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RESULTS ON (ENGEL, SOLVABLE, NILPOTENT) FUZZY SUBPOLYGROUPS

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Abstract. In this paper, first we define the notion of an Engel polygroup, to get further properties on Engel fuzzy subpolygroups. Moreover, we prove that every normal fuzzy subpolygroup of an Engel polygroup is Engel. Furthermore, we introduce the notions of solvable and nilpotent fuzzy subpolygroups and we get some of their properties. Finally we investigate the relations among solvable and nilpotent fuzzy subpolygroups with Engel fuzzy subpolygroups.

1. Introduction

Researches on Engel groups have centered mainly on the question, whether n-Engel groups are nilpotents. Clearly every 1-Engel group is Abelian. Levi [14] proved that 2-Engel groups are nilpotent of class at most 3. Heineken in [12] showed that every 3-Engel group G is nilpotent of class at most 4 if G has no element of order 2 or 5. L. Kappe and W. Kappe [13] gave a characterization of 3-Engel groups which is analogous to Levi's theorem on 2-Engel groups. Moreover, the study of fuzzy Engel groups was investigated in [2, 16, 17].

On the other hand, hyperstructure theory was first initiated by Marty [15] in 1934 when he defined hypergroups and started to analyze their properties. Since there are extensive applications in many branches of mathematics and applied sciences, the theory of algebraic hyperstructures has nowadays become a well-established branch in algebraic theory. Fuzzy subsets have been introduced in (1965) by L. A. Zadeh [22] as an extension of the classical notion of set. With appropriate definitions in the fuzzy setting most of the elementary results of group theory have been superseded

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with a starling generalized effect. Specially, the study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work on the connections between fuzzy sets and hyperstructures. Fuzzy hyperstructures is a direct extension of the concept of fuzzy algebras. This approach can be extended to fuzzy hypergroups. In [23], the concept of a fuzzy subpolygroup is introduced. In [7], Borzooei and et. al introduced the notion of Engel (nilpotent) fuzzy subpolygroups and various properties of Engel fuzzy subpolygroups were proved.

Now, in this paper, first we introduce and study Engel polygroups and solvable fuzzy subpolygroups. Then, we investigate the important properties of such fuzzy hyperstructure. Moreover, we obtain a necessary and sufficient condition between solvable fuzzy subpolygroups and the solvable group $P/\sim$, the group of equivalence classes derived from a fuzzy subpolygroup of P. Finally, by the relation between these notions we get some interesting results on Engel fuzzy subpolygroups.

2. Preliminary

Let $X_1, X_2, \dots, X_n$ be non-empty subsets of group G. Define the *commutator subgroup* of X_1 and X_2 by

$$[X_1, X_2] = \langle [x_1, x_2] \mid x_1 \in X_1, x_2 \in X_2 \rangle.$$

More generally, define

$$[X_1, \dots, X_n] = [[X_1, \dots, X_{n-1}], X_n],$$

where $n \geq 2$ and $[X_1] = \langle X_1 \rangle$. Also, recall that $X_1^{X_2} = \langle x_1^{x_2} \mid x_1 \in X_1, x_2 \in X_2 \rangle$ [19]. Let G be any group and $x, y \in G$. Define the *n-commutator* $[x,_n y]$, for any $n \in \mathbb{N}$ and $x, y \in G$, by $[x,_0 y] = x$, $[x,_1 y] = xyx^{-1}y^{-1}$ and $[x,_n y] = [[x,_{n-1} y], y]$. Now, a group G is called an *Engel group* if for each $x, y \in G$, there is a positive integer $n = n(x, y)$, such that $[x,_n y] = e$, where e is the identity of the group G. Suppose $n = n(x, y)$ can be chosen independently of any $x, y \in G$, then we say that G is an *n-Engel group.*

We recall the notion of a nilpotent group. Let G be a group. *Lower central series* of G is defined by $G = l_1(G) \geq l_2(G) \geq \cdots$, where $l_1(G) = G$ and for each integer $n > 1$, $l_n(G) = [l_{n-1}(G), G]$. Then G is called *nilpotent* if there exists a non-negative integer m, such that $l_m(G) = \{e\}$. The smallest such integer is called the class of G. Also, *derived series* of G is defined by $\cdots \subseteq G^n \subseteq \cdots \subseteq G^0 = G$; where for each integer $n > 1$, $G^n = [G^{n-1}, G^{n-1}]$. Now, G is called *solvable* if there exists a non-negative integer m, such that $G^m = \{e\}$. The smallest such integer is called the class of G (see [19]).

Definition 2.1 ([9])**.** A *polygroup* is an algebraic structure $(P, \cdot, ^{-1}, e)$, where " $\cdot$ " is a hyperoperation on P, "$^{-1}$" is an unitary operation on P and $e \in P$, such that the following axioms hold:

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(ii) $e \cdot x = x \cdot e = x$;
(iii) $x \in y \cdot z \Rightarrow y \in x \cdot z^{-1} \Rightarrow z \in y^{-1} \cdot x$,

for any $x, y, z \in P$.

A non-empty subset K of a polygroup P is called a *subpolygroup* of P, if $a, b \in K$ implies $a \cdot b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. A subpolygroup N of a polygroup P is called *normal*, if $a^{-1}Na \subseteq N$, for any $a \in P$ (see [9]). The *commutator* of two elements in a polygroup $\langle P, \cdot, e, ^{-1} \rangle$, is defined by $[x, y] = \{t \mid t \in x \cdot y \cdot x^{-1} \cdot y^{-1}\}$. If $A \subseteq P$, then $[A, y] = \{t \mid t \in A \cdot y \cdot A^{-1} \cdot y^{-1}\}$. Therefore,

$$[[x, y], y] = \{t \mid t \in [x, y] \cdot y \cdot [x, y]^{-1} \cdot y^{-1}\}$$

and, inductively, we define

$$[x,_n y] = [[x,_{n-1} y], y] = \{t \mid t \in [x,_{n-1} y] \cdot y \cdot [x,_{n-1} y]^{-1} \cdot y^{-1}\}.$$

Also, $A^x = \{t \mid t \in x \cdot A \cdot x^{-1}\}$ (see [3]).

Definition 2.2 ([1,3])**.** Let P be a polygroup. For any $s \in P$ and $k \geq 0$, we define:

(i) $L_{0,s}(P) = P$;
(ii) $L_{k+1,s}(P) = \{h \in P \mid x \cdot s \cap h \cdot s \cdot x \neq \phi, x \in L_{k,s}(P)\}$;
(iii) $L_0(P) = P$;
(iv) $L_{k+1}(P) = \{h \mid x \cdot y \cap h \cdot y \cdot x \neq \phi, x \in L_k(P) \ \ and \ \ y \in P\}$;
(v) $l_{0,s}(P) = P$;
(vi) $l_{k+1,s}(P) = \langle \{h \in P \mid h \in [x, s], x \in l_{k,s}(P)\} \rangle$;
(vii) $l_0(P) = P$;
(viii) $l_{k+1}(P) = \langle \{h \in P \mid h \in [x, y], x \in l_k(P), y \in P\} \rangle$;
(ix) $i_0(P) = P$, $i_{k+1}(P) = \langle \{h \in P \mid h \in [x, y], x, y \in i_k(P)\} \rangle$.

Theorem 2.1 ([3])**.** *Let P be a polygroup. Then for any $s \in P$ and $k \geq 0$*

$$L_{k+1,s}(P) = \{h \in P \mid h \in [x, s], x \in L_{k,s}(P)\}.$$

Let P be a polygroup and $\rho \subseteq P \times P$ be an equivalence relation on P. For non-empty subsets A and B of P, we define $A\overline{\overline{\rho}}B \Leftrightarrow$ (for all $a \in A$ and for all $b \in B$ we get $a\rho b$). Then the relation ρ is called a strongly regular on the left (on the right) if $x\rho y \Rightarrow a \cdot x\overline{\overline{\rho}}a \cdot y(x \cdot a\overline{\overline{\rho}}y \cdot a)$ for any $x, y, a \in P$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 2.2 ([8])**.** *If P is a polyrgroup and ρ is a strongly regular relation on P, then $(P/\rho, \otimes)$ is a group, where $\rho(x) \otimes \rho(y) = \rho(z)$ for any $z \in x \cdot y$.*

For any $n \geq 1$, we define the relation β_n on a polygroup P, as follows:

$$a\beta_n b \Leftrightarrow (\exists (x_1, \ldots, x_n) \in P^n) \ \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and we let $\beta = \cup_{n\geq 1}\beta_n$. Suppose that β^* is the *transitive closure* of β. Then β^* is a *strongly regular relation* on P [8].

Let $(H, \cdot)$ and $(H', \star)$ be two polygroups. A function $f : H \to H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \star f(b)$ for any $a, b \in H$. We say that f is a *good homomorphism* if $f(a \cdot b) = f(a) \star f(b)$ for any $a, b \in H$.

Definition 2.3 ([9])**.** A polygroup P is said to be *nilpotent* if there exists $n \in \mathbb{N}$ such that $l_n(P) \subseteq w$ or equivalently $l_n(P).w = w$, where w is the kernel of $f : P \to \frac{P}{\beta^*}$. The smallest integer n such that $l_n(P).w = w$ is called the *nilpotency class* or for simplicity the class of P. Also, a polygroup P is said to be *solvable* if there exists $n \in \mathbb{N}$ such that $i_n(P) \subseteq w$. The smallest such integer is called the class of P.

A *fuzzy subset* μ of X is a function $\mu : X \to [0,1]$. Let f be a function from X into Y, and μ, ν be two fuzzy subsets of X, Y, respectively. Defined the fuzzy subset $f(\mu)$ of Y, by

$$(f(\mu))(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise}, \end{cases}$$

for any $y \in Y$, and fuzzy subset $f^{-1}(\nu)$ of X by $(f^{-1}(\nu))(x) = \nu(f(x))$ for any $x \in X$. The *intersection* $\mu_1 \cap \mu_2$ of fuzzy subsets μ_1 and μ_2 of X, is defined by $(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}$ for any $x \in X$. (Note that $\mu_1 \cap \mu_2$, is the largest fuzzy subset of X contained in the both of μ_1 and μ_2). Also $\mu_1 \times \mu_2$ is a fuzzy subset of $X \times X$, which is defined by $(\mu_1 \times \mu_2)(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}$ for any $x_1, x_2 \in X$ (see [20, 22]).

Definition 2.4 ([20])**.** Let μ be a fuzzy subset of a group G. Then μ is called a *fuzzy subgroup* of G, if $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and $\mu(x^{-1}) \geq \mu(x)$ for any $x, y \in G$. A fuzzy subgroup μ of G is called *normal* if $\mu(xy) = \mu(yx)$ for any $x, y \in G$.

Definition 2.5 ([23])**.** Let $(P, \cdot)$ be a polygroup and μ be a fuzzy subset of P. Then μ is called a *fuzzy subpolygroup* of P, when $z \in x \cdot y$ implies $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(x^{-1}) \geq \mu(x)$ for any $x, y \in P$. Moreover, a fuzzy subpolygroup μ of P is called *normal* if $z \in x \cdot y$ and $z^{'} \in y \cdot x$, then $\mu(z) = \mu(z^{'})$ for any $x, y \in P$.

Theorem 2.3 ([23])**.** *Let μ be a fuzzy subpolygroup of polygroup P. Then $\mu(e) \geq \mu(x)$ and $\mu(x^{-1}) = \mu(x)$, for any $x \in P$. Moreover, μ is a normal fuzzy subpolygroup of P if and only if $\mu_t = \{x \mid \mu(x) \geq t\}$ is a normal subpolygroup of P for any $t \in [0, \mu(e)]$.*

Theorem 2.4 ([10])**.** *Let μ be a fuzzy subpolygroup of a polygroup P. Then the following conditions are equivalent, for any $x, y \in P$:*

(i) *μ is a normal fuzzy subpolygroup of P;*
(ii) *for any $z \in y \cdot x \cdot y^{-1}$, $\mu(z) = \mu(x)$;*
(iii) *for any $z \in y \cdot x \cdot y^{-1}$, $\mu(z) \geq \mu(x)$;*
(iv) *for any $z \in y^{-1} \cdot x^{-1} \cdot y \cdot x$, $\mu(z) \geq \mu(x)$.*

Theorem 2.5 ([10])**.** *Let P and $P^{'}$ be two polygroups, μ be a fuzzy subpolygroup of P, λ be a fuzzy subpolygroup of $P^{'}$ and $f : P \longrightarrow P^{'}$ be a function. If f is a good homomorphism, then $f^{-1}(\lambda)(f(\mu))$ is a fuzzy subpolygroup of $P(P')$.*

Theorem 2.6 ([10])**.** *Let P_1 and P_2 be two polygroups and μ and ν be two fuzzy subpolygroups of P_1 and P_2, respectively. If $\mu(e_1) = \nu(e_2) = 1$ and $\mu \times \nu$ is a fuzzy subpolygroup of $P_1 \times P_2$, then μ and ν are fuzzy subpolygroups of P_1 and P_2, respectively.*

Notation. From now on, in this paper we let $(P, \cdot, ^{-1}, e)$ be a polygroup and $n \in \mathbb{N}$. For simplicity of notations, sometimes we may write xy instead of $x.y$.

3. Engel Polygroups

In this section, we introduce the notion of Engel polygroup and we obtain some results on Engel polygroups that are used in the other sections.

Definition 3.1. A polygroup P is said to be *n-Engel*$(n \in \mathbb{N})$ if $l_{n,s}(P) \subseteq \omega$ or equivalently $l_{n,s}(P).\omega = \omega$ for any $s \in P$, where ω is the heart of P and

$$l_{0,s}(P) = P,$$
$$l_{k+1,s}(P) = \langle \{h \in P \mid h \in [x, s], x \in l_{k,s}(P)\} \rangle.$$

Example 3.1. Let P be a polygroup by the following table:

.	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a, b\}$

Then $[e, a] = e$, $[a, a] = e$, $[b, a] = P$ and so, $l_{1,a}(P) = \langle \{h \in P \mid h \in [x, a], x \in P\} \rangle = P$. Similarly, we see that $l_{1,b}(P) = P = l_{1,e}(P)$. Therefore, for any $s \in P$, $l_{1,s}(P) = P = \omega$. Consequently, P is an 1-Engel polygroup.

Theorem 3.1. *Every polygroup of order less than* 7 *is* 1*-Engel.*

Proof. Suppose that P is a proper polygroup of order less than 7. Then $\frac{P}{\beta^*}$ is an Abelian group of order less than 6. Now, let $h \in l_{1,s}(P)$ where $s \in P$. Then there exists $x \in P$ such that $h \in [x, s]$. Thus,

$$\beta^*(h) = \beta^*([x, s]) = [\beta^*(x), \beta^*(s)] = \beta^*(e),$$

which implies that $h \in w$. Therefore, P is 1-Engel. □

Theorem 3.2 ([9]). *Let* $(G, \cdot)$ *be a group and* $P_G = G \cup \{a\}$, *where* $a \notin G$. *Then* $(P_G, \circ)$ *is a polygroup, where operation " $\circ$ " is defined as follows*

(1) $a \circ a = e$;
(2) $e \circ x = x \circ e = x$ *for every* $x \in P_G$;
(3) $x \circ x^{-1} = \{e, a\}$, *for every* $x \in P_G \setminus \{e, a\}$;
(4) $a \circ x = x \circ a = x$, *for every* $x \in P_G \setminus \{e, a\}$;
(5) $x \circ y = x \cdot y$, *for every* $(x, y) \in G^2$ *such that* $y \neq x^{-1}$.

Theorem 3.3. *Let* G *be an* 1*-Engel group. Then* $\langle P_G, \circ, e, -1 \rangle$ *is an* 1*-Engel polygroup.*

Proof. Let G be an 1-Engel group. By (1), $[a, a] = \{t \mid t \in a \circ a \circ a^{-1} \circ a^{-1} = e\}$. Then $e \in [a, a]$. Using (3) and (4), we have $e \in [a, y]$ in which $a \neq y \in P_G \setminus \{e, a\}$. Hence, $e \in [a, y]$ for any $y \in G \cup \{a\}$.

(I) Also, by hypotheses, for any $x, y \in G$ in which $y \neq x^{-1}$, we have $e = [x, y]$.

(II) So, by (I) and (II), $e \in [x, y]$ for any $x, y \in G \cup \{a\}$.

(III) Now, let $h \in l_{1,s}(P)$ where $s \in P_G$. Then $h \in [x, s]$ for some $x \in P_G$ and so by (III), $\beta^*(h) = [\beta^*(x), \beta^*(s)] = \beta^*(e)$, which implies that $h \in w$. Therefore, P_G is an 1-Engel polygroup. □

Now, in the following theorem we give a method to construct a 1-Engel polygroup of order $n \in \mathbb{N}$.

Theorem 3.4. *For every $n \in \mathbb{N}$, there is a nontrivial 1-Engel polygroup of order $n + 1$.*

Proof. For $n \in \mathbb{N}$, consider the Abelian group $\mathbb{Z}_n$. Clearly, $\mathbb{Z}_n$ is 1-Engel. Then by Theorem 3.3, $(P_{\mathbb{Z}_n}, \circ)$ is an 1-Engel polygroup of order $n + 1$. □

Theorem 3.5. *Let P_1 and P_2 be two polygroups. Then for any $k \geq 0$*

$$i_k(P_1 \times P_2) = i_k(P_1) \times i_k(P_2).$$

Proof. We prove our claim by induction on k. For $k = 0$, the proof is obvious. Now suppose that $(a, b) \in i_{k+1}(P_1 \times P_2)$. Then there exist $(u, v), (s, t) \in i_k(P_1 \times P_2)$ such that

$$(a, b) \in [(u, v), (s, t)] = [u, s] \times [v, t].$$

By using the hypotheses of induction, we conclude that $(u, v), (s, t) \in i_k(P_1) \times i_k(P_2)$. Thus for any $u, s \in i_k(P_1)$, we get $a \in [u, s]$ and for any $v, t \in i_k(P_2)$, we get $b \in [v, t]$. Hence $(a, b) \in i_{k+1}(P_1) \times i_{k+1}(P_2)$. Similarly, we obtain the converse. Therefore,

$$i_k(P_1 \times P_2) = i_k(P_1) \times i_k(P_2).$$ □

Theorem 3.6. *Let P be a polygroup, $s \in P$ and N be a normal subpolygroup of P. Then*

$$l_{n,sN}\left(\frac{P}{N}\right) = \frac{l_{n,s}(P)N}{N}, \quad i_n\left(\frac{P}{N}\right) = \frac{i_n(P)N}{N}.$$

Proof. By induction on n we show that $l_{n,sN}\left(\frac{P}{N}\right) \subseteq \frac{l_{n,s}(P)N}{N}$ and $l_{n,sN}(\frac{P}{N}) \supseteq \frac{l_{n,s}(P)N}{N}$. For $n = 0$, the inclusions are obvious. Now, suppose that $yN \in l_{n+1,sN}(\frac{P}{N})$. Hence, there exists $aN \in l_{n,sN}(\frac{P}{N})$ such that $yN \in [aN, sN]$. By hypotheses of induction, we have $aN \in \frac{l_{n,s}(P)N}{N}$. Hence, there exists $a^{'} \in l_{n,s}(P)$ such that $aN = a^{'}N$. Thus, $yN \in [a^{'}N, sN] = [a^{'}, s]N$. So, there exist $a^{'} \in l_{n,s}(P)$ and $y^{'} \in [a^{'}, s]$ such that $yN = y^{'}N$. Hence, $yN \in \frac{l_{n+1,s}(P)N}{N}$. Conversely, if $yN \in \frac{l_{n+1,s}(P)N}{N}$, then there exists $y^{'} \in l_{n+1,s}(P)$ such that $yN = y^{'}N$. Therefore, $y^{'} \in [a, s]$, for some $a \in l_{n,s}(P)$. Thus, by hypotheses of induction, $aN \in \frac{l_{n,s}(P)N}{N} = l_{n,sN}\left(\frac{P}{N}\right)$ and $yN = y^{'}N \in [aN, sN]$ implies that $yN \in l_{n+1,sN}\left(\frac{P}{N}\right)$. Therefore, $l_{n,sN}\left(\frac{P}{N}\right) = \frac{l_{n,s}(P)N}{N}$. Similarly, we can prove that $i_n(\frac{P}{N}) = \frac{i_n(P)N}{N}$. □

Corollary 3.1. (*i*) *If P is an n-Engel polygroup and N is a normal subpolygroup of P, then $\frac{P}{N}$ is n-Engel.*

(ii) If P is a solvable polygroup and N is a normal subpolygroup of P, then $\frac{P}{N}$ is solvabel.

Theorem 3.7. *Let P_1 and P_2 be two polygroups and $\phi : P_1 \to P_2$ be a good homomorphism. If ϕ is one to one and K is an n- Engel subpolygroup of P_1, then $\phi(K)$ is an n-Engel subpolygroup of P_2.*

Proof. By induction on n, we show that $l_{n,y}(\phi(K)) = \phi(l_{n,b}(K))$, where $\phi(b) = y$ and b, y are fix elements of K and $\phi(K)$, respectively. For $n = 0$, the proof is obvious. Now, let $z \in l_{n+1,y}(\phi(K))$. Then there exists $x \in l_{n,y}(\phi(K))$ such that $z \in [x, y]$. By hypotheses of induction, $x \in \phi(l_{n,b}(K))$. Also there exist $c, a \in K$ such that $z = \phi(c)$ and $x = \phi(a)$. Hence,

$$\phi(c) = z \in [\phi(a), \phi(b)] = \phi[a, b], \quad x = \phi(a) \in \phi(l_{n,b}(K)).$$

Thus for $a \in l_{n,b}(K)$, we get $c \in [a, b]$ that implies that $c \in l_{n+1,b}(K)$. Conversely, let $z \in \phi(l_{n+1,b}(K))$. Then for some $c \in l_{n+1,b}(K)$, $z = \phi(c)$. Using hypotheses of induction, $z = \phi(c) \in \phi[a, b] = [\phi(a), \phi(b)]$, where $a \in l_{n,b}(K)$, $y = \phi(b)$ and $\phi(a) \in l_{n,y}(\phi(K))$. Therefore, $z \in l_{n+1,y}(\phi(K))$. □

4. Results on Engel Fuzzy Subpolygroups

In this section, by considering the notion of Engel fuzzy subpolygroup, which is defined in [7], we state and prove some new related results.

Definition 4.1 ([7]). Let μ be a fuzzy subpolygroup of P and $n \in \mathbb{N}$. If for any $x, y \in P$ and $z \in [x,_n y]$, we have $\mu(z) = \mu(e)$, then μ is called an *n-Engel fuzzy subpolygroup* of P.

Theorem 4.1 ([7]). *Let P and P' be two polygroups with the identity elements e_1 and e_2, respectively, μ and λ be two n-Engel fuzzy subpolygroup of P and P', respectively, and $f : P \to P'$ be a function.*

(i) If f is a good homomorphism, then $f^{-1}(\lambda)$ is an n-Engel fuzzy subpolygroup of P.

(ii) If f is an onto good homomorphism, then $f(\mu)$ is an n-Engel fuzzy subpolygroup of P'.

Proposition 4.1 ([7]). *Let μ be a normal fuzzy subpolygroup of P and relation $\sim$ on P is defined as follows:*

$$x \sim y \Leftrightarrow (\exists a \in xy^{-1}) \ st. \ \mu(a) = \mu(e).$$

Then $\sim$ is a strongly regular relation on P.

Suppose that for any $x \in P$, $\mu[x]$ is the equivalence class containing x with respect to strongly regular relation $\sim$ on P and $\frac{P}{\sim}$ denoted the set of all equivalence classes $\mu[x]$, i.e., $\frac{P}{\sim} = \{\mu[x] \mid x \in P\}$.

Theorem 4.2 ([7])**.** $\left(\frac{P}{\sim}, \odot, ^{-1}, \mu[e]\right)$ *is a group, where*

$$\mu[x]^{-1} = \mu[x^{-1}], \quad \mu[x] \odot \mu[y] = \{\mu[z] \mid z \in xy\},$$

for any $x, y \in P$.

Theorem 4.3 ([7])**.** *Let μ be a normal fuzzy subpolygroup of a polygroup P. Then μ is a n-Engel fuzzy subpolygroup of P if and only if $\frac{P}{\sim}$ is a n-Engel group.*

Let μ be a normal fuzzy subpolygroup of P. Then $\{\mu(x) \mid x \in P\}$ is called the *order of μ*.

Theorem 4.4. *Any normal fuzzy subpolygroup of order less than* 6*, is an* 1*-Engel fuzzy subpolygroup of P.*

Proof. Let μ be a normal fuzzy subpolygroup of order less than 6. Then $\frac{P}{\sim}$ is a group of order less than 6. Hence it is Abelian, which implies that $\frac{P}{\sim}$ is an 1-Engel group. Now, by Theorem 4.3, μ is a 1-Engel fuzzy subpolygroup of P. □

Let $\mu_* = \{x \mid \mu(x) = \mu(e)\}$. Clearly, μ_* is a normal subpolygroup of P.

Theorem 4.5. *If P is an n-Engel polygroup, then any normal fuzzy subpolygroups of P is n-Engel.*

Proof. Let P be n-Engel and μ be a normal fuzzy subpolygroup of P. First we show that $\frac{P}{\sim} \approx \frac{P}{\mu_*}$. Define

$$f : \frac{P}{\sim} \to \frac{P}{\mu_*} \text{ by } f(\mu[x]) = \mu_* x, \quad x \in P.$$

If $\mu[x] = \mu[y]$ for $x, y \in P$, then $x \sim y$ and so there exists $r \in xy^{-1}$ such that $\mu(r) = \mu(e)$, where e is the identity element of P. Now, we show that for any $x, y \in P$, if $x \sim y$, then $\mu(r) = \mu(e)$ for any $r \in xy^{-1}$. If $x \sim y$, then by the definition of $\sim$, there exists $a \in xy^{-1}$ such that $\mu(a) = \mu(e)$. Now, let $r \in xy^{-1}$ be an arbitrary element of P. Since μ is normal, we have $\mu(e) = \mu(a) = \mu(r)$ which implies that for any $r \in xy^{-1}$, $\mu(e) = \mu(r)$. Hence, $xy^{-1} \subseteq \mu_*$. Thus, $\mu_* x = \mu_* y$.

Conversely, if $\mu_* x = \mu_* y$, then $xy^{-1} \subseteq \mu_*$ and so for any $r \in xy^{-1}$, $\mu(e) = \mu(r)$, which implies that $x \sim y$. Consequently, f is an isomorphism by the fact that

$$\mu_* x \odot \mu_* y = \{\mu_* z \mid z \in xy\}, \quad \mu[x] \odot \mu[y] = \{\mu[z] \mid z \in xy\}.$$

Hence, $\frac{P}{\sim} \approx \frac{P}{\mu_*}$. Since P is n-Engel, by Corollary 3.1, $\frac{P}{\mu_*}$ is n-Engel and so $\frac{P}{\sim}$ is n-Engel. Therefore, by Theorem 4.3, μ is n-Engel. □

Example 4.1. Let $P = \{e, a, b, c, d, f, g\}$. Then P with the following hyperoperation is a polygroup

.	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e,a\}$	g	f	d	c
c	c	c	f	$\{e,a\}$	g	b	d
d	d	d	g	f	$\{e,a\}$	c	b
f	f	f	c	d	b	g	$\{e,a\}$
g	g	g	d	b	c	$\{e,a\}$	f

.

Now, we define the fuzzy set μ on P, by

$$\mu(x) = \begin{cases} 0.75, & x \in \{e, a, f, g\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, P is not an n-Engel polygroup. But, we show that μ is a normal n-Engel fuzzy subpolygroup of P. Since, for any $t \in [0,1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e, a, f, g\}$ or P, hence, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. Now, for any $z \in [x,_n s]$ where $x, s \in P$ we get $z \in l_n(P) = \{e, a, f, g\}$ and so $\mu(z) = \mu(e)$, which implies that μ is a normal n-Engel fuzzy subpolygroup of P.

Theorem 4.6. *Let μ be a normal fuzzy subpolygroup of $(P, \cdot, ^{-1}, e_1)$. Then $\left(\frac{P}{\mu_*}, \cdot, ^{-1}, e_2\right)$ is an n-Engel polygroup if and only if μ is an n-Engel fuzzy subpolygroup of P.*

Proof. Let $\frac{P}{\mu_*}$ be an n-Engel polygroup and $\pi : P \to \frac{P}{\mu_*}$ be the natural epimorphism. Since $z \in \pi^{-1}(\pi(x))$, we get $\pi(z) = \pi(x)$ and so $\pi(e_1) \in \pi(z^{-1} \cdot z) = \pi(z^{-1} \cdot x)$. Thus, there exists $r \in z^{-1} \cdot x$ such that $e_2 = \pi(e_1) = \pi(r)$, which implies that $r \in \ker \pi = \mu_*$. Therefore, $\mu(r) = \mu(e_1)$ and so $z \sim x$. Hence, for any $x \in P$

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z \in \pi^{-1}(\pi(x))} \mu(z) = \bigvee_{z \sim x} \mu(z) \geq \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) \supseteq \mu$. Now, since $\frac{P}{\mu_*}$ is an n-Engel polygroup and $\pi(\mu)$ is a fuzzy subpolygroup of $\frac{P}{\mu_*}$, by Theorem 4.5, $\pi(\mu)$ is n-Engel and by Theorem 4.1, $\pi^{-1}(\pi(\mu))$ is an n-Engel. Now, we show that μ is n-Engel. For this, let $x \in [t,_n s]$, where $t \in P$, $s \in P$ and $f : \frac{P}{\sim} \to \frac{P}{\mu_*}$ be as in the proof of Theorem 4.5. Since $\pi^{-1}(\pi(\mu))$ is an n-Engel fuzzy subpolygroup of P, so $\pi^{-1}(\pi(\mu))(x) = \pi^{-1}(\pi(\mu))(e_1)$. Hence, $\bigvee_{z \sim x} \mu(z) = \mu(e_1)$. Then $x \sim e_1$ and so $\mu[x] = \mu[e_1]$. Hence by $f(\mu[x]) = \mu_* x$ we have $\mu_* x = \mu_* e_1$. Thus, $x \in \mu_*$, which implies that $\mu(x) = \mu(e_1)$. Therefore, μ is an n-Engel fuzzy subpolygroup of P.

Conversely, let μ be a normal n-Engel fuzzy subpolygroup of P. By Theorem 4.3, $\frac{P}{\sim}$ is an n-Engel group also, $\frac{P}{\sim} \cong \frac{P}{\mu_*}$ and so $\frac{P}{\mu_*}$ is an n-Engel group. □

Example 4.2. Let $D_3 = \langle a, b; a^3 = b^2 = e, ba = a^2 b \rangle$ be the dihedral group with six elements and $t_0, t_1 \in [0,1]$ such that $t_0 > t_1$. Define a fuzzy subgroup μ of D_3 as follows:

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in < a >, \\ t_1 & \text{if } x \notin < a > . \end{cases}$$

Then $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a \rangle$. Thus, μ_* is a normal subgroup of D_3. Also, $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian, hence it is 1-Engel and so by Theorem 4.6, μ is an 1-Engel fuzzy subpolygroup of D_3.

Theorem 4.7. *Let μ and ν be two fuzzy subpolygroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is an n-Engel fuzzy subpolygroup of P, then ν is an n-Engel fuzzy subpolygroup of P, too.*

Proof. Let μ and ν be two fuzzy subgroups of P, where $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Now let μ be an n-Engel and $x \in [h,_n s]$, where $h \in P$ and $s \in P$. Then, $\mu(x) = \mu(e) = \nu(e)$ and so by hypotheses $\nu(e) = \mu(x) \leq \nu(x)$. Thus, $\nu(x) = \nu(e)$, which implies that ν is an n-Engel fuzzy subpolygroup of P. □

Definition 4.2 ([6])**.** Let μ be a fuzzy set on P. Then the lower level subset of μ is defined by,

$$\overline{\mu}_t = \{x \in P; \mu(x) \leq t\}, \quad \text{where } t \in [0, 1].$$

Now the fuzzy set $A_{\overline{\mu}_t}$ is defined by

$$A_{\overline{\mu}_t}(x) = \begin{cases} \mu(x), & \text{if } x \in \overline{\mu}_t, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $A_{\overline{\mu}_t} \subseteq \mu$.

Corollary 4.1. *Let $A_{\overline{\mu}_t}$ be an n-Engel fuzzy subpolygroup of P. Then μ is an n-Engel fuzzy subpolygroup of P, too.*

Proof. Let μ be an Engel fuzzy subpolygroup of P. Clearly, $A_{\overline{\mu}_t}$ is a fuzzy supolygroup of P. Since $A_{\overline{\mu}_t} \subseteq \mu$, by Theorem 4.7, $A_{\overline{\mu}_t}$ is Engel fuzzy subpolygroup of P. □

Suppose that μ is a fuzzy subset of P. Support of μ is defined by $\text{supp}(\mu) = \{x \in P \mid \mu(x) > 0\}$.

Definition 4.3 ([4])**.** Let μ and ν be fuzzy subpolygroups of P and H, respectively. Then a good isomorphism $f : \text{supp}(\mu) \to \text{supp}(\nu)$ is called a *fuzzy good isomorphism* from μ to ν, if there exists a positive real number k such that $\mu(x) = k\nu(f(x))$ for any $x \in \text{supp}(\mu) \setminus \{e\}$. In this case we write $\mu \simeq \nu$.

Theorem 4.8. *Let μ and ν be two fuzzy subpolygroups of $(P, \cdot,^{-1}, e_1)$ and $(H, \cdot,^{-1}, e_2)$, respectively, and $\mu \simeq \nu$. If μ is n-Engel, then ν is an Engel fuzzy subpolygroup of* $\text{supp}(\nu)$.

Proof. Let $z \in [x,_n y]$, where $x, y \in \text{supp}(\nu)$. Since $\mu \simeq \nu$, then there exists a positive real number k such that $\mu(x) = k\nu(f(x))$ for any $x \in \text{supp}(\mu) \setminus \{e_1\}$ and $x = f(a)$, $y = f(b)$ for some $a, b \in \text{supp}(\mu)$. So, $z \in [x,_n y] = [f(a),_n f(b)] = f[a,_n b]$. Therefore, $z = f(c)$, for some $c \in [a,_n b]$ and so, by hypotheses $\mu(c) = \mu(e_1)$. Thus,

$k\nu(z) = k\nu(f(c)) = \mu(c) = \mu(e_1) = k\nu(f(e_1)) = k\nu(e_2)$ and so, $\nu(z) = \nu(e_2)$, which implies that ν is n-Engel. □

5. Nilpotent Fuzzy Subpolygroups

In this section, by considering the notion of nilpotent fuzzy subpolygroup, we state and prove some results on this structure.

Definition 5.1 ([7])**.** Let μ be a fuzzy subpolygroup of P. Then μ is called a *nilpotent fuzzy subpolygroup of class n* ($n \in \mathbb{N}$), if $z \in l_n(P)$ implies that $\mu(z) = \mu(e)$.

