{\gamma},
$$

that is,

$$
x^{\Delta}(t) \leq \left(\frac{r(\tau_j)}{r(t)}\right)^{\frac{1}{\gamma}} x^{\Delta}(\tau_j^{+}) \prod_{\tau_j < \tau_k < t} b_k^{*} = -\alpha \left(\frac{r(\tau_j)}{r(t)}\right)^{\frac{1}{\gamma}} \prod_{\tau_j < \tau_k < t} b_k^{*}. \tag{2.1}
$$

For $k = j+1, j+2, \ldots$, we also have $x(\tau_k^{+}) \leq a_k x(\tau_k)$. By (2.1) and since $x(\tau_k^{+}) \leq a_k x(\tau_k)$, $k = j+1, j+2, \ldots$, it follows from Lemma 2.2 that

$$
\begin{aligned}
x(t) &\leq x(\tau_j^{+}) \prod_{\tau_j < \tau_k < t} a_k - \int_{\tau_j}^{t} \prod_{s < \tau_k < t} a_k \left[\alpha \left(\frac{r(\tau_j)}{r(t)}\right)^{\frac{1}{\gamma}} \prod_{\tau_j < \tau_k < s} b_k^{*}\right] \Delta s \\
&\leq \prod_{\tau_j < \tau_k < t} a_k \left[x(\tau_j^{+}) - \alpha \, (r(\tau_j))^{\frac{1}{\gamma}} \int_{\tau_j}^{t} \left(\frac{1}{r(s)}\right)^{\frac{1}{\gamma}} \prod_{\tau_j < \tau_k < s} \frac{b_k^{*}}{a_k} \Delta s\right]
\end{aligned}
$$

$\to -\infty$ as $t \to \infty$.

Due to (A_4), a contradiction to the fact that $x(t) > 0$ eventually. Hence, our assertation holds, that is, $x^\Delta(\tau_k) \geq 0$ for $\tau_k \geq T$ and hence $x^\Delta(t) > x^\Delta(\tau_k^+)$. Since $[r(t)(x^\Delta(t))^\gamma]^\Delta \leq 0$ for any $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $\tau_k \geq T$, then

$$x^\Delta(t) \geq \left(\frac{r(\tau_{k+1})}{r(t)}\right)^{\frac{1}{\gamma}} x^\Delta(\tau_{k+1}) \geq 0, \quad t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}.$$

Therefore, $x^\Delta(\tau_k^+) > 0$ and $x^\Delta(t) > 0$ for $t \in (\tau_k, \tau_{k+i})]_{\mathbb{T}}$, $t \geq t_2$, and the lemma is proved. ☐

Remark 2.1. If $x(t)$ is an eventually negative solution of (E). Then, using (A_1)-(A_3), it is easy to prove that $x^\Delta(\tau_k^+) \leq 0$ and $x^\Delta(t) \leq 0$, for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$ and $\tau_k \geq T \geq t_0$.

3. Sufficient Conditions for Oscillation

Theorem 3.1. *Let all conditions of Lemma* 2.3 *hold. Furthermore, assume that*

(A_5) $\int_{t_0}^{\infty} \prod_{t_0<\tau_k<s} \frac{1}{b_k^\gamma} q(s)\Delta s = \infty.$

Then every solution of (E) *oscillates.*

Proof. Suppose on the contrary that $x(t)$ is a nonoscillatory solution of (E). Without loss of generality, assume that $x(t) > 0$, $x(\sigma(t) - \delta) > 0$ for $t \geq t_1$. Hence, by Lemma 2.3, there exists $t_2 > t_1$ such that $x^\Delta(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$ and $\tau_k \geq t_2$. Indeed, $x^\Delta(t - \delta) > 0$ for $t \geq t_3 \geq t_2 + \delta$. Let

$$w(t) = \frac{r(t)(x^\Delta(t))^\gamma}{x(t-\delta)}. \tag{3.1}$$

Then $w(\tau_k^+) \geq 0$ and $w(t) \geq 0$ for $\tau_k \geq t_3$. From (3.1), for $t \neq \tau_k$ we have

$$\begin{aligned} w^\Delta(t) &= \frac{[r(t)(x^\Delta(t))^\gamma]^\Delta x(t-\delta) - r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma x^\Delta(t-\delta)}{x(t-\delta)x(\sigma(t)-\delta)} \\ &\leq \frac{[r(t)(x^\Delta(t))^\gamma]^\Delta}{x(\sigma(t)-\delta)} - \frac{r(\sigma(t))(x^\Delta(\sigma(t)))^\gamma x^\Delta(t-\delta)}{x(t-\delta)x(\sigma(t)-\delta)} \\ &\leq -q(t), \end{aligned}$$

that is,

$$w^\Delta(t) \leq -q(t), \quad t \neq \tau_k. \tag{3.2}$$

We note that

$$w(\tau_k^+) = \frac{r(\tau_k^+)(x^\Delta(\tau_k^+))^\gamma}{x(\tau_k^+ - \delta)} \leq \frac{b_k^\gamma r(\tau_k)(x^\Delta(\tau_k))^\gamma}{x(\tau_k - \delta)} = b_k^\gamma w(\tau_k).$$

Now, we have the following impulsive dynamics inequalities

$$\begin{aligned} w^\Delta(t) &\leq -q(t), \quad t \neq \tau_k \\ w(\tau_k^+) &\leq b_k^\gamma w(\tau_k), \quad k \in \mathbb{N}, \end{aligned}$$

and, by Lemma 2.2, it follows that

$$\begin{aligned} w(t) &\leq w(t_3) \prod_{t_3<\tau_k<t} b_k^\gamma - \int_{t_3}^{t} \prod_{s<\tau_k<t} b_k^\gamma q(s)\Delta s \\ &\leq \prod_{t_3<\tau_k<t} b_k^\gamma \left[w(t_3) - \int_{t_3}^{t} \prod_{t_3<\tau_k<s} \frac{1}{b_k^\gamma} q(s)\Delta s \right] \\ &\to -\infty \text{ as } t \to \infty. \end{aligned}$$

Due to (A_5), a contradiction to the fact that $w(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$. This completes the proof of the theorem. □

Theorem 3.2. *Let all conditions of Lemma* 2.3 *hold. Furthermore, assume that* $\tau_{k+1} - \tau_k = \delta$ *and*

(A_6) $\int_{t_0}^{\infty} \prod_{t_0<\tau_k<s} \frac{1}{d_k} q(s)\Delta s = \infty$,

where

$$d_k = \begin{cases} b_1^\gamma, & \text{if } k = 1, \\ d\dfrac{b_k^\gamma}{a_{k-1}^*}, & \text{if } k = 2, 3, \dots, \end{cases}$$

hold. Then every solution of (E) *oscillates.*

Proof. Proceed as in the proof Theorem 3.1 to obtain that $x^\Delta(t) > 0$ and $x^\Delta(\tau_k^+) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$, $t \geq t_2$. Indeed, $x^\Delta(t-\delta) > 0$ for $t \geq t_3 \geq t_2 + \delta$. Define $w(t)$ as in (3.1), we get (3.2) holds for $\tau_k \geq t_3$ and $t \neq \tau_k$. Now, if $k = 1$ we have

$$w(\tau_1^+) = \frac{r(\tau_1^+)(x^\Delta(\tau_1^+))^\gamma}{x(\tau_1^+ - \delta)} \leq \frac{b_1^\gamma r(\tau_1)(x^\Delta(\tau_1))^\gamma}{x(\tau_1 - \delta)} = d_1 w(\tau_1).$$

If $k = 2, 3, \dots$, then

$$\begin{aligned} w(\tau_k^+) &= \frac{r(\tau_k^+)(x^\Delta(\tau_k^+))^\gamma}{x(\tau_k^+ - \delta)} \leq \frac{b_k^\gamma r(\tau_k)(x^\Delta(\tau_k))^\gamma}{x(\tau_{k-1}^+ - \delta)} \leq \frac{b_k^\gamma r(\tau_k)(x^\Delta(\tau_k))^\gamma}{a_{k-1}^* x(\tau_{k-1} - \delta)} \\ &\leq \frac{b_k^\gamma r(\tau_k)(x^\Delta(\tau_k))^\gamma}{a_{k-1}^* x(\tau_k - \delta)} = d_k w(\tau_k). \end{aligned}$$

Consider the following impulsive dynamic inequality

$$\begin{cases} w^\Delta(t) \leq -q(t), & t \neq \tau_k, t \geq t_3 \\ w(\tau_k^+) \leq d_k w(\tau_k), & k \in \mathbb{N}. \end{cases}$$

Therefore, by Lemma 2.2, we get

$$w(t) \leq w(t_3) \prod_{t_3<\tau_k<t} d_k - \int_{t_3}^{t} \prod_{u<\tau_k<t} d_k q(u)\Delta u.$$

Then proceeding as in the proof of Theorem 3.1 and using (A_6), we get a contradiction to the fact that $w(t) > 0$ for $t \in (\tau_k, \tau_{k+1}]_{\mathbb{T}}$, $k \in \mathbb{N}$. This completes the proof of the theorem. □

Corollary 3.1. *Let all conditions of Lemma* 2.3 *hold. Assume that there exists a positive integer k_0 such that $a_k^* \geq 1$, $b_k \leq 1$ for $k \geq k_0$. Furthermore, assume that*

(A_7) $\int_{t_0}^{\infty} q(s)\Delta s = \infty$

holds, then every solution of (E) oscillates.

Proof. Without loss of generality, we assume that $k_0 = 1$. Since $b_k \leq 1$, then $\frac{1}{b_k^\gamma} \geq 1$. Therefore,

$$\int_{t_0}^{t} \prod_{t_0 \leq \tau_k < s} \frac{1}{b_k^\gamma} q(s)\Delta s \geq \int_{t_0}^{t} q(s)\Delta s.$$

Letting $t \to \infty$ and in view of Theorem 3.1, We get every solution of (E) is oscillatory. This completes the proof. □

Corollary 3.2. *Let all conditions of Lemma* 2.3 *hold. Assume that there exists a positive integer k_0 and a positive constant α such that $a_k^* \geq 1$ and $\frac{1}{b_k} \geq \left(\frac{\tau_{k+1}}{\tau_k}\right)^\alpha$ for $k \geq k_0$. Furthermore, assume that*

(A_8) $\int_{t_0}^{\infty} s^\alpha q(s)\Delta s = \infty$

holds, then every solution of (E) oscillates.

Proof. Without loss of generality, we assume that $k_0 = 1$. Now

$$\begin{aligned}
\int_{t_0}^{t} \prod_{t_0 < \tau_k < s} \frac{1}{b_k^\gamma} q(s)\Delta s &= \sum_{i=1}^{n} \prod_{t_0 < \tau_k < \tau_{i+1}} \frac{1}{b_k^\gamma} \int_{\tau_i}^{\tau_{i+1}} q(s)\Delta s \\
&\geq \frac{1}{\tau_1^\alpha} \sum_{i=1}^{n} \tau_{i+1}^\alpha \int_{\tau_i}^{\tau_{i+1}} q(s)\Delta s \\
&\geq \frac{1}{\tau_1^\alpha} \sum_{i=1}^{n} \int_{\tau_i}^{\tau_{i+1}} s^\alpha q(s)\Delta s \\
&= \frac{1}{\tau_1^\alpha} \int_{\tau_1}^{\tau_{n+1}} s^\alpha q(s)\Delta s.
\end{aligned}$$

Letting $t \to \infty$ and in view of Theorem 3.1, we get every solution of (E) is oscillatory. This completes the proof. □

Corollary 3.3. *Let all conditions of Lemma* 2.3 *hold. Assume that there exists a positive integer k_0 and a positive constant α such that $a_k^* \geq 1$ and $\frac{1}{d_k} \geq \left(\frac{\tau_{k+1}}{\tau_k}\right)^\alpha$ for $k \geq k_0$. If (A_8) hold, then every solution of (E) oscillates.*

Proof. The proof of the corollary can be be follows from Corollary 3.2 and Theorem 3.2. Hence, details are omitted. □

Next, we present some new oscillation criteria for (E), by using an integral averaging condition of Kamenev type.

Theorem 3.3. *Let all the conditions of Lemma* 2.3 *and $b_k \geq 1$ hold. Furthermore, assume that*

(A_9) $\limsup_{k\to\infty} \frac{1}{t^m}\int_{t_0}^{\tau_{k+1}}(t-s)^m q(s)\Delta s = \infty$,
then every solution of (E) *oscillates.*

Proof. Proceeding as in the proof of Theorem 3.1, we get

$$w^{\Delta}(t) \le -q(t), \quad \text{for } t \ne \tau_k.$$

Multiplying $(t-s)^m$ to both side of the preceding inequality and integrating from τ_k to τ_{k+1}, we get

$$\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)ds \le -\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s.$$

Indeed,

$$\begin{aligned}
&\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)\Delta s\\
=&(t-s)^m u(s)|_{\tau_k}^{\tau_{k+1}} - \int_{\tau_k}^{\tau_{k+1}}((t-s)^m)^{\Delta_s} w(s)\Delta s\\
=&\int_{\tau_k}^{\tau_{k+1}} m(t-s)^{m-1}w(s)\Delta s + (t-\tau_{k+1})^m w(\tau_{k+1}) - (t-\tau_k)^m w(\tau_k^+),
\end{aligned}$$

because $((t-s)^m)^{\Delta_s} = -m(t-s)^{m-1}$. As a result,

$$\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)\Delta s \ge -(t-\tau_k)^m w(\tau_k^+).$$

Therefore,

$$\begin{aligned}
\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s &\le -\int_{\tau_k}^{\tau_{k+1}}(t-s)^m w^{\Delta}(s)\Delta s\\
&\le (t-\tau_k)^m w(\tau_k^+)\\
&\le b_k(t-\tau_k)^m w(\tau_k),
\end{aligned}$$

that is,

$$\frac{1}{t^m}\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s \le b_k\left(\frac{t-\tau_k}{t}\right)^m w(\tau_k),$$

and hence,

$$\limsup_{k\to\infty}\frac{1}{t^m}\int_{\tau_k}^{\tau_{k+1}}(t-s)^m q(s)\Delta s < \infty,$$

a contradiction to (A_9). This completes the proof of the theorem. □

4. Appendix: Time Scale Preliminaries

We will briefly recall some basic definitions and facts from the time scale calculus that we will use in the sequel. For more details see [2,3,19]. On any time scale $\mathcal{T}$, we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : Vs < t\},$$

where $\inf\phi = \sup\mathbb{T}$, $\sup\phi = \inf\mathbb{T}$, and ϕ denotes the empty set. A nonmaximal element $t \in \mathbb{T}$ is called right-dense if $\sigma(t) = t$ and right-scattered if $\sigma(t) > t$. A nonminimal element $t \in \mathbb{T}$ is said to be left-dense if $\rho(t) = t$ and left-scattered if $\rho(t) > t$. The graininess μ of the time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$.

A mapping $f : \mathbb{T} \to \mathbb{X}$ is said to be differentiable at $t \in \mathbb{T}$, if there exists $f^{\Delta}(t) \in \mathbb{X}$ such that for any $\epsilon > 0$, there exists a neighborhood U of t satisfying

$$|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]| \leq \epsilon|\sigma(t) - s|,$$

for all $s \in U$. We say that f is delta differentiable (or in short: differentiable) on $\mathbb{T}$ provided $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T}, \mathbb{R})$.

The derivative and forward jump operator σ are related by the formula

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t).$$

Let f be a differentiable function on $[a, b]_{\mathbb{T}}$. Then f is increasing, decreasing, nondecreasing and nonincreasing on $[a, b]_{\mathbb{T}}$ if $f^{\Delta} > t$, $f^{\Delta} < t$, $f^{\Delta} \geq t$, $f^{\Delta} \leq t$ for all $t \in [a, b)_{\mathbb{T}}$, respectively. We will make use of the following product fg and quotient $\frac{f}{g}$ rules for the derivative of two differentiable functions f and g

$$\begin{aligned}(fg)^{\Delta} =& f^{\Delta}g + f^{\sigma}g^{\Delta} = fg^{\Delta} + f^{\Delta}g^{\sigma},\\ \left(\frac{f}{g}\right)^{\Delta} =& \frac{f^{\Delta}g - fg^{\Delta}}{gg^{\sigma}},\end{aligned}$$

where $f^{\sigma} = fo\sigma$, $gg^{\sigma} \neq 0$. The integration by parts formula reads

$$\int_a^b f^{\Delta}(t)g(t) = f(t)g(t)|_a^b - \int_a^b f^{\sigma}(t)g^{\Delta}(t)\Delta t.$$

Chain Rule. Assume $g : \mathbb{T} \to \mathbb{R}$ is $\Delta-$ differentiable on $\mathbb{T}$ and $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then $fog : \mathbb{T} \to \mathbb{R}$ is $\Delta-$ differentiable and satisfies

$$(fog)^{\Delta}(t) = \left\{ \int_0^1 f'(g(t) + h\mu(t)g^{\Delta}(t))dh \right\} g^{\Delta}(t).$$

Regressive. A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if for all $t \in \mathbb{T}$, $1+\mu(t)p(t) \neq 0$.

The set of all function $p : \mathbb{T} \to \mathbb{R}$, which are regressive and rd-continuous will be denoted by $\mathcal{R}$. We define the set $\mathcal{R}^+$ of all positively regressive elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Exponential Function. If $p \in \mathcal{R}$, then general exponential function e_p on $\mathbb{T}$ is defined as

$$e_p(t, s) = \exp\left(\int_s^t \frac{1}{\mu(z)} \log(1 + \mu(z)p(z))\Delta z \right),$$

with $\mu(z) \neq 0$ and $s, t \in \mathbb{T}$.

5. Examples

Example 5.1. Consider the impulsive dynamic equation

$$\left\{\begin{array}{l} x^{\Delta\Delta}(t) + \dfrac{1}{t}x(t-\dfrac{1}{2}) = 0, \quad t > \dfrac{1}{2}, t \neq \tau_k, \\ x(\tau_k^+) = \dfrac{k+1}{k}x(\tau_k), \quad x^{\Delta}(\tau_k^+) = x^{\Delta}(\tau_k), \quad k \in \mathbb{N}, \end{array}\right. \tag{5.1}$$

where $\gamma = 1$, $r(t) = 1$, $\delta = \frac{1}{2}$, $q(t) = \frac{1}{t} \geq 0$, $a_k^* = a_k = \frac{k+1}{k}$, $b_k^* = b_k = 1$, $\tau_k = 3k$, $\tau_{k+1} - \tau_k = 3 > 2$, $k \in \mathbb{N}$. Then, from (A_4)

$$\begin{aligned} &\int_T^{\infty} \prod_{T<\tau_k<s} \frac{b_k^*}{a_k} \Delta s \\ =&\int_2^{\infty} \prod_{2<\tau_k<s} \frac{k}{k+1} ds \\ =&\int_2^{\tau_1} \prod_{2<\tau_k<s} \frac{k}{k+1}\Delta s + \int_{\tau_1^+}^{\tau_2} \prod_{2<\tau_k<s} \frac{k}{k+1}\Delta s + \int_{\tau_2^+}^{\tau_3} \prod_{2<\tau_k<s} \frac{k}{k+1}\Delta s + \cdots \\ =&\frac{1}{2}(\tau_1 - 2) + \frac{1}{2} \times \frac{2}{3}(\tau_2 - \tau_1) + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4}(\tau_3 - \tau_2) + \cdots \\ =&\frac{1}{2} \times 2 + \frac{1}{3} \times 3 + \frac{1}{4} \times 3 + \frac{1}{5} \times 3 + \cdots \\ \geq&\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots = \sum_{i=2}^{\infty} \frac{1}{i} = \infty, \end{aligned}$$

and from (A_5)

$$\int_2^{\infty} \prod_{\frac{1}{2}<\tau_k<s} \frac{1}{b_k^{\gamma}} \frac{1}{s}\Delta s = \int_2^{\infty} \frac{1}{s}\Delta s \to \infty.$$

Therefore, all conditions of Theorem 3.1 are satisfied and hence (5.1) has an oscillatory solution.

Example 5.2. Consider the impulsive dynamic equation

$$\left\{\begin{array}{l} x^{\Delta\Delta}(t) + \dfrac{1}{t^3}x(t-1) = 0, \quad t > 1, t \neq \tau_k, \\ x(\tau_k^+) = \dfrac{k-1}{k}x(\tau_k), \quad k \in \mathbb{N}, k > k_0, \\ x^{\Delta}(\tau_k^+) = \dfrac{1}{k}x^{\Delta}(\tau_k), \quad k \in \mathbb{N}, k > k_0, \end{array}\right. \tag{5.2}$$

where $\gamma = 1$, $\delta = 1$, $r(t) = 1$, $q(t) = \frac{1}{t^3} \geq 0$, $a_k^* = a_k = \frac{k-1}{k}$, $b_k^* = b_k = \frac{1}{k}$, $\tau_k = 3k$, $\tau_{k+1} - \tau_k = 3 > 1$, $k \in \mathbb{N}$, $k > k_0 = 1$. Clearly, from (A_4) we have

$$\int_T^{\infty} \prod_{T<\tau_k<s} \frac{b_k^*}{a_k} \Delta s$$

$$
\begin{aligned}
&=\int_1^{\infty} \prod_{1<\tau_k<s} \frac{1}{k-1}\Delta s\\
&=\int_1^{\tau_2} \prod_{1<\tau_k<s} \frac{1}{k-1}\Delta s+\int_{\tau_2^+}^{\tau_3} \prod_{1<\tau_k<s} \frac{1}{k-1}\Delta s+\int_{\tau_3^+}^{\tau_4} \prod_{1<\tau_k<s} \frac{1}{k-1}\Delta s+\cdots\\
&=(\tau_2-1)+\frac{1}{2}\times(\tau_3-\tau_2)+\frac{1}{2}\times\frac{1}{3}\times(\tau_4-\tau_3)+\cdots\\
&=2+\frac{1}{2}\times 2^2+\frac{1}{2}\times\frac{1}{3}\times 2^3+\frac{1}{2}\times\frac{1}{3}\times\frac{1}{4}\times 2^4+\cdots\\
&\geq 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots=1+\sum_{i=2}^{\infty}\frac{1}{i}=\infty.
\end{aligned}
$$

Let $\alpha = 1$. Then

$$\frac{1}{b_k}=k\geq\left(\frac{\tau_{k+1}}{\tau_k}\right)^{\alpha}=\frac{k+1}{k}.$$

Also, from (A_8) we have

$$\int_1^{\infty} s^{\alpha}q(s)\Delta s=\int_1^{\infty} s^3\frac{1}{s^3}\Delta s=\int_1^{\infty}\Delta s=\infty.$$

All conditions of Corollary 3.2 are satisfied for (5.2) and hence, (5.2) has an oscillatory solution.

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EXISTENCE OF SOLUTIONS FOR A CLASS OF CAPUTO FRACTIONAL q-DIFFERENCE INCLUSION ON MULTIFUNCTIONS BY COMPUTATIONAL RESULTS

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Abstract. In this paper, we study a class of fractional q-differential inclusion of order $0 < q < 1$ under L^1-Caratheodory with convex-compact valued properties on multifunctions. By the use of existence of fixed point for closed valued contractive multifunction on a complete metric space which has been proved by Covitz and Nadler, we provide the existence of solutions for the inclusion problem via some conditions. Also, we give a couple of examples to elaborate our results and to present the obtained results by some numerical computations.

1. Introduction

Fractional calculus is an important branch in mathematical analysis. However, after Leibniz and Newton invented differential calculus, it has numerous applications in different sciences such as mechanics, electricity, biology, control theory, signal and image processing (for example, see [4,6,40]). In recent years the fractional differential equations and the fractional differential inclusions were developed intensively (for more information, see [8,10,19,22,38]). Also, it has been appeared many work on fractional differential inclusions [11,14–16,23,25,27,28]

In 1910, the subject of q-difference equations introduce by Jackson [33]. Later, at the beginning of the last century, studies on q-difference equation, appeared in so many works especially in Carmichael [26], Mason [39], Adams [3], Trjitzinsky [45]. It has been proven that these cases of equations have numerous applications in

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diverse domains and thus have evolved into multidisciplinary subjects (for example, see [1, 2, 7, 18, 30, 32, 47] and references therein).

In this paper, motivated by [9, 44] and among these achievements, we wish to discuss the existence of solutions for a problem of fractional q-derivative inclusions via the integral boundary value conditions given by

$$\begin{cases} {}^cD_q^\alpha x(t) \in F\left(t, x(t), x'(t), {}^cD_q^\beta x(t)\right), \\ x(0) + x'(0) + {}^cD_q^\beta x(0) = \int_0^\eta x(s)\, ds, \\ x(1) + x'(1) + {}^cD_q^\beta x(1) = \int_0^\nu x(s)\, ds, \end{cases} \tag{1.1}$$

for real number t in $[0,1]$, where F maps $[0,1] \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a compact valued multifunction, ${}^cD_q^\alpha$ is the fractional Caputo type q-derivative operator of order $\alpha \in (1,2]$ with q belongs to $(0,1)$, and

$$\Gamma_q(2-\beta)(\eta^2\nu - \nu^2\eta - \eta^2 + \nu^2 + 4\eta - 2\nu - 2) + 2(1-\eta) \neq 0,$$

for $\eta, \nu, \beta \in (0,1)$, such that $\alpha - \beta > 1$.

In 2012, Ahmad, Ntouyas and Purnaras investigated the q-difference equation:

$$\begin{cases} \left({}^cD_q^\alpha y\right)(x) = f(x, y(x)), \\ \alpha_1 y(0) - \beta_1 D_q y(0) = \gamma_1 y(e_1), \quad \alpha_2 y(1) + \beta_2 D_q y(1) = \gamma_2 y(e_2), \end{cases}$$

where $0 \leq x \leq 1$, $1 < \alpha \leq 2$ and $\alpha_i, \beta_i, \gamma_i, e_i \in \mathbb{R}$ for all i (see [17]). In 2013, Zhao, Chen and Zhang reviewed the nonlinear fractional q-difference equation:

$$\begin{cases} (D_q^\alpha y)(x) = f(x, y(x)), \\ y(0) = 0, \quad y(1) = \mu I_q^\beta y(e), \end{cases}$$

where $0 < x < 1$, $1 < \alpha \leq 2$, $0 < \beta \leq 2$ and $\mu > 0$ [46]. In 2015, Etemad, Ettefagh, and Rezapour investigated the q-differential equation:

$$\begin{cases} \left({}^cD_q^\alpha y\right)(x) = f(x, y(x), D_q y(x)), \\ \lambda_1 y(0) + \mu_1 D_q y(0) = e_1 I_q^\beta y(x_1), \quad \lambda_2 y(1) + \mu_2 D_q y(1) = e_2 I_q^\beta y(x_2), \end{cases}$$

where $0 \leq x \leq 1$, $1 < \alpha \leq 2$, $q \in (0,1)$, $\beta \in (0,2]$, $x_1, x_2 \in (0,1)$, with $x_1 < x_2$, $\lambda_i, \mu_i, e_i, \in \mathbb{R}$ for $i = 1,2$, and real value map f from $[0,1] \times \mathbb{R}^2$ is continuous [13]. Also, in the same year, Agarwal, Baleanu, Hedayati, and Rezapour founded results for the inclusion Caputo fractional differential:

$$\begin{cases} {}^cD^\alpha f(t) \in T\left(t, f(t), {}^cD^\beta f(t)\right), \\ f(0) = 0, \quad f(1) + f'(1) = \int_0^e f(s) ds, \end{cases}$$

such that $0 < e < 1$, $1 < \alpha \leq 2$, $0 < \beta < 1$, with $\alpha - \beta > 1$, and multifunction T define on $[0,1] \times \mathbb{R}^2$ has a compact valued in $2^{\mathbb{R}}$ [9]. Also, they investigate the existence of solutions for the Caputo fractional differential inclusion ${}^cD^\alpha x(t) \in F(t, x(t))$ such that $x(0) = a \int_0^\nu x(s) ds$ and $x(1) = b \int_0^\eta x(s) ds$, where $0 < \nu, \eta < 1$, $1 < \alpha \leq 2$ and

$a, b \in \mathbb{R}$ [9]. In 2016, Abdeljawad, Alzabut, and Baleanu stated and proved a new discrete q-fractional version of Gronwall inequality:

$$\begin{cases} {}_qC_a^\alpha f(t) = T\left(t, f(t)\right), \\ f(a) = \gamma, \end{cases}$$

such that $\alpha \in (0,1]$, $a \in \mathbb{T}_q = \{q^n \mid n \in \mathbb{Z}\}$, t belongs to $\mathbb{T}_a = [0,\infty)_q = \{q^{-i}a \mid i = 0,1,2,\dots\}$, ${}_qC_a^\alpha$ means the Caputo fractional difference of order α, and $T(t,x)$ fulfills a Lipschitz condition for all t and x [2]. Later, in 2017, Zhou, Alzabut, and Yang provide existence criteria for the solutions of p-Laplacian fractional Langevin differential equations with ansi-periodic boundary conditions:

$$\begin{cases} D_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \lambda)x(t)] = f(t, x(t), D_{0^+}^\alpha x(t)), \\ x(0) = -x(1), \quad D_{0^+}^\alpha x(0) = -D_{0^+}^\alpha x(1), \end{cases}$$

and

$$\begin{cases} {}_qD_{0^+}^\beta \phi_p[(D_{0^+}^\alpha + \lambda)x(t)] = g(t, x(t), {}_qD_{0^+}^\alpha x(t)), \\ x(0) = -x(1), \quad {}_qD_{0^+}^\alpha x(0) = -{}_qD_{0^+}^\alpha x(1), \end{cases}$$

for all $0 \le t \le 1$, where $0 < \alpha, \beta \le 1$, λ is more than or equal to zero, $1 < \alpha + \beta < 2$, $q \in (0,1)$ and $\phi_p(s) = |s|^{p-2}s$, with $p \in (1,2]$ [47]. In this manuscript, by using idea of the works, we study the existence of solutions for the fractional q-derivative inclusions via the integral and q-derivative boundary value conditions.

2. Preliminaries

Here, we recall some discovered facts on fractional q-calculus and their derivatives and integral. For more details on this, we refer the reader to the references [20, 34].

Let $q \in (0,1)$, $a \in \mathbb{R}$, and $\alpha \neq 0$ be a real number. Define $[a]_q = \frac{1-q^a}{1-q}$ (see [33]). The q-analogue of the power function $(a-b)^n$, with $n \in \mathbb{N}_0$, is $(a-b)_q^{(n)} = \prod_{k=0}^{n-1}(a - bq^k)$ and $(a-b)_q^{(0)} = 1$, where a and b in $\mathbb{R}$ and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$ (see [43]). Also, for $\alpha \in \mathbb{R}$ and $a \neq 0$, we have

$$(a-b)_q^{(\alpha)} = a^\alpha \prod_{k=0}^{\infty} \frac{a - bq^k}{a - bq^{\alpha+k}}.$$

If $b = 0$, then it is clear that $a^{(\alpha)} = a^\alpha$ (Algorithm 1). The q-Gamma function is given by $\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}$, where x belongs to $\mathbb{R}\backslash\{0,-1,-2,\dots\}$ (see [33]). Note that, $\Gamma_q(x+1) = [x]_q\Gamma_q(x)$. A simplified analysis can be performed to estimate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in summation. To this aim, we consider a pseudo-code description of the method for calculated q-Gamma function of order n which show in Algorithm 2. For function f, the q-derivative is defined by $(D_qf)(x) = \frac{f(x)-f(qx)}{(1-q)x}$ and $(D_qf)(0) = \lim_{x\to 0}(D_qf)(x)$ (see [3]). Also, the higher order q-derivative of a function f is defined by $(D_q^nf)(x) = D_q(D_q^{n-1}f)(x)$ for all $n \ge 1$, where $(D_q^0f)(x) = f(x)$ (see [3]). The

q-integral of a function f define on $[0, b]$ by

$$I_q f(x) = \int_0^x f(s) d_q s = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k),$$

for $x \in [0, b]$, provided that the sum converges absolutly [3]. If $a \in [0, b]$, then

$$\int_a^b f(u) d_q u = I_q f(b) - I_q f(a) = (1-q) \sum_{k=0}^{\infty} q^k \left[b f(bq^k) - a f(aq^k) \right],$$

whenever the series exists. The operator I_q^n is given by $I_q^0 f(x) = f(x)$ and $I_q^n f(x) = I_q(I_q^{n-1} f)(x)$ for all $n \geq 1$ (see [3]). It has been proved that $(D_q I_q f)(x) = f(x)$ and $(I_q D_q f)(x) = f(x) - f(0)$ whenever f is continuous at $x = 0$ (see [3]). The fractional Riemann-Liouville type q-integral of the function f on $[0, 1]$ is given by

$$I_q^\alpha f(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qs)^{(\alpha - 1)} f(s) d_q s,$$

whenever $\alpha > 0$ and $I_q^0 f(x) = f(x)$ whenever $\alpha = 0$, where $x \leq 1$ is a real number [13]. Also, the fractional Caputo type q-derivative of the function f is given by

$$\begin{aligned} \left({}^c D_q^\alpha f\right)(x) &= \left(I_q^{[\alpha]-\alpha} D_q^{[\alpha]} f\right)(x) \\ &= \frac{1}{\Gamma_q([\alpha]-\alpha)} \int_0^x (x - qs)^{([\alpha]-\alpha-1)} \left(D_q^{[\alpha]} f\right)(s) d_q s, \end{aligned}$$

for $x \in [0, 1]$ and $\alpha > 0$ (see [13]). It has been proved that $\left(I_q^\beta I_q^\alpha f\right)(x) = \left(I_q^{\alpha+\beta} f\right)(x)$, and $\left(D_q^\alpha I_q^\alpha f\right)(x) = f(x)$, where $\alpha, \beta \geq 0$ (see [29]). By using Algorithm 2, we can calculate $(I_q^\alpha f)(x)$ which is shown in Algorithm 3.

It is well recognized that the Pompeiu-Hausdorff metric H_d maps $2^X \times 2^X$ into $\mathbb{R}^{\geq 0}$ on metric space (X, d) is defined by

$$H_d(A, B) = \max \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(A, b) \right\},$$

where $d(A, b) = \inf_{a \in A} d(a, b)$ (also, see [12, 31]). Denote the set of bounded and closed subsets of X, the set of closed subsets of X and the set of compact and convex subsets of X by $CB(X)$, $C(X)$ and $P_{cp,cv}(X)$, respectively. Thus, $(CB(X), H_d)$ and $(C(X), H_d)$ are a metric space and a generalized metric space, respectively (for more details, see [35]). An element x belongs to X is called an fixed point of multifunction T maps X into 2^X whenever x in $T(x)$ (for more information, see [31]). If $\gamma \in (0, 1)$ exists somehow that $H_d(N(x), N(y))$ is less than or equal to $\gamma d(x, y)$ for all x and y in X, then a multifunction T maps X to $C(X)$ is called a contraction.

In 1970, Covitz and Nadler prove that there is a fixed point for each closed valued contractive multifunction on a complete metric space has a fixed point [27]. Let $J = [0, 1]$. A multifunction $G : J \to P_{cl}(\mathbb{R})$ is said to be measurable whenever the function $t \mapsto d(y, G(t))$ is measurable for all y belongs to $\mathbb{R}$ [28]. We say that F maps $J \times \mathbb{R}^3$ into $2^{\mathbb{R}}$ is a Caratheodory multifunction whenever $t \mapsto F(t, x, y, z)$ is measurable for all x, y, and z in $\mathbb{R}$ and $(x, y, z) \mapsto F(t, x, y, z)$ is upper semi-continuous for all t belongs to J [21, 28, 35]. Also, a Caratheodory multifunction F defines on $J \times \mathbb{R}^3$ to $2^{\mathbb{R}}$ is called L^1-Caratheodory whenever for each ρ more than zero, there exists $\phi_\rho \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t, x, y, z)\| = \sup_{v \in F(t,x,y,z)} |v| \leq \phi_\rho(t),$$

for all $|x|, |y|, |z| \leq \rho$ and for $t \in J$ (for more details, see [21, 28]). Denote by $AC[0, 1]$ the space of all the absolutely continuous functions defined on J. By using main idea of [15, 16, 41], we define the set of selections of F by

$$S_{F,x} := \left\{ v \in AC(J, \mathbb{R}) \,\middle|\, v(t) \in F\left(t, x(t), {}^c D_q^\beta x(t), x'(t)\right) \text{ for all } t \in J \right\},$$

for all x belongs to $C(J, \mathbb{R})$. Let E be a nonempty closed subset of a Banach space X and G maps E into 2^X a multifunction with nonempty closed values. We say that the multifunction G is lower semi-continuous whenever the set $\{y \in E \mid G(y) \cap B \neq \emptyset\}$ is open for all open set $B \subset X$ [31]. Furthermore, It has been proved that each completely continuous multifunction is lower semi-continuous [31]. Let $AC^2[0, 1] = \{w \in C^1[0, 1] \mid w' \in L[0, 1]\}$. The following lemmas will be used in the sequel.

Lemma 2.1 ([37])**.** *For Banach space X, consider multifunction F maps $J \times X$ into $P_{cp,cv}(X)$ and function Θ maps $L^1(J, X)$ into $C(J, X)$ such that are L^1-Caratheodory and linear continuous, respectively. The operator*

$$\begin{cases} \Theta o S_F : C(J, X) \to P_{cp,cv}(C(J \times X)), \\ (\Theta o S_F)(x) = \Theta(S_{F,x}), \end{cases}$$

is a closed graph operator in $C(J, X) \times C(J, X)$.

