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PERFECT NILPOTENT GRAPHS

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ABSTRACT. Let R be a commutative ring with identity. The nilpotent graph of R, denoted by $\Gamma_N(R)$, is a graph with vertex set $Z_N(R)^*$, and two vertices x and y are adjacent if and only if xy is nilpotent, where $Z_N(R) = \{x \in R \mid xy \text{ is nilpotent, for some } y \in R^*\}$. A perfect graph is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph. In this paper, we characterize all rings whose $\Gamma_N(R)$ is perfect. In addition, it is shown that for a ring R, if R is Artinian, then $\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = |\mathrm{Nil}(R)^*| + |\mathrm{Max}(R)|$.

1. INTRODUCTION

The theory of graphs associated with rings was started by Beck [4] in 1981 and has grown a lot since then. Anderson and Livingston [2] modified Beck's definition and introduced the notion of zero-divisor graph. Surely, this is the most important graph associated with a ring and not only zero-divisor graphs but also various generalizations of it have attracted many researchers, see for instance [9,11] and [10]. The *zero-divisor graph* of a ring R, denoted by $\Gamma(R)$, is a graph with the vertex set $Z(R)^*$ and two distinct vertices x and y are joined by an edge if and only if $xy = 0$, where $Z(R)$ is set of zero-divisors of R. In [6], Chen defined a kind of graph structure of rings. He let all the elements of ring R be the vertices of the graph and two vertices x and y are adjacent if and only if xy is nilpotent. However, in 2010, Li and Li [10] modified and studied the *nilpotent graph* $\Gamma_N(R)$ of R is a graph with vertex set $Z_N(R)^*$, and two vertices x and y are adjacent if and only if xy is nilpotent, where $Z_N(R) = \{x \in R \mid xy \text{ is nilpotent, for some } y \in R^*\}$. Note that the usual zero-divisor graph $\Gamma(R)$ is a subgraph of the graph $\Gamma_N(R)$. B. Smith determine all values of n for which zero-divisor graph of $\mathbb{Z}_n$ is perfect [13]. Also, Patil et al. [12] characterize

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various algebraic and order structures whose zero-divisor graphs are perfect graph. Therefore, this paper is devoted to study the perfect of a super graph of zero-divisor graphs. First let us recall some necessary notation and terminology from ring theory and graph theory.

Throughout this paper, all rings are assumed to be commutative with identity. We denote by $Z(R)$, $\mathrm{U}(R)$, $\mathrm{Max}(R)$ and $\mathrm{Nil}(R)$, the set of all zero-divisors, the set of all unit elements of R, the set of all maximal ideals of R and the set of all nilpotent elements of R, respectively. For a subset A of a ring R, we let $A^* = A \setminus \{0\}$. The ring R is said to be *reduced* if it has no non-zero nilpotent element. Some more definitions about commutative rings can be find in [3, 5, 15].

We use the standard terminology of graphs following [7, 14]. Let $G = (V, E)$ be a graph, where $V = V(G)$ is the set of vertices and $E = E(G)$ is the set of edges. By $\bar{G}$, we mean the complement graph of G. We write $u - v$, to denote an edge with ends u, v. A graph $H = (V_0, E_0)$ is called a *subgraph* of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called an *induced subgraph* by V_0, denoted by $G[V_0]$, if $V_0 \subseteq V$ and $E_0 = \{u, v \in E | u, v \in V_0\}$. Also G is called a *null graph* if it has no edge. A complete graph of n vertices is denoted by K_n. An n-partite graph is one whose vertex set can be partitioned into n subsets, so that no edge has both ends in any one subset. A complete n-partite graph is one in which each vertex is jointed to every vertex that is not in the same subset. A *clique* of G is a maximal complete subgraph of G and the number of vertices in the largest clique of G, denoted by $\omega(G)$, is called the clique number of G. For a graph G, let $\chi(G)$ denote the *chromatic number* of G, i.e., the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. Note that for every graph G, $\omega(G) \leq \chi(G)$. A graph G is said to be *weakly perfect* if $\omega(G) = \chi(G)$. A *perfect* graph G is a graph in which the chromatic number of every induced subgraph equals the size of the largest clique of that subgraph.