Theorem 5.1. *Any nilpotent fuzzy subpolygroup of class $n = 1$ is a normal fuzzy subpolygroup.*

Proof. Let μ be a nilpotent fuzzy subpolygroup of class $n = 1$. Then for any $z \in l_1(P)$, $\mu(e) = \mu(z)$. Now, the proof follows by Theorem 2.4. □

By the following example we see that the converse of Theorem 5.1, is not true in general.

Example 5.1. Let $P = \{e, a, b, c, d, f, g\}$. Then P with the following hyperoperation is a polygroup

$\cdot$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e, a\}$	g	f	d	c
c	c	c	f	$\{e, a\}$	g	b	d
d	d	d	g	f	$\{e, a\}$	c	b
f	f	f	c	d	b	g	$\{e, a\}$
g	g	g	d	b	c	$\{e, a\}$	f

.

We define the fuzzy set μ on P, by

$$\mu(x) = \begin{cases} 0.75, & x \in \{e, a\}, \\ 0.5, & x \in \{f, g\}, \\ 0, & \text{otherwise.} \end{cases}$$

We show that, μ is a normal fuzzy subpolygroup of P which is not nilpotent of class $n = 1$. First for any $t \in [0, 1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e, a, f, g\}$, $\{e, a\}$ or P and since for any $x \in P$, $x^{-1}\{e, a\}x \subseteq \{e, a\}$, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. But for $g = [c, f] \in l_1(P) = \{e, a, f, g\}$ we get $\mu(g) \neq \mu(e)$ which implies that μ is not nilpotent of class $n = 1$.

Theorem 5.2. *Let P_1 and P_2 be two polygroups with the identity elements e_1 and e_2, respectively. Suppose that μ and λ be two nilpotent fuzzy subpolygroups of P_1 and P_2, respectively, and $\phi : P_1 \to P_2$ be a function.*

(i) *If ϕ is a good homomorphism, then $\phi^{-1}(\lambda)$ is a nilpotent fuzzy subpolygroup of P_1.*

(ii) *If ϕ is an isohomomorphism, then $\phi(\mu)$ is a nilpotent fuzzy subpolygroup of P_2.*

Proof. (i) The proof is clear. (ii) First note that $l_n(\phi(P_1)) = \phi(l_n(P_1))$ (see [9]). Now, let μ be a nilpotent fuzzy subpolygroup of P_1 and $y \in l_{n,}(P_2)$. Then,

$$y \in l_n(P_2) = l_n(\phi(P_1)) = \phi(l_n(P_1))$$

and so there exists $z \in l_n(P_1)$ such that $y = \phi(z)$. By hypotheses, $\mu(z) = \mu(e_1)$. Thus,

$$\phi(\mu)(y) = \bigvee_{x \in \phi(y)} \mu(x) = \mu(z) = \mu(e_1) = \phi(\mu)(e_2).$$

Hence, $\phi(\mu)(y) = \phi(\mu)(e_2)$, which implies that $\phi(\mu)$ is nilpotent. □

Theorem 5.3 ([7]). *Let μ be a fuzzy subpolygroup of P. Then μ is a nilpotent fuzzy subpolygroup of P if and only if $\frac{P}{\sim}$ is a nilpotent group.*

Note that if P is a nilpotent polygroup and N is a normal subpolygroup of P, then $\frac{P}{N}$ is nilpotent (see [9]).

Theorem 5.4. *If P is a nilpotent polygroup, then any normal fuzzy subpolygroup of P is nilpotent.*

Proof. Let P be a nilpotent of class n and μ be a fuzzy subpolygroup of P. Since $\frac{P}{\sim} \approx \frac{P}{\mu_*}$ and P is nilpotent, $\frac{P}{\mu_*}$ is nilpotent and so $\frac{P}{\sim}$ is nilpotent. Therefore, by Theorem 5.3, μ is nilpotent. □

Example 5.2. Let μ be as Example 4.1. We show that, P is not nilpotent and μ is a nilpotent normal fuzzy subpolygroup of P. First note that $l_n(P) = \{e, a, f, g\}$ (see [9]) and so P is not nilpotent. Also, for any $t \in [0, 1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e, a, f, g\}$ or P. Therefore, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. But for any $z \in l_n(P)$, $\mu(z) = \mu(e)$, which implies that μ is nilpotent.

Theorem 5.5. *Let μ be a normal fuzzy subpolygroup of $(P, \cdot, ^{-1}, e_1)$. Then $\left(\frac{P}{\mu_*}, \cdot, ^{-1}, e_2\right)$ is a nilpotent polygroup if and only if μ is a nilpotent fuzzy subpolygroup of P.*

Proof. Let $\frac{P}{\mu_*}$ be a nilpoten polygroup and $\pi : P \to \frac{P}{\mu_*}$ be the natural epimomorphism. Since $z \in \pi^{-1}(\pi(x))$, we have $\pi(z) = \pi(x)$ and so $\pi(e_1) \in \pi(z^{-1} \cdot z) = \pi(z^{-1} \cdot x)$. Then, there exists $r \in z^{-1} \cdot x$ such that $e_2 = \pi(e_1) = \pi(r)$, which implies that $r \in \ker \pi = \mu_*$. Thus, $\mu(r) = \mu(e_1)$ and so $z \sim x$. Hence, for any $x \in P$

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z \in \pi^{-1}(\pi(x))} \mu(z) = \bigvee_{z \sim x} \mu(z) \geq \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) \supseteq \mu$. Now since $\frac{P}{\mu_*}$ is a nilpotent polygroup and $\pi(\mu)$ is a fuzzy subpolygroup of $\frac{P}{\mu_*}$, then by Theorem 5.4, $\pi(\mu)$ is nilpotent and by Theorem 5.2, $\pi^{-1}(\pi(\mu))$ is nilpotent. Now, we show that μ is nilpotent. For this, let $x \in l_n(p)$ and $f : \frac{P}{\sim} \to \frac{P}{\mu_*}$ be as in the proof of Theorem 4.5. Since $\pi^{-1}(\pi(\mu))$ is nilpotent, so $\pi^{-1}(\pi(\mu))(x) = \pi^{-1}(\pi(\mu))(e_1)$. Hence, $\bigvee_{z \sim x} \mu(z) = \mu(e_1)$ and so $x \sim e_1$. Then

$\mu[x] = \mu[e_1]$ and by $f(\mu[x]) = \mu_* x$, we have $\mu_* x = \mu_* e_1$. Thus, $x \in \mu_*$, which implies that $\mu(x) = \mu(e_1)$. Therefore, μ is a nilpotent fuzzy subpolygroup of P.

Conversely, let μ be a normal nilpotent fuzzy subpolygroup of P. By Theorem 5.3, $\frac{P}{\sim}$ is nilpotent. Also, $\frac{P}{\sim} \cong \frac{P}{\mu_*}$ and so $\frac{P}{\mu_*}$ is nilpotent. □

Example 5.3. In Example 4.2, $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a \rangle$. Thus μ_* is a normal subgroup of D_3. Also $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian, hence it is nilpotent and so by Theorem 5.5, μ is a nilpotent fuzzy subpolygroup.

Theorem 5.6. *Let μ and ν be two fuzzy subpolygroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is a nilpotent fuzzy subpolygroup of class n, then ν is a nilpotent fuzzy subpolygroup of class n.*

Proof. Let μ and ν be two fuzzy subgroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Now let μ be nilpotent of class n and $x \in l_n(P)$. Therefore, $\mu(x) = \mu(e) = \nu(e)$ and so by hypotheses $\nu(e) = \mu(x) \leq \nu(x)$. Thus, $\nu(x) = \nu(e)$, which implies that ν is nilpotent of class at most n. □

Corollary 5.1. *Let $A_{\overline{\mu}_t}$ be a nilpotent fuzzy subpolygroup of P. Then μ is nilpotent, too.*

Proof. Let $A_{\overline{\mu}_t}$ be a nilpotent fuzzy subpolygroup of P. Since $A_{\overline{\mu}_t} \subseteq \mu$, by Theorem 5.6, μ is nilpotent. □

6. Solvable Fuzzy Subpolygroups

In this section, we introduce the notion of solvable fuzzy subpolygroup on a polygroup and we state and prove some new results on it. Specially, we get the relation between solvable fuzzy subpolygroups and Engel fuzzy subpolygroups (nilpotent fuzzy subpolygroups).

Definition 6.1. Let μ be a fuzzy subpolygroup of P. Then μ is called a *solvable fuzzy subpolygroup* of P if there exists $n \in \mathbb{N}$ such that for any $z \in i_n(P)$, $\mu(z) = \mu(e)$.

In the following example we have a solvable fuzzy subpolygroup.

Example 6.1. Let $P = \{e, a, b, c, d\}$. Then P with the following hyperoperation is a polygroup

.	e	a	b	c	d
e	e	a	b	c	d
a	a	e	b	c	d
b	b	b	{ e,a }	d	c
c	c	c	d	{ e,a }	b
d	d	d	c	b	{ e,a }

.

We define the fuzzy subset μ on P, by

$$\mu(x) = \begin{cases} 0.75, & x \in \{e, a\}, \\ 0.5, & x = b, \\ 0, & \text{otherwise.} \end{cases}$$

Then we show that, μ is a solvable fuzzy subpolygroup. First for any $t \in [0,1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e,a,b\}$, $\{e,a\}$ or P. Hence, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. Since for any $x, y \in P$, $[x,y] = e$ *or* $\{e,a\}$ then for any $z \in i_1(P)$, $\mu(z) = \mu(e)$ and so, μ is solvable.

In the following, we are ready to obtain a necessary and sufficient condition between solvable fuzzy subpolygroups and the solvable group $P/\sim$, the group of equivalence classes derived from the fuzzy subpolygroup of P. Now, we use notation $i_k(H)$ instead of derived series G^k, where $k \in N$ and H is a group. Also, for simplity we write $\mu[x]\mu[y]$ instead of $\mu[x] \odot \mu[y]$.

Lemma 6.1. *For any* $0 \leq k$

$$i_k\left(\frac{P}{\sim}\right) = \langle \{\mu[t] \mid t \in i_k(P)\}\rangle.$$

Proof. We do the proof by induction on k. For $k = 0$, we have

$$i_0\left(\frac{P}{\sim}\right) = \frac{P}{\sim} = \langle \{\mu[t] \mid t \in i_0(P) = P\}\rangle.$$

Now, let it is true for k. We claim that

$$i_{k+1}\left(\frac{P}{\sim}\right) \supseteq \langle \{\mu[t] \mid t \in i_{k+1}(P)\}\rangle.$$

For this, suppose that $\mu[a] \in \langle \{\mu[t] \mid t \in i_{k+1}(P)\}\rangle$. Then $a \in i_{k+1}(P)$ and so there exist $x, s \in i_k(P)$ such that $a \in [x,s]$. By hypotheses of induction we conclude that $\mu[x], \mu[s] \in i_k(\frac{P}{\sim})$. Thus, $\mu[a] = [\mu[x], \mu[s]]$ in which $\mu[x], \mu[s] \in i_k(\frac{P}{\sim})$. Hence, $\mu[a] \in i_{k+1}\left(\frac{P}{\sim}\right)$. Also,

$$i_{k+1}\left(\frac{P}{\sim}\right) \subseteq \langle \{\mu[t] \mid t \in i_{k+1}(P)\}\rangle.$$

Since for $\mu[a] \in \frac{P}{\sim} \in i_{k+1}(\frac{P}{\sim})$, we have $\mu[a] = [\mu[x], \mu[s]]$ in which $\mu[x], \mu[s] \in i_k\left(\frac{P}{\sim}\right)$. Using hypotheses of induction $x, s \in i_k(P)$ (1). Thus $\mu[a] = \mu[x]\mu[s](\mu[x])^{-1}(\mu[s])^{-1}$, which implies that $\mu[x]\mu[s] = \mu[a]\mu[s]\mu[x]$. Thus, there exist $c \in xs$ and $d \in asx$ such that $\mu[c] = \mu[d]$. Since P is a polygroup, then there exists $u \in P$ such that $c \in xs \cap usx$ (2). Then

$$\mu[a]\mu[s]\mu[x] = \mu[d] = \mu[c] = \mu[x]\mu[s] = \mu[c] = \mu[u]\mu[s]\mu[x].$$

Hence, $\mu[a] = \mu[u]$ (3). By (2) and (1), we have $u \in i_{k+1}(P)$. Now, using (3) and previous relation we have

$$\mu[a] = \mu[u] \in \langle \{\mu[t] \mid t \in i_{k+1}(P)\}\rangle. \qquad \square$$

Theorem 6.1. *Let* μ *be a normal fuzzy subpolygroup of a polygroup* P. *Then* μ *is a solvable fuzzy subpolygroup if and only if* $\frac{P}{\sim}$ *is a solvable group.*

Proof. ($\Rightarrow$) Suppose that μ is a solvable fuzzy subpolygroup of P and $k \in \mathbb{N}$. Then by Lemmas 6.1, it is enough to show that $\langle \{\mu[t] \mid t \in i_k(P)\}\rangle = \{\mu[e]\}$. If $t \in i_k(P)$, then by hypotheses $\mu(t) = \mu(e)$ and so $t \sim e$, which implies that $\mu[t] = \mu[e]$. Therefore, $\frac{P}{\sim}$ is a solvable group.

($\Leftarrow$) Let $\frac{P}{\sim}$ is solvable. We show that if $z \in i_k(P)$, then $\mu(z) = \mu(e)$. If $z \in i_k(P)$, then $z \in [x, s]$ where $x, s \in i_{k-1}(P)$. Hence, $\mu[z] = [\mu[x], \mu[s]]$, which by hypotheses implies that $\mu[z] = \mu[e]$ and so $z \sim e$. Then there exists $r \in ze^{-1}$ such that $\mu(r) = \mu(e)$ and so $\mu(z) = \mu(r) = \mu(e)$. Therefore, μ is an a solvable fuzzy subpolygroup. ☐

Theorem 6.2. *Let P be a solvable polygroup. Then any normal fuzzy subpolygroup of P is solvable.*

Proof. Let P be solvable polygroup and μ be a fuzzy subpolygroup of P. Since $\frac{P}{\sim} \approx \frac{P}{\mu_*}$ and P is solvable, by Corollary 3.1, $\frac{P}{\mu_*}$ is solvable and so $\frac{P}{\sim}$ is solvable, too. Therefore, by Theorem 6.1, μ is solvable. ☐

Example 6.2. Let A_5 be the alternating group of degree 5 and $P = A_5 \cup \{a\}$ be a polygroup as Theorem 3.2. We define the fuzzy subset μ on P, by $\mu(x) = 1$, for any $x \in P$. It is clear that P is not solvable (see [9]). But, for any $t \in [0, 1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to P. Hence, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. Now, since for any $z \in P$, $\mu(z) = \mu(e)$, we get that μ is solvable.

By the same manipulation of Theorem 5.2, we have the following theorem.

Theorem 6.3. *Let P_1 and P_2 be two polygroups with the identity elements e_1 and e_2, respectively. Suppose that μ and λ be two solvable fuzzy subpolygroup of P_1 and P_2, respectively, and $\phi : P_1 \to P_2$ be a function.*

(i) *If ϕ is a good homomorphism, then $\phi^{-1}(\lambda)$ is a solvable fuzzy subpolygroup of P_1.*
(ii) *If ϕ is an isohomomorphism, then $\phi(\mu)$ is a solvable fuzzy subpolygroup of P_2.*

Theorem 6.4. *Let μ be a normal fuzzy subpolygroup of $(P, \cdot, ^{-1}, e_1)$. Then $(\frac{P}{\mu_*}, \cdot, ^{-1}, e_2)$ is a solvable polygroup if and only if μ is a solvable fuzzy subpolygroup.*

Proof. Let $\frac{P}{\mu_*}$ be a solvable polygroup and $\pi : P \to \frac{P}{\mu_*}$ be the natural epimorphism. Since $z \in \pi^{-1}(\pi(x))$, we have $\pi(z) = \pi(x)$ and so $\pi(e_1) \in \pi(z^{-1} \cdot z) = \pi(z^{-1} \cdot x)$. Thus, there exists $r \in z^{-1} \cdot x$ such that $e_2 = \pi(e_1) = \pi(r)$ which implies that $r \in \ker \pi = \mu_*$. Hence, $\mu(r) = \mu(e_1)$ and so $z \sim x$. Then, for any $x \in P$,

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z \in \pi^{-1}(\pi(x))} \mu(z) = \bigvee_{z \sim x} \mu(z) \geq \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) \supseteq \mu$. Now, since $\frac{P}{\mu_*}$ is a solvable polygroup and $\pi(\mu)$ is a fuzzy subpolygroup of $\frac{P}{\mu_*}$, by Theorem 6.2, $\pi(\mu)$ is solvable and by Theorem 6.3, $\pi^{-1}(\pi(\mu))$ is solvable. Now, we show that μ is solvable. For this let $x \in i_n(p)$ and $f : \frac{P}{\sim} \longrightarrow \frac{P}{\mu_*}$ be as in the proof of Theorem 4.5. Since $\pi^{-1}(\pi(\mu))$ is solvable, so $\pi^{-1}(\pi(\mu))(x) =$

$\pi^{-1}(\pi(\mu))(e_1)$. Then $\bigvee_{z\sim x}\mu(z) = \mu(e_1)$ and so $x \sim e_1$. Hence, $\mu[x] = \mu[e_1]$. Now, by $f(\mu[x]) = \mu_*x$ we have $\mu_*x = \mu_*e_1$. Thus $x \in \mu_*$ which implies that $\mu(x) = \mu(e_1)$. Therefore, μ is a solvable fuzzy subpolygroup.

Conversely, let μ be a normal solvable fuzzy subpolygroup of P. By Theorem 6.1, $\frac{P}{\sim}$ is solvable also, $\frac{P}{\sim} \cong \frac{P}{\mu_*}$ and so $\frac{P}{\mu_*}$ is solvable. □

Example 6.3. In Example 4.2, $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a\rangle$. Thus μ_* is a normal subgroup of D_3. Also, $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian, it is solvable and so by Theorem 6.4, μ is a solvable fuzzy subgroup.

Theorem 6.5. *Let μ and ν be two fuzzy subpolygroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is a solvable fuzzy subpolygroup, then ν is a solvable fuzzy subpolygroup.*

Proof. Let μ and ν be two fuzzy subgroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Now let μ be solvable and $x \in i_n(P)$. Hence, $\mu(x) = \mu(e) = \nu(e)$ and so by hypotheses $\nu(e) = \mu(x) \leq \nu(x)$. Therefore, $\nu(x) = \nu(e)$, which implies that ν is solvable. □

Corollary 6.1. *If $A_{\overline{\mu}_t}$ is a solvable fuzzy subpolygroup of P, then μ is solvable, too.*

Proof. Let $A_{\overline{\mu}_t}$ be a solvable fuzzy subpolygroup of P. Since $A_{\overline{\mu}_t} \subseteq \mu$, by Theorem 6.5, μ is solvable. □

Theorem 6.6. *Let μ be a nilpotent fuzzy subpolygroup of P. Then μ is a solvable fuzzy subpolygroup.*

Proof. First we prove that $i_j(P) \subseteq l_j(P)$, for any non negative integer j. We do the proof by induction on j. The proof is clear for $j = 0$. Now let $i_j(P) \subseteq l_j(P)$, for any $j \leq n$ and $x \in i_n(P)$. Then $x \in [a, b]$, for some $a, b \in i_{n-1}(P)$. By hypotheses of induction, $a \in l_{n-1}(P)$ and $b \in P$. Thus, $x \in l_n(P)$. Hence $i_j(P) \subseteq l_j(P)$, for any non negative integer j. Now, let $x \in i_n(P)$ and μ be a nilpotnt fuzzy subpolygroup of class $n \in \mathbb{N}$. Since $x \in i_n(P) \subseteq l_n(P)$ so by hypotheses $\mu(x) = \mu(e)$. Therefore, μ is solvable. □

We recall that if G is a group and $a \in G$, then the order of a is the least positive integer n such that $a^n = e$. Also, a group G is of exponent n ($n \in \mathbb{N}$), if the order of any $x \in G$ is n.

Definition 6.2. If μ is a fuzzy subpolygroup of P and $a \in P$, then the *order of a with respect to μ* is the least positive integer n such that for any $r \in a^n$, $\mu(r) = \mu(e)$. We denote the order of a with respect to μ by $\circ(\mu(a))$. Also, μ is of exponent n, if the order of any $a \in P$ is n.

Theorem 6.7. *Let μ be a fuzzy polygroup of P and $x \in P$. If for any $r \in x^m$ we have $\mu(r) = \mu(e)$ for some integer m, then $\circ(\mu(a)) \mid m$.*

Proof. Let $\circ(\mu(a)) = n$. By the Euclidean algorithm, there exist integers s and t such that $m = ns + t$, where $0 \leq t < n$. Then for $r \in x^t = x^m \cdot (x^n)^{-s}$, there exist

$h \in x^m$ and $g \in (x^n)^{-s}$ such that $r \in hg$ and so $\mu(r) \geq \mu(h) \wedge \mu(g) \geq \mu(e) \wedge \mu(g) = \mu(g)$. Since $g \in (x^n)^{-s} = (x^n)^{-1} \cdot (x^n)^{-1} \cdots (x^n)^{-1}$, we get $g \in p_1.p_2 \ldots p_s$, in which $p_1, p_2, \ldots, p_s \in (x^n)^{-1}$ and so by hypotheses $\mu(g) \geq \mu(e)$. Consequently, $\mu(r) = \mu(e)$. Hence, $t = 0$, by the minimality of n. □

Theorem 6.8 ([11,14])**.** (i) *Every* 3*-Engel group of exponent* 4*, is solvable.*
(ii) *Each group of exponent* 3 *is* 2*-Engel.*

Theorem 6.9. (i) *Let μ be a* 3*-Engel normal fuzzy subpolygroup of exponent 4. Then μ is solvable.*
(ii) *Each normal fuzzy subpolygroup of exponent* 3 *is* 2*-Engel.*

Proof. (i) Let μ be a 3-Engel normal fuzzy subpolygroup on P such that for any $z \in x^4$, $\mu(z) = \mu(e)$. Then, by Theorem 4.3, $\underset{\sim}{P}$ is a 3-Engel group and $\mu[e] = \mu[z] = (\mu[x])^4$. Therefore, by Theorem 6.8 (i), $\underset{\sim}{P}$ is solvable and so, by Theorem 6.1, μ is solvable.

(ii) By Theorem 4.2, $\mu[e] = \mu[z] = (\mu[x])^3$. Thus, $\underset{\sim}{P}$ is of exponent 3 and so by Theorem 6.8(ii), $\underset{\sim}{P}$ is 2-Engel. Therefore, by Theorem 4.3, μ is 2-Engel. □

Theorem 6.10 ([18])**.** *Every* 3*-Engel solvable group with no element of order* 2*, is nilpotent.*

Theorem 6.11. *Let μ be a* 3*-Engel solvabel normal fuzzy subpolygroup on P such that for any $z \in x^2$, $\mu(z) \neq \mu(e)$. Then μ is nilpotent.*

Proof. Let μ be a 3-Engel solvable normal fuzzy subpolygroup of P such that for any $z \in x^2$, $\mu(z) \neq \mu(e)$. Then, by Theorems 4.3 and 6.1 $\underset{\sim}{P}$ is a 3-Engel solvabel group and $\mu(e) \neq \mu(z) = (\mu(x))^2$. Therefore, by Theorem 6.10, $\underset{\sim}{P}$ is nilpotent and so, by Theorem 5.3, μ is nilpotent. □

7. Conclusions

In this paper, we defined the notion of Engel polygroups. This help us to get usefull results on Engel fuzzy subpolygroups. On the other hand, we prove that every normal fuzzy subpolygroup of an Engel polygroup is Engel. Also, some connections between Engel (nilpotent, solvable) fuzzy subpolygroups and Engel (nilpotent, solvable) groups are stablished and studied. Finally, we prove some results on 3-Engel fuzzy subpolygroups. Specially, we prove that every 3-Engel normal fuzzy subpolygroup of exponent 4, is solvable.

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VERTEX-EDGE ROMAN DOMINATION

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Abstract. A vertex-edge Roman dominating function (or just ve-RDF) of a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ such that for each edge $e = uv$ either $\max\{f(u), f(v)\} \neq 0$ or there exists a vertex w such that either $wu \in E$ or $wv \in E$ and $f(w) = 2$. The weight of a ve-RDF is the sum of its function values over all vertices. The vertex-edge Roman domination number of a graph G, denoted by $\gamma_{veR}(G)$, is the minimum weight of a ve-RDF G. In this paper, we initiate a study of vertex-edge Roman dominaton. We first show that determining the number $\gamma_{veR}(G)$ is NP-complete even for bipartite graphs. Then we show that if T is a tree different from a star with order n, l leaves and s support vertices, then $\gamma_{veR}(T) \geq (n - l - s + 3)/2$, and we characterize the trees attaining this lower bound. Finally, we provide a characterization of all trees with $\gamma_{veR}(T) = 2\gamma'(T)$, where $\gamma'(T)$ is the edge domination number of T.

1. Introduction

Let $G = (V, E)$ be a simple graph with order $n = |V|$. For every vertex $v \in V$, the *open neighborhood* $N(v)$ is the set $\{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is the set $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is the cardinality of its open neighborhood, denoted $d_G(v) = |N(v)|$. By $\delta(G) = \delta$ we denote the *minimum degree* of a graph G. A vertex of degree one is called a *leaf* and its neighbor is called a *support vertex*. A support vertex is *strong* (*weak*, respectively) if it is adjacent to at least two leaves (exactly one leaf, respectively). An edge incident with a leaf is called a *pendant edge*. A *star* of order $n \geq 2$, denoted by $K_{1,n-1}$, is a tree with at least $n - 1$ leaves. A *double star* is a tree that contains exactly two vertices that are not leaves. A double star with respectively r and s leaves attached to each support vertex is denoted by $D_{r,s}$.

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Let D be a nonempty subset of E. The *subgraph* of G whose vertex set is the set of ends of edges in D and whose edge set is D is called the subgraph of G induced by D and is denoted by $\langle D\rangle$. The subgraph $\langle D\rangle$ is called *edge induced subgraph* of G. The *distance* between two vertices u and v in a connected graph G is the number of edges in a shortest between u and v. The *diameter,* diam(G), of a graph G is the greatest distance between any pair of vertices.

A set S of vertices is a *dominating set* of G if every vertex not in S is adjacent to some vertex in S. A subset X of E is an *edge dominating* set (or just EDS) of G if every edge not in X is adjacent to some edge in X. The *edge domination number* $\gamma'(G)$ of G is the minimum cardinality of an edge dominating set. An edge dominating set of G of minimum cardinality is called a $\gamma'(G)$-set. Edge domination was introduced by Mitchell and Hedetniemi [7].

A vertex v *ve-dominates* every edge incident to v, as well as, every edge adjacent to these incident edges, that is, a vertex v *ve*-dominates every edge incident to a vertex in $N[v]$. A set $S\subseteq V$ is a *vertex-edge dominating set* (or simply, a *ve-dominating set*) if for every edge $e\in E$, there exists a vertex $v\in S$ such that v *ve*-dominates e. The minimum cardinality of a *ve*-dominating set of G is called the *ve-domination number* $\gamma_{ve}(G)$. The concept of vertex-edge domination was introduced by Peters [8] in 1986 and studied further in [1, 5, 6].

A function $f: V(G)\to\{0,1,2\}$ is a *Roman dominating function* (or just RDF) if every vertex u for which $f(u)=0$ is adjacent to at least one vertex v for which $f(v)=2$. The *weight* of an RDF f is $f(V(G))=\sum_{u\in V(G)} f(u)$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF on G. For more information on Roman domination, see [3, 4].

A variation of Roman dominating function, say, vertex-edge Roman dominating function was defined in [9]. A vertex-edge Roman dominating function (ve-RDF) is a function $f: V(G)\to\{0,1,2\}$ such that each edge $e=vu$ is either incident with a vertex having function value at least one or uv is *ve*-dominated by some vertex w with $f(w)=2$. The *vertex-edge Roman domination number* $\gamma_{veR}(G)$ equals the minimum weight of all ve-RDF on G.

2. Complexity

We show that the Vertex-edge Roman domination problem (VERD-Dom) is NP-complete for bipartite graphs by proposing a polynomial reduction from the well-known NP-complete problem, Exact cover by 3-sets (X3C).

Vertex-Edge Roman Domination (VERD)
INSTANCE. Graph $G=(V,E)$, positive integer $k\leq|V|$.
QUESTION. Does G have an vertex-edge Roman dominating function of weight at most k?

Exact cover by 3-sets(X3C)
INSTANCE. A finite set X with $|X| = 3q$ and a collection C of 3-element subsets of X.
QUESTION. Does C contain an exact cover for X, that is, a sub collection $C' \subseteq C$ such that for every element in X belongs to exactly one member of C'?

Theorem 2.1. *VERD problem in NP-complete for bipartite graphs.*

Proof. VERD problem is a member of *NP*, since we can check in polynomial time that a function $f : V \to \{0, 1, 2\}$ has a weight at most k and that is a vertex-edge Roman dominating function. Now let us show how to transform any instance of X3C into an instance G of VERD, so that one of them has a solution if and only if the other one has a solution. Let $X = \{x_1, x_2, \ldots, x_{3q}\}$ and $C = \{C_1, C_2, \ldots, C_t\}$ be an arbitrary instance of X3C.

For each $x_i \in X$, we create a path $P_6^i = x_i y_i z_i a_i b_i p_i$ and for each C_j we create a single vertex c_j. To obtain the graph G, we add edges $c_j x_i$ if $x_i \in C_j$. Clearly, G is bipartite graph. Let $Y = \{c_1, c_2, \ldots, c_t\}$ and $W = \{x_1, x_2, \ldots, x_{3q}\}$. Let H be the subgraph of G induced by all paths P_6^i's. Set $k = 8q$. Observe that for any vertex-edge Roman dominating function f on G, $f(V(P_6^i)) \geq 2$.

Suppose that the instance X, C of $X3C$ has a solution C'. We construct a vertex-edge Roman dominating function of G with weight k as follows. For each $i \in \{1, 2, \ldots, 3q\}$, we assign a 0 to every vertex of $\{x_i, y_i, z_i, b_i, p_i\}$ and we assign a 2 to every a_i. For every $j \in \{1, 2, \ldots, t\}$, we assign a 2 to c_j if $C_j \in C'$ and a 0 if $C_j \notin C'$. Note that since C' exists, its cardinality is precisely q and so the number of c_j's with weight 2 is q, having disjoint neighborhoods in W. Since C' is a solution for X3C, the edges incident with W are *ve*-Roman dominated by the c_j's. Hence it is straightforward to see that f is a vertex-edge Roman dominating set of G with cardinality $8q = k$.

Conversely, suppose that G has a vertex-edge Roman dominating function $f = (V_0, V_1, V_2)$ with weight at most k. As seen above we may assume, without loss of generality, that $a_i \in V_2$ and every vertex of $\{p_i, b_i, z_i, y_i\}$ is in V_0. Since $\sum_{i=1}^{3q} f(a_i) = 6q$, we deduce that $f(W \cup Y) \leq 2q$. If some x_i belongs to V_2, then we can substitue it by a vertex of $N(x_i) \cap Y$. Hence $W \cap V_2 = \emptyset$. Now if there are two vertices x_i and x_r assigned a 1 and have a common neighbor, say c_j, then we can reassign a 0 to each of x_i and x_r and a 2 to c_j. So all vertices of $V_1 \cap W$ have no common neighbors. Suppose x_i and x_j are assigned a 1. The vertices adjacent to $(N(x_i) \cap Y) \setminus \{x_i\}$ are assigned 0. To dominates the edges incident with these vertices, the vertex in $N(x_i) \cap Y$ are assigned weight 2. Since $|W| = 3q$, we must have $W \cap V_0 = \emptyset$, implying that $C \cap V_2 \neq \emptyset$. Let $y = |C \cap V_2|$. Clearly $y \leq 2q$ and using the fact that every c_j has exactly three neighbors in W, we deduce that $f(C) \geq 2q$. Now, combining all these facts with $f(V(G)) \leq k = 8q$, we obtain $y \geq q$ and hence $y = q$. Hence, $C' = \{C_j \mid f(c_j) = 2\}$ is an exact cover for C. □

3. Bounds

We present in this section some sharp bounds on the vertex-edge Roman domination number. We begin with the following observation.

Observation 3.1. Let $f = (V_0, V_1, V_2)$ be an minimum vertex-edge Roman dominating function of a graph G. Then

(a) $|V_0| \geq 1$;
(b) no edge of G joins V_1 and V_2;
(c) $V_1 \cup V_2$ is a vertex edge dominating set of G.

In the following, we give a lower bound on the vertex-edge Roman domination for every graph in terms of the order and maximum degree.

Proposition 3.1. *If G is a connected graph of order $n \geq 2$, then $\gamma_{veR}(G) \geq \left\lceil \frac{2n}{(\Delta+1)^2} \right\rceil$, and the bound is sharp.*

Proof. Let $f = (V_0, V_1, V_2)$ be an $\gamma_{veR}(G)$-function. From the Observation 3.1, we have $|V_0| \geq 1$. The edge of G are *ve*-dominated by the vertices in $V_1 \cup V_2$. Therefore $|V_0| \leq \Delta^2|V_2| + \Delta|V_1|$. From $n = |V_0| + |V_1| + |V_2| \leq \Delta^2|V_2| + \Delta|V_1| + |V_1| + |V_2|$, we obtain $\frac{2n}{(\Delta+1)^2} \leq 2|V_2| + \frac{2|V_1|}{\Delta+1} \leq 2|V_2| + |V_1| = \gamma_{veR}(G)$. Since $\gamma_{veR}(G)$ is an integer, we get $\gamma_{veR}(G) \geq \left\lceil \frac{2n}{(\Delta+1)^2} \right\rceil$. The bound is sharp as it is attained for stars $K_{1,n}$. □

Every Roman dominating function is a vertex-edge roman dominating function, we have the following.

Proposition 3.2. *If G is connected graph of order $n \geq 2$ with maximum degree Δ, then $\gamma_{veR}(G) \leq n - \Delta + 1$ and the bound is sharp.*

We now present an upper bound of vertex-edge Roman domination in terms of edge domination number.