Lemma 2.2 ([31])**.** *Suppose that C a closed convex subset of Banach space E, $U \subset C$ is an open such that $0 \in U$. Also, let $F : \overline{U} \to P_{cp,cv}(C)$ is a upper semi-continuous compact map, where $P_{cp,cv}(C)$ denotes the family of nonempty, compact convex subsets of C. Then either F has a fixed point in $\overline{U}$ or there exist $u \in \partial U$ and $\lambda \in (0, 1)$ such that $u \in \lambda F(u)$.*

3. Main Results

Now, we would be ready to give theorems for the solution of the q-derivative inclusion problem (1.1). Define $x_v(t) = I_q^\alpha v(t) - c_{0v} - c_{1v} t$, where

$$\begin{aligned} c_{1v} = &-\frac{(1-\nu)t}{\gamma \Gamma_q(\alpha)} \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} v(m) d_q m ds \\ &- \frac{(1-\eta)t}{\gamma \Gamma_q(\alpha)} \int_0^1 (1 - qs)^{(\alpha-1)} v(s) d_q s \\ &- \frac{(\eta-1)t}{\gamma \Gamma_q(\alpha)} \int_0^\nu \int_0^s (s - qm)^{(\alpha-1)} v(m) d_q m ds \end{aligned}$$

$$
\begin{aligned}
&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\
&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs
\end{aligned}
$$

and

$$
\begin{aligned}
c_{0v}=&-\frac{1}{\Gamma_q(\alpha)(1-\eta)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(2-\eta^2)(\nu-1)}{2\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\\
&+\frac{(2-\eta^2)(1-\eta)}{2\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\
&+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{\alpha-2}v(s)d_qs.
\end{aligned}
$$

Clearly, $x_v\in AC^2[0,1]$ is well-define and x_v', ${}^cDx_v$ and $\int_0^\eta x_v(s)\,ds$ exist whenever v belongs to $AC[0,1]$ (for more details, see [36]).

Lemma 3.1. *Let v belongs to $AC[0,1]$, q,β,η and ν in $(0,1)$, $1<\alpha\leq 2$, with $\alpha-\beta>1$, and*

$$
\Gamma_q(2-\beta)(\eta^2\nu-\nu^2\eta-\eta^2+\nu^2+4\eta-2\nu-2)+2(1-\eta)\neq 0. \tag{3.1}
$$

Then, $x_v(t)$ is the unique solution for the problem ${}^cD_q^\alpha x(t)=v(t)$ with the integral boundary value conditions

$$
\begin{cases}
x(0)+x'(0)+{}^cD_q^\beta x(0)=\int_0^\eta x(s)ds,\\
x(1)+x'(1)+{}^cD_q^\beta x(1)=\int_0^\nu x(s)ds.
\end{cases} \tag{3.2}
$$

Proof. It is observed that the general solution of the equation $v(t)={}^cD_q^\alpha x(t)$ is

$$
x(t)=I_q^\alpha v(t)-a_0-a_1t=\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}v(s)d_qs-a_0-a_1t,
$$

where a_0 and a_1 are arbitrary constants and t in J (see [42]). Thus,

$$
\begin{aligned}
{}^cD_q^\beta x(t)&=I_q^{\alpha-\beta}v(t)-\frac{t^{1-\beta}a_1}{\Gamma_q(2-\beta)}\\
&=\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}v(s)d_qs-\frac{t^{1-\beta}a_1}{\Gamma_q(2-\beta)}
\end{aligned}
$$

and

$$x'(t) = I_q^{\alpha-1} v(t) - a_1 = \frac{1}{\Gamma_q(\alpha-1)} \int_0^t (t-qs)^{(\alpha-2)} v(s) d_q s - a_1.$$

Hence, by using an easy calculation, we get $x(0) + {}^c D_q^{\beta} x(0) + x'(0) = -a_0 - a_1$ and

$$\begin{aligned} x(1) + {}^c D_q^{\beta} x(1) + x'(1) &= \frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_q s \\ &+ \left(\frac{1}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} v(s) d_q s \right) \\ &\times \left(\frac{1}{\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s \right) \\ &- \frac{\Gamma_q(2) a_1}{\Gamma_q(2-\beta)} - 2a_1 - a_0. \end{aligned}$$

By using the boundary conditions (3.2), we obtain

$$a_0(\eta - 1) - a_1 \left(\frac{\eta^2}{2} - 1 \right) = \frac{1}{\Gamma_q(\alpha)} \int_0^{\eta} \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds$$

and

$$\begin{aligned} a_0(\nu - 1) + a_1 \left(\frac{\nu^2}{2} - 2 - \frac{\Gamma_q(2)}{\Gamma_q(2-\beta)} \right) &= -\frac{1}{\Gamma_q(\alpha)} \int_0^1 (1-qs)^{\alpha-1} v(s) d_q s \\ &- \frac{1}{\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} v(s) d_q s \\ &- \frac{1}{\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} v(s) d_q s \\ &+ \frac{1}{\Gamma_q(\alpha)} \int_0^{\nu} \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds. \end{aligned}$$

Thus,

$$\begin{aligned} a_0 = c_{0v} = &- \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^{\eta} \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds \\ &+ \frac{(2-\eta^2)(\nu-1)}{2\gamma\Gamma_q(\alpha)} \int_0^{\eta} \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} v(s) d_q s \\ &+ \frac{(2-\eta^2)(1-\eta)}{2\gamma\Gamma_q(\alpha)} \int_0^{\nu} \int_0^s (s-qm)^{(\alpha-1)} v(m) d_q m ds \\ &+ \frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} v(s) d_q s \end{aligned}$$

$$+\frac{(2-\eta^2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs$$

and

$$\begin{aligned}a_1=c_{1v}=&-\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\\&-\frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\&-\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs,\end{aligned}$$

where

$$\gamma=(\nu-1)\left(\frac{\eta^2}{2}-1\right)+(\eta-1)\left(\frac{\eta^2}{2}-2-\frac{\Gamma_q(2)}{\Gamma_q(2)-\beta}\right). \tag{3.3}$$

Hence,

$$\begin{aligned}x(t)=x_vt=&\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qs)^{(\alpha-1)}v(s)d_qs\\&+\frac{1}{\Gamma_q(\alpha)(1-\eta)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\&+\frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\&+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\\&+\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\&+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\&+\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs\\&+\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)}\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds\\&+\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs\end{aligned}$$

$$+\frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)}\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}v(m)d_qmds$$
$$+\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs$$
$$+\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs=I_q^\alpha v(t)-c_{0v}-c_{1v}t.$$

Conversely, it is clear that

$$\begin{cases} x_v'(t)=I_q^{\alpha-1}v(t)+c_{1v}, \\ x_v''(t)=\left(I_q^{\alpha-1}v(t)\right)'={}^RD_q^{2-\alpha}v(t), \end{cases}$$

for almost all $t\in J$. Because, $2-\alpha$ belongs to $(0,1]$, we get

$${}^cD_q^\alpha x_v(t)=I_q^{2-\alpha}x_v''(t)=I_q^{2-\alpha}\left({}^RD_q^{2-\alpha}v(t)\right)=v(t).$$

Similar to last part, we obtain

$$x_v(0)+x_v'(0)+{}^cD_q^\beta x_v(0)=-c_{0v}-c_{1v}=\int_0^\eta x(s)ds$$

and

$$\begin{aligned} x_v(1)+x_v'(1)+{}^cD_q^\beta x_v(1)&=\frac{1}{\Gamma_q(\alpha)}\int_0^1(1-qs)^{(\alpha-1)}v(s)d_qs \\ &+\left(\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^1(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\right) \\ &\times\left(\frac{1}{\Gamma_q(\alpha-1)}\int_0^1(1-qs)^{(\alpha-2)}v(s)d_qs\right) \\ &-\frac{\Gamma_q(2)a_1}{\Gamma_q(2-\beta)}-2c_{1v}-c_{0v}=\int_0^\nu x(s)ds. \end{aligned}$$

This finishes the proof. □

A solution of the inclusion problem (1.1) is an element $x\in AC^2([0,1],\mathbb{R})$ such that it satisfies the integral boundary conditions and there exists a function $v\in S_{F,x}$ such that $x(t)=I_q^\alpha v(t)-c_{0v}-c_{1v}t$ for all $t\in J$. Suppose that

$$\mathcal{X}=\left\{x\,|\,x,x',{}^cD_q^\beta x\in C(J,\mathbb{R})\text{ for all }\beta\in(0,1)\right\}, \tag{3.4}$$

endowed with the norm

$$\|x\|=\sup_{t\in J}|x(t)|+\sup_{t\in J}|x'(t)|+\sup_{t\in J}\left|{}^cD_q^\beta x(t)\right|. \tag{3.5}$$

Then, $(\mathcal{X},\|.\|)$ is a Banach space [24].

For investigation of the inclusion problem (1.1), we provide two different methods. In the first method which is used in Theorem 3.1, we showed a compact map F is upper semi-continuous and so by using fixed point theorem in Lemma 2.2, and in the second method which is presented in Theorem 3.2, by using fixed point theorem of

Covitz and Nadler, and consider three conditions, respectively, we found a solution for the inclusion problem (1.1).

Theorem 3.1. *Let $F : J \times \mathbb{R}^3 \to P_{cp,cv}(\mathbb{R})$ is a L^1-Caratheodory multifunction and there exist a bounded continuous increasing self map ψ define on $[0,\infty)$ and a continuous function p maps J into $(0,\infty)$ such that*

$$\begin{aligned}\left\|F\left(t, x(t), x'(t), {}^cD_q^\beta x(t)\right)\right\| &= \sup\left\{|v| \mid v \in F\left(t, x(t), x'(t), {}^cD_q^\beta x(t)\right)\right\}\\ &\leq p(t)\psi(\|x\|),\end{aligned}$$

for all $t \in J$ and $x \in \mathfrak{X}$. Then the inclusion problem (1.1) has at least one solution.

Proof. First, define the operator $N : \mathfrak{X} \to 2^{\mathfrak{X}}$ by

$$N(x) = \left\{h \in \mathfrak{X} \mid \text{exists } v \in S_{F,x} : h(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t,\ t \in J\right\}.$$

In the following, prove that the operator N has a fixed point.

Step I. We show that N maps bounded sets of $\mathfrak{X}$ into bounded sets. Let $r > 0$ and $B_r = \{x \in \mathfrak{X} \mid \|x\| \leq r\}$. Suppose that $x \in B_r$ and $h \in N(x)$. We can choose $v \in S_{F,x}$ such that $h(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t$ for almost all $t \in J$. Thus,

$$\begin{aligned}|h(t)| \leq & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} |v(s)| d_q s\\
&+ \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} |v(m)| d_q m ds\\
&+ \left|\frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma_q(\alpha)}\right| \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} |v(m)| d_q m ds\\
&+ \left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\right| \int_0^1 (1 - qs)^{(\alpha-1)} |v(s)| d_q s\\
&+ \left|\frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma_q(\alpha)}\right| \int_0^\nu \int_0^s (s - qm)^{(\alpha-1)} |v(m)| d_q m ds\\
&+ \left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)}\right| \int_0^1 (1 - qs)^{(\alpha-\beta-1)} |v(s)| d_q s\\
&+ \left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)}\right| \int_0^1 (1 - qs)^{(\alpha-2)} \left|v(s)\right| d_q s\\
&+ \left|\frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)}\right| \int_0^\eta \int_0^s (s - qm)^{(\alpha-1)} |v(m)| d_q m ds\\
&+ \left|\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)}\right| \int_0^1 (1 - qs)^{(\alpha-1)} |v(s)| d_q s\\
&+ \left|\frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)}\right| \int_0^\nu \int_0^s (s - qm)^{(\alpha-1)} |v(m)| d_q m ds\end{aligned}$$

$$
\begin{aligned}
&+\left|\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)}\right|\int_0^1(1-qs)^{(\alpha-2)}|v(s)|d_qs\\
&\le\Lambda_1\left\|p\right\|_\infty\psi\left(\|x\|\right),
\end{aligned}
$$

$$
\begin{aligned}
\left|{}^cD_q^\beta h(t)\right|\le&\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t(t-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\nu)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta-1)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-\beta)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-1)\Gamma_q(2-\beta)}\right|\int_0^1(1-qs)^{(\alpha-2)}|v(s)|d_qs\\
\le&\Lambda_2\left\|p\right\|_\infty\psi\left(\|x\|\right)
\end{aligned}
$$

and

$$
\begin{aligned}
|h'(t)|\le&\frac{1}{\Gamma_q(\alpha-1)}\int_0^t(t-qs)^{(\alpha-2)}|v(s)|d_qs\\
&+\left|\frac{(1-\nu)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\eta\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)}\right|\int_0^1(1-qs)^{(\alpha-1)}|v(s)|d_qs\\
&+\left|\frac{(\eta-1)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\nu\int_0^s(s-qm)^{(\alpha-1)}|v(m)|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1(1-qs)^{(\alpha-\beta-1)}|v(s)|d_qs\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-1)}\right|\int_0^1(1-qs)^{(\alpha-2)}|v(s)|d_qs\\
\le&\Lambda_3\left\|p\right\|_\infty\psi\left(\|x\|\right),
\end{aligned}
$$

for all $t\in J$, where $\|p\|_\infty=\sup_{t\in J}|p(t)|$,

$$
\Lambda_1=\left[\frac{1}{\Gamma_q(\alpha+1)}+\frac{\eta^{\alpha+1}}{\Gamma_q(\alpha+2)(1-\eta)}+\left|\frac{(\eta^2-2)(\nu-1)\eta^{\alpha+1}}{2\gamma\Gamma_q(\alpha+2)}\right|\right. \tag{3.6}
$$

$$+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha+1)}\right|+\left|\frac{(\eta^2-2)(1-\eta)\nu^{\alpha+1}}{2\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta+1)}\right|$$
$$+\left|\frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)}\right|+\left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha+1)}\right|$$
$$+\left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)}\right|\Bigg],$$

$$\begin{aligned}\Lambda_2=\Bigg[&\frac{1}{\Gamma_q(\alpha-\beta+1)}+\left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)}\right|\\&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha+1)\Gamma_q(2-\beta)}\right|+\left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)}\right|\\&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)\Gamma_q(2-\beta)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\Bigg],\end{aligned}\tag{3.7}$$

and

$$\begin{aligned}\Lambda_3=\Bigg[&\frac{1}{\Gamma_q(\alpha)}+\left|\frac{(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha+1)}\right|+\left|\frac{(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)}\right|\\&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)}\right|+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha)}\right|\Bigg].\end{aligned}\tag{3.8}$$

Hence,

$$\|h\|=\max_{t\in J}|h(t)|+\max_{t\in J}\left|{}^cD_q^{\beta}h(t)\right|+\max_{t\in J}|h'(t)|$$

is less than equal to $(\Lambda_1+\Lambda_2+\Lambda_3)\|p\|_\infty\psi(\|x\|)$.

Step II. We demonstrate that N maps bounded sets into equicontinuous subsets of $\mathcal{X}$. Let $x\in B_r$ and $t_1,t_2\in J$, with $t_1<t_2$. After that, for all $h\in N(x)$, we have

$$\begin{aligned}|h(t_2)-h(t_1)|=\Bigg|&\frac{1}{\Gamma_q(\alpha)}\int_0^{t_2}(t_2-qs)^{(\alpha-1)}v(s)d_qs\\&-\frac{1}{\Gamma_q(\alpha)}\int_0^{t_1}(t_1-qs)^{(\alpha-1)}v(s)d_qs\\&+\frac{(1-\nu)t_2}{\gamma\Gamma_q(\alpha)}\int_0^{\eta}\int_0^{s}(s-qm)^{(\alpha-1)}v(m)d_qmds\\&-\frac{(1-\nu)t_1}{\gamma\Gamma_q(\alpha)}\int_0^{\eta}\int_0^{s}(s-qm)^{(\alpha-1)}v(m)d_qmds\\&+\frac{(1-\eta)t_2}{\gamma\Gamma_q(\alpha)}\int_0^{1}(1-qs)^{(\alpha-1)}v(s)d_qs\\&-\left(\frac{(1-\eta)t_1}{\gamma\Gamma_q(\alpha)}\int_0^{1}(1-qs)^{(\alpha-1)}v(s)d_qs\right)\end{aligned}$$

$$
\begin{aligned}
&\times\left(\frac{(\eta-1)t_2}{\gamma\Gamma_q(\alpha)}\int_0^{\nu}\int_0^{s}(s-qm)^{(\alpha-1)}v(m)d_qmds\right)\\
&-\frac{(\eta-1)t_1}{\gamma\Gamma_q(\alpha)}\int_0^{\nu}\int_0^{s}(s-qm)^{(\alpha-1)}v(m)d_qmds\\
&+\frac{(1-\eta)t_2}{\gamma\Gamma_q(\alpha-\beta)}\int_0^{1}(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\
&-\frac{(1-\eta)t_1}{\gamma\Gamma_q(\alpha-\beta)}\int_0^{1}(1-qs)^{(\alpha-\beta-1)}v(s)d_qs\\
&+\frac{(1-\eta)t_2}{\gamma\Gamma_q(\alpha-1)}\int_0^{1}(1-qs)^{(\alpha-2)}v(s)d_qs\\
&-\frac{(1-\eta)t_1}{\gamma\Gamma_q(\alpha-1)}\int_0^{1}(1-qs)^{(\alpha-2)}v(s)d_qs\Bigg|\\
\le&\|p\|_\infty\psi\left(\|x\|\right)\left[\left|\frac{t_2^\alpha-t_1^\alpha}{\Gamma_q(\alpha+1)}\right|+\left|\frac{(1-\nu)\eta^{\alpha+1}(t_2-t_1)}{\gamma\Gamma_q(\alpha+2)}\right|\right.\\
&+\left|\frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma_q(\alpha+1)}\right|+\left|\frac{(\eta-1)\nu^{\alpha+1}(t_2-t_1)}{\gamma\Gamma_q(\alpha+2)}\right|\\
&\left.+\left|\frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma_q(\alpha-\beta+1)}\right|+\left|\frac{(1-\eta)(t_2-t_1)}{\gamma\Gamma_q(\alpha)}\right|\right],
\end{aligned}
$$

$|h'(t_2)-h'(t_1)|\le\|p\|_\infty\psi\left(\|x\|\right)\frac{\left(t_2^{\alpha-1}-t_1^{\alpha-1}\right)}{\Gamma_q(\alpha)}$, and

$$
\begin{aligned}
\left|{}^cD_q^\beta h(t_2)-{}^cD_q^\beta h(t_1)\right|\le\|p\|_\infty\psi\left(\|x\|\right)&\left[\left|\frac{t_2^{\alpha-\beta}-t_1^{\alpha-\beta}}{\Gamma_q(\alpha-\beta+1)}\right|\right.\\
&+\left|\frac{\left(t_2^{1-\beta}-t_1^{1-\beta}\right)(1-\nu)\eta^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)}\right|\\
&+\left|\frac{\left(t_2^{1-\beta}-t_1^{1-\beta}\right)(1-\eta)}{\gamma\Gamma_q(\alpha+1)\Gamma_q(2-\beta)}\right|+\left|\frac{\left(t_2^{1-\beta}-t_1^{1-\beta}\right)(\eta-1)\nu^{\alpha+1}}{\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta)}\right|\\
&\left.+\left|\frac{\left(t_2^{1-\beta}-t_1^{1-\beta}\right)(1-\eta)}{\gamma\Gamma_q(\alpha-\beta+1)\Gamma_q(2-\beta)}\right|+\left|\frac{\left(t_2^{1-\beta}-t_1^{1-\beta}\right)(1-\eta)}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\right].
\end{aligned}
$$

Hence,

$$\lim_{t_2\to t_1}|h(t_2)-h(t_1)|=\lim_{t_2\to t_1}|h'(t_2)-h'(t_1)|=\lim_{t_2\to t_1}\left|{}^cD_q^\beta h(t_2)-{}^cD_q^\beta h(t_1)\right|=0,$$

and so by using the Arzela-Ascoli theorem, N is completely continuous.

Step III. Now, we show that N has a closed graph. Let $x_n\to x_0$, $h_n\in N(x_n)$ for all n and $h_n\to h_0$. We prove that $h_0\in N(x_0)$. For each n, choose $v_n\in S_{F,x_n}$ such that $h_n(t)=I_q^\alpha v_n(t)-c_{0v_n}-c_{1v_n}t$ for all $t\in J$. Consider the continuous linear operator

$$\begin{cases} \theta : L^1(J,\mathbb{R}) \to \mathfrak{X}, \\ \theta(v)(t) = I_q^\alpha v(t) - c_{0v} - c_{1v}t. \end{cases}$$

It can be seen, by Lemma 2.1, $\theta o S_F$ is a closed graph operator. Since $x_n \to x_0$ and $h_n \in \theta(S_{F,x_n})$ for all n, there exists $v_0 \in S_{F,x_0}$ such that $h_0(t) = I_q^\alpha v_0(t) - c_{0v} - c_{1v_0}t$. Thus, N has a closed graph.

Step IV. In this level, we show that $N(x)$ is convex for all $x \in \mathfrak{X}$. Let $h_1, h_2 \in N(x)$ and $0 \le w \le 1$. Choose $v_1, v_2 \in S_{F,x}$ such that $h_i(t) = I_q^\alpha v_i(t) - c_{0v_i} - c_{1v_i}t$, for almost all $t \in J$ and $i = 1, 2$. Then,

$$\begin{aligned}
&[wh_1 + (1-w)h_2](t) \\
&= \frac{1}{\Gamma_q(\alpha)} \int_0^t (t-qs)^{(\alpha-1)} [wv_1(s) + (1-w)v_2(s)] d_q s \\
&+ \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} [wv_1(m) + (1-w)v_2(m)] d_q m ds \\
&+ \frac{(\eta^2-2)(\nu-1)}{2\gamma\Gamma_q(\alpha)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} [wv_1(m) + (1-w)v_2(m)] d_q m ds \\
&+ \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} [wv_1(s) + (1-w)v_2(s)] d_q s \\
&+ \frac{(\eta^2-2)(1-\eta)}{2\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} [wv_1(m) + (1-w)v_2(m)] d_q m ds \\
&+ \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} [wv_1(s) + (1-w)v_2(s)] d_q s \\
&+ \frac{(\eta^2-2)(\eta-1)}{2\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} [wv_1(s) + (1-w)v_2(s)] d_q s \\
&+ \frac{(1-\nu)t}{\gamma\Gamma_q(\alpha)} \int_0^\eta \int_0^s (s-qm)^{(\alpha-1)} [wv_1(m) + (1-w)v_2(m)] d_q m ds \\
&+ \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha)} \int_0^1 (1-qs)^{(\alpha-1)} [wv_1(s) + (1-w)v_2(s)] d_q s \\
&+ \frac{(\eta-1)t}{\gamma\Gamma_q(\alpha)} \int_0^\nu \int_0^s (s-qm)^{(\alpha-1)} [wv_1(m) + (1-w)v_2(m)] d_q m ds \\
&+ \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-\beta)} \int_0^1 (1-qs)^{(\alpha-\beta-1)} [wv_1(s) + (1-w)v_2(s)] d_q s \\
&+ \frac{(1-\eta)t}{\gamma\Gamma_q(\alpha-1)} \int_0^1 (1-qs)^{(\alpha-2)} [wv_1(s) + (1-w)v_2(s)] d_q s,
\end{aligned}$$

for $t \in J$. Since F has convex values, $S_{F,x}$ is convex and so $wh_1 + (1-w)h_2$ belongs to $N(x)$. If there exists $\lambda \in (0,1)$ such that $x \in \lambda N(x)$, then there exists $v \in S_{F,x}$

such that $x(t) = I_q^{\alpha} v(t) - c_{0v} - c_{1v}t$, for all $t \in J$. Choose $L > 0$ such that

$$\frac{L}{(\Lambda_1 + \Lambda_2 + \Lambda_3)\|p\|_\infty \psi(\|x\|)} > 1,$$

for all $x \in \mathcal{X}$. Thus, $\|x\| < L$. Now, put $U = \{x \in \mathcal{X} \mid \|x\| < L + 1\}$. Note that, there are no $x \in \partial U$ and $0 < \lambda < 1$ such that $x \in \lambda N(x)$ and the operator $N : \overline{U} \to P_{cp,cv}(\overline{U})$ is upper semi-continuous, because it is completely continuous. Therefore, by using Lemma 2.2, N has a fixed point in $\overline{U}$ which is a solution of the inclusion problem (1.1). This completes the proof. $\square$

Here, by changing values of multifunction in the assumption Theorem 3.1, we provide another result about the existence of solutions for the problem (1.1).

Theorem 3.2. *Let $m \in C(J, \mathbb{R}^+)$ be such that $\|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) < 1$ and consider an integrable bounded multifunction $F : J \times \mathbb{R}^3 \to P_{cp}(\mathbb{R})$ such that the map $t \mapsto F(t, x, y, z)$ is measurable and*

$$H_d\left(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)\right) \leq m(t)\left(|x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|\right), \tag{3.9}$$

for $t \in J$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Then the problem (1.1) *has a solution.*

Proof. Note that, the multivalued map $t \mapsto F\left(t, x(t), x'(t), {}^cD_q^B x(t)\right)$, for $x \in \mathcal{X}$, is measurable and closed valued. Hence, it has a measurable selection and so the set $S_{F,x}$ is nonempty. Now, consider the operator $N : \mathcal{X} \to 2^{\mathcal{X}}$ defined by

$$N(x) = \left\{h \in \mathcal{X} \mid \text{exists } v \in S_{F,x} : h(t) = I_q^{\alpha} v(t) - c_{0v} - c_{1v}t\right\},$$

for all $t \in J$.

Step I. We show that $N(x)$ is a closed subset of $\mathcal{X}$ for all $x \in \mathcal{X}$. Let $x \in \mathcal{X}$ and $\{u_n\}_{n\geq 1}$ be a sequence in $N(x)$ with $u_n \to u$. For each n, choose $v_n \in S_{F,x}$ such that $u_n(t) = I_q^{\alpha} v_n(t) - c_{0v_n} - c_{1v_n}t$ for $t \in J$. From being compacted values F, $\{v_n\}_{n\geq 1}$ has a subsequence which converges to some $v \in L^1(J, \mathbb{R})$. Again the subsequence denote by $\{v_n\}_{n\geq 1}$. It is easy to check that $v \in S_{F,x}$ and $u_n(t) \to u(t) = I_q^{\alpha} v(t) - c_{0v} - c_{1v}t$ for all $t \in J$. This implies that $u \in N(x)$. Thus, the multifunction N has closed values.

Step II. In this level, we show that N is a contractive multifunction with constant $l := \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) < 1$. Let $x, y \in \mathcal{X}$ and $h_1 \in N(y)$. Choose $v_1 \in S_{F,y}$ such that $h_1(t) = I^{\alpha} v_1(t) - c_{0v_1} - c_{1v_1}t$ for almost all $t \in J$. Put

$$\begin{aligned} A_x &= F\left(t, x(t), x'(t), {}^cD_q^{\beta} x(t)\right), \\ A_y &= F\left(t, y(t), y'(t), {}^cD_q^{\beta} y(t)\right). \end{aligned}$$

By assumption, if

$$H_d\left(A_x, A_y\right) \leq m(t)\left(|x(t) - y(t)| + |x'(t) - y'(t)| + \left|{}^cD_q^{\beta} x(t) - {}^cD_q^{\beta} y(t)\right|\right),$$

for all $t \in J$, then there exists $w \in F\left(t, x(t), x'(t), {}^cD_q^{\beta} x(t)\right)$ such that

$$|v_1(t) - w| \leq m(t)\left(|x(t) - y(t)| + |x'(t) - y'(t)| + \left|{}^cD_q^{\beta} x(t) - {}^cD_q^{\beta} y(t)\right|\right), \tag{3.10}$$

for almost all $t \in J$. For the multifunction $U : J \to 2^{\mathbb{R}}$, define $U(t)$ by the set of all $w \in \mathbb{R}$ where satisfies in (3.10) for $t \in J$. It is easy to check that the multifunction

$$U(\cdot) \cap F\left(\cdot, x(\cdot), x'(\cdot), {}^{c}D_q^{\beta} x(\cdot)\right),$$

is measurable. Therefore, we can choose $v_2 \in S_{F,x}$ such that

$$|v_1(t) - v_2(t)| \leq m(t)\left(|x(t) - y(t)| + |x'(t) - y'(t)| + \left|{}^{c}D_q^{\beta} x(t) - {}^{c}D_q^{\beta} y(t)\right|\right),$$

for almost all $t \in J$. Now, define $h_2 \in N(x)$ by $h_2(t) = I_q^{\alpha} v(t) - c_{0v_2} - c_{1v_2} t$. Hence, we get

$$\begin{aligned}
|h_1(t) - h_2(t)| \leq & \frac{1}{\Gamma_q(\alpha)} \int_0^t (t - qs)^{(\alpha-1)} |v_1(s) - v_2(s)| \, d_q s \\
& + \frac{1}{\Gamma_q(\alpha)(1-\eta)} \int_0^{\eta} \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)| \, d_q m ds \\
& + \left| \frac{(\eta^2 - 2)(\nu - 1)}{2\gamma \Gamma_q(\alpha)} \right| \int_0^{\eta} \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)| \, d_q m ds \\
& + \left| \frac{(\eta^2 - 2)(\eta - 1)}{2\gamma \Gamma_q(\alpha)} \right| \int_0^1 (1 - qs)^{(alpha-1)} |v_1(s) - v_2(s)| \, d_q s \\
& + \left| \frac{(\eta^2 - 2)(1 - \eta)}{2\gamma \Gamma_q(\alpha)} \right| \int_0^{\nu} \int_0^s (s - qm)^{\alpha-1} |v_1(m) - v_2(m)| \, d_q m ds \\
& + \left| \frac{(\eta^2 - 2)(\eta - 1)}{2\gamma \Gamma_q(\alpha - \beta)} \right| \int_0^1 (1 - qs)^{(\alpha-\beta-1)} |v(s)| \, d_q s \\
& + \left| \frac{(\eta^2 - 2)(\eta - 1)}{2\gamma \Gamma_q(\alpha - 1)} \right| \int_0^1 (1 - qs)^{(\alpha-2)} |v_1(s) - v_2(s)| \, d_q s \\
& + \left| \frac{(1 - \nu)t}{\gamma \Gamma_q(\alpha)} \right| \int_0^{\eta} \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)| \, d_q m ds \\
& + \left| \frac{(1 - \eta)t}{\gamma \Gamma_q(\alpha)} \right| \int_0^1 (1 - qs)^{(\alpha-1)} |v_1(s) - v_2(s)| \, d_q s \\
& + \left| \frac{(\eta - 1)t}{\gamma \Gamma_q(\alpha)} \right| \int_0^{\nu} \int_0^s (s - qm)^{(\alpha-1)} |v_1(m) - v_2(m)| \, d_q m ds \\
& + \left| \frac{(1 - \eta)t}{\gamma \Gamma_q(\alpha - \beta)} \right| \int_0^1 (1 - qs)^{(\alpha-\beta-1)} |v_1(s) - v_2(s)| \, d_q s \\
& + \left| \frac{(1 - \eta)t}{\gamma \Gamma_q(\alpha - 1)} \right| \int_0^1 (1 - qs)^{(\alpha-2)} |v_1(s) - v_2(s)| \, d_q s \\
\leq & \Lambda_1 \|m\|_{\infty} \|x - y\|,
\end{aligned}$$

$$|h_1'(t) - h_2'(t)| \leq \frac{1}{\Gamma_q(\alpha - 1)} \int_0^t (t - qs)^{(\alpha-2)} |v_1(s) - v_2(s)| \, d_q s$$

$$
\begin{aligned}
&+\left|\frac{(1-\nu)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\eta\int_0^s (s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma+q(\alpha)}\right|\int_0^1 (1-qs)^{(\alpha-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(\eta-1)}{\gamma\Gamma_q(\alpha)}\right|\int_0^\nu\int_0^s (s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-\beta)}\right|\int_0^1 (1-qs)^{(\alpha-\beta-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(1-\eta)}{\gamma\Gamma_q(\alpha-1)}\right|\int_0^1 (1-qs)^{\alpha-2}\left|v_1(s)-v_2(s)\right|d_qs\\
\leq\ &\Lambda_3\|m\|_\infty\|x-y\|
\end{aligned}
$$

and

$$
\begin{aligned}
&\left|{}^cD^\beta h_1(t)-{}^cD^\beta h_2(t)\right|\\
\leq&\frac{1}{\Gamma_q(\alpha-\beta)}\int_0^t (t-qs)^{(\alpha-\beta-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(1-\nu)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\eta\int_0^s (s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^1 (1-qs)^{(\alpha-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(\eta-1)t^{1-\beta}}{\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta)}\right|\int_0^\nu\int_0^s (s-qm)^{(\alpha-1)}\left|v_1(m)-v_2(m)\right|d_qmds\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-\beta)\Gamma_q(2-\beta)}\right|\int_0^1 (1-qs)^{(\alpha-\beta-1)}\left|v_1(s)-v_2(s)\right|d_qs\\
&+\left|\frac{(1-\eta)t^{1-\beta}}{\gamma\Gamma_q(\alpha-1)\Gamma_q(2-\beta)}\right|\int_0^1 (1-qs)^{(\alpha-2)}\left|v_1(s)-v_2(s)\right|d_qs\\
\leq&\Lambda_2\|m\|_\infty\|x-y\|.
\end{aligned}
$$

So,

$$\|h_1-h_2\|\leq(\Lambda_1+\Lambda_2+\Lambda_3)\,\|m\|_\infty\|x-y\|=l\|x-y\|.$$

This implies that the multifunction N is a contraction with closed values. Thus by using the result of Covitz and Nadler, N has a fixed point which is a solution for the inclusion problem (1.1). $\square$

Here, we provide two examples for the results.

Example 3.1. Put $q=\frac{1}{3}$, $\alpha=\frac{5}{2}$, $\beta=\frac{1}{2}$, $\eta=\frac{1}{2}$, $\nu=\frac{1}{3}$, consider the fractional q-derivative inclusion

$$ {}^cD^{\frac{5}{2}}_{\frac{1}{3}}x(t)\in F\left(t,x(t),x'(t),{}^cD^{\frac{1}{2}}_{\frac{1}{3}}x(t)\right), \tag{3.11}$$

with the boundary value conditions

$$\begin{cases} x(0) + x'(0) + {}^cD^{\frac{1}{2}}_{\frac{1}{3}} x(0) = \int_0^{\frac{1}{2}} x(s) ds, \\ x(1) + x'(1) + {}^cD^{\frac{1}{2}}_{\frac{1}{3}} x(1) = \int_0^{\frac{1}{3}} x(s) ds, \end{cases} \tag{3.12}$$

and consider the multifunction $F : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ defined by

$$F(t, x_1, x_2, x_3) = \left[\cos t + \frac{e^{-\sin^2 x_1}}{1 + e^{\cos^2 x_1}} + \sin x_2, 4 + t^2 + \frac{t+1}{2 + e^{|x_3|}} \right].$$

Note that, $\|F(t, x_1, x_2, x_3)\| = \sup\{|y| \mid y \in F(t, x_1, x_2, x_3)\} \le 6$. If $p(t) = 1$ and $\psi(t) = 6$, then one can check that the assumptions of Theorem 3.1 hold and so the inclusion problem (3.11) has at least one solution.

Next example illustrates last result.

Example 3.2. Put $q = \frac{1}{3}$, $\frac{1}{2}$ and $\frac{2}{3}$, $\alpha = \frac{7}{3}$, $\beta = \frac{1}{3}$, $\eta = \frac{1}{2}$, $\nu = \frac{1}{3}$, consider the inclusion problem

$${}^cD^{\frac{7}{3}}_{\frac{1}{2}} x(t) \in F\left(t, x(t), x'(t), {}^cD^{\frac{1}{3}}_{\frac{1}{2}} x(t)\right), \tag{3.13}$$

with the boundary value conditions

$$\begin{cases} x(0) + x'(0) + {}^cD^{\frac{1}{3}}_{\frac{1}{2}} x(0) = \int_0^{\frac{1}{2}} x(s) ds, \\ x(1) + x'(1) + {}^cD^{\frac{1}{3}}_{\frac{1}{2}} x(1) = \int_0^{\frac{1}{3}} x(s) ds, \end{cases} \tag{3.14}$$

and consider the multifunction $F : J \times \mathbb{R}^3 \to 2^{\mathbb{R}}$ defined by

$$F(t, x_1, x_2, x_3) = \left[0, \frac{t \sin^2 x_1}{12(4 + 3t^2)} + \frac{(t+1)|x_2|}{100(2 + |x_2|)} + \frac{|x_3|}{100(1 + |x_3|)}\right].$$

It is easy to understand that

$$H_d\left(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)\right) \le \left(\frac{t}{12(4 + 3t^2)} + \frac{t+1}{100} + \frac{1}{100}\right) \sum_{i=1}^{3} |x_i - y_i|,$$

for all $t \in J = [0, 1]$ and $x_1, x_2, x_3, y_1, y_2, y_3 \in \mathbb{R}$. Thus, if

$$m(t) = \frac{t}{12(4 + 3t^2)} + \frac{t+1}{100} + \frac{1}{100},$$

for all $t \in J$, then

$$H_d\left(F(t, x_1, x_2, x_3), F(t, y_1, y_2, y_3)\right) \le m(t) \sum_{i=1}^{3} |x_i - y_i|.$$

On the other side, we have three cases for q:

$q := \frac{1}{3}$:

$$L = \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 0.0508(3.0182 + 2.0213 + 2.1289) \simeq 0.3643 < 1,$$

$q := \frac{1}{2}$:

$$L = \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 0.0508(2.6576 + 1.8297 + 1.9831) \simeq 0.3289 < 1,$$

$q := \frac{2}{3}$:

$$L = \|m\|_\infty(\Lambda_1 + \Lambda_2 + \Lambda_3) \leq 0.0508(2.3812 + 1.6771 + 1.1.8681) \simeq 0.3012 < 1.$$

These values calculate by Algorithm 4, 5 and 6 which present in Table 5, 6 and 7. Consequently, the assumptions of Theorem 3.2 hold and then the inclusion problem (3.13) have at least one solution.