Using the Strong Perfect Graph Theorem, in Section 2 we completely determine all Artinian rings for which the nilpotent graph of R is perfect, leading to our main theorem. In Section 3 for an Artinian ring R, it is shown that the graph $\Gamma_N(R)$ is weakly perfect. Moreover, the exact value of the $\chi(\Gamma_N(R))$ is given.

2. On Perfect Graph

We start with some properties of the nilpotent elements of a ring. The following remark is useful in our proofs.

Remark 2.1. ([10, Remark 2, 3]). Let R be a non-reduced ring. Then the following statements hold.

(1) For every $x \in \mathrm{Nil}(R)^*$, x is adjacent to all non-zero elements of R and so $Z_N(R) = R$.
(2) $\Gamma_N(R)[\mathrm{Nil}(R)^*]$ is a (induced) complete subgraph of $\Gamma_N(R)$.

To prove our main results we need the following celebrate theorem.

Theorem 2.1 (The Strong Perfect Graph Theorem [7])**.** *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd cycle of length at least 5.*

The following result, which is proved in [1, Corollary 2.2], will be helpful in our main results and used frequently in the sequel.

Corollary 2.1. *Let G be a graph and $\{V_1, V_2\}$ be a partition of $V(G)$. If $G[V_i]$ is a complete graph, for every $1 \leq i \leq 2$, then G is a perfect graph.*

The following lemmas have a key role in this paper.

Lemma 2.1. *Let n be a positive integer and $R \cong R_1 \times R_2 \times \cdots \times R_n$, where R_i is a ring, for every $1 \leq i \leq n$. If $\Gamma_N(R)$ contains no induced odd cycle of length at least 5, then $n \leq 4$.*

Proof. Suppose that $n \geq 5$. Then we can easily get

$$(1,0,0,1,0,0,\ldots,0) - (0,1,0,0,1,0,\ldots,0) - (1,0,1,0,0,0,\ldots,0)$$
$$- (0,0,0,1,1,0,\ldots,0) - (0,1,1,0,0,0,\ldots,0) - (1,0,0,1,0,0,\ldots,0)$$

is a cycle of length 5. Thus, Theorem 2.1 lead to a contradiction. So, $n \leq 4$. □

Before proving first main result of this paper, we bring the following remark, which shows that Artinian rings share the following nice property.

Remark 2.2. Let $R \cong R_1 \times \cdots \times R_n$, $a = (x_1, x_2, \ldots, x_n)$ and $b = (y_1, y_2, \ldots, y_n)$, where n is a positive integer, every R_i is an Artinian local ring and $x_i, y_i \in R_i$ for every $1 \leq i \leq n$. Then

(1) a is adjacent to b in $\Gamma_N(R)$ if and only if $x_i y_i \in \mathrm{Nil}(R_i)$ for all $1 \leq i \leq n$;
(2) a is not adjacent to b in $\Gamma_N(R)$ if and only if $x_j y_j \in \mathrm{U}(R_j)$ for some $1 \leq j \leq n$;
(3) a is adjacent to b in $\overline{\Gamma_N(R)}$ if and only if $x_i y_i \in \mathrm{U}(R_i)$ for some $1 \leq i \leq n$;
(4) a is not adjacent to b in $\overline{\Gamma_N(R)}$ if and only if $x_j y_j \in \mathrm{Nil}(R_j)$ for all $1 \leq j \leq n$.

By using a similar way as used in the proof of [1, Lemma 2.3], one can prove the following result.

Lemma 2.2. *Let S_1, S_2, S_3, S_4 be rings such that $S_1 \cong R_1$, $S_2 \cong R_1 \times R_2$, $S_3 \cong R_1 \times R_2 \times R_3$ and $S_4 \cong R_1 \times R_2 \times R_3 \times R_4$, where R_i is a ring for every $1 \leq i \leq 4$. Then, if $\Gamma_N(S_4)$ is a perfect graph, then $\Gamma_N(S_3)$, $\Gamma_N(S_2)$ and $\Gamma_N(S_1)$ are perfect graphs.*

We are now in a position to state our first main result in this section.