Proposition 3.3. *For any graph G, $\gamma_{veR}(G) \leq 2\gamma'(G)$.*

Proof. Let D be a $\gamma'(G)$-set. Define a function f on $V(G)$ by assigning a 1 to the vertices incident with the edges in D and a 0 to the remaining vertices. It is easy to see that f is a *veR*-dominating function of G, and thus, $\gamma_{veR}(G) \leq 2\gamma'(G)$. □

3.1. **Trees.** In this section we provide a lower bound of the vertex-edge Roman domination number for trees with diameter at least three in terms of order n, number of leaves l and support vertices s. We shall show that vertex-edge Roman domination number of a tree with diameter at least three of order n with l leaves and s support vertices bounded below by $(n - l - s + 3)/2$. Let T^* be the tree obtained from $K_{1,3}$ by subdividing two edges and α be the leaf which is incident to the edge which is not subdivided. Moreover, for the purpose of characterizing the trees attaining this bound, we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let

$T_1 = P_5$ or P_7. If k is a positive integer, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a vertex by joining it to any support vertex of T_i.
- Operation $\mathcal{O}_2$: Attach a path P_2 by joining one of its vertices to a vertex of T_i adjacent to mP_2 where $m \geq 2$.
- Operation $\mathcal{O}_3$: Attach a tree T^* by joining the vertex α to a leaf of T_i.
- Operation $\mathcal{O}_4$: Attach a path P_4 by joining one of its leaves to a vertex of T_i is a leaf or adjacent to P_2 or P_4

Lemma 3.1. *If $T \in \mathcal{T}$, then $\gamma_{veR}(T) = (n - \ell - s + 3)/2$.*

Proof. We use induction on the number k of operations performed to construct the tree T. If T is P_5, then obviously $\gamma_{veR}(T) = 2 = (n - \ell - s + 3)/2$. Let k be a positive integer. Assume the result is true for $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let $T = T_{k+1}$ be a tree constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let v be a support vertex and x be a leaf adjacent to v in T'. Let the tree T is obtained from T' by attaching a vertex y to v. We have $n = n' + 1$, $l = l' + 1$ and $s' = s$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of T'. If $f_1(x) = 1$ then $f_1(v) = 0$. Replacing the weight of x and v, we get f_1 is a veR-dominating function of tree T. If $f_1(x) = 2$ or 0 then the vertex which dominates the edge vx dominates vy. The function f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Let f be a γ_{veR}-dominating function of tree T. If $f(y) = 0$ then $f|_{V(T')}$ is a veR-dominating function of T'. Let $f(y) = 1$ then $f(x) = 1$. The function $f|_{V(T')}$ is a veR-dominating function of T'. Assume $f(y) = 2$ then $f(x) = 0$. Replacing the weight of x and y, we get $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T)$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - l - s + 3)/2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. Let u be the vertex in T' which is adjacent to many P_2. Let the tree T is obtained from T' by attaching the path $P_2 = xy$ by joining x to u. We have $n' = n - 2, l' = l - 1$ and $s' = s - 1$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of tree T'. To dominate the edges incident to vertices in $V(T_u)$, the vertex u is assigned weight two. The function

$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 0, & \text{otherwise,} \end{cases}$$

is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges incident to vertices in $V(T_u)$, to the vertex u is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T)$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - 2 - l + 1 - s + 1 + 3)/2 = (n - l - s + 3)/2$.

Now assume that T is obtained from T' by operation $\mathcal{O}_3$. Let d be the leaf in T'. Let the tree T is obtained from T' by attaching a tree T^* by the vertex α. We have $n = n' + 6$, $l = l' + 1$ and $s = s' + 1$. Let f_1 a $\gamma_{veR}(T')$-dominating function of tree T'.

The function
$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 2, & \text{if Child of } \alpha, \\ 0, & \text{otherwise ,} \end{cases}$$
is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges incident to the vertices in $V(T_\alpha)$, to the child of α is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T) - 2$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 2 = (n - 6 - l + 1 - s + 1 + 3)/2 + 2 = (n - l - s + 3)/2$.

Now, assume that T is obtained from T' by operation $\mathcal{O}_4$. Let d be the leaf in T'. Let the tree T is obtained from T' by attaching a path $P_4 = wuvt$ by joining w to d. We have $n = n' + 4, l' = l$ and $s' = s$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of tree T'. The function
$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = u, \\ 0, & \text{otherwise,} \end{cases}$$
is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges tv, vu, uw and wd, to the vertex u is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T) - 2$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 2 = (n - 4 - l - s + 3)/2 + 2 = (n - l - s + 3)/2$.

Now, d is adjacent to a path P_2 or P_4. Let the tree T is obtained from T' by attaching a path $P_4 = wuvt$ by joining w to d. We have $n = n' + 4$, $l = l' + 1$ and $s = s' + 1$. Let f_1 be a $\gamma_{veR}(T')$-dominating function of tree T'. Thus, the weight of d is two in T'. Then the
$$f(a) = \begin{cases} f_1(a), & \text{if } a \in V(T'), \\ 1, & \text{if } a = u, \\ 0, & \text{otherwise,} \end{cases}$$
is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 1$. Let f be a $\gamma_{veR}(T)$-dominating function of T. To dominate the edges tv, vu, uw and wd, the vertex d is assigned the weight two and v is assigned the weight one. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T) - 1$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 1 = (n - 4 - l + 1 - s + 1 + 3)/2 + 1 = (n - l - s + 3)/2$. ☐

We now ready to establish the lower bound.

Theorem 3.1. *If T is a tree with* $\operatorname{diam}(T) \geq 3$ *of order n with l leaves and s support vertices, then $\gamma_{veR}(T) \geq (n - l - s + 3)/2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $T \in \mathcal{T}$, then by Lemma 3.1, $\gamma_{veR}(T) = (n - l - s + 3)/2$. If $\operatorname{diam}(T) = 3$, then T is a double star. We have $l = n - 2$ and $s = 2$. Consequently, $(n - l - s + 3)/2 = (n - n + 2 - 2 + 3)/4 = 3/2 < 2 = \gamma_{veR}(T)$. Now, assume that $\operatorname{diam}(T) \geq 4$. Thus, the order n of the tree is at least five. We obtain the result by induction on the number n. Assume that the theorem is true for every tree T' of order $n' < n$ with l' leaves and s' support vertices.

Assume any support vertex of T, say y, is strong. Let x and t be the leaves adjacent to y. Let $T' = T - x$. We have $n' = n - 1$ and $l' = l - 1$. Let f be a $\gamma_{veR}(T)$-dominating function of a tree T. If $f(x) = 0$ then $f|_{V(T')}$ is a veR-dominating function of T'. If $f(t) = 1$ then $f(x) = 1$. The function $f|_{V(T')}$ is a veR-dominating function of T'. Assume $f(x) = 2$ then $f(t) = 0$. Replacing the weight of x and t, we get $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') \geq (n'-l'-s'+3)/2 = (n-l-s+3)/2$. If $\gamma_{veR}(T) = (n-l-s+3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_1$. Therefore, $T \in \mathcal{T}$. Henceforth, we can assume that every support vertex of T is weak.

Let $x_0x_1x_2 \ldots x_{d-1}x_d$ be the longest path in tree T. We now root the tree at a vertex x_d. Clearly $d_T(x_0) = d_T(x_d) = 1$. From the previous paragraph, we can assume $d_T(x_1) = d_T(x_{d-1}) = 2$.

Now, assume that x_2 is adjacent to a leaf y_1. Let $T' = T - y_1$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edge x_0x_1 and x_1x_2, to the vertex x_2 is assigned the weight two. Clearly $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - 1 - l + 1 - s + 1 + 3)/2 > (n - l - s + 3)/2$.

Now, assume that x_2 is adjacent to paths $P_i = y_{1_i}y_{2_i}$ where $i = 1, 2, \ldots, m$ $(m \geq 2)$ other than x_1x_0. Let $T' = T - T_{x_1}$. We have $n' = n-2$, $l' = l-1$ and $s' = s-1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edges x_2x_1, x_1x_0, $x_2y_{1_i}$ and $y_{1_i}y_{2_i}$, to the vertex x_2 is assigned the weight two. It is obvious that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_2$. Therefore, $T \in \mathcal{T}$.

Assume that x_2 is adjacent to a path $P_2 = y_1y_2$ other than x_1x_0. If $d_T(x_2) = 2$, then $T = P_5$, we have $\gamma_{veR}(P_5) = 2 = (n - l - s + 3)/2$. Thus, $T \in \mathcal{T}$. Assume $\deg(x_2) = 3$. Let us consider some child of x_3 say t is not a leaf. It suffices to consider x_3 is adjacent to isomorphic copy of T_{x_2}. Let $T' = T - T_{x_2}$. We have $n' = n - 5$, $l' = l - 2$ and $s' = s - 2$. To dominate the edges incident to vertices in $V(T_t)$, to the vertex t is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 \geq (n' - l' - s' + 3)/2 + 2 \geq (n - 5 - l + 2 - s + 2 + 3)/2 + 2 > (n - l - s + 3)/2$.

Assume x_3 is adjacent to path $P_3 : tuv$. Let $T' = T - T_t$. We have $n' = n - 3$, $l' = l-1$ and $s' = s-1$. To dominate the edge x_0x_1, x_1x_2, to the vertex x_2 is assigned the weight two. It is easy to see that the vertex x_2 dominates the edge x_3t. To dominate the edge tu and uv, to the vertex u is assigned the weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 \geq (n' - l' - s' + 3)/2 + 1 \geq (n - 3 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Assume x_3 is adjacent to path $P_2 : tu$. Let $T' = T - T_t$. We have $n' = n - 2$, $l' = l - 1$ and $s' = s - 1$. To dominate the edge x_0x_1, x_1x_2, to the vertex x_2 is assigned the weight two. It is clear that the vertex x_2 dominates the edge x_3t. To dominate the edge tu, to the vertex u is assigned the weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 \geq (n' - l' - s' + 3)/2 + 1 \geq (n - 2 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Assume x_3 is a support vertex. Let t be a child of x_3 other than x_2. From operation $\mathcal{O}_1$, it suffices to consider $d_T(x_3) = 3$. Let $T' = T - T_t$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. To dominate the edge x_3t, to the vertex x_2 is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') \geq (n' - l' - s' + 3)/2 \geq (n - 1 - l + 1 - s + 1 + 3)/2 > (n - l - s + 3)/2$.

Suppose $\deg(x_3) = 2$. Now assume that $d_T(x_4) \geq 3$. Let $T' = T - T_{x_3}$. We have $n' = n - 6, l' = l - 2$ and $s' = s - 2$. To dominate the edges incident to $V(T_{x_3})$, to the vertex x_2 is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 \geq (n' - l' - s' + 3)/2 + 2 \geq (n - 6 - l + 2 - s + 2 + 3)/2 + 2 > (n - l - s + 3)/2$.

Now $\deg(x_4) = 2$. Let $T' = T - T_{x_3}$. We have $n' = n - 6$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges incident to the vertices in $V(T_{x_3})$, to the vertex x_2 is assigned the weight two. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 \geq (n' - l' - s' + 3)/2 + 2 \geq (n - 6 - l + 1 - s + 1 + 3)/2 + 2 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_3$. Therefore, $T \in \mathcal{T}$.

Now, assume $d_T(x_2) = 2$. Suppose that x_3 is adjacent to a path $P_3 = y_2y_1y_0$ other than $x_0x_1x_2$. Let x_3 be adjacent to y_2. Let $d_T(x_3) = 2$. We have $T = P_7$. It is easy to see that $\gamma_{veR}(P_7) = (n - l - s + 3)/2$. Thus, $T \in \mathcal{T}$. Now assume that $d_T(x_3) \geq 3$. Let $T' = T - T_{x_2}$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges y_0y_1, y_1y_2, y_2x_3 and x_3x_2, to the vertex y_2 is assigned the weight two. To dominate the edges x_2x_1 and x_1x_0, to the vertex x_1 is assigned weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 = (n' - l' - s' + 3)/2 + 1 = (n - 3 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Assume that x_3 is adjacent to a path $P_2 = y_2y_1$ with x_3 adjacent to y_2. Let $T' = T - T_{x_2}$. We have $n' = n - 3$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges y_1y_2, y_2x_3, x_2x_1 and x_3x_2, to the vertex x_3 is assigned the weight two. To dominate the edge x_1x_0, either x_1 or x_0 is assigned weight one. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 = (n' - l' - s' + 3)/2 + 1 = (n - 3 - l + 1 - s + 1 + 3)/2 + 1 > (n - l - s + 3)/2$.

Now, assume that x_3 is a support vertex. Let x be the leaf adjacent to x_3. Let $T' = T - T_x$. We have $n' = n - 1$, $l' = l - 1$ and $s' = s - 1$. To dominate the edges x_0x_1, x_2x_1, x_2x_3 and x_3x, to the vertex x_2 is assigned the weight two. It is clear that the function $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') = (n' - l' - s' + 3)/2 = (n - 1 - l + 1 - s + 1 + 3)/2 > (n - l - s + 3)/2$.

Assume that some child of x_4, say y_1 other than x_3 such that distance of d to the most distance vertex of T_{y_1} is 2 or 4. It suffices to consider the case when T_x is $P_2 = y_1y_2$ or $P_4 = y_1y_2y_3y_4$. Let $T' = T - T_{x_3}$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edges x_4x_3, x_3x_2, x_2x_1, x_1x_0, x_4y_1 and y_1y_2, to the vertices x_4 and x_1 are assigned the weights 2 and 1 respectively. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 1 = (n' - l' - s' + 3)/2 + 1 = (n - 4 - l + 1 - s + 1 + 3)/2 + 1 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_4$. Therefore, $T \in \mathcal{T}$.

Assume that some child of x_4, say x other than x_3 such that distance of d to the most distance vertex of T_x is one or three. It suffices to consider the case when T_x is $P_1 = y_1$ or $P_3 = y_1y_2y_3$. Let $T' = T - T_{x_3}$. We have $n' = n - 4$, $l' = l - 1$ and $s' = s - 1$. Let f be a $\gamma_{veR}(T)$-dominating function. To dominate the edges x_4x_3 ,x_3x_2, x_2x_1 and x_1x_0, to the vertex x_2 is assigned the weight two. Thus, $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 2 = (n - 4 - l + 1 - s + 1 + 3)/2 + 2 > (n - l - s + 3)/2$.

Now, $d_T(x_4) = 2$. Let $T' = T - T_{x_3}$. We have $n' = n - 4$, $l' = l$ and $s' = s$. To dominate the edges x_4x_3 ,x_3x_2, x_2x_1 and x_1x_0, to the vertex x_2 is assigned the weight two. Thus, $f|_{V(T')}$ is a veR-dominating function of T'. It is easy to see that $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T) \geq \gamma_{veR}(T') + 2 = (n' - l' - s' + 3)/2 + 1 = (n - 4 - l - s + 3)/2 + 1 = (n - l - s + 3)/2$. If $\gamma_{veR}(T) = (n - l - s + 3)/2$, we have $\gamma_{veR}(T') = (n' - l' - s' + 3)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T is obtained from T' by operation $\mathcal{O}_4$. Therefore, $T \in \mathcal{T}$. □

4. Trees T with $\gamma_{veR}(T) = 2\gamma'(T)$

In this section we provide a constructive characterization of trees with equal vertex-edge Roman domination number and twice edge domination number. For the purpose of characterizing the trees with equal vertex-edge Roman domination number and twice edge domination number, we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 = P_4$. If $k \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the following operations.

- Operation $\mathcal{O}_5$: Attach a vertex by joining it to any support vertex of T_i.
- Operation $\mathcal{O}_6$: Attach a path $P_4 = pqrs$ by joining the vertex q of a vertex w of T_i adjacent to path $P_4 = xuvt$ with w adjacent to u.
- Operation $\mathcal{O}_7$: Attach a double star $D_{r,s}(r, s \geq 2)$ by joining one of its leaf to a vertex of T_i adjacent to a path P_4 or P_3 or P_2 or P_1 or double star.

Lemma 4.1. *If $T \in \mathcal{F}$, then $\gamma_{veR}(T) = 2\gamma'(T)$.*

Proof. We use induction on the number k of operations performed to construct the tree T. If T is P_5, then obviously $\gamma_{veR}(T) = 2 = 2\gamma'(T)$. Let k be a positive integer.

Assume the result is true for $T' = T_k$ of the family $\mathcal{F}$ constructed by $k-1$ operations. Let $T = T_{k+1}$ be a tree constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_5$. Let u be a support vertex and x be a leaf adjacent to u in the graph T'. The graph T is obtained from T' by adding a vertex y to u. Let D be a $\gamma'(T)$-set. To dominate the edges ux and uy, an edge incident with u other than ux and uy is in D. It is obvious that D is an EDS of T'. Thus, $\gamma'(T') \leq \gamma'(T)$. Let D' be a $\gamma'(T')$-set. The edge which dominates ux dominates the edge uy in graph T. Thus, $\gamma'(T) \leq \gamma'(T')$. We have $\gamma'(T) = \gamma'(T')$. Let f_1 be a $veR(T')$-dominating function of T'. If the vertex x has weight one, then the vertex u has weight zero. Replace the weight of these two vertices. The function f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Let f be a γ_{veR}-dominating function of T. To dominate the edges ux and yu, the vertex u is assigned with weight one or a vertex in $N(u)$ is assigned with weight two. If the leaf y is assigned weight two, then the vertex x has weight zero. Replace the weight of x from zero to two. The function f is a veR-dominating function of T'. If the vertex u is assigned with weight one then f is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T)$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') = 2\gamma'(T') = 2\gamma'(T)$.

Now, assume that T is obtained from T' by operation $\mathcal{O}_6$. Let the vertex $w \in T'$ be adjacent to path $P_4 = xuvt$ with u adjacent to w. The graph T is obtained from T' by adding another path $P_4 = pqrs$ with q adjacent to w. Let D be a $\gamma'(T')$-set of T'. It is clear that $D \cup \{qr\}$ is an EDS of T. Thus, $\gamma'(T) \leq \gamma'(T) + 1$. Let D' be a $\gamma'(T)$-set. To dominate the edges rs and vt, the edges $qr, uv \in D'$. It is easy to verify that $D' \setminus \{qr\}$ is an EDS of the graph T'. Thus, $\gamma'(T') \leq \gamma'(T) - 1$. We have $\gamma'(T) = \gamma'(T') + 1$. Let f be a γ_{veR}-function of T'. To dominate the edges vt, uv and ux, the vertex u is assigned with weight two. Define a function f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = r, \\ 0, & \text{if } a = p, q, s. \end{cases}$$

Clearly, f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. Let f_1 be a $\gamma_{veR}(T)$-dominating function. As in the previous case, the vertex r and u are assigned a weight two. The function $f|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T) - 2$. We have $\gamma_{veR}(T') = \gamma_{veR}(T) - 2$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = 2\gamma'(T') + 2 = 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$.

Now, assume that T is obtained from T' by operation $\mathcal{O}_7$. Let d be a vertex of T' with $d_{T'}(d) \geq 3$. Let d be adjacent to P_4 or P_3 or P_2 or P_1 or $D_{r,s}$, $r, s \geq 2$. The graph T is obtained from T' by joining a leaf of $D_{r,s}$, $r, s \geq 2$, to d. Let the support vertices of $D_{r,s}$ be u and v. Let the leaves of u be w and w_1 and the leaves of v be t and t_1. Let w be adjacent to d. Let D be a $\gamma'(T')$-set. The vertex d is adjacent to P_4 or P_3 or P_2 or P_1 or $D_{r,s}(r, s \geq 2)$, an edge incident with d is in D. It is easy to see that $D \cup \{uv\}$ is an EDS of the graph T. Thus, $\gamma'(T) \leq \gamma'(T') + 1$. Let D' be a $\gamma'(T)$-set. To dominate the edges vt, uw and uw_1, the edge uv is in D'. It is obvious that $D' \setminus \{uv\}$

is an EDS of graph T'. Thus, $\gamma'(T') \leq \gamma'(T) - 1$. We have $\gamma'(T') = \gamma'(T) - 1$. Let f_1 be a γ_{veR}-dominating function of T. To dominate the edges vt and uv, the vertex u is assigned with weight two. It is obvious that $f_1|_{V(T')}$ is a veR-dominating function of T'. Thus, $\gamma_{veR}(T') \leq \gamma_{veR}(T) - 2$. Let f be a $\gamma_{veR}(G)$-dominating function of T'. Define f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'), \\ 2, & \text{if } a = u, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') + 2$. We have $\gamma_{veR}(T) = \gamma_{veR}(T') + 2$. We get $\gamma_{veR}(T) = \gamma_{veR}(T') + 2 = 2\gamma'(T') + 2 = 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$. □

The following theorem gives a characterization of trees for which $\gamma_{veR}(T) = 2\gamma'(T)$.

Theorem 4.1. *Let T be a nontrivial tree. Then $\gamma_{veR}(T) = 2\gamma'(T)$ with equality if and only if $T \in \mathcal{F}$.*

Proof. If $T \in \mathcal{F}$, then by Lemma 4.1, $\gamma_{veR}(T) = 2\gamma'(T)$. If $\operatorname{diam}(T) = 1$ or 2, then T is P_2 or star. We have $\gamma_{veR}(T) = 1 < 2 = 2\gamma'(T)$. Assume $\operatorname{diam}(T) = 3$. If T is P_4. We have $\gamma_{veR}(T) = 2\gamma'(T)$. If T is a double star other than P_4, then T can be obtained from P_4 by applying operation $\mathcal{O}_1$. The result is proved by induction on order n. Assume that the result is true for all tree T' of order $n' < n$.

Let u be a strong support vertex. Let u be adjacent to two leaves x and y. Let $T' = T - x$. Let D be a any $\gamma'(T')$-set. To dominate the edges ux and uy, an edge incident with u other than ux and uy is in D. It is easy to see that D is an EDS of T'. Thus, $\gamma'(T') \leq \gamma'(T)$. Let f_1 be a $veR(T')$-dominating function of G. If the vertex x has weight one, then the vertex u has weight zero. Replace the weight of these two vertices. The function f_1 is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T')$. Thus, $\gamma_{veR}(T) \leq \gamma_{veR}(T') \leq 2\gamma'(T') \leq 2\gamma'(T)$. If $\gamma_{veR}(T) = 2\gamma'(T)$, then $\gamma_{veR}(T') = 2\gamma'(T')$. By the inductive hypothesis $T' \in \mathcal{F}$. The tree T is obtained from T' by operation $\mathcal{O}_5$. Thus, $T \in \mathcal{F}$. Henceforth, we can assume that every support vertex of T is weak.

Let $u_1u_2u_3 \ldots u_k$ be the longest path in the tree T. Then $k \geq 4$ and $d_T(u_1) = d_T(u_k) = 1$. The vertices u_2 and u_{k-1} are support vertices, we can assume $d_T(u_2) = d_T(u_{k-1}) = 2$.

Assume that u_3 is adjacent to a path $P_2 = pq$ other than u_2u_1. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and pq, the edges u_2u_3, pu_3 is in D. Define a function f on $V(T)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_2, u_3, p\}$, assigning weight two to u_3 and zero to all other vertices. It is clear that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 2 < 2\gamma'(T)$. Hence, the vertex u_3 is a support vertex. By operation $\mathcal{O}_5$, it suffices to consider $d_T(u_3) = 3$. Let x be a leaf adjacent to u_3.

Assume that u_4 is adjacent to a path $P_3 = pqr$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and rq, the edges u_2u_3, pq is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_2, p\}$, assigning weight two to u and zero to all other vertices. It is easy to observe that f is a veR-dominating function of G. Thus, $\gamma_{veR}(T) \le 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume that u_4 is adjacent to a path $P_2 = pq$. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and pq, the edges u_2u_3, pu_4 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_4, u_3, p\}$, assigning weight two to u_4 and zero to all other vertices. It is easy to observe that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \le 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume that u_4 is a support vertex. Let y be the leaf adjacent to u_4. Let $d_T(u_4) = 2$. We have T is G_1, where G_1 is obtained from P_5 by attaching a leaf adjacent to vertex of P_5 with minimum eccentricity. We have $\gamma_{veR}(G_1) = 2 < 4 = 2\gamma'(G_1)$. Assume $d_T(u_4) \ge 3$. Let d be a vertex adjacent to u_4 other than u_3 and y. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and u_4y, the edges u_3u_2, du_4 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_4, d\}$, assigning weight two to u_4 and zero to all other vertices. It is easy to observe that f is a veR-dominating function of G. Thus, $\gamma_{veR}(T) \le 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume that u_4 is adjacent to $P_4 = pqrs$ with q adjacent to u_4. Let $T' = T - T_q$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and rs, the edges $u_3u_2, qr \in D'$. It is easy to verify that $D \setminus \{qr\}$ is an EDS of the graph T'. Thus, $\gamma'(T') \le \gamma'(T) - 1$. Let f be a γ_{veR}-function of T. To dominate the edges u_1u_2, u_2u_3 and u_3x, the vertex u is assigned with weight two. Define a function f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'),\\ 2, & \text{if } a = q,\\ 0, & \text{if } a = p, r, s. \end{cases}$$

Clearly, f_1 is a veR-dominating function of H. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2$. We get $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2 \le 2\gamma'(T') + 2 \le 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$. If $\gamma_{veR}(T) = 2\gamma'(T)$, then $\gamma_{veR}(T') = 2\gamma'(T')$. By inductive hypothesis $T' \in \mathcal{F}$. The tree T is obtained from T' by operation $\mathcal{O}_6$. Thus, $T \in \mathcal{F}$.

Assume $d_T(u_4) = 2$. Let $d_T(u_5) \ge 3$. Let $T' = T - T_{u_4}$. Let D be a $\gamma'(T)$-set. To dominate the edges u_4u_3, u_3x and u_2u_1, the edge u_3u_2 is in D. It is obvious that $D \setminus \{u_3v_2\}$ is an EDS of graph G. Thus, $\gamma'(T') \le \gamma'(T) - 1$. Let f be a $\gamma_{veR}(T')$-dominating function. Define f_1 on $V(T)$ as

$$f_1(a) = \begin{cases} f(a), & \text{if } a \in V(T'),\\ 2, & \text{if } a = u_3,\\ 0, & \text{otherwise}. \end{cases}$$

Clearly, f_1 is a veR-dominating function of H. Thus, $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2$. We get $\gamma_{veR}(T) \le \gamma_{veR}(T') + 2 \le 2\gamma'(T') + 2 \le 2(\gamma'(T) - 1) + 2 = 2\gamma'(T)$. If $\gamma_{veR}(T) = 2\gamma'(T)$, then $\gamma_{veR}(T') = 2\gamma'(T')$. By inductive hypothesis $T' \in \mathcal{F}$. The tree T is obtained from T' by operation $\mathcal{O}_7$. Thus, $T \in \mathcal{F}$.

Assume $d_T(u_5) = 2$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and u_5u_4, the edges u_2u_3, u_5u_6 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_2, u_3, u_5\}$, assigning weight two to u_3 and zero to all other vertices. It is clear that f is a veR-dominating function of T. Thus, $\gamma_{veR}(G) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Now, assume $d_T(u_3) = 2$. Assume the vertex u_4 is adjacent to path $P_3 = pqr$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and rq, the edges u_2u_3, pq is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_2, p\}$, assigning weight two to u_3 and zero to all other vertices. It is easy to observe that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume the vertex u_4 is adjacent to path $P_2 = pq$. Let D be a $\gamma'(T)$-set. To dominate the edges u_2u_1 and pq, the edges u_3u_2, pu_4 is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_4, p\}$, assigning weight two to u_4 and zero to all other vertices. It is clear that f is a veR-dominating function of G. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Assume the vertex u_4 is a support vertex. Let x be the leaf adjacent to u_4. Assume that $d_T(u_4) = 2$. We have $T = P_5$ and $\gamma_{veR}(T) = 2 < 4 = 2\gamma'(T)$. Now assume $d_T(u_4) \geq 3$. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and xu_4, the edges u_3u_2 and an edge incident with u_4, say u_4d, other than u_4u_3 and u_4x is in D. Define a function f on $V(G)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_4, d\}$, assigning weight two to u_4 and zero to all other vertices. It is easy to see that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$.

Now, $d_T(u_4) = 2$. Let $d_G(u_5) = 1$. Then T is P_5. We have $\gamma_{veR}(T) = 2 < 4 = 2\gamma'(T)$. Assume $d_T(u_5) \geq 2$. Let D be a $\gamma'(T)$-set. To dominate the edges u_1u_2 and u_4u_5, the edges u_3u_2, u_4u_6 is in D. Define a function f on $V(T)$ by assigning weight one to the vertices in $V(\langle D\rangle) \setminus \{u_3, u_5, u_6\}$, assigning weight two to the vertex u_5 and zero to all other vertices. It is obvious that f is a veR-dominating function of T. Thus, $\gamma_{veR}(T) \leq 2(\gamma'(T) - 2) + 3 < 2\gamma'(T)$. □

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DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR A NEW DIFFERENTIAL OPERATOR CONTAINING MITTAG-LEFFLER FUNCTION

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Abstract. Owning to the importance and great interest of linear operators, a generalisation of linear derivative operator $\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)$ is newly introduced in this study. The main objective of this paper is to investigate various subordination and superordination related to the aforementioned generalised linear derivative operator. Additionally, the resultant sandwich-type of this operator is also considered.

1. Definition and Preliminaries

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and $\mathcal{H} = \mathcal{H}(\Delta)$ indicate the family of analytic functions within Δ. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ containing the functions of the form

$$\mathcal{H}[a,n] = \left\{ f \in \mathcal{H}(\Delta) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\}, \quad z \in \Delta.$$

Furthermore, let $\mathcal{A}(p)$ indicate the subclass of $\mathcal{H}$ containing the functions having the following form

$$f(z) = z^p + \sum_{i=p+1}^{\infty} a_i z^i, \quad p \in \mathbb{N}, \tag{1.1}$$

which are analytic and p-valent in Δ. For clarity, we write $\mathcal{A}(1) = \mathcal{A}$.

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The convolution (or Hadamard product) $f * g$ for two analytic functions f defined by (1.1) and

$$g(z) = z^p + \sum_{i=p+1}^{\infty} b_i z^i$$

is given by

$$f(z) * g(z) = z^p + \sum_{i=p+1}^{\infty} a_i b_i z^i.$$

For the two analytic functions f and g in $\mathcal{H}(\Delta)$, we are saying that $f(z)$ is subordinate to $g(z)$ usually denoted by $f(z) \prec g(z)$ in case if there is a Schwarz function ω with $\omega(z) = 0$, $|\omega(z)| < 1$, $z \in \Delta$, such that $f(z) = g(\omega(z))$ for all $z \in \Delta$.

Especially, if $g(z)$ is univalent in Δ, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\Delta) \subseteq g(\Delta)$.

Let $\mathcal{S}^*_\alpha(p)$ and $\mathcal{K}_\alpha(p)$ denote the familiar subclasses of the class $\mathcal{A}(p)$ consisting of the functions which are p-valently starlike and p-valently convex of order α in Δ, respectively,

$$\mathcal{S}^*_\alpha(p) = \left\{ f \in \mathcal{A}(p) : \operatorname{Re}\left\{ \frac{zf^{'}(z)}{f(z)} \right\} > \alpha,\ z \in \Delta \right\},$$

$$\mathcal{K}_\alpha(p) = \left\{ f \in \mathcal{A}(p) : \operatorname{Re}\left\{ 1 + \frac{zf^{''}(z)}{f^{'}(z)} \right\} > \alpha,\ z \in \Delta \right\}.$$

The method of differential subordinations ,which is additionally called the admissible functions method, was maybe the first one presented by Miller and Mocanu in 1978 [13]. From that point onward and roughly in 1981 [14] the theory started to proliferate and progressively develop. Relevant details are epitomized in a book written by Miller and Mocanu [15].

Definition 1.1 (see [15])**.** Let $\varphi : \mathbb{C}^3 \times \Delta \to \mathbb{C}$ and $h(z)$ be univalent in Δ. If $\zeta(z)$ is analytic function in Δ and also satisfies the second-order differential subordination

$$\varphi(\zeta(z), z\zeta^{'}(z), z^2\zeta^{''}(z); z \in \Delta) \prec h(z), \quad z \in \Delta, \tag{1.2}$$

then $\zeta(z)$ is defined as a solution of the differential subordination (1.2). A univalent function $q(z)$ is called a dominant if $\zeta(z) \prec q(z)$ for all $\zeta(z)$ satisfying (1.2). A dominant $\tilde{\mathrm{q}}$ is called the best dominant when $\tilde{\mathrm{q}} \prec q$ for all dominants q of (1.2).

Definition 1.2 (see [16])**.** Let $\phi : \mathbb{C}^3 \times \Delta \to \mathbb{U}$ let $h(z)$ be analytic function in Δ. If $\zeta(z)$ and $\phi(\zeta(z), z\zeta^{'}(z), z^2\zeta^{''}(z); z)$ are univalent in Δ and $\zeta(z)$ satisfies the (second-order) differential subordination

$$h(z) \prec \phi(\zeta(z), z\zeta^{'}(z), z^2\zeta^{''}(z)), \quad z \in \Delta, \tag{1.3}$$

then $\zeta(z)$ is defined as a solution of the differential subordination (1.3). An analytic function $q(z)$ is called a a subordinates, if $q(z) \prec \zeta(z)$ for all $\zeta(z)$ satisfying (1.3). A univalent subordinate $\tilde{\mathrm{q}}$ is called the best subordinate when $q \prec \tilde{\mathrm{q}}$ for all subordinates q of (1.3).

Definition 1.3 (see [16])**.** Let G denote the set of functions f which are analytic and injective on $\overline{\Delta}\backslash B(f)$, where

$$B(f)=\left\{\xi\in\partial\Delta:\lim_{z\to\xi}f(z)=\infty\right\},$$

and $f^{'}(\xi)\neq 0$, $\xi\in\partial\Delta\backslash B(f)$.

In 1999, Dziok and Srivastava [6] introduced the function $g_p(a_1,\dots,a_r,b_1,\dots,b_s;z)$, which defined by generalized hypergeometric function as following

$$g_p(a_1,\dots,a_r,b_1,\dots,b_s;z)=z^p+\sum_{i=p+1}^{\infty}\frac{(a_1)_{i-p}\cdots(a_r)_{i-p}}{(b_1)_{i-p}\cdots(b_s)_{i-p}}\frac{z^i}{(i-p)!},\quad p\in\mathbb{N}, \tag{1.4}$$

where $a_k\in\mathbb{C}$, $b_n\in\mathbb{C}\setminus\{0,-1,\dots\}$, $k=1,\dots,r$, $n=1,\dots,s$ and $r\leq 1+s$, $r,s\in\mathbb{N}_0$ and $(v)_i$ is the Pochhammer symbol defined by

$$(v)_i=\frac{\Gamma(v+i)}{\Gamma(v)}=\begin{cases} v(v+1)\cdots(v+i-1), & i=1,2,\dots,\\ 1, & i=0.\end{cases}$$

For convenience, we write $g_p(a_1,\dots,a_r,b_1,\dots,b_s;z)=\mathcal{G}_p(a_1,b_1;z)$.

The well known Mittag-Leffler function $E_\alpha(z)$ which is introduced by Mittag- Leffler [17] and [18] is defined hereunder. Similarly, the first two parametric generalization $E_{\alpha,\beta}(z)$ of the same function by Wiman [27] is defined as well

$$E_\alpha(z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+1)}$$

and

$$E_{\alpha,\beta}(z)=\sum_{i=0}^{\infty}\frac{z^i}{\Gamma(\alpha i+\beta)},$$

where $\alpha,\beta\in\mathbb{C}$, $\mathrm{Re}(\alpha)>0$ and $\mathrm{Re}(\beta)>0$.

