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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF KIRRCHOFF ELLIPTIC SYSTEMS WITH RIGHT HAND SIDE DEFINED AS A MULTIPLICATION OF TWO SEPARATE FUNCTIONS

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Abstract. The paper deals with the study of existence of weak positive solutions for a new class of Kirrchoff elliptic systems in bounded domains with multiple parameters, where the right hand side defined as a multiplication of two separate functions.

1. Introduction

In this paper, we consider the following system of differential equations

$$
\begin{cases}
-A\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = \lambda_1 \alpha\left(x\right) f\left(v\right) h\left(u\right) \ \text{ in } \Omega, \\
-B\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v = \lambda_2 \beta\left(x\right) g\left(u\right) \tau\left(v\right) \ \text{ in } \Omega, \\
u = v = 0 \text{ on } \partial\Omega,
\end{cases} \tag{1.1}
$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a bounded smooth domain with C^2 boundary $\partial\Omega$, and $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions, $\alpha, \beta \in C\left(\overline{\Omega}\right)$, λ_1 and λ_2 are nonnegative parameters.

Since the first equation in (1.1) contains an integral over Ω, it is no longer a pointwise identity, therefore, it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends

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on the average of itself, such as the population density, see [9]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho \frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

presented by Kirchhoff in 1883 (see [10]). This equation is an extension of the classical d'Alembert's wave equation by considering the effect of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to ([3–5, 7, 9, 11]), in which the authors have used different methods to get the existence of solutions for Kirchhoff type equations. Our paper is motivated by the recent results in ([1, 2]). In the paper [2], Azzouz and Bensedik studied the existence of a positive weak solution for the nonlocal problem of the form

$$\begin{cases} -M\left(\int_\Omega |\nabla u|^2 dx \right) \triangle u = |u|^{p-2} u + \lambda f(x) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $p > 1$, with a sign-changing function f.

Using the sub-supersolution method combining a comparison principle introduced in [1], the authors established the existence of a positive solution for (1.3), where the parameter $\lambda > 0$ is small enough. In the present paper, we consider system (1.1) in the case when the nonlinearities are "sublinear" at infinity, see the condition $(H\ 3)$. Under suitable conditions on f, g, h and τ, we shall show that system (1.1) has a positive solution for $\lambda > \lambda^*$ large enough. To our best knowledge, this is a new research topic for nonlocal problems, see [8]. In current paper, motivated by previous works in ([2], [6]) and by using sub-super solutions method, we study of existence of weak positive solutions for a new class of Kirrchoff elliptic systems in bounded domains with multiple parameters, where the right hand side defined as a multiplication of two separate functions. Our results extend and improve our recent results in [3] and [11].

2. Existence Result

Lemma 2.1 ([2]). *Assume that $M : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous and nonincreasing function satisfying $\lim_{t\to 0^+} M(t) = m_0$, where m_0 is a positive constant. Suppose further that function $H(t) := tM(t^2)$ is increasing on $\mathbb{R}$.*

Assume that u, v *are two non-negative functions such that*

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u \geq -M\left(\int\limits_{\Omega} |\nabla v|^2 \, dx\right) \triangle v \ \textit{in } \Omega, \\ u = v = 0 \ \textit{on } \partial\Omega, \end{cases}$$

then $u \geq v$ *a.e. in* Ω.

Lemma 2.2 ([1])**.** *If* M *verifies the conditions of Lemma* 2.1*, then for each* $f \in L^2(\Omega)$ *there exists a unique solution* $u \in H_0^1(\Omega)$ *to the* M*-linear problem*

$$\begin{cases} -M\left(\int\limits_{\Omega} |\nabla u|^2 \, dx\right) \triangle u = f(x) \ \textit{ in } \Omega, \\ u = 0 \ \textit{in } \partial\Omega. \end{cases}$$

Lemma 2.3 ([6])**.** *Let* w *solve* $\Delta w = g$ *in* Ω*. If* $g \in C(\Omega)$*, then* $w \in C^{1,\alpha}(\Omega)$ *for any* $\alpha \in (0,1)$*, so particularly* w *is continuous in* Ω*.*

In this section, we shall state and prove the main result of this paper. Let us assume the following assumptions.