Theorem 2.2. *Let R be a non-reduced Artinian ring. Then $\Gamma_N(R)$ is a perfect graph if and only if $|\mathrm{Max(R)}| \leq 4$.*

Proof. For one direction assume that $|\mathrm{Max(R)}| \leq 4$. This together with [3, Theorem 8.7] implies that there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$ and $n \leq 4$. By Theorem 2.1, it is enough to show that $\Gamma_N(R)$ and $\overline{\Gamma_N(R)}$ contains no induced odd cycle of length at least 5. By Lemma 2.2, we need to prove the case $n = 4$. So let $R \cong R_1 \times R_2 \times R_3 \times R_4$, where R_i is an Artinian local ring. We have the following two claims.

Claim 1. $\Gamma_N(R)$ contains no induced odd cycle of length at least 5. Note that if R is an Artinian non-reduced ring, then $Z_N(R) = R = U(R) \cup Z(R)$, where $U(R) = U(R_1) \times \cdots \times U(R_4)$. We consider the following partition for non-zero zero-divisors of R:

$$A = \{\{(x_1, x_2, x_3, x_4) \mid x_i \in \mathrm{Nil}(R_i) \text{ for all } i\} \backslash \{(0, 0, 0, 0)\}\},$$
$$B = \{(x_1, x_2, x_3, x_4) \mid \text{for some } i, x_i \notin \mathrm{Nil}(R_i)\}.$$

Thus $A \cap B = \varnothing, A \cap U(R) = \varnothing, B \cap U(R) = \varnothing$ and $V(\Gamma_N(R)) = A \cup B \cup U(R)$. Also we consider the following partition for B:

$$B_1 =\{(x, y, z, w) \in B \mid x \in \mathrm{U}(R_1)\},$$
$$B_2 =\{(x, y, z, w) \in B \mid x \in \mathrm{Nil}(R_1) \text{ and } y \in \mathrm{U}(R_2)\},$$
$$B_3 =\{(x, y, z, w) \in B \mid x \in \mathrm{Nil}(R_1),\ y \in \mathrm{Nil}(R_2) \text{ and } z \in \mathrm{U}(R_3)\},$$
$$B_4 =\{(x, y, z, w) \in B \mid x \in \mathrm{Nil}(R_1),\ y \in \mathrm{Nil}(R_2),\ z \in \mathrm{Nil}(R_3) \text{ and } w \in \mathrm{U}(R_4)\}.$$

It is easy to see that $B = \cup_{i=1}^{4} B_i$ and $B_i \cap B_j = \varnothing$ for every $i \neq j$. The elements of $V(\Gamma_N(R))$ have form $a_i = (x_i, y_i, z_i, w_i)$, where $x_i \in R_1, y_i \in R_2, z_i \in R_3$ and $w_i \in R_4$ for each $i \in \mathbb{N}$. Now, assume to the contrary that $a_1 - a_2 - \cdots - a_n - a_1$ is an induced odd cycle of length at least 5 in $\Gamma_N(R)$. We have the following cases.

Case 1. $\{a_1, \ldots, a_n\} \cap \mathrm{U}(R) = \varnothing$. Assume to the contrary and with no loss of generality that $a_1 = (x_1, y_1, z_1, w_1) \in \mathrm{U}(R)$. Then a_2 and a_n must be in $\mathrm{Nil}(R)^*$. Therefore, a_n is adjacent to a_2, which is a contradiction.

Case 2. $\{a_1, \ldots, a_n\} \cap A = \varnothing$. Let $a_i \in \{a_1, \ldots, a_n\} \cap A$, for some $1 \leq i \leq n$. Then by Remark 2.1, a_i is adjacent to all other vertices, a contradiction. Thus, $\{a_1, \ldots, a_n\} \cap A = \varnothing$.