4. Computational Results

A simplified analysis can be performed to estimate the value of q-Gamma function, $\Gamma_q(x)$, for input values q and x by counting the number of sentences n in summation. To this aim, we consider a pseudo-code description of the method for calulated q-Gamma functiuon of order n in Algorithm 2.

Algorithm 1 The proposed method for calculated $(a-b)^{(\alpha)}$

Input: a, b, α, n, q

$s \leftarrow 1$
if $n = 0$ **then**
  $p \leftarrow 1$
else
  for $k = 0$ to n **do**
    $s \leftarrow s * \frac{a-b*a^k}{a-b*q^{\alpha+k}}$
  end for
  $p \leftarrow a^\alpha * s$
end if

Output: $(a-b)^{(\alpha)}$

Algorithm 2 The proposed method for calculated $\Gamma_q(x)$

Input: n, $q \in (0,1)$, $x \in \mathbb{R}\backslash\{0, -1, 2, \cdots\}$

$p \leftarrow 1$
for $k = 0$ to n **do**
  $p \leftarrow p(1-q^{k+1})(1-q^{x+k})$
end for
$\Gamma_q(x) \leftarrow p/(1-q)^{x-1}$

Output: $\Gamma_q(x)$

Algorithm 3 The proposed method for calculated $(I_q^\alpha f)(x)$

Input: $q \in (0,1)$, α, n, $f(x)$, x

$s \leftarrow 0$
for $i = 0$ to n **do**
  $pf \leftarrow (1-q^{i+1})^{\alpha-1}$
  $s \leftarrow s + pf * q^i * f(x * q^i)$
end for
$g \leftarrow \frac{x^\alpha * (1-q) * s}{\Gamma_q(x)}$

Output: $(I_q^\alpha f)(x)$

Table 1 shows that when q is constant, the q-Gamma function is an increasing function. Also, for smaller values of x, an approximate result is obtained with less values of n. It has been shown by underlined rows. Table 2 shows that the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns. They have been underlined in line 8 of the first column, line 17 of the second column and line 29 of third column of Table 2. Also, Table 3 is the same as Table 2, but x values increase in 3. Similarly, the q-Gamma function for values q near to one is obtained with more values of n in comparison with other columns.

Now, we investigate the computational complexity of Example 3.2 of Algorithm 4, 5 and 6. First, Table 4 shows the values of γ for $q \in (0,1)$, an approximate result is obtained with less than four decimal places indicated by underline. Furthermore, Tables 5, 6, 7 show valued calculations of Λ_1, Λ_2 and Λ_3 for $q = \frac{1}{3}$, $q = \frac{1}{2}$ and $q = \frac{2}{3}$, respectively.

Algorithm 4 The proposed method for calculated Λ_1

Input: n, $q \in (0,1)$, α, η, ν

for $k = 0$ to n **do**
  $\gamma \leftarrow (\nu-1)(\eta^2/2-1) + (\eta-1)(\eta^2/2-2-\Gamma_q(2)/(\Gamma_q(2)-\beta))$
  $\Lambda_{1_1} \leftarrow 1/\Gamma_q(\alpha+1) + \eta^{\alpha+1}/(\Gamma_q(\alpha+2)(1-\eta))$
  $\Lambda_{1_2} \leftarrow |((\eta^2-2)(\nu-1)\eta^{\alpha+1})/(2\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_3} \leftarrow |((\eta^2-2)(\eta-1))/(2\gamma\Gamma_q(\alpha+1))|$
  $\Lambda_{1_4} \leftarrow |((\eta^2-2)(1-\eta)\nu^{\alpha+1})/(2\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_5} \leftarrow |((\eta^2-2)(\eta-1))/(2\gamma\Gamma_q(\alpha-\beta+1))|$
  $\Lambda_{1_6} \leftarrow |((\eta^2-2)(\eta-1))/(2\gamma\Gamma_q(\alpha))| + |((1-\nu)\eta^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_7} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha+1))| + |((\eta-1)\nu^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{1_8} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha-\beta+1))| + |(1-\eta)/(\gamma\Gamma_q(\alpha))|$
  $\Lambda_1 = \Lambda_{1_1} + \Lambda_{1_2} + \Lambda_{1_3} + \Lambda_{1_4} + \Lambda_{1_5} + \Lambda_{1_6} + \Lambda_{1_7} + \Lambda_{1_8}$
end for

Output: Λ_1

Algorithm 5 The proposed method for calculated Λ_2

Input: n, $q \in (0,1)$, α, η, ν
for $k = 0$ to n **do**
  $\gamma \leftarrow (\nu - 1)(\eta^2/2 - 1) + (\eta - 1)(\eta^2/2 - 2 - \Gamma_q(2)/(\Gamma_q(2) - \beta))$
  $\Lambda_{2_1} \leftarrow 1/\Gamma_q(\alpha - \beta + 1)$
  $\Lambda_{2_2} \leftarrow |((1-\nu)\eta^{\alpha+1})/(\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta))|$
  $\Lambda_{2_3} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha+1)\Gamma_q(2-\beta))|$
  $\Lambda_{2_4} \leftarrow |((\eta-1)\nu^{\alpha+1})/(\gamma\Gamma_q(\alpha+2)\Gamma_q(2-\beta))|$
  $\Lambda_{2_5} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha-\beta+1)\Gamma_q(2-\beta))|$
  $\Lambda_{2_6} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha)\Gamma_q(2-\beta))|$
  $\Lambda_2 = \Lambda_{2_1} + \Lambda_{2_2} + \Lambda_{2_3} + \Lambda_{2_4} + \Lambda_{2_5} + \Lambda_{2_6}$
end for
Output: Λ_2

Algorithm 6 The proposed method for calculated Λ_3

Input: n, $q \in (0,1)$, α, η, ν
for $k = 0$ to n **do**
  $\gamma \leftarrow (\nu - 1)(\eta^2/2 - 1) + (\eta - 1)(\eta^2/2 - 2 - \Gamma_q(2)/(\Gamma_q(2) - \beta))$
  $\Lambda_{3_1} \leftarrow 1/\Gamma_q(\alpha) + |((1-\nu)\eta^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{3_2} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha+1))|$
  $\Lambda_{3_3} \leftarrow |((\eta-1)\nu^{\alpha+1})/(\gamma\Gamma_q(\alpha+2))|$
  $\Lambda_{3_4} \leftarrow |(1-\eta)/(\gamma\Gamma_q(\alpha-\beta+1))| + |(1-\eta)/(\gamma\Gamma_q(\alpha))|$
  $\Lambda_3 = \Lambda_{3_1} + \Lambda_{3_2} + \Lambda_{3_3} + \Lambda_{3_4}$
end for
Output: Λ_3

All routines are written in "Matalab" software with the "Digits" 16 (Digits environment variable controls the number of digits in Matlab) and run on a PC with 2.90 GHz of Core 2 CPU and 4 GB of RAM.

TABLE 1. Some numerical results for calculation of $\Gamma_q(x)$, with $q = \frac{1}{3}$ that is constant, $x = 4.5, 8.4, 12.7$ and $n = 1, 2, \ldots, 15$, of Algorithm 2.

n	$x = 4.5$	$x = 8.4$	$x = 12.7$	n	$x = 4.5$	$x = 8.4$	$x = 12.7$
1	2.472950	11.909360	68.080769	9	$\underline{2.340263}$	11.257158	64.351366
2	2.383247	11.468397	65.559266	10	2.340250	$\underline{11.257095}$	64.351003
3	2.354446	11.326853	64.749894	11	2.340245	11.257074	$\underline{64.350881}$
4	2.344963	11.280255	64.483434	12	2.340244	11.257066	64.350841
5	2.341815	11.264786	64.394980	13	2.340243	11.257064	64.350828
6	2.340767	11.259636	64.365536	14	2.340243	11.257063	64.350823
7	2.340418	11.257921	64.355725	15	2.340243	11.257063	64.350822
8	2.340301	11.257349	64.352456				

TABLE 2. Some numerical results for calculation of $\Gamma_q(x)$, with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$, $x = 5$ and $n = 1, 2, \ldots, 35$, of Algorithm 2.

n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$
1	3.016535	6.291859	18.937427	18	2.853224	4.921884	8.476643
2	2.906140	5.548726	14.154784	19	2.853224	4.921879	8.474597
3	2.870699	5.222330	11.819974	20	2.853224	4.921877	8.473234
4	2.859031	5.069033	10.537540	21	2.853224	4.921876	8.472325
5	2.855157	4.994707	9.782069	22	2.853224	4.921876	8.471719
6	2.853868	4.958107	9.317265	23	2.853224	4.921875	8.471315
7	2.853438	4.939945	9.023265	24	2.853224	4.921875	8.471046
8	$\underline{2.853295}$	4.930899	8.833940	25	2.853224	4.921875	8.470866
9	2.853247	4.926384	8.710584	26	2.853224	4.921875	8.470747
10	2.853232	4.924129	8.629588	27	2.853224	4.921875	8.470667
11	2.853226	4.923002	8.576133	28	2.853224	4.921875	8.470614
12	2.853224	4.922438	8.540736	29	2.853224	4.921875	$\underline{8.470578}$
13	2.853224	4.922157	8.517243	30	2.853224	4.921875	8.470555
14	2.853224	4.922016	8.501627	31	2.853224	4.921875	8.470539
15	2.853224	4.921945	8.491237	32	2.853224	4.921875	8.470529
16	2.853224	4.921910	8.484320	33	2.853224	4.921875	8.470522
17	2.853224	$\underline{4.921893}$	8.479713	34	2.853224	4.921875	8.470517

TABLE 3. Some numerical results for calculation of $\Gamma_q(x)$, with $x = 8.4$, $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 40$, of Algorithm 2.

n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$
1	11.909360	63.618604	664.767669	21	11.257063	49.065390	260.033372
2	11.468397	55.707508	474.800503	22	11.257063	49.065384	260.011354
3	11.326853	52.245122	384.795341	23	11.257063	49.065381	259.996678
4	11.280255	50.621828	336.326796	24	11.257063	49.065380	259.986893
5	11.264786	49.835472	308.146441	25	11.257063	49.065379	259.980371
6	11.259636	49.448420	290.958806	26	11.257063	49.065379	259.976023
7	11.257921	49.256401	280.150029	27	11.257063	49.065379	259.973124
8	11.257349	49.160766	273.216364	28	11.257063	49.065378	259.971192
9	11.257158	49.113041	268.710272	29	11.257063	49.065378	259.969903
10	$\underline{11.257095}$	49.089202	265.756606	30	11.257063	49.065378	259.969044
11	11.257074	49.077288	263.809514	31	11.257063	49.065378	259.968472
12	11.257066	49.071333	262.521127	32	11.257063	49.065378	259.968090
13	11.257064	49.068355	261.666471	33	11.257063	49.065378	259.967836
14	11.257063	49.066867	261.098587	34	11.257063	49.065378	259.967666
15	11.257063	49.066123	260.720833	35	11.257063	49.065378	259.967553
16	11.257063	$\underline{49.065751}$	260.469369	36	11.257063	49.065378	259.967478
17	11.257063	49.065564	260.301890	37	11.257063	49.065378	259.967427
18	11.257063	49.065471	260.190310	38	11.257063	49.065378	$\underline{259.967394}$
19	11.257063	49.065425	260.115957	39	11.257063	49.065378	259.967371
20	11.257063	49.065402	260.066402	40	11.257063	49.065378	259.967357

TABLE 4. Some numerical results for calculation of γ, with $q = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}$ and $n = 1, 2, \ldots, 20$, of Example 3.2.

n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$	n	$q=\frac{1}{3}$	$q=\frac{1}{2}$	$q=\frac{2}{3}$
1	2.257197	2.226716	2.174059	11	2.270833	2.270788	2.268911
2	2.266232	2.248106	2.203418	12	2.270833	$\underline{2.270810}$	2.269551
3	2.269293	2.259295	2.224501	13	2.270833	2.270822	2.269978
4	2.270319	2.265019	2.239296	14	2.270833	2.270828	2.270263
5	2.270662	2.267915	2.249509	15	2.270833	2.270830	2.270453
6	2.270776	2.269371	2.256481	16	2.270833	2.270832	2.270580
7	$\underline{2.270814}$	2.270102	2.261204	17	2.270833	2.270833	2.270664
8	2.270827	2.270467	2.264386	18	2.270833	2.270833	$\underline{2.270721}$
9	2.270831	2.270650	2.266523	19	2.270833	2.270833	2.270758
10	2.270833	2.270742	2.267954	20	2.270833	2.270833	2.270783

TABLE 5. Some numerical results for calculattion of $\Lambda_1, \Lambda_2, \Lambda_3$, with $q = \frac{1}{3}$ and $n = 1, 2, \ldots, 20$, of Example 3.2.

n	Λ_1	Λ_2	Λ_3	$\sum_{i=1}^{3} \Lambda_i$
1	2.793328	1.846027	1.990304	6.629659
2	2.942153	1.961611	2.082118	6.985882
3	2.992794	2.001290	2.113262	7.107345
4	3.009790	2.014645	2.123703	7.148138
5	3.015468	2.019112	2.127190	7.161770
6	3.017362	2.020602	2.128353	7.166318
7	3.017993	2.021099	2.128741	7.167834
8	3.018204	2.021265	2.128870	7.168339
9	3.018274	2.021320	2.128913	$\underline{7.168508}$
10	3.018298	2.021339	2.128928	7.168564
11	3.018305	2.021345	2.128933	7.168583
12	3.018308	2.021347	2.128934	7.168589

TABLE 6. Some numerical results for calculation of $\Lambda_1, \Lambda_2, \Lambda_3$, with $q = \frac{1}{2}$ and $n = 1, 2, \ldots, 20$, of Example 3.2.

n	Λ_1	Λ_2	Λ_3	$\sum_{i=1}^{3} \Lambda_i$
1	1.980443	1.311532	1.552811	4.844787
2	2.303542	1.554800	1.759966	5.618308
3	2.476635	1.688162	1.869507	6.034304
4	2.566137	1.757911	1.925802	6.249851
5	2.611636	1.793570	1.954335	6.359541
6	2.634573	1.811598	1.968699	6.414870
7	2.646088	1.820662	1.975905	6.442655
8	2.651858	1.825206	1.979514	6.456578
9	2.654746	1.827482	1.981320	6.463547
10	2.656191	1.828620	1.982223	6.467034
11	2.656913	1.829190	1.982675	6.468778
12	2.657274	1.829474	1.982901	6.469650
13	2.657455	1.829617	1.983014	6.470086
14	2.657545	1.829688	1.983070	6.470304
15	2.657591	1.829724	1.983098	6.470413
16	2.657613	1.829741	1.983113	6.470467
17	2.657624	1.829750	1.983120	6.470494
18	2.657630	1.829755	1.983123	$\underline{6.470508}$
19	2.657633	1.829757	1.983125	6.470515
20	2.657634	1.829758	1.983126	6.470518

TABLE 7. Some numerical results for calculation of $\Lambda_1, \Lambda_2, \Lambda_3$, with $q = \frac{2}{3}$ and $n = 1, 2, \ldots, 30$, of Example 3.2.

n	Λ_1	Λ_2	Λ_3	$\sum_{i=1}^{3} \Lambda_i$
1	1.051016	0.687483	0.979592	2.718091
2	1.419580	0.948096	1.237258	3.604934
3	1.705375	1.157875	1.429740	4.292990
4	1.914775	1.315447	1.567753	4.797976
5	2.063077	1.428895	1.664216	5.156188
6	2.165905	1.508420	1.730549	5.404873
7	2.236244	1.563214	1.775683	5.575140
8	2.283940	1.600547	1.806181	5.690669
9	2.316097	1.625798	1.826697	5.768592
10	2.337695	1.642794	1.840456	5.820945
11	2.352165	1.654198	1.849665	5.856027
12	2.361843	1.661832	1.855820	5.879496
13	2.368310	1.666936	1.859931	5.895177
14	2.372627	1.670345	1.862675	5.905648
15	2.375508	1.672621	1.864506	5.912635
16	2.377430	1.674139	1.865727	5.917296
17	2.378712	1.675152	1.866541	5.920405
18	2.379567	1.675827	1.867084	5.922478
19	2.380137	1.676277	1.867446	5.923861
20	2.380517	1.676578	1.867688	5.924783
21	2.380770	1.676778	1.867849	5.925397
22	2.380939	1.676911	1.867956	5.925807
23	2.381052	1.677000	1.868028	5.926080
24	2.381127	1.677060	1.868075	5.926262
25	2.381177	1.677099	1.868107	5.926384
26	2.381211	1.677126	1.868128	5.926464
27	2.381233	1.677143	1.868142	<u>5.926518</u>
28	2.381248	1.677155	1.868152	5.926554
29	2.381258	1.677163	1.868158	5.926578
30	2.381264	1.677168	1.868162	5.926594

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FRACTIONAL ORDER OPERATIONAL MATRIX METHOD FOR SOLVING TWO-DIMENSIONAL NONLINEAR FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. This article presents a numerical method for solving nonlinear two-dimensional fractional Volterra integral equation. We derive the Hat basis functions operational matrix of the fractional order integration and use it to solve the two-dimensional fractional Volterra integro-differential equations. The method is described and illustrated with numerical examples. Also, we give the error analysis.

1. Introduction

Fractional differential and integral equations involving the Caputo fractional operator or the Riemann-Liouville fractional operator has been paid more and more attention. There are several numerical methods for solving fractional integro-differential equations. Such as Haar wavelet method [24], CAS wavelets [25], Bernstein polynomials [1], collocation method [23], fractional differential transform method [3], Block pulse operational matrix [20, 28].

Integro-differential equation of fractional order has been proved to be valuable tools to model the dynamics of many processes in various fields of science and engineering through strongly anomalous media. Indeed, we can find numerous applications in electro-chemistry, viscoelasticity, signal processing, economies, electromagnetic, etc. [9, 10, 18, 22].

Hat functions (HFs) are a powerful mathematical tool for solving various kinds of equations. The solution of stochastic Ito-Volterra integral equations based on stochastic operational matrix [11], E. Babolian et al. have applied this method for

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solving systems of nonlinear integral equations [5], M. H. Heydari et al. have applied Hat functions for solving nonlinear stochastic Ito integral equations [11,13]. F. Mirzaee and E. Hadadiyan have used two-dimensional Hat functions for solving space-time integral equations [17]. M. P. Tripathi et al. have applied HFs for solving fractional differential equations [27].

The operational matrix of integration has been determined for several types of orthogonal polynomials, such as Legendre polynomials [21], Laguerre series [12], and Block-pulse functions [4, 7], Triangular functions [15]. The operational matrix of fractional derivatives has been determined for some types of orthogonal polynomials, such as Legendre polynomials [26], Chebyshev polynomials [6], Triangular functions [8, 14].

In this paper, two dimensional Hat functions (2DHFs) will be used to solve the following nonlinear two-dimensional fractional integral equation

$$D_x^{\alpha}u(x,y) = f(x,y) + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{(\beta-1)}G(x,y,s,t,u(s,t))dsdt, \tag{1.1}$$

with the initial conditions

$$\frac{\partial^i}{\partial x^i}u(0,y) = \delta_i, \quad i=0,1,\dots,\rho-1, \rho-1<\alpha\le\rho,\ \rho\in\mathbb{N}, \tag{1.2}$$

where $(\alpha,\beta)\in(0,\infty)\times(0,\infty)$, $u\in L^1(\Omega)$, $\Omega:=[0,a]\times[0,b]$, are known functions, (1.1) is the Caputo fractional differentiation operator and the unknown function $u(x,y)$ to be determined. In this work, we consider that, the nonlinear function has the following form $G(x,y,s,t,u) = k(x,y,s,t,)[u(s,t)]^P$, where p is positive integer. In this paper, we introduce a new operational method to solve nonlinear two dimensional fractional Volterra integro-differential equations. The method is based on reducing the equation to the system of algebraic equation by expanding the solution as Hat functions.

2. Riemann-Liouville and Caputo Fractional Derivatives

There are various types of definition for the fractional derivative. The most commonly used definitions are Riemann-Liouville and Caputo formulas. Riemann-Liouville fractional integration of order α is defined as

$$I_{x_0}^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_{x_0}^x (x-t)^{\alpha-1}f(t)dt, \quad \alpha>0, x>0. \tag{2.1}$$

The following equations define Riemann-Liouville and Caputo fractional derivatives of order α, respectively,

$$\begin{aligned} D_{x_0}^{\alpha}f(x) =& \frac{d^m}{dx^m}[I_{x_0}^{m-\alpha}f(x)], \\ D_{*x_0}^{\alpha}f(x) =& I_{x_0}^{m-\alpha}\left[\frac{d^m}{dx^m}f(x)\right], \end{aligned} \tag{2.2}$$

where $m-1 \leq \alpha < m$ and $n \in \mathbb{N}$. From (2.1) and (2.2), we have

$$D^{\alpha}_{x_0} f(x) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dx^m} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f(t)dt, \quad x > x_0.$$

Lemma 2.1. *If $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, then $D^{\alpha}_x I^{\alpha} u(x,t) = u(x,t)$, and*

$$I^{\alpha} D^{\alpha}_x \mathrm{u}(x,t) = \mathrm{u}(x,t) - \sum_{k=0}^{n-1} \frac{\partial^k u(0^+,t)}{\partial x^k} \frac{x^k}{k!}, \quad x > 0.$$

Definition 2.1 ([2])**.** Let $(\alpha,\beta) \in (0,\infty) \times (0,\infty)$, $\theta = (0,0)$, $\Omega := [0,a] \times [0,b]$, and $u \in L^1(\Omega)$. The left-sided mixed Riemann-Liouille integral of order (α,β) of u is defined by

$$(I^{(\alpha,\beta)}_{\theta} u)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^y \int_0^x (y-s)^{(\alpha-1)} (x-t)^{(\beta-1)} u(s,t) dsdt.$$

In particular

1. $(I^{(\alpha,\beta)}_{\theta} u)(x,y) = u(x,y)$;
2. $(I^{(\alpha,\beta)}_{\theta} u)(x,y) = \int_0^x \int_0^y u(s,t)dtds,\ (x,y) \in \Omega,\ \sigma = (1,1)$;
3. $(I^{(\alpha,\beta)}_{\theta} u)(x,0) = (I^{(\alpha,\beta)}_{\theta})(0,y) = 0,\ x \in [0,a],\ y \in [0,b]$;
4. $I^{\alpha,\beta}_{\theta} x^{\lambda} y^{\omega} = \frac{\Gamma(1+\lambda)\times\Gamma(1+\omega)}{\Gamma(1+\lambda+\alpha)\times\Gamma(1+\omega+\beta)} x^{\lambda+\alpha} y^{\omega+\beta},\ (x,y) \in \Omega,\ \lambda,\omega \in (-1,\infty)$.

3. Review of Hat Functions and Their Properties

A set of HFs is usually defined on $[0,1]$ as:

$$\phi_0(t) = \begin{cases} \dfrac{h-t}{h}, & 0 \leq t < h, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_i(t) = \begin{cases} \dfrac{t-(i-1)h}{h}, & (i-1)h \leq t < ih, \\ \dfrac{(i+1)h-t}{h}, & ih \leq t < (i+1)h,\ i = 1,2,\ldots,n-1, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_n(t) = \begin{cases} \dfrac{t-(1-h)}{h}, & T-h \leq t < T, \\ 0, & \text{otherwise}, \end{cases}$$

where $h = \frac{1}{n}$ and n is an arbitrary positive integer. Indeed, the unit interval [0,1] is divided into n equidistant subintervals. According to the definition of HFs, we have

$$\phi_i(jh) = \delta_{ij}, \tag{3.1}$$

where δ denotes the Kronecker delta function. By generalizing the definition of one-dimensional HFs, 2DHFs can be defined as follows

$$\Phi_{i,j}(x,y) = \Phi_i(x)\Phi_j(x), \quad i,j = 0,1,\ldots,n. \tag{3.2}$$

By substituting (3.1) and (3.2), we have $\Phi_{i,j}(kh, lh) = \delta_{jl}\delta_{ik}$. Now, for the 2DHFs, we have

$$\phi_{i,j}(x,y)\phi_{k,l}(x,y) = 0, \quad |i-j| \geq 2 \text{ or } |j-l| \geq 2 \tag{3.3}$$

and

$$\sum_{i=0}^{n}\sum_{j=0}^{n}\phi_{i,j}(x,y) = 1.$$

An arbitrary function $U(x,y)$ can be expanded in vector form as:

$$U(x,y) \simeq U^T\Phi(x,y) = \Phi^T(x,y)U, \tag{3.4}$$

where $U = [u_0, u_1, \ldots, u_n]^T$,

$$\Phi(x,y) = [\phi_{0,0}(x,y), \ldots, \phi_{0,m}(x,y), \phi_{1,0}(x,y), \ldots, \phi_{1,0}(x,y)]^T$$

and $u_{i,j} = u(ih, jh)$, $i, j = 0, 1, \ldots, n$. The positive integer powers of $u(x,y)$ may be approximated by HFs as $[u(x,y)]^P \simeq C_P^T \cdot \Phi(x,y)$. Now, let $k(x,y,s,t)$ be an arbitrary function of two variables defined on $L^2([0,1] \times [0,1])$. It can be expanded by HFs as: $k(x,y,s,t) \simeq \Phi^T(x,y)K\Phi(s,t)$, where $\Phi(x,y)$ and $\Phi(s,t)$ are 2DHFs vectors of dimention $(n+1)^2$, and K is 2DHFs coefficients matrix of dimention $(n_1+1)^2 \times (n+1)^2$ with entries $a_{ij}, i = 0, 1, \ldots, n_1,\ j = 0, 1, \ldots, n_2$, as $a_{ij} = k(ih, jh)$. In this paper, for convenience, we put $n_1 = n_2 = n$. Moreover, from (3.3) follows:

$$\Phi(x,y)\Phi^T(x,y)$$
$$= \begin{pmatrix} \phi_0^2(x) & \phi_0(x)\phi_1(x) & & & \\ \phi_0(x)\phi_1(x) & \phi_1^2(x) & \phi_1(x)\phi_2(x) & & \\ & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \phi_{n-1}(x)\phi_n(x) \\ & & & \phi_{n-1}(x)\phi_n(x) & \phi_n^2(x) \end{pmatrix}$$
$$\otimes \begin{pmatrix} \phi_0^2(x) & \phi_0(x)\phi_1(x) & & & \\ \phi_0(x)\phi_1(x) & \phi_1^2(x) & \phi_1(x)\phi_2(x) & & \\ & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \phi_{n-1}(x)\phi_n(x) \\ & & & \phi_{n-1}(x)\phi_n(x) & \phi_n^2(x) \end{pmatrix}$$

and

$$P_1 = \int_0^1\int_0^1 \Phi(x,y)\Phi^T(x,y)dxdy = \Upsilon_1 \otimes \Upsilon_1,$$

where P_1 is the following $(n+1)\times(n+1)$ matrix

$$P_1 = \frac{h}{6}\begin{pmatrix} 2 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots \\ & & 1 & 4 & 1 \\ & & & 1 & 2 \end{pmatrix}.$$

By considering (3.1), and expanding entries of $\Phi(x,y)\Phi^T(x,y)$ by 2DHFs, we have $\Phi(x,y)\Phi^T(x,y) \simeq \text{diag}(\Phi(x,y))$. Now, suppose that Λ is a vector $(n+1)^2$. We obtain

$$\Phi(x,y)\Phi^T(x,y)\Lambda \simeq \tilde{\Lambda}\Phi(x,y), \tag{3.5}$$

where $\tilde{\Lambda} = \text{diag}(\Lambda)$ is an $(n+1)^2 \times (n+1)^2$-diagonal matrix. Furthermore, if A is an $(n+1)^2 \times (n+1)^2$-matrix, we have

$$\Phi^T(x,y)A\Phi(x,y) \simeq \Phi^T(x,y)\hat{A}, \tag{3.6}$$

where $\hat{A}$ is an $(n+1)^2$-vector with elements equal to diagonal entries of matrix A. Now, we have

$$\begin{aligned}\int_0^y\int_0^x \Phi(s,t)dydt &= \int_0^y\int_0^x \Phi(s)\otimes\Phi(t)dsdt = \left(\int_0^y \Phi(s)ds\right)\otimes\left(\int_0^x \Phi(t)dt\right)\\ &\simeq (\Upsilon_1\Phi(x))\otimes(\Upsilon_2\Phi(y)) = (\Upsilon_1\otimes\Upsilon_2)\Phi(x,y) = P_2\Phi(x,y),\end{aligned}$$

where P_2 is the following $(n+1)\times(n+1)$ matrix

$$P_2 = \frac{h}{2}\begin{pmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 2 & 2 & \cdots & 2 \\ 0 & 0 & 1 & 2 & \cdots & 2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

3.1. **Operational matrix of the fractional order integration (OMFI).** Our goal is to get, to derive the Hat OMFI. For this purpose, Block pulse fractional matrix for the one-dimensional case is presented as follows:

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}b(\tau)d\tau = F^\alpha b(t),$$

where $\alpha \in \mathbb{R}$ is the order of the integration and $\Gamma(\alpha)$ is the Gamma function. Also, we define an m-set of Block Pulse Functions (BPFs) as

$$b_i(x) = \begin{cases} 1, & \dfrac{i}{m} \le x < \dfrac{(i+1)}{m}, \\ 0, & \text{otherwise,} \end{cases}$$

where $i = 0, 1, 2, \dots, m-1$. The function $b_i(x)$ is disjoint and orthogonal, that is

$$b_j(x)b_i(x) = \begin{cases} b_j, & j = i, \\ 0, & j \neq i, \end{cases}$$

where F^α is the $m \times m$ fractional operational matrix of integration of order α for the BPFs (see [16]) where

$$(I^\alpha B_m)(x) \simeq F^\alpha B_m(x),$$

$$F^\alpha = \frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

and $\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$. Our aim is to derive the Hat OMFI. For this purpose, we used the Riemann-Liouville fractional order integration, as following:

$$\begin{aligned}(I^\alpha u)(x,y) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{\beta-1}u(s,t)dsdt \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}x^{\beta-1} * u(x,y),\end{aligned}$$

where $\alpha, \beta \in \mathbb{R}$ are the order of the integration, $\Gamma(\alpha)$ and $\Gamma(\beta)$ are the Gamma functions and $y^{\alpha-1} * u(x,y)$, $x^{\beta-1} * u(x,y)$ denote the convolution products of $y^{\alpha-1}$, $x^{\beta-1}$ and $u(x,y)$. Now if $u(x,y)$ is expanded in HFs, as shown in (3.4), the Riemann-Liouville fractional integration becomes

$$(I^\alpha u)(x,y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}x^{\beta-1} * u(x,y) \approx C^T\frac{1}{\Gamma(\alpha)\Gamma(\beta)}y^{\alpha-1}x^{\beta-1} * \Phi(x,y).$$

Thus, if $y^{\alpha-1} * u(x,y)$ and $x^{\beta-1} * u(x,y)$ can be integrated, then by expanding the Hat functions, the Riemann-Liouville fractional order integration solve the HFs. Also, we define an m-set of BPF as

$$b_{i_1,i_2}(x,y) = \begin{cases} 1, & (i_1-1)h_1 \leqslant x < i_1h_1 \text{ and } (i_2-1)h_2 \leqslant y < i_2h_2, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 0, 1, 2, \dots, m-1$. The function $b_{i,j}(t)$ is disjoint and orthogonal, that is

$$b_{i_1,i_2}(x,y)b_{j_1,j_2}(x,y) = \begin{cases} b_{i_1,i_2}(x,y), & i_1 = j_1 \text{ and } i_2 = j_2, \\ 0, & \text{otherwise}. \end{cases}$$

The HFs can be expanded in to m-set of BPs functions as

$$\Phi(x,y) = \Psi_{m\times m}B_m(x,y), \tag{3.7}$$

where $B_m(x) = (b_0(x), b_1(x), \dots, b_i(x), \dots, b_{m-1}(x))^T$ (see [24,25]) and Ψ is an $MN \times MN$ product operational matrix. Next, we derive the Hat OMFI. We have the two dimensional BPFs operational matrix of fractional integration as:

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{\beta-1}U(s,t)dsdt = F^{\alpha,\beta}U(x,y),$$

where

$$F^{\alpha,\beta} = \frac{1}{m^\alpha m^\beta}\frac{1}{\Gamma(\alpha+2)\Gamma(\beta+2)}$$
$$\times\begin{bmatrix} 1 & \xi_1 & \xi_2 & \xi_3 & \dots & \xi_{m-1} \\ 0 & 1 & \xi_1 & \xi_2 & \dots & \xi_{m-1} \\ 0 & 0 & 1 & \xi_1 & \dots & \xi_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \xi_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & \eta_1 & \eta_2 & \eta_3 & \dots & \eta_{m-1} \\ 0 & 1 & \eta_1 & \eta_2 & \dots & \eta_{m-1} \\ 0 & 0 & 1 & \eta_1 & \dots & \eta_{m-3} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ 0 & 0 & \dots & 0 & 1 & \eta_1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix},$$

$\xi_k = (k+1)^{\alpha+1} - 2k^{\alpha+1} + (k-1)^{\alpha+1}$ and $\eta_k = (k+1)^{\beta+1} - 2k^{\beta+1} + (k-1)^{\beta+1}$.

Fractional integration of the BPFs is given as the following

$$(I^{\alpha,\beta}B_m)(x,y) \approx F^{\alpha,\beta}B_m(x,y). \tag{3.8}$$

Now, we derive the HFs operational matrix of the fractional order integration. Let

$$(I^{\alpha,\beta}\Phi)(x,y) \approx P^{\alpha,\beta}_{m\times m}\Phi(x,y), \tag{3.9}$$

where matrix $P^{\alpha,\beta}_{m\times m}$ is called the Hat functions OMFI. Using (3.7) and (3.8), we have

(3.10)
$$(I^{\alpha,\beta}\Phi)(x,y) \approx (I^{\alpha,\beta}\Psi_{m\times m}B_m)(x,y) = \Psi_{m\times m}(I^{\alpha}B_m)(x,y) \approx \Psi_{m\times m}F^{\alpha,\beta}B_m(x,y).$$

By (3.9) and (3.10) we get

$$P^{\alpha,\beta}_{m\times m}\Phi(x,t) = \Psi_{m\times m}F^{\alpha,\beta}B_m(x,y) = \Psi_{m\times m}F^{\alpha,\beta}\Phi_{m\times m}\Psi^{-1}_{m\times m}.$$

Then, the Hat functions OMFI $P^{\alpha,\beta}_{m\times m}$ is given by

$$P^{\alpha,\beta}_{m\times m} = \Psi_{m\times m}F^{\alpha,\beta}\Psi^{-1}_{m\times m}. \tag{3.11}$$

4. Applying the Method

In this section, 2DHFs fractional operational matrix are applied to solving (1.1). Now, let

$$D^\alpha_* u(x,y) \simeq C^T\Phi(x,y). \tag{4.1}$$

By using (4.1) and (3.9) and Lemma 2.1, we have

$$u(x,y) = C^T P^\alpha_{m\times m}\Phi(x,y) + \sum_{k=0}^{m-1}\frac{\partial^k u(0^+,y)}{\partial x^k}\frac{x^k}{k!}, \quad x>0.$$

So, by replacing the supplementary initial conditions (1.2), in the above summation in the above equations and approximating it by Hat functions, we have

$$u(x,y) \cong (C^T P^\alpha_{m\times m} + C^T_p)\Phi(x,y),$$

where C_p is a column m-vector. Define $e = [e_0, e_1, \ldots, e_{m-1}] = (C^T P^{\alpha}_{m\times m} + C_p^T)$, so, $u(x, y) \cong e\Phi(x, y)$. We could easily check out the correctness of the expression with induction $[u(x, y)]^q \cong [e_0^q, e_1^q, \ldots, e_{m-1}^q]\Phi(x, y) = e_q\Phi_{m\times m}$, where $\tilde{e}_q = [e_0^q, e_1^q, \ldots, e_{m-1}^q]$. The function $u(x, y), k(x, y, s, t)$ and $f(x, y)$ can be approximated by

$$
\begin{aligned}
u(x, y) =& U^T\Phi(x, y) = U\Phi^T(x, y),\\
F(x, y) =& F^T\Phi(x, y) = F\Phi^T(x, y),\\
[u(x, y)]^p =& \Phi^T(x, y)C_p,\\
k(x, y, s, t) =& \Phi^T(x, y) \cdot K \cdot \Phi(s, t).
\end{aligned} \tag{4.2}
$$

Now, with substituting (4.2) in (1.1), we have

$$
D_x^{\alpha}u(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y - s)^{\alpha-1}(x - t)^{(\beta-1)}G(x, y, s, t, u(s, t))dsdt + f(x, y).
$$

Using (3.5), (3.6), (3.9), and (3.11), we have

$$
\begin{aligned}
&C\Phi^T(x, y)\\
=&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y - s)^{\alpha-1}(x - t)^{(\beta-1)}k(x, y, s, t)[u(s, t)]^p dsdt + F\Phi^T(x, y)\\
=&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y - s)^{\alpha-1}(x - t)^{(\beta-1)}\Phi^T(x, t)K\Phi(s, t)\Phi^T(x, y)C_p dsdt + F\Phi^T(x, y)\\
=&\Phi^T(x, y)K\tilde{C}_p\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y - s)^{\alpha-1}(x - t)^{(\beta-1)}\Phi(s, t)dsdt + F\Phi^T(x, y)\\
=&\Phi^T(x, y)K\tilde{C}_p P^{\alpha,\beta}_{m\times m}\Phi(x, y) = \left(\widehat{K\tilde{c}pP^{\alpha,\beta}_{m\times m}}\right)^T \cdot \Phi(x, y) + F\Phi^T(x, y)\\
=&\left(\widehat{K\tilde{c}pP^{\alpha,\beta}_{m\times m}}\right) \cdot \Phi^T(x, y) + F\Phi^T(x, y).
\end{aligned}
$$

Set

$$
B = \left(\widehat{K\tilde{c}pP^{\alpha,\beta}_{m\times m}}\right),
$$

so,

$$
C\Phi^T(x, y) = B\Phi^T(x, y) + F\Phi^T(x, y),
$$

hence, we have

$$
C = B + F, \tag{4.3}
$$

which is a system of algebraic equations. By solving this system, we can obtain the approximate solution of (1.1) according to (4.3).