The above mentioned resulted in plenty of valuable work has been made by numerous authors in an endeavor to clarify Mittag-Leffler function and its first two parametric generalization, see for instance [4, 8–10, 20, 23, 25] and [26].

Now, we define the function $\mathcal{F}_{\alpha,\beta}(z)$ by

$$\mathcal{F}_{\alpha,\beta}(z)=z\Gamma(\beta)E_{\alpha,\beta}(z)=z+\sum_{i=2}^{\infty}\frac{\Gamma(\beta)}{\Gamma(\alpha(i-1)+\beta)}z^i.$$

Having use of the function $\mathcal{F}_{\alpha,\beta}(z)$, Elhaddad et al. [7] defined the differential operator $\mathcal{D}_\delta^m(\alpha,\beta)f:\mathcal{A}\longrightarrow\mathcal{A}$ as illustrated below:

$$\mathcal{D}_\delta^m(\alpha,\beta)f(z)=z+\sum_{i=2}^{\infty}[1+(i-1)\delta]^m\frac{\Gamma(\beta)}{\Gamma(\alpha(i-1)+\beta)}a_iz^i, \tag{1.5}$$

where $m\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$, $\delta>0$.

Now, we define the operator $\mathcal{D}_{\delta}^{m}(\alpha,\beta)f(z)$ in (1.5) of a function $f \in \mathcal{A}(p)$ given by (1.1) as below:

$$\mathcal{D}_{\delta,p}^{m}(\alpha,\beta)f(z) = z^p + \sum_{i=p+1}^{\infty}\left[\frac{p+(i-p)\delta}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(i-p)+\beta)}a_i z^i, \quad p \in \mathbb{N}, \tag{1.6}$$

where $m \in \mathbb{N}_0$, $\delta > 0$.

Corresponding to $\mathcal{G}_p(a_1, b_1; z)$ which defined in (1.4), $\mathcal{D}_{\delta,p}^{m}(\alpha,\beta)f(z)$ defined in (1.6) and utilizing Hadamard product, we define a new generalized derivative operator $\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)$ as follows.

Definition 1.4. Let $f \in \mathcal{A}(p)$, then the generalized derivative operator $\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z) : \mathcal{A}(p) \to \mathcal{A}(p)$ is given by

$$\begin{aligned}
&\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\\
=&\mathcal{G}_p(a_1,b_1;z) * \mathcal{D}_{\delta,p}^{m}(\alpha,\beta)f(z)\\
=&z^p + \sum_{i=p+1}^{\infty}\left[\frac{p+(i-p)\delta}{p}\right]^m \frac{\Gamma(\beta)}{\Gamma(\alpha(i-p)+\beta)}\frac{(a_1)_{i-p}\cdots(a_r)_{i-p}}{(b_1)_{i-p}\cdots(b_s)_{i-p}}\frac{a_i z^i}{(i-p)!}.
\end{aligned}\tag{1.7}$$

We can easily verify from (1.7) that

$$\begin{aligned}
p\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z) =&(p-p\delta)\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)\\
&+\delta z(\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z))'.
\end{aligned}\tag{1.8}$$

Remark 1.1.
- For $s = 0,\ r = 1,\ a_1 = 1,\ \alpha = 0,\ \beta = 1$ and $p = 1$, we get Al-Oboudi operator [1].
- For $s = 0,\ r = 1,\ a_1 = 1,\ \beta = 1,\ \alpha = 0,\ \delta = 1$ and $p = 1$, we get Sălăgean operator [22].
- For $s = 0,\ r = 1,\ a_1 = 1,\ m = 0$ and $p = 1$, we get $\mathbb{E}_{\alpha,\beta}(z)$ [25].
- For $m = 0,\ \alpha = 0$ and $\beta = 1$, we get the operator studied by Dziok and Srivastava [6].
- For $m = 0,\ \alpha = 0,\ p = 1,\ r = 1,\ s = 0,\ a_1 = \lambda + 1$ and $\beta = 1$, we get the operator examined byby Ruscheweyh [21].
- For $m = 0,\ \alpha = 0,\ p = 1,\ r = 2,\ s = 1$ and $\beta = 1$, we get the operator which was introduced by Hohlov [11].
- For $m = 0,\ \alpha = 0,\ p = 1,\ r = 2,\ s = 1,\ a_2 = 1$ and $\beta = 1$, we get the operator investigated by Carlson and Shaffer [5].

So as to demonstrate and approve above results, following primer results are required.

Lemma 1.1 (see [24])**.** *Let $g(z)$ be convex function within the open unit disk Δ and let ν and μ be complex numbers, $\nu \in \mathbb{C}$ and $\mu \in \mathbb{C}/\{0\}$, with*

$$\operatorname{Re}\left\{\frac{zg''(z)}{g'(z)}+1\right\} > \max\left\{-\operatorname{Re}\left(\frac{\nu}{\mu}\right), 0\right\}.$$

If $h(z)$ is analytic within Δ and

$$\nu h(z) + \mu z h'(z) \prec \nu g(z) + \mu z g'(z). \tag{1.9}$$

Thus, $h(z) \prec g(z)$, $z \in \Delta$, and $g(z)$ is the best dominant of (1.9).

Lemma 1.2 (see [16])**.** *Let μ be a complex number with* $\mathrm{Re}(\mu) > 0$ *and g be a convex function within Δ. If $h(z) \in \mathcal{H}[g(0), 1] \cap G$ and $h(z) + \mu z h'(z)$ is univalent in Δ, thus*

$$g(z) + \mu z g'(z) \prec h(z) + \mu z h'(z), \tag{1.10}$$

consequently, $g(z) \prec h(z)$ and $g(z)$ is the best subordinant of (1.10).

2. Main Results

Theorem 2.1. *Let $m \in \mathbb{N}_0 = \mathbb{N} \bigcup \{0\}$, $\delta > 0$, $\sigma \in \mathbb{C}/\{0\}$ and $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)$ defined by* (1.7)*. Let $g(z)$ be univalent in Δ, with $g(0) = 1$, and assume that*

$$\mathrm{Re}\left\{\frac{z g''(z)}{g'(z)} + 1\right\} > \max\left\{-\frac{p}{\delta}\mathrm{Re}\left(\frac{1}{\sigma}\right), 0\right\}. \tag{2.1}$$

If f in the class $\mathcal{A}(p)$ satisfies the subordination condition

$$\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p}\right) \prec g(z) + \frac{\sigma\delta}{p} z g'(z), \tag{2.2}$$

then

$$\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p} \prec g(z)$$

and $g(z)$ is the best dominant of (2.2).

Proof. Define the function $\zeta(z)$ by

$$\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p} = \zeta(z). \tag{2.3}$$

Differentiating (2.3) logarithmically with respect to z, we have

$$\frac{z\zeta'(z)}{\zeta(z)} = \frac{z(\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z))'}{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)} - p. \tag{2.4}$$

Using (1.8) in the resulting equation (2.4), we get

$$\begin{aligned}
\frac{z\zeta'(z)}{\zeta(z)} = &\left(\frac{p}{\delta}\right)\left\{\frac{z(\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z))'}{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)} - 1\right\}\\
=&\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^{m}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p}\right)\\
=&\zeta(z) + \frac{\sigma\delta}{p} z\zeta'(z),
\end{aligned}$$

then the differential subordination from hypothesis (2.2) is equivalent to

$$\zeta(z) + \frac{\sigma\delta}{p} z\zeta'(z) \prec g(z) + \frac{\sigma\delta}{p} zg'(z).$$

To prove our result, we need to use Lemma 1.1. For that purpose, let $\nu = 1$, $\mu = \frac{\sigma\delta}{p}$. We get

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p} \prec g(z),$$

which is the required result. □

Setting $g(z) = \frac{1+Cz}{1+Dz}$ in Theorem 2.1, where $-1 \leq D < C \leq 1$. Then, the condition (2.1) turn into

$$\text{Re}\left\{\frac{1 - Dz}{1 + Dz}\right\} > \max\left\{0, -\frac{p}{\delta}\text{Re}\left(\frac{1}{\sigma}\right)\right\}, \quad z \in \Delta. \tag{2.5}$$

The function

$$\Psi(\gamma) = \frac{1 - \gamma}{1 + \gamma}, \quad |\gamma| < |D|,$$

is convex in Δ and since $\Psi(\overline{\gamma}) = \overline{\Psi(\gamma)}$ for all $|\gamma| < |D|$, then the image $\Psi(\Delta)$ is a convex domain symmetric with respect to the real axis. Thus,

$$\inf\left\{\text{Re}\left(\frac{1 - Dz}{1 + Dz}\right), z \in \Delta\right\} = \frac{1 - |D|}{1 + |D|} > 0.$$

Then, the relation (2.5) is identical to

$$\frac{p}{\delta}\text{Re}\left(\frac{1}{\sigma}\right) \geq \frac{|D| - 1}{|D| + 1},$$

as a result, we get the following corollary.

Corollary 2.1. *Let $m \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $\delta > 0$, $-1 \leq D < C \leq 1$ and $\sigma \in \mathbb{C}/\{0\}$ with*

$$\max\left\{0, -\frac{p}{\delta}\text{Re}\left(\frac{1}{\sigma}\right)\right\} \leq \frac{1 - |D|}{1 + |D|}.$$

If f in the class $\mathcal{A}(p)$ and

$$\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p}\right) + (1 - \sigma)\left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p}\right) \prec \frac{1 + Cz}{1 + Dz} + \frac{\sigma\delta}{p}\frac{(C - D)z}{(1 + D)^2} z, \tag{2.6}$$

then

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha, \beta, a_1, b_1) f(z)}{z^p} \prec \frac{1 + Cz}{1 + Dz}$$

and $\dfrac{1 + Cz}{1 + Dz}$ is the best dominant of (2.6).

Theorem 2.2. *Let* $m \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $\delta > 0$, $\sigma \in \mathbb{C}/\{0\}$ *and* $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)$ *defined by* (1.7). *Let* $h(z)$ *be a convex function in* Δ, *with* $h(0) = 1$. *Let* f *in the class* $\mathcal{A}(p)$ *and*

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p} \in \mathcal{H}[1,1] \cap G.$$

If

$$\sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)$$

in univalent in Δ, *and*

$$h(z) + \frac{\sigma\delta}{p}zh'(z) \prec \sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right), \tag{2.7}$$

then

$$h(z) \prec \frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}$$

and $h(z)$ *is the best subordinant of* (2.7).

Proof. Define the function $\chi(z)$ by

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p} = \chi(z). \tag{2.8}$$

From the presumption of Theorem 2.2, we note that the function χ is analytic in the open unit disk Δ. Differentiating (2.8) logarithmically with respect to z, we get

$$\frac{z\chi'(z)}{\chi(z)} = \frac{z(\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z))'}{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)} - p. \tag{2.9}$$

Using (1.8) in (2.9) and after some calculations, we get

$$\chi(z) + \frac{\sigma\delta}{p}z\chi'(z) = \sigma\left(\frac{\widetilde{\mathcal{H}}^{m+1}_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right) + (1-\sigma)\left(\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)$$

and presently, by utilizing Lemma 1.2, we have the specified result. □

Setting $h(z) = \frac{1+Cz}{1+Dz}$ in Theorem 2.2, where $-1 \leq D < C \leq 1$, we get the following result.

Corollary 2.2. *Let* $m \in \mathbb{N}_0 = \mathbb{N}\bigcup\{0\}$, $\delta > 0$, $\sigma \in \mathbb{C}/\{0\}$, $-1 \leq D < C \leq 1$ *and* $\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)$ *defined by* (1.7). *Let* f *in the class* $\mathcal{A}(p)$ *and*

$$\frac{\widetilde{\mathcal{H}}^m_{\delta,p}(\alpha,\beta,a_1,b_1)f(z)}{z^p} \in \mathcal{H}[1,1] \cap G.$$

If

$$\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)$$

is univalent in Δ*, and*

$$\frac{1+Cz}{1+Dz}+\frac{\sigma\delta}{p}\frac{(C-D)z}{(1+Dz)^2}\prec\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right), \tag{2.10}$$

then

$$\frac{1+Cz}{1+Dz}\prec\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}$$

and $\frac{1+Cz}{1+Dz}$ *is the best subordinant of* (2.10).

Combining Theorem 2.1 and Theorem 2.2, we get the following sandwich result.

Theorem 2.3. *Let* $m\in\mathbb{N}_0=\mathbb{N}\bigcup\{0\}$, $\delta>0$, $\sigma\in\mathbb{C}/\{0\}$ *and* $\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)$ *defined by* (1.7)*. Let* $h(z)$ *and* $g(z)$ *be a convex function in* Δ*, with* $h(0)=g(z)=1$*. Let* f *in the class* $\mathcal{A}(p)$ *and*

$$\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\in\mathcal{H}[1,1]\cap G.$$

If

$$\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)$$

is univalent in Δ *and*

$$h(z)+\frac{\sigma\delta}{p}zh'(z)\prec\sigma\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m+1}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)+(1-\sigma)\left(\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\right)\prec g(z)+\frac{\sigma\delta}{p}zg'(z), \tag{2.11}$$

then

$$h(z)\prec\frac{\widetilde{\mathcal{H}}_{\delta,p}^{m}(\alpha,\beta,a_1,b_1)f(z)}{z^p}\prec g(z),$$

and $h(z)$ *and* $g(z)$ *is the best subordinant and the best dominant respectively of* (2.11).

We skip the proofing because it is the same as in the proof of the last theorem.

Remark 2.1. Other work associated with the derivative and integral operator for different issues can be determined in [2,3,12] and [19].

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SIMPSON'S TYPE INEQUALITIES VIA THE KATUGAMPOLA FRACTIONAL INTEGRALS FOR s-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce some Simpson's type integral inequalities via the Katugampola fractional integrals for functions whose first derivatives at certain powers are s-convex (in the second sense). The Katugampola fractional integrals are generalizations of the Riemann–Liouville and Hadamard fractional integrals. Hence, our results generalize some results in the literature related to the Riemann–Liouville fractional integrals. Results related to the Hadamard fractional integrals could also be derived from our results.

1. Introduction

The inequality below is known in the literature as the Simpson's inequality:

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty,$$

where $f : [a,b] \to \mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)}\left|f^{(4)}(t)\right| < \infty$.

This inequality has been studied and generalized by many authors over the years. For more information on recent results about the Simpson's inequality, we refer the interested reader to the papers [1, 2, 6–8, 11, 14, 15].

Definition 1.1 ([3]). A function $f : [0,\infty) \to \mathbb{R}$ is said to be s-convex (in the second sense), for $s \in (0,1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

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for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

Remark 1.1. If $s = 1$ in Definition 1.1, then we have the definition of convex functions.

Recently, Cheng and Huang [5] obtained the following Simpson's type inequalities for s-convex functions via the Riemann–Liouville fractional integrals.

Theorem 1.1 ([5])**.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a diferentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + J^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2^{s+1}}\Big(|f'(a)| + |f'(b)|\Big) I(\alpha, s),$$

where

$$I(\alpha, s) = \int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| [(1+t)^s + (1-t)^s] dt,$$

$J^\alpha_{b^-} f(x)$ and $J^\alpha_{a^+} f(x)$ denotes the right- and left-sided Riemann–Liouville fractional integrals of f at x respectively (see Definition 1.2*).*

Theorem 1.2 ([5])**.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a diferentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + J^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right|^r dt\right)^{\frac{1}{r}} \left[\left(\frac{(2^{s+1}-1)|f'(b)|^q + |f'(a)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

Theorem 1.3 ([5])**.** *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + J^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2} I_5(\alpha, s)\left\{I_6(\alpha, s)^{\frac{1}{q}} + I_7(\alpha, s)^{\frac{1}{q}}\right\},$$

where

$$I_5(\alpha, s) = \left(\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| dt\right)^{1-\frac{1}{q}},$$

$$I_6(\alpha, s) = \int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| \left[\left(\frac{1+t}{2}\right)^s |f'(b)|^q + \left(\frac{1-t}{2}\right)^s |f'(a)|^q\right] dt$$

and

$$I_7(\alpha, s) = \int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| \left[\left(\frac{1+t}{2}\right)^s |f'(a)|^q + \left(\frac{1-t}{2}\right)^s |f'(b)|^q\right] dt.$$

The goal in this paper is to provide some Simpson's type inequalities for s-convex functions in the second sense via the Katugampola fractional integrals. Our results generalizes Theorems 1.1, 1.2, 1.3 and also some results in [11]. We complete this section with the definitions of the Riemann–Liouville, Hadamard and Katugampola fractional integrals.

Definition 1.2 ([12])**.** The left- and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ of f are defined by

$$J_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt$$

and

$$J_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt,$$

with $a < x < b$ and $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} dt, \quad \operatorname{Re}(x) > 0,$$

with the property that $\Gamma(x+1) = x\Gamma(x)$.

Definition 1.3 ([13])**.** The left- and right-sided Hadamard fractional integrals of order $\alpha > 0$ of f are defined by

$$H_{a+}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} \frac{f(t)}{t} dt$$

and

$$H_{b-}^\alpha f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} \frac{f(t)}{t} dt.$$

In what follows, $X_c^p(a,b)$, $c \in \mathbb{R}$, $1 \leq p \leq \infty$, denotes the set of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t}\right)^{1/p}, \quad 1 \leq p < \infty,$$

and, for $p = \infty$, $\|f\|_{X_c^\infty} = \operatorname{esssup}_{a \leq t \leq b} |t^c f(t)|$.

In 2011, Katugampola [9] introduced a new fractional integral operator which generalizes the Riemann–Liouville and Hadamard fractional integrals as follows.

Definition 1.4. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by

$$ {}^\rho I_{a+}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t)dt $$

and

$$ {}^\rho I_{b-}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t)dt, $$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Remark 1.2. It is shown in [9] that the Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$.

Theorem 1.4 ([9]). *Let $\alpha > 0$ and $\rho > 0$. Then, for $x > a$,*

(a) $\lim_{\rho \to 1} {}^\rho I_{a+}^\alpha f(x) = J_{a+}^\alpha f(x)$;

(b) $\lim_{\rho \to 0^+} {}^\rho I_{a+}^\alpha f(x) = H_{a+}^\alpha f(x)$.

Similar results also hold for right-sided operators.

For more information about the Katumgapola fractional integrals and related results, we refer the interested reader to the papers [4, 9, 10].

2. Main Results

To obtain our main results, we need the following lemma which is a generalization of [5, Lemma 2.1] and [11, Lemma 5].

Lemma 2.1. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. Then the following identity holds:*

$$ \begin{aligned} &\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho) \right] \\ &- \frac{2^{\alpha-1}\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} \left[{}^\rho I_{a+}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I_{b-}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) \right] \\ =& \frac{\rho(b^\rho - a^\rho)}{2} \left[\int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right) dt \right. \\ &\left. - \int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1-t^\rho}{2}a^\rho + \frac{1+t^\rho}{2}b^\rho\right) dt \right]. \end{aligned} $$

Proof. We start by considering the following computations which follows from change of variables and using the definiton of the Katugampola fractional integrals.

$$ \begin{aligned} &\int_0^1 t^{\alpha\rho-1} f\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right) dt \\ =& \int_0^1 t^{(\alpha-1)\rho} t^{\rho-1} f\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right) dt \end{aligned} $$

$$
\begin{aligned}
&= \frac{2^{\alpha}}{(b^{\rho}-a^{\rho})^{\alpha}} \int_{a}^{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{\frac{1}{\rho}}} \left(\frac{a^{\rho}+b^{\rho}}{2} - u^{\rho}\right)^{\alpha-1} u^{\rho-1} f\left(u^{\rho}\right) du \\
&= \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha)}{(b^{\rho}-a^{\rho})^{\alpha}} {}^{\rho}I_{a+}^{\alpha} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \qquad (2.1)
\end{aligned}
$$

and, by similar argument as above, we have

$$
\int_{0}^{1} t^{\alpha\rho-1} f\left(\frac{1-t^{\rho}}{2}a^{\rho} + \frac{1+t^{\rho}}{2}b^{\rho}\right) dt = \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha)}{(b^{\rho}-a^{\rho})^{\alpha}} {}^{\rho}I_{b-}^{\alpha} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right). \qquad (2.2)
$$

Now, by using integration by parts and (2.1), we obtain

$$
\begin{aligned}
I_1 &= \int_{0}^{1} \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{2}{\rho(a^{\rho}-b^{\rho})} \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) f\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right)\Bigg|_{0}^{1} \\
&\quad + \frac{2\alpha\rho}{\rho(a^{\rho}-b^{\rho})} \int_{0}^{1} \frac{t^{\alpha\rho-1}}{2} f\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{1}{3\rho(b^{\rho}-a^{\rho})} f(a^{\rho}) + \frac{2}{3\rho(b^{\rho}-a^{\rho})} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \\
&\quad - \frac{\alpha}{b^{\rho}-a^{\rho}} \int_{0}^{1} t^{\alpha\rho-1} f\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{1}{3\rho(b^{\rho}-a^{\rho})} f(a^{\rho}) + \frac{2}{3\rho(b^{\rho}-a^{\rho})} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \\
&\quad - \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} {}^{\rho}I_{a+}^{\alpha} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right). \qquad (2.3)
\end{aligned}
$$

Similarly, by using integration by parts and (2.2), we obtain

$$
\begin{aligned}
I_2 &= \int_{0}^{1} \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1-t^{\rho}}{2}a^{\rho} + \frac{1+t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{-1}{3\rho(b^{\rho}-a^{\rho})} f(b^{\rho}) - \frac{2}{3\rho(b^{\rho}-a^{\rho})} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \\
&\quad + \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} {}^{\rho}I_{b-}^{\alpha} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right). \qquad (2.4)
\end{aligned}
$$

Using (2.3) and (2.4), we obtain

$$
\begin{aligned}
I_1 - I_2 &= \frac{1}{3\rho(b^{\rho}-a^{\rho})} \left[f(a^{\rho}) + 4f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) + f(b^{\rho})\right] \\
&\quad - \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} \left[{}^{\rho}I_{a+}^{\alpha} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) + {}^{\rho}I_{b-}^{\alpha} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)\right]. \qquad (2.5)
\end{aligned}
$$

The desired identity is obtained by multiplying both sides of (2.5) by $\frac{\rho(b^\rho-a^\rho)}{2}$. This completes the proof. □

Theorem 2.1. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \le a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|$ is s-convex for $s \in (0,1]$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right]-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\right.\\
&\left.\times\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right|\\
\le&\frac{b^\rho-a^\rho}{2^{s+1}}\mathcal{C}(\alpha,s)\Big(|f'(a^\rho)|+|f'(b^\rho)|\Big)\\
\le&\frac{b^\rho-a^\rho}{3(s+1)}\Big(|f'(a^\rho)|+|f'(b^\rho)|\Big),
\end{aligned}\tag{2.6}$$

where

$$\mathcal{C}(\alpha,s)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\Big((1+u)^s+(1-u)^s\Big)du.$$

Proof. Using Lemma 2.1 and the s-convexity of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right|\\
\le&\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(\left|f'\left(\frac{1+t^\rho}{2}a^\rho+\frac{1-t^\rho}{2}b^\rho\right)\right|\right.\\
&\left.+\left|f'\left(\frac{1-t^\rho}{2}a^\rho+\frac{1+t^\rho}{2}b^\rho\right)\right|\right)dt\\
\le&\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(\frac{(1+t^\rho)^s}{2^s}|f'(a^\rho)|+\frac{(1-t^\rho)^s}{2^s}|f'(b^\rho)|\right.\\
&\left.+\frac{(1-t^\rho)^s}{2^s}|f'(a^\rho)|+\frac{(1+t^\rho)^s}{2^s}|f'(b^\rho)|\right)dt\\
=&\frac{(b^\rho-a^\rho)}{2^{s+1}}\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\Big((1+u)^s+(1-u)^s\Big)\Big(|f'(a^\rho)|+|f'(b^\rho)|\Big)du\\
=&\frac{(b^\rho-a^\rho)}{2^{s+1}}\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|((1+u)^s+(1-u)^s)\,du\,(|f'(a^\rho)|+|f'(b^\rho)|)\\
=&\frac{(b^\rho-a^\rho)}{2^{s+1}}\mathcal{C}(\alpha,s)\,(|f'(a^\rho)|+|f'(b^\rho)|)\,,
\end{aligned}$$

where

$$\mathcal{C}(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| \left((1+u)^s + (1-u)^s\right) du.$$

This proves the first inequality in (2.6). To obtain the second inequality in (2.6), we observe that $\left|\frac{1}{3} - \frac{u^\alpha}{2}\right| \leq \frac{1}{3}$ for all $u \in [0,1]$. Thus,

$$\mathcal{C}(\alpha, s) \leq \frac{1}{3}\int_0^1 \bigg((1+u)^s + (1-u)^s\bigg) du = \frac{2^{s+1}}{3(s+1)}.$$

This completes the proof. □

Remark 2.1. If $\rho = 1$, then the first inequality in Theorem 2.1 coincides with the inequality in Theorem 1.1 and the second inequality coincides with the inequality in Corollary 8 in [11].

Corollary 2.1. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|$ is convex, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right]\right. \\
&\left.\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b^-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\right| \\
\leq& \frac{b^\rho - a^\rho}{4}\mathcal{C}(\alpha, 1)\left(|f'(a^\rho)| + |f'(b^\rho)|\right) \\
\leq& \frac{b^\rho - a^\rho}{6}\left(|f'(a^\rho)| + |f'(b^\rho)|\right).
\end{aligned}$$

Proof. The result follows directly if we take $s = 1$ in Theorem 2.1. □

Theorem 2.2. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is s-convex for $s \in (0,1]$ and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right]\right. \\
&\left.\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b^-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\right| \\
\leq& \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{2^{s+1}-1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$\le \frac{b^\rho - a^\rho}{6}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right.$$

$$\left. + \left(\frac{1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{2^{s+1}-1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right], \tag{2.7}$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

Proof. Using Lemma 2.1, the Hölder's inequality and the s-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right]\right. \\
&\left. - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\right| \\
\le & \frac{\rho(b^\rho - a^\rho)}{2}\left[\int_0^1 \left|\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left|f'\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right)\right| dt\right. \\
&\left. + \int_0^1 \left|\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left|f'\left(\frac{1-t^\rho}{2}a^\rho + \frac{1+t^\rho}{2}b^\rho\right)\right| dt\right] \\
= & \frac{b^\rho - a^\rho}{2}\left[\int_0^1 \left|\frac{1}{3} - \frac{u^{\alpha}}{2}\right|\left|f'\left(\frac{1+u}{2}a^\rho + \frac{1-u}{2}b^\rho\right)\right| du\right. \\
&\left. + \int_0^1 \left|\frac{1}{3} - \frac{u^{\alpha}}{2}\right|\left|f'\left(\frac{1-u}{2}a^\rho + \frac{1+u}{2}b^\rho\right)\right| du\right] \\
\le & \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \left|\frac{1}{3} - \frac{u^{\alpha}}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\int_0^1 \left|f'\left(\frac{1+u}{2}a^\rho + \frac{1-u}{2}b^\rho\right)\right|^q du\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 \left|f'\left(\frac{1-u}{2}a^\rho + \frac{1+u}{2}b^\rho\right)\right|^q du\right)^{\frac{1}{q}}\right] \\
\le & \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \left|\frac{1}{3} - \frac{u^{\alpha}}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\int_0^1 \left(\frac{(1+u)^s}{2^s}|f'(a^\rho)|^q + \frac{(1-u)^s}{2^s}|f'(b^\rho)|^q\right) du\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\int_0^1 \left(\frac{(1-u)^s}{2^s}|f'(a^\rho)|^q + \frac{(1+u)^s}{2^s}|f'(b^\rho)|^q\right) du\right)^{\frac{1}{q}}\right] \\
= & \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \left|\frac{1}{3} - \frac{u^{\alpha}}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{2^{s+1}-1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This proves the first inequality of (2.7). The second inequality follows from the first inequality by using the fact that $\left|\frac{1}{3}-\frac{u^\alpha}{2}\right| \leq \frac{1}{3}$ for all $u \in [0,1]$. □

Remark 2.2. If $\rho = 1$, then the first inequality in Theorem 2.2 coincides with the inequality in Theorem 1.2 and the second inequality concides with the inequality in Corollary 12 in [11].

Corollary 2.2. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is convex and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right] \\
&\quad -\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\Bigg| \\
\leq&\frac{b^\rho-a^\rho}{2}\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\frac{3|f'(a^\rho)|^q+|f'(b^\rho)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(b^\rho)|^q+|f'(a^\rho)|^q}{4}\right)^{\frac{1}{q}}\right] \\
\leq&\frac{b^\rho-a^\rho}{6}\left[\left(\frac{3|f'(a^\rho)|^q+|f'(b^\rho)|^q}{4}\right)^{\frac{1}{q}}+\left(\frac{3|f'(b^\rho)|^q+|f'(a^\rho)|^q}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{r}+\frac{1}{q}=1$.

Proof. The result follows directly if we take $s = 1$ in Theorem 2.2. □

Theorem 2.3. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is s-convex for $s \in (0,1]$ and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right] \\
&\quad -\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\Bigg| \\
\leq&\frac{b^\rho-a^\rho}{2}\Big(\mathcal{M}_0(\alpha)\Big)^{\frac{1}{r}}\left[\left(\frac{1}{2^s}\Big(\mathcal{M}_1(\alpha,s)|f'(a^\rho)|^q+\mathcal{M}_2(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2^s}\Big(\mathcal{M}_2(\alpha,s)|f'(a^\rho)|^q+\mathcal{M}_1(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right] \\
\leq&\frac{b^\rho-a^\rho}{2}\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\left(\frac{1}{2^s}\left(\frac{2^{s+1}-1}{3(s+1)}|f'(a^\rho)|^q+\frac{1}{3(s+1)}|f'(b^\rho)|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2^s}\left(\frac{1}{3(s+1)}|f'(a^\rho)|^q+\frac{2^{s+1}-1}{3(s+1)}f'(b^\rho)|^q\right)\right)^{\frac{1}{q}}\right], \qquad (2.8)
\end{aligned}$$

where $\frac{1}{r}+\frac{1}{q}=1$, *with*

$$\mathcal{M}_0(\alpha)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|du,$$
$$\mathcal{M}_1(\alpha,s)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|(1+u)^s du$$

and

$$\mathcal{M}_2(\alpha,s)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|(1-u)^s du.$$

Proof. Using Lemma 2.1, the Hölder's inequality and the s-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right] \\
&\quad-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\Bigg| \\
\le&\frac{\rho(b^\rho-a^\rho)}{2}\left[\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left|f'\left(\frac{1+t^\rho}{2}a^\rho+\frac{1-t^\rho}{2}b^\rho\right)\right|dt\right. \\
&\left.+\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left|f'\left(\frac{1-t^\rho}{2}a^\rho+\frac{1+t^\rho}{2}b^\rho\right)\right|dt\right] \\
=&\frac{b^\rho-a^\rho}{2}\left[\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left|f'\left(\frac{1+u}{2}a^\rho+\frac{1-u}{2}b^\rho\right)\right|du\right. \\
&\left.+\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left|f'\left(\frac{1-u}{2}a^\rho+\frac{1+u}{2}b^\rho\right)\right|du\right] \\
\le&\frac{b^\rho-a^\rho}{2}\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|du\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left(\frac{(1+u)^s}{2^s}|f'(a^\rho)|^q\right.\right.\right. \\
&\left.\left.+\frac{(1-u)^s}{2^s}|f'(b^\rho)|^q\right)du\right)^{\frac{1}{q}} \qquad (2.9)\\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left(\frac{(1-u)^s}{2^s}|f'(a^\rho)|^q+\frac{(1+u)^s}{2^s}|f'(b^\rho)|^q\right)du\right)^{\frac{1}{q}}\right] \\
=&\frac{b^\rho-a^\rho}{2}\Big(\mathcal{M}_0(\alpha)\Big)^{\frac{1}{r}}\left[\left(\frac{1}{2^s}\Big(\mathcal{M}_1(\alpha,s)|f'(a^\rho)|^q+\mathcal{M}_2(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right. \\
&\left.+\left(\frac{1}{2^s}\Big(\mathcal{M}_2(\alpha,s)|f'(a^\rho)|^q+\mathcal{M}_1(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\mathcal{M}_0(\alpha) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| du,$$

$$\mathcal{M}_1(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1+u)^s du$$

and

$$\mathcal{M}_2(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1-u)^s du.$$

This proves the first inequality of (2.8). For the second inequality, since $\left|\frac{1}{3} - \frac{u^\alpha}{2}\right| \leq \frac{1}{3}$ for all $u \in [0,1]$, it follows that

$$\mathcal{M}_0(\alpha) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| du \leq \frac{1}{3},$$

$$\mathcal{M}_1(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1+u)^s du \leq \frac{1}{3}\int_0^1 (1+u)^s du = \frac{2^{s+1}-1}{3(s+1)}$$

and

$$\mathcal{M}_2(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1-u)^s du \leq \frac{1}{3}\int_0^1 (1-u)^s du = \frac{1}{3(s+1)}.$$

This completes the proof of the theorem. ☐

Remark 2.3. If $\rho = 1$, then the first inequality in Theorem 2.3 coincides with the inequality in Theorem 1.3.

Corollary 2.3. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is convex and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right]\right. \\
&\left.\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+} f\left(\frac{a^\rho+b^\rho}{2}\right) + {}^\rho I^\alpha_{b^-} f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right| \\
\leq & \frac{b^\rho - a^\rho}{2}\Big(\mathcal{M}_0(\alpha)\Big)^{\frac{1}{r}}\left[\left(\frac{1}{2}\Big(\mathcal{M}_1(\alpha,1)|f'(a^\rho)|^q + \mathcal{M}_2(\alpha,1)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{1}{2}\Big(\mathcal{M}_2(\alpha,1)|f'(a^\rho)|^q + \mathcal{M}_1(\alpha,1)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

Proof. The result follows directly if we take $s = 1$ in Theorem 2.3. ☐

3. Conclusion

We have introduced some new integral inequalities of Simpson's type for s-convex functions using the Katugampola fractional integrals. Our results generalize some results in the literature related to the Riemann–Liouville fractional integrals as pointed out in the paper. We have new results for the case $\rho \neq 1$. In particular, if we take the limit as $\rho \to 0^+$, then our results could be stated using the Hadamard fractional integrals. The details are left for the interested reader.

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WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS OF SASAKIAN MANIFOLDS

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Abstract. Recently, B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds. By using the notion of pointwise slant submanifolds, we investigate the geometry of pointwise semi-slant submanifolds and their warped products in Sasakian manifolds. We give non-trivial examples of such submanifolds and obtain several fundamental results, including a characterization for warped product pointwise semi-slant submanifolds of Sasakian manifolds.