Case 3. $\{a_1, \ldots, a_n\} \cap B_4 = \varnothing$. To show this, for a contradiction assume that $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Since a_2 and a_n are adjacent to a_1 and

$$a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4),$$

we see that the fourth components of a_2 and a_n must be in $\mathrm{Nil}(R_4)$. Now since x_3x_1, y_1y_3 and z_1z_3 are nilpotent elements and a_3 is not adjacent to a_1, by Part (2) of Remark 2.2, we conclude that the fourth component of a_3 must be in $\mathrm{U}(R_4)$. This together with the fact that a_4 is adjacent to a_3 imply that the fourth component of a_4 is nilpotent element and so $a_4a_1 \in \mathrm{Nil}(R)$. Therefore, a_4 is adjacent to a_1, which is a contradiction. So the assertion is proved.

Case 4. $\{a_1, \ldots, a_n\} \cap B_1 = \varnothing$. Assume to the contrary and with no loss of generality, $a_1 = (x_1, y_1, z_1, w_1) \in B_1$. It is easy to see that for every $1 \leq i \leq 4$, there is no edge between any two vertices of B_i. This together with the above cases imply that a_n and a_2 are in $B_2 \cup B_3$. We distinguish the following three subcases.

Subcase 4.1. $\{a_n, a_2\} \subset B_3$. In this case, we have

$$\{a_n, a_2\} \subset \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times R_4.$$

Then the third components of a_1 and a_3 must be in $\mathrm{Nil}(R_3)$. Also, since a_n is not adjacent to a_3, by Part (2) of Remark 2.2, the fourth components of a_n and a_3 must be in $\mathrm{U}(R_4)$. This yields

$$\begin{aligned} a_1 &\in \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times R_4, \\ a_3 &\in R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4), \\ a_n &\in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4). \end{aligned}$$

Then the fourth components of a_1 and a_2 must be in $\mathrm{Nil}(R_4)$. Hence, we find that

$$\begin{aligned} a_1 &\in \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4), \\ a_2 &\in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4). \end{aligned}$$

Now, since a_2 is not adjacent to a_4, the third components of a_4 must be in $\mathrm{U}(R_3)$. This implies that a_4 is not adjacent to a_n and so $n \geq 7$. It is easy to see that the third component of a_5 must be in $\mathrm{Nil}(R_3)$ and so $a_5a_2 \in \mathrm{Nil}(R)$. This implies that $a_5 - a_2$, a contradiction. So, in this case the assertion is proved.

Subcase 4.2. $\{a_n, a_2\} \subset B_2$. By a similar way as used in Subcase (4.1), we get a contradiction.

Subcase 4.3. $a_n \in B_2$ and $a_2 \in B_3$. By a similar way as used in Subcase (4.1), we get a contradiction. Thus $\{a_1, \dots, a_n\} \cap B_1 = \varnothing$.

By the above cases, $\{a_1, \dots, a_n\} \subseteq B_2 \cup B_3$, but this is contradicts the fact $\Gamma_N(R)[B_2 \cup B_3]$ is a bipartite graph, and thus, $\Gamma_N(R)$ contains no induced odd cycle of length at least 5.

In Claim 2, $\mathrm{U}(R)$, A, B and B_i are sets that mentioned in Claim 1.

Claim 2. $\overline{\Gamma_N(R)}$ contains no induced odd cycle of length at least 5. We show that $\overline{\Gamma_N(R)}$ contains no induced odd cycle at least 5. Assume to the contrary that

$$a_1 - a_2 - \cdots - a_n - a_1$$

is an induced odd cycle of length at least 5 in $\overline{\Gamma_N(R)}$. It is clear that $\overline{\Gamma_N(R)}[A]$ is a null graph and so $\{a_1, \dots, a_n\} \cap A = \varnothing$. Also, we show that

$$\{a_1, \dots, a_n\} \cap \mathrm{U}(R) = \varnothing.$$

Assume to the contrary and with no loss of generality that $a_1 \in \mathrm{U}(R)$. Obviously, a_1 is just adjacent to all of vertices of $Z_N(R) \setminus \mathrm{Nil}(R)$. This together with the fact that $\{a_1, \dots, a_n\} \subset Z_N(R) \setminus \mathrm{Nil}(R)$ imply that a_1 is adjacent to all other vertices, a contradiction. Thus $\{a_1, \dots, a_n\} \cap \mathrm{U}(R) = \varnothing$. We claim that

$$\{a_1, \dots, a_n\} \cap B_4 = \varnothing.$$

Indeed, if not, there would exist an $a_i \in B_4$. Without loss of generality, we may assume that $a_1 = (x_1, y_1, z_1, w_1) \in B_4$. Then $a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4)$. This together with Part (3) of Remark 2.2 implies that the forth components of a_2 and a_n must be in $\mathrm{U}(R_4)$ and so we have

$$a_n \in R_1 \times R_2 \times R_3 \times \mathrm{U}(R_4),$$
$$a_2 \in R_1 \times R_2 \times R_3 \times \mathrm{U}(R_4).$$