5. Convergence and Error Analysis

In this section, we obtain an error bound for the approximate solution, then from which we conclude convergence of the method. We define the error function as

$$
e_n(x, y) = u(x, y) - \hat{u}(x, y),
$$

where $u(x, y)$ and $\hat{u}(x, y)$ denote the exact and approximate solutions, respectively.

Theorem 5.1. *Suppose $u(x, y) \in I$ and $e_n(x, y) = u(x, y) - u_n(x, y)$, $(x, y) \in I = [0, T) \times [0, T)$, where $u_n(x, y) = \sum_{i=0}^{n} u(ih, jh)\phi_{i,j}(x, y)$ is the generalized hat function expansion of $u(x, y)$. Then, we have*

$$\|e_n(x, y)\| \leq \frac{T^2}{2n^2}\|u''(x, y)\|, \tag{5.1}$$

and so the convergence is of order two, that is $\|e_n(x, y)\| = O\left(\frac{1}{n^2}\right)$.

Proof. See [17]. □

Theorem 5.2. *Suppose $u(x, y)$ as an exact solution of fractional integral* (1.1) *and $\hat{u}(x, y)$ show the approximate solution by Hat functions. If $|(x-s)^{\alpha-1}(y-t)^{\beta-1}k(x, y, s, t)| < N$, $u(x, y)$ and $k(x, y, s, t)$ are continuous functions and also, $G(u) = (u(x, t))^p$ satisfies Lipschitz condition $|G(u) - G(\hat{u})| \leq L|u - \hat{u}|$, then*

$$\|u - \hat{u}\| = \sup_{0 \leq x, y \leq 1} |u(x, y) - \hat{u}(x, y)| = O\left(\frac{1}{n^2}\right).$$

Proof. We have

$$\begin{aligned}
&|u(x, y) - \hat{u}(x, y)| \\
=&\left|\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x (y-s)^{\alpha-1}(x-t)^{\beta-1}k(x, y, s, t)(u(s, t) - \hat{u}(s, t))dtds\right| \\
\leq&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^x\int_0^y |(x-s)^{\alpha-1}(y-t)^{\beta-1}k(x, y, s, t)(u(s, t) - \hat{u}(s, t))|dsdt \\
\leq&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x |(y-s)^{\alpha-1}(x-t)^{\beta-1}k(x, y, s, t)||(u(s, t) - \hat{u}(s, t))|dsdt \\
\leq&\frac{N}{\Gamma(\alpha)\Gamma(\beta)}\int_0^y\int_0^x |(u(s, t) - \hat{u}(s, t))|dsdt.
\end{aligned}$$

From (5.1), we conclude that

$$|u(x, y) - \hat{u}(x, y)| \leq \frac{NLT^2xy}{2n^2\Gamma(\alpha)\Gamma(\beta)} \leq \frac{NLT^2}{2n^2\Gamma(\alpha)\Gamma(\beta)}.$$

This completes the proof. □

Theorem 5.3. *The solving systems of partial 2DFVIE by using 2D-HFs converge if $0 < \theta < 1$, where $\theta = \frac{NLT^2}{2n^2\Gamma(\alpha)\Gamma(\beta)}$.*

Proof. If we assume $G(u) = D_x^{\alpha}u(x, y)$, we have $\|G(u) - G(u_m)\|_{\infty} \leq \|u - u_m\|_{\infty}$. From Theorem 5.2, we have

$$\|G(u) - G(u_m)\|_{\infty} \leq \frac{NLT^2}{2n^2\Gamma(\alpha)\Gamma(\beta)}\|u - u_m\|_{\infty}. \tag{5.2}$$

Inequality (5.2) implies that if $0 < \theta < 1$, then we have $\lim_{m\to\infty} \|G(u)-G(u_m)\|_\infty = 0$ and $\lim_{m\to\infty} \|u - u_m\|_\infty = 0$. $\square$

6. Numerical Examples

To illustrate the effectiveness of the proposed method in the present paper, some test examples are carried out in this section.

Example 6.1. Consider the fractional partial volterra integro-differential equation [19]

$$D_x^{0.75}u(x,y) = \int_0^y \int_0^x (y+t)u(s,t)dsdt = \frac{6.4}{\Gamma(0.25)}yx^{5/4} - \frac{5}{18}x^3y^3,$$

where the exact solution is known and it is given by $u(x,y) = x^2y$, for $x, y \in [0,1]$ and with supplementary condition $u(0,y) = 0$. Numerical results are presented in Table 1.

Table 1. The absolute errors for Example 1.

	$m=n=4$	$m=n=4$	$m=n=5$	$m=n=5$
(x,y)	u_{2DLWs} [19]	u_{2DHFs}	u_{2DLWs} [19]	u_{2DHFs}
$(0.0, 0.7)$	0.1404×10^{-2}	0.1404×10^{-2}	0.3508×10^{-3}	0.2327×10^{-3}
$(0.1, 0.3)$	0.1636×10^{-3}	0.2584×10^{-2}	0.1342×10^{-3}	0.4158×10^{-3}
$(0.3, 0.8)$	0.1456×10^{-2}	0.3651×10^{-3}	0.8962×10^{-3}	0.1001×10^{-4}
$(0.4, 0.2)$	0.1087×10^{-3}	0.6521×10^{-3}	0.2700×10^{-4}	0.5057×10^{-4}
$(0.6, 0.6)$	0.3248×10^{-3}	0.1421×10^{-3}	0.6759×10^{-3}	0.5884×10^{-4}
$(0.7, 0.5)$	0.8878×10^{-3}	0.6250×10^{-3}	0.5285×10^{-4}	0.1019×10^{-4}
$(0.8, 0.4)$	0.7061×10^{-3}	0.7247×10^{-3}	0.4090×10^{-4}	0.1018×10^{-4}
$(0.9, 0.9)$	0.5898×10^{-3}	0.1997×10^{-3}	0.1974×10^{-3}	0.4108×10^{-4}

Example 6.2. Consider the linear two-dimensional fractional integro-differential equation [19]

$$D_x^{0.5}u(x,y) = \int_0^y \int_0^x (x^2y+s)u(s,t)dsdt = 4y\sqrt{\frac{x}{\pi}} - \frac{1}{2}x^4y^3 - \frac{1}{3}x^3y^2,$$

where the exact solution is known and given by $u(x,y) = 2xy$, for $x, y \in [0,1]$ and with supplementary condition $u(0,y) = 0$. Numerical results are presented in the Table 2.

Example 6.3. Consider the linear two-dimensional fractional integro-differential equation [19]

$$D_x^{0.5}u(x,y) = \int_0^y \int_0^x (x\cos(s)+yt)u(s,t)dsdt = f(x,y),$$

Table 2. The absolute errors for Example 2.

	$m = n = 4$	$m = n = 4$	$m = n = 5$	$m = n = 5$
(x, y)	u_{2DLWs} [19]	u_{2DHFs}	u_{2DLWs} [19]	u_{2DHFs}
$(0.1, 0.8)$	0.1173×10^{-3}	0.1853×10^{-3}	0.1250×10^{-3}	0.4141×10^{-3}
$(0.2, 0.6)$	0.1805×10^{-3}	0.9461×10^{-3}	0.2751×10^{-4}	0.4258×10^{-3}
$(0.3, 0.8)$	0.9276×10^{-4}	0.9276×10^{-4}	0.1189×10^{-4}	0.1104×10^{-4}
$(0.4, 0.6)$	0.2710×10^{-4}	0.3621×10^{-4}	0.1395×10^{-5}	0.1245×10^{-5}
$(0.5, 0.5)$	0.7309×10^{-5}	0.1001×10^{-4}	0.4065×10^{-5}	0.7412×10^{-5}
$(0.6, 0.5)$	0.3884×10^{-4}	0.3621×10^{-4}	0.1174×10^{-4}	0.3241×10^{-5}
$(0.7, 0.3)$	0.3548×10^{-4}	0.5200×10^{-3}	0.9798×10^{-5}	0.4142×10^{-4}
$(0.8, 0.4)$	0.9069×10^{-4}	0.3247×10^{-4}	0.2406×10^{-4}	0.3258×10^{-4}
$(0.9, 0.9)$	0.6179×10^{-3}	0.1657×10^{-3}	0.1607×10^{-3}	0.4741×10^{-4}

where

$$\begin{aligned} f(x,y) =& \frac{2\sin(y)\sqrt{x}}{\sqrt{0.5}} + x\cos(x) - x^2\sin(x) - x\cos(y) + x\cos(x)\cos(y) \\ &+ x^2\sin(x)\cos(y) - \frac{1}{2}x^2 y\sin(y) + \frac{1}{2}x^2y^2\cos(y), \end{aligned}$$

where the exact solution is known and given by $u(x, y) = x\sin(y)$, for $x, y \in [0, 1]$ and with supplementary condition $u(0, y) = 0$. Numerical results are presented in the Table 3.

Table 3. The absolute errors for Example 3.

	$m = n = 3$	$m = n = 3$	$m = n = 4$	$m = n = 4$
(x, y)	u_{2DLWs} [19]	u_{2DHFs}	u_{2DLWs} [19]	u_{2DHFs}
$(0.1, 0.1)$	0.1599×10^{-3}	0.2514×10^{-2}	0.5398×10^{-4}	0.9841×10^{-3}
$(0.2, 0.2)$	0.2155×10^{-3}	0.6251×10^{-3}	0.5185×10^{-4}	0.4625×10^{-4}
$(0.3, 0.3)$	0.1566×10^{-3}	0.5210×10^{-3}	0.6503×10^{-4}	0.1984×10^{-4}
$(0.4, 0.4)$	0.2122×10^{-3}	0.9654×10^{-3}	0.7688×10^{-4}	0.1962×10^{-4}
$(0.5, 0.5)$	0.2477×10^{-3}	0.2014×10^{-3}	0.8809×10^{-4}	0.7620×10^{-4}
$(0.6, 0.6)$	0.2971×10^{-3}	0.6521×10^{-3}	0.9899×10^{-4}	0.3021×10^{-4}
$(0.7, 0.7)$	0.3662×10^{-3}	0.6214×10^{-3}	0.1226×10^{-3}	0.4142×10^{-4}
$(0.8, 0.8)$	0.4738×10^{-3}	0.2147×10^{-3}	0.1599×10^{-3}	0.3108×10^{-4}
$(0.9, 0.9)$	0.6344×10^{-3}	0.9651×10^{-3}	0.2246×10^{-3}	0.4748×10^{-3}

Example 6.4. Consider the two-dimensional fractional Volterra integral equation [1]

$$u(x,y) - \frac{1}{\Gamma(\frac{7}{2})\Gamma(\frac{5}{2})}\int_0^y\int_0^x (y-s)^{\frac{5}{2}}(x-t)^{\frac{3}{2}}(y^2+s)e^{-t}u(s,t)dsdt = f(x,y),$$

where

$$f(x,y) = x^2e^y - \frac{1024x^{\frac{11}{2}}y^{\frac{5}{2}}(6x+13y^2)}{2027025\pi},$$

where the exact solution is known and it is given by $u(x,y) = x^2e^y$. To solve this equation, we implement the HFs method for $\alpha = \frac{7}{2}$ and $\beta = \frac{5}{2}$. Numerical results are presented in Table 4 and Figure 1.

Table 4. The absolute errors for Example 4.

	$m=n=2$	$m=n=2$	$m=n=4$	$m=n=4$
$x=y$	u_{2DBPOM} [1]	u_{2DHFs}	u_{2DBPOM} [1]	u_{2DHFs}
0.0	2.090×10^{-4}	2.125×10^{-4}	4.086×10^{-4}	5.237×10^{-5}
0.1	2.532×10^{-4}	2.635×10^{-4}	4.181×10^{-4}	4.258×10^{-5}
0.2	6.967×10^{-5}	5.689×10^{-4}	4.471×10^{-4}	4.125×10^{-4}
0.3	2.602×10^{-4}	3.070×10^{-4}	4.970×10^{-4}	4.157×10^{-4}
0.4	3.346×10^{-4}	4.325×10^{-4}	5.656×10^{-4}	4.984×10^{-4}
0.5	2.778×10^{-4}	3.215×10^{-3}	6.474×10^{-4}	6.259×10^{-4}
0.6	1.701×10^{-3}	2.587×10^{-3}	7.316×10^{-4}	7.147×10^{-4}
0.7	2.090×10^{-3}	2.090×10^{-3}	7.817×10^{-4}	7.548×10^{-4}
0.8	3.542×10^{-3}	3.985×10^{-3}	6.788×10^{-4}	7.214×10^{-4}
0.9	1.137×10^{-3}	2.087×10^{-3}	1.004×10^{-4}	2.587×10^{-4}

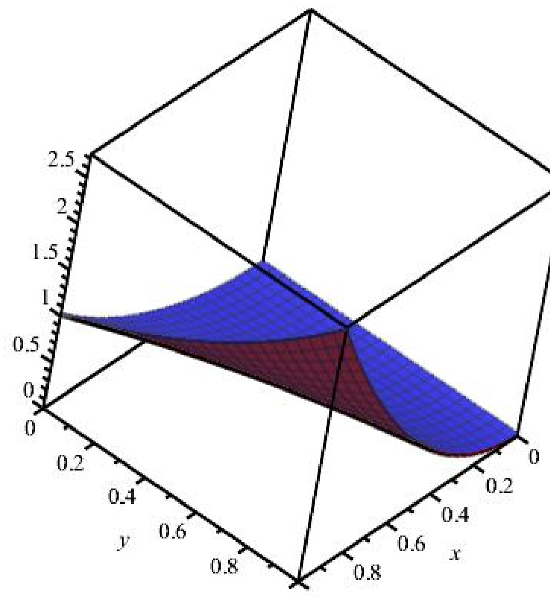

Figure 1. Exact and approximation solutions of Example 4.

Example 6.5. Consider the two-dimensional nonlinear fractional Volterra equation [20]

$$u(x,y) - \frac{1}{\Gamma(\frac{3}{2})\Gamma(\frac{5}{2})}\int_0^y\int_0^x (y-s)^{\frac{1}{2}}(x-t)^{\frac{3}{2}}\sqrt{xyt}[u(s,t)]^2dsdt = f(x,y),$$

where

$$f(x,y) = \sqrt{y}\left(\frac{-1}{180}x^3 y^{\frac{7}{2}} + \sqrt{\frac{x}{3}}\right).$$

The exact solution is known and it is given by $u(x,y) = \frac{\sqrt{3xy}}{3}$. This example has been solved, for $\alpha = \frac{3}{2}$ and $\beta = \frac{5}{2}$. Numerical results for this a solution are presented in Table 5 and Figure 2.

Table 5. The numerical results for Example 5.

	Exact solution	$m = 32$	$m = 32$
$x = y$		u_{2DBPFs} [20]	u_{2DHFs}
0.0	0	0.009386	0.002541
0.1	0.05773	0.042121	0.042541
0.2	0.11547	0.124282	0.138744
0.3	0.17323	0.156905	0.144871
0.4	0.23094	0.239179	0.235487
0.5	0.28867	0.274574	0.275487
0.6	0.34641	0.354075	0.344872
0.7	0.40414	0.389848	0.404151
0.8	0.46188	0.468971	0.469874
0.9	0.50702	0.507021	0.507210

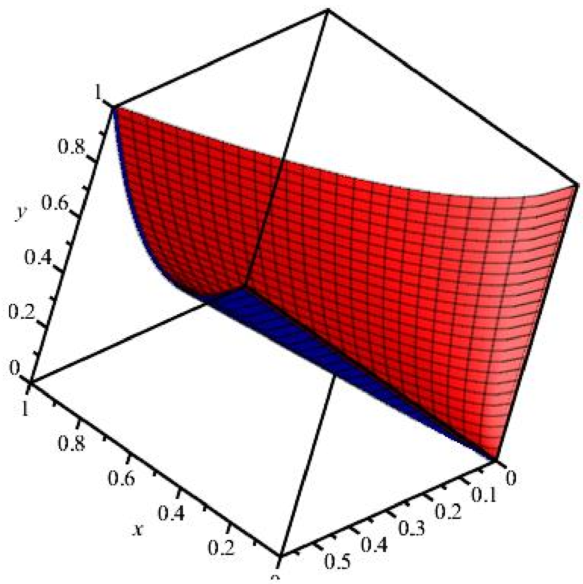

Figure 2. Comparison the exact solution and the presented method for Example 5.

7. Conclusion

In this paper, a Hat operational matrix of fractional order integration is obtained and it is used to solve the two-dimensional nonlinear fractional Volterra integro-differential equations. By properties of 2DHFs and using of operational matrices the possibility of reducing these equations to a system of algebraic equations are provided. Moreover, a general procedure of forming this matrix $P^{\alpha,\beta}_{m\times m}$ is summarized. For more investigation, some examples are presented. As the numerical results showed, the proposed method is an accurate and effective method for solving a fractional two-dimensional integral equation.

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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF KIRRCHOFF ELLIPTIC SYSTEMS WITH RIGHT HAND SIDE DEFINED AS A MULTIPLICATION OF TWO SEPARATE FUNCTIONS

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Abstract. The paper deals with the study of existence of weak positive solutions for a new class of Kirrchoff elliptic systems in bounded domains with multiple parameters, where the right hand side defined as a multiplication of two separate functions.

1. Introduction

In this paper, we consider the following system of differential equations

$$\begin{cases} -A\left(\int\limits_{\Omega}|\nabla u|^2\,dx\right)\triangle u=\lambda_1\alpha\left(x\right)f\left(v\right)h\left(u\right) \text{ in } \Omega, \\ -B\left(\int\limits_{\Omega}|\nabla v|^2\,dx\right)\triangle v=\lambda_2\beta\left(x\right)g\left(u\right)\tau\left(v\right) \text{ in } \Omega, \\ u=v=0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N$, $N\geq 3$, is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $A,B:\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions, $\alpha,\beta\in C\left(\overline{\Omega}\right)$, λ_1 and λ_2 are nonnegative parameters.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity, therefore, it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends

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on the average of itself, such as the population density, see [9]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{P_0}{h}+\frac{E}{2L}\int_0^L\left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2}=0, \tag{1.2}$$

presented by Kirchhoff in 1883 (see [10]). This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to ([3–5, 7, 9, 11]), in which the authors have used different methods to get the existence of solutions for Kirchhoff type equations. Our paper is motivated by the recent results in ([1, 2]). In the paper [2], Azzouz and Bensedik studied the existence of a positive weak solution for the nonlocal problem of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2\,dx\right)\triangle u=|u|^{p-2}u+\lambda f(x) \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N\geq 3$ and $p>1$, with a sign-changing function f.

Using the sub-supersolution method combining a comparison principle introduced in [1], the authors established the existence of a positive solution for (1.3), where the parameter $\lambda>0$ is small enough. In the present paper, we consider system (1.1) in the case when the nonlinearities are "sublinear" at infinity, see the condition (H 3). Under suitable conditions on f, g, h and τ, we shall show that system (1.1) has a positive solution for $\lambda>\lambda^*$ large enough. To our best knowledge, this is a new research topic for nonlocal problems, see [8]. In current paper, motivated by previous works in ([2], [6]) and by using sub-super solutions method, we study of existence of weak positive solutions for a new class of Kirrchoff elliptic systems in bounded domains with multiple parameters, where the right hand side defined as a multiplication of two separate functions. Our results extend and improve our recent results in [3] and [11].

2. Existence Result

Lemma 2.1 ([2])**.** *Assume that* $M:\mathbb{R}^+\to\mathbb{R}^+$ *is a continuous and nonincreasing function satisfying* $\lim_{t\to 0^+}M(t)=m_0$, *where* m_0 *is a positive constant. Suppose further that function* $H(t):=tM(t^2)$ *is increasing on* $\mathbb{R}$.

Assume that u, v are two non-negative functions such that

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u \geq -M\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v \ \text{in } \Omega, \\ u = v = 0 \ \text{on } \partial\Omega, \end{cases}$$

then $u \geq v$ a.e. in Ω.

Lemma 2.2 ([1])**.** *If M verifies the conditions of Lemma 2.1, then for each $f \in L^2(\Omega)$ there exists a unique solution $u \in H_0^1(\Omega)$ to the M-linear problem*

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = f(x) \ \text{ in } \Omega, \\ u = 0 \ \text{in } \partial\Omega. \end{cases}$$

Lemma 2.3 ([6])**.** *Let w solve $\Delta w = g$ in Ω. If $g \in C(\Omega)$, then $w \in C^{1,\alpha}(\Omega)$ for any $\alpha \in (0,1)$, so particularly w is continuous in Ω.*

In this section, we shall state and prove the main result of this paper. Let us assume the following assumptions.

$(H1)$ Assume that $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the same conditions as M in Lemma 1, and there exists $a_i, b_i > 0, i = 1, 2$, such that

$$a_1 \leq A(t) \leq a_2, \quad b_1 \leq B(t) \leq b_2, \quad \text{for all } t \in \mathbb{R}^+.$$

$(H2)$ $\alpha, \beta \in C\left(\overline{\Omega}\right)$ and

$$\alpha(x) \geq \alpha_0 > 0, \quad \beta(x) \geq \beta_0 > 0,$$

for all $x \in \Omega$.

$(H3)$ f, g, h, and τ are C^1 on $(0, +\infty)$, and increasing functions such that

$$\lim_{t \to +\infty} f(t) = +\infty, \quad \lim_{t \to +\infty} g(t) = +\infty, \quad \lim_{t \to +\infty} h(t) = +\infty = \lim_{t \to +\infty} \tau(t) = +\infty.$$

$(H4)$ Exists $\gamma > 0$ such that

$$\lim_{t \to +\infty} \frac{h(t) f(k[g(t)^\gamma])}{t} = 0, \quad \text{for all } k > 0,$$

and

$$\lim_{t \to +\infty} \frac{\tau(kt^\gamma)}{t^{\gamma - 1}} = 0, \quad \text{for all } k > 0.$$

We present below an example where hypotheses $(H3)$ and $(H4)$ hold

$$\tau(t) = \ln(t), \quad h(t) = \sqrt{t}, \quad f(t) = \ln(t), \quad g(t) = t, \quad \gamma = 2.$$

Theorem 2.1. *Assume that the conditions $(H1)$-$(H4)$ hold. Then for $\lambda_1\alpha_0$ and $\lambda_2\beta_0$ large the problem (1.1) has a large positive weak solution.*

We give the following two definitions before we give our main result.

Definition 2.1. Let $(u,v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$, (u,v) is said to be a weak solution of (1.1) if it satisfies

$$
\begin{aligned}
A\left(\int_\Omega |\nabla u|^2 \,\mathrm{d}x\right)\int_\Omega \nabla u \nabla \phi \mathrm{d}x =& \lambda_1 \int_\Omega \alpha(x) f(v) h(u) \phi \,\mathrm{d}x \text{ in } \Omega,\\
B\left(\int_\Omega |\nabla v|^2 \,\mathrm{d}x\right)\int_\Omega \nabla v \nabla \psi \mathrm{d}x =& \lambda_2 \int_\Omega \beta(x) g(u) \tau(v) \psi \,\mathrm{d}x \text{ in } \Omega,
\end{aligned}
$$

for all $(\phi,\psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Definition 2.2. A pair of nonnegative functions $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ in $(H_0^1(\Omega) \times H_0^1(\Omega))$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\underline{u},\underline{v}), (\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$

$$
\begin{aligned}
A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u} \nabla \phi \mathrm{d}x \leq& \lambda_1 \int_\Omega \alpha(x) f(\underline{v}) h(\underline{u}) \phi \mathrm{d}x \text{ in } \Omega,\\
B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v} \nabla \psi \mathrm{d}x \leq& \lambda_2 \int_\Omega \beta(x) g(\underline{u}) \tau(\underline{v}) \psi \mathrm{d}x \text{ in } \Omega
\end{aligned}
$$

and

$$
\begin{aligned}
A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\int_\Omega \nabla \overline{u} \nabla \phi \mathrm{d}x \geq& \lambda_1 \int_\Omega \alpha(x) f(\overline{v}) h(\overline{u}) \phi \mathrm{d}x \text{ in } \Omega,\\
B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla \overline{v} \nabla \psi \mathrm{d}x \geq& \lambda_2 \int_\Omega \beta(x) g(\overline{u}) \tau(\overline{v}) \psi \mathrm{d}x \text{ in } \Omega,
\end{aligned}
$$

for all $(\phi,\psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Proof of Theorem 1. Let σ be the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction, with $\|\phi_1\| = 1$. Let $m_0, \delta > 0$ be such that $|\nabla\phi_1|^2 - \sigma\phi_1^2 \geq m_0$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x,\partial\Omega) \leq \delta\}$.

For each $\lambda_1\alpha_0$ and $\lambda_2\beta_0$ large, let us define

$$
\underline{u} = \left(\frac{\lambda_1\alpha_0}{2a_1}\right)\phi_1^2 \quad \text{and} \quad \underline{v} = \left(\frac{\lambda_2\beta_0}{2b_1}\right)\phi_1^2,
$$

where a_1, b_1 are given by the condition $(H1)$. We shall verify that $(\underline{u},\underline{v})$ is a weak subsolution of problem (1.1), for $\lambda_1\alpha_0$ and $\lambda_2\beta_0$ large enough. Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. By $(H1)$-$(H3)$, a simple calculation shows that

$$
A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 dx\right)\int_{\overline{\Omega}_\delta} \nabla \underline{u}.\nabla\phi dx = A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 dx\right)\frac{\lambda_1\alpha_0}{a_1}\int_{\overline{\Omega}_\delta} \phi_1 \nabla\phi_1.\nabla\phi dx
$$

$$
\begin{aligned}
&= \frac{\lambda_1 \alpha_0}{a_1} A \left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \\
&\quad \times \left\{ \int_{\overline{\Omega}_\delta} \nabla \phi_1 \nabla \left(\phi_1 . \phi \right) dx - \int_{\overline{\Omega}_\delta} |\nabla \phi_1|^2 \, \phi dx \right\} \\
&= \frac{\lambda_1 \alpha_0}{a_1} A \left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\overline{\Omega}_\delta} \left(\sigma \phi_1^2 - |\nabla \phi_1|^2 \right) \phi dx.
\end{aligned}
$$

On $\overline{\Omega}_\delta$ we have $|\nabla \phi_1|^2 - \sigma \phi_1^2 \geq m_0$, then $\sigma \phi_1^2 - |\nabla \phi_1|^2 < 0$. So,

$$
A \left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx < 0,
$$

by $(H3)$ for $\lambda_1 \alpha_0$ and $\lambda_2 \beta_0$ large enough we get $f(\underline{v}) h(\underline{u}) > 0$. And then

$$
A \left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx \ \leq \lambda_1 \int_{\overline{\Omega}_\delta} \alpha(x) f(\underline{v}) h(\underline{u}) \phi dx. \tag{2.1}
$$

Next, on $\Omega \backslash \overline{\Omega}_\delta$ we have $\phi_1 \geq r$ for some $r > 0$, and therefore, by the conditions $(H1)$-$(H3)$ and the definition of $\underline{u}$ and $\underline{v}$, it follows that

$$
\begin{aligned}
\lambda_1 \int_{\Omega \backslash \overline{\Omega}_\delta} \alpha(x) f(\underline{v}) h(\underline{u}) \phi dx &\geq \frac{\lambda_1 \alpha_0 a_2}{a_1} \sigma \int_{\Omega \backslash \overline{\Omega}_\delta} \phi dx \\
&\geq \frac{\lambda_1 \alpha_0}{a_1} A \left(\int_{\Omega \backslash \overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega \backslash \overline{\Omega}_\delta} \sigma \phi dx \\
&\geq \frac{\lambda_1 \alpha_0}{a_1} A \left(\int_{\Omega \backslash \overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega \backslash \overline{\Omega}_\delta} \left(\sigma \phi_1^2 - |\nabla \phi_1|^2 \right) \phi dx \\
&= A \left(\int_{\Omega \backslash \overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega \backslash \overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx,
\end{aligned} \tag{2.2}
$$

for $\lambda_1 \alpha_0 > 0$ large enough.

Relations (2.1) and (2.2) imply that

$$
A \left(\int_{\Omega} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega} \nabla \underline{u} \nabla \phi dx \leq \lambda_1 \int_{\Omega} \alpha(x) f(\underline{v}) h(\underline{u}) \phi dx \text{ in } \Omega, \tag{2.3}
$$

for $\lambda_1 \alpha_0 > 0$ large enough and any $\phi \in H_0^1(\Omega)$, with $\phi \geq 0$ in Ω.

Similarly,

$$B\left(\int_{\Omega}|\nabla \underline{v}|^{2}\,dx\right)\int_{\Omega}\nabla \underline{v}\nabla\psi dx\leq\lambda_{2}\int_{\Omega}\beta\left(x\right)g\left(\underline{u}\right)\tau\left(\underline{v}\right)\psi dx \text{ in } \Omega, \tag{2.4}$$

for $\lambda_2\beta_0>0$ large enough and any $\psi\in H_0^1(\Omega)$, with $\psi\geq 0$ in Ω. From (2.3) and (2.4), $(\underline{u},\underline{v})$ is a subsolution of problem (1.1). Moreover, we have $\underline{u}>0$ and $\underline{v}>0$ in Ω, $\underline{u}\to+\infty$ and $\underline{v}\to+\infty$ as $\lambda_1\alpha_0\to+\infty$ and $\lambda_2\beta_0\to+\infty$.

Next, we shall construct a weak supersolution of problem (1.1). Let e be the solution of the following problem

$$\begin{cases} -\triangle e=1 \text{ in } \Omega,\\ e=0 \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

Let

$$\overline{u}=Ce,\quad \overline{v}=\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\right)\left[g\left(C\|e\|_\infty\right)\right]^{\gamma}e,$$

where γ is given by (H_4) and $C>0$ is a large positive real number to be chosen later. We shall verify that $(\overline{u},\overline{v})$ is a supersolution of problem (1.1). Let $\phi\in H_0^1(\Omega)$ with $\phi\geq 0$ in Ω. Then we obtain from (2.5) and the condition $(H1)$ that

$$\begin{aligned} A\left(\int_{\Omega}|\nabla\overline{u}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{u}.\nabla\phi dx =&A\left(\int_{\Omega}|\nabla\overline{u}|^{2}\,dx\right)C\int_{\Omega}\nabla e.\nabla\phi dx\\ =&A\left(\int_{\Omega}|\nabla\overline{u}|^{2}\,dx\right)C\int_{\Omega}\phi dx\\ \geq& a_1C\int_{\Omega}\phi dx. \end{aligned}$$

By $(H4)$, we can choose C large enough so that

$$a_1C\geq\lambda_1\|\alpha\|_\infty f\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\|e\|_\infty\left[g\left(C\|e\|_\infty\right)\right]^{\gamma}\right)h\left(C\|e\|_\infty\right).$$

Therefore,

$$\begin{aligned} &A\left(\int_{\Omega}|\nabla\overline{u}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{u}.\nabla\phi dx\\ \geq&\lambda_1\|\alpha\|_\infty f\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\|e\|_\infty\left[g\left(C\|e\|_\infty\right)\right]^{\gamma}\right).h\left(C\|e\|_\infty\right)\int_{\Omega}\phi dx\\ \geq&\lambda_1\int_{\Omega}\|\alpha\|_\infty f\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\|e\|_\infty\left[g\left(C\|e\|_\infty\right)\right]^{\gamma}\right).h\left(C\|e\|_\infty\right)\phi dx\\ \geq&\lambda_1\int_{\Omega}\alpha\left(x\right)f\left(\overline{v}\right)h\left(\overline{u}\right)\phi\,\mathrm{d}x. \end{aligned} \tag{2.6}$$

Also,

$$B\left(\int_\Omega |\nabla \overline{v}|^2\,dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx \geq \lambda_2\|\beta\|_\infty \int_\Omega [g(C\|e\|_\infty)]^\gamma \psi dx. \tag{2.7}$$

Again by $(H4)$ for C large enough we have

$$[g(C\|e\|_\infty)]^\gamma \geq g(C\|e\|_\infty)\,\tau\left(\frac{\lambda_2\|\beta\|_\infty\|e\|_\infty}{b_1}[g(C\|e\|_\infty)]^\gamma\right). \tag{2.8}$$

From (2.7) and (2.8), we have

$$B\left(\int_\Omega |\nabla \overline{v}|^2\,dx\right)\int_\Omega \nabla\overline{v}\nabla\psi dx \geq \lambda_2\int_\Omega \beta(x)\,g(\overline{u})\,\tau(\overline{v})\,\psi dx. \tag{2.9}$$

From (2.6) and (2.9) we have$(\overline{u},\overline{v})$ is a weak supersolution of problem (1.1), with $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for C large.

In order to obtain a weak solution of problem (1.1) we define the sequence

$$\{(u_n,v_n)\}\subset E=\left(H_0^1(\Omega)\times H_0^1(\Omega)\right)\cap\left(C(\Omega)\times C(\Omega)\right)$$

as follows: $(u_0,v_0):=(\overline{u},\overline{v})\in E$ and (u_n,v_n) is the unique solution of the system

$$\begin{cases} -A\left(\int_\Omega |\nabla u_n|^2\,dx\right)\triangle u_n=\lambda_1\alpha(x)\,f(v_{n-1})\,h(u_{n-1}) \text{ in }\Omega,\\ -B\left(\int_\Omega |\nabla v_n|^2\,dx\right)\triangle v_n=\lambda_2\beta(x)\,g(u_{n-1})\,\tau(v_{n-1}) \text{ in }\Omega,\\ u_n=v_n=0 \text{ on }\partial\Omega.\end{cases} \tag{2.10}$$

Problem (2.10) is (A,B)-linear in the sense that, if $(u_{n-1},v_{n-1})\in E$ is a given, the right hand sides of (2.10) is independent of u_n,v_n.