$(H1)$ Assume that $A, B : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy the same conditions as M in Lemma 1, and there exists $a_i, b_i > 0, i = 1, 2$, such that

$$a_1 \leq A(t) \leq a_2, \quad b_1 \leq B(t) \leq b_2, \quad \text{for all } t \in \mathbb{R}^+.$$

$(H2)$ $\alpha, \beta \in C\left(\overline{\Omega}\right)$ and

$$\alpha(x) \geq \alpha_0 > 0, \quad \beta(x) \geq \beta_0 > 0,$$

for all $x \in \Omega$.

$(H3)$ f, g, h, and τ are C^1 on $(0, +\infty)$, and increasing functions such that

$$\lim_{t\to+\infty} f(t) = +\infty, \quad \lim_{t\to+\infty} g(t) = +\infty, \quad \lim_{t\to+\infty} h(t) = +\infty = \lim_{t\to+\infty} \tau(t) = +\infty.$$

$(H4)$ Exists $\gamma > 0$ such that

$$\lim_{t\to+\infty} \frac{h(t) f(k[g(t)^\gamma])}{t} = 0, \quad \text{for all } k > 0,$$

and

$$\lim_{t\to+\infty} \frac{\tau(kt^\gamma)}{t^{\gamma-1}} = 0, \quad \text{for all } k > 0.$$

We present below an example where hypotheses $(H3)$ and $(H4)$ hold

$$\tau(t) = \ln(t), \quad h(t) = \sqrt{t}, \quad f(t) = \ln(t), \quad g(t) = t, \quad \gamma = 2.$$

Theorem 2.1. *Assume that the conditions* $(H1)$*-*$(H4)$ *hold. Then for* $\lambda_1\alpha_0$ *and* $\lambda_2\beta_0$ *large the problem* (1.1) *has a large positive weak solution.*

We give the following two definitions before we give our main result.

Definition 2.1. Let $(u,v) \in (H_0^1(\Omega) \times H_0^1(\Omega))$, (u,v) is said to be a weak solution of (1.1) if it satisfies

$$A\left(\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\int_\Omega \nabla u \nabla \phi \mathrm{d}x = \lambda_1 \int_\Omega \alpha(x) f(v) h(u) \phi \,\mathrm{d}x \text{ in } \Omega,$$
$$B\left(\int_\Omega |\nabla v|^2 \mathrm{d}x\right)\int_\Omega \nabla v \nabla \psi \mathrm{d}x = \lambda_2 \int_\Omega \beta(x) g(u) \tau(v) \psi \,\mathrm{d}x \text{ in } \Omega,$$

for all $(\phi,\psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Definition 2.2. A pair of nonnegative functions $(\underline{u},\underline{v}), (\overline{u},\overline{v})$ in $(H_0^1(\Omega) \times H_0^1(\Omega))$ are called a weak subsolution and supersolution of (1.1) if they satisfy $(\underline{u},\underline{v}), (\overline{u},\overline{v}) = (0,0)$ on $\partial\Omega$

$$A\left(\int_\Omega |\nabla \underline{u}|^2 dx\right)\int_\Omega \nabla \underline{u} \nabla \phi \mathrm{d}x \leq \lambda_1 \int_\Omega \alpha(x) f(\underline{v}) h(\underline{u}) \phi \mathrm{d}x \text{ in } \Omega,$$
$$B\left(\int_\Omega |\nabla \underline{v}|^2 dx\right)\int_\Omega \nabla \underline{v} \nabla \psi \mathrm{d}x \leq \lambda_2 \int_\Omega \beta(x) g(\underline{u}) \tau(\underline{v}) \psi \mathrm{d}x \text{ in } \Omega$$