It is easy to see that a_2 is adjacent to a_n, a contradiction, and so,

$$\{a_1, \ldots, a_n\} \cap B_4 = \varnothing.$$

Finally to complete the proof, we prove that $\{a_1, \ldots, a_n\} \cap B_3 = \varnothing$. To get a contradiction, let $a_1 = (x_1, y_1, z_1, w_1) \in B_3$. Then

$$a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times R_4.$$

Since $a_1 - a_n$, $a_1 - a_2$ and a_2 is not adjacent to a_n, we consider the following two cases.

Case 1.

$$a_1 \in \mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4),$$
$$a_2 \in R_1 \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4),$$
$$a_n \in R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4).$$

Since a_3 is not adjacent to a_1, the third and the fourth components a_3 must be nilpotent. On the other hand, a_3 is adjacent to a_2. This implies that $x_3x_2 \in \mathrm{U}(R_1)$ or $y_2y_3 \in \mathrm{U}(R_2)$.

First suppose that $x_3x_2 \in \mathrm{U}(R_1)$. Now, we know that

$$a_3 \in \mathrm{U}(R_1) \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4),$$
$$a_2 \in \mathrm{U}(R_1) \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4).$$

This together with that a_3 is adjacent to a_4 implies that $x_3x_4 \in \mathrm{U}(R_1)$ or $y_3y_4 \in \mathrm{U}(R_2)$. If $x_3x_4 \in \mathrm{U}(R_1)$, then we have $x_2x_4 \in \mathrm{U}(R_1)$. Therefore, a_4 is adjacent to a_2, which is a contradiction. Thus, we conclude that $y_3y_4 \in \mathrm{U}(R_2)$. This yields

$$a_3 \in \mathrm{U}(R_1) \times \mathrm{U}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4),$$
$$a_4 \in \mathrm{Nil}(R_1) \times \mathrm{U}(R_2) \times R_3 \times R_4.$$

Since a_4 is not adjacent to a_1, we have

$$a_4 \in \mathrm{Nil}(R_1) \times \mathrm{U}(R_2) \times \mathrm{Nil}(R_3) \times \mathrm{Nil}(R_4).$$

Thus a_4 is not adjacent to a_n and so $n \geq 7$. On the other hand, since $a_4 - a_5$, the second components of a_5 must be unit and so a_5 is adjacent to a_2, which is a contradiction.

So, suppose that $y_2y_3 \in \mathrm{U}(R_2)$. Similarly, we get a contradiction. Thus in this case the assertion is proved.

Case 2.

$$\begin{aligned} a_1 \in &\mathrm{Nil}(R_1) \times \mathrm{Nil}(R_2) \times \mathrm{U}(R_3) \times \mathrm{U}(R_4), \\ a_2 \in &R_1 \times R_2 \times \mathrm{Nil}(R_3) \times \mathrm{U}(R_4), \\ a_n \in &R_1 \times R_2 \times \mathrm{U}(R_3) \times \mathrm{Nil}(R_4). \end{aligned}$$

By similar argument that of Case 1, we get a contradiction.

This means that $\{a_1, \dots, a_n\} \subseteq B_2 \cup B_1$. Clearly, $\overline{\Gamma_N(R)}[B_1]$, $\overline{\Gamma_N(R)}[B_2]$ are complete, and thus by Corollary 2.1, $\overline{\Gamma_N(R)}[B_1 \cup B_2]$ is a perfect graph, a contradiction. Hence $\overline{\Gamma_N(R)}$ contain no induced odd cycle of length at least 5. Therefore, by Claim 1, Claim 2 and Theorem 2.1, $\Gamma_N(R)$ is a perfect graph.