Set $\widetilde{A}(t)=tA(t^2)$, $\widetilde{B}(t)=tB(t^2)$. Then, since $\widetilde{A}(\mathbb{R})=\mathbb{R}$, $\widetilde{B}(\mathbb{R})=\mathbb{R}$, $f(v_0)$, $h(u_0)$, $g(u_0)$ and $\tau(v_0)\in C(\Omega)\subset L^2(\Omega)$ (in x), we deduce from Lemma 2.2 that system (2.10), with $n=1$ has a unique solution $(u_1,v_1)\in(H_0^1(\Omega)\times H_0^1(\Omega))$. And by observing that

$$\begin{cases} -\triangle u_1=\dfrac{\lambda_1}{A\left(\int_\Omega |\nabla u_1|^2\,dx\right)}\alpha f(v_0)\,h(u_0)\in C(\Omega),\\ -\triangle v_1=\dfrac{\lambda_2}{B\left(\int_\Omega |\nabla v_1|^2\,dx\right)}\beta g(u_0)\,\tau(v_0)\in C(\Omega),\\ u_1=v_1=0 \text{ on }\partial\Omega.\end{cases}$$

We deduce from Lemma 2.3 that $(u_1, v_1) \in C(\Omega) \times C(\Omega)$. Consequently $(u_1, v_1) \in E$. By the same way we construct the following elements $(u_n, v_n) \in E$ of our sequence. From (2.10) and the fact that (u_0, v_0) is a weak supersolution of (1.1), we have

$$\begin{cases} -A\left(\int\limits_{\Omega} |\nabla u_0|^2 \, dx\right) \triangle u_0 \geq \lambda_1 \alpha(x) f(v_0) h(u_0) = -A\left(\int\limits_{\Omega} |\nabla u_1|^2 \, dx\right) \triangle u_1, \\ -B\left(\int\limits_{\Omega} |\nabla v_0|^2 \, dx\right) \triangle v_0 \geq \lambda_2 \beta(x) g(u_0) \tau(v_0) = -B\left(\int\limits_{\Omega} |\nabla v_1| \, dx\right) \triangle v_1, \end{cases}$$

and by Lemma 1, $u_0 \geq u_1$ and $v_0 \geq v_1$. Also, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$ and the monotonicity of f, h, g, and τ one has

$$\begin{aligned} -A\left(\int\limits_{\Omega} |\nabla u_1|^2 \, dx\right) \triangle u_1 =& \lambda_1 \alpha(x) f(v_0) h(u_0) \\ \geq& \lambda_1 \alpha(x) f(\underline{v}) h(\underline{u}) \geq -A\left(\int\limits_{\Omega} |\nabla \underline{u}|^2 \, dx\right) \triangle \underline{u}, \\ -B\left(\int\limits_{\Omega} |\nabla v_1|^2 \, dx\right) \triangle v_1 =& \lambda_2 \beta(x) g(u_0) \tau(v_0) \\ \geq& \lambda_2 \beta(x) g(\underline{u}) \tau(\underline{v}) \geq -B\left(\int\limits_{\Omega} |\nabla \underline{v}|^2 \, dx\right) \triangle \underline{v}, \end{aligned}$$

from which, according to Lemma 1, $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$, for u_2, v_2 we write

$$\begin{aligned} -A\left(\int\limits_{\Omega} |\nabla u_1|^2 \, dx\right) \triangle u_1 =& \lambda_1 \alpha(x) f(v_0) h(u_0) \\ \geq& \lambda_1 \alpha(x) f(v_1) h(u_1) = -A\left(\int\limits_{\Omega} |\nabla u_2|^2 \, dx\right) \triangle u_2, \\ -B\left(\int\limits_{\Omega} |\nabla v_1| \, dx\right) \triangle v_1 =& \lambda_2 \beta(x) g(u_0) \tau(v_0) \\ \geq& \lambda_2 \beta(x) g(u_1) \tau(v_1) = -B\left(\int\limits_{\Omega} |\nabla v_2|^2 \, dx\right) \triangle v_2, \end{aligned}$$

and then $u_1 \geq u_2$, $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned} -A\left(\int\limits_{\Omega} |\nabla u_2|^2 \, dx\right) \triangle u_2 =& \lambda_1 \alpha(x) f(v_1) h(u_1) \\ \geq& \lambda_1 \alpha(x) f(\underline{v}) h(\underline{u}) \geq -A\left(\int\limits_{\Omega} |\nabla \underline{u}|^2 \, dx\right) \triangle \underline{u}, \end{aligned}$$

$$
\begin{aligned}
-B\left(\int_{\Omega}|\nabla v_2|^2\,dx\right)\triangle v_2 =&\lambda_2\beta(x)\,g(u_1)\,\tau(v_1)\\
&\geq\lambda_2\beta(x)\,g(\underline{u})\,\tau(\underline{v})\geq -B\left(\int_{\Omega}|\nabla\underline{v}|^2\,dx\right)\triangle\underline{v}.
\end{aligned}
$$

Repeating this argument we get a bounded monotone sequence $\{(u_n,v_n)\}\subset E$ satisfying

$$
\begin{aligned}
\overline{u} =&u_0\geq u_1\geq u_2\geq\cdots\geq u_n\geq\cdots\geq\underline{u}>0,\\
\overline{v} =&v_0\geq v_1\geq v_2\geq\cdots\geq v_n\geq\cdots\geq\underline{v}>0.
\end{aligned}
$$

Using the continuity of the functions $f,h,g,$ and τ and the definition of the sequences $\{u_n\},\{v_n\}$, there exist constants $C_i>0$, $i=1,\ldots,4$, independent of n such that

$$
|f(v_{n-1})|\leq C_1,\quad |h(u_{n-1})|\ \leq C_2,\quad |g(u_{n-1})|\leq C_3 \tag{2.11}
$$

and

$$
|\tau(u_{n-1})|\leq C_4,\quad \text{for all } n.
$$

From (2.11), multiplying the first equation of (2.10) by u_n, integrating, using the Hölder inequality and Sobolev embedding we can show that

$$
\begin{aligned}
a_1\int_{\Omega}|\nabla u_n|^2\,dx \leq&A\left(\int_{\Omega}|\nabla u_n|^2\,dx\right)\int_{\Omega}|\nabla u_n|^2\,dx\\
=&\lambda_1\int_{\Omega}\alpha(x)\,f(v_{n-1})\,h(u_{n-1})\,u_n dx\\
\leq&\lambda_1\|\alpha\|_{\infty}\int_{\Omega}|f(v_{n-1})|\cdot|h(u_{n-1})|\cdot|u_n|\,dx\\
\leq&C_1C_2\|\alpha\|_{\infty}\lambda_1\int_{\Omega}\ |u_n|\,dx\\
\leq&C_5\|u_n\|_{H_0^1(\Omega)}\,.
\end{aligned}
$$

Then

$$
\|u_n\|_{H_0^1(\Omega)}\leq C_5,\quad \text{for all } n, \tag{2.12}
$$

where $C_5>0$ is a constant independent of n. Similarly, there exists $C_6>0$ independent of n such that

$$
\|v_n\|_{H_0^1(\Omega)}\leq C_6,\quad \text{for all } n. \tag{2.13}
$$

From (2.12) and (2.13), we infer that $\{(u_n,v_n)\}$ has a subsequence which weakly converges in $H_0^1(\Omega,\mathbb{R}^2)$ to a limit (u,v) with the properties $u\geq\underline{u}>0$ and $v\geq\underline{v}>0$. Being monotone and also using a standard regularity argument, $\{(u_n,v_n)\}$ converges itself to (u,v). Now, letting $n\to+\infty$ in (2.10), we deduce that (u,v) is a positive weak solution of system (1.1). The proof of theorem is now completed. $\square$

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ON n-ABSORBING IDEALS IN A LATTICE

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Abstract. Let L be a lattice, and let n be a positive integer. In this article, we introduce n-absorbing ideals in L. We give some properties of such ideals. We show that every n-absorbing ideal I of L has at most n minimal prime ideals. Also, we give some properties of 2-absorbing and weakly 2-absorbing ideals in L. In particular we show that in every non-zero distributive lattice L, 2-absorbing and weakly 2-absorbing ideals are equivalent.

1. Introduction

The concept of a 2-absorbing ideal in a commutative ring with identity, which is a generalization of prime ideals, was defined in [2] by Badawi. Anderson and Badawi [1] generalized the concept of a 2-absorbing ideal to an n-absorbing ideal. According to their definition, a proper ideal I of commutative ring R is called an n-absorbing ideal whenever $a_1a_2\cdots a_{n+1} \in I$, then there are n of the a_i's whose product is in I for every $a_1, \dots, a_{n+1} \in R$. Badawi and Darani [3] studied weakly 2-absorbing ideals which are generalizations of weakly prime ideals. The concepts of 2-absorbing, weakly 2-absorbing, 2-absorbing primary and weakly 2-absorbing primary elements in multiplicative lattices are studied in [10] and [5] as generalizations of prime and weakly prime elements. The concepts of φ-prime, φ-primary ideals are recently introduced in [4, 7], and generalizations of these are studied in [12]. Celikel et al. in [6] extended the concepts of 2-absorbing elements to φ-2-absorbing elements and investigated some characterizations in some special lattices. In [16], Wasadikar and Gaikwad introduced 2-absorbing and weakly 2-absorbing ideals in lattices and studied their properties.

This article is organized as follows. In Section 2, we review some basic notions and properties from lattice theory. In Section 3, we study some basic properties of

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2-absorbing and weakly 2-absorbing ideal in a lattice. For example in Proposition 3.4, we show that 2-absorbing and weakly 2-absorbing ideals are equivalent in a distributive lattice. Also, we show that in a distributive lattice, an ideal I is a 2-absorbing ideal if and only if $I_i \wedge I_j \subseteq I$ for some ideals I_1, I_2, I_3 of I where $I_1 \wedge I_2 \wedge I_3 \subseteq I$. In Section 4, we introduce the concept of an n-absorbing ideal in a lattice and give some basic properties of these ideals. For example, we show that an n-absorbing ideal is m-absorbing for every $m \geq n$. In a major result of this section (Proposition 4.5) we show that a n-absorbing ideal has at most n minimal prime ideals.

2. Preliminaries

In this section, we recall some concepts from lattice theory, see [8]. A partially ordered set $(L; \leq)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$. A nonempty subset I of a lattice L is called an ideal if it is a sublattice of L and $x \in I$ and $a \in L$ imply that $x \wedge a \in I$. An ideal I of L is proper if $I \neq L$. A proper ideal I of L is prime if $a \wedge b \in I$ implies that $a \in I$ or $b \in I$, and it is weakly prime if $0 \neq a \wedge b \in I$ implies that either $a \in I$ or $b \in I$. A prime ideal P of L is said to be a minimal prime ideal if there is no prime ideal which is properly contained in P. Also, a prime ideal P of L is said to be a minimal prime ideal belonging to an ideal I, if $I \subseteq P$ and there are no prime ideals strictly contained in P that contain I. If an ideal I of a lattice L is contained in a prime ideal P of a lattice L, then P contains a minimal prime ideal belonginig to I. Note that a minimal prime ideal belonging to the zero ideal of L is a minimal prime ideal of L. The set of minimal prime ideals belonging to the ideal I of L denoted by $\mathrm{Min}(I)$. Let I be an ideal of a distributive lattice L with 0, and let P be a prime ideal such that $P \supseteq I$. The prime ideal P is a element of $\mathrm{Min}(I)$ if and only if for each $x \in P$ there is a $y \notin P$ such that $x \wedge y \in I$. All these results can be found in [15].

For basic facts concerning the fractions of a lattice we refer to [9]. Let L be a non-empty distributive lattice with 0, and let S be a non-empty subset of L which is a complete sublattice. Define a binary relation $\sim_S$ on $L \times S$ by

$$(a, b) \sim_S (c, d) \Leftrightarrow (\exists t \in S)(a \wedge d) \wedge t = (b \wedge c) \wedge t.$$

The relation $\sim_S$ on $L \times S$ is an equivalence relation. The set of all equivalence classes of $\sim_S$ is denoted by $L/\sim_S$. In other words, $L/\sim_S = \{[(a, b)]_{\sim_S} : a \in L, b \in S\}$. Let $m = \bigwedge_{x \in S} x$, then $(a, m) \sim_S (b, m) \Leftrightarrow (a, m) \sim_{\{m\}} (b, m)$ and $L/\sim_S = L/\sim_{\{m\}}$

From now on, $L/\sim_S$ will be denoted by $S^{-1}L$ and it is called the fractions of L with respect to S. Any element $[(a, b)]_{\sim_S} \in S^{-1}L$ is shown by $\frac{a}{b}$. We can consider every S as a singleton $\{m\}$, where $m = \bigwedge_{x \in S} x$. Therefore, from now on we assume S to be the singleton $\{m\}$. So, we can write $\frac{a}{m}$ for $\frac{a}{b}$. For $\frac{a_1}{m}$ and $\frac{a_2}{m} \in S^{-1}L$, we have $\frac{a_1}{m} = \frac{a_2}{m}$ if and only if $a_1 \wedge m = a_2 \wedge m$. $(S^{-1}L, \leq)$ is a partially ordered set, where $\leq$ is defined as follows:

$$\frac{a}{m} \leq \frac{b}{m} \Leftrightarrow a \wedge m \leq b \wedge m.$$

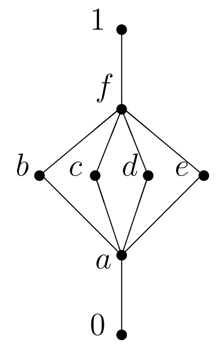

Figure 1.

The well-defined binary operations $\vee, \wedge : S^{-1}L \times S^{-1}L \to S^{-1}L$ are given by

$$\frac{a_1}{m} \wedge \frac{a_2}{m} = \frac{(a_1 \wedge a_2)}{m}$$

and

$$\frac{a_1}{m} \vee \frac{a_2}{m} = \frac{(a_1 \vee a_2)}{m}.$$

3. 2-Absorbing Ideals

In this section, we give some properties of 2-absorbing and weakly 2-absorbing ideals. We recall that from [16], a proper ideal I of lattice L is said to be a 2-absorbing ideal if for any $a_1, a_2, a_3 \in L$, $a_1 \wedge a_2 \wedge a_3 \in I$ implies $a_i \wedge a_j \in I$ for some $i, j \in \{1, 2, 3\}$ and weakly 2-absorbing ideal if for any $a_1, a_2, a_3 \in L$, $0 \neq a_1 \wedge a_2 \wedge a_3 \in I$ implies $a_i \wedge a_j \in I$ for some $i, j \in \{1, 2, 3\}$. Let I be a weakly 2-absorbing ideal of a lattice L and $a_1, a_2, a_3 \in L$. We say that (a_1, a_2, a_3) is a triple-zero of I if $a_1 \wedge a_2 \wedge a_3 = 0$ and for every $i, j \in \{1, 2, 3\}$, $a_i \wedge a_j \notin I$.

Example 3.1. Let $L = \{0, a, b, c, d, e, f, 1\}$ be a lattice, whose Hasse diagram is given in the Figure 1.

Consider the ideal $I = \downarrow a$. It is clear that I is a 2-absorbing ideal of L, but I is not a prime ideal of L.

Definition 3.1. Let I be an ideal of a lattice L. The radical of I, denoted by $\operatorname{Rad} I$, is the intersection all prime ideals P which contain I. If the set of prime ideals containing I is empty, then $\operatorname{Rad} I$ is defined to be L.

Proposition 3.1. *Every ideal I of a distributive lattice with* 0 *is the intersection of all prime ideals containing it, i.e.,* $\operatorname{Rad} I = I$.

Proof. See Page 64, Corollary 18 of [8]. $\square$

Proposition 3.2. *Let I be a* 2*-absorbing ideal of distributive lattice L. Then there are at most* 2 *prime ideals of L minimal over I.*

Proof. Suppose that $\operatorname{Min}(I)$ has at least there elements. Let P_1, P_2 be two distinct prime ideals of L that are minimal over I. Hence, there is a $x_1 \in P_1 \setminus P_2$ and a $x_2 \in P_2 \setminus P_1$. First we show that $x_1 \wedge x_2 \in I$. By Lemma 3.1 of [11], there is $c_1 \in L \setminus P_2$ and $c_2 \in L \setminus P_1$ such that $x_1 \wedge c_2 \in I$ and $x_2 \wedge c_1 \in I$. Then $x_1 \wedge c_2 \wedge x_2 \in I$ and $x_2 \wedge c_1 \wedge x_1 \in I$, which implies that $(c_1 \vee c_2) \wedge x_1 \wedge x_2 \in I$. Since I is a 2-absorbing ideal of L, we conclude that $(c_1 \vee c_2) \wedge x_1 \in I$ or $(c_1 \vee c_2) \wedge x_2 \in I$ or $x_1 \wedge x_2 \in I$. If $(c_1 \vee c_2) \wedge x_1 \in I$, since $I \subseteq P_2$ and P_2 is a prime ideal, we have $x_1 \in P_2$ or $c_1 \vee c_2 \in P_2$, which is a contradiction. Therefore, $(c_1 \vee c_2) \wedge x_1 \notin I$. Similarity, $(c_1 \vee c_2) \wedge x_2 \notin I$ and so, $x_1 \wedge x_2 \in I$.

Now, suppose that there is a $P_3 \in \operatorname{Min}(I)$ such that P_3 is neither P_1 nor P_2. Then we can chose $y_1 \in P_1 \setminus (P_2 \cup P_3)$, $y_2 \in P_2 \setminus (P_1 \cup P_3)$, and $y_3 \in P_3 \setminus (P_1 \cup P_2)$. By the previous argument $y_1 \wedge y_2 \in I$. Since $I \subseteq P_1 \cap P_2 \cap P_3$ and $y_1 \wedge y_2 \in I$, we conclude that either $y_1 \in P_3$ or $y_2 \in P_3$, which is a contradiction. Hence, $\operatorname{Min}(I)$ contains at most two elements. □

Corollary 3.1. *Let I be a 2-absorbing ideal of a distributive lattice L. If I is not a prime ideal of L, then $|\operatorname{Min}(I)| = 2$.*

Proof. Let $|\operatorname{Min}(I)| \neq 2$. Then by Proposition 3.2, $|\operatorname{Min}(I)| = 1$. Let P be a minimal prime ideal of L such that $I \subseteq P$. Therefore by Proposition 3.1, $P = \operatorname{Rad} I = I$ and so I is a prime ideal which is a contradiction. Thus $|\operatorname{Min}(I)| = 2$. □

Proposition 3.3. *Suppose that I is a proper ideal of a distributive lattice L. Then the following statements are equivalent:*

(1) *I is a 2-absorbing ideal of L;*
(2) *If $I_1 \wedge I_2 \wedge I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L, then $I_i \wedge I_j \subseteq I$ for some $i, j \in \{1, 2, 3\}$.*

Proof. (1) $\Rightarrow$ (2). If I is a prime ideal, it is clear. Now, let I be not a prime ideal, by Corollary 3.1, we conclude that $\operatorname{Min}(I) = \{P_1, P_2\}$. Then by Proposition 3.1, $I = P_1 \cap P_2$. Now, let $I_1 \wedge I_2 \wedge I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of L. Then, $I_1 \wedge I_2 \wedge I_3 \subseteq P_i$ for $i = 1, 2$ and so, there exists $1 \leq i_1, i_2 \leq 3$ such that $I_{i_1} \subseteq P_1$ and $I_{i_2} \subseteq P_2$. Therefore, $I_{i_1} \cap I_{i_2} \subseteq P_1 \cap P_2 = I$.

(2) $\Rightarrow$ (1). It is obvious. □

Proposition 3.4. *For every proper ideal $I \neq \{0\}$ in distributive lattice L, the following statements are equivalent:*

(1) *I is a 2-absorbing ideal;*
(2) *I is a weakly 2-absorbing ideal.*

Proof. (1) $\Rightarrow$ (2). It is evident.

(2) $\Rightarrow$ (1). Let I be a weakly 2-absorbing ideal of L that is not a 2-absorbing ideal. Then there exist $a_1, a_2, a_3 \in L$ such that $a_1 \wedge a_2 \wedge a_3 \in I$ and $a_i \wedge a_j \notin I$ for all $i \neq j \in \{1, 2, 3\}$. Consider $0 \neq a \in I$. Since $0 \neq (a_1 \vee a) \wedge (a_2 \vee a) \wedge (a_3 \vee a) \in I$, we

conclude that there exist $i, j \in \{1, 2, 3\}$ such that $(a_i \vee a) \wedge (a_j \vee a) \in I$. So $a_i \wedge a_j \in I$, for some $i, j \in \{1, 2, 3\}$, which is a contradiction. $\square$

For an ideal I of a lattice L and $a, b \in L$, we define $a \wedge b \wedge I = \{a \wedge b \wedge i : i \in I\}$.

Proposition 3.5. *Let I be a weakly 2-absorbing ideal of distributive lattice L, and let (a_1, a_2, a_3) be a triple-zero of I for some $a_1, a_2, a_3 \in L$. Then the following statements hold:*

(1) $a_1 \wedge a_2 \wedge I = a_2 \wedge a_3 \wedge I = a_1 \wedge a_3 \wedge I = \{0\}$;
(2) $a_1 \wedge I = a_2 \wedge I = a_3 \wedge I = \{0\}$.

Proof. (1) See Theorem 3.1 of [16].

(2) Suppose that $a_1 \wedge a \neq 0$ for some $a \in I$. Then, by (1), we have

$$\begin{aligned} a_1 \wedge (a_2 \vee a) \wedge (a_3 \vee a) =& a_1 \wedge \big((a_2 \wedge a_3) \vee a)\big) \\ =& (a_1 \wedge a_2 \wedge a_3) \vee (a_1 \wedge a) \\ =& 0 \vee (a_1 \wedge a) \\ =& a_1 \wedge a \\ \neq& 0. \end{aligned}$$

Then, by Proposition 3.4, we have $a_1 \wedge a_2 \in I$ or $a_1 \wedge a_3 \in I$ or $a_2 \wedge a_3 \in I$, which is a contradiction. Thus $a_1 \wedge I = \{0\}$. Similarly, $a_2 \wedge I = a_3 \wedge I = \{0\}$. $\square$

4. n-Absorbing Ideals

In this section, we introduce the concept of an n-absorbing ideal in a lattice and give some basic properties of them.

Definition 4.1. Let n be a positive integer. A proper ideal I of a lattice L is an n-absorbing ideal of L whenever $a_1 \wedge a_2 \wedge \cdots \wedge a_{n+1} \in I$, then there are n of the a_i's whose meet is in I for every $a_1, a_2, \ldots, a_{n+1} \in L$.

It is easy to see that if I is an n-absorbing ideal of L, then I is an m-absorbing ideal of L for all $m \geq n$. Also, a proper ideal I of L is n-absorbing if and only if whenever $a_1 \wedge a_2 \wedge \cdots \wedge a_m \in I$ for $a_1, \ldots, a_m \in I$ with $m \geq n$ then there are n of a_i's whose meet is in I.

Proposition 4.1. *If I_j is an n_j-absorbing ideal of L for each $1 \leq j \leq m$, then $\bigcap_{i=1}^m I_j$ is an n-absorbing ideal, where $n = \sum_{i=1}^m n_j$.*

Proof. Let $I_1, \ldots, I_m$ be proper ideals of L such that I_j is an n_j-absorbing and $k > n_1 + \cdots + n_m$. Suppose that $\bigwedge_{i=1}^k x_i \in \bigcap_{j=1}^m I_j$. Since for all j, I_j is n_j-absorbing ideal, a meet of n_j of these k elements belongs to I_j. Let the collection of those elements be denoted A_j and $A = \bigcup_{j=1}^m A_j$. Thus A has at most $n_1 + \cdots + n_m$ elements. Now since I_j is an ideal, the meet of all element of A must be in I_j for every $1 \leq j \leq m$. So $\bigcap_{j=1}^m I_j$ contains a meet of at most $n_1 + \cdots + n_m$ elements. Thus, the intersections of the I_j's is an $(n_1 + \cdots + n_m)$-absorbing ideal. $\square$

Proposition 4.2. *If $\{I_\lambda\}_{\lambda\in\Lambda}$ is a non-empty chain of n-absorbing ideals of L, then $\bigcap_{\lambda\in\Lambda} I_\lambda$ is an n-absorbing ideal.*

Proof. Let $a_1,\dots,a_{n+1}\in L$ such that $\bigwedge_{i=1}^{n+1} a_i\in J$ and $J=\bigcap_{\lambda\in\Lambda} I_\lambda$. Let $\widehat{a_i}=\bigwedge_{j\neq i} a_j$ and $\widehat{a_i}\notin J$ for all $1\le i\le n$. Then for each $1\le i\le n$, there exists an n-absorbing ideal I_{λ_i} such that $\widehat{a_i}\notin I_{\lambda_i}$. We may assume that $I_{\lambda_1}\subseteq\cdots\subseteq I_{\lambda_n}$. Consider $\mu\in\Lambda$. If $I_\mu\subseteq I_{\lambda_1}\subseteq\cdots\subseteq I_{\lambda_n}$, then $\widehat{a_i}\notin I_\mu$ for each $1\le i\le n$. Now since $\bigwedge_{i=1}^{n+1} a_i\in J$ and I_μ is an n-absorbing ideal of L, we have $\widehat{a_{n+1}}\in I_\mu$. If there exists $1\le j\le n$ such that $I_{\lambda_1}\subseteq\cdots\subseteq I_{\lambda_{j-1}}\subseteq I_\mu\subseteq I_{\lambda_j}\subseteq\cdots\subseteq I_{\lambda_n}$, then $\widehat{a_i}\in I_{\lambda_1}$ for each $1\le i\le n$. Now since $\bigwedge_{i=1}^{n+1} a_i\in I_{\lambda_1}$ and I_{λ_1} is an n-absorbing ideal of L, we conclude that $\widehat{a_{n+1}}\in I_{\lambda_1}$ and so $\widehat{a_{n+1}}\in I_\mu$ for every $\mu\in\Lambda$. Therefore, $\widehat{a_{n+1}}\in J$. □

Proposition 4.3. *If I is an ideal of distributive lattice L such that $L\setminus I$ is closed under meet of $n+1$ elements, then I is an n-absorbing ideal.*

Proof. Let $a_1,\dots,a_{n+1}\in L$ such that $\bigwedge_{i=1}^{n+1} a_i\in I$ and $\widehat{a_i}=\bigwedge_{j\neq i} a_j$ for each $1\le i\le n+1$. Assume that $\widehat{a_i}\notin I$ for each $1\le i\le n+1$. Since $L\setminus I$ is closed under the meet of $n+1$ elements, we have $\bigwedge_{i=1}^{n+1} a_i=\bigwedge_{i=1}^{n+1}\widehat{a_i}\in L\setminus I$ which is a contradiction. Which implies that I is an n-absorbing ideal. □

Let S be a non-empty subset of a lattice L. We say that S is a multiplicatively closed subset of L if $x\wedge y\in S$ for all x and y of S.

Proposition 4.4. *If S is a multiplicatively closed subset of L which does not meet the ideal I, then I is contained in an ideal M which is maximal with respect to the property of not meeting S and M is an n-absorbing ideal.*

Proof. Let $\mathcal{F}=\{J\mid \text{J is an ideal of L which does not meet } S \text{ and } I\subseteq J\}$. Since $I\in\mathcal{F}$, $\mathcal{F}\neq\emptyset$. Hence, by Zorn's Lemma, $(\mathcal{F},\subseteq)$ has a maximal element say M. We show that M is an n-absorbing ideal. Let $a_1,\dots,a_{n+1}\in L$ and for every $1\le i\le n+1$, $\widehat{a_i}=\bigwedge_{j\neq i} a_j\notin M$. Then $(M\vee\downarrow\widehat{a_i})\cap S\neq\emptyset$. Let $x_i\in(M\vee\downarrow\widehat{a_i})\cap S$ for each $1\le i\le n+1$. Since S is a multiplicatively closed subset of L, $\bigwedge_{i=1}^{n+1} x_i\in S$ and $\bigwedge_{i=1}^{n+1} x_i\in\bigcup_{i=1}^{n+1}(M\vee\downarrow\widehat{a_i})$. If $\bigwedge_{i=1}^{n+1} a_i\in M$, then $\bigwedge_{i=1}^{n+1} x_i\in M\cap S$ which is not true as $M\in\mathcal{F}$. Therefore, $\bigwedge_{i=1}^{n+1} a_i\notin M$ and so M is an n-absorbing ideal. □

Proposition 4.5. *Let I be an n-absorbing ideal of L. Then there are at most n prime ideals of L minimal over I.*

Proof. We may assume that $n\ge 2$, since an 1-absorbing ideal is a prime ideal. Suppose that $P_1,P_2,\dots,P_n,P_{n+1}$ are distinct prime ideals of L minimal over I. Thus for each $1\le i\le n$, there is an element x_i of $P_i\setminus\bigcup_{\substack{1\le k\le n+1\\ k\neq i}} P_k$. For each $1\le i\le n$, there is an element $c_i\in L\setminus P_i$ such that $x_i\wedge c_i\in I$ and hence $x_1\wedge\cdots\wedge x_n\wedge c_i\in I$. Therefore, $x_1\wedge x_2\wedge\cdots\wedge x_n\wedge(c_1\vee c_2\vee\cdots\vee c_n)\in I$. Since $x_i\in P_i\setminus\bigcup_{\substack{1\le k\le n+1\\ k\neq i}} P_k$ and $x_i\wedge c_i\in I\subseteq P_1\cap P_2\cap\cdots\cap P_n$ for each $1\le i\le n$, we conclude that $c_i\in(\bigcap_{\substack{1\le k\le n\\ k\neq i}} P_k)\setminus P_i$

for each $1 \leq i \leq n$, and thus $c_1 \vee c_2 \vee \cdots \vee c_n \notin P_i$ for each $1 \leq i \leq n$. Hence,

$$(c_1 \vee c_2 \vee \cdots \vee c_n) \wedge \bigwedge_{\substack{1 \leq k \leq n \\ k \neq i}} x_k \notin P_i,$$

and so, $(c_1 \vee c_2 \vee \cdots \vee c_n) \wedge \bigwedge_{\substack{1 \leq k \leq n \\ k \neq i}} x_k \notin I$ for each $1 \leq i \leq n$. Since I is an n-absorbing ideal of L, we conclude that $x_1 \wedge \cdots \wedge x_n \in I \subseteq P_{n+1}$. Then $x_i \in P_{n+1}$ for some $1 \leq i \leq n$, which is a contradiction. Hence there are at most n prime ideals of L minimal over I. $\square$

Let L be a distributive lattice and $S := \{m\} \subseteq L$. We recall from [9] that if I is an ideal of L, then $S^{-1}I$ is an ideal of $S^{-1}L$. Moreover, every ideal of $S^{-1}L$ can be represented as $S^{-1}I$, where I is an ideal of L.

Proposition 4.6. *Let I be an ideal of distributive lattice L and $S := \{m\} \subseteq L$. Then I is an n-absorbing ideal of L if and only if $S^{-1}I$ is an n-absorbing ideal of $S^{-1}L$.*

Proof. Let $\frac{a_1}{m}, \ldots, \frac{a_{n+1}}{m} \in S^{-1}L$ such that $\bigwedge_{i=1}^{n+1} \frac{a_i}{m} \in S^{-1}I$. Then $\frac{\bigwedge_{i=1}^{n+1} a_i}{m} \in S^{-1}I$ and so $\bigwedge_{i=1}^{n+1} a_i \in I$. Since I is a 2-absorbing ideal, we conclude that there exists an element i in $\{1, 2, \ldots, n+1\}$ such that $\widehat{a_i} \in I$, which implies that $\frac{\bigwedge a_j}{m} = \frac{\widehat{a_i}}{m} \in S^{-1}I$, where $\widehat{a_i} = \bigwedge_{j \neq i} a_j$. Hence $S^{-1}I$ is an 2-absorbing ideal of $S^{-1}L$.

Conversely, let $a_1, \ldots, a_{n+1} \in L$ such that $\bigwedge_{i=1}^{n+1} a_i \in I$. Then, $\bigwedge_{i=1}^{n+1} \frac{a_i}{m} = \frac{\bigwedge_{i=1}^{n+1} a_i}{m} \in S^{-1}I$. Since $S^{-1}I$ is an n-absorbing ideal of $S^{-1}L$, we infer that $\frac{\bigwedge_{i=1}^{n} a_i}{m} \in S^{-1}I$, and so $\bigwedge_{i=1}^{n} a_i \in I$. $\square$

Let I be an n-absorbing ideal of a lattice L. Then I is a m-absorbing ideal for all integers $m \geq n$. Now, we put $\omega_L(L) = 0$ and if I is an n-absorbing ideal for some $n \in \mathbb{N}$, then we define $\omega_L(I) = \min\{n \in \mathbb{N} \mid I \text{ is an } n\text{-absorbing ideal of } L\}$, otherwise, set $\omega_L(I) = \infty$. Thus for any ideal I of L, we have $\omega(I) \in \mathbb{N} \cup \{0, \infty\}$ with $\omega(I) = 1$ if and only if I is a prime ideal of L, and $\omega(I) = 0$ if and only if $I = L$.

Proposition 4.7. *Let $f : L \to M$ be a homomorphism of lattices. Then the following statements hold.*

(1) *If $f : L \to M$ is an epimorphism, and J is an n-absorbing ideal of M, then $f^{-1}(J)$ is an n-absorbing ideal of L. Moreover, $\omega_L(f^{-1}(J)) < \omega_M(J)$.*
(2) *If f is an isomorphism, and I is an n-absorbing ideal of L, then $f(I)$ is an n-absorbing ideal of M.*

Proof. (1). Let $x_1, x_2, \ldots, x_{n+1} \in L$ such that $x_1 \wedge \cdots \wedge x_{n+1} \in f^{-1}(J)$, then

$$f(x_1) \wedge \cdots \wedge f(x_{n+1}) = f(x_1 \wedge \cdots \wedge x_{n+1}) \in J.$$

Then there is a meet of n of the $f(x_i)$'s that is in J, which implies that there is a meet of n of the x_i's that is in $f^{-1}(J)$. Then $f^{-1}(J)$ is an n-absorbing ideal of L.
(2). It is straightforward. $\square$

Proposition 4.8. *Let I_1 be an m-absorbing ideal of a distributive bounded lattice L_1, and let I_2 be an n-absorbing ideal of a distributive bounded lattice L_2. Then $I_1 \times I_2$ is an $(m+n)$-absorbing ideal of the lattice $L_1 \times L_2$. Moreover $\omega_{L_1 \times L_2}(I_1 \times I_2) = \omega_{L_1}(I_1) + \omega_{L_2}(I_2)$.*

Proof. Let $L = L_1 \times L_2$. First we show that $I_1 \times I_2$ is an $(m+n)$-absorbing ideal. Let $\bigwedge_{i=1}^{n+m+1}(x_i, y_i) \in I_1 \times I_2$ for some $(x_1, y_1), \ldots, (x_{n+m+1}, y_{n+m+1}) \in I_1 \times I_2$. Since $\bigwedge_{i=1}^{n+m+1} x_i \in I_1$ and $\bigwedge_{i=1}^{n+m+1} y_i \in I_2$, we conclude that there exist

$$\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, n+m+1\},$$

such that $\bigwedge_{k=1}^{m} x_{i_k} \in I_1$ and $\bigwedge_{l=1}^{n} y_{j_l} \in I_2$, which implies that

$$(x_{i_1}, 1) \wedge \cdots \wedge (x_{i_m}, 1) \wedge (1, y_{j_1}) \wedge \cdots \wedge (1, y_{j_n}) = \left(\bigwedge_{k=1}^{m} x_{i_k}, \bigwedge_{l=1}^{n} y_{j_l}\right) \in I_1 \times I_2.$$

Now, we show that $\omega_L(I_1 \times I_2) = \omega_{L_1}(I_1) +_{L_2} (I_2)$. Let $\omega_{L_1}(I_1) = m < \infty$ and $\omega_{L_2}(I_2) = n < \infty$. Then, there are $x_1, \ldots, x_m \in L_1$ and $y_1, \ldots, y_n \in L_2$ such that satisfies the following statements:

- $x_1 \wedge \cdots \wedge x_m \in I_1$ and $y_1 \wedge \cdots \wedge y_n \in I_2$;
- for every $X \subsetneq \{x_1, \ldots, x_m\}$, $\bigwedge X \notin I_1$;
- for every $Y \subsetneq \{y_1, \ldots, y_n\}$, $\bigwedge Y \notin I_2$.

Thus,

$$(x_1, 1) \wedge \cdots \wedge (x_m, 1) \wedge (1, y_1) \wedge \cdots \wedge (1, y_n) = (x_1 \wedge \cdots \wedge x_m, y_1 \wedge \cdots \wedge y_n)$$

is an element of $I_1 \times I_2$, and also for proper subset S of

$$\{(x_1, 1), \ldots, (x_m, 1), (1, y_1), \ldots, (1, y_n)\},$$

$\bigwedge S \notin I_1 \times I_2$, which implies that $\omega_L(I_1 \times I_2) \geq m + n = \omega_{L_1}(I_1) + \omega_{L_2}(I_2)$.

Consider $N = m + n + 1$ and suppose that $(x_1, y_1), \ldots, (x_N, y_N) \in L$ such that $(x_1, y_1) \wedge \cdots \wedge (x_N, y_N) \in I_1 \times I_2$. Then $x_1 \wedge \cdots \wedge x_N \in I_1$ and $y_1 \wedge \cdots \wedge y_N \in I_2$, which implies that there are $\{i_1, \ldots, i_m\}, \{j_1, \ldots, j_n\} \subseteq \{1, \ldots, N\}$ such that $x_{i_1} \wedge \cdots \wedge x_{i_m} \in I_1$ and $y_{j_i} \wedge \cdots \wedge y_{j_m} \in I_2$. Let $K = \{i_1, \ldots, i_m\} \cup \{j_1, \ldots, j_n\}$, then $|K| \leq m + n$ and $\bigwedge_{k \in K}(x_k, y_k) \in I_1 \times I_2$, where $x_k = 1$ for every $k \notin \{i_1, \ldots, i_m\}$ and $y_k = 1$ for every $k \notin \{j_1, \ldots, j_n\}$. Hence, $\omega_L(I_1 \times I_2) \leq m + n = \omega_{L_1}(I_1) + \omega_{L_2}(I_2)$. Therefore, $\omega_L(I_1 \times I_2) = \omega_{L_1}(I_1) +_{L_2} (I_2)$. □

Corollary 4.1. *Let I_k be an ideal of a lattice L_k for each integer $1 \leq k \leq n$, and let $L = L_1 \times \cdots \times L_n$. Then $\omega_L(I_1 \times \cdots \times L_n) = \omega_{L_1}(I_1) + \cdots + \omega_{L_n}(I_n)$.*

Proof. By induction on n and Proposition 4.8, it is clear. □

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ON CO-FILTERS IN SEMIGROUPS WITH APARTNESS

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Abstract. The logical environment of this research is the Intuitionistic Logic and principled-philosophical orientation of the Bishop's Constructive Mathematics. In this paper, basing our consideration on the sets with the apartness relation, we analyze the lattices of all co-filters of an ordered semigroup under a co-quasiorder as a continuation of our article [19]. We prove a number of results related to co-filters in a semigroup with apartness and the lattice of all co-filters of such semigroups.

1. Introduction

The setting of this research is Bishop's constructive mathematics [2–5, 11, 20], mathematics developed on the Intuitionistic logic [20]. In our text [19] we talked about co-ideals and co-filters in sets with apartness ordered under a co-quasiorder relation (co-order relation). In this text, the word will be about the co-filters of the semigroups with apartness ordered under a co-quasiorder relation (co-order relation).

We refer the reader to look at our previously published texts [6, 7, 12, 16, 18] for more details on semigroups with apartness. In these articles, the concept of co-order relations and the concept of co-quasiorder relations in such semigroups have been introduced and analyzed. Additionally, these relations are left and right cancellative with respect to apartness.

In this text, we are interested in the left and right classes of co-quasiorder (co-order) relation generated by a subset of a semigroup with apartness ordered under the co-quasiorder (co-order) relation. Concepts of co-quasiorder and co-order on sets with apartness are investigated by this author in many of his articles. (See for example [14–18]).