1. Introduction

In [7], B.-Y. Chen introduced the notion of slant submanifolds of almost Hermitian manifolds as a natural generalization of holomorphic (invariant) and totally real (anti-invariant) submanifolds. Afterwards, the geometry of slant submanifolds became an active topic of research in differential geometry. Later, A. Lotta [20] has extended this study for almost contact metric manifolds. J. L. Cabrerizo et al. investigated slant submanifolds of a Sasakian manifold [6]. N. Papaghiuc introduced in [22] a class of submanifolds, called semi-slant submanifolds of almost Hermitian manifolds, which are the generalizations of slant and CR-submanifolds. Later on, Cabrerizo et al. [5] extended this idea for semi-slant submanifolds of contact metric manifolds and provided many examples of such submanifolds.

Next, as an extension of slant submanifolds of an almost Hermitian manifold, F. Etayo [16] introduced the notion of pointwise slant submanifolds of almost Hermitian manifolds. B.-Y. Chen and O. J. Garay [14] studied pointwise slant submanifolds of almost Hermitian manifolds. They have obtained several fundamental results, in

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particular, a characterization of these submanifolds. K. S. Park [23] has extended this study. B. Sahin studied pointwise semi-slant submanifolds and warped product pointwise semi-slant submanifolds by using the notion of pointwise slant submanifolds [26]. In [31], the authors considered pointwise slant submanifolds of an almost contact metric manifold such that the structure vector field ξ is tangent to the submanifold. They have obtained a simple characterization for such submanifolds and studied warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In 1969, R. L. Bishop and B. O'Neill [3] introduced and studied warped product manifolds. 30 years later, around the beginning of this century, B.-Y. Chen initiated in [9, 10] the study of warped product CR-submanifolds of Kaehler manifolds. Chen's work in this line of research motivated many geometers to study the geometry of warped product submanifolds by using his idea for different structures on manifolds (see, for instance, [2, 17, 21] and [27]). For a detailed survey on warped product submanifolds we refer to Chen's books [11, 13] and his survey article [12] as well.

In [24], B. Sahin showed that there exists no proper warped product semi-slant submanifold of Kaehler manifolds. Then, he introduced the notion of warped product hemi-slant submanifolds of Kaehler manifolds [25]. He defined and studied warped product pointwise semi-slant submanifolds and showed that there exists a non-trivial warped product pointwise semi-slant submanifold of the form $M_T \times_f M_\theta$ in a Kaehler manifold $\tilde{M}$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively [26]. For almost contact metric manifolds, we have seen in [19] and [1] that there are no proper warped product semi-slant submanifolds in cosymplectic and Sasakian manifolds. Then, we have considered warped product pseudo-slant submanifolds (warped product hemi-slant submanifolds [25], in the same sense of almost Hermitian manifolds) of cosymplectic [28] and Sasakian manifolds [29].

K. S. Park [23] studied warped product pointwise semi-slant submanifolds. He proved that there do not exist warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ such that M_θ and M_T are proper pointwise slant and invariant submanifolds, respectively. Then he provided many examples and obtained several results for warped products by reversing these two factors, including sharp estimations for the squared norm of the second fundamental form in terms of the warping functions. Later, we also extended this idea in [31] to warped product pointwise pseudo-slant submanifolds of Sasakian manifolds.

In this paper, we study warped product pointwise semi-slant submanifolds of the form $M_T \times M_\theta$ of Sasakian manifolds.

The present paper is organized as follows: in Section 2, we give basic definitions and formulas needed for this paper. Section 3 is devoted to the study of pointwise semi-slant submanifolds of Sasakian manifolds; we define pointwise semi-slant submanifolds and in the definition of pointwise semi-slant submanifolds we assume that the structure vector field ξ is always tangent to the submanifold. We give two non-trivial examples of such submanifolds for the justification of our definition and a result which is useful to the next section. In Section 4, we study warped product pointwise semi-slant submanifolds of Sasakian manifolds. In [1], we have seen that there are no warped product semi-slant submanifolds of the form $M_T \times_f M_\theta$ in a Sasakian manifold other than contact CR-warped products, but if we assume that M_θ is a proper pointwise slant submanifold then there exists a non-trivial class of such warped products. In this section, we obtain several new results which are generalizations of warped product semi-slant submanifolds and contact CR-warped product submanifolds. In Section 5, we provide nontrivial examples of Riemannian product and warped product pointwise semi-slant submanifolds in Euclidean spaces.

2. Preliminaries

An *almost contact structure* (φ, ξ, η) on a $(2n+1)$-dimensional manifold $\tilde{M}$ is defined by a $(1,1)$ tensor field φ, a vector field ξ, called *characteristic* or *Reeb vector field*, and a 1-form η satisfying the following conditions

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \eta \circ \xi = 0, \quad \eta(\xi) = 1, \tag{2.1}$$

where $I : T\tilde{M} \to T\tilde{M}$ is the identity map [4]. There always exists a Riemannian metric g on an almost contact manifold $\tilde{M}$ satisfying the following compatibility condition

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

for any $X, Y \in \Gamma(T\tilde{M})$, the Lie algebra of vector fields on $\tilde{M}$. This metric g is called a *compatible metric* and the manifold $\tilde{M}$ together with the structure (φ, ξ, η, g) is called an *almost contact metric manifold*. As an immediate consequence of (2.2), one has $\eta(X) = g(X, \xi)$ and $g(\varphi X, Y) = -g(X, \varphi Y)$. If ξ is a Killing vector field with respect to g, then the contact metric structure is called a *K-contact structure*. A normal contact metric manifold is said to be a *Sasakian manifold*. In terms of the covariant derivative of φ, the Sasakian condition can be expressed by

$$(\tilde{\nabla}_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.3}$$

for all $X, Y \in \Gamma(T\tilde{M})$, where $\tilde{\nabla}$ is the Levi-Civita connection of g. From the formula (2.3), it follows that

$$\tilde{\nabla}_X \xi = -\varphi X, \tag{2.4}$$

for any $X \in \Gamma(T\tilde{M})$.

Let M be a Riemannian manifold isometrically immersed in $\tilde{M}$ and denote by the same symbol g the Riemannian metric induced on M. Let $\Gamma(TM)$ be the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$ the set of all vector fields normal to M. The Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.5}$$

and

$$\tilde{\nabla}_X N = -A_N X + \nabla^\perp_X N, \tag{2.6}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where ∇ is the Levi-Civita connection on M, $\nabla^\perp$ is the normal connection in the normal bundle $T^\perp M$ and A_N is the shape operator of M with respect to the normal vector N. Moreover, $h : TM \times TM \to T^\perp M$ is the second fundamental form of M in $\tilde{M}$. Furthermore, A_N and h are related by [32]

$$g(h(X,Y),N) = g(A_N X, Y), \tag{2.7}$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$.

For any X tangent to M, we write

$$\varphi X = PX + FX, \tag{2.8}$$

where PX and FX are the tangential and normal components of φX, respectively. Then P is an endomorphism of the tangent bundle TM and F is a normal bundle valued 1-form on TM. Similarly, for any vector field N normal to M, we put

$$\varphi N = tN + fN, \tag{2.9}$$

where tN and fN are the tangential and normal components of φN, respectively. Moreover, from (2.2) and (2.8), we have

$$g(PX, Y) = -g(X, PY), \tag{2.10}$$

for any $X, Y \in \Gamma(TM)$.

Throughout this paper, we assume the structure field ξ is tangent to M; otherwise M is a C-totally real submanifold [20]. Let M be a Riemannian manifold isometrically immersed in an almost contact metric manifold $(\tilde{M}, \varphi, \xi, \eta, g)$. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *slant* [6], if for each non-zero vector X tangent to M at $p \in M$ such that X is not proportional to ξ_p, the angle $\theta(X)$ between φX and T_pM is constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_pM - \langle \xi_p \rangle$.

A slant submanifold is said to be *proper slant* if neither $\theta = 0$ nor $\theta = \frac{\pi}{2}$. We note that on a slant submanifold if $\theta = 0$, then it is an invariant submanifold and if $\theta = \frac{\pi}{2}$, then it is an anti-invariant submanifold. A slant submanifold is said to be *proper slant* if it is neither invariant nor anti-invariant.

As a natural extension of slant submanifolds, F. Etayo [16] introduced pointwise slant submanifolds of an almost Hermitian manifold under the name of quasi-slant submanifolds. Later on, B.-Y. Chen and O. J. Garay studied pointwise slant submanifolds of almost Hermitian manifolds and obtained many interesting results [14]. In [31], the authors studied pointwise slant submanifolds of almost contact metric manifolds tangent to the structure vector field ξ.

A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be *pointwise slant* if for any nonzero vector X tangent to M at $p \in M$, such that X is not propotional to ξ_p, the angle $\theta(X)$ between φX and $T_p^*M = T_pM - \{0\}$ is independent of the choice of nonzero vector $X \in T_p^*M$. In this case, θ can be regarded as a function on M, which is called the *slant function* of the pointwise slant submanifold.

We note that every slant submanifold is a pointwise slant submanifold, but the converse is not true. We also note that a pointwise slant submanifold is *invariant* (respectively, *anti-invariant*) if for each point $p \in M$, the slant function $\theta = 0$ (respectively, $\theta = \frac{\pi}{2}$). A pointwise slant submanifold is slant if and only if the slant function θ is constant on M. Moreover, a pointwise slant submanifold is proper if neither $\theta = 0, \frac{\pi}{2}$ nor θ is constant.

In [31], we have obtained the following characterization theorem.

Theorem 2.1 ([31]). *Let M be a submanifold of an almost contact metric manifold $\tilde{M}$ such that $\xi \in \Gamma(TM)$. Then, M is pointwise slant if and only if*

$$P^2 = \cos^2\theta\,(-I + \eta \otimes \xi)\,, \tag{2.11}$$

for some real valued function θ defined on the tangent bundle TM of M.

The following relations are immediate consequences of Theorem 2.1.

Let M be a pointwise slant submanifold of an almost contact metric manifold $\tilde{M}$. Then, we have

$$g(PX, PY) = \cos^2\theta\,[g(X,Y) - \eta(X)\eta(Y)], \tag{2.12}$$

$$g(FX, FY) = \sin^2\theta\,[g(X,Y) - \eta(X)\eta(Y)], \tag{2.13}$$

for any $X, Y \in \Gamma(TM)$.

The next useful relations for a pointwise slant submanifold of an almost contact metric manifold was obtained in [31]

$$tFX = \sin^2\theta\,(-X + \eta(X)\xi), \quad fFX = -FPX, \tag{2.14}$$

for any $X \in \Gamma(TM)$.

3. Pointwise Semi-Slant Submanifolds

B. Sahin [26] defined and studied pointwise semi-slant submanifolds of Kaehler manifolds. In this section, we define and study pointwise semi-slant submanifolds of Sasakian manifolds.

Definition 3.1. A submanifold M of an almost contact metric manifold $\tilde{M}$ is said to be a *pointwise semi-slant* submanifold if there exists a pair of orthogonal distributions $\mathfrak{D}$ and $\mathfrak{D}^\theta$ on M such that

(i) the tangent bundle TM admits the orthogonal direct decomposition $TM = \mathfrak{D} \oplus \mathfrak{D}^\theta \oplus \langle\xi\rangle$;
(ii) the distribution $\mathfrak{D}$ is invariant under φ, i.e., $\varphi(\mathfrak{D}) = \mathfrak{D}$;
(iii) the distribution $\mathfrak{D}^\theta$ is pointwise slant with slant function θ.

Note that the normal bundle $T^\perp M$ of a pointwise semi-slant submanifold M is decomposed as

$$T^\perp M = F\mathfrak{D}^\theta \oplus \nu, \quad F\mathfrak{D}^\theta \perp \nu,$$

where ν is an invariant normal subbundle of $T^\perp M$ under φ.

If we denote the dimensions of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by m_1 and m_2, respectively, then we have the following.

(i) If $m_1 = 0$, then M is a pointwise slant submanifold.
(ii) If $m_2 = 0$, then M is an invariant submanifold.
(iii) If $m_1 = 0$ and $\theta = \frac{\pi}{2}$, then M is an anti-invariant submanifold.
(iv) If $m_1 \neq 0$ and $\theta = \frac{\pi}{2}$, then M is a contact CR-submanifold.
(v) If θ is constant on M, then M is a semi-slant submanifold with slant angle θ.

We also note that a pointwise semi-slant submanifold is *proper* if neither $m_1, m_2 = 0$ nor $\theta = 0, \frac{\pi}{2}$ and θ should not be a constant.

Now, we provide the following non-trivial examples of pointwise semi-slant submanifolds of an almost contact metric manifold.

Example 3.1. Let $(\mathbf{R}^7, \varphi, \xi, \eta, g)$ be an almost contact metric manifold with cartesian coordinates $(x_1, y_1, x_2, y_2, x_3, y_3, z)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial z}\right) = 0, \quad 1 \le i, j \le 3,$$

where $\xi = \frac{\partial}{\partial z}$, $\eta = dz$ and g is the standard Euclidean metric on $\mathbf{R}^7$. Then (φ, ξ, η, g) is an almost contact metric structure on $\mathbf{R}^7$. Consider a submanifold M of $\mathbf{R}^7$ defined by $\psi(u, v, w, t, z) = (u + v, -u + v, t\cos w, t\sin w, w\cos t, w\sin t, z)$, such that w, t $(w \neq t)$ are non-zero real numbers. Then the tangent space TM is spanned by the following vector fields

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \\ X_3 &= -t\sin w\frac{\partial}{\partial x_2} + t\cos w\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial x_3} + \sin t\frac{\partial}{\partial y_3}, \\ X_4 &= \cos w\frac{\partial}{\partial x_2} + \sin w\frac{\partial}{\partial y_2} - w\sin t\frac{\partial}{\partial x_3} + w\cos t\frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial z}. \end{aligned}$$

Thus, we observe that $\mathfrak{D} = \mathrm{Span}\{X_1, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}((t - w)/\sqrt{(t^2+1)(w^2+1)})$. Hence, M is a pointwise semi-slant submanifold of $\mathbf{R}^7$ such that $\xi = \frac{\partial}{\partial z}$ is tangent to M.

Example 3.2. Consider a submanifold of $\mathbf{R}^7$ with almost contact structure φ given in Example 3.1. If the immersion $\psi : \mathbf{R}^5 \to \mathbf{R}^7$ is given by

$$\psi(u_1, u_2, u_3, u_4, t) = \left(u_1, \frac{u_3^2 + u_4^2}{2}, \cos u_4, -u_2, \frac{u_3^2 - u_4^2}{2}, \sin u_4, t\right), \quad u_4 \neq 0,$$

then the tangent space TM is spanned by X_1, X_2, X_3, X_4 and X_5, where

$$X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = -\frac{\partial}{\partial y_1}, \quad X_3 = u_3\frac{\partial}{\partial x_2} + u_3\frac{\partial}{\partial y_2},$$

$$X_4 = u_4\frac{\partial}{\partial x_2} - u_4\frac{\partial}{\partial y_2} - \sin u_4\frac{\partial}{\partial x_3} + \cos u_4\frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial t}.$$

Therefore, M is a pointwise semi-slant submanifold such that $\mathfrak{D} = \mathrm{Span}\{X_1,\, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3,\, X_4\}$ is a pointwise slant distribution with pointwise slant function $\theta = \cos^{-1}\left(\sqrt{2}\,u_4/\sqrt{1+2u_4^2}\right)$.

Now, we obtain the following useful results for semi-slant submanifolds of a Sasakian manifold.

Lemma 3.1. *Let M be a pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. Then, we have*

(i) $\sin^2\theta\, g(\nabla_X Y, Z) = g(h(X, \varphi Y), FZ) - g(h(X, Y), FPZ),$

(ii) $\sin^2\theta\, g(\nabla_Z W, X) = g(h(X, Z), FPW) - g(h(\varphi X, Z), FW),$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z, W \in \Gamma(\mathfrak{D}^\theta)$.

Proof. The first and second parts of the lemma can be proved in a similar way. For any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $Z \in \Gamma(\mathfrak{D}^\theta)$, we have

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X Y, Z) = g(\varphi\tilde{\nabla}_X Y, \varphi Z).$$

From the covariant derivative formula of φ, we derive

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi Y, \varphi Z) - g((\tilde{\nabla}_X \varphi)Y, \varphi Z).$$

Then, from (2.3), (2.8) and the orthogonality of the two distributions, we find

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\tilde{\nabla}_X \varphi Y, PZ) + g(\tilde{\nabla}_X \varphi Y, FZ) \\ &= -g(\tilde{\nabla}_X PZ, \varphi Y) + g(h(X, \varphi Y), FZ) \\ &= g(\varphi\tilde{\nabla}_X PZ, Y) + g(h(X, \varphi Y), FZ). \end{aligned}$$

Again, from the covariant derivative formula of φ, we get

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X \varphi PZ, Y) - g((\tilde{\nabla}_X \varphi)PZ, Y) + g(h(X, \varphi Y), FZ).$$

Using (2.3), (2.8) and the orthogonality of vector fields, we obtain

$$g(\nabla_X Y, Z) = g(\tilde{\nabla}_X P^2 Z, Y) + g(\tilde{\nabla}_X FPZ, Y) + g(h(X, \varphi Y), FZ).$$

Then, from (2.11) and (2.6), we have

$$\begin{aligned} g(\nabla_X Y, Z) = &-\cos^2\theta\, g(\tilde{\nabla}_X Z, Y) + \sin 2\theta\, X(\theta)\, g(Y, Z) - g(h(X, Y), FPZ) \\ &+ g(h(X, \varphi Y), FZ). \end{aligned}$$

From the orthogonality of the two distributions the above equation takes the form

$$g(\nabla_X Y, Z) = \cos^2\theta\, g(\tilde{\nabla}_X Y, Z) - g(h(X, Y), FPZ) + g(h(X, \varphi Y), FZ).$$

Hence, (i) follows from the above relation. In a similar way we can prove (ii). □

4. Warped Product Pointwise Semi-Slant Submanifolds

In this section, we study warped product submanifolds of Sasakian manifolds, by considering that one factor is a pointwise slant submanifold. In the following, first we give a brief introduction on warped product manifolds.

In [3], R. L. Bishop and B. O'Neill introduced the notion of warped product manifolds as follows: Let M_1 and M_2 be two Riemannian manifolds with Riemannian metrics g_1 and g_2, respectively, and a positive differentiable function f on M_1. Consider the product manifold $M_1 \times M_2$ with its projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$. Then their warped product manifold $M = M_1 \times_f M_2$ is the Riemannian manifold $M_1 \times M_2 = (M_1 \times M_2, g)$ equipped with the Riemannian metric

$$g(X, Y) = g_1(\pi_{1\star} X, \pi_{1\star} Y) + (f \circ \pi_1)^2 g_2(\pi_{2\star} X, \pi_{2\star} Y),$$

for any vector field X, Y tangent to M, where $\star$ is the symbol for the tangent maps. A warped product manifold $M = M_1 \times_f M_2$ is said to be *trivial* or simply a *Riemannian product manifold* if the warping function f is constant.

Let X be a vector field tangent to M_1 and Z be an another vector field on M_2; then from Lemma 7.3 of [3], we have

$$\nabla_X Z = \nabla_Z X = X(\ln f) Z, \tag{4.1}$$

where ∇ is the Levi-Civita connection on M. If $M = M_1 \times_f M_2$ is a warped product manifold then the base manifold M_1 is totally geodesic in M and the fiber M_2 is totally umbilical in M [3, 9].

By analogy to CR-warped products which are introduced by B.-Y. Chen in [9], one defines the warped product pointwise semi-slant submanifolds as follows.

Definition 4.1. A warped product of an invariant and a pointwise slant submanifolds, say M_T and M_θ of a Sasakian manifold $\tilde{M}$ is called a *warped product pointwise semi-slant submanifold*.

A warped product pointwise semi-slant submanifold is called *proper* if M_θ is a proper pointwise slant submanifold and M_T is an invariant submanifold of $\tilde{M}$.

The non-existence of warped product pointwise semi-slant submanifolds of the form $M_\theta \times_f M_T$ in Kaehler manifolds is proved in [26]. A similar result holds in Sasakian manifolds. On the other hand, there exist non-trivial warped product pointwise semi-slant submanifolds of the form $M_T \times M_\theta$ of Kaehler manifolds [26] and contact metric manifolds.

Note that a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product contact CR-submanifold if the slant function $\theta = \frac{\pi}{2}$. Similarly, the warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$ is a warped product semi-slant submanifold if θ is constant on M, i.e., M_θ is a proper slant submanifold.

In this section, we study the warped product pointwise semi-slant submanifold of the form $M = M_T \times_f M_\theta$ of a Sasakian manifold $\tilde{M}$. To fill the gap in the earlier study, we obtain some results as a generalization.

On a warped product pointwise semi-slant submanifold $M = M_T \times_f M_\theta$, if we consider the structure vector field ξ tangent to M, then either $\xi \in \Gamma(TM_T)$ or $\xi \in \Gamma(TM_\theta)$. When ξ is tangent to M_θ, then it is easy to check that warped product is trivial (see [27]); therefore we always consider $\xi \in \Gamma(TM_T)$.

First, we prove the following useful results.

Lemma 4.1. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$. Then, we have*

$$g(h(X,W),FPZ) - g(h(X,PZ),FW) = \sin 2\theta\, X(\theta)\, g(Z,W), \tag{4.2}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. For any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, we have

$$g(\tilde{\nabla}_X Z, W) = X(\ln f)\, g(Z,W). \tag{4.3}$$

On the other hand, we can obtain $g(\tilde{\nabla}_X Z, W) = g(\varphi\tilde{\nabla}_X Z, \varphi W)$. Using the covariant derivative formula of φ, we get

$$g(\tilde{\nabla}_X Z, W) = g(\tilde{\nabla}_X \varphi Z, \varphi W) - g((\tilde{\nabla}_X \varphi)Z, \varphi W).$$

The second term in the right hand side of above relation is identically zero by using (2.3) and the orthogonality of vector fields. Then, from (2.5), (2.8), (4.1) and the orthogonality of vector fields, we find

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& g(\tilde{\nabla}_X PZ, PW) + g(\tilde{\nabla}_X PZ, FW) + g(\tilde{\nabla}_X FZ, \varphi W)\\ =& X(\ln f)\, g(PZ,PW) + g(h(X,PZ),FW) - g(\varphi\tilde{\nabla}_X FZ, W)\\ =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) - g(\tilde{\nabla}_X \varphi FZ, W)\\ &+ g((\tilde{\nabla}_X \varphi)FZ, W). \end{aligned}$$

Again, the last term in the above equation is zero by using (2.3) and the orthogonality of vector fields. Then, from (2.9) and (2.14), we derive

$$\begin{aligned} g(\tilde{\nabla}_X Z, W) =& \cos^2\theta\, X(\ln f)\, g(Z,W) + g(h(X,PZ),FW) + \sin^2\theta\, g(\tilde{\nabla}_X Z, W)\\ &+ \sin 2\theta\, X(\theta)\, g(Z,W) + g(\tilde{\nabla}_Z FPX, Y). \end{aligned} \tag{4.4}$$

Hence, the result follows from (4.3) and (4.4) by using (2.6)–(2.7) and (4.1). ☐

Lemma 4.2. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

(i) *$g(PZ,W) = -\xi(\ln f)\, g(Z,W)$;*
(ii) *$g(h(X,Y),FZ) = 0$;*

(iii) $g(h(X,Z),FW) = X(\ln f)\,g(PZ,W) - \varphi X(\ln f)\,g(Z,W) - \eta(X)\,g(Z,W)$,

for any $X,Y \in \Gamma(TM_T)$ *and* $Z,W \in \Gamma(TM_\theta)$.

Proof. From (2.4), (2.5) and (2.8), we have $\nabla_Z \xi = -PZ$, for any $Z \in \Gamma(TM_\theta)$. Using (4.1) and taking the inner product with $W \in \Gamma(TM_\theta)$, we get (i). For the other parts of the lemma, considering any $X,Y \in \Gamma(TM_T)$ and $Z \in \Gamma(TM_\theta)$, we have

$$g(h(X,Y),FZ) = g(\tilde{\nabla}_X Y, FZ) = g(\tilde{\nabla}_X Y, \varphi Z) - g(\tilde{\nabla}_X Y, PZ).$$

From (2.2) and (4.1), we get

$$g(h(X,Y),FZ) = -g(\varphi\tilde{\nabla}_X Y, Z) + X(\ln f)\,g(Y,PZ).$$

By covariant derivative formula of φ and the orthogonality of vector fields, we find

$$g(h(X,Y),FZ) = g((\tilde{\nabla}_X \varphi)Y, Z) - g(\tilde{\nabla}_X \varphi Y, Z).$$

Using (2.3) and the fact that $\xi \in \Gamma(TM_T)$, the first term in the right hand side of above equation vanishes identically and then by using (4.1) and the orthogonality of vector fields, we find (ii). Now, for any $X \in \Gamma(TM_T)$ and $Z,W \in \Gamma(TM_\theta)$, we have

$$g(h(X,Z),FW) = g(\tilde{\nabla}_Z X, FW) = g(\tilde{\nabla}_X Z, \varphi W) - g(\tilde{\nabla}_X Z, PW).$$

Again, using the covariant derivative formula of the Riemannain connection and (4.1), we get

$$g(h(X,Z),FW) = g((\tilde{\nabla}_Z \varphi)X, W) - g(\tilde{\nabla}_Z \varphi X, W) - X(\ln f)\,g(Z,PW).$$

Then from (2.3), (2.5) and (4.1), we derive

$$g(h(X,Z),FW) = -\eta(X)\,g(Z,W) - \varphi X(\ln f)\,g(Z,W) - X(\ln f)\,g(Z,PW),$$

which is the third part of the lemma. Hence, the proof is complete. ☐

Lemma 4.3. *Let* $M = M_T \times_f M_\theta$ *be a warped product pointwise semi-slant submanifold of a Sasakian manifold* $\tilde{M}$ *such that* $\xi \in \Gamma(TM_T)$*, where* M_T *is an invariant submanifold and* M_θ *is a pointwise slant submanifold of* $\tilde{M}$*. Then*

$$g(h(\varphi X,Z),FW) = X(\ln f)\,g(Z,W) - \eta(X)\,g(Z,PW) - \varphi X(\ln f)\,g(Z,PW), \tag{4.5}$$

for any $X \in \Gamma(TM_T)$ *and* $Z,W \in \Gamma(TM_\theta)$.

Proof. Interchanging X by φX, for any $X \in \Gamma(TM_T)$ in Lemma 4.2 (iii) and using the first part of Lemma 4.2, we get the required result. ☐

Lemma 4.4. *Let* $M = M_T \times_f M_\theta$ *be a warped product pointwise semi-slant submanifold of a Sasakian manifold* $\tilde{M}$ *such that* $\xi \in \Gamma(TM_T)$*, where* M_T *and* M_θ *are invariant and pointwise slant submanifolds of* $\tilde{M}$*, respectively. Then, we have*

(4.6)

$$g(h(X,PZ),FW) = \varphi X(\ln f)\,g(Z,PW) - \eta(X)\,g(PZ,W) - \cos^2\theta\,X(\ln f)\,g(Z,W),$$

for any $X \in \Gamma(TM_T)$ *and* $Z,W \in \Gamma(TM_\theta)$.

Proof. Interchange Z by PZ, for any $Z \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii) and after using (2.12), we get (4.6). $\square$

Similarly, if we interchange W by PW, for any $W \in \Gamma(TM_\theta)$ in Lemma 4.2 (iii), then we can obtain the following result.

Lemma 4.5. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then*

(4.7)
$$g(h(X,Z),FPW) = \cos^2\theta\, X(\ln f)\, g(Z,W) - \varphi X(\ln f)\, g(Z,PW) - \eta(X)\, g(Z,PW),$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Lemma 4.6. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and proper pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

$$g(A_{FW}\varphi X, Z) - g(A_{FPW}X, Z) = \sin^2\theta\, X(\ln f)\, g(Z,W), \tag{4.8}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Subtracting (4.7) from (4.5), we get (4.8). $\square$

A warped product submanifold $M = M_1 \times_f M_2$ of a Sasakian manifold $\tilde{M}$ is said to be *mixed totally geodesic* if $h(X,Z) = 0$, for any $X \in \Gamma(TM_1)$ and $Z \in \Gamma(TM_2)$, where M_1 and M_2 are any Riemannian submanifolds of $\tilde{M}$.

From Lemma 4.6, we obtain the following result.

Theorem 4.1. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is warped product of invariant submanifolds or the warping function f is constant on M.*

Proof. From (4.8) and the mixed totally geodesic condition, we have

$$\sin^2\theta\, X(\ln f)\, g(Z,W) = 0.$$

Since g is a Riemannian metric, then either $\sin^2\theta = 0$ or $X(\ln f) = 0$. Therefore, either M is warped product of invariant submanifolds or f is constant on M, thus, the proof is complete. $\square$

Lemma 4.7. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi \in \Gamma(TM_T)$, where M_T and M_θ are invariant and pointwise slant submanifolds of $\tilde{M}$, respectively. Then, we have*

$$g(A_{FPZ}W, X) - g(A_{FW}PZ, X) = 2\cos^2\theta\, X(\ln f)\, g(Z,W), \tag{4.9}$$

for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$.

Proof. Interchanging Z and W in (4.7) and using (2.10), we get

$$g(h(X,W),FPZ)=\cos^2\theta\, X(\ln f)\, g(Z,W)+\varphi X(\ln f)\, g(Z,PW)+\eta(X)\, g(Z,PW), \tag{4.10}$$

for any $X\in\Gamma(TM_T)$ and $Z,W\in\Gamma(TM_\theta)$. Subtracting (4.6) from (4.10), we find (4.9). □

Also, with the help of Lemma 4.7, we find the following result.

Theorem 4.2. *Let $M=M_T\times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. If M is mixed totally geodesic, then either M is a contact CR-warped product of the form $M_T\times_f M_\perp$ or the warping function f is constant on M.*

Proof. From (4.9) and the mixed totally geodesic condition, we have

$$\cos^2\theta\, X(\ln f)\, g(Z,W)=0.$$

Since g is a Riemannian metric, then either $\cos^2\theta=0$ or $X(\ln f)=0$. Therefore, either M is a contact CR-warped product or f is constant on M, which ends the proof. □

From Theorem 4.1 and Theorem 4.2, we conclude the following result.

Corollary 4.1. *There does not exist any mixed totally geodesic proper warped product pointwise semi-slant submanifold $M=M_T\times_f M_\theta$ of a Sasakian manifold.*

Also, from Lemma 4.1 and Lemma 4.7, we have the following result.

Theorem 4.3. *Let $M=M_T\times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$ such that $\xi\in\Gamma(TM_T)$, where M_T is an invariant submanifold and M_θ is a pointwise slant submanifold of $\tilde{M}$. Then, either M is a contact CR-warped product of the form $M=M_T\times_f M_\perp$ or $\nabla(\ln f)=\tan\theta\,\nabla\theta$, for any $X\in\Gamma(TM_T)$, where ∇f is the gradient of f.*

Proof. From Lemma 4.1 and Lemma 4.7, we have

$$\cos^2\theta\{X(\ln f)-\tan\theta\, X(\theta)\}\, g(Z,W)=0.$$

Since g is a Riemannian metric, therefore, we conclude that either $\cos^2\theta=0$ or $X(\ln f)-\tan\theta\, X(\theta)=0$. Consequently, either $\theta=\frac{\pi}{2}$ or $X(\ln f)=\tan\theta\, X(\theta)$, which proves the theorem completely. □

As an application of Theorem 4.3, we have the following consequence.

Remark 4.1. If we consider that the slant function θ is constant, i.e., M_θ is a proper slant submanifold in Theorem 4.3, then $Z(\ln f)=0$, i.e., there are no warped product semi-slant submanifolds of the form $M_T\times_f M_\theta$ in Sasakian manifolds. Hence, Theorem 3.3 of [1] is a special case of Theorem 4.3.

In order to give a characterization result for pointwise semi-slant submanifolds of a Sasakian manifold, we need the following well-known result of Hiepko [18].

Theorem 4.4 (Hiepko's Theorem)**.** *Let $\mathfrak{D}_1$ and $\mathfrak{D}_2$ be two orthogonal distribution on a Riemannian manifold M. Suppose that both $\mathfrak{D}_1$ and $\mathfrak{D}_2$ are involutive such that $\mathfrak{D}_1$ is a totally geodesic foliation and $\mathfrak{D}_2$ is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where M_1 and M_2 are integral manifolds of $\mathfrak{D}_1$ and $\mathfrak{D}_2$, respectively.*

Theorem 4.5. *Let M be a pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. Then M is locally a non-trivial warped product submanifold of the form $M_T \times_f M_\theta$, where M_T is an invariant submanifold and M_θ is a proper pointwise slant submanifold of $\tilde{M}$ if and only if*

$$A_{FW}\varphi X - A_{FPW}X = \sin^2\theta\, X(\mu)W, \quad \textit{for all } X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle), W \in \Gamma(\mathfrak{D}^\theta), \tag{4.11}$$

for some smooth function μ on M satisfying $Z(\mu) = 0$ for any $Z \in \Gamma(\mathfrak{D}^\theta)$.

Proof. Let $M = M_T \times_f M_\theta$ be a warped product pointwise semi-slant submanifold of a Sasakian manifold $\tilde{M}$. Then for any $X \in \Gamma(TM_T)$ and $Z, W \in \Gamma(TM_\theta)$, from Lemma 4.2 (ii) we have

$$g(A_{FW}X, Y) = 0. \tag{4.12}$$

Interchanging X by φX in (4.12), we get $g(A_{FW}\varphi X, Y) = 0$, which means that $A_{FW}\varphi X$ has no component in TM_T. Similarly, if we interchange W by PW in (4.12) then, we get $g(A_{FPW}X, Y) = 0$, i.e., $A_{FPW}X$ also has no component in TM_T. Therefore, $A_{FW}\varphi X - A_{FPW}X$ lies in TM_θ, using this fact with Lemma 4.6, we find (4.11).