For the other direction, since $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$, then by Theorem 2.1 and Lemma 2.1, $n \leq 4$, as desired. □

3. The Nilpotent Graph of an Artinian Ring is Weakly Perfect

The main goal of this section is to study the coloring of the nilpotent graphs of Artinian rings. For an Artinian ring R, it is shown that the graph $\Gamma_N(R)$ is weakly perfect. Moreover, the exact value of the $\chi(\Gamma_N(R))$ is given.

Theorem 3.1. *Let R be an Artinian ring. Then*

$$\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = |\mathrm{Nil}(R)^*| + |\mathrm{Max}(R)|.$$

Proof. First let R be an Artinian local ring. One may easily check that $V(\Gamma_N(R)) = \mathrm{Nil}(R) \cup \mathrm{U}(R)$ and so $\{\mathrm{Nil}(R), \mathrm{U}(R)\}$ is a partition of $V(\Gamma_N(R))$. By Remark 2.1, we have $\Gamma_N(R)[\mathrm{Nil}(R)^*]$ is a complete subgraph of $\Gamma_N(R)$ and every vertex $x \in \mathrm{Nil}(R)^*$ is adjacent to all other vertices. This together with this fact that there is no adjacency between two vertices of $\mathrm{U}(R)$ imply that $\Gamma_N(R) = \Gamma_N(R)[\mathrm{Nil}(R)^*] \vee \Gamma_N(R)[\mathrm{U}(R)]$ and so

$$\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = \omega(\Gamma_N(R)[\mathrm{Nil}(R)^*]) + \omega(\Gamma_N(R)[\mathrm{U}(R)]) = |\mathrm{Nil}(R)^*| + 1.$$

Now, let R be an Artinian non-local ring. By [3, Theorem 8.7], one can deduce that there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. We put:

$$\begin{aligned} A = &\{\{(x_1, \dots, x_n) \mid x_i \in \mathrm{Nil}(R_i) \text{ for all } 1 \leq i \leq n\} \setminus \{(0,0,0,0)\}\}, \\ B = &\{(x_1, \dots, x_n) \mid \text{for some } i, x_i \notin \mathrm{Nil}(R_i)\}, \\ \mathrm{U}(R) = &\{(x_1, \dots, x_n) \mid x_i \in \mathrm{U}(R_i) \text{ for all } 1 \leq i \leq n\}. \end{aligned}$$

One may easily check that $V(\Gamma_N(R)) = A \cup B \cup \mathrm{U}(R)$, $A \cap B = \varnothing, A \cap \mathrm{U}(R) = \varnothing, B \cap \mathrm{U}(R) = \varnothing$ and so $\{A, B, \mathrm{U}(R)\}$ is a partition of $V(\Gamma_N(R))$. It is clear that $\Gamma_N(R)[U(R)] = \overline{K_{|\mathrm{U}(R)|}}$ and there is no adjacency between two vertices of B and $\mathrm{U}(R)$. To complete the proof, we prove that

$$\begin{aligned} \Gamma_N(R)[A \cup B] = &\Gamma_N(R)[A] \vee \Gamma_N(R)[B], \\ \Gamma_N(R)[A \cup \mathrm{U}(R)] = &\Gamma_N(R)[A] \vee \Gamma_N(R)[\mathrm{U}(R)], \end{aligned}$$

where $\Gamma_N(R)[A]$ is a complete subgraph of $\Gamma_N(R)$ and $\Gamma_N(R)[B]$ is an n-partite subgraph of $\Gamma_N(R)$, which is not an $(n-1)$-partite subgraph of $\Gamma_N(R)$. To see this, by Part (1) of Remark 2.1, we have $\Gamma_N(R)[A]$ is a complete subgraph of $\Gamma_N(R)$ and every vertex $x \in A$ is adjacent to all other vertices.