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Let $(S, =, \neq)$ be a set with apartness. Any strongly extensional total function

$$w : S \times S \ni (x, y) \mapsto xy \in S$$

is an internal binary operation in S. If we are speaking in the language of classical algebra, it can be said that the function w is left and right cancellative with respect to apartness relation

$$(\forall x, y, z \in S)((xz \neq yz \vee zx \neq zy) \Rightarrow x \neq y).$$

System $S = ((S, =, \neq), w)$ is called a grupoid with apartness. Additionally, if the operation w is associative, then the system S is a semigrop with apartness. A relation $\nleqslant$ in S is a co-quasiorder relation (co-order relation) in S if it is consistent $\nleqslant \subseteq \neq$, co-transitive $\nleqslant \subseteq \nleqslant * \nleqslant$ (and $\neq \subseteq \nleqslant \cup \nleqslant^{-1}$) and if the following holds:

$$(\forall x, y, z \in S)((xz \nleqslant yz \vee zx \nleqslant zy) \Rightarrow x \nleqslant y).$$

In this text we will accept the following assumption

$$(\forall x, y \in S)(\neg(x \nleqslant xy) \wedge \neg(y \nleqslant xy)). \tag{1.1}$$

The usual term used to indicate this kind of relations is that the relation $' \nleqslant '$ is a *negatively defined ordered relation* [10].

2. Co-Filters of Semigroups with Apartness

Juhasz and Vernitski [10] expressed a statement: "There was no systematic study of filters in semigroups". By reviewing the available literature on the internet, we found a very small number of texts in which the filters were researched in ordered semigroups: for example [1, 9, 10]. In older semigroup theory literature, filters (also known as faces and under several other names) were introduced as subsemigroups whose complement is an ideal. Some results concerning such filters were obtained in the 1960s and 1970s, see, for instance, [1,8]. Our intention is to introduce and analyze the concept of co-filters in semigroup with apartness ordered under a co-quasiorder (co-order). The concept of co-filters as a substructure in such semigroups is a dual of the concept of filters in the classical theory of ordered semigroups.

We will start this section with the following statement.

Proposition 2.1. *Let $\nleqslant$ be a co-quasiorder on a semigroup S. Then the left class $L(a)$ and the right class $R(b)$ are strongly extensional subsets of S such that $a \rhd L(a)$ and $b \rhd R(b)$ for any $a, b \in S$. Moreover, the following implications hold:*
(0) *classes $L(a)$ and $R(b)$ are co-subsemigroups of semigroup S;*
(1) *$y \in L(a) \wedge x \in S \Rightarrow x \in L(a) \vee x \nleqslant y$;*
(2) *$y \in R(b) \wedge x \in S \Rightarrow x \in R(b) \vee y \nleqslant x$;*
(3) *$a \nleqslant b \Rightarrow L(a) \cup R(b) = S$;*
(4) *$a \neq b \Rightarrow b \in L(a) \vee a \in R(b)$ if $\nleqslant$ is an co-order relation.*

Proof. For illustration we will show how some of these claims are proven. Other assertions are proved by analogy.

Let $x \in S$ and $y \in L(a)$. Then, by co-transitivity of $\nleqslant$, we have $a \nleqslant x$ or $x \nleqslant y$. So, $x \in L(a)$ or, by consistency of $\nleqslant$, $x \neq y$. Thus, $L(a)$ is a strongly extensional subset of S and $a \rhd L(a)$ holds.

(0) The subset $L(a)$ is a co-subsemigroup of semigroup S. Indeed, suppose $x, y \in L(a)$. Then from $a \nleqslant xy$ follows $a \nleqslant x$ or $x \nleqslant xy$. Thus $x \in L(a)$ by hypothesis (1.1). Also, from $a \nleqslant xy$ we have $a \nleqslant y$ or $y \nleqslant xy$. Thus $y \in L(a)$ by hypothesis (1.1) too.

(1) From $y \in L(a) \wedge x \in S$, i.e., from $a \nleqslant y \wedge x \in S$ we have $a \nleqslant x \vee x \nleqslant y$. Then $x \in L(a) \vee x \nleqslant y$.

In a similar manner we can prove that $R(b)$ is a strongly extensional subsemigroup of S with $b \rhd R(b)$ and that the implication $y \in R(b) \wedge x \in S \Rightarrow x \in R(b) \vee y \nleqslant x$ holds. □

The concept of co-filters in an ordered semigroup S is introduced by the following definition.

Definition 2.1. For subset G of S we say that it is a co-filter in S if

$$(\forall x, y \in S)(xy \in G \Rightarrow (x \in G \vee y \in G))$$

and

$$(\forall x, y \in S)(y \in G \wedge x \in S \Rightarrow (x \in G \vee x \nleqslant y)).$$

According to the first property, the co-filter is a co-subsemigroup in a semigroup S. From another property, immediately follows that a co-filter in semigroup S is a strongly extensional subset in S. So, the subset $L(a)$ is a *principal co-filter* of S generated by the element a. In addition, the sets $\emptyset$ and S are trivial co-filters of S.

Remark 2.1. Since $\nleqslant$ is a negatively defined co-quasiorder in S, for any co-filter G in S we have $xy \in G \Rightarrow (x \in G \wedge y \in G)$. Indeed. Let for elements x and y holds $xy \in G$. Thus $xy \in G \Rightarrow (x \in G \vee x \nleqslant xy)$ and $xy \in G \Rightarrow (y \in G \vee y \nleqslant xy)$. According the hypothesis (1.1), we finally have $x \in G$ and $y \in G$.

In the following statement we show that a strongly complement of a co-filter is a filter.

Theorem 2.1. *If G is a co-filter of ordered semigroup S, then $G^{\rhd}$ is a filter in ordered semigroup S under quasi-order $\nleqslant^{\rhd}$.*

Proof. Let $x \in G^{\rhd}$ and $y \in G^{\rhd}$ and let u be an arbitrary element in G. By strongly extensionality of G follows $u \neq xy$ or $xy \in G$. Since the second option leads to contradiction, we conclude $xy \neq u \in G$. So, the subset $G^{\rhd}$ is a subsemigroup in S.

Let G be a co-filter of S. Then $\nleqslant^{\rhd}$ is a quasi-order on semigroup S. Suppose that $x \in G^{\lhd}$ and $x \nleqslant^{\lhd} y$. Let u be an arbitrary element of G. Thus, from the implication $u \in G \Rightarrow x \in G \vee x \nleqslant u$ follows $x \nleqslant u$ because $x \rhd G$. Further on, by co-transitivity of $\nleqslant$, we have $x \nleqslant y \vee y \nleqslant u$. Hence, we conclude $y \neq u \in G$ because $x \nleqslant^{\triangleright} y$. Finally, $y \in G^{\triangleright}$. So, the subset $G^{\triangleright}$ is a filter in S. □

Theorem 2.2. *Let $f : (S, \nleqslant_S) \Rightarrow (T, \nleqslant_T)$ be a reverse isotone homomorphism between two ordered semigroups with apartness under co-quasiordereds. If G is a co-filter in T, then the set $f^{-1}(G) = \{a \in S : f(a) \in G\}$ is a co-filter in S.*

Proof. Let $x, y \in S$ be arbitrary elements such that $xy \in f^{-1}(G)$. Then $f(xy) \in G$ and $f(x)f(y) \in G$. Thus $f(x) \in G$ or $f(y) \in G$. Therefore, $x \in f^{-1}(G)$ or $y \in f^{-1}(G)$ and the subset $f^{-1}(G)$ is a cosubsemigroup of semigroup S.

Let $y \in f^{-1}(G)$ and $x \in S$ be arbitrary elements. Thus, $f(y) \in G$ and $f(x) \in T$. Hence $f(x) \in G$ or $f(x) \nleqslant_T f(y)$. Therefore, we have $x \in f^{-1}(G)$ or $x \nleqslant_S y$ because f is a reverse isotone homomorphism.

Finally, the subset $f^{-1}(G)$ is a co-filter in S. □

In following text, we represent some properties of the union of co-filters. In the following theorem, we prove that the union of any family of co-filters is a co-filter again.

Theorem 2.3. *If $\{G_j\}_{j \in J}$ be a family of co-filters in S, then $\bigcup_{j \in J} G_j$ is a co-filter too.*

Proof. If $xy \in \bigcup_{j \in J} G_j$, then there exists an index $j \in J$ such that $xy \in G_j$. Thus $x \in G_j$ or $y \in G_j$. So, $x \in \bigcup_{j \in J} G_j$ or $y \in \bigcup_{j \in J} G_j$. Therefore, $\bigcup_{j \in J} G_j$ is a co-subsemigroup in S. Let $y \in \bigcup_{j \in J} G_j$ and $x \in S$. Thus, there exists an index $j \in J$ such that $y \in G_j$. Hence, by definition of co-filter, we have $x \in G_j$ or $x \nleqslant y$. Finally, we conclude $x \in \bigcup_{j \in J} G_j$ or $x \nleqslant y$. Therefore, the union $\bigcup_{j \in J} G_j$ is a co-filter in S too. □

Corollary 2.1. *Let S is an ordered semigroup with apartness under co-quasiorder $\nleqslant$. Then the family $\mathfrak{G}_S$ of all co-filters in S forms a join semi-lattice. The greatest element in this semi-lattice is S.*

Let T be a subset of a semigroup S. Then, by previous theorem, $T^R = \bigcup_{t \in T} L(t)$ is a co-filter in S.

Definition 2.2. For a subset T of a semigroup S the co-filter T^R is called ordered co-filter generated by subset T.

Particularly, for each element $a \in S$ the set $\{a\}^R$ is the principal ordered co-filter generated by element a and, in addition, $\{a\}^R = L(a)$ holds.

Theorem 2.4. *If $\{G_j\}_{j \in J}$ be a family of ordered co-filters in semigroup S, then $\bigcup_{j \in J} G_j$ is an ordered co-filter too.*

Proof. Let $\{G_j\}_{j \in J}$ be a family of ordered co-filters in semigroup S. Then for any $j \in J$ there exists a subset T_j of S such that $G_j = T_j^R$. Since $(\bigcup_{j \in J} T_j)^R = \bigcup_{j \in J} T_j^R$ holds, it is directly verified that $\bigcup_{j \in J} G_j$ is an ordered co-filter in S generated by subset $\bigcup_{j \in J} T_j$. □

Corollary 2.2. *The family $\mathfrak{O}_S$ of all ordered co-filters form join semi-lattice.*

In what follows we will represent our findings concerning the intersection of co-filters in semigroup with apartness.

Theorem 2.5. *If G_1 and G_2 are co-filters in a semigroup S, then the intersection $G_1 \cap G_2$ is also co-filter in S.*

Proof. Let x and y be arbitrary element of S such that $xy \in G_1 \cap G_2$. It means $xy \in G_1$ and $xy \in G_2$. Thus $x \in G_1 \wedge y \in G_1$ and $x \in G_2 \wedge y \in G_2$ by Remark 2.1. So, we have $x \in G_1 \cap G_2 \wedge y \in G_1 \cap G_2$. Therefore, the intersection $G_1 \cap G_2$ is a co-subsemigroup in S.

From $y \in G_1 \cap G_2$ and $x \in S$, i.e., from $y \in G_1 \wedge y \in G_2 \wedge x \in S$ follows $x \in G_1 \vee x \nleq y$ and $x \in G_2 \vee x \nleq y$. Thus, $x \in G_1 \cap G_2 \vee x \nleq y$.

So, the intersection $G_1 \cap G_2$ is a co-filter in S. □

Corollary 2.3. *The family $\mathfrak{G}_S$ of all co-filters in S forms a lattice. The smallest and the greatest elements in this lattice are the empty set $\emptyset$ and S.*

Remark 2.2. Let us note that if G_1 and G_2 be two order co-filters, than the intersection $G_1 \cap G_2$ is not an ordered co-filter in general case. For example, the intersection of two ordered co-filters $G_1 = A^R$ and $G_2 = B^R$ is an ordered co-filter if the following holds

$$(\forall a \in A)(\forall b \in B)(\exists c \in A \cap B)(c \nleq^{\triangleright} a \wedge c \nleq^{\triangleright} b).$$

Indeed, for arbitrary elements $y \in A^R \cap B^R$ and $x \in S$ there exist elements $a \in A$ and $b \in B$ such that $y \nleq a$ and $y \nleq b$. There exists an element $c \in A \cap B$ such that $c \nleq^{\triangleright} a$ and $c \nleq^{\triangleright} b$ by hypothesis. Thus, we have $y \nleq c$. Further, from this follows $y \nleq x$ or $x \nleq c$ and finally we have $y \nleq x$ or $x \in (A \cap B)^R$.

The previous analysis is the motivation for the introduction of the following definition.

Definition 2.3. An ordered semigroup S is called directed if the following holds

$$(\forall a, b \in S)(\exists c \in S)(c \nleq^{\triangleright} a \wedge c \nleq^{\triangleright} b).$$

Corollary 2.4. *The family $\mathfrak{O}_S$ of all ordered co-filters in directed semigroup S forms lattice.*

Proof. Let a and b be arbitrary elements of semigroup S. Then there exists an element $c \in S$ such that $c \nleq^{\triangleright} a$ and $c \nleq^{\triangleright} b$. Thus, $L(c) \subseteq L(a) \cap L(b)$. Indeed. Suppose $c \nleq s$. Thus $c \nleq a \vee a \nleq s$ and $c \nleq b \vee b \nleq s$. Then $a \nleq s$ and $b \nleq s$. Therefore, $L(c) \subseteq L(a) \cap L(b)$. □

Corollary 2.5. *The family of all principal co-filters in directed band S forms lattice. Every finitely generated ordered co-filter is a principal co-filter.*

Proof. In a directed semigroup S for any elements a and b there exists an element $c \in S$ such that $L(c) \subseteq L(a) \cap L(b)$, by previous corollary.

Since $' \nleqslant '$ is a negatively defined relation in semigroup S with apartness, we have $L(a) \cup L(b) \subseteq L(ab)$ for any elements $a, b \in S$. Let s be element in S such that $ab \nleqslant s$. Thus, $ab \nleqslant as \vee as \nleqslant ss \vee ss \nleqslant s$ and $b \nleqslant s \vee a \nleqslant s \vee ss \nleqslant s$. Therefore, $b \nleqslant s \vee a \nleqslant s$ because $s = ss$. Finally, we have $L(ab) \subseteq L(a) \cup L(b)$ and $L(ab) = L(a) \cup L(b)$. □

As it has already been said, for each element $a \in S$ the set $\{a\}_R = L(a)$ is the principal co-filter generated by a. If T is a finite set then, by Theorem 2.5, $T_R = \bigcap_{a \in A} L(a)$ is a co-filter in S also. This is the motive to introduce the following concept.

Definition 2.4. Let $T \subseteq S$ be a finite subset of semigroup with apartness. Subsets of the form $T_R = \{z \in S : (\forall t \in T)(t \nleqslant z)\} = \bigcap_{a \in A} L(a)$ are a normal co-filter in S.

Remark 2.3. Let G be a normal co-filter of semigroup S ordered under co-quasiorder $\nleqslant$. Then there exists a finitely subset T of S such that $G = T_R$. If z is an arbitrary element of G, we have $(\forall t \in T)(z \nleqslant t)$ and $z \rhd T$ because $\nleqslant$ is a consistent relation. So, we have $(\forall z \in G)(z \rhd T)$.

Theorem 2.6. *If $\{G_j\}_{j \in J}$ is a finitely family of normal co-filters in semigroup S, then $\bigcap_{j \in J} G_j$ is a normal co-filter too.*

Proof. Let $\{G_j\}_{j \in J}$ be a finitely family of normal co-filters in semigroup S. Then for each $j \in J$ there exists a subset T_j of S such that $G_j = (T_j)_R$. Since $\bigcap_{j \in J}(T_j)_R = (\bigcup_{j \in J} R_j)_R$ holds, we conclude that the intersection $\bigcap_{j \in J} G_j$ is a normal co-filter of S. □

Corollary 2.6. *The family $\mathfrak{N}_{\mathfrak{S}}$ of all normal co-filters forms meet semi-lattice.*

Final observation. As one of the answers to the question: "Why should the content of this text be mathematically acceptable?" we can offer the next reflection. Why is a text that contains thoughts about some algebraic concepts acceptable by the mathematical community? According to the usual standards, the text is mathematically acceptable because the author(s) expound and prove in an acceptable way some new logical possibility of the mentioned algebraic objects. To the first question raised above, we need to offer a completely identical answer according to the usual standards, by our opinion.

Why should this be interesting for a significant number of mathematicians? This is another question that naturally appears. Well, it does not need to be. The questions and answers to the observation and expression of logical possibilities in the constructive algebra are interesting only to interested logicians and mathematicians.

Many aspects of constructive mathematics are not just logical hygiene: avoid indirect proofs in favor of explicit constructions, detect and eliminate needless uses of the axiom of choice and so on. Of course, constructivism goes deeper than that.

By accepting the non-existence of the TND principle, it is possible to have the multilayered properties of algebraic objects and processes with them. In this article, this two-stratification is shown on the example of filters and co-filters in semi-groups with apartness.

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ON BERNSTEIN-TYPE INEQUALITIES FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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ABSTRACT. In this paper, we shall use a parameter β and obtain some Bernstein-type inequalities for rational functions with prescribed poles which generalize the results of Qasim and Liman and Li, Mohapatra and Rodriguez and others.

1. INTRODUCTION

Let $\mathbb{P}_n$ denote the class of all complex polynomials of degree at most n. If $P \in \mathbb{P}_n$, then concerning the estimate of $|P'(z)|$ on $|z| = 1$, we have

$$|P'(z)| \leq n \sup_{|z|=1} |P(z)|. \tag{1.1}$$

Inequality (1.1) is a famous result due to Bernstein [2], who proved it in 1912. Later, in 1969 (see [10]), Malik improved the above inequality (1.1) and established that if $P \in \mathbb{P}_n$, then for $|z| = 1$, we have

$$|P'(z)| + |Q'(z)| \leq n \sup_{|z|=1} |P(z)|, \tag{1.2}$$

where $Q(z) = z^n \overline{P(\frac{1}{\bar{z}})}$.

It is worth mentioning that equality holds in (1.1) if and only if $P(z)$ has all its zeros at the origin, so it is natural to seek improvements under appropriate assumption on the zeros of $P(z)$. If we restrict ourselves to the class of polynomials

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$P(z)$ having no zeros in $|z| < 1$, then (1.1) can be replaced by

$$\sup_{|z|=1} |P'(z)| \leq \frac{n}{2} \sup_{|z|=1} |P(z)|, \tag{1.3}$$

whereas if $P(z)$ has no zeros in $|z| > 1$, then

$$\sup_{|z|=1} |P'(z)| \geq \frac{n}{2} \sup_{|z|=1} |P(z)|. \tag{1.4}$$

Inequality (1.3) was conjectured by Erdös and later verified by Lax [9], whereas inequality (1.4) is due to Turán [12]. Li, Mohapatra and Rodriguez [14] gave a new perspective to the above inequalities and extended them to rational functions with prescribed poles. Essentially, in the inequalities referred to, they replaced the polynomial $P(z)$ by a rational function $r(z)$ with prescribed poles $a_1, a_2, \dots, a_n$ and z^n by a Blaschke product $B(z)$. Before proceeding towards their results, let us introduce the set of rational functions involved.

For $a_j \in \mathbb{C}$ with $j = 1, 2, \dots, n$, let

$$W(z) := \prod_{j=1}^{n} (z - a_j)$$

and let

$$B(z) := \prod_{j=1}^{n} \left(\frac{1 - \overline{a}_j z}{z - a_j} \right), \quad \mathcal{R}_n := \mathcal{R}_n(a_1, a_2, \dots, a_n) = \left\{ \frac{P(z)}{W(z)} : \ P \in \mathbb{P}_n \right\}.$$

Then $\mathcal{R}_n$ is the set of rational functions with poles $a_1, a_2, \dots, a_n$ at most and with finite limit at ∞. Note that $B(z) \in \mathcal{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. For $r(z) = \frac{P(z)}{W(z)} \in \mathcal{R}_n$, the conjugate transpose r^* of r is defined by $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$. The rational function $r \in \mathcal{R}_n$ is called self-inversive if $r^*(z) = \lambda r(z)$ for some λ with $|\lambda| = 1$.

As an extension of (1.2) to rational functions, Li, Mohapatra and Rodriguez [14, Theorem 2] showed that if $r \in \mathcal{R}_n$, then

$$|r'(z)| + |(r^*(z))'| \leq |B'(z)| \sup_{|z|=1} |r(z)|, \quad \text{for } |z| = 1. \tag{1.5}$$

Equality holds in (1.5) for $r(z) = \alpha B(z)$ with $|\alpha| = 1$.

For $r \in \mathcal{R}_n$ to be self-inversive, Li, Mohapatra and Rodriguez [14, Corollary 4] proved that

$$|r'(z)| \leq \frac{|B'(z)|}{2} \sup_{|z|=1} |r(z)|. \tag{1.6}$$

In the same paper, Li, Mohapatra and Rodriguez [14] showed that inequality (1.6) also holds for rational functions $r \in \mathcal{R}_n$ having no zeros in $|z| < 1$ with prescribed poles. The latest development of further results along this line can be found in the monographs and papers [3–5, 7, 8, 11].

More recently, Qasim and Liman [6] proved several results by considering a specialized class of rational functions $r(t(z))$, defined by

$$(r \circ t)(z) = r(t(z)) := \frac{P(t(z))}{W(t(z))},$$

where $t(z)$ is a polynomial of degree m and $r \in \mathcal{R}_n$, so that $r(t(z)) \in \mathcal{R}_{mn}$, and

$$W(t(z)) = \prod_{j=1}^{mn} \Big(z - a_j\Big).$$

Also the Blaschke product is given by

$$B(z) = \frac{\Big(W(t(z))\Big)^*}{W(t(z))} = \frac{z^{mn}\overline{W(t(\frac{1}{\overline{z}}))}}{W(t(z))} = \prod_{j=1}^{mn} \Big(\frac{1 - \overline{a}_j z}{z - a_j}\Big).$$

Assume that the mn poles of $r(t(z))$ are denoted by a_j, $j = 1, 2, \ldots, mn$, and $|a_j| > 1$. They proved the following Bernstein-type inequality for rational functions $r(t(z)) \in \mathcal{R}_{mn}$ with restricted zeros.

Theorem 1.1. *If $r(t(z)) \in \mathcal{R}_{mn}$ and all the mn zeros of $r(t(z))$ lie in $|z| \geq 1$, then for $|z| = 1$*

$$|r'(t(z))| \leq \frac{|B'(z)|}{2m\mu} \sup_{|z|=1} |r(t(z))|, \tag{1.7}$$

where $t(z)$ has all its zeros in $|z| \leq 1$ and $\mu = \inf_{|z|=1} |t(z)|$.

2. Lemmas

For the proofs of our theorems we need the following lemmas.

Lemma 2.1. *If $r \in \mathcal{R}_n$ has n zeros all lie in $|z| \leq 1$, then*

$$|r'(z)| \geq \frac{1}{2}|B'(z)||r(z)|, \quad \textit{for } |z| = 1.$$

The above lemma is due to Li, Mohapatra and Rodriguez [14].

Lemma 2.2. *Let A and B be any two complex numbers, then*

(i) if $|A| \geq |B|$ and $B \neq 0$, then $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$;

(ii) conversely, if $A \neq \delta B$ for all complex numbers δ satisfying $|\delta| < 1$, then $|A| \geq |B|$.

The above lemma is due to Li [13].

Lemma 2.3. *If $r(t(z))$, $s(t(z)) \in \mathcal{R}_{mn}$ and all the mn zeros of $s(t(z))$ lie in $|z| \leq 1$ and $|r(t(z))| \leq |s(t(z))|$ for $|z| = 1$. Then for every $\beta \in \mathbb{C}$, with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq \left|B(z)s'(t(z))t'(z) + \frac{\beta}{2}B'(z)s(t(z))\right|. \tag{2.1}$$

The result is sharp and equality holds in (2.1) for $r(t(z)) = \alpha s(t(z))$, with $|\alpha| = 1$.

Proof. The proof of this lemma is identical to the proof of Theorem 3.2 of Li [13], but for the sake of completeness we give the brief outlines of its proof. First assume that no zero of $s(t(z))$ are on the unit circle $|z| = 1$ and therefore, all the mn zeros of $s(t(z))$ are in $|z| < 1$. By Rouche's theorem, the rational function $\lambda r(t(z)) + s(t(z))$ has all its zeros in $|z| < 1$ for $|\lambda| < 1$ and has no poles in $|z| \leq 1$. On applying Lemma 2.1 to $\lambda r(t(z)) + s(t(z))$, we get on $|z| = 1$

$$2|B(z)|\left|\lambda(r(t(z)))' + (s(t(z)))'\right| \geq |B'(z)|\left|\lambda r(t(z)) + s(t(z))\right|. \tag{2.2}$$

Now, note that $B'(z) \neq 0$ (e.g. see formula (14) in [14]). So, the right hand side of (2.2) is non zero. Thus, by using (i) of Lemma 2.2, we have for all $\beta \in \mathbb{C}$, with $|\beta| < 1$,

$$2B(z)\Big(\lambda r'(t(z))t'(z) + s'(t(z))t'(z)\Big) \neq -\beta B'(z)\Big(\lambda r(t(z)) + s(t(z))\Big),$$

for $|z| = 1$. Equivalently, for $|z| = 1$,

$$\lambda\Big(2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z))\Big) \neq -\Big(2B(z)s'(t(z))t'(z) + \beta B'(z)s(t(z))\Big),$$

for $|\lambda| < 1$ and $|\beta| < 1$. Using (ii) of Lemma 2.2, we have

$$|2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z))| \leq |2B(z)s'(t(z))t'(z) + \beta B'(z)s(t(z))| \tag{2.3}$$

for $|z| = 1$ and $|\beta| < 1$. Now, using the continuity in zeros and β, we can obtain the (2.3), when some zeros of $s(t(z))$ lie on the unit circle $|z| = 1$ and $|\beta| \leq 1$. ☐

Applying Lemma 2.3 to the rational function $r(t(z))$ and $B(z)\sup_{|z|=1}|r(t(z))|$, we get the following.

Lemma 2.4. *If $r(t(z)) \in \mathcal{R}_{mn}$, then for all $\beta \in \mathbb{C}$, with $|\beta| \leq 1$ and $|z| = 1$, we have*

$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq |B(z)|\left|1 + \frac{\beta}{2}\right| \sup_{|z|=1}|r(t(z))|.$$

Lemma 2.5. *If $P(z)$ is a polynomial of degree n having all zeros in $|z| \leq 1$, then*

$$\inf_{|z|=1}|P'(z)| \geq n \inf_{|z|=1}|P(z)|. \tag{2.4}$$

The result is best possible and equality in (2.4) holds for polynomials, having all zeros at the origin.

The above lemma is due to Aziz and Dawood [1].

3. Main Results

In this note, we shall use a parameter β and obtain generalizations of (1.5), (1.6) and (1.7). We shall always assume that all the poles of $r(t(z)) \in \mathcal{R}_{mn}$ lie in $|z| > 1$.

Theorem 3.1. *If $r(t(z)) \in \mathcal{R}_{mn}$ and $|z| = 1$, then for every β, with $|\beta| \leq 1$,*

$$\begin{aligned}&\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| + \left|B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2}B'(z)(r(t(z)))^*\right| \\ &\leq |B'(z)|\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \sup_{|z|=1} |r(t(z))|. \end{aligned} \tag{3.1}$$

Proof. Let $M := \sup_{|z|=1} |r(t(z))|$. Therefore, for every $\lambda \in \mathbb{C}$, with $|\lambda| > 1$, $|r(t(z))| < |\lambda M B(z)|$ for $|z| = 1$.

By Rouche's theorem, all the mn zeros of $G(z) = r(t(z)) + \lambda M B(z)$ lie in $|z| < 1$. If $H(z) = B(z)\overline{G(\frac{1}{\bar{z}})}$, then $|H(z)| = |G(z)|$ for $|z| = 1$ and hence, for any γ, with $|\gamma| < 1$, the rational function $\gamma H(z) + G(z)$ has all mn zeros in $|z| < 1$. By applying Lemma 2.1 to $\gamma H(z) + G(z)$, we have

$$2\Big|B(z)\Big(\gamma H'(z) + G'(z)\Big)\Big| \geq |B'(z)|\Big|\gamma H(z) + G(z)\Big|, \quad \text{for } |z| = 1. \tag{3.2}$$

Since $B'(z) \neq 0$ therefore, the right hand side of (3.2) is non zero. Thus, by using (i) of Lemma 2.2, we have for all $\beta \in \mathbb{C}$, with $|\beta| < 1$,

$$2B(z)\Big(\gamma H'(z) + G'(z)\Big) \neq -\beta B'(z)\Big(\gamma H(z) + G(z)\Big), \quad \text{for } |z| = 1.$$

Equivalently, for $|z| = 1$,

$$-\gamma\Big(2B(z)H'(z) + \beta B'(z)H(z)\Big) \neq -\Big(2B(z)G'(z) + \beta B'(z)G(z)\Big), \tag{3.3}$$

for $|\gamma| < 1$, $|\beta| < 1$. Using (ii) of Lemma 2.2 in (3.3), we have

$$\Big|2B(z)G'(z) + \beta B'(z)G(z)\Big| \leq \Big|2B(z)H'(z) + \beta B'(z)H(z)\Big|, \tag{3.4}$$

for $|z| = 1$, $|\beta| < 1$. Now, using $G(z) = r(t(z)) + \lambda M B(z)$ and since

$$H(z) = B(z)\overline{G\left(\frac{1}{\bar{z}}\right)} = B(z)\left(\overline{r\left(t\left(\frac{1}{\bar{z}}\right)\right)} + \bar{\lambda} M \overline{B\left(\frac{1}{\bar{z}}\right)}\right) = (r(t(z)))^* + \bar{\lambda} M,$$

for $|z| = 1$ in (3.4), we get, for $|\beta| < 1$ and $|z| = 1$,

$$\begin{aligned}&\Big|2B(z)\Big[(r(t(z)))^*\Big]' + \beta B'(z)(r(t(z)))^* + \bar{\lambda}\beta M B'(z)\Big| \\ &\leq \Big|2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z)) + \lambda B(z)B'(z)\Big(2 + \beta\Big)M\Big|.\end{aligned} \tag{3.5}$$

By choosing a suitable argument of λ and applying Lemma 2.4 on the right hand side of (3.5), we get, for $|z| = 1$ and $|\beta| < 1$,

$$\begin{aligned}&\Big|2B(z)\Big[(r(t(z)))^*\Big]' + \beta B'(z)(r(t(z)))^*\Big| - |\lambda|\Big|\beta B'(z)\Big|M\\ &\le|\lambda|\Big|B(z)B'(z)\Big(2+\beta\Big)\Big|M - \Big|2B(z)r'(t(z))t'(z) + \beta B'(z)r(t(z))\Big|.\end{aligned}\tag{3.6}$$

Note that $|B(z)| = 1$ for $|z| = 1$. Making $|\lambda| \to 1$ and using continuity for $|\beta| = 1$ in (3.6), we get (3.1) and this proves the desired result. □

For $t(z) = z$, Theorem 3.1 reduces to the following result.

Corollary 3.1. *If $r \in \mathcal{R}_n$ and $|z| = 1$, then for every β, with $|\beta| \le 1$,*

$$\begin{aligned}&\left|B(z)r'(z) + \frac{\beta}{2}B'(z)r(z)\right| + \left|B(z)(r^*(z))' + \frac{\beta}{2}B'(z)r^*(z)\right|\\ &\le|B'(z)|\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\sup_{|z|=1}|r(z)|.\end{aligned}\tag{3.7}$$

Remark 3.1. For $\beta = 0$, (3.7) reduces to (1.5).

Theorem 3.2. *If $r(t(z)) \in \mathcal{R}_{mn}$ is self-inversive and $|z| = 1$, then for every β with $|\beta| \le 1$, we have*

$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \le \frac{|B'(z)|}{2}\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\sup_{|z|=1}|r(t(z))|.\tag{3.8}$$

Proof. Since $r(t(z))$ is self-inversive, therefore, we have $(r(t(z)))^* = \lambda r(t(z))$ with $|\lambda| = 1$. Hence, for all $\beta \in \mathbb{C}$,

$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| = \left|B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2}B'(z)(r(t(z)))^*\right|.\tag{3.9}$$

Combining Theorem 3.1 and (3.9), we have for every β, with $|\beta| \le 1$ and $|z| = 1$,

$$\begin{aligned}2\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| =&\left|B'(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right|\\ &+\left|B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2}B'(z)(r(t(z)))^*\right|\\ \le&|B'(z)|\left\{\left|1+\frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\}\sup_{|z|=1}|r(t(z))|,\end{aligned}$$

which proves Theorem 3.2 completely. □

Remark 3.2. If we take $\beta = 0$ in inequality (3.8) and make use of the Lemma 2.5, after supposing that $t(z)$ has all its zeros in $|z| \le 1$, we get the following result.

Corollary 3.2. *If $r(t(z)) \in \mathcal{R}_{mn}$ is self-inversive, where $t(z)$ has all its zeros in $|z| \leq 1$, then for $|z| = 1$,*

$$|r'(t(z))| \leq \frac{|B'(z)|}{2m\mu} \sup_{|z|=1} |r(t(z))|, \tag{3.10}$$

where $\mu = \inf_{|z|=1} |t(z)|$.

Remark 3.3. For $t(z) = z$, (3.10) reduces to (1.6).

We end this section by proving the following interesting generalization of (1.7).

Theorem 3.3. *Suppose $r(t(z)) \in \mathcal{R}_{mn}$ and all the mn zeros of $r(t(z))$ lie in $|z| \geq 1$. Then for every β, with $|\beta| \leq 1$ and $|z| = 1$, we have*

(3.11)
$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq \frac{|B'(z)|}{2}\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \sup_{|z|=1} |r(t(z))|.$$

Proof. Since $r(t(z)) \in \mathcal{R}_{mn}$ has all its mn zeros in $|z| \geq 1$ and $(r(t(z)))^* = B(z)\overline{r(t(\frac{1}{\bar{z}}))}$, therefore, all the zeros of $(r(t(z)))^*$ lie in $|z| \leq 1$. Also, $|r(t(z))| = |(r(t(z)))^*|$ for $|z| = 1$. Hence, by Lemma 2.3, it follows for every β, with $|\beta| \leq 1$ and $|z| = 1$,

(3.12)
$$\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq \left|B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2}B'(z)(r(t(z)))^*\right|.$$

Combining Theorem 3.1 and (3.12), we have for every β, with $|\beta| \leq 1$ and $|z| = 1$,

$$\begin{aligned}2\left|B(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \leq& \left|B'(z)r'(t(z))t'(z) + \frac{\beta}{2}B'(z)r(t(z))\right| \\ &+ \left|B(z)\Big[(r(t(z)))^*\Big]' + \frac{\beta}{2}B'(z)(r(t(z)))^*\right| \\ \leq& |B'(z)|\left\{\left|1 + \frac{\beta}{2}\right| + \left|\frac{\beta}{2}\right|\right\} \sup_{|z|=1} |r(t(z))|,\end{aligned}$$

which is equivalent to (3.11) and this completes the proof of Theorem 3.3. □

Remark 3.4. If we take $\beta = 0$ in (3.11) and assume that $t(z)$ has all its zeros in $|z| \leq 1$, we get (1.7) by virtue of Lemma 2.5.

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EXISTENCE OF POSITIVE SOLUTIONS FOR A PERTUBED FOURTH-ORDER EQUATION

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Abstract. In this paper, a special type of fourth-order differential equations with a perturbed nonlinear term and some boundary conditions is considered which is very important in mechanical engineering. Therefore, the existence of a non-trivial solution for such equations is very important. Our goal is to ensure at least three weak solutions for a class of perturbed fourth-order problems by applying certain conditions to the functions that are available in the differential equation (problem (1.1)). Our approach is based on variational methods and critical point theory. In fact, using a fundamental theorem that is attributed to Bonanno, we get some important results. Finally, for some results, an example is presented.

1. Introduction

In the present paper, the following fourth-order problem

$$\begin{cases} u^{(iv)}(x) = \lambda f(x, u(x)) + h(u(x)), & x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu\, g(u(1)), \end{cases} \tag{1.1}$$

is studied, where λ and μ are positive parameters, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is non-negative L^1-Carathéodory function, $h : \mathbb{R} \to \mathbb{R}$ is a non-negative Lipschitz continuous function with the Lipschitz constant $0 < L < 1$, i.e.,

$$|h(t_1) - h(t_2)| \leq L|t_1 - t_2|,$$

for every $t_1, t_2 \in \mathbb{R}$, and $h(0) = 0$ and $g : \mathbb{R} \to \mathbb{R}$ is a non-positive continuous function. It is clear that for function h we have $h(t) \leq L|t|$ for each $t \in \mathbb{R}$.

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The problem (1.1) is related to the deflections of elastic beams based on nonlinear elastic. In relation with the problem (1.1), there is an interesting physical description.

An elastic beam of length $d = 1$, which is clamped at its left side $x = 0$, and resting on a kind of elastic bearing at its right side $x = 1$ which is given by μg . Along its length, a load $\lambda f + h$, is added to cause deformations. If $u = u(x)$ denotes the configuration of the deformed beam, then since $u'''(1)$ represents the shear force at $x = 1$, the condition $u'''(1) = \mu g(u(1))$ means that the vertical force is equal to $\mu g(u(1))$, which denotes a relation, possibly nonlinear, between the vertical force and the displacement $u(1)$.

Different models and their applications for problems such as (1.1) can be derived from [9]. Studying fourth-order differential equations are very important in engineering sciences. Therefore, several results are known concerning the existence of multiple solutions for fourth-order boundary value problems. For example, in [7] the author obtained the existence of at least two positive solutions for the problem

$$\begin{cases} u^{(iv)}(x) = f(x, u(x)), \quad x \in [0, 1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = g(u(1)), \end{cases} \tag{1.2}$$

based on variational methods and maximum principle.