Conversely, if M is a pointwise semi-slant submanifold such that (4.11) holds, then from Lemma 3.1 (i), we have

$$g(\nabla_X Y, W) = \csc^2\theta\, g(A_{FW}\varphi Y - A_{FPW}Y, X),$$

for any $X, Y \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$ and $W \in \Gamma(\mathfrak{D}^\theta)$. From (4.11), we arrive at

$$g(\nabla_X Y, W) = Y(\mu)g(X, W) = 0,$$

which means that the leaves of the distribution $\mathfrak{D} \oplus \langle\xi\rangle$ are totally geodesic in M. Also, from Lemma 3.1 (ii), we have

$$g(\nabla_Z W, X) = \csc^2\theta\, g(A_{FPW}X - A_{FW}\varphi X, Z), \tag{4.13}$$

for any $Z, W \in \Gamma(\mathfrak{D}^\theta)$ and $X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle)$. By polarization, we derive

$$g(\nabla_W Z, X) = \csc^2\theta\, g(A_{FPZ}X - A_{FZ}\varphi X, W). \tag{4.14}$$

Substracting (4.14) from (4.13), we get

$$\sin^2\theta\, g([Z, W], X) = g(A_{FZ}\varphi X - A_{FPZ}X, W) - g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.11), we get

$$\sin^2\theta\, g([Z,W],X) = X(\mu)\, g(Z,W) - X(\mu)\, g(W,Z) = 0.$$

Since M is proper pointwise semi-slant, then $\sin^2\theta \neq 0$, thus we conclude that the pointwise slant distribution $\mathfrak{D}^\theta$ is integrable. Let us consider M_θ to be a leaf of $\mathfrak{D}^\theta$ and h^θ is the second fundamental form of M_θ in M. Then from (4.14), we have

$$g(h^\theta(Z,W),X) = g(\nabla_Z W, X) = -\csc^2\theta\, g(A_{FW}\varphi X - A_{FPW}X, Z).$$

Using (4.11), we find that

$$g(h^\theta(Z,W),X) = -X(\mu)\, g(Z,W).$$

Then from the definition of the gradient of a function, we arrive at

$$h^\theta(Z,W) = -(\vec{\nabla}\mu)\, g(Z,W).$$

Hence, M_θ is a totally umbilical submanifold of M with the mean curvature vector $H^\theta = -\vec{\nabla}\mu$, where $\vec{\nabla}\mu$ is the gradient of the function μ. Since $Z(\mu) = 0$, for any $Z \in \Gamma(\mathfrak{D}^\theta)$, then we can show that $H^\theta = -\vec{\nabla}\mu$ is parallel with respect to the normal connection, say D^n of M_θ in M (see [25, 26], [28]). Thus, M_θ is a totally umbilical submanifold of M with a non vanishing parallel mean curvature vector $H^\theta = -\vec{\nabla}\mu$, i.e., M_θ is an extrinsic sphere in M. Then from Heipko's Theorem [18], we conclude that M is a warped product manifold of M_T and M_θ, where M_T and M_θ are integral manifolds of $\mathfrak{D} \oplus \langle\xi\rangle$ and $\mathfrak{D}^\theta$, respectively. Thus, the proof is complete. □

As an application of Theorem 4.5, if we consider $\theta = \frac{\pi}{2}$ in Theorem 4.5, then by interchanging X by φX in (4.11), we get the condition (74) of Theorem 3.2 in [21]; thus the Theorem 4.5 is true for contact CR-warped product submanifolds of the form $M_T \times_f M_\perp$. Hence, Theorem 3.2 of [21] is a special case of Theorem 4.5 as follows.

Corollary 4.2 (Theorem 3.2 of [21])**.** *A strictly proper CR-submanifold M of a Sasakian manifold $\tilde{M}$ tangent to the structure vector field ξ is locally a contact CR-warped product if and only if*

$$A_{\varphi Z}X = (\eta(X) - \varphi X(\mu))\, Z, \quad X \in \Gamma(\mathfrak{D} \oplus \langle\xi\rangle), Z \in \Gamma(\mathfrak{D}^\perp), \tag{4.15}$$

for some function μ on M satisfying $W\mu = 0$, for all $W \in \Gamma(\mathfrak{D}^\perp)$.

5. Examples

In this section, we provide the following non-trivial examples of Riemannian products and warped product pointwise semi-slant submanifolds in Euclidean spaces.

Example 5.1. Let M be a submanifold of Euclidean 7-space $\mathbb{R}^7$ with its cartesian coordinates $(x_1, \ldots, x_3, y_1, \ldots, y_3, t)$ and the almost contact structure

$$\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \leq i,j \leq 3.$$

If M is given by the equations

$$
\begin{aligned}
x_1 &= u_1, \quad x_2 = u_3 \cos u_4, \quad x_3 = \frac{u_3^2}{2}, \quad y_1 = u_2, \quad y_2 = u_3 \sin u_4, \\
y_3 &= u_4, \quad t = t,
\end{aligned}
$$

for any non-zero function u_3 on M, then tangent space TM of M is spanned by X_1, X_2, X_3, X_4 and X_5, where

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial x_1}, \quad X_2 = \frac{\partial}{\partial y_1}, \quad X_3 = \cos u_4 \frac{\partial}{\partial x_2} + u_3 \frac{\partial}{\partial x_3} + \sin u_4 \frac{\partial}{\partial y_2}, \\
&X_4 = -u_3 \sin u_4 \frac{\partial}{\partial x_2} + u_3 \sin u_4 \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}, \quad X_5 = \frac{\partial}{\partial t}.
\end{aligned}
$$

Then, M is a pointwise semi-slant submanifold with invariant distribution $\mathfrak{D} = \mathrm{Span}\{\mathrm{X}_1, \mathrm{X}_2\}$ and the pointwise slant distribution $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$. Clearly, the slant function is $\theta = \cos^{-1}(2u_3/\sqrt{1+u_3^2})$. Moreover, $\mathfrak{D}$ and $\mathfrak{D}^\theta$ are integrable. If M_T and M_θ are integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$, respectively, then, $M = M_T \times M_\theta$ is a Riemannian product of M_T and M_θ in $\mathbb{R}^9$.

Example 5.2. Consider the Euclidean 9-space $\mathbb{R}^9$ with its Cartesian coordinates $(x_1, \ldots, x_4, y_1, \ldots, y_4, t)$ and the almost contact structure

$$
\varphi\left(\frac{\partial}{\partial x_i}\right) = -\frac{\partial}{\partial y_i}, \quad \varphi\left(\frac{\partial}{\partial y_j}\right) = \frac{\partial}{\partial x_j}, \quad \varphi\left(\frac{\partial}{\partial t}\right) = 0, \quad 1 \le i, j \le 4.
$$

Let M be a submanifold of $\mathbb{R}^9$ defined by the immersion ψ as follows:

$$
\psi(u, v, w, s, t) = \left(u + v, \frac{1}{2}w^2, s\cos w, s\sin w, -u + v, \frac{1}{2}s^2, -w\sin s, w\cos s, t\right),
$$

for any non-zero real numbers w and s. The tangent space of M is spanned by the following vectors

$$
\begin{aligned}
&X_1 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial y_1}, \quad X_2 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1}, \\
&X_3 = w\frac{\partial}{\partial x_2} - s\sin w\,\frac{\partial}{\partial x_3} + s\cos w\,\frac{\partial}{\partial x_4} - \sin s\,\frac{\partial}{\partial y_3} + \cos v\,\frac{\partial}{\partial y_4}, \\
&X_4 = \cos w\,\frac{\partial}{\partial x_3} + \sin w\,\frac{\partial}{\partial x_4} + s\frac{\partial}{\partial y_2} - w\cos s\,\frac{\partial}{\partial y_3} - w\sin s\,\frac{\partial}{\partial y_4}, \quad X_5 = \frac{\partial}{\partial t}.
\end{aligned}
$$

Then, M is a pointwise semi-slant submanifold such that the structure vector field $\xi = \frac{\partial}{\partial t}$ is tangent to M and $\mathfrak{D} = \mathrm{Span}\{X_1, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{(1-ws)\sin(w-s)-ws}{1+w^2+s^2}\right)$. It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then M is a Riemannian product of invariant and pointwise slant submanifolds in $\mathbb{R}^9$.

Example 5.3. Let M be a submanifold of $\mathbb{R}^{13}$ given by the immersion $\psi : \mathbb{R}^5 \to \mathbb{R}^{13}$ as follows:

$$\begin{aligned}\psi(u_1, v_1, u_2, v_2, t) =&(u_1 - v_1,\ u_1\cos(u_2+v_2),\ u_1\sin(u_2+v_2),\ v_2,\ u_1\cos(u_2-v_2),\\ &u_1\sin(u_2-v_2),\ u_1+v_1,\ v_1\cos(u_2+v_2),\ v_1\sin(u_2+v_2),\ u_2,\\ &v_1\cos(u_2-v_2),\ v_1\sin(u_2-v_2),\ t),\end{aligned}$$

for non-zero functions u_1 and v_1. We use the almost contact structure from Example 5.2. Then, we have

$$\begin{aligned}X_1 =&\frac{\partial}{\partial x_1} + \cos(u_2+v_2)\frac{\partial}{\partial x_2} + \sin(u_2+v_2)\frac{\partial}{\partial x_3} + \cos(u_2-v_2)\frac{\partial}{\partial x_5}\\ &+ \sin(u_2-v_2)\frac{\partial}{\partial x_6} + \frac{\partial}{\partial y_1},\\ X_2 =& -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial y_1} + \cos(u_2+v_2)\frac{\partial}{\partial y_2} + \sin(u_2+v_2)\frac{\partial}{\partial y_3} + \cos(u_2-v_2)\frac{\partial}{\partial y_5}\\ &+ \sin(u_2-v_2)\frac{\partial}{\partial y_6},\\ X_3 =& -u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2} + u_1\cos(u_2+v_2)\frac{\partial}{\partial x_3} - u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\ &+ u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6} - v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2},\ + v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}\\ &+ \frac{\partial}{\partial y_4} - v_1\sin(u_2-v_2)\frac{\partial}{\partial y_5} + v_1\cos(u_2-v_2)\,\frac{\partial}{\partial y_6},\\ X_4 =& -u_1\sin(u_2+v_2)\frac{\partial}{\partial x_2} + u_1\cos(u_2+v_2)\,\frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + u_1\sin(u_2-v_2)\frac{\partial}{\partial x_5}\\ &- u_1\cos(u_2-v_2)\,\frac{\partial}{\partial x_6} - v_1\sin(u_2+v_2)\,\frac{\partial}{\partial y_2},\ + v_1\cos(u_2+v_2)\,\frac{\partial}{\partial y_3}\\ &+ v_1\sin(u_2-v_2)\frac{\partial}{\partial y_5} - v_1\cos(u_2-v_2)\,\frac{\partial}{\partial y_6},\\ X_5 =&\frac{\partial}{\partial t}.\end{aligned}$$

By easy and direct computations we find that $\mathfrak{D} = \mathrm{Span}\{X_1, X_2\}$ is an invariant distribution and $\mathfrak{D}^\theta = \mathrm{Span}\{X_3, X_4\}$ is a pointwise slant distribution with slant function $\theta = \cos^{-1}\left(\frac{1}{1+2u_1^2+2v_1^2}\right)$. Hence, M is a pointwise semi-slant submanifold of $\mathbb{R}^{13}$. It is easy to observe that both the distributions are integrable. If we denote the integral manifolds of $\mathfrak{D}$ and $\mathfrak{D}^\theta$ by M_T and M_θ, respectively, then the product metric structure of M is given by

$$g = 4(du_1^2 + dv_1^2) + (1 + 2u_1^2 + 2v_1^2)(du_2^2 + dv_2^2) = g_{M_T} + f^2 g_{M_\theta}.$$

Hence, $M = M_T \times_f M_\theta$ is a warped product submanifold in $\mathbb{R}^{13}$ with warping function $f = \sqrt{1 + 2u_1^2 + 2v_1^2}$.

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CONVERGENCE ESTIMATES FOR GUPTA-SRIVASTAVA OPERATORS

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Dedicated to Prof. Vijay Gupta

Abstract. The Grüss-Voronovskaya-type approximation results for the modified Gupta-Srivastava operators are considered. Moreover, the magnitude of differences of two linear positive operators defined on an unbounded interval has been estimated. Quantitative type results are established as we initially obtain the moments of generalized discrete operators and then estimate the difference of these operators with the Gupta-Srivastava operators.

1. Introduction

For $f \in C[0,\infty)$, $n \in \mathbb{N}$, $c \in \mathbb{N} \cup \{0\} \cup \{-1\}$ and l an integer, the generalized form of the discrete operators are given by (cf. [5, 15]):

$$M_{n,l,c}(f,x) = \sum_{k=0}^{\infty} p_{n+lc,k}(x,c) f\left(\frac{k}{n}\right), \tag{1.1}$$

where $p_{n+lc,k}(x,c) = \frac{\left(\frac{n}{c}+l\right)_k}{k!} \cdot \frac{(cx)^k}{(1+cx)^{\frac{n}{c}+l+k}}$, the rising factorial given by

$$(\gamma)_k = \gamma(\gamma+1)(\gamma+2)\cdots(\gamma+k-1), \quad (\gamma)_0 = 1.$$

These operators (1.1) reproduce only the constant function unlike other exponential functions. In case $l = 0$, we immediately get Szász-Mirakyan operators for $l = 0$, $c = 0$; classical Baskakov operators for $l = 0$, $c = 1$, and Bernstein polynomials for $l = 0$, $c = -1$. In these special cases, these operators reproduce linear function too.

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A family of linear positive operators for locally integrable functions was defined in the year 2003 [18]. Durrmeyer variants of many hybrid operators have been extensively studied in literature since then (cf. [6, 9, 14, 17]). Varied approximation properties of these operators have been studied and investigated (cf. [1, 2, 4, 8, 12, 13, 16, 19, 20], etc.). For c, an integer and $x \in [0,\infty)$, V. Gupta and H. M. Srivastava [10] introduced a modification of these family of operators as:

$$\begin{aligned} R_{n,l,c}(f,x) =& [n+(l+1)c] \sum_{k=1}^{\infty} p_{n+lc,k}(x,c) \int_0^{\infty} p_{n+(l+2)c,k-1}(t,c) f(t)dt \\ & + p_{n+lc,0}(x,c) f(0), \end{aligned} \tag{1.2}$$

where $p_{n+lc,k}(x,c)$ is as defined previously above. For $c = 0$, we get the Phillips operators preserving linear functions and for $c = 1$, we immediately obtain the Baskakov-Durrmeyer type operators. For $l = 0$, the operators (1.2) reduce to the operators defined in [8, Example 2]. Very recently, Gupta [7] established a general estimate for the difference of linear positive operators as follows.

Theorem A.([7]). Let $f^{(s)} \in C_B[0,\infty)$, $s \in \{0,1,2\}$ (the class of bounded continuous functions defined on the interval $[0,\infty)$) and $x \in [0,\infty)$, then for $n \in \mathbb{N}$, we have

$$|(G_n - V_n)(f,x)| \quad \le \quad \|f''\|\alpha(x) + \omega(f'',\delta_1)(1+\alpha(x)) + 2\omega(f,\delta_2(x)),$$

where $\|\cdot\| = \sup\limits_{x\in[0,\infty)} |f(x)| < \infty$, $\alpha(x) = \frac{1}{2}\sum_{k=0}^{\infty} p_{n,k}(x,c)(\mu_2^{G_{n,k}} + \mu_2^{H_{n,k}})$ and

$$\delta_1^2 = \frac{1}{2}\sum_{k=0}^{\infty} p_{n,k}(x,c)(\mu_4^{G_{n,k}} + \mu_4^{H_{n,k}}), \quad \delta_2^2 = \sum_{k=0}^{\infty} p_{n,k}(x,c)(b^{G_{n,k}} - b^{H_{n,k}})^2.$$

We consider a family of functions $G_{n,k} : D \to \mathbb{R}$, ($k$ being a non-negative integer), which are positive linear functionals defined on a subspace D of $C[0,\infty)$, which contains polynomials upto degree 6 and $C_2[0,\infty)$, such that, $G_{n,k}(e_0) = 1$, $b^{G_{n,k}} := G_{n,k}(e_1)$, $\mu_r^{G_{n,k}} := G_{n,k}(e_1 - b^{G_{n,k}} e_0)^r$, $r \in \mathbb{N}$. Also, let $H_{n,k}$ be a similar family of functions.

We extend the studies of [7] as we study a quantitative Voronovskaya type theorem in terms of weighted modulus of continuity and estimate the difference of the two operators having the same basis function, viz. the generalized Baskakov operators and the genuine Gupta-Srivastava operators.

2. Moments

In this section, we give the moments of generalized operators (1.1) with the help of a recurrence formula.

Lemma 2.1. *For $m \in \mathbb{N}$, the operators* (1.1) *satisfy the following recurrence relation:*

$$M_{n,l,c}(e_{m+1},x) = \frac{x(1+cx)}{n} M'_{n,l,c}(e_m,x) + \left(1+\frac{lc}{n}\right) x M_{n,l,c}(e_m,x),$$

where $e_m(y) = y^m$.

Proof. On taking the derivative of the operators $M_{n,l,c}$, we get

$$M'_{n,l,c}(f,x) = \sum_{k=0}^{\infty} p'_{n+lc,k}(x,c) f\left(\frac{k}{n}\right)^m,$$

which implies that

$$x(1+cx)M'_{n,l,c}(f,x) = \sum_{k=0}^{\infty} x(1+cx)p'_{n+lc,k}(x,c) f\left(\frac{k}{n}\right)^m.$$

Now, using the identity $x(1+cx)p'_{n+lc,k}(x,c) = [k-(n+lc)x]p_{n+lc,k}(x,c)$, we obtain

$$\begin{aligned} x(1+cx)M'_{n,l,c}(f,x) =& \sum_{k=0}^{\infty}[k-(n+lc)x]p_{n+lc,k}(x,c) f\left(\frac{k}{n}\right)^m \\ =& n\sum_{k=0}^{\infty} p_{n+lc,k}(x,c)\left(\frac{k}{n}\right)^{m+1} - (n+lc)xp_{n+lc,k}(x,c)\left(\frac{k}{n}\right)^m \\ =& nM_{n,l,c}(e_{m+1},x) - (n-lc)xM_{n,l,c}(e_m,x), \end{aligned}$$

which derives the recurrence relation. □

Remark 2.1. Using Lemma 2.1, first few moments of the operators (1.1) are given by

$$\begin{aligned} M_{n,l,c}(e_0,x) =& 1, \\ M_{n,l,c}(e_1,x) =& x\left(1+\frac{lc}{n}\right), \\ M_{n,l,c}(e_2,x) =& x^2\left(1+\frac{c^2l}{n^2}+\frac{c^2l^2}{n^2}+\frac{c}{n}+\frac{2cl}{n}\right)+x\left(\frac{cl}{n^2}+\frac{1}{n}\right), \\ M_{n,l,c}(e_3,x) =& x^3\left(1+\frac{2c^3l}{n^3}+\frac{3c^3l^2}{n^3}+\frac{c^3l^3}{n^3}+\frac{2c^2}{n^2}+\frac{6c^2l}{n^2}+\frac{3c^2l^2}{n^2}+\frac{3c}{n}+\frac{3cl}{n}\right) \\ &+x^2\left(\frac{3c^2l}{n^3}+\frac{3c^2l^2}{n^3}+\frac{3c}{n^2}+\frac{6cl}{n^2}+\frac{3}{n}\right)+x\left(\frac{cl}{n^3}+\frac{1}{n^2}\right), \\ M_{n,l,c}(e_4,x) =& x^4\Bigg(1+\frac{6c^4l}{n^4}+\frac{11c^4l^2}{n^4}+\frac{6c^4l^3}{n^4}+\frac{c^4l^4}{n^4}+\frac{6c^3}{n^3}+\frac{22c^3l}{n^3}+\frac{18c^3l^2}{n^3}+\frac{4c^3l^3}{n^3} \\ &+\frac{11c^2}{n^2}+\frac{18c^2l}{n^2}+\frac{6c^2l^2}{n^2}+\frac{6c}{n}+\frac{4cl}{n}\Bigg) \\ &+x^3\Bigg(\frac{12c^3l}{n^4}+\frac{18c^3l^2}{n^4}+\frac{6c^3l^3}{n^4}+\frac{12c^2}{n^3}+\frac{36c^2l}{n^3}+\frac{18c^2l^2}{n^3}+\frac{18c}{n^2}+\frac{18cl}{n^2} \\ &+\frac{6}{n}\Bigg)+x^2\left(\frac{7c^2l}{n^4}+\frac{7c^2l^2}{n^4}+\frac{7c}{n^3}+\frac{14cl}{n^3}+\frac{7}{n^2}\right)+x\left(\frac{cl}{n^4}+\frac{1}{n^3}\right), \\ M_{n,l,c}(e_5,x) =& x^5\Bigg(1+\frac{24c^5l}{n^5}+\frac{50c^5l^2}{n^5}+\frac{35c^5l^3}{n^5}+\frac{10c^5l^4}{n^5}+\frac{c^5l^5}{n^5}+\frac{24c^4}{n^4}+\frac{100c^4l}{n^4} \end{aligned}$$

$$+\frac{105c^4l^2}{n^4}+\frac{40c^4l^3}{n^4}+\frac{5c^4l^4}{n^4}+\frac{50c^3}{n^3}+\frac{105c^3l}{n^3}+\frac{60c^3l^2}{n^3}+\frac{10c^3l^3}{n^3}$$
$$+\frac{35c^2}{n^2}+\frac{40c^2l}{n^2}+\frac{10c^2l^2}{n^2}+\frac{10c}{n}+\frac{5cl}{n}\Bigg)$$
$$+x^4\Bigg(\frac{60c^4l}{n^5}+\frac{110c^4l^2}{n^5}+\frac{60c^4l^3}{n^5}+\frac{10c^4l^4}{n^5}+\frac{60c^3}{n^4}+\frac{220c^3l}{n^4}+\frac{180c^3l^2}{n^4}$$
$$+\frac{40c^3l^3}{n^4}+\frac{110c^2}{n^3}+\frac{180c^2l}{n^3}+\frac{60c^2l^2}{n^3}+\frac{60c}{n^2}+\frac{40cl}{n^2}+\frac{10}{n}\Bigg)$$
$$+x^3\Bigg(\frac{50c^3l}{n^5}+\frac{75c^3l^2}{n^5}+\frac{25c^3l^3}{n^5}+\frac{50c^2}{n^4}+\frac{150c^2l}{n^4}+\frac{75c^2l^2}{n^4}+\frac{75c}{n^3}$$
$$+\frac{75cl}{n^3}+\frac{25}{n^2}\Bigg)$$
$$+x^2\left(\frac{15c^2l}{n^5}+\frac{15c^2l^2}{n^5}+\frac{15c}{n^4}+\frac{30cl}{n^4}+\frac{15}{n^3}\right)+x\left(\frac{cl}{n^5}+\frac{1}{n^4}\right),$$

$$M_{n,l,c}(e_6,x)=x^6\Bigg(1+\frac{120c^6l}{n^6}+\frac{274c^6l^2}{n^6}+\frac{225c^6l^3}{n^6}+\frac{85c^6l^4}{n^6}+\frac{15c^6l^5}{n^6}+\frac{c^6l^6}{n^6}+\frac{120c^5}{n^5}$$
$$+\frac{548c^5l}{n^5}+\frac{675c^5l^2}{n^5}+\frac{340c^5l^3}{n^5}+\frac{75c^5l^4}{n^5}+\frac{6c^5l^5}{n^5}+\frac{274c^4}{n^4}+\frac{675c^4l}{n^4}$$
$$+\frac{510c^4l^2}{n^4}+\frac{150c^4l^3}{n^4}+\frac{15c^4l^4}{n^4}+\frac{225c^3}{n^3}+\frac{340c^3l}{n^3}+\frac{150c^3l^2}{n^3}+\frac{20c^3l^3}{n^3}$$
$$+\frac{85c^2}{n^2}+\frac{75c^2l}{n^2}+\frac{15c^2l^2}{n^2}+\frac{15c}{n}+\frac{6cl}{n}\Bigg)$$
$$+x^5\Bigg(\frac{360c^5l}{n^6}+\frac{750c^5l^2}{n^6}+\frac{525c^5l^3}{n^6}+\frac{150c^5l^4}{n^6}+\frac{15c^5l^5}{n^6}+\frac{360c^4}{n^5}$$
$$+\frac{1500c^4l}{n^5}+\frac{1575c^4l^2}{n^5}+\frac{600c^4l^3}{n^5}+\frac{75c^4l^4}{n^5}+\frac{750c^3}{n^4}+\frac{1575c^3l}{n^4}$$
$$+\frac{900c^3l^2}{n^4}+\frac{150c^3l^3}{n^4}+\frac{525c^2}{n^3}+\frac{600c^2l}{n^3}+\frac{150c^2l^2}{n^3}+\frac{150c}{n^2}+\frac{75cl}{n^2}$$
$$+\frac{15}{n}\Bigg)+x^4\Bigg(\frac{390c^4l}{n^6}+\frac{715c^4l^2}{n^6}+\frac{390c^4l^3}{n^6}+\frac{65c^4l^4}{n^6}+\frac{390c^3}{n^5}+\frac{1430c^3l}{n^5}$$
$$+\frac{1170c^3l^2}{n^5}+\frac{260c^3l^3}{n^5}+\frac{715c^2}{n^4}+\frac{1170c^2l}{n^4}+\frac{390c^2l^2}{n^4}+\frac{390c}{n^3}+\frac{260cl}{n^3}$$
$$+\frac{65}{n^2}\Bigg)+x^3\Bigg(\frac{180c^3l}{n^6}+\frac{270c^3l^2}{n^6}+\frac{90c^3l^3}{n^6}+\frac{180c^2}{n^5}+\frac{540c^2l}{n^5}+\frac{270c^2l^2}{n^5}$$
$$+\frac{270c}{n^4}+\frac{270cl}{n^4}+\frac{90}{n^3}\Bigg)+x^2\left(\frac{31c^2l}{n^6}+\frac{31c^2l^2}{n^6}+\frac{31c}{n^5}+\frac{62cl}{n^5}+\frac{31}{n^4}\right)$$

$$+ x\left(\frac{cl}{n^6} + \frac{1}{n^5}\right).$$

Remark 2.2. Denote $\mu_{n,m}^{l,c}(x) := R_{n,l,c}((t-x)^m, x)$. Then, using [10, (6)], the central moments are given by $\mu_{n,0}^{l,c}(x) = 1$, $\mu_{n,1}^{l,c}(x) = 0$, and $\mu_{n,2}^{l,c}(x) = \frac{2x(1+cx)}{n+(m-1)c}\cdot$ Higher central moments can be obtained easily.

3. Grüss-Voronovskaya-Type Approximation Results

The Voronovskaya theorem in quantitative form for a class of sequences of linear positive operators is one of the most significant pointwise results. We obtain these by making using of Taylor series expansion. Let us see at some notations.

Let $C[0,\infty)$ be the set of all continuous functions f defined on $[0,\infty)$ and $B_2[0,\infty) := \{f : |f(x)| \leq M_f(1+x^2) \text{ with } M_f > 0\}$. Also, let $C_2[0,\infty)$ denote the subspace of all continuous functions in $B_2[0,\infty)$. Further $C_2^*[0,\infty)$ denotes the closed subspace of $C_2[0,\infty)$ for which $\lim\limits_{x\to\infty} |f(x)|(1+x^2)^{-1} < C$ for some constant C and $\|\cdot\|_2 = \sup_{x\in[0,\infty)} |f(x)|(1+x^2)^{-1}$. In [11], Ispir considered for each $f \in C_2[0,\infty)$, the following weighted modulus of continuity:

$$\Omega(f,\delta) = \sup_{x\geqslant 0, |h|<\delta} \frac{|f(x+h) - f(x)|}{(1+x^2)(1+h^2)}.$$

The quantitative Voronovskaya-type theorem in weighted spaces is as follows.

Theorem 3.1. *If $f \in C[0,\infty)$ and $f'' \in C_2^*[0,\infty)$, then, for $x \in [0,\infty)$, we have*

$$\left|R_{n,l,c}(f,x) - f(x) - \frac{x(1+cx)}{[n+(m-1)c]} f''(x)\right| \leq 16(1+x^2)\Omega\left(f'', \left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right) \times \mu_{n,2}^{l,c}(x).$$

Proof. Using the Taylor series expansion of f, we can write

$$f(t) = f(x) + (t-x)f'(x) + (t-x)^2 \frac{f''(x)}{2!} + H(t,x),$$

where $H(t,x) = \frac{(t-x)^2}{2!}(f''(\xi) - f''(x))$, ξ is a number lying between t and x.

Applying the operators $R_{n,l,c}$ to the above expansion, we have

$$R_{n,l,c}(f,x) - f(x) - f'(x)\mu_{n,1}^{l,c}(x) + \frac{f''(x)}{2!}\mu_{n,2}^{l,c}(x) = R_{n,l,c}(H(t,x),x).$$

Using Remark 2.2, we obtain

$$\left|R_{n,l,c}(f,x) - f(x) + \frac{f''(x)}{2!}\mu_{n,2}^{l,c}(x)\right| \leq R_{n,l,c}(|H(t,x)|,x). \tag{3.1}$$

Now, using the property of weighted modulus of continuity given in [11], it follows that

$$\begin{aligned}\left|\frac{f''(\xi)-f''(x)}{2!}\right| \leq & \frac{1}{2}\Omega(f'',|\xi-x|)(1+(\xi-x)^2)(1+x^2)\\ \leq & \frac{1}{2}\Omega(f'',|t-x|)(1+(t-x)^2)(1+x^2)\\ \leq & \left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f'',\delta)(1+(t-x)^2)(1+x^2).\end{aligned}$$

Moreover,

$$\left|\frac{f''(\xi)-f''(x)}{2!}\right| \leq \begin{cases} 2(1+\delta^2)(1+x^2)\Omega(f'',\delta), & |t-x|<\delta,\\ 2(1+\delta^2)(1+x^2)\dfrac{(t-x)^4}{\delta^4}\Omega(f'',\delta), & |t-x|\geqslant\delta.\end{cases}$$

For $0<\delta<1$, we get

$$\left|\frac{f''(\xi)-f''(x)}{2!}\right| \leq 8(1+x^2)\left(1+\frac{(t-x)^4}{\delta^4}\right)\Omega(f'',\delta).$$

So, we have

$$|H(t,x)| \leq 8(1+x^2)\left((t-x)^2+\frac{(t-x)^6}{\delta^4}\right)\Omega(f'',\delta).$$

Thus, (3.1) implies

$$\begin{aligned}&\left|R_{n,l,c}(f,x)-f(x)+\frac{f''(x)}{2!}\left(\frac{2x(1+cx)}{n+(m-1)c}\right)\right|\\ \leq & 8(1+x^2)\left(\mu_{n,2}^{l,c}(x)+\frac{1}{\delta^4}\mu_{n,6}^{l,c}(x)\right)\Omega(f'',\delta)\\ \leq & 8(1+x^2)\mu_{n,2}^{l,c}(x)\left(1+\frac{1}{\delta^4}\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)\Omega(f'',\delta).\end{aligned}$$

Choosing $\delta=\left(\dfrac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}$, we get the conclusion. □

Following is the Grüss-Voronovskaya-type result.

Theorem 3.2. *If $f,g\in C[0,\infty)$ and $f'',g''\in C_2^*[0,\infty)$, such that, $fg\in C[0,\infty)$ and $(fg)''\in C_2^*[0,\infty)$. Then for $x\in[0,\infty)$, we have*

$$\begin{aligned}&n\left|R_{n,l,c}(fg,x)-R_{n,l,c}(f,x)R_{n,l,c}(g,x)-\mu_{n,2}^{l,c}(x)f'(x)g'(x)\right|\\ \leq & 16(1+x^2)n\mu_{n,2}^{l,c}(x)\left\{\Omega\left((fg)'',\left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right)\right.\end{aligned}$$

$$+\|f\|_2(1+x^2)\Omega\left(g'',\left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right)$$
$$+\|g\|_2(1+x^2)\Omega\left(f'',\left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right)\Bigg\}+nS_n(f)S_n(g),$$

where $S_n(f)=\|f''\|_2\frac{(1+x^2)}{2}\left(2\mu_{n,2}^{l,c}(x)+\frac{2x}{1+x^2}\mu_{n,3}^{l,c}(x)+\frac{1}{1+x^2}\mu_{n,4}^{l,c}(x)\right).$

Proof. Applying Taylor expansion of f, using the fact that $R_{n,l,c}(e_i,x)=e_i$, $e_i(y)=y^i$ for $i=0,1$, and $(fg)''(x)=f''(x)g(x)+2f'(x)g'(x)+g''(x)f(x)$, we have

$$\begin{aligned}
&R_{n,l,c}(fg,x)-R_{n,l,c}(f,x)R_{n,l,c}(g,x)-R_{n,l,c}((t-x)^2,x)f'(x)g'(x)\\
=&\left[R_{n,l,c}(fg,x)-f(x)g(x)-\frac{(fg)''(x)}{2!}R_{n,l,c}((t-x)^2,x)\right]\\
&-f(x)\left[R_{n,l,c}(g,x)-g(x)-\frac{g''(x)}{2!}R_{n,l,c}((t-x)^2,x)\right]\\
&-g(x)\left[R_{n,l,c}(f,x)-f(x)-\frac{f''(x)}{2!}R_{n,l,c}((t-x)^2,x)\right]\\
&+(g(x)R_{n,l,c}(g,x))\cdot(R_{n,l,c}(f,x)-f(x))\\
:=&S_1+S_2+S_3+S_4.
\end{aligned}$$

Next,

$$\begin{aligned}
&\left|R_{n,l,c}(fg,x)-R_{n,l,c}(f,x)R_{n,l,c}(g,x)-R_{n,l,c}((t-x)^2,x)f'(x)g'(x)\right|\\
\leq&|S_1|+|S_2|+|S_3|+|S_4|.
\end{aligned}$$

By Theorem 3.1, we have the following estimates

$$\begin{aligned}
|S_1|\leq&16(1+x^2)\Omega\left((fg)'',\left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right)\mu_{n,2}^{l,c}(x),\\
|S_2|\leq&|f(x)|16(1+x^2)\Omega\left(g'',\left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right)\mu_{n,2}^{l,c}(x),\\
|S_3|\leq&|g(x)|16(1+x^2)\Omega\left(f'',\left(\frac{\mu_{n,6}^{l,c}(x)}{\mu_{n,2}^{l,c}(x)}\right)^{1/4}\right)\mu_{n,2}^{l,c}(x).
\end{aligned}$$

Now, as $f''\in C_2^*[0,\infty)$,

$$R_{n,l,c}(f,x)-f(x)=f'(x)\mu_{n,1}^{l,c}(x)+\frac{1}{2}R_{n,l,c}\left(f''(\xi)(t-x)^2,x\right).$$

So,

$$\begin{aligned}|R_{n,l,c}(f,x)-f(x)| \leq & \frac{1}{2}R_{n,l,c}\left(|f''(\xi)|(t-x)^2,x\right)\\ \leq & \|f''\|_2\frac{1}{2}R_{n,l,c}\left((1+\xi^2)(t-x)^2,x\right),\end{aligned}$$

where ξ is a number between t and x. There are two possible cases now.