Now, for every $1 \leq i \leq n$, let $B_i = \{(x_1, \ldots, x_n) \in B \mid x_i \in \mathrm{U}(R_i)$ and $x_j \in \mathrm{Nil}(R_j)$ for every $1 \leq j \leq i\}$. It is easy to see that for every $1 \leq i \leq n$, there is no adjacency between two vertices of B_i. This together with this fact that the set $\{(1,0,0,\ldots,0),(0,1,0,\ldots,0),\ldots,(0,0,0,\ldots,1)\}$ is a clique of $\Gamma_N(R)[B]$ imply that $\Gamma_N(R)[B]$ is an n-partite subgraph of $\Gamma_N(R)$, which is not an $(n-1)$-partite subgraph of $\Gamma_N(R)$. Therefore,

$$\Gamma_N(R)[A \cup B] = \Gamma_N(R)[A] \vee \Gamma_N(R)[B],$$
$$\Gamma_N(R)[A \cup \mathrm{U}(R)] = \Gamma_N(R)[A] \vee \Gamma_N(R)[\mathrm{U}(R)]$$

and so

$$\omega(\Gamma_N(R)) = \chi(\Gamma_N(R)) = \omega(\Gamma_N(R)[A]) + \omega(\Gamma_N(R)[B]) = |\mathrm{Nil}(R)^*| + |\mathrm{Max}(R)|$$

and the proof is complete. □

We close this paper with the following result.

Theorem 3.2. *Let R be a non-reduced ring. Then the following statements are equivalent:*

(1) $\omega(\Gamma_N(R)) = 2$;
(2) $\chi(\Gamma_N(R)) = 2$;
(3) *either* $\Gamma_N(R) \cong K_{1,2}$ *or* $\Gamma_N(R) \cong K_1 \vee \overline{K_\infty}$.

Proof. (3) $\Rightarrow$ (1), (2) are clear. (2) $\Rightarrow$ (3) is obtained by similar argument to that proof of (1) $\Rightarrow$ (3). (1) $\Rightarrow$ (3) is only thing to prove.

(1) $\Rightarrow$ (3). Suppose that $\omega(\Gamma_N(R)) = 2$. First we show that $|\mathrm{Nil}(R)^*| = 1$. To see this, consider $A = \{a, b, c\}$ where $a, b \in \mathrm{Nil}(R)^*$ and $c \in \mathrm{U}(R)$. Then the subgraph induced by A is isomorphic to K_3, a contradiction. Thus, $|\mathrm{Nil}(R)^*| = 1$.

Now, we have two following cases.

Case 1. $Z(R) = \mathrm{Nil}(R)$. Since $|Z(R)^*| = 1 < \infty$, R is an Artinian (indeed R is finite). By [3, Theorem 8.7] there exists a positive integer n such that $R \cong R_1 \times \cdots \times R_n$, where each R_i, $1 \leq i \leq n$, is an Artinian local ring. If $n \geq 2$, then $Z(R)^* \geq 2$, a contradiction. So we may assume that R is an Artinian local ring. This, together [8, Example 1.5], implies that $R \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$ and so $\Gamma_N(R) \cong K_{1,2}$.

Case 2. $Z(R) \neq \mathrm{Nil}(R)$. Since $\omega(\Gamma_N(R)) = 2$ and by Remark 2.1, every $x \in \mathrm{Nil}(R)^*$, x is adjacent to all non-zero elements of R, we have only to show that $|Z(R)| = \infty$. To get a contradiction, let $|Z(R)| < \infty$. Then by [3, Theorem 8.7], we may write $R \cong R_1 \times \cdots \times R_n$, where R_i is an Artinian local ring, for every $1 \leq i \leq n$. Since $Z(R) \neq \mathrm{Nil}(R)$, we have $n \geq 2$. Also, since R is non-reduced, without loss of generality, we can suppose that $a \in \mathrm{Nil}(R_1)^*$. Consider $\phi = \{x, y, z\}$, where $x = (a, 0, \ldots, 0), y = (1, 0, \ldots, 0), z = (0, 1, 0, \ldots, 0)$. Then the subgraph induced by ϕ in $\Gamma_N(R)$ is isomorphic to K_3, a contradiction. Thus, $|Z(R)| = \infty$ and so $\Gamma_N(R) \cong K_1 \vee \overline{K_\infty}$ and the proof is complete. $\square$

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