and

$$A\left(\int_\Omega |\nabla \overline{u}|^2 dx\right)\int_\Omega \nabla \overline{u} \nabla \phi \mathrm{d}x \geq \lambda_1 \int_\Omega \alpha(x) f(\overline{v}) h(\overline{u}) \phi \mathrm{d}x \text{ in } \Omega,$$
$$B\left(\int_\Omega |\nabla \overline{v}|^2 dx\right)\int_\Omega \nabla \overline{v} \nabla \psi \mathrm{d}x \geq \lambda_2 \int_\Omega \beta(x) g(\overline{u}) \tau(\overline{v}) \psi \mathrm{d}x \text{ in } \Omega,$$

for all $(\phi,\psi) \in (H_0^1(\Omega) \times H_0^1(\Omega))$.

Proof of Theorem 1. Let σ be the first eigenvalue of $-\triangle$ with Dirichlet boundary conditions and ϕ_1 the corresponding positive eigenfunction, with $\|\phi_1\| = 1$. Let $m_0, \delta > 0$ be such that $|\nabla\phi_1|^2 - \sigma\phi_1^2 \geq m_0$ on $\overline{\Omega}_\delta = \{x \in \Omega : d(x,\partial\Omega) \leq \delta\}$.

For each $\lambda_1\alpha_0$ and $\lambda_2\beta_0$ large, let us define

$$\underline{u} = \left(\frac{\lambda_1\alpha_0}{2a_1}\right)\phi_1^2 \quad \text{and} \quad \underline{v} = \left(\frac{\lambda_2\beta_0}{2b_1}\right)\phi_1^2,$$

where a_1, b_1 are given by the condition $(H1)$. We shall verify that $(\underline{u},\underline{v})$ is a weak subsolution of problem (1.1), for $\lambda_1\alpha_0$ and $\lambda_2\beta_0$ large enough. Indeed, let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. By $(H1)$-$(H3)$, a simple calculation shows that

$$A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 dx\right)\int_{\overline{\Omega}_\delta} \nabla \underline{u}.\nabla\phi dx = A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 dx\right)\frac{\lambda_1\alpha_0}{a_1}\int_{\overline{\Omega}_\delta} \phi_1\nabla\phi_1.\nabla\phi dx$$

$$
\begin{aligned}
&= \frac{\lambda_1 \alpha_0}{a_1} A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \\
&\quad \times \left\{ \int_{\overline{\Omega}_\delta} \nabla \phi_1 \nabla (\phi_1 . \phi) \, dx - \int_{\overline{\Omega}_\delta} |\nabla \phi_1|^2 \phi dx \right\} \\
&= \frac{\lambda_1 \alpha_0}{a_1} A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\overline{\Omega}_\delta} \left(\sigma \phi_1^2 - |\nabla \phi_1|^2 \right) \phi dx.
\end{aligned}
$$

On $\overline{\Omega}_\delta$ we have $|\nabla \phi_1|^2 - \sigma \phi_1^2 \geq m_0$, then $\sigma \phi_1^2 - |\nabla \phi_1|^2 < 0$. So,

$$
A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx < 0,
$$

by $(H3)$ for $\lambda_1 \alpha_0$ and $\lambda_2 \beta_0$ large enough we get $f(\underline{v}) h(\underline{u}) > 0$. And then

$$
A\left(\int_{\overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx \ \leq \lambda_1 \int_{\overline{\Omega}_\delta} \alpha(x) f(\underline{v}) h(\underline{u}) \phi dx. \tag{2.1}
$$