If $t<\xi<x$, then $1+\xi^2\leq 1+x^2$. So, we get

$$|R_{n,l,c}(f,x)-f(x)| \leq \|f''\|_2\frac{(1+x^2)}{2}\mu_{n,2}^{l,c}(x).$$

If $x<\xi<t$, then $1+\xi^2\leq 1+t^2$. So, we get

$$\begin{aligned}|R_{n,l,c}(f,x)-f(x)| \leq & \|f''\|_2\frac{1}{2}R_{n,l,c}\left((1+t^2)(t-x)^2,x\right)\\ = & \|f''\|_2\frac{1}{2}\left((1+x^2)\mu_{n,2}^{l,c}(x)+2x\mu_{n,3}^{l,c}(x)+\mu_{n,4}^{l,c}(x)\right).\end{aligned}$$

Combining these two cases, we obtain

$$\begin{aligned}|R_{n,l,c}(f,x)-f(x)| \leq & \|f''\|_2\frac{(1+x^2)}{2}\left(2\mu_{n,2}^{l,c}(x)+\frac{2x}{1+x^2}\mu_{n,3}^{l,c}(x)+\frac{1}{1+x^2}\mu_{n,4}^{l,c}(x)\right)\\ := & S_n(f).\end{aligned}$$

Similarly, we can obtain $|R_{n,l,c}(g,x)-g(x)|\leq S_n(g)$ and hence, we get the desired result. □

4. Difference of Operators

We compute the magnitude of difference of the two operators having the same basis function, viz. the generalized Baskakov operators and the genuine Gupta-Srivastava operators in this section. Varied researchers have studied in this direction (cf. [3, 7] and references therin).

Consider

$$M_{n,l,c}(f,x)=\sum_{k=0}^{\infty}p_{n+lc,k}(x,c)G_{n,k}(f)$$

and

$$R_{n,l,c}(f,x)=\sum_{k=0}^{\infty}p_{n+lc,k}(x,c)H_{n,k}(f),$$

where $G_{n,k}(f)=f\left(\frac{k}{n}\right)$ and $H_{n,k}(f)=[n+(l+1)c]\int\limits_0^{\infty}p_{n+(l+2)c,k-1}(t,c)f(t)dt$, $1\leq k<\infty$, $H_0(f)=f(0)$.

Remark 4.1. By simple computation, we have $b^{G_{n,k}}:=G_{n,k}(e_1)=\frac{k}{n}$ and

$$\mu_2^{G_{n,k}}:=G_{n,k}(e_1-b^{G_{n,k}}e_0)^2=0 \quad\text{and}\quad \mu_4^{G_{n,k}}:=G_{n,k}(e_1-b^{G_{n,k}}e_0)^4=0.$$

Now,

$$\begin{aligned}
H_{n,k}(e_r) =&[n+(l+1)c]\int_0^\infty p_{n+(l+2)c,k-1}(t,c)t^r dt\\
=&[n+(l+1)c]\binom{\frac{n}{c}+l+k}{k-1}\int_0^\infty \frac{(ct)^{k-1}}{(1+ct)^{\frac{n}{c}+l+k+1}}\, t^r dt\\
=&\frac{[n+(l+1)c]}{c^r}\binom{\frac{n}{c}+l+k}{k-1}\int_0^\infty \frac{(ct)^{k+r-1}}{(1+ct)^{\frac{n}{c}+l+k+1}} dt\\
=&\frac{[n+(l+1)c]}{c^{r+1}}\binom{\frac{n}{c}+l+k}{k-1}B\left(k+r,\frac{n}{c}+l-r+1\right)\\
=&\frac{[n+(l+1)c]}{c^{r+1}}\binom{\frac{n}{c}+l+k}{k-1}\frac{\Gamma(k+r)\Gamma\left(\frac{n}{c}+l-r+1\right)}{\Gamma\left(\frac{n}{c}+l+k+1\right)}\\
=&\frac{[n+(l+1)c]}{c^{r+1}}\frac{(k+r-1)!}{(k-1)!}\frac{\Gamma\left(\frac{n}{c}+l-r+1\right)}{\Gamma\left(\frac{n}{c}+l+2\right)}.
\end{aligned}$$

Remark 4.2. $b^{H_{n,k}} := H_{n,k}(e_1) = \frac{k}{n+lc}$ and we have

$$\begin{aligned}
\mu_2^{H_{n,k}} :=& H_{n,k}(e_1 - b^{H_{n,k}}e_0)^2 = H_{n,k}(e_2) + \left(\frac{k}{n+lc}\right)^2 - 2H_{n,k}(e_1)\left(\frac{k}{n+lc}\right)\\
=& \frac{k[n+c(l+k)]}{(n+lc)^2[n+(l-1)c]}
\end{aligned}$$

and

$$\begin{aligned}
\mu_4^{H_{n,k}} :=&H_{n,k}(e_1 - b^{H_{n,k}}e_0)^4\\
=&H_{n,k}(e_4) - 4\left(\frac{k}{n+lc}\right)H_{n,k}(e_3) + 6\left(\frac{k}{n+lc}\right)^2 H_{n,k}(e_2)\\
&-4\left(\frac{k}{n+lc}\right)^3 H_{n,k}(e_1) + \left(\frac{k}{n+lc}\right)^4\\
=&\frac{k\left[\begin{array}{c}-3c^3k^3(l-1)(l-2)(l-3)+(k+1)(k+2)(k+3)lc^3-\\ (k+1)(k+2)(k-9)lc^2n+\left(18+17k+k^3\right)lcn^2+3(2+k)n^3\\ +3c^2k^2\left(2(k+1)(l-2)(l-3)lc+\left(12+k+2(k-5)l-(k-2)l^2\right)n\right)\\ +ck\Big(-4(k+1)(k+2)(l-3)lc^2+2(k+1)(24-3k-8l+2kl)lcn\\ +\left(6(k+4)-\left(k^2+8\right)l\right)n^2\Big)\end{array}\right]}{(n+lc)^4[n+(l-1)c][n+(l-2)c][n+(l-3)c]}.
\end{aligned}$$

As an application of Theorem A, we have the following quantitative estimate for the difference between the operators $M_{n,l,c}$ and $R_{n,l,c}$.

Theorem 4.1. *Let* $f^{(s)} \in C_B[0,\infty)$, $s \in \{0,1,2\}$ *and* $x \in [0,\infty)$, *then for* $n, c \in \mathbb{N}$, *we have*

$$|(M_{n,l,c} - R_{n,l,c})(f,x)| \leq \|f''\|\alpha(x) + \omega(f'', \delta_1(x))(1+\alpha(x)) + 2\omega(f, \delta_2(x)),$$

where

$$\alpha(x) := \frac{1}{2}\sum_{k=0}^{\infty} p_{n+lc,k}(x,c)\left(\mu_2^{G_{n,k}} + \mu_2^{H_{n,k}}\right) = \frac{x(1+cx)[n+(l+1)c]}{2(n+lc)[n+(l-1)c]},$$

$$\begin{aligned}\delta_1^2(x) :=& \frac{1}{2}\sum_{k=0}^{\infty} p_{n+lc,k}(x,c)\left(\mu_4^{G_{n,k}} + \mu_4^{H_{n,k}}\right)\\ =& \frac{1}{2}(n+lc)x\\ &\times \begin{bmatrix} 6(n+lc)^3 - (8c(l-3) - 11lc - 3n)(n+lc)^2(1+(n+(l+1)c)x) \\ +6(n+lc)\left(c^2(l-2)(l-3) + lc^2 + c(-2(l-3)lc+n)\right) \\ (1+(n+(l+1)c)x(3+c(l+2)x+nx)) \\ -\Big(3c^3(l-1)(l-2)(l-3) + c(2lc-n)(2(l-3)lc-ln) \\ -lc\left(lc^2 - lcn + n^2\right) - 3c^2(n+(l-2)(2(l-3)lc-ln))\Big) \\ (1+(n+(l+1)c)x(7+(c(l+2)+n)x(6+c(l+3)x+nx))) \end{bmatrix}\end{aligned}$$

and

$$\delta_2^2(x) := \sum_{k=0}^{\infty} p_{n+lc,k}(x,c)\left(b^{G_{n,k}} - b^{H_{n,k}}\right)^2 = \frac{lcx}{(n+lc)}\left(1+\frac{lc}{n}\right).$$

Proof. The proof immediately follows using Remark 2.1, 4.1 and 4.2. We omit the details. □

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EIGENVALUES OF CIRCULANT MATRICES AND A CONJECTURE OF RYSER

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ABSTRACT. We prove that there is no circulant Hadamard matrix H with first row $[h_1, \dots, h_n]$ of order $n > 4$, under some linear conditions on the h_i's. All these conditions hold in the known case $n = 4$, so that our results can be thought as characterizations of properties that only hold when $n = 4$. Our first conditions imply that some eigenvalue λ of H is a sum of $\sqrt{n}$ terms $h_j\omega^j$, where ω is a primitive n-th root of 1. The same conclusion holds also if some complex arithmetic means associated to λ are algebraic integers (second conditions). Moreover, our third conditions, related to the recent notion of *robust* Hadamard matrices, implies also the nonexistence of these circulant Hadamard matrices. If some of the conditions fail, it appears (to us) very difficult to be able to prove the result.

1. INTRODUCTION

A matrix of order n is a square matrix with n rows. A *circulant* matrix $A := \operatorname{circ}(a_1, \dots, a_n)$ of order n is a matrix of order n of first row $[a_1, \dots, a_n]$ in which each row after the first is obtained by a cyclic shift of its predecessor by one position. For example, the second row of A is $[a_n, a_1, \dots, a_{n-1}]$. A *Hadamard* matrix H of order n is a matrix of order n with entries in $\{-1, 1\}$ such that $K := \frac{H}{\sqrt{n}}$ is an orthogonal matrix. A *circulant Hadamard* matrix of order n is a circulant matrix that is Hadamard. The 10 known circulant Hadamard matrices are $H_1 := \operatorname{circ}(1),\ H_2 := -H_1,\ H_3 := \operatorname{circ}(1, -1, -1, -1),\ H_4 := -H_3,\ H_5 := \operatorname{circ}(-1, 1, -1, -1),\ H_6 := -H_5,\ H_7 := \operatorname{circ}(-1, -1, 1, -1),\ H_8 := -H_7,\ H_9 := \operatorname{circ}(-1, -1, -1, 1),\ H_{10} := -H_9$.

If $H = \operatorname{circ}(h_1, \dots, h_n)$ is a circulant Hadamard matrix of order n then its *representer* polynomial is the polynomial $R(x) := h_1 + h_2x + \cdots + h_nx^{n-1}$.

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No one has been able, despite several deep computations (see [1, 14]), to discover any other circulant Hadamard matrix. Ryser proposed in 1963 (see [16], [3, p. 97]) the conjecture of the non-existence of these matrices when $n > 4$. Preceding work on the conjecture includes [4, 5, 8–11, 13, 15, 18]. Ryser's conjecture above has been studied with several different methods. The first special case done by Brualdi [2] in 1965, assumed that all eigenvalues of $H := \text{circ}(h_1, \dots, h_n)$, a circulant Hadamard matrix of order $n > 4$, were real, i.e., that H is symmetric, or equivalently that

$$h_{n-k} - h_{k+2} = 0, \quad \text{for } k = 1, \dots, \frac{n}{2} - 2. \tag{1.1}$$

Besides Brualdi's result, all other known results are only partial results for particular n's , generally obtained by deep methods: see Turyn's work [18] and e.g., [15]. For example, the known case where n has two prime factors, i.e., $n = 4p^{2m}$ for some odd prime number p, is a consequence of some results of Turyn. These results permitted some computer calculations (e.g., in citations above) that proved the result for increasing numerical values of n. However, these methods seem to be unable to produce general proofs (say a proof of the conjecture for an infinity of n's with more than two prime factors).

The object of the present paper is to prove the conjecture in some new special cases related to some properties of eigenvalues of a possible new circulant Hadamard matrix, generalizing some properties of the 8 circulant Hadamard matrices of order 4. Indeed, we prove that these properties hold only for $n = 4$ assuming that they hold for $n \geq 4$. Essentially we prove that circulant Hadamard matrices of order $n > 4$ cannot "inherit" some "linear" and "count" properties of the known circulant Hadamard matrices of order 4. To prove the full conjecture is equivalent to find a procedure that do not depends on conditions. Thus, we (and many other people in this area) are far from attaining this goal.

In practice, and more precisely, first, we prove (in Theorem 1.1 below) the result by replacing the equalities (1.1) on the h_i's by an upper bound on the number of similar equalities.

Theorem 1.1. *Let $H = \text{circ}(h_1, \dots, h_n)$ be a circulant Hadamard matrix of order $n \geq 4$. Then $n = 4$ provided the number r of i's between 1 and $n/2$ such that $h_i + h_{n/2+i} = 0$ does not exceed $\sqrt{n}/2$.*

Remark 1.1. When $n = 4$ the condition of Theorem 1.1 holds, with $r = 1$, for all 8 circulant Hadamard matrices $H_3, \dots, H_{10}$.

In our second result we replace the condition of Theorem 1.1 by a property of some appropriate (complex) arithmetic mean related to the eigenvalues of H.

Theorem 1.2. *Let $H = \text{circ}(h_1, \dots, h_n)$ be a circulant Hadamard matrix of order $n \geq 4$. Let $\omega := \exp(2\pi i/n)$. Then $n = 4$ provided both statements* (a) *and* (b) *below hold.*

(a) *There exists* $k \in \{1, \ldots, n\}$ *such that* $k \notin \{n, n/2\}$, *and for* $v := \omega^k$ *there exists an* n*-tuple* $S := (\epsilon_1, \ldots, \epsilon_n)$, *depending on* k, *where* $\epsilon_j \in \{-1, 1\}$, *such that*

$$a := \frac{\sum_{j=1}^{n} \epsilon_j h_j v^{j-1}}{n} \in \mathbb{Z}[\omega].$$

(b) *The set* $T := \{1 \leq j \leq n : \epsilon_j = -1\}$ *satisfies* $r := \text{card(T)} \leq \sqrt{n}/2$.

Remark 1.2. When $n = 4$ and $H = \text{circ}(h_1, h_2, h_3, h_4)$, the conditions on Theorem 1.2 hold with:

$$\omega := i, \quad \text{so that } \omega^2 = -1, \quad k = 1, \quad S = (h_1, h_2, h_3, h_4),$$

so that $r = 1$ for all 8 circulant Hadamard matrices of order 4, namely for $H_3, \ldots, H_{10}$.

Finally, in our third main result, we consider properties of the circulant Hadamard matrices (namely: (-1) robust, say type 1 Hadamard matrices) related to the recent notion [6] of *robust* Hadamard matrices More precisely, 4 of the 8 known circulant Hadamard matrices of order 4 are indeed (-1) robust Hadamard matrices while the other 4 (call them *weak* Hadamard, say type 2 Hadamard matrices) have a strong opposite property on their principal minors, (see definitions of *robust* Hadamard matrices and of both types of Hadamard matrices in section 2) and see more details in Remark 1.3 below. We show then in the following theorem that, under some mild conditions, these properties hold for $n = 4$ and not for $n > 4$. Observe also (see again Theorem 1.3) that there is no circulant Hadamard matrices that are robust. This is the reason why we defined the related notions discussed above. Given any $n \times n$ matrix $M = (M_{i,j})$, with $n \geq 2$, we denote, in all the paper, by $m(1, k)$ the principal 2×2 minor of M, i.e., the determinant of the 2×2 submatrix S of M such that $S_{1,1} = M_{1,1},\ S_{2,1} = M_{1,k},\ S_{1,2} = M_{k,1}$ and $S_{2,2} = M_{k,k}$. Moreover, in all the paper H^* means the (complex) conjugate transpose of the matrix H, so that H^* coincides with the transpose H^T when H has real coefficients.

Theorem 1.3. *Let* $H = \text{circ}(h_1, \ldots, h_n)$ *be a circulant Hadamard matrix of order* $n \geq 4$. *Then statement* (a) *holds, and one has* $n = 4$ *provided any of statements* (b) *or* (c) *below hold. We can assume without loss of generality that* $h_1 = 1$.

(a) *The matrix* H *cannot be robust.*
(b) *The matrix* H *is* (-1) *robust.*
(c) *The matrix* H *is weak,* $h_1 + h_{n/2+1} = 0$, *the number* n_1 *of* 1*'s in the entries* $h_1, \ldots, h_{n/2}$ *of* H, *equals* $\frac{n+\sqrt{n}+2}{4}$ *and the number* n_{-1} *of* -1*'s inside the same entries equals* $\frac{n-\sqrt{n}-2}{4}$.

Remark 1.3. When $n = 4$ the four (-1) robust Hadamard circulants are H_5, H_6, H_9 and H_{10}. Thus the 4 weak circulant Hadamard are H_3, H_4, H_7 and H_8.

Remark 1.4. For a general regular Hadamard matrix $H = \text{circ}(h_1, \ldots, h_n)$, say with $h_1 = 1$, it is known (see Lemma 2.1) that the number of 1's in any row equals $r_1 := \frac{n+\sqrt{n}}{2}$. Since we consider type 2 matrices in part (c) of our last theorem it is natural to think, (but it is not proved, and might be difficult to prove), and has been nevertheless used as an hypothesis, that we should have about $r_1/2$ entries equal to 1 in the first $\frac{n}{2}$ entries of the first row of H. The condition on Theorem 1.3, part (c) comes from this consideration, since it matches exactly the case of the 4 circulant matrix $H_8 := \text{circ}(1, 1, -1, 1)$ where we have two 1's and so zero -1 in the first two entries of the first row. The other 3 circulant Hadamard matrices of order 4 and type 2, are obtained by shifts of length 2 of the first row of H_8, (see details, as before, in Remark 1.3).

The necessary tools for the proof of all three theorems are given in Section 2. The proof of Theorem 1.1 is presented in Section 3, the proof of Theorem 1.2 is presented in Section 4, and the proof of Theorem 1.3 is presented in Section 5.

2. Tools

The following is well known. See, e.g., [7, p. 1193], [12, p. 234], [18, p. 329–330].

Lemma 2.1. *Let H be a regular Hadamard matrix of order $n \geq 4$, i.e., a Hadamard matrix whose row and column sums are all equal. Then $n = 4h^2$ for some positive integer h. Moreover, the row and column sums are all equal to $\pm 2h$ and each row has $2h^2 \pm h$ positive entries and $2h^2 \mp h$ negative entries. Finally, if H is circulant then h is odd.*

Lemma 2.2. *Let H be a circulant Hadamard matrix of order n, let $w = \exp(2\pi i/n)$ and let $R(x)$ be its representer polynomial. Then all the eigenvalues $R(v)$ of H, where $v \in \{1, w, w^2, \ldots, w^{n-1}\}$, satisfy $|R(v)| = \sqrt{n}$.*

We recall here the definition of *robust* Hadamard matrices from [6] and define the notions of (-1) *robust* and of *weak* Hadamard matrix.

Definition 2.1. Let H be an Hadamard matrix of order n.

(a) We say that H is *robust* if all 2×2 principal minors of H are in $\{-2, 2\}$.
(b) We say that H is (-1) *robust* if all 2×2 principal minors, but the minor $m(1, n-1)$ of H, that equals 0, are in $\{-2, 2\}$.
(c) We say that H is *weak* if all 2×2 principal minors of H equal 0.

Remark 2.1. An Hadamard matrix H is robust if and only if every principal 2×2 submatrix of H is also an Hadamard matrix. An Hadamard matrix H is weak if and only if every principal 2×2 submatrix of H is singular. In order that a circulant Hadamard $H := \text{circ}(h_1, \ldots, h_n)$ matrix be robust (resp. weak) it is necessary and sufficient that the principal 2×2 submatrices with first column $[h_1, h_k]^T$ (where the T means "transpose") be Hadamard (resp. be singular).

The next lemma (see [17, Lemma 8.6]) is frequently used in the theory of group representations. Here, it is useful for the proof of Lemma 2.4.

Lemma 2.3. *Let $c_1, \dots, c_\ell$ be ℓ complex numbers of absolute value 1. If $|c_1 + \cdots + c_\ell| = \ell$, then $c_1 = \cdots = c_\ell$.*

The next lemma is about some complex arithmetic means.

Lemma 2.4. *Let $\omega := \exp(2\pi i/n)$. Let $c_1, \dots, c_n$ be n elements of $\mathbb{Z}[\omega]$ of absolute value 1. If*

$$\frac{c_1 + \cdots + c_n}{n} \in \mathbb{Z}[\omega],$$

then either $c_1 = \cdots = c_n$ or $c_1 + \cdots + c_n = 0$.

Proof. Put $a := \frac{c_1+\cdots+c_n}{n}$. The hypothesis implies that $|a| \le 1$. If at least two of the c_j's are distinct, then by Lemma 2.3 (with $\ell = n$) we get $|a| < 1$ so that $|\sigma(a)| < 1$ for any $\sigma \in G$, where $G := Gal(\mathbb{Q}(\omega)/\mathbb{Q})$ is the Galois group of the cyclotomic field $\mathbb{Q}(\omega)$ over $\mathbb{Q}$. Thus $P := \prod_{\sigma \in G} \sigma(a) \in \mathbb{Z}$ satisfies $0 \le |P| < 1$. It follows that $P = 0$, so that $a = 0$. □

3. Proof of Theorem 1.1

Put $w := \exp(2\pi i/n)$. Observe that H is regular in terms of Lemma 2.1 since H is circulant. In particular, Lemma 2.1 implies that $n = 4h^2$ for some positive integer h. Write $H = \operatorname{circ}(h_1, \dots, h_n)$ and let $R(x)$ be the representer polynomial of H. By Lemma 2.2 one has $R(w) = 2ha$ where a is a complex number in the unit circle. Let $W := \left\{ j = 1, \dots \frac{n}{2} : h_j = -h_{n/2+j} \right\}$ and let

$$t := \sum_{j \in W} h_j \omega^{j-1}. \tag{3.1}$$

Then one has

$$2ha - 2t = z_1 + \cdots + z_n, \tag{3.2}$$

where

$$z_j := h_j \omega^{j-1}, \quad \text{for all } j = 1, \dots, \frac{n}{2} \text{ such that } j \notin W, \tag{3.3}$$

and

$$z_j := -h_j \omega^{j-1}, \quad \text{for all } j = 1, \dots, \frac{n}{2} \text{ such that } j \in W, \tag{3.4}$$

and

$$z_{n/2+j} := h_{n/2+j} \omega^{n/2+j-1}, \quad \text{for all } j = 1, \dots, \frac{n}{2}. \tag{3.5}$$

Since $\omega^{n/2} = -1$, we see that (3.3), (3.4) and (3.5) guarantee that

$$z_{n/2+j} = -z_j, \quad \text{for all } j = 1, \dots, \frac{n}{2}. \tag{3.6}$$

More precisely, if $j \notin W$ then $z_j = h_j \omega^{j-1}$, while $z_{n/2+j} = h_{n/2+j} \omega^{n/2+j-1} = h_j \omega^{n/2} \omega^{j-1} = -h_j \omega^{j-1} = -z_j$. If $j \in W$ then $z_j = -h_j \omega^{j-1}$, while $z_{n/2+j} = h_{n/2+j} \omega^{n/2+j-1} = -h_j \omega^{n/2} \omega^{j-1} = h_j \omega^{j-1} = -z_j$.

Since $\omega \notin \mathbb{R}$ and ω has multiplicative order equal to n it follows from (3.6) that we have $z_i \neq z_j$ for all $i \neq j$, $1 \leq i, j \leq n$.

It then follows from (3.6) that

$$z_1 + z_2 + \cdots + z_n = 0. \tag{3.7}$$

But by (3.2), we see that (3.7) implies

$$\sqrt{n}a = 2t. \tag{3.8}$$

But $|a| = 1$, and by hypothesis card(W) $\leq \sqrt{n}/2$, thus it follows from (3.8) and from the definition of t in (3.1) that

$$\frac{\sqrt{n}}{2} = |t| \leq \text{card(W)} \leq \frac{\sqrt{n}}{2},$$

so that

$$\frac{\sqrt{n}}{2} = |t| = \left| \sum_{j \in W} h_j \omega^{j-1} \right| = \text{card(W)}. \tag{3.9}$$

Put for every $j \in W$, $d_j := h_j \omega^{j-1}$. Since $|d_j| = 1$ for all these j's, it follows from (3.9) and from Lemma 2.3 (with $\ell = \sqrt{n}/2$) that

$$d_i = d_j, \quad \text{for all } i, j \in W. \tag{3.10}$$

Assume now that $n > 4$. Then (3.10) is impossible since $\omega^{i-1} \neq \pm\, \omega^{j-1}$ when $i \neq j$ for any $i, j \in \left\{1, 2, \ldots, \frac{n}{2}\right\}$. Therefore, $n = 4$. This finishes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

We refer to notations in Theorem 1.2. From Lemma 2.2, λ defined by

$$\lambda := h_1 + h_2 v + \cdots + h_n v^{n-1}, \tag{4.1}$$

where $v = \omega^k$, is an eigenvalue of H. By the same Lemma 2.2, λ satisfies $|\lambda| = \sqrt{n}$.

Observe that T is not empty, since $T = \varnothing$ implies $a = \lambda/n$ so that $|a| = 1/\sqrt{n}$ since by Lemma 2.2 $|\lambda| = \sqrt{n}$. But hypothesis (a) implies that the complex conjugate $\overline{a} \in \mathbb{Z}[\omega]$ so that $1/n = |a|^2 = a\overline{a} \in \mathbb{Z}[\omega]$. Therefore, we get the contradiction that $n = 1$. One has by hypothesis (a) and by (4.1)

$$\lambda - na = 2 \sum_{i \in T} h_i v^{i-1}. \tag{4.2}$$

Putting $c_j = \epsilon_j h_j v^{j-1}$ for all $j = 1 \ldots n$, it is clear that $na = c_1 + \cdots + c_n$, $c_j \in \mathbb{Z}[\omega]$, and that $|c_j| = 1$ for all these j's. Moreover, $k \notin \{n, n/2\}$ implies that $v \notin \mathbb{R}$ so that $c_1 \neq c_2$.

It follows then from Lemma 2.4 that $a = 0$. Thus, from (4.2) we get

$$\lambda = 2s, \quad \text{where } s = \sum_{i \in T} h_i v^{i-1}. \tag{4.3}$$

Now, Lemma 2.2 and (4.3) imply that

$$|s| = \frac{\sqrt{n}}{2}. \tag{4.4}$$

But from the definition of s in (4.3) and the triangular inequality one has

$$|s| \leq \sum_{i \in T} |h_i v^{i-1}| = \sum_{i \in T} 1 = \text{card(T)}. \tag{4.5}$$

From (4.5), (4.4) and hypothesis (b) we obtain

$$|s| = \text{card(T)} = \frac{\sqrt{n}}{2}. \tag{4.6}$$

Putting $d_j := h_j v^{j-1}$ for all $j \in T$, it is clear that $|d_j| = 1$ for all these $\frac{\sqrt{n}}{2}$ values of j. Thus from (4.6) and from Lemma 2.3 (with $\ell = \sqrt{n}/2$) we obtain that

$$d_i = d_j, \quad \text{for all } j \in T. \tag{4.7}$$

Remember that, by Lemma 2.1, $n = 4h^2$ with odd h. By (4.6), $h = \text{card(T)}$. Thus, if $\text{card(T)} > 1$ then $h = \text{card(T)} \geq 3$ so that (4.7) cannot hold since $\omega^{n/2} = -1$ implies that for $i, j \in T$, with $i \leq j$

$$d_i = d_j \iff i = j \text{ or } j = i + \frac{n}{2}. \tag{4.8}$$

In other words, (4.8) says that there cannot exist three elements $i, j, k \in T$ that are 2 by 2 distinct and for which $d_i = d_j = d_k$. Thus, $\text{card(T)} = 1$, that is, from (4.6), we have $n = 4$. This proves the theorem.

5. Proof of Theorem 1.3

Part (a). Assume, to the contrary, that H is robust. It follows from the following equality (see [6, Formula (3.5) in proof of Lemma 3.6, Subsection 3.3]) that:

$$HD^* + DH^* = 2I, \tag{5.1}$$

where D is the diagonal matrix containing the diagonal elements of H, i.e., in our case $D = I$ so that (5.1) becomes

$$H + H^* = 2I. \tag{5.2}$$

But, multiplying both sides of (5.2) by the eigenvector $v := R(1) = [1, 1, \ldots, 1]^*$ of H, (see Lemma 2.2) we get $2\sqrt{n}\, v = 2v$, i.e., we get the contradiction $n = 1$.

The following observation is useful for the proof of parts (b) and (c): $H := (h_{i,j}) = \text{circ}(h_1, \ldots, h_n)$ if and only if the following condition on the indices (mod n) holds

$$h_{i,j} = h_{j-i+1 \pmod{n}}. \tag{5.3}$$

Part (b). Assume to the contrary, that $n > 4$. Observe that by Lemma 2.1 we can assume that

$$n \geq 36. \tag{5.4}$$

Since H is (-1) robust one has $m(1,j)=2$ for all $j=2,\dots,n-2$, $m(1,n-1)=0$ and $m(1,n)=2$. In other words, (and by using (5.3)) we have $h_j h_{n-j+2}=-1$ for all $j=2,\dots,n-2$, $h_3h_{n-1}=1$ and $h_2h_n=-1$. This can also be written as: $h_{n-j+2}=-hj$ for all $j=2,\dots,n-2$, $h_{n-1}=h_3$ and $h_n=-h_2$. Thus we can write the relation $\sqrt{n}=R(1)$ as follows

$$\sqrt{n}=h_1+\sum_{j=2,j\neq 3}^{n/2+1} h_j-\left(\sum_{t=2,t\neq 3}^{n/2} h_t\right)+h_3+h_3. \tag{5.5}$$

Writing (5.5) in the following form

$$\sqrt{n}=h_1+h_{n/2+1}+\sum_{j=2,j\neq 3}^{n/2} h_j-\left(\sum_{t=2,t\neq 3}^{n/2} h_t\right)+h_3+h_3,$$

it is clear that we get

$$\sqrt{n}=h_1+h_{n/2+1}+2h_3,$$

so that

$$\sqrt{n}=|h_1+h_{n/2+1}+2h_3|\leq 4. \tag{5.6}$$

But, (5.6) contradicts (5.4), thereby finishing the proof of part (b).

Part (c). Assume, to the contrary, that $n>4$. Let $s:=\sum_{k=2}^{n/2} h_k$. Proceeding as before we get now

$$\sqrt{n}=R(1)=h_1+h_{n/2+1}+2s, \tag{5.7}$$

since now we have $h_{n-j-2}=h_j$ for all $j=2,\dots,n$. Let us compute now s by using our hypothesis on the number of 1's and -1's in the h_j's, with $j=1,\dots\frac{n}{2}$,

$$s=m_1-m_{-1}=\frac{2\sqrt{n}+4}{4},$$

thus (5.7) becomes

$$\sqrt{n}=h_1+h_{n/2+1}+\sqrt{n}+2. \tag{5.8}$$

We have then from (5.8)

$$h_1=h_{n/2+1}=-1. \tag{5.9}$$

But, (5.9) contradicts our hypothesis $h_1+h_{n/2+1}=0$, thereby proving part (c). This proves the theorem.

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SOLVABILITY FOR MULTI-POINT BVP OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE WITH THREE DIMENSIONAL KERNELS

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Abstract. This work deals with the BVP multi-point existence of solutions of a nonlinear fractional differential equations at resonance, where the kernel's dimension of the fractional differential operator is equal to three. Our results are based on Mawhin's theory of coincidence. As application, we give an example to illustrate our results.

1. Introduction

The present work concerns a kind of fractional differential equation which can be written as $Lx = Nx$, where L is a linear Fredholm operator of index zero, and N is a nonlinear operator. It is well known that if the kernel of the linear part contains only zero, the corresponding boundary value problem is called non-resonant. In this case, L is invertible, the equation can be reduced to a fixed point problem for the $L^{-1}N$ operator. Otherwise, if L is a non-invertible, i.e., $\dim\ker L \geq 1$, then the problem is said to be at resonance, and then the problem can be solved by using the coincidence degree theory. The higher value of dim $kerL$ is the more difficult. Recently, many authors investigated the existence of solutions for fractional differential equations at resonance. For instance, see [3–6, 9–11, 15, 16, 18, 19, 32] and the references therein.

The case of $\dim\ker L = 1$ has been discussed by many authors [3, 4, 6, 9–11, 16, 18, 19, 32]. In [6], Z. Bai and Y. Zhang investigated the boundary value problem for a

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fractional differential equation with nonlinear growth with $\dim\ker L = 1$

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\Big(t, u(t), D_{0^+}^{\alpha-1} u(t)\Big), & t \in [0,1], \\ u(0) = 0, \quad u(1) = \sigma u(\eta), \end{cases}$$

where $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville derivative, $1 < \alpha \leq 2$, $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous and $\sigma \in (0, \infty)$, $\eta \in (0, 1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$.

Z. Hu et al. in [10] prove the existence of solutions of two-point boundary value problem for a fractional differential equation at resonance with $\dim\ker L = 1$

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\Big(t, u(t), u'(t)\Big), & t \in [0,1], \\ u(0) = 0, \quad u(1) = u'(1), \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Caputo fractional derivative, $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory conditions.