Moreover, in [8] authors considered iterative solutions for problem (1.2) with nonlinear boundary conditions. In particular, by using a variational methods the existence of non-zero solutions for problem (1.1) in the case of $h(t) \equiv 0$ has been established in [2]. In [6], using a critical points theorem obtained in [3], multiplicity results for the problem (1.1) were discussed. Also based on variational methods, existence and multiplicity results for this kind of problems were considered in [4, 5].

In the present paper, using a three critical points theorem obtained in [1] we will establish the existence of at least three weak solutions for the problem (1.1).

2. Preliminaries

Our main tool is a three critical points theorem that we recall here in a appropriate form. This theorem has been established in [1]. In this theorem a suitable sign hypothesis is assumed.

Theorem 2.1. ([1, Corollary 3.1]). *Let X be a reflexive real Banach space, $\Phi : X \to \mathbb{R}$ be a convex, coercive and continuously Gâteaux differentiable functional whose derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose derivative is compact, such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

Assume that there are two positive constants r_1, r_2 and $w \in X$, with $2r_1 < \Phi(w) < \frac{r_2}{2}$, such that

(b_1) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(w)}{\Phi(w)};$

(b_2) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(w)}{\Phi(w)}$;

(b_3) *for each*

$$\lambda \in \Lambda_{r_1,r_2} := \left(\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right)$$

and for every $u_1, u_2 \in X$, which are local minimum for the functional $\Phi - \lambda\Psi$ and such that $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, one has

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

Then, for each $\lambda \in \Lambda_{r_1,r_2}$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which lie in $\Phi^{-1}(]-\infty, r_2[)$.

Now we give some preliminary definitions and basic concepts. Denote

$$X := \{u \in H^2[0,1] \mid u(0) = u'(0) = 0,\ u(1) \geq 0\},$$

where $H^2[0,1]$ is the Sobolev space of all functions $u : [0,1] \to \mathbb{R}$ such that u and its distributional derivative u' are absolutely continuous and u'' belongs to $L^2[0,1]$. Obviously, X is a Hilbert space with the usual norm

$$\|u\|_X = \left(\int_0^1 (|u''(x)|^2 + |u'(x)|^2 + |u(x)|^2)dx\right)^{1/2},$$

which is equivalent to the norm

$$\|u\| = \left(\int_0^1 |u''(x)|^2 dx\right)^{1/2}.$$

The embedding $X \hookrightarrow C^1[0,1]$ is compact and also

$$\|u\|_{C^1([0,1])} = \max\{\|u\|_\infty, \|u'\|_\infty\} \leq \|u\|, \tag{2.1}$$

for each $u \in X$ (see [10]). We assume that the Lipschitz constant L of the function h satisfies $L < 1$.

Definition 2.1. We mean by a (weak) solution of the problem (1.1), any function $u \in X$ such that

(2.2)
$$\int_0^1 u''(x)v''(x)dx - \lambda\int_0^1 f(x,u(x))v(x)dx + \mu g(u(1))v(1) - \int_0^1 h(u(x))v(x)dx = 0,$$

holds for every $v \in X$.

Here, we note that if f is continuous function, then every weak solution u of the problem (1.1) is a classical solution (see [10, Lemma 2.1]).

Proposition 2.1. *If $u_0 \not\equiv 0$ is a weak solution for problem (1.1), then u_0 is non-negative.*

Proof. Let $A = \{x \in [0,1] \mid u_0(x) < 0\}$. Since u_0 is a weak solution for problem (1.1), then from (2.2) we have

$$\int_{A\cup A^c} u_0''(x)v''(x)dx - \lambda \int_{A\cup A^c} f(x,u_0(x))v(x)dx + \mu g(u_0(1))v(1)$$
$$- \int_{A\cup A^c} h(u_0(x))v(x)dx = 0,$$

for every $v \in X$. Choosing $v(x) = \bar{u}_0 = \max\{-u_0(x), 0\}$. Since u_0 is a weak solution for problem (1.1), then $u_0(1) \geq 0$ and hence $v(1) = 0$. So, one has

$$- \int_A v''(x)v''(x)dx + \lambda \int_A f(x,u_0(x))u_0(x)dx + \int_A h(u_0(x))u_0(x)dx = 0,$$

that is

$$- \int_A v''(x)v''(x)dx = -\lambda \int_A f(x,u_0(x))u_0(x)dx - \int_A h(u_0(x))u_0(x)dx \geq 0,$$

which means that $-\|v\|^2 \geq 0$ and one has, $v = 0$. Hence, $-u_0 \leq 0$, that is, $u_0 \geq 0$ and the proof is complete. $\square$

Put

$$F(x,t) = \int_0^t f(x,\xi)d\xi, \quad \text{for all } (x,t) \in [0,1]\times\mathbb{R},$$
$$G(t) = \int_0^t g(\xi)d\xi, \quad \text{for all } t \in \mathbb{R},$$
$$G_\eta = \min_{|t|\leq\eta} G(t) = \inf_{|t|\leq\eta} G(t), \quad \text{for all } \eta > 0,$$

and

$$H(t) = \int_0^t h(\xi)d\xi, \quad \text{for all } t \in \mathbb{R}.$$

We state the following proposition which will be used in the next sections.

Proposition 2.2. ([6, Proposition 2.2]) *Let $T : X \to X^*$ be the operator defined by*

$$T(u)(v) = \int_0^1 u''(x)v''(x)dx - \int_0^1 h(u(x))v(x)dx,$$

for each $u, v \in X$. Then T admits a continuous inverse on X^*.

Now, we introduce the functional $I_\lambda : X \to \mathbb{R}$ associated with (1.1), $I_\lambda(u) := \Phi(u) - \lambda\,\Psi(u)$ for all $u \in X$, where

$$\Phi(u) = \frac{1}{2}\int_0^1 |u''(x)|^2dx - \int_0^1 H(u(x))dx$$

and

$$\Psi(u) = \int_0^1 F(x,u(x))dx - \frac{\mu}{\lambda}G(u(1)),$$

for each $u \in X$. It is well known that Ψ is a continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_0^1 f(x,u(x))v(x)dx - \frac{\mu}{\lambda}g(u(1))v(1)$$

and furthermore, $\Psi' : X \to X^*$ is a compact operator (see [10, page 1602]). Moreover, Φ is continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)dx - \int_0^1 h(u(x))v(x)dx,$$

for every $v \in X$. Also according to Proposition 2.2, functional Φ whose derivative admits a continuous inverse on X and moreover Φ is coercive and convex.

Definition 2.2. Let Φ and Ψ be defined as above. Put $I_\lambda = \Phi - \lambda\Psi$, $\lambda > 0$. We say that $u \in X$ is a critical point of I_λ when $I'_\lambda(u) = 0_{\{X^*\}}$, that is, $I'_\lambda(u)(v) = 0$ for all $v \in X$.

Remark 2.1. We note that, the weak solutions of the problem (1.1) are exactly the critical points of the functional I_λ.

3. Main Results

To get our result, fix three positive constants θ_1, θ_2 and δ such that

$$\frac{12(1+L)(\frac{2}{3})^3\pi^4\delta^2}{\int_{\frac{3}{4}}^1 F(x,\delta)dx} < (1-L)\min\left\{\frac{{\theta_1}^2}{\int_0^1 \sup_{|t|\le\theta_1} F(x,t)dx}, \frac{{\theta_2}^2}{2\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx}\right\}$$

and take

$$\lambda \in \Lambda := \left]\frac{6(1+L)(\frac{2}{3})^3\pi^4\delta^2}{\int_{\frac{3}{4}}^1 F(x,\delta)dx}, \min\left\{\frac{(1-L){\theta_1}^2}{2\int_0^1 \sup_{|t|\le\theta_1} F(x,t)dx}, \frac{(1-L){\theta_2}^2}{4\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx}\right\}\right[$$

and set $\eta_{\lambda,g}$ given by

(3.1)

$$\eta_{\lambda,g} := \min\left\{\frac{2\lambda\int_0^1 \sup_{|t|\le\theta_1} F(x,t)dx - (1-L){\theta_1}^2}{2G_{\theta_1}}, \frac{4\lambda\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx - (1-L){\theta_2}^2}{4G_{\theta_2}}\right\},$$

where G_{θ_1} and G_{θ_2} are assumed to be negative. It is easy to show that $\eta_{\lambda,g} > 0$. Our main result is the following theorem.

Theorem 3.1. *Suppose that there exist three positive constants θ_1, θ_2 and δ, with $\frac{3}{4\pi^2}\sqrt{\frac{3}{2}}\theta_1 < \delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\theta_2$, such that*

(A_1) $12\pi^4(1+L)(\frac{2}{3})^3\delta^2\int_0^1 \sup_{|t|\le\theta_1} F(x,t)dx < (1-L){\theta_1}^2\int_{\frac{3}{4}}^1 F(x,\delta)dx$;

(A_2) $24\pi^4(1+L)(\frac{2}{3})^3\delta^2\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx < (1-L){\theta_2}^2\int_{\frac{3}{4}}^1 F(x,\delta)dx$.

Then, for every $\lambda \in \Lambda$ and for each non-positive continuous function $g : \mathbb{R} \to \mathbb{R}$ there exists $\eta_{\lambda,g} > 0$ given by (3.1) such that, for every $\mu \in]0, \eta_{\lambda,g}[$, the problem (1.1) admits at least three weak solutions u_i for $i = 1, 2, 3$, in X such that $0 \le u_i(x) < \theta_2$ for all $x \in [0,1]$, $i = 1, 2, 3$.

Proof. Our aim is to apply Theorem 2.1, to problem (1.1). For this purpose, fix $\lambda \in \Lambda$ and $\mu \in]0, \eta_{\lambda,g}[$. Let $\Phi,\ \Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\int_0^1 |u''(x)|^2 dx - \int_0^1 H(u(x))dx$$

and

$$\Psi(u) = \int_0^1 F(x, u(x))dx - \frac{\mu}{\lambda} G(u(1)),$$

for every $u \in X$. As seen before, the functionals Φ and Ψ satisfy the regularity assumptions requested in Theorem 2.1. Put

$$r_1 := \frac{(1-L)}{2}\theta_1^2, \quad r_2 := \frac{(1-L)}{2}\theta_2^2 \tag{3.2}$$

and

$$w(x) := \begin{cases} 0, & \text{if } x \in \left[0, \frac{3}{8}\right], \\ \delta \cos^2\left(\frac{4\pi x}{3}\right), & \text{if } x \in \left]\frac{3}{8}, \frac{3}{4}\right[, \\ \delta, & \text{if } x \in \left[\frac{3}{4}, 1\right]. \end{cases} \tag{3.3}$$

We see that $w \in X$ and

$$\|w\|^2 = 8\pi^4\delta^2\left(\frac{2}{3}\right)^3.$$

Now, according to (2.1), for every $u \in X$

$$\frac{(1-L)}{2}\|u\|^2 \le \Phi(u) \le \frac{(1+L)}{2}\|u\|^2$$

holds and in particular

$$4(1-L)\pi^4\delta^2\left(\frac{2}{3}\right)^3 \le \Phi(w) \le 4(1+L)\pi^4\delta^2\left(\frac{2}{3}\right)^3. \tag{3.4}$$

Now, using $\frac{3}{4\pi^2}\sqrt{\frac{3}{2}}\theta_1 < \delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\theta_2$ and (3.4) we have $2r_1 < \Phi(w) < \frac{r_2}{2}$. Since, $\frac{(1-L)}{2}\|u\|^2 \le \Phi(u)$ for each $u \in X$ and for $i = 1, 2$, we see that

$$\begin{aligned}\Phi^{-1}(]-\infty, r_i]) =& \{u \in X \mid \Phi(u) \le r_i\} \\ &\subseteq \left\{u \in X \mid \frac{(1-L)}{2}\|u\|^2 \le r_i\right\} \\ &\subseteq \{u \in X \mid |u(x)| \le \theta_i \text{ for each } x \in [0,1]\}\end{aligned}$$

and it follows that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} = \frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\left(\int_0^1 F(x,u(x))dx - \frac{\mu}{\lambda}G(u(1))\right)}{\frac{(1-L)}{2}\theta_1^2}$$

$$\leq \frac{\int_0^1 \sup_{|t|\leq\theta_1} F(x,t)dx - \frac{\mu}{\lambda}G_{\theta_1}}{\frac{(1-L)}{2}\theta_1^2}. \tag{3.5}$$

On the other hand, since $w(x)\in[0,\delta]$ for each $x\in[0,1]$, we have

$$\Psi(w) = \int_0^1 F(x,w(x))dx - \frac{\mu}{\lambda}G(w(1)) \geq \int_{\frac{3}{4}}^1 F(x,\delta)dx - \frac{\mu}{\lambda}G(\delta).$$

Hence, we have

$$\frac{\Psi(w)}{\Phi(w)} \geq \frac{\int_{\frac{3}{4}}^1 F(x,\delta)dx - \frac{\mu}{\lambda}G(\delta)}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3}. \tag{3.6}$$

Now, since $\mu < \eta_{\lambda,g}$ and $\lambda\in\Lambda$ one has

$$\frac{\int_0^1 \sup_{|t|\leq\theta_1} F(x,t)dx - \frac{\mu}{\lambda}G_{\theta_1}}{\frac{(1-L)}{2}{\theta_1}^2} \leq \frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^1 F(x,\delta)dx}{6(1+L)\pi^4\delta^2(\frac{2}{3})^3}$$

$$\leq \frac{2}{3}\frac{\int_{\frac{3}{4}}^1 F(x,\delta)dx - \frac{\mu}{\lambda}G(\delta)}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3}. \tag{3.7}$$

So, from (3.5), (3.6) and (3.7), one has

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}$$

and hence, (b_1) of Theorem 2.1 is established. As in the above process, we will have

$$\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} \leq \frac{2\left(\int_0^1 \sup_{|t|\leq\theta_2} F(x,t)dx - \frac{\mu}{\lambda}G_{\theta_2}\right)}{\frac{(1-L)}{2}{\theta_2}^2} \leq \frac{1}{\lambda} < \frac{\int_{\frac{3}{4}}^1 F(x,\delta)dx}{6(1+L)\pi^4\delta^2(\frac{2}{3})^3}$$

$$\leq \frac{2}{3}\frac{\int_{\frac{3}{4}}^1 F(x,\delta)dx - \frac{\mu}{\lambda}G(\delta)}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3} \leq \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}, \tag{3.8}$$

that is,

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(w)}{\Phi(w)}$$

and hence, (b_2) of Theorem 2.1 is established.

Finally, we will prove that $\Phi-\lambda\Psi$ satisfies the assumption (b_3) of Theorem 2.1. Let u_1 and u_2 be two local minima for $\Phi-\lambda\Psi$. Then u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1.1). According to Proposition 2.1 one has $u_1(x) \geq 0$ and $u_2(x) \geq 0$ for every $x \in [0,1]$. Hence, it follows that

$$\inf_{s\in[0,1]} \Psi(su_1 + (1-s)u_2) \geq 0.$$

From Theorem 2.1, for every

$$\lambda \in \Lambda \subseteq \Lambda_{r_1,r_2} = \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ has at least three distinct critical points u_i, in X such that $0 \leq u_i(x) < \theta_2$, for all $x \in [0,1]$, $i = 1,2,3$, which are the weak solutions of (1.1). $\square$

Remark 3.1. If in Theorem 3.1 we assume $f(x,0) \neq 0$, then problem (1.1) has at least three distinct non-trivial and non-negative weak solutions.

Now, we present a variant of Theorem 3.1, which will be achieved by reversing the role of λ and μ.

Theorem 3.2. *Suppose that there exist three positive constants θ_1, θ_2 and δ, with $\frac{3}{4\pi^2}\sqrt{\frac{3}{2}}\theta_1 < \delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\theta_2$, such that*

(B_1) $G(\delta)(1-L){\theta_1}^2 < 12\, G_{\theta_1}(1+L)^4\delta^2(\frac{2}{3})^3$;
(B_2) $G(\delta)(1-L){\theta_2}^2 < 24\ G_{\theta_2}(1+L)\pi^4\delta^2(\frac{2}{3})^3$.

Then, for each

$$\mu \in \Lambda' := \left]\frac{6(1+L)\pi^4\delta^2(\frac{2}{3})^3}{-G(\delta)}, \min\left\{\frac{(1-L){\theta_1}^2}{-2G_{\theta_1}}, \frac{(1-L){\theta_2}^2}{-4G_{\theta_2}}\right\}\right[$$

and for each non-negative L^1-Carathéodory function $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ there exists ${\eta'}_{\lambda,g} > 0$, where

$${\eta'}_{\lambda,g} = \min\left\{\frac{(1-L){\theta_1}^2 + 2\mu\ G_{\theta_1}}{2\displaystyle\int_0^1 \sup_{|t|\leq\theta_1} F(x,t)dx}, \frac{(1-L){\theta_2}^2 + 4\mu\ G_{\theta_2}}{4\displaystyle\int_0^1 \sup_{|t|\leq\theta_2} F(x,t)dx}\right\},$$

such that, for all $\lambda \in]0, {\eta'}_{\lambda,g}[$, (1.1) admits at least three weak solutions in X.

Proof. Fix $\mu \in \Lambda'$ and $\lambda \in]0, {\eta'}_{\lambda,g}[$. Let $\hat{\Psi} : X \to \mathbb{R}$ be defined by

$$\hat{\Psi}(u) = \frac{\lambda}{\mu}\int_0^1 F(x,u(x))dx - G(u(1)),$$

for each $u \in X$. We observe that $\Phi(u) - \lambda\Psi(u) = \Phi(u) - \mu\hat{\Psi}(u)$ for every $u \in X$. Choose r_1, r_2 and w as given in (3.2) and (3.3). Now, we have

$$\begin{aligned}\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\hat{\Psi}(u)}{r_1} &= \frac{\frac{\lambda}{\mu}\int_0^1 \sup_{|t|\le\theta_1} F(x,t)dx - G_{\theta_1}}{\frac{(1-L)}{2}{\theta_1}^2} \le \frac{1}{\mu} < \frac{-G(\delta)}{6(1+L)\pi^4\delta^2(\frac{2}{3})^3}\\ &\le \frac{2}{3}\frac{\frac{\lambda}{\mu}\int_{\frac{3}{4}}^1 F(x,\delta)dx - G(\delta)}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3} \le \frac{2}{3}\frac{\hat{\Psi}(w)}{\Phi(w)},\end{aligned}$$

that is,

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r_1)}\hat{\Psi}(u)}{r_1} < \frac{2}{3}\frac{\hat{\Psi}(w)}{\Phi(w)}$$

and

$$\begin{aligned}\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\hat{\Psi}(u)}{r_2} &= \frac{2\Big(\frac{\lambda}{\mu}\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx - G_{\theta_2}\Big)}{\frac{(1-L)}{2}{\theta_2}^2} \le \frac{1}{\mu} < \frac{-G(\delta)}{6(1+L)\pi^4\delta^2(\frac{2}{3})^3}\\ &\le \frac{2}{3}\frac{\frac{\lambda}{\mu}\int_{\frac{3}{4}}^1 F(x,\delta)dx - G(\delta)}{4(1+L)\pi^4\delta^2(\frac{2}{3})^3} \le \frac{2}{3}\frac{\hat{\Psi}(w)}{\Phi(w)},\end{aligned}$$

that is,

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r_2)}\hat{\Psi}(u)}{r_2} < \frac{1}{3}\frac{\hat{\Psi}(w)}{\Phi(w)}.$$

Therefore, since for each

$$\mu \in \Lambda' \subseteq \left]\frac{3}{2}\frac{\Phi(w)}{\hat{\Psi}(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\hat{\Psi}(u)},\ \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\hat{\Psi}(u)}\right\}\right[,$$

the assumptions of Theorem 2.1 are fulfilled, so the desired result is achieved from Theorem 2.1. □

Now we will give a special case of Theorem 3.1 that the function f depends only on t.

Corollary 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$\lim_{t\to 0^+}\frac{f(t)}{t} = 0$$

and

$$\int_0^{100} f(\xi)d\xi < \frac{625(1-L)}{6(1+L)\pi^4(\frac{2}{3})^3}\int_0^1 f(\xi)d\xi.$$

Also, suppose that, $1 < \frac{300}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}$. *Then, for every*

$$\lambda \in \left] \frac{24(1+L)\pi^4(\frac{2}{3})^3}{\int_0^1 f(\xi)d\xi}, \frac{2500(1-L)}{\int_0^{100} f(\xi)d\xi} \right[$$

and for every non-positive function $g : \mathbb{R} \to \mathbb{R}$ *there exists* $\delta^*_{\lambda,g} > 0$ *such that, for each* $\mu \in [0, \delta^*_{\lambda,g}[$, *the problem*

$$\begin{cases} u^{(iv)}(x) = \lambda f(u(x)) + h(u(x)), \quad x \in [0,1], \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \quad u'''(1) = \mu\, g(u(1)), \end{cases}$$

admits at least three classical solutions.

Proof. Our aim is to employ Theorem 3.1 by choosing $\theta_2 = 100$ and $\delta = 1$. Hence, we have

$$\frac{6(1+L)(\frac{2}{3})^3\pi^4\delta^2}{\displaystyle\int_{\frac{3}{4}}^1 F(x,\delta)dx} = \frac{24(1+L)\pi^4(\frac{2}{3})^3}{\int_0^1 f(\xi)d\xi}$$

and

$$\frac{(1-L){\theta_2}^2}{4\displaystyle\int_0^1 \sup_{|t|\le\theta_2} F(x,t)dx} = \frac{2500(1-L)}{\int_0^{100} f(\xi)d\xi}.$$

Also, according to the condition $1 < \frac{300}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}$, we have

$$\delta < \frac{3}{8\pi^2}\sqrt{\frac{3(1-L)}{2(1+L)}}\theta_2.$$

Moreover, since $\lim_{t\to 0^+} \frac{f(t)}{t} = 0$, one has

$$\lim_{t\to 0^+} \frac{\displaystyle\int_0^t f(\xi)d\xi}{t^2} = 0.$$

Then, there exists a positive constant $\theta_1 < \frac{4\pi^2}{3}\sqrt{\frac{2}{3}}$ such that

$$\frac{\displaystyle\int_0^{\theta_1} f(\xi)d\xi}{\theta_1^2} < \frac{1-L}{48(1+L)(\frac{2}{3})^3\pi^4}\int_0^1 f(\xi)d\xi$$

and

$$\frac{\theta_1^2}{\displaystyle\int_0^{\theta_1} f(\xi)d\xi} > \frac{5000}{\displaystyle\int_0^{100} f(\xi)d\xi}.$$

Finally, a simple computation shows that all the circumstances of the Theorem 3.1 hold and so the desired result is achieved. $\square$

Remark 3.2. If we consider

$$f(t) := \begin{cases} 18\,t^2, & \text{if } t \leq 1, \\ -18000\,t + 18018, & \text{if } 1 < t \leq 1.001, \\ 0, & \text{if } t > 1.001, \end{cases}$$

and $h(t) = \frac{1}{2}|t|$ for all $t \in \mathbb{R}$, then we can consider $L = \frac{1}{2}$. In this case, a simple calculation reveals that, all the conditions of Corollary 3.1 are established.

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THE MAXIMUM NORM ANALYSIS OF SCHWARZ METHOD FOR ELLIPTIC QUASI-VARIATIONAL INEQUALITIES

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Abstract. In this paper, we present a maximum norm analysis of an overlapping Schwartz method on non matching grids for a quasi-variational inequality, where the obstacle and the second member depend on the solution. Our result improves and generalizes some previous results.

1. Introduction

Historically, Schwarz method has been introduced by Herman Amondus Schawarz, in order to resolve a purely theoretical matters. The Schawarz alternating method has been used to solve the stationary or evolutionary boundary valued problems, on domain which consists of two or more overlapping sub-domains, see for example [6, 7]. The solution is approximated by an infinite sequence of function, the result which is the resolution of a sequence of stationary or evolutionary boundary valued problems, in each of sub-domain.

In this work, we are interested in the analysis of error estimates in uniform norm for the quasi-variational inequality. Our goal is to generalize and improve some previous results given in [2–4, 10, 11] which concerning analysis of error estimates in uniform norm for the elliptic quasi-variational inequality. As in [2] they got the following approximation:

$$\|u_i - u_{ih}^{n+1}\|_\infty \le Ch^2|\log h|^3, \quad i = 1, 2,$$

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for the problem

$$\begin{cases} a(u, v-u) \geq (f, v-u) \text{ in } \Omega, & \text{for all } v \in K, \\ u \leq \psi, \quad v \leq \psi, \end{cases}$$

where K is a convex, closed and not empty set. In [4], they have obtained the same approximation for the following problem:

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) \text{ in } \Omega, & \text{for all } v \in K(u), \\ u \leq \psi, \quad v \leq \psi, \end{cases}$$

also, for the non-coercive variational inequality, it has been reached in [11], the same approximation mentioned above. In [10], the authors studied a quasi-variational inequality related to control ergodic problem

$$\begin{cases} b(u_\alpha, v-u_\alpha) \geq (f + r u_\alpha, v-u_\alpha), & \alpha \in (0,1), \\ u_\alpha \leq M u_\alpha, \quad v \leq M u_\alpha, \end{cases}$$

and they got the following result:

$$\|u_{\alpha_i} - u_{\alpha_i h}^{n+1}\|_\infty \leq C\alpha^{-2} h^2 |\log h|^4, \quad i = 1, 2.$$

Finally in [3], the authors studied the following problem:

$$\begin{cases} a(u, v-u) \geq (f, v-u), & \text{for all } v \in K, \\ u \leq Mu, \quad Mu \geq 0, \\ Mu = k + \inf\limits_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u(x+\varepsilon), \\ \dfrac{\partial u}{\partial \eta} = \varphi \text{ in } \Gamma_0 \text{ and } u = 0 \text{ in } \Gamma/\Gamma_0, \end{cases}$$

and they obtained the following result:

$$\|u_i - u_{ih}^{n+1}\|_\infty \leq C h^2 |\log h|^3, \quad i = 1, 2.$$

For our work, we claim about the general problem where the second member and the obstacle are related to the solution

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) \text{ in } \Omega, & \text{for all } v \in K_g(u), \\ u \leq Mu, \quad v \leq Mu, \\ u = g \text{ on } \partial\Omega. \end{cases}$$

The outline of the paper, is as follows: in the second section, we will mention the same notations and assumptions, in the third section we will give our continuous problem, analogously in section four, we will define the discrete problem. Section five, is devoted to the L^∞-error analysis of the method.

2. Notation and Assumptions

Let Ω be an open in $\mathbb{R}^n$, with sufficiently smooth boundary $\partial\Omega$. For $u, v \in H^1(\Omega)$, consider the bilinear form as follows:

$$a(u,v) = \int_\Omega \left(\sum_{1\leq i,j\leq n} a_{ij}(x) \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + \sum_{1\leq i\leq n} a_i(x)\frac{\partial u}{\partial x_i} v + a_0(x) u.v \right) dx, \tag{2.1}$$

where $a_{ij}(x), a_i(x), a_0(x), x \in \overline{\Omega}$, $1 \leq i, j \leq n$, are sufficiently smooth coefficients and satisfying the following conditions:

$$\begin{aligned} \sum_{1\leq i,j\leq n} a_{ij}\xi_i\xi_j \geq& \nu|\xi|^2, \quad \xi \in \mathbb{R}^n, \nu > 0, \\ a_0(x) \geq& \beta > 0, \end{aligned}$$

where β is a constant. The operator M is given by $Mu = k + \inf_{\varepsilon\geq 0, x+\varepsilon\in\overline{\Omega}} u(x+\varepsilon)$, where $k > 0$ and M satisfies

$$Mu \in W^{2,\infty}(\Omega), \quad Mu \geq 0 \text{ on } \partial\Omega : 0 \leq g \leq Mu, \tag{2.2}$$

where g is a regular function defined on $\partial\Omega$. Let f be a Lipschitzian non decreasing nonlinear function with rate α satisfying $\frac{\alpha}{\beta} < 1$ and $f \in L^\infty(\Omega)$, and $K_g(u)$ is an implicit convex and non empty set which defined as follows:

$$K_g(u) = \{v \in H^1(\Omega),\ v = g \text{ on } \partial\Omega,\ v \leq Mu \text{ in } \Omega\}.$$

3. The Continuous Problem

We consider the following problem: Find $u \in k_g(u)$ the solution of

$$\begin{cases} a(u, v-u) \geq (f(u), v-u) \text{ in } \Omega, \quad \text{for all } v \in K_g(u), \\ u \leq Mu, \quad v \leq Mu, \\ u = g \text{ on } \partial\Omega. \end{cases} \tag{3.1}$$

We will present some results for our problem as the existence, uniqueness and other optimal properties which given in previous papers where we need them in the sequel.

Theorem 3.1 ([5]). *Under the previous conditions the problem* (3.1) *has an unique solution $u \in K_g(u)$. Moreover, we have*

$$u \in W^{2,p}(\Omega), \quad 2 \leq p \leq \infty.$$

Lemma 3.1 ([6]). *For all u and $\tilde{u} \in K_g(u)$, we have*

(a) *if $u \leq \tilde{u}$, then $Mu \leq M\tilde{u}$ and $M(u+\lambda) = M(u) + \lambda$ for all $\lambda \in \mathbb{R}$;*
(b) $\|Mu - M\tilde{u}\|_{L^\infty(\Omega)} \leq \|u - \tilde{u}\|_{L^\infty(\Omega)}$.

3.1. **The continuous Schwarz sequences.** We decompose Ω in two sub-domains Ω_1, Ω_2 such that $\Omega = \Omega_1 \cup \Omega_2$ and u satisfies the local regularity condition:

$$u/_{\Omega_i} \in W^{2,p}(\Omega_i), \quad i = 1, 2, \text{ and } 2 \leq p < \infty,$$

denote by $\partial\Omega_i$ the boundary of Ω_i and $\Gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\Gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\Gamma_1 \cap \Gamma_2 = \emptyset$.

We define the following process. Choose $u_0 = k$ to be given, and define the alternating Schwarz sequences (u_1^{n+1}) on Ω_1 such that $u_1^{n+1} \in K(u_1^n)$ is solution of the following problem:

$$\begin{cases} a_1(u_1^{n+1}, v - u_1^{n+1}) \geq (f_1(u_1^n), v - u_1^{n+1}), \\ u_1^{n+1} \leq Mu_1^n, \\ u_1^{n+1} = u_2^n \text{ on } \Gamma_1, \quad v = u_2^n \text{ on } \Gamma_1, \end{cases} \tag{3.2}$$

and (u_2^{n+1}) on Ω_2 such that $u_2^{n+1} \in K(u_2^n)$ solution of the following problem:

$$\begin{cases} a_2(u_2^{n+1}, v - u_2^{n+1}) \geq (f_2(u_2^n), v - u_2^{n+1}), \\ u_2^{n+1} \leq Mu_2^n, \\ u_2^{n+1} = u_1^n \text{ on } \Gamma_2, \quad v = u_1^n \text{ on } \Gamma_2, \end{cases} \tag{3.3}$$

where $f_i = f/_{\Omega_i}$, $i = 1, 2$, and $(a_i(u, v)$ the form bilinear which defined in (2).

3.2. **Geometrical convergence.**

Theorem 3.2 ([3])**.** *The sequences (u_1^{n+1}), (u_2^{n+1}), $n \geq 0$, produced by the Schawarz alternating method converge geometrically to the solution u of the problem* (3.1)*, more precisely, there exist two constants $K_1, K_2 \in (0, 1)$ such that for all $n \geq 0$, we have*

$$\begin{aligned} \|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} &\leq K_1^n K_2^n \|u^0 - u\|_{L^\infty(\Gamma_1)}, \\ \|u_2 - u_2^{n+1}\|_{L^\infty(\Omega_2)} &\leq K_1^{n+1} K_2^n \|u^0 - u\|_{L^\infty(\Gamma_2)}. \end{aligned}$$

We will show an important proposition, which give the continuous dependence to the second member, the data g and the obstacle. We note that $u = \sigma(f(u), Mu, g)$, $\tilde{u} = \sigma(f(\tilde{u}), M\tilde{u}, \tilde{g})$, where $u, \tilde{u} \in K_g(u)$.

Proposition 3.1. *Under the previous hypotheses and notations, we have*

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \leq \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)},$$

where $\Gamma_i = \partial\Omega_i \cap \Omega_j$, $i, j = 1, 2$, and $i \neq j$.

Proof. Setting

$$\Phi = \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)},$$

we have

$$f(u) \leq f(\tilde{u}) + f(u) - f(\tilde{u}) \leq f(\tilde{u}) + \|f(u) - f(\tilde{u})\| \leq f(\tilde{u}) + \Phi.$$

Similarly, we have $g \leq \tilde{g} + \Phi$ and $Mu \leq M\tilde{u} + \phi$.

Now, making use of Lemma 3.2, we obtain

$$\begin{aligned}\sigma(f(u), Mu, g) \leq& \sigma(f(\tilde{u}) + \Phi, M\tilde{u} + \Phi, \tilde{g} + \Phi)\\ \leq& (f(\tilde{u}), M\tilde{u}, \tilde{g}) + \Phi,\end{aligned}$$

so, $\sigma(f(u), Mu, g) - \sigma(f(\tilde{u}), M\tilde{u}, \tilde{g}) \leq \Phi$. Since $(f(u), Mu, g)$ and $(f(\tilde{u}), M\tilde{u}, \tilde{g})$ are symmetrical, we have $\sigma(f(\tilde{u}), M\tilde{u}, \tilde{g}) - \sigma(f(u), Mu, g) \leq \Phi$, and then

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \leq \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|Mu - M\tilde{u}\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)}. \qquad \square$$

Remark 3.1. If $Mu = M\tilde{u}$, we have

$$\|u - \tilde{u}\|_{L^\infty(\Omega_i)} \leq \|f(u) - f(\tilde{u})\|_{L^\infty(\Omega_i)} + \|g - \tilde{g}\|_{L^\infty(\Gamma_i)}.$$

4. The Discrete Problem

We denote by V_h the standard piecewise linear finite element space, we consider the discrete quasi-variational inequality. Find $u_h \in K_{gh}(u_h)$ such that:

$$\left\{\begin{array}{l} a(u_h, v - u_h) \geq (f(u_h), v - u_h), \quad \text{for all } u_h, v \in K_{gh}(u_h),\\ u_h \leq r_h M u_h,\\ u_h = \pi_h g \text{ on } \partial\Omega, \end{array}\right. \tag{4.1}$$

where $f \in L^\infty(\Omega)$; $Mu_h = k + \inf_{\varepsilon \geq 0, x+\varepsilon \in \overline{\Omega}} u_h(x + \varepsilon)$ and

$$K_{gh}(u_h) = \{v \in V_h : v = \pi_h g \text{ on } \partial\Omega, v \leq r_h M u_h \text{ in } \Omega\}.$$

We denote π_h the interpolation operator on $\partial\Omega$ and r_h is the usual finite element restriction operator in Ω.

4.1. **The discrete maximum principle.**

We assume that the respective matrices resulting from the discretization of problems (3.2), (3.1) are M-matrice [9].

Theorem 4.1 ([1])**.** *Let u and u_h be the solutions of problem* (3.1) *and* (4.1) *respectively, there exists a constant C_1 independent of h such that*

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C_1 h^2 \log |h|^2.$$

Similarly, for the continuous case we will establish the discrete version of the lemma.

Lemma 4.1. *For all u_h and $\tilde{u_h} \in K_g(u_h)$ we have*

(a) *if $u_h \leq \tilde{u_h}$, then $Mu_h \leq M\tilde{u_h}$ and $M(u_h + \lambda) = M(u_h) + \lambda$ for all $\lambda \in \mathbb{R}$;*
(b) $\|Mu_h - M\tilde{u_h}\|_{L^\infty(\Omega)} \leq \|u_h - \tilde{u_h}\|_{L^\infty(\Omega)}$.

4.2. **The discrete Schwarz sequences.** For $i = 1, 2$, let $V_{h_i} = V_h(\Omega_i)$ be the space of continuous picewise linear function on τ_{h_i}, which vanish on $\partial\Omega\cap\partial\Omega_i$. For $w \in C(\Gamma_i)$, we define

$$V_{h_i}^{(w)} = \{v \in V_{h_i}, v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega, v = \pi_{h_i}(w) \text{ on } \Gamma_2\},$$

where τ_{h_i} be a standard regular finite element triangulation in Ω_i, h_i being the mesh size. We suppose that the two triangulation are mutually independent on Ω_1, Ω_2, a triangle belonging to one triangulation does not necessarily belong to the other.

We now define the discrete countreparts of the continuous Schwarz sequences defined in (3.2) and (3.1) respectively, by $(u_{1h}^{n+1}) \in V_{h_1}^{(u_{2h}^n)}$, where (u_{1h}^{n+1}) is the solution of

$$\text{(4.2)}\qquad \begin{cases} a_1(u_{1h}^{n+1}, v - u_{1h}^{n+1}) \geq (f_1(u_{1h}^n), v - u_{1h}^{n+1}), \quad \text{for all } v \in V_{h_1}^{(u_{2h}^n)}, \\ u_{1h}^{n+1} \leq r_h M u_{1h}^n, \quad v \leq r_h m u_{1h}^n, \\ u_{1h}^{n+1} = u_{2h}^n \text{ on } \Gamma_1, \quad v = u_{2h}^n \text{ on } \Gamma_1, \end{cases}$$

and $(u_{2h}^{n+1}) \in V_{h2}^{(u_{1h}^{n+1})}$ such that (u_{2h}^{n+1}) is the solution of

$$\text{(4.3)}\qquad \begin{cases} a_2(u_{2h}^{n+1}, v - u_{2h}^{n+1}) \geq (f_2(u_{2h}^n), v - u_{2h}^{n+1}), \quad \text{for all } v \in V_{h_2}^{(u_{1h}^n)}, \\ u_{2h}^{n+1} \leq r_h M u_{2h}^n, \quad v \leq r_h M u_{2h}^n, \\ u_{2h}^{n+1} = u_{1h}^n \text{ on } \Gamma_2, \quad v = u_{2h}^n \text{ on } \Gamma_2. \end{cases}$$

We will finish this section by the discrete version of Proposition 3.1, this version plays an important role in the sequel.