Next, on $\Omega \backslash \overline{\Omega}_\delta$ we have $\phi_1 \geq r$ for some $r > 0$, and therefore, by the conditions $(H1)$-$(H3)$ and the definition of $\underline{u}$ and $\underline{v}$, it follows that

$$
\begin{aligned}
\lambda_1 \int_{\Omega \backslash \overline{\Omega}_\delta} \alpha(x) f(\underline{v}) h(\underline{u}) \phi dx &\geq \frac{\lambda_1 \alpha_0 a_2}{a_1} \sigma \int_{\Omega \backslash \overline{\Omega}_\delta} \phi dx \\
&\geq \frac{\lambda_1 \alpha_0}{a_1} A\left(\int_{\Omega \backslash \overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega \backslash \overline{\Omega}_\delta} \sigma \phi dx \\
&\geq \frac{\lambda_1 \alpha_0}{a_1} A\left(\int_{\Omega \backslash \overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega \backslash \overline{\Omega}_\delta} \left(\sigma \phi_1^2 - |\nabla \phi_1|^2 \right) \phi dx \\
&= A\left(\int_{\Omega \backslash \overline{\Omega}_\delta} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega \backslash \overline{\Omega}_\delta} \nabla \underline{u} \nabla \phi dx,
\end{aligned} \tag{2.2}
$$

for $\lambda_1 \alpha_0 > 0$ large enough.

Relations (2.1) and (2.2) imply that

$$
A\left(\int_{\Omega} |\nabla \underline{u}|^2 \, dx \right) \int_{\Omega} \nabla \underline{u} \nabla \phi dx \leq \lambda_1 \int_{\Omega} \alpha(x) f(\underline{v}) h(\underline{u}) \phi dx \text{ in } \Omega, \tag{2.3}
$$

for $\lambda_1 \alpha_0 > 0$ large enough and any $\phi \in H_0^1(\Omega)$, with $\phi \geq 0$ in Ω.

Similarly,

$$B\left(\int_{\Omega}|\nabla \underline{v}|^{2}\,dx\right)\int_{\Omega}\nabla \underline{v}\nabla\psi dx\leq\lambda_{2}\int_{\Omega}\beta\left(x\right)g\left(\underline{u}\right)\tau\left(\underline{v}\right)\psi dx \text{ in } \Omega, \tag{2.4}$$

for $\lambda_2\beta_0 > 0$ large enough and any $\psi \in H_0^1(\Omega)$, with $\psi \geq 0$ in Ω. From (2.3) and (2.4), $(\underline{u}, \underline{v})$ is a subsolution of problem (1.1). Moreover, we have $\underline{u} > 0$ and $\underline{v} > 0$ in Ω, $\underline{u} \to +\infty$ and $\underline{v} \to +\infty$ as $\lambda_1\alpha_0 \to +\infty$ and $\lambda_2\beta_0 \to +\infty$.

Next, we shall construct a weak supersolution of problem (1.1). Let e be the solution of the following problem

$$\begin{cases} -\triangle e = 1 \text{ in } \Omega, \\ e = 0 \text{ on } \partial\Omega. \end{cases} \tag{2.5}$$

Let

$$\overline{u} = Ce, \quad \overline{v} = \left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\right)\left[g\left(C\|e\|_\infty\right)\right]^\gamma e,$$

where γ is given by (H_4) and $C > 0$ is a large positive real number to be chosen later. We shall verify that $(\overline{u}, \overline{v})$ is a supersolution of problem (1.1). Let $\phi \in H_0^1(\Omega)$ with $\phi \geq 0$ in Ω. Then we obtain from (2.5) and the condition $(H1)$ that

$$\begin{aligned} A\left(\int_{\Omega}|\nabla\overline{u}|^2 dx\right)\int_{\Omega}\nabla\overline{u}.\nabla\phi dx =& A\left(\int_{\Omega}|\nabla\overline{u}|^2 dx\right)C\int_{\Omega}\nabla e.\nabla\phi dx \\ =& A\left(\int_{\Omega}|\nabla\overline{u}|^2 dx\right)C\int_{\Omega}\phi dx \\ \geq& a_1 C\int_{\Omega}\phi dx. \end{aligned}$$

By $(H4)$, we can choose C large enough so that

$$a_1 C \geq \lambda_1\|\alpha\|_\infty f\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\|e\|_\infty\left[g\left(C\|e\|_\infty\right)\right]^\gamma\right)h\left(C\|e\|_\infty\right).$$

Therefore,

$$\begin{aligned} & A\left(\int_{\Omega}|\nabla\overline{u}|^2 dx\right)\int_{\Omega}\nabla\overline{u}.\nabla\phi dx \\ \geq& \lambda_1\|\alpha\|_\infty f\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\|e\|_\infty\left[g\left(C\|e\|_\infty\right)\right]^\gamma\right).h\left(C\|e\|_\infty\right)\int_{\Omega}\phi dx \\ \geq& \lambda_1\int_{\Omega}\|\alpha\|_\infty f\left(\frac{\lambda_2\|\beta\|_\infty}{b_1}\|e\|_\infty\left[g\left(C\|e\|_\infty\right)\right]^\gamma\right).h\left(C\|e\|_\infty\right)\phi dx \\ \geq& \lambda_1\int_{\Omega}\alpha(x)f(\overline{v})h(\overline{u})\phi\,\mathrm{d}x. \end{aligned} \tag{2.6}$$

Also,

$$B\left(\int_{\Omega}|\nabla\overline{v}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\lambda_{2}\,\|\beta\|_{\infty}\int_{\Omega}\left[g\left(C\,\|e\|_{\infty}\right)\right]^{\gamma}\psi dx. \tag{2.7}$$

Again by $(H4)$ for C large enough we have

$$\left[g\left(C\,\|e\|_{\infty}\right)\right]^{\gamma}\geq g\left(C\,\|e\|_{\infty}\right)\tau\left(\frac{\lambda_{2}\,\|\beta\|_{\infty}\,\|e\|_{\infty}}{b_{1}}\left[g\left(C\,\|e\|_{\infty}\right)\right]^{\gamma}\right). \tag{2.8}$$

From (2.7) and (2.8), we have

$$B\left(\int_{\Omega}|\nabla\overline{v}|^{2}\,dx\right)\int_{\Omega}\nabla\overline{v}\nabla\psi dx\geq\lambda_{2}\int_{\Omega}\beta\left(x\right)g\left(\overline{u}\right)\tau\left(\overline{v}\right)\psi dx. \tag{2.9}$$

From (2.6) and (2.9) we have$(\overline{u},\overline{v})$ is a weak supersolution of problem (1.1), with $\underline{u}\leq\overline{u}$ and $\underline{v}\leq\overline{v}$ for C large.

In order to obtain a weak solution of problem (1.1) we define the sequence

$$\{(u_n,v_n)\}\subset E=\left(H_0^1\left(\Omega\right)\times H_0^1\left(\Omega\right)\right)\cap\left(C\left(\Omega\right)\times C\left(\Omega\right)\right)$$

as follows: $(u_0,v_0):=(\overline{u},\overline{v})\in E$ and (u_n,v_n) is the unique solution of the system

$$\begin{cases}-A\left(\int_{\Omega}|\nabla u_n|^{2}\,dx\right)\triangle u_n=\lambda_1\alpha\left(x\right)f\left(v_{n-1}\right)h\left(u_{n-1}\right)\ \text{ in }\Omega,\\ -B\left(\int_{\Omega}|\nabla v_n|^{2}\,dx\right)\triangle v_n=\lambda_2\beta\left(x\right)g\left(u_{n-1}\right)\tau\left(v_{n-1}\right)\ \text{ in }\Omega,\\ u_n=v_n=0\text{ on }\partial\Omega.\end{cases} \tag{2.10}$$

Problem (2.10) is (A,B)-linear in the sense that, if $(u_{n-1},v_{n-1})\in E$ is a given, the right hand sides of (2.10) is independent of u_n,v_n.