Proposition 4.1. *Using the notations*

$$\begin{aligned} u_h =& \sigma(f(u_h), Mu_h, \pi_h g), \\ \tilde{u_h} =& \sigma_h(f(\tilde{u_h}, M\tilde{u_h}, \pi_h \tilde{g}), \end{aligned}$$

where $u_h, \tilde{u_h} \in K_g(u_h)$, *we have*

$$\|u_h - \tilde{u_h}\|_{L^\infty(\Omega_i)} \leq \|f(u_h) - f(\tilde{u_h})\|_{L^\infty(\Omega_i)} + \|Mu_h - M\tilde{u_h}\|_{L^\infty(\Omega_i)} + \|\pi_h g - \pi_h \tilde{g}\|_{L^\infty(\Gamma_i)},$$

$\Gamma_i = \partial\Omega_i \cap \Omega_j$, $i, j = 1, 2$, *and* $i \neq j$.

Proof. Similar for the continuous case. □

Remark 4.1. If $Mu_h = M\tilde{u_h}$, we obtain

$$\|u_h - \tilde{u_h}\|_{L^\infty(\Omega_i)} \leq \|f(u_h) - f(\tilde{u_h})\|_{L^\infty(\Omega_i)} + \|\pi_h g - \pi_h \tilde{g}\|_{L^\infty(\Gamma_i)}.$$

5. L^∞-Error Estimate

We will use the algorithmic approach, which was used in [2, 4], but our problem is more complicated because the second member and the obstacle are related to the solution.

5.1. **Auxiliary sequences.** We introduce two discrete auxiliary sequences. Starting from $w_{ih}^0 = u_{ih}^0 = r_h M u_h^0 = k$, $i = 1,2$, define the sequences (w_{1h}^{n+1}) such that $w_{1h}^{n+1} \in V_{h_1}^{u_2^n}$

$$(5.1) \qquad \begin{cases} a_1(w_{1h}^{n+1}, v - w_{1h}^{n+1}) \geq (f_1(u_{1h}^n), v - w_{1h}^{n+1}), & \text{for all } v \in V_{h_1}^{(u_2^n)}, \\ w_{1h}^{n+1} \leq r_h M u_{1h}^n, \quad v \leq r_h M u_{1h}^n, \end{cases}$$

and (w_{2h}^{n+1}) such that $w_{2h}^{n+1} \in V_{h_2}^{(u_1^{n+1})}$ is a solution of

$$(5.2) \qquad \begin{cases} a_2(w_{2h}^{n+1}, v - w_{2h}^{n+1}) \geq (f_2(^n_{2h}), v - w_{2h}^{n+1}), & \text{for all } v \in V_{h_2}^{(u_1^{n+1})}, \\ w_{2h}^{n+1} \leq r_h M u_{2h}^n, \quad v \leq r_h M u_{2h}^n. \end{cases}$$

Note that w_{ih}^{n+1} is the finite element approximation of u_i^{n+1} which defined in (3.2) and (3.1). The following lemma will play a crucial role in proving the main result of this paper. The demonstration of the lemma is an adaptation of the one in [2], given for the problem of variational inequality.

Lemma 5.1. *We have the following inequalities:*

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \leq \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2,$$
$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \leq \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1.$$

Proof. In order to simplify the notation, we will adopt the following notations:

$$\begin{aligned} |\cdot|_1 =& \|\cdot\|_{L^\infty(\Gamma_1)}, \quad |\cdot|_2 = \|\cdot\|_{L^\infty(\Gamma_2)}, \\ \|\cdot\|_1 =& \|\cdot\|_{L^\infty(\Omega_1)}, \quad \|\cdot\|_2 = \|\cdot\|_{L^\infty(\Omega_2)}, \\ \pi_{h_1} =& \pi_{h_2} = \pi_h, \quad h_1 = h_2 = h. \end{aligned}$$

Started for $n = 0$, using the Remark 4.1, we get

$$\begin{aligned} \|u_1^1 - u_{1h}^1\|_1 \leq& \|u_1^1 - w_{1h}^1\|_1 + \|w_{1h}^1 - u_{1h}^1\|_1 \\ \leq& \|u_1^1 - w_{1h}^1\|_1 + \|f_1(u_1^0) - f_1(u_{1h}^0)\|_1 + |\pi_h M u_2^0 - \pi_h M u_{2h}^0|_1 \\ \leq& \|u_1^1 - w_{1h}^1\|_1 + |M u_2^0 - M u_{2h}^0|_1, \\ \|u_1^1 - u_{1h}^1\|_1 \leq& \|u_1^1 - w_{1h}^1\|_1 + \|M u_2^0 - M u_{2h}^0\|_2, \end{aligned}$$

and, from Lemma 4.1, we obtain

$$(5.3) \qquad \|u_1^1 - u_{1h}^1\|_1 \leq \|u_1^1 - w_{1h}^1\|_1 + \|u_2^0 - u_{2h}^0\|_2.$$

Similarly, we obtain

$$\begin{aligned}\|u_2^1-u_{2h}^1\|_2 \leq&\|u_2^1-w_{2h}^1\|_2+\|w_{2h}^1-u_{2h}^1\|_2\\ \leq&\|u_2^1-w_{2h}^1\|_2+\|f_2(u_{2h}^0)-f_2(u_{2h}^0)\|_2+|\pi_hMu_1^1-\pi_hMu_{1h}^1|_2\\ \leq&\|u_2^1-w_{2h}^1\|_2+|Mu_1^1-Mu_{1h}^1|_2\\ \leq&\|u_2^1-w_{2h}^1\|_2+\|Mu_1^1-Mu_{1h}^1\|_1\end{aligned}$$

and

$$\|u_2^1-u_{2h}^1\|_2 \leq \|u_2^1-w_{2h}^1\|_2+\|u_1^1-u_{1h}^1\|_1.$$

From (5.3), we get

$$\|u_2^1-u_{2h}^1\|_2 \leq \|u_1^1-w_{1h}^1\|_1+\|u_2^0-u_{2h}^0\|_2+\|u_2^1-w_{2h}^1\|_2, \tag{5.4}$$

so

$$\begin{aligned}\|u_1^1-u_{1h}^1\|_1 \leq&\sum_{p=1}^{1}\|u_1^p-w_{1h}^p\|_1+\sum_{p=0}^{0}\|u_2^0-u_{2h}^0\|_2,\\ \|u_2^1-u_{2h}^1\|_2 \leq&\sum_{p=0}^{1}\|u_2^p-w_{2h}^p\|_2+\sum_{p=1}^{1}\|u_1^p-w_{1h}^p\|_1.\end{aligned}$$

For $n=1$, we have

$$\begin{aligned}\|u_1^2-u_{1h}^2\|_1 \leq&\|u_1^2-w_{1h}^2\|_1+\|w_{1h}^2-u_{1h}^2\|_1\\ \leq&\|u_1^2-w_{1h}^2\|_1+\|f(u_{1h}^1)-f(u_{1h}^1)\|_1+|\pi_hMu_2^1-\pi_hMu_{2h}^1|_1,\\ \leq&\|u_1^2-w_{1h}^2\|_1+|Mu_2^1-Mu_{2h}^1|_1\\ \leq&\|u_1^2-w_{1h}^2\|_1+\|u_2^1-u_{2h}^1\|_2.\end{aligned}$$

From (5.4), we get

$$\|u_1^2-u_{1h}^2\|_1 \leq \|u_2^1-w_{1h}^2\|_1+\|u_2^1-w_{2h}^1\|_2+\|u_1^1-w_{1h}^1\|_1+\|u_2^0-u_{2h}^0\|_2. \tag{5.5}$$

Similarly, we obtain

$$\begin{aligned}\|u_2^2-u_{2h}^2\|_2 \leq&\|u_2^2-w_{2h}^2\|_2+\|w_{2h}^2-u_{2h}^2\|_2\\ \leq&\|u_2^2-w_{2h}^2\|_2+\|f(u_{2h}^2)-f(u_{2h}^2)\|_2+|\pi_hMu_1^2-\pi_hMu_{1h}^2|_2\\ \leq&\|u_2^2-w_{2h}^2\|_2+\|u_1^2-u_{1h}^2\|_1.\end{aligned}$$

From (5.5), we get

$$\|u_2^2-u_{2h}^2\|_2 \leq \|u_2^2-w_{2h}^2\|_2+\|u_2^1-w_{1h}^2\|_1+\|u_2^1-w_{2h}^1\|_2+\|u_1^1-w_{1h}^1\|_1+\|u_2^0-u_{2h}^0\|_2,$$

where

$$\|u_1^2-u_{1h}^2\|_1 \leq \sum_{p=1}^{2}\|u_1^p-w_{1h}^p\|_1+\sum_{p=0}^{1}\|u_2^p-w_{2h}^p\|_2$$

and

$$\|u_2^2-u_{2h}^2\|_2 \leq \sum_{p=0}^{2}\|u_1^p-w_{2h}^p\|_2+\sum_{p=1}^{2}\|u_1^p-w_{1h}^p\|_1.$$

We go to the second step. Suppose that

$$\|u_2^n - u_{2h}^n\|_2 \le \sum_{p=0}^{n} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n} \|u_1^p - w_{1h}^p\|_1. \tag{5.6}$$

We claim the first inequality, for $i = 1$,

$$\begin{aligned}
\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|w_{1h}^{n+1} - u_{1h}^{n+1}\|_1 \\
\le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|f_1(u_{1h}^n) - f_1(u_{1h}^n)\|_1 + |\pi_h M u_2^n - \pi_h M u_{2h}^n|_1 \\
\le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|M u_2^n - M u_{2h}^n\|_2 \\
\le& \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \|u_2^n - u_{2h}^n\|_2.
\end{aligned}$$

From (5.6), we get

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \le \|u_1^{n+1} - w_{1h}^{n+1}\|_1 + \sum_{p=0}^{n} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=1}^{n} \|u_1^p - w_{1h}^p\|_1.$$

Consequently,

$$\|u_1^{n+1} - u_{1h}^{n+1}\|_1 \le \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n} \|u_2^p - w_{2h}^p\|_2. \tag{5.7}$$

For the second inequality, $i = 2$, we have

$$\begin{aligned}
\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|w_{2h}^{n+1} - u_{2h}^{n+1}\|_2 \\
\le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + |\ f_2(u_{2h}^n) - f_2(u_{2h}^n)\|_2 + |\pi_h M u_1^{n+1} - \pi_h M u_{1h}^{n+1}|_2 \\
\le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|M u_1^{n+1} - M u_{1h}^{n+1}\|_1 \\
\le& \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \|u_1^{n+1} - u_{1h}^{n+1}\|_1.
\end{aligned}$$

From (5.7), we get

$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \le \|u_2^{n+1} - w_{2h}^{n+1}\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n} \|u_2^p - w_{2h}^p\|_2.$$

Consequently,

$$\|u_2^{n+1} - u_{2h}^{n+1}\|_2 \le \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2 + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1. \qquad \square$$

5.2. L^∞ **error estimate.** The main result is given as follows.

Theorem 5.1. *Setting $h = \max\{h_1, h_2\}$, so there exists a constant C independent of h and n such that*

$$\|u_i - u_{ih}^{n+1}\|_{L^\infty(\Omega_i)} \le Ch^2 |\log h|^3, \quad i = 1, 2.$$

Proof. Indeed, let $K = \max\{k_1, k_2\}$, for $i = 1$ we have

$$\begin{aligned}\|u_1 - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)} \leq& \|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} + \|u_1^{n+1} - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)}\\ \leq& \|u_1 - u_1^{n+1}\|_{L^\infty(\Omega_1)} + \sum_{p=1}^{n+1} \|u_1^p - w_{1h}^p\|_1 + \sum_{p=0}^{n+1} \|u_2^p - w_{2h}^p\|_2\\ \leq& K^{2n}\|u^0 - u\|_{L^\infty(\Gamma_1)} + 2(n+1)C_1 h^2 |\log h|^2,\end{aligned}$$

where we used Lemma 4.1 and Theorem 3.1, respectively. Now, setting $K^{2n} \leq h^2$ we get $\|u_1 - u_{1h}^{n+1}\|_{L^\infty(\Omega_1)} \leq Ch^2|\log h|^3$. Similarly, we obtain the same result for $i = 2$. □

Remark 5.1. Confirmation for what we mentioned previously that this result is a generalization to the previous works, we note that:

(a) if the second member and the obstacle are not related to the solution, we get [2];
(b) if only the obstacle is related to the solution, we get [3];
(c) if only the second member is related to the solution, we get [4, 10, 11].

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SOME NEW INEQUALITIES ON GENERALIZATION OF HERMITE-HADAMARD AND BULLEN TYPE INEQUALITIES, APPLICATIONS TO TRAPEZOIDAL AND MIDPOINT FORMULA

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Abstract. In this paper, we give a new general identity for differentiable functions. A consequence of the identity is that we obtain some new general inequalities containing all of the Hermite-Hadamard and Bullen type for functions whose derivatives in absolute value at certain power are convex. Some applications to special means of real numbers are also given. Finally, some error estimates for the trapezoidal and midpoint formula are addressed.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following double inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

is known in the literature as the Hadamard inequality for convex mapping. Inequality (1.1) holds in the reversed direction if f is concave. More information on these inequalities can be found in several papers and monographs (see [2, 3]).

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, +\infty) \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Key words and phrases. Convex functions, Hermite-Hadamard type inequality, Bullen type inequality, general integral ineqaulities, trapezoidal and Midpoint formula.

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Theorem 1.1. *Suppose that $f : [a,b] \to \mathbb{R}$ is a convex function on $[a,b]$. Then we have the inequalities:*

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right)+f\left(\frac{a+3b}{4}\right)\right] \\
&\leq \frac{1}{b-a}\int_a^b f(x)dx \\
&\leq \frac{1}{2}\left[f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2}\right] \leq \frac{f(a)+f(b)}{2}. \qquad (1.2)
\end{aligned}
$$

The third inequality in (1.2) is known in the literature as Bullen's inequality.

Lemma 1.1 ([1])**.** *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $a, b \in I^\circ$, with $a < b$. If $f' \in L[a,b]$, then*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt. \qquad (1.3)$$

In [1] Dragomir and Agarwal established inequalities for differentiable convex functions which are related to Hadamard's inequlity as follows.

Theorem 1.2. ([1, Theorem 2.2]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$, with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a\left(|f'(a)|+|f'(b)|\right)}{8}. \qquad (1.4)$$

Theorem 1.3. ([1, Theorem 2.3]). *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$, with $a < b$ and let $p > 1$. If the new mapping $|f'|^{p/p-1}$ is convex on $[a,b]$, then the following inequality holds*

(1.5)
$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\frac{|f'(a)|^{p/p-1}+|f'(b)|^{p/p-1}}{2}\right]^{(p-1)/p}.$$

In [5], the above inequalities were generalized.

Theorem 1.4. ([5, Theorem 1 and 2]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$, with $a < b$ and $q \geq 1$. If $|f'|^q$ is convex on $[a,b]$, then the following inequalities hold*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}} \qquad (1.6)$$

and

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left(\frac{|f'(a)|^q+|f'(b)|^q}{2}\right)^{\frac{1}{q}}.$$

In [4], the above inequalities were further generalized.

Theorem 1.5. ([4, Theorem 2.3 and 2.4]). *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I^\circ$, with $a < b$ and $p > 1$. If $|f'|^{\frac{p}{p-1}}$ is convex on $[a, b]$, then the following inequalities hold*

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}} \times \left\{\left[|f'(a)|^{p/p-1} + 3|f'(b)|^{p/p-1}\right]^{(p-1)/p} + \left[3|f'(a)|^{p/p-1} + |f'(b)|^{p/p-1}\right]^{(p-1)/p}\right\}$$

and

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}(|f'(a)| + |f'(b)|).$$

Lemma 1.2 ([7]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $a, b \in I$, with $a < b$. If $f' \in L[a, b]$, then*

$$\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx$$

$$=\frac{b-a}{4}\int_0^1 \left(\frac{1}{2}-t\right)\left[f'\left(ta + (1-t)\frac{a+b}{2}\right) + f'\left(t\frac{a+b}{2} + (1-t)b\right)\right]dt. \tag{1.7}$$

Corollary 1.1. ([7, Corollary 3.4]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $a, b \in I$, with $a < b$, and $f' \in L[a, b]$. If $|f'|$ is convex on $[a, b]$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{16}(|f'(a)| + |f'(b)|). \tag{1.8}$$

Corollary 1.2. ([6, Corollary 3.3]). *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $a, b \in I$, with $a < b$ and $f' \in L[a, b]$. If $|f'|^q$ is convex on $[a, b]$ for $q \ge 1$, then*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right|$$

$$\le\frac{b-a}{16}\left(\frac{1}{12}\right)^{\frac{1}{q}}\left[\left(9|f'(a)|^q + 3|f'(b)|^q\right)^{\frac{1}{q}} + \left(3|f'(a)|^q + 9|f'(b)|^q\right)^{\frac{1}{q}}\right]. \tag{1.9}$$

2. Main Results

In order to establish our main results, we first establish the following lemma.

Lemma 2.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $a, b \in I^\circ$, with $a < b$. If $f' \in L[a,b]$, then the following equality holds*

$$\begin{aligned} I_n(f,a,b) =& \sum_{i=0}^{n-1} \frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right) + f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\ &- \frac{1}{b-a}\int_a^b f(x)dx \\ =& \sum_{i=0}^{n-1} \frac{b-a}{2n^2}\left[\int_0^1 (1-2t)f'\left(t\frac{(n-i)a+ib}{n}\right.\right. \\ &\left.\left. + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt\right]. \end{aligned} \tag{2.1}$$

Proof. If we take $n \in \mathbb{N}$ arbitrarily, then for $i \in \{1, 2, \ldots, n-1\}$ by integration by parts, we have

$$\begin{aligned} I_i =& \int_0^1 (1-2t)f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt \\ =& \frac{n}{a-b}(1-2t)f\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\Bigg|_0^1 \\ &+ \frac{2n}{a-b}\int_0^1 f\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)dt. \end{aligned}$$

By making use of the substitutions $x = t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n}$

$$\begin{aligned} I_i =& -\frac{n}{a-b}\left[f\left(\frac{(n-i)a+ib}{n}\right) + f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\ &- \frac{2n^2}{(a-b)^2}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(x)dx. \end{aligned}$$

Multiplying the both sides by $\frac{b-a}{2n^2}$, we have

$$\begin{aligned} \frac{b-a}{2n^2}I_i =& \frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right) + f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\ &- \frac{1}{b-a}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(x)dx. \end{aligned}$$

Finally, we have

$$\begin{aligned}
\sum_{i=0}^{n-1}\frac{b-a}{2n^2}I_i =& \sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\
&-\frac{1}{b-a}\sum_{i=0}^{n-1}\int_{\frac{(n-i)a+ib}{n}}^{\frac{(n-i-1)a+(i+1)b}{n}} f(x)dx \\
=& \sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)a+ib}{n}\right)+f\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right] \\
&-\frac{1}{b-a}\int_a^b f(x)dx. \qquad \square
\end{aligned}$$

Remark 2.1. If we choose n=1 in Lemma 2.1, then (2.1) reduces to (1.3).

Remark 2.2. If we choose n=2 in Lemma 2.1, then (2.1) reduces to (1.7).

Theorem 2.1. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, where $a, b \in I^\circ$, with $a < b$. If $|f'|^q$ is convex on $[a, b]$ for some fixed $q \geq 1$, then the following inequality is satisfied*

$$|I_n(f,a,b)| \leq \sum_{i=0}^{n-1}\frac{b-a}{4n^2}\left[\left(\frac{2n-2i-1}{2n}\right)|f'(a)|^q+\left(\frac{2i+1}{2n}\right)|f'(b)|^q\right]^{\frac{1}{q}}. \tag{2.2}$$

Proof. From Lemma 2.1 and by using the well known Power-mean inequality, we have

$$\begin{aligned}
&|I_n(f,a,b)| \\
&\leq \sum_{i=0}^{n-1}\frac{b-a}{2n^2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 |1-2t|\left|f'\left(t\frac{(n-i)a+ib}{n}+(1-t)\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

By using the convexity of $|f'|^q$, we have

$$\begin{aligned}
&|I_n(f,a,b)| \\
&\leq \sum_{i=0}^{n-1}\frac{b-a}{2n^2}\left[\int_0^1 |1-2t|\,dt\right]^{1-\frac{1}{q}}\left[\int_0^1 |1-2t|\left(t\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q\right.\right. \\
&\left.\left.+(1-t).\left|f'\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right)dt\right]^{\frac{1}{q}}
\end{aligned}$$

$$=\sum_{i=0}^{n-1}\frac{b-a}{2n^2}\left(\int_0^1 |1-2t|\,dt\right)^{1-\frac{1}{q}}\left(\int_0^1 |1-2t|\,t.\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q dt\right.$$

$$\left.+\int_0^1 |1-2t|\,(1-t)\left|f'\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q dt\right)^{\frac{1}{q}}$$

$$=\sum_{i=0}^{n-1}\frac{b-a}{2n^2}\left[\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\frac{\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q+f'\left|\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q}{4}\right)^{\frac{1}{q}}\right]$$

$$=\sum_{i=0}^{n-1}\frac{b-a}{n^2(2)^{2+\frac{1}{q}}}\left(\left|f'\left(\frac{(n-i)a+ib}{n}\right)\right|^q+\left|f'\left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right|^q\right)^{\frac{1}{q}}$$

Finally, by using the convexity of $|f'|^q$, we have

$$|I_n(f,a,b)|\leq\sum_{i=0}^{n-1}\frac{b-a}{n^2(2)^{2+\frac{1}{q}}}\left[\left(\frac{2n-2i-1}{n}\right)|f'(a)|^q+\left(\frac{2i+1}{n}\right)|f'(b)|^q\right]^{\frac{1}{q}}. \qquad \square$$

Remark 2.3. If we choose $n=1$ in Theorem 2.1, then (2.2) reduces to (1.6).

Remark 2.4. If we choose $n=1$ and $q=1$ in Theorem 2.1, then (2.2) reduces to (1.4).

Remark 2.5. If we choose $n=2$ in Theorem 2.1, then (2.2) reduces to (1.9).

Remark 2.6. If we choose $n=2$ and $q=1$ in Theorem 2.1, then (2.2) reduces to (1.8).

Corollary 2.1. *If we choose $n=3$ in Theorem 2.1, then we obtain*

$$\left|\frac{1}{6}\left[f(a)+f(b)+2f\left(\frac{2a+b}{3}\right)+2f\left(\frac{a+2b}{3}\right)\right]-\frac{1}{b-a}\int_0^1 f(x)dx\right|$$

$$\leq\frac{b-a}{6^{2+\frac{1}{q}}}\left[\left(5\,|f'(a)|^q+|f'(b)|^q\right)^{\frac{1}{q}}+\left(3\,|f'(a)|^q+3\,|f'(b)|^q\right)^{\frac{1}{q}}+\left(|f'(a)|^q+5\,|f'(b)|^q\right)^{\frac{1}{q}}\right].$$

Theorem 2.2. *Let $f: I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I°, where $a,b\in I^\circ$, with $a<b$. If $|f'|^q$ is convex on $[a,b]$ for some fixed $q>1$, then the following inequality is satisfied*

(2.3)

$$|I_n(f,a,b)|\leq\sum_{i=0}^{n-1}\frac{b-a}{n^2 2^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{2n-2i-1}{n}\right)|f'(a)|^q+\left(\frac{2i+1}{n}\right)|f'(b)|^q\right]^{\frac{1}{q}},$$

where $\frac{1}{q}+\frac{1}{p}=1$.

Proof. From Lemma 2.1 and by using the Hölder inequality, we have

$$|I_n(f,a,b)| \leq \sum_{i=0}^{n-1} \frac{b-a}{2n^2} \left[\left(\int_0^1 |1-2t|^p \, dt \right)^{\frac{1}{p}} \right.$$
$$\left. \times \left(\int_0^1 \left| f'\left(t\frac{(n-i)a+ib}{n} + (1-t)\frac{(n-i-1)a+(i+1)b}{n} \right) \right|^q dt \right)^{\frac{1}{q}} \right]$$

By using the convexity of $|f^{'}|^q$, we have

$$\begin{aligned}
&|I_n(f,a,b)| \\
\leq &\sum_{i=0}^{n-1} \frac{b-a}{2n^2} \left[\left(\int_0^1 |1-2t|^p \, dt \right)^{\frac{1}{p}} \right. \\
&\left. \times \left(\int_0^1 \left(t\left| f'\left(\frac{(n-i)a+ib}{n} \right) \right|^q + (1-t)\left| f'\left(\frac{(n-i-1)a+(i+1)b}{n} \right) \right|^q \right) dt \right)^{\frac{1}{q}} \right] \\
= &\sum_{i=0}^{n-1} \frac{b-a}{2^{1+\frac{1}{q}} n^2} \left[\left(\frac{1}{p+1} \right)^{\frac{1}{p}} \right. \\
&\left. \times \left(\left| f'\left(\frac{(n-i)a+ib}{n} \right) \right|^q + \left| f'\left(\frac{(n-i-1)a+(i+1)b}{n} \right) \right|^q \right)^{\frac{1}{q}} \right].
\end{aligned}$$

By using the convexity of $|f'|^q$, we have

$$|I_n(f,a,b)| \leq \sum_{i=0}^{n-1} \frac{b-a}{n^2 2^{1+\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(\frac{2n-2i-1}{n} \right) |f'(a)|^q + \left(\frac{2i+1}{n} \right) |f'(b)|^q \right]^{\frac{1}{q}} . \square$$

Remark 2.7. If we choose $n = 1$ in Theorem 2.2, then (2.3) reduces to (1.5).

Corollary 2.2. *If we choose $n = 2$ in Theorem 2.2, then we have*

$$\begin{aligned}
&\left| \frac{1}{2} \left[\frac{f(a)+f(b)}{2} + f\left(\frac{a+b}{2} \right) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\
= &\frac{b-a}{2^{3+\frac{2}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[(3|f'(a)|^q + |f'(b)|)^{\frac{1}{q}} + (|f'(a)|^q + 3|f'(b)|^q)^{\frac{1}{q}} \right].
\end{aligned}$$

Corollary 2.3. *If we choose $n = 3$ in Theorem 2.2, then we have*

$$\begin{aligned}
&\left| \frac{1}{6} \left[f(a) + f(b) + 2f\left(\frac{2a+b}{3} \right) + 2f\left(\frac{a+2b}{3} \right) \right] - \frac{1}{b-a} \int_0^1 f(x)dx \right| \\
\leq &\frac{b-a}{18 \cdot 6^{\frac{1}{q}}} \left(\frac{1}{p+1} \right)^{\frac{1}{p}} \left[\left(5\,|f'(a)|^q + |f'(b)|^q \right)^{\frac{1}{q}} + \left(3\,|f'(a)|^q + 3\,|f'(b)|^q \right)^{\frac{1}{q}} \right.
\end{aligned}$$

$$+\left(|f'(a)|^q+5|f'(b)|^q\right)^{\frac{1}{q}}\Bigg].$$

3. Applications to Special Means

We consider some special means, for which will get new inequalities. Let $a, b \in \mathbb{R}$.

(i) The arithmetic mean: $A = A(a,b) := \dfrac{a+b}{2}$, $a, b \geq 0$.

(ii) The harmonic mean:

$$H = H(a,b) := \frac{2ab}{a+b}, \quad a, b > 0.$$

(iii) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a, & \text{if } a = b, \\ \dfrac{b-a}{\ln b - \ln a}, & \text{if } a \neq b, \end{cases} \quad a, b > 0.$$

(iv) The p-logarithmic mean

$$L_p = L_p(a,b) := \begin{cases} a, & \text{if } a = b, \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, & \text{if } a \neq b, \end{cases} \quad p \in \mathbb{R}\backslash\{-1, 0\}, a, b > 0.$$

Proposition 3.1. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $t \in \mathbb{N}$, $t \geq 2$. Then, for all $q \geq 1$, the following inequality holds*

$$\left|\sum_{i=0}^{n-1}\frac{1}{n}A\left(\left(\frac{(n-i)a+ib}{n}\right)^t, \left(\frac{(n-i-1)a+(i+1)b}{n}\right)^t\right) - L_t^t(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1}\frac{(b-a)t}{n^2(2)^{2+\frac{1}{q}}}\left[\left(\frac{(2n-2i-1)}{n}\right)a^{(t-1)q} + \left(\frac{2i+1}{n}\right)b^{(t-1)q}\right]^{\frac{1}{q}}.$$

Proof. The proof is immediate from (2.2) in Theorem 2.1, with $f(x) = x^t$, $x \in [a,b]$, $t \in \mathbb{N}$, $t \geq 2$. □

Remark 3.1. (a) If we choose $n = 1$, in the Proposition 3.1, we have [5, Proposition 1] for positive real numbers.

(b) If we choose $n = 1$ and $q = 1$, in the Proposition 3.1, we have [1, Proposition 3.1] for positive real numbers.

Proposition 3.2. *Let $a, b \in \mathbb{R}$, $0 < a < b$ and $t \in \mathbb{N}$, $t \geq 2$. Then, for all $q > 1$, the following inequality holds*

$$\left|\sum_{i=0}^{n-1}\frac{1}{n}A\left(\left(\frac{(n-i)a+ib}{n}\right)^t, \left(\frac{(n-i-1)a+(i+1)b}{n}\right)^t\right) - L_t^t(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1}\frac{(b-a)t}{n^2 2^{1+\frac{1}{q}}}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\left(\frac{(2n-2i-1)}{n}\right)a^{(t-1)q} + \left(\frac{2i+1}{n}\right)b^{(t-1)q}\right]^{\frac{1}{q}}.$$

Proof. The proof is immediate from (2.3) in Theorem 2.2, with $f(x) = x^t$, $x \in [a,b]$, $t \in \mathbb{N}$, $t \geq 2$. □

Remark 3.2. If we choose $n = 1$, in the Proposition 3.2, we have [1, Proposition 3.2] for positive real numbers.

Proposition 3.3. *Suppose $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q \geq 1$, the following inequality holds*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} H^{-1}\left(\left(\frac{(n-i)a+ib}{n}\right), \left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right) - L^{-1}(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)}{n^2(2)^{2+\frac{1}{q}}} \left[\left(\frac{(2n-2i-1)}{n}\right)a^{-2q} + \left(\frac{2i+1}{n}\right)b^{-2q}\right]^{\frac{1}{q}}.$$

Proof. The proof is immediate from (2.2) in Theorem 2.1, with $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

Remark 3.3. (a) If we choose $n = 1$, in the Proposition 3.3, we have [5, Proposition 2] for positive real numbers.

(b) If we choose $n = 1$ and $q = 1$, in the Proposition 3.3, we have [1, Proposition 3.3] for positive real numbers.

Proposition 3.4. *Let $a, b \in \mathbb{R}$, $0 < a < b$. Then, for all $q > 1$, the following inequality holds*

$$\left|\sum_{i=0}^{n-1} \frac{1}{n} H^{-1}\left(\left(\frac{(n-i)a+ib}{n}\right), \left(\frac{(n-i-1)a+(i+1)b}{n}\right)\right) - L^{-1}(a,b)\right|$$
$$\leq \sum_{i=0}^{n-1} \frac{(b-a)}{n^2 2^{1+\frac{1}{q}}} \left(\frac{1}{p+1}\right)^{\frac{1}{p}} \left[\left(\frac{(2n-2i-1)}{n}\right)a^{-2q} + \left(\frac{2i+1}{n}\right)b^{-2q}\right]^{\frac{1}{q}}.$$

Proof. The proof is immediate from (2.3) in Theorem 2.2, with $f(x) = \frac{1}{x}$, $x \in [a,b]$. □

Remark 3.4. If we choose $n = 1$, in the Proposition 3.4, we have [1, Proposition 3.4] for positive real numbers.

4. Applications to Trapezoided and Midpoint Formulas

Throughout this section, let $f : [a,b] \to \mathbb{R}$ be integrable and let $I_t : a = x_0 < x_1 < \cdots < x_t = b$ be a partition of $[a,b]$ and $l_k = x_{k+1} - x_k$, $k = 0, 1, \ldots, t-1$. Tseng et al. described the following notations in [8]:

- the trapezoidal formula

$$T(f, I_t) = \sum_{k=0}^{t-1} \frac{f(x_k) + f(x_{k+1})}{2} l_k;$$

- the midpoint formula

$$M(f, I_t) = \sum_{k=0}^{t-1} f\left(\frac{x_k + x_{k+1}}{2}\right) l_k;$$

- the approximation error of $\int_a^b f(x)dx$ by $T(f, I_t)$

$$E(f, I_t) = \int_a^b f(x)dx - T(f, I_t);$$

- the approximation error of $\int_a^b f(x)dx$ by $M(f, I_t)$

$$F(f, I_t) = \int_a^b f(x)dx - M(f, I_t).$$

In [5] Pearce and Pecaric established the following proposition which is approximation errors for the trapezoidal and midpoint formulas.

Proposition 4.1. *Under the conditions of Theorem* 1.4*, we have the following inequalities*

$$|E(f, I_t)| \leq \frac{1}{4}\sum_{k=0}^{t-1}\left(\frac{|f'(x_k)|^q + |f'(x_{k+1})|^q}{2}\right)^{\frac{1}{q}} l_k^2 \leq \frac{\max\{|f'(a)|, |f'(b)|\}}{4}\sum_{k=0}^{t-1} l_k^2 \tag{4.1}$$

and

$$|F(f, I_t)| \leq \frac{1}{4}\sum_{k=0}^{t-1}\left(\frac{|f'(x_k)|^q + |f'(x_{k+1})|^q}{2}\right)^{\frac{1}{q}} l_k^2 \leq \frac{\max\{|f'(a)|, |f'(b)|\}}{4}\sum_{k=0}^{t-1} l_k^2.$$

We have the following proposition which reduce (4.1) in Propositions 4.1 as $n = 1$ on $[a, b]$.

Proposition 4.2. *Under the conditions of Theorem* 2.1*, we have the following inequalities*

$$\begin{aligned} &\left|\sum_{k=0}^{t-1}\sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)x_k + ix_{k+1}}{n}\right) + f\left(\frac{(n-i-1)x_k + (i+1)x_{k+1}}{n}\right)\right](x_{k+1} - x_k)\right. \\ &\left. -\int_a^b f(x)dx\right| \leq \sum_{k=0}^{t-1}\frac{(x_{k+1} - x_k)^2}{4n^2}\sum_{i=0}^{n-1}\left[\left(\frac{2n-2i-1}{2n}\right)|f'(x_k)|^q + \left(\frac{2i+1}{2n}\right)|f'(x_{k+1})|^q\right]^{\frac{1}{q}} \\ &\leq \frac{1}{4n}\max\{|f'(a)|, |f'(b)|\}\sum_{k=0}^{t-1}(x_{k+1} - x_k)^2. \end{aligned} \tag{4.2}$$

Proof. Apply Theorem 2.1 on $[x_k, x_{k+1}]$, $k = 0, 1, \ldots, t-1$, we get

$$\begin{aligned} &\left|\sum_{i=0}^{n-1}\frac{1}{2n}\left[f\left(\frac{(n-i)x_k + ix_{k+1}}{n}\right) + f\left(\frac{(n-i-1)x_k + (i+1)x_{k+1}}{n}\right)\right](x_{k+1} - x_k)\right. \\ &\left. - \int_{x_k}^{x_{k+1}} f(x)dx\right| \leq \frac{(x_{k+1} - x_k)^2}{4n^2}\sum_{i=0}^{n-1}\left[\left(\frac{2n-2i-1}{2n}\right)|f'(x_k)|^q + \left(\frac{2i+1}{2n}\right)|f'(x_{k+1})|^q\right]^{\frac{1}{q}}. \end{aligned}$$

Taking into account that $|f'|^q$ is convex, we deduce, by the triangle inequality, that

$$\left[\left(\frac{2n-2i-1}{2n}\right)|f'(x_k)|^q + \left(\frac{2i+1}{2n}\right)|f'(x_{k+1})|^q\right] \leq \max\{|f'(a)|^q, |f'(b)|^q\}.$$

Finally, summing over k from 0 to $t-1$, we have (4.2). $\square$

Remark 4.1. If we choose $n=1$ on $[a,b]$, then the (4.2) reduce to (4.1).

Corollary 4.1. *If we choose $n=2$ in Proposition 4.2, we get*

$$\begin{aligned}
&|E(f,I_t)+F(f,I_t)| \\
\leq &\sum_{k=0}^{t-1}\frac{(x_{k+1}-x_k)^2}{8}\left[\left(\frac{3}{4}\left|f'(x_k)\right|^q+\frac{1}{4}\left|f'(x_{k+1})\right|\right)^{\frac{1}{q}}+\left(\frac{1}{4}\left|f'(x_k)\right|^q+\frac{3}{4}\left|f'(x_{k+1})\right|^q\right)^{\frac{1}{q}}\right] \\
\leq &\frac{1}{4}\max\{|f'(a)|,|f'(b)|\}\sum_{k=0}^{t-1}(x_{k+1}-x_k)^2.
\end{aligned}$$

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