Set $\widetilde{A}\left(t\right)=tA\left(t^2\right),\ \widetilde{B}\left(t\right)=tB\left(t^2\right).$ Then, since $\widetilde{A}\left(\mathbb{R}\right)=\mathbb{R},\ \ \widetilde{B}\left(\mathbb{R}\right)=\mathbb{R},\ f\left(v_0\right),$ $h\left(u_0\right)$, $g\left(u_0\right)$ and $\tau\left(v_0\right)\in C\left(\Omega\right)\subset L^2\left(\Omega\right)$ (in x), we deduce from Lemma 2.2 that system (2.10), with $n=1$ has a unique solution $(u_1,v_1)\in\left(H_0^1\left(\Omega\right)\times H_0^1\left(\Omega\right)\right).$ And by observing that

$$\begin{cases}-\triangle u_1=\dfrac{\lambda_1}{A\left(\int_{\Omega}|\nabla u_1|^{2}\,dx\right)}\alpha f\left(v_0\right)h\left(u_0\right)\in C\left(\Omega\right),\\ -\triangle v_1=\dfrac{\lambda_2}{B\left(\int_{\Omega}|\nabla v_1|^{2}\,dx\right)}\beta g\left(u_0\right)\tau\left(v_0\right)\in C\left(\Omega\right),\\ u_1=v_1=0\text{ on }\partial\Omega.\end{cases}$$

We deduce from Lemma 2.3 that $(u_1, v_1) \in C(\Omega) \times C(\Omega)$. Consequently $(u_1, v_1) \in E$. By the same way we construct the following elements $(u_n, v_n) \in E$ of our sequence. From (2.10) and the fact that (u_0, v_0) is a weak supersolution of (1.1), we have

$$\begin{cases} -A\left(\int_\Omega |\nabla u_0|^2\, dx\right) \triangle u_0 \geq \lambda_1 \alpha(x) f(v_0) h(u_0) = -A\left(\int_\Omega |\nabla u_1|^2\, dx\right) \triangle u_1, \\ -B\left(\int_\Omega |\nabla v_0|^2\, dx\right) \triangle v_0 \geq \lambda_2 \beta(x) g(u_0) \tau(v_0) = -B\left(\int_\Omega |\nabla v_1|\, dx\right) \triangle v_1, \end{cases}$$

and by Lemma 1, $u_0 \geq u_1$ and $v_0 \geq v_1$. Also, since $u_0 \geq \underline{u}$, $v_0 \geq \underline{v}$ and the monotonicity of f, h, g, and τ one has

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2\, dx\right) \triangle u_1 =& \lambda_1 \alpha(x) f(v_0) h(u_0) \\ \geq& \lambda_1 \alpha(x) f(\underline{v}) h(\underline{u}) \geq -A\left(\int_\Omega |\nabla \underline{u}|^2\, dx\right) \triangle \underline{u}, \\ -B\left(\int_\Omega |\nabla v_1|^2\, dx\right) \triangle v_1 =& \lambda_2 \beta(x) g(u_0) \tau(v_0) \\ \geq& \lambda_2 \beta(x) g(\underline{u}) \tau(\underline{v}) \geq -B\left(\int_\Omega |\nabla \underline{v}|^2\, dx\right) \triangle \underline{v}, \end{aligned}$$

from which, according to Lemma 1, $u_1 \geq \underline{u}$, $v_1 \geq \underline{v}$, for u_2, v_2 we write

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_1|^2\, dx\right) \triangle u_1 =& \lambda_1 \alpha(x) f(v_0) h(u_0) \\ \geq& \lambda_1 \alpha(x) f(v_1) h(u_1) = -A\left(\int_\Omega |\nabla u_2|^2\, dx\right) \triangle u_2, \\ -B\left(\int_\Omega |\nabla v_1|\, dx\right) \triangle v_1 =& \lambda_2 \beta(x) g(u_0) \tau(v_0) \\ \geq& \lambda_2 \beta(x) g(u_1) \tau(v_1) = -B\left(\int_\Omega |\nabla v_2|^2\, dx\right) \triangle v_2, \end{aligned}$$

and then $u_1 \geq u_2$, $v_1 \geq v_2$. Similarly, $u_2 \geq \underline{u}$ and $v_2 \geq \underline{v}$ because

$$\begin{aligned} -A\left(\int_\Omega |\nabla u_2|^2\, dx\right) \triangle u_2 =& \lambda_1 \alpha(x) f(v_1) h(u_1) \\ \geq& \lambda_1 \alpha(x) f(\underline{v}) h(\underline{u}) \geq -A\left(\int_\Omega |\nabla \underline{u}|^2\, dx\right) \triangle \underline{u}, \end{aligned}$$

$$
\begin{aligned}
-B\left(\int_\Omega |\nabla v_2|^2\,dx\right)\triangle v_2 =& \lambda_2\beta(x)\,g(u_1)\,\tau(v_1)\\
\geq& \lambda_2\beta(x)\,g(\underline{u})\,\tau(\underline{v}) \geq -B\left(\int_\Omega |\nabla \underline{v}|^2\,dx\right)\triangle \underline{v}.
\end{aligned}
$$

Repeating this argument we get a bounded monotone sequence $\{(u_n,v_n)\}\subset E$ satisfying

$$
\begin{aligned}
\overline{u} =& u_0 \geq u_1 \geq u_2 \geq \cdots \geq u_n \geq \cdots \geq \underline{u} > 0,\\
\overline{v} =& v_0 \geq v_1 \geq v_2 \geq \cdots \geq v_n \geq \cdots \geq \underline{v} > 0.
\end{aligned}
$$

Using the continuity of the functions $f, h, g,$ and τ and the definition of the sequences $\{u_n\}, \{v_n\}$, there exist constants $C_i > 0$, $i = 1, \ldots, 4$, independent of n such that

$$
|f(v_{n-1})| \leq C_1, \quad |h(u_{n-1})| \leq C_2, \quad |g(u_{n-1})| \leq C_3 \tag{2.11}
$$

and

$$
|\tau(u_{n-1})| \leq C_4, \quad \text{for all } n.
$$

From (2.11), multiplying the first equation of (2.10) by u_n, integrating, using the Hölder inequality and Sobolev embedding we can show that

$$
\begin{aligned}
a_1\int_\Omega |\nabla u_n|^2\,dx \leq& A\left(\int_\Omega |\nabla u_n|^2\,dx\right)\int_\Omega |\nabla u_n|^2\,dx\\
=& \lambda_1\int_\Omega \alpha(x)\,f(v_{n-1})\,h(u_{n-1})\,u_n dx\\
\leq& \lambda_1\|\alpha\|_\infty\int_\Omega |f(v_{n-1})|\cdot|h(u_{n-1})|\cdot|u_n|\,dx\\
\leq& C_1C_2\|\alpha\|_\infty\lambda_1\int_\Omega |u_n|\,dx\\
\leq& C_5\|u_n\|_{H_0^1(\Omega)}.
\end{aligned}
$$

Then

$$
\|u_n\|_{H_0^1(\Omega)} \leq C_5, \quad \text{for all } n, \tag{2.12}
$$

where $C_5 > 0$ is a constant independent of n. Similarly, there exists $C_6 > 0$ independent of n such that

$$
\|v_n\|_{H_0^1(\Omega)} \leq C_6, \quad \text{for all } n. \tag{2.13}
$$

From (2.12) and (2.13), we infer that $\{(u_n,v_n)\}$ has a subsequence which weakly converges in $H_0^1(\Omega,\mathbb{R}^2)$ to a limit (u,v) with the properties $u \geq \underline{u} > 0$ and $v \geq \underline{v} > 0$. Being monotone and also using a standard regularity argument, $\{(u_n,v_n)\}$ converges itself to (u,v). Now, letting $n \to +\infty$ in (2.10), we deduce that (u,v) is a positive weak solution of system (1.1). The proof of theorem is now completed. □

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