L. Hu et al. studied in [11] a two-point boundary value problem for fractional differential equation at resonance with $\dim\ker L = 1$:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\Big(t, u(t),\, D_{0^+}^{\alpha-1} u(t),\, D_{0^+}^{\alpha-2} u(t), \dots,\, D_{0^+}^{\alpha-(N-1)} u(t)\Big), \\ u(0) = D_{0^+}^{\alpha-2} u(0) = \cdots = D_{0^+}^{\alpha-(N-1)} u(0) = 0, \quad D_{0^+}^{\alpha-1} u(0) = D_{0^+}^{\alpha-1} u(1), \end{cases}$$

where $0 < t < 1$, $N - 1 < \alpha \leq N$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

For the case $\dim\ker L = 2$, Bai and Zhang established in [5] the existence of at least one solution for the m-point boundary value problem for fractional differential equation at resonance with $\dim\ker L = 2$

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\Big(t, u(t), D_{0^+}^{\alpha-2} u(t), D_{0^+}^{\alpha-1} u(t)\Big), & t \in (0,1), \\ I_{0^+}^{\alpha-1} u(0) = 0, \quad D_{0^+}^{\alpha-1} u(0) = D_{0^+}^{3-\alpha}(\eta), \quad u(1) = \sum_{i=1}^{m} \alpha_i u(\eta_i), \end{cases}$$

where $2 < \alpha < 3$, $0 < \eta \leq 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $m \geq 2$, $\sum_{i=1}^{m} \alpha_i \eta_i^{\alpha-1} = \sum_{i=1}^{m} \alpha_i \eta_i^{\alpha-2} = 1$. $D_{0^+}^{\alpha}$ and $I_{0^+}^{\alpha}$ are the standard Riemann-Liouville fractional derivative and fractional integral respectively and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Caratheodory conditions. The results are obtained under the assumption that:

$$R = \frac{1}{\alpha}\eta^{\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-2}\right] - \frac{1}{\alpha-1}\eta^{\alpha-1}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left[1 - \sum_{i=1}^{m} \alpha_i \eta_i^{2\alpha-1}\right] \neq 0.$$

W. Jiang showed in [15] an existence result for the boundary value problem of fractional differential equation at resonance with $\dim\ker L = 2$:

$$\begin{cases} D_{0^+}^{\alpha} u(t) = f\Big(t, u(t), D_{0^+}^{\alpha-1} u(t)\Big), & t \in J = [0,1], \\ u(0) = 0, \quad D_{0^+}^{\alpha-1} u(0) = \sum_{i=1}^{m} a_i D_{0^+}^{\alpha-1}(\xi_i), \quad D_{0^+}^{\alpha-2} u(0) = \sum_{j=1}^{n} b_j D_{0^+}^{\alpha-2}(\eta_j), \end{cases}$$

where $2 < \alpha < 3$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^{m} a_i = 1$, $\sum_{j=1}^{n} b_j = 1, \sum_{j=1}^{n} b_j \eta_j = 1$,

$f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory conditions. The results are obtained under the assumption that

$$\frac{1}{3}\Big(1 - \sum_{j=1}^{n} b_j \eta_j^3\Big) \sum_{i=1}^{m} a_i \xi_i - \frac{1}{2}\Big(1 - \sum_{j=1}^{n} b_j \eta_j^2\Big) \sum_{i=1}^{m} a_i \xi_i^2 \neq 0.$$

Motivated by the results cited above, we investigate the solvability of multi-point boundary value problem of nonlinear fractional differential equation at resonance with $\dim \ker L = 3$

$$\begin{cases} \Big(\phi(t)^C D_{0+}^{\alpha} u(t)\Big)' = f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^C D_{0+}^{\alpha} u(t)\Big), & t \in I, \\ u(0) = 0, \quad {}^C D_{0+}^{\alpha} u(0) = 0, \quad u'''(0) = \sum_{i=1}^{m} a_i u'''(\xi_i), \\ u''(0) = \sum_{j=1}^{l} b_j u''(\eta_j), \quad u'(1) = \sum_{k=1}^{n} c_k u'(\rho_k), \end{cases} \tag{1.1}$$

where ${}^C D_{0+}^{\alpha}$ is the Caputo fractional derivative, $3 < \alpha \leq 4$, $0 < \xi_1 < \cdots < \xi_m < 1$, $0 < \eta_1 < \cdots < \eta_l < 1$, $0 < \rho_1 < \cdots < \rho_n < 1$, $a_i, b_j, c_k \in \mathbb{R}$, $i = 1, \ldots, m$, $j = 1, \ldots, l$, $k = 1, \ldots, n$, $I = [0,1]$, $\phi(t) \in C^1([0,1])$, $\mu = \min_{t \in I} \phi(t) > 0$ and $f : [0,1] \times \mathbb{R}^5 \to \mathbb{R}$ is a Caratheodory function, that is,

(i) for each $x \in \mathbb{R}^5$, the function $x \to f(t,x)$ is Lebesgue measurable;
(ii) for almost every $t \in [0,1]$, the function $t \to f(t,x)$ is continuous on $\mathbb{R}^5$;
(iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1\Big([0,1], \mathbb{R}\Big)$ such that, for a.e. $t \in [0,1]$ and every $|x| \leq r$, we have $|f(t,x)| \leq \varphi_r(t)$.

In this work, we will always suppose that the following conditions hold.

(H_1) $\sum_{i=1}^{m} a_i = \sum_{j=1}^{l} b_j = \sum_{k=1}^{n} c_k = 1$, $\sum_{j=1}^{l} b_j \eta_j = 0$, $\sum_{k=1}^{n} c_k \rho_k = \sum_{k=1}^{n} c_k \rho_k^2 = 1$.
(H_2)

$$\Delta = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} \neq 0,$$

where for $\nu = 1, 2, 3$, we define

$$d_{\nu 1} = \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{s^{\nu}(\xi_i - s)^{\alpha-4}}{\nu \phi(s)} ds, \quad d_{\nu 2} = \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{s^{\nu}(\eta_j - s)^{\alpha-3}}{\nu \phi(s)} ds,$$
$$d_{\nu 3} = \int_0^1 \frac{s^{\nu}(1-s)^{\alpha-2}}{\nu \phi(s)} ds - \sum_{k=1}^{n} c_k \int_0^{\rho_k} \frac{s^{\nu}(\rho_k - s)^{\alpha-2}}{\nu \phi(s)} ds.$$

The rest of this work is organized as follows. In Section 2, we introduce some notations, definitions and lemmas which will be used later. In Section 3, we present and prove our main results by applying the coincidence degree continuation theorem. Finally, in Section 4 we provide an example.

2. Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and properties can be found in recent literature, see for example [17, 26–28, 30].

Definition 2.1. Let $\alpha > 0$, and u a function $u : (0, \infty) \to \mathbb{R}$. The Riemann-Liouville fractional integral of order α of u is defined by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Remark 2.1. The notation $I_{0+}^{\alpha} u(t) \mid_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \varepsilon)$, $\varepsilon > 0$, of 0 as follows:

$$I_{0+}^{\alpha} u(t) \mid_{t=0} = \lim_{t \to 0+} I_{0+}^{\alpha} u(t).$$

Generally $[I_{0+}^{\alpha} u(t) \mid_{t=0}]$ is not necessarily zero. For instance, let $\alpha \in (0,1)$, $u(t) = t^{-\alpha}$. Then

$$I_{0+}^{\alpha} t^{-\alpha}|_{t=0} = \lim_{t \to 0+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \lim_{t \to 0+} \Gamma(1-\alpha) = \Gamma(1-\alpha).$$

Definition 2.2. Let $\alpha > 0$. The Caputo fractional derivative of order α of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$${}^{C}D_{0+}^{\alpha} u(t) = I_{0+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *Let $\alpha, \eta > 0$, $n = [\alpha] + 1$, then the following relations hold*

$${}^{C}D_{0+}^{\alpha} t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} t^{\eta-\alpha}, \quad \eta > n-1,$$

and

$${}^{C}D_{0+}^{\alpha} t^{k} = 0, \quad k = 0, \dots, n-1.$$

Lemma 2.2. *Let $\alpha, \beta \geq 0$ and $u \in L^1\big([0,1]\big)$. Then $I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t)$ and ${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$ for all $t \in [0,1]$*

Lemma 2.3. *Let $\alpha > 0$, $n = [\alpha] + 1$. Then*

$$I_{0+}^{\alpha} \left({}^{C}D_{0+}^{\alpha} u(t) \right) = u(t) + \sum_{k=0}^{n-1} \delta_k t^k, \quad \delta_k \in \mathbb{R}.$$

Lemma 2.4. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If ${}^{C}D_{0+}^{\alpha} u(t) \in C[0,1]$, then $u(t) \in C^{n-1}([0,1])$.*

Proof. Let $h(t) \in C[0,1]$, such that ${}^{C}D^{\alpha}_{0^+}u(t) = h(t)$, then, from Lemma 2.2, we have

$$u(t) = I^{\alpha}_{0^+}h(t) + \sum_{k=0}^{n-1} \delta_k t^k, \quad \delta_k \in \mathbb{R}.$$

It is easy to check that $u(t) \in C^{n-1}([0,1])$. □

Lemma 2.5. *Let $\alpha > 0$, $u \in L^1\Big([0,1],\mathbb{R}\Big)$. Then, for all $t \in [0,1]$, we have*

$$I^{\alpha+1}_{0^+}u(t) \le \|I^{\alpha}_{0^+}u\|_{L^1}.$$

Proof. Let $u \in L^1\Big([0,1],\mathbb{R}\Big)$, from Lemma 2.2, we have

$$I^{\alpha+1}_{0^+}u(t) = I^{1}_{0^+}I^{\alpha}_{0^+}u(t) = \int_0^t I^{\alpha}_{0^+}u(s)ds \le \int_0^1 |I^{\alpha}_{0^+}u(s)|ds = \|I^{\alpha}_{0^+}u\|_{L^1}.$$ □

Lemma 2.6 ([30])**.** *The fractional integral $I^{\alpha}_{0^+}$, $\alpha > 0$, is bounded in $L^1([0,1],\mathbb{R})$ with*

$$\|I^{\alpha}_{0^+}u\|_{L^1} \le \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.$$

Now, let us recall some notations about the coincidence degree continuation theorem. For more details see [25].

Definition 2.3. Let X and Y be real Banach spaces. A linear operator $L : \operatorname{dom} L \subset X \to Y$ is said to be a Fredholm operator of index zero if

(1) $\operatorname{Im} L$ is a closed subset of Y;
(2) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

It follows from Definition 2.3 that there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\ker L = \operatorname{Im} P, \quad \operatorname{Im} L = \ker Q, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L_p = L\,|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible. We denote the inverse of this map by K_p.

Definition 2.4. Let L be a Fredholm operator of index zero. If Ω is an open bounded subset of X and $\operatorname{dom} L \bigcap \Omega \neq \emptyset$. The map $N : \overline{\Omega} \to X$ will be called L-compact on $\overline{\Omega}$ if

(1) $QN(\overline{\Omega})$ is bounded;
(2) $K_{P,Q}N = K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1. *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [(\operatorname{dom} L \backslash \ker L) \cap \partial\Omega] \times (0,1)$;*
(2) *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*
(3) *$\deg(QN\,|_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Then, the abstract equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

For our purpose and according to Lemma 2.4, the adequate functional space is:

$$X = \Big\{u : {}^{C}D_{0^+}^{\alpha} u \in C([0,1], \mathbb{R}),\ u \text{ satisfies boundary value conditions of } (1.1)\Big\}$$

endowed with the norm:

$$\|u\|_X = \sum_{i=0}^{3} \|u^{(i)}\|_\infty + \|{}^{C}D_{0^+}^{\alpha} u\|_\infty, \quad \text{where } \|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

By means of the functional analysis theory, we can prove that $(X, \|.\|_X)$ is a Banach space.

Let $Y = L^1[0,1]$ be the Lebesgue space of real measurable functions $t \mapsto y(t)$ defined on $[0,1]$ and such that $t \mapsto |y(t)|$ is Lebesgue integrable. Y is a Banach space with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$. Define L to be the linear operator from $\operatorname{dom} L \cap X$ to Y

$$Lu = \Big(\phi\, {}^{C}D_{0^+}^{\alpha} u\Big)', \quad u \in \operatorname{dom} L,$$

where

$$\operatorname{dom} L = \Big\{u \in X \ :\ {}^{C}D_{0^+}^{\alpha} u(t) \text{ is absolutely continuous on } [0,\ 1]\Big\}$$

and define the operator $N : X \to Y$ as:

$$Nu(t) = f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^{C}D_{0^+}^{\alpha} u(t)\Big), \quad t \in [0,1].$$

Then the boundary value problem (1.1) can be written in abstract form as:

$$Lu = Nu, \quad u \in \operatorname{dom} L.$$

To study the compactness of operator N, we need the following lemma.

Lemma 2.7. *$U \subset X$ is a relatively compact set in X if and only if U is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$ such that for every $u \in U$*

$$\|u\|_X = \sum_{i=0}^{3} \|u^{(i)}\|_\infty + \|{}^{C}D_{0^+}^{\alpha} u\|_\infty \le M,$$

and equicontinuous means that for all $\varepsilon > 0$, exists $\delta > 0$, such that

$$|u^{(i)}(t_1) - u^{(i)}(t_2)| < \varepsilon, \quad \textit{for all } u \in U,\ t_1, t_2 \in I,\ |t_1 - t_2| < \delta,\ i \in \{0,1,2,3\},$$

and

$$|{}^{C}D_{0^+}^{\alpha} u(t_1) - {}^{C}D_{0^+}^{\alpha} u(t_2)| < \varepsilon, \quad \textit{for all } u \in U,\ t_1, t_2 \in I,\ |t_1 - t_2| < \delta.$$

3. Main Results

In this section we shall present and prove our main result.

Lemma 3.1. *Let $y \in Y$, $\phi \in C^1[0,1]$, $\min_{t\in I}\phi(t) > \mu > 0$, and suppose that (H_1) holds. Then $u \in X$ is the solution of the following fractional differential equation:*

$$\begin{cases} \left(\phi(t)\,{}^C D_{0+}^{\alpha} u(t)\right)' = y(t), \quad t \in I = [0,1], \\ u(0) = 0, \quad {}^C D_{0+}^{\alpha} u(0) = 0, \quad u'''(0) = \sum_{i=1}^{m} a_i u'''(\xi_i), \\ u''(0) = \sum_{j=1}^{l} b_j u''(\eta_j),\ u'(1) = \sum_{k=1}^{n} c_k u'(\rho_k), \end{cases} \tag{3.1}$$

where u is given by

$$u(t) = \sum_{i=1}^{3} \delta_i t^i + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,dr\,ds, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}, \tag{3.2}$$

and

$$T_1(y) = T_2(y) = T_3(y) = 0, \tag{3.3}$$

where T_1, T_2, $T_3 : Y \to Y$ are three linear operators defined as follow:

$$\begin{aligned} T_1(y) &= \sum_{i=1}^{m} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-4}}{\phi(s)} \int_0^s y(r)\,dr\,ds, \\ T_2(y) &= \sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)\,dr\,ds, \\ T_3(y) &= \int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,dr\,ds - \sum_{k=1}^{n} c_k \int_0^{\rho_k} \frac{(\rho_k - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)\,dr\,ds. \end{aligned}$$

Proof. Let u be a solution of problem (3.1). Then we have

$$\phi(t)\,{}^C D_{0+}^{\alpha} u(t) = \delta + \int_0^t y(s)\,ds, \quad \delta \in \mathbb{R}.$$

The hypothesis ${}^C D_{0+}^{\alpha} u(0) = 0$ and $\min_{t\in I}\phi(t) > 0$, allow us to write

$${}^C D_{0+}^{\alpha} u(t) = \frac{1}{\phi(t)} \int_0^t y(s)\,ds.$$

By Lemma 2.3, we get that

$$u(t) = \sum_{i=0}^{3} \delta_i t^i + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,dr\,ds, \quad \delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}.$$

$u(0) = 0$, implies that

$$u(t) = \sum_{i=1}^{3} \delta_i t^i + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)\,dr\,ds, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}.$$

By $u'''(0)=\sum_{i=1}^{m}a_iu'''(\xi_i)$ and $\sum_{i=1}^{l}a_i=1$, we obtain

$$\sum_{i=1}^{l}a_i\int_0^{\xi_i}\frac{(\xi_i-s)^{\alpha-4}}{\phi(s)}\int_0^s y(r)drds=0,$$

From the conditions $u''(0)=\sum_{j=1}^{l}b_ju''(\eta_j)$ and $\sum_{j=1}^{l}b_j=1$, $\sum_{j=1}^{l}b_j\eta_j=0$, we get

$$\sum_{j=1}^{l}b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{\alpha-3}}{\phi(s)}\int_0^s y(r)drds=0.$$

Combining $u'(1)=\sum_{k=1}^{n}c_ku'(\rho_k)$, $\sum_{k=1}^{n}c_k=1$ and $\sum_{k=1}^{n}c_k\rho_k=1$, $\sum_{k=1}^{n}c_k\rho_k^2=1$, we find

$$\int_0^1\frac{(1-s)^{\alpha-2}}{\phi(s)}\int_0^s y(r)drds-\sum_{k=1}^{n}c_k\int_0^{\rho_k}\frac{(\rho_k-s)^{\alpha-2}}{\phi(s)}\int_0^s y(r)drds=0.$$

Thus,

$$T_1(y)=T_2(y)=T_3(y)=0.$$

On the other hand, we let

$$u(t)=\sum_{i=1}^{3}\delta_it^i+\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)drds,$$

where $\delta_1,\delta_2,\delta_3$ are arbitrary constants. It is clear that $u(0)=0$, in view of Lemmas 2.1 and 2.2, we obtain

$$^{C}D_{0^+}^{\alpha}u(t)=\frac{1}{\phi(t)}\int_0^t y(s)ds.$$

Thus, $^{C}D_{0^+}^{\alpha}u(0)=0$ and $\left(\phi(t)^{C}D_{0^+}^{\alpha}u(t)\right)'=y(t)$ for all $t\in[0,1]$.

If (3.3) holds, we can calculate the following equations

$$u'''(0)-\sum_{i=1}^{m}a_iu'''(\xi_i)=\frac{T_1(y)}{\Gamma(\alpha-3)}=0,\quad u''(0)-\sum_{j=1}^{l}b_ju''(\eta_j)=\frac{T_2(y)}{\Gamma(\alpha-2)}=0,$$

$$u'(1)-\sum_{k=1}^{n}c_ku'(\rho_k)=\frac{T_3(y)}{\Gamma(\alpha-1)}=0,$$

so, u is the solution of the problem (3.1), this completes the proof. □

Lemma 3.2. *Assume that (H_1) and (H_2) hold. Let $\phi\in C^1([0,1])$, $\min_{t\in[0,1]}\phi(t)>\mu>0$, then $L:\operatorname{dom}L\subset X\to Y$ is a Fredholm operator of index zero, and the inverse linear operator $K_p=L_p^{-1}:\operatorname{Im}L\to\operatorname{dom}L\cap\ker P$ is defined by*

$$(K_py)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)drds. \tag{3.4}$$

It satisfies

$$\|K_py\|_X\le\frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|y\|_{L^1}. \tag{3.5}$$

Proof. It is clear that $\ker L = \Big\{u \,:\, u(t) = \sum_{k=1}^{3} \delta_k t^k, \delta_1, \delta_2, \delta_3 \in \mathbb{R}\Big\}$. Furthermore, Lemma 3.1 implies

$$\text{Im} L = \Big\{y \in Y \,:\, T_1(y) = T_2(y) = T_3(y) = 0\Big\}. \tag{3.6}$$

Consider continuous linear mapping $Q : Y \to Y$ defined by

$$Qy = Q_1(y) + Q_2(y)t + Q_3(y)t^2, \tag{3.7}$$

where Q_1, Q_2, $Q_3 : Y \to Y$ are three linear operators defined as follows

$$Q_1(y) = \frac{1}{\Delta}\Big(e_{11}T_1(y) + e_{12}T_2(y) + e_{13}T_3(y)\Big),$$
$$Q_2(y) = \frac{1}{\Delta}\Big(e_{21}T_1(y) + e_{22}T_2(y) + e_{23}T_3(y)\Big),$$
$$Q_3(y) = \frac{1}{\Delta}\Big(e_{31}T_1(y) + e_{32}T_2(y) + e_{33}T_3(y)\Big),$$

e_{ij}, $i, j = 1, 2, 3$, are the algebraic complements of d_{ij}.

We will prove that $\ker Q = \text{Im}\, L$. Obviously, $\text{Im}\, L \subset \ker Q$. As well, if $y \in \ker Q$, then

$$\begin{cases} e_{11}T_1(y) + e_{12}T_2(y) + e_{13}T_3(y) = 0, \\ e_{21}T_1(y) + e_{22}T_2(y) + e_{23}T_3(y) = 0, \\ e_{31}T_1(y) + e_{32}T_2(y) + e_{33}T_3(y) = 0. \end{cases} \tag{3.8}$$

The determinant of coefficients for (3.8) is $\Delta^2 \neq 0$. We find $T_1(y) = T_2(y) = T_3(y) = 0$ and that implies $y \in \text{Im}\, L$. So, $\ker Q \subset \text{Im}\, L$. Now, we prove $Q^2 y = Qy$, $y \in Y$. For $y \in Y$, we have

$$\begin{aligned} Q_1^2(y) &= \frac{1}{\Delta}\Big[e_{11}T_1\Big(Q_1(y)\Big) + e_{12}T_2\Big(Q_1(y)\Big) + e_{13}T_3\Big(Q_1(y)\Big)\Big] \\ &= \frac{1}{\Delta}\Big(e_{11}d_{11} + e_{12}d_{21} + e_{13}d_{31}\Big)Q_1 y \\ &= Q_1 y, \\ Q_1\Big(Q_2(y)t\Big) &= \frac{1}{\Delta}\Big[e_{11}T_1\Big(Q_2(y)t\Big) + e_{12}T_2\Big(Q_2(y)t\Big) + e_{13}T_3\Big(Q_2(y)t\Big)\Big] \\ &= \frac{1}{\Delta}\Big(e_{11}d_{12} + e_{12}d_{22} + e_{13}d_{32}\Big)Q_2 y \\ &= 0, \\ Q_1\Big(Q_3(y)t^2\Big) &= \frac{1}{\Delta}\Big[e_{11}T_1\Big(Q_3(y)t^2\Big) + e_{12}T_2\Big(Q_3(y)t^2\Big) + e_{13}T_3\Big(Q_3(y)t^2\Big)\Big] \\ &= \frac{1}{\Delta}\Big(e_{11}d_{13} + e_{12}d_{23} + e_{13}d_{33}\Big)Q_3 y \\ &= 0. \end{aligned}$$

Similarly, we obtain

$$Q_2\Big(Q_1(y)\Big) = 0, \quad Q_2\Big(Q_2(y)t\Big) = Q_2 y, \quad Q_2\Big(Q_3(y)t^2\Big) = 0,$$

$$Q_3\Big(Q_1(y)\Big) = 0, \quad Q_3\Big(Q_2(y)t\Big) = 0, \quad Q_3\Big(Q_3(y)t^2\Big) = Q_3 y.$$

Therefore, we get

$$\begin{aligned}Q^2 g =& Q_1\Big(Q_1(y)\Big) + Q_1\Big(Q_2(y)t\Big) + Q_1\Big(Q_3(y)t^2\Big) + Q_2\Big(Q_1(y)\Big)t + Q_2\Big(Q_2(y)t\Big)t\\ &+ Q_2\Big(Q_3(y)t^2\Big)t + Q_3\Big(Q_1(y)\Big)t^2 + Q_3\Big(Q_2(y)t\Big)t^2 + Q_3\Big(Q_3(y)t^2\Big)t^2\\ =& Q_1(y) + Q_2(y)t + Q_3(y)t^2\\ =& Qg.\end{aligned}$$

This implies that the operator Q is a projector.

Take $y \in Y$ in the form $y = (y - Qy) + Qy$. Then $(y - Qy) \in \ker Q = \operatorname{Im} L$ and $Qy \in \operatorname{Im} Q$. Thus, $Y = \operatorname{Im} Q + \operatorname{Im} L$. And for any $y \in \operatorname{Im} Q \cap \operatorname{Im} L$ from $y \in \operatorname{Im} Q$, there exist constants $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ such that $y(t) = \sum_{k=1}^{3} \delta_k t^k$, from $y \in \operatorname{Im} L$, we obtain

$$\begin{cases} d_{11}\delta_1 + d_{12}\delta_2 + d_{13}\delta_3 = 0,\\ d_{21}\delta_1 + d_{22}\delta_2 + d_{23}\delta_3 = 0,\\ d_{31}\delta_1 + d_{32}\delta_2 + d_{33}\delta_3 = 0.\end{cases} \tag{3.9}$$

The determinant of coefficients for (3.9) is $\Delta \neq 0$. Therefore, (3.9) has an unique solution $\delta_1 = \delta_2 = \delta_3 = 0$, which implies $\operatorname{Im} Q \cap \operatorname{Im} L = 0$. Then, we have

$$Y = \operatorname{Im} Q \oplus \ker Q = \operatorname{Im} Q \oplus \operatorname{Im} L. \tag{3.10}$$

Thus, $\dim \ker L = 3 = \dim \operatorname{Im} Q = \operatorname{codim} \ker Q = \operatorname{codimIm} L$, this means that L is a Fredholm operator of index zero.

Let $P : X \to X$ be a mapping defined by

$$Pu(t) = \sum_{k=1}^{3} \frac{u^{(k)}(0)}{k!} t^k. \tag{3.11}$$

We note that P is a linear continuous projector and $\operatorname{Im} P = \ker L$. It follows from $u = (u - Pu) + Pu$ that $X = \ker P + \ker L$. By simple calculation, we obtain that $\ker L \cap \ker P = \{0\}$. Hence,

$$X = \ker L \oplus \ker P. \tag{3.12}$$

Define $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ as follows:

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r) dr ds.$$

Now, we will prove that K_p is the inverse of $L\,|_{\operatorname{dom} L \cap \ker P}$. In fact, for $u \in \operatorname{dom} L \cap \ker P$, we have

$$(K_p L)u(t) = I_{0^+}^{\alpha} \left(\frac{I_{0^+}^{1}\Big(\phi\, {}^{C}D_{0^+}^{\alpha} u\Big)'}{\phi} \right)(t) = I_{0^+}^{\alpha}\, {}^{C}D_{0^+}^{\alpha} u(t) = u(t) + \sum_{k=0}^{3} \frac{u^{(k)}(0)}{k!} t^k.$$

In view of $u \in \operatorname{dom} L \cap \ker P$, $u(0) = 0$ and $Pu = 0$. Thus,

$$(K_p L)u(t) = u(t), \tag{3.13}$$

and for $y \in \operatorname{Im} L$, we find

$$(LK_p)y(t) = L(K_p y)(t) = \left[\phi(t)\, {}^C D^{\alpha}_{0^+}\left(I^{\alpha}_{0^+}\Big(\frac{I^1_{0^+}y}{\phi}\Big)(t)\right)\right]' = y(t).$$

Thus, $K_p = \left(L\,|_{\operatorname{dom} L \cap \ker P}\right)^{-1}$. Again for each $y \in \operatorname{Im} L$, and from Lemmas 2.2, 2.5 and 2.6, we have

$$\begin{aligned}
\|K_p y\|_X &= \sum_{i=0}^{3} \max_{t\in[0,1]} \left|(K_p y)^{(i)}(t)\right| + \max_{t\in[0,1]} \left|{}^C D^{\alpha}_{0^+}(K_p y)(t)\right| \\
&= \sum_{i=0}^{3} \max_{t\in[0,1]} \left|I^{\alpha-i}_{0^+}\Big(\frac{I^1_{0^+}y}{\phi}\Big)(t)\right| + \max_{t\in[0,1]} \left|{}^C D^{\alpha}_{0^+} I^{\alpha}_{0^+}\Big(\frac{I^1_{0^+}y}{\phi}\Big)(t)\right| \\
&\le \sum_{i=0}^{3} \|y\|_{L^1} \max_{t\in[0,1]} \left|I^{\alpha-i}_{0^+}\frac{1}{\phi}(t)\right| + \|y\|_{L^1} \max_{t\in[0,1]} \left|I^1_{0^+}\frac{1}{\phi}(t)\right| \\
&\le \sum_{i=0}^{3} \|y\|_{L^1} \max_{t\in[0,1]} \left|I^{\alpha-i}_{0^+}\frac{1}{\mu}(t)\right| + \|y\|_{L^1} \max_{t\in[0,1]} \left|I^1_{0^+}\frac{1}{\mu}(t)\right| \\
&\le \sum_{i=0}^{3} \frac{\|y\|_{L^1}}{\mu\Gamma(\alpha+1-i)} + \frac{\|y\|_{L^1}}{\mu} \\
&\le \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|y\|_{L^1}.
\end{aligned}$$

□

Lemma 3.3. *Suppose that* Ω *is an open bounded subset of* X *such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. *Then,* N *is* L*-compact on* $\overline{\Omega}$.

Proof. It is clear that $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are bounded, due to the fact that f realize the caratheodory conditions.

Using the Lebesgue dominated convergence theorem, we can easily find that QN and $K_{P,Q}N = K_p(I-Q)N : \overline{\Omega} \to X$ are continuous. By the hypothesis (iii) on the function f, there exists a constant $A > 0$, such that $|(I-Q)N(u(t))| \le A$, for all $u \in \Omega$, $t \in [0,1]$. For $i = 0,1,2,3$, $0 \le t_1 \le t_2 \le 1$, and $u \in \Omega$, we put $M(t) = (I-Q)Nu(t)$. One has

$$\begin{aligned}
&\left|\Big(K_{P,Q}Nu\Big)^{(i)}(t_2) - \Big(K_{P,Q}Nu\Big)^{(i)}(t_1)\right| \\
=&\frac{1}{\Gamma(\alpha-i)}\left|\int_0^{t_2}\frac{(t_2-s)^{\alpha-i-1}}{\phi(s)}\int_0^s M(r)drds - \int_0^{t_1}\frac{(t_1-s)^{\alpha-i-1}}{\phi(s)}\int_0^s M(r)drds\right| \\
\le&\frac{A}{\mu\Gamma(\alpha-i)}\left\{\int_0^{t_1}(t_2-s)^{\alpha-i-1} - (t_1-s)^{\alpha-i-1}ds + \int_{t_1}^{t_2}(t_2-s)^{\alpha-i-1}ds\right\} \\
=&\frac{A}{\mu\Gamma(\alpha+1-i)}(t_2^{\alpha-i} - t_1^{\alpha-i}),
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|{}^{C}D^{\alpha}_{0^+}K_{P,Q}\,Nu(t_2) - {}^{C}D^{\alpha}_{0^+}K_{P,Q}\,Nu(t_1)\right| \\
=&\left|\frac{1}{\phi(t_2)}\int_0^{t_2} M(s)ds - \frac{1}{\phi(t_1)}\int_0^{t_1} M(s)ds\right| \\
=&\left|\left(\frac{1}{\phi(t_2)} - \frac{1}{\phi(t_1)}\right)\int_0^{t_1} M(s)ds + \frac{1}{\phi(t_2)}\int_{t_1}^{t_2} M(s)ds\right| \\
\leq&\frac{A}{\mu^2}\Big|\phi(t_2) - \phi(t_1)\Big| + \frac{A}{\mu}\Big(t_2 - t_1\Big).
\end{aligned}$$

Since $t^{\alpha-i}$ and $\phi(t)$ are uniformly continuous on $[0,1]$, we get that $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. The lemma is proved. ☐

Theorem 3.1. *Let f be a Caratheodory function, $\phi \in C^1[0,1]$, $\min_{t\in[0,1]}\phi(t) > \mu > 0$. (H_1) and (H_2) hold. In addition, assume that the following conditions hold.*

(H_3) *There exist non-negative functions $\theta_i(t) \in Y$, $i = 0, \dots, 5$, such that*

$$\left|f\Big(t, x_0, x_1, x_2, x_3, x_4\Big)\right| \leq \sum_{i=0}^{4}\theta_i(t)|x_i| + \theta_5(t),$$

where

$$\Lambda = \frac{22 + \Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\sum_{i=0}^{4}\|\theta_i\|_{L^1} < 1.$$

(H_4) *There exists a constant $M > 0$ such that for $u \in \operatorname{dom} L\backslash \ker L$, if $|u^{'}(t)| > M$ or $|u^{''}(t)| > M$ or $|u^{'''}(t)| > M$ for all $t \in [0,1]$, then $T_1(Nu) \neq 0$ or $T_2(Nu) \neq 0$ or $T_3(Nu) \neq 0$.*

(H_5) *There exists a constant $M^* > 0$ such that for any $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$, if $|\delta_1| > M^*$, $|\delta_2| > M^*$, $|\delta_3| > M^*$, then either*

$$\sum_{i=1}^{3} T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right) < 0$$

or

$$\sum_{i=1}^{3} T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right) > 0.$$

Then (1.1) has at least one solution.

Proof. Consider the set

$$\Omega_1 = \{u \in \operatorname{dom} L\backslash \ker L \,:\, Lu = \lambda Nu,\ \lambda \in [0,1]\}.$$

Then for $u \in \Omega_1$, $Lu = \lambda Nu$, thus $\lambda \neq 0$, $Nu \in \operatorname{Im} L = \ker Q \subset Y$. Hence, $Q(Nu) = 0$ that is, $T_1(Nu) = T_2(Nu) = T_3(Nu) = 0$. We get from (H_4) the existence of $t_1, t_2, t_3 \in [0,1]$, such that $|u'(t_1)| \leq M$, $|u''(t_2)| \leq M$, $|u'''(t_3)| \leq M$.

If $t_1 = t_2 = t_3 = 0$, we have that $|u'(0)| \leq M$, $|u''(0)| \leq M$, $|u'''(0)| \leq M$. Otherwise, if $\max\{t_1, t_2, t_3\} \neq 0$, by $Lu = \lambda Nu$, we obtain

$$u(t) = \sum_{k=1}^{3} \frac{u^{(k)}(0)}{k!} t^k + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s Nu(r)drds.$$

Then

$$u'''(t) = u'''(0) + \frac{\lambda}{\Gamma(\alpha-3)} \int_0^t \frac{(t-s)^{\alpha-4}}{\phi(s)} \int_0^s Nu(r)drds.$$

If $t_3 \neq 0$, we get

$$u'''(t_3) = u'''(0) + \frac{\lambda}{\Gamma(\alpha-3)} \int_0^{t_3} \frac{(t_3-s)^{\alpha-4}}{\phi(s)} \int_0^s Nu(r)drds,$$

together with $|u'''(t_3)| \leq M$, we have

$$|u'''(0)| \leq |u'''(t_3)| + \frac{1}{\Gamma(\alpha-3)} \int_0^{t_3} \frac{(t_3-s)^{\alpha-4}}{\phi(s)} \int_0^s |Nu(r)|drds \leq M + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}.$$

Therefore,

$$|u'''(0)| \leq M + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.14}$$

If $t_2 \neq 0$, then

$$u''(t_2) = u''(0) + u'''(0)t_2 + \frac{\lambda}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s Nu(r)drds,$$

from (3.14) and $|u''(t_2)| \leq M$, we find

$$\begin{aligned} |u''(0)| &\leq |u''(t_2)| + |u'''(0)| + \frac{1}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s |Nu(r)|drds \\ &\leq 2M + \frac{2\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \end{aligned}$$

Consequently,

$$|u''(0)| \leq 2M + \frac{2\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.15}$$

If $t_1 \neq 0$, then

$$u'(t_1) = u'(0) + u''(0)t_1 + \frac{u'''(0)}{2} t_1^2 + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s Nu(r)drds,$$

according to (3.14), (3.15) and $|u'(t_1)| \leq M$, we get

$$\begin{aligned} |u'(0)| &\leq |u'(t_1)| + |u''(0)| + |u'''(0)| + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s |Nu(r)|drds \\ &\leq 4M + \frac{4\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \end{aligned}$$

So,

$$|u'(0)| \leq 4M + \frac{4\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.16}$$

Again for $u \in \Omega_1$, we get

$$\begin{aligned}
\|Pu\|_X &= \sum_{i=0}^{3} \max_{t\in[0,1]} \left|(Pu)^{(i)}(t)\right| + \max_{t\in[0,1]} \left|{}^C D^{\alpha}_{0^+}(Pu)(t)\right| \\
&\leq 2|u'(0)| + 3|u''(0)| + 4|u'''(0)|.
\end{aligned}$$

From (3.14), (3.15) and (3.16), we obtain

$$\|Pu\|_X \leq 18M + \frac{18\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.17}$$

Again for all $u \in \Omega_1$, we have $(I-P)u \in \operatorname{dom} L \cap \ker P$. Thus, by (3.13) and (3.5), we find

$$\begin{aligned}
\|(I-P)u\|_X = \|K_p L(I-P)u\|_X &\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|L(I-P)u\|_{L^1} \\
&\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Lu\|_{L^1} \\
&\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Nu\|_{L^1}.
\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\|u\|_X \leq \|Pu\|_X + \|(I-P)u\|_X \leq 18M + \frac{22+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Nu\|_{L^1}. \tag{3.19}$$

On the other hand, from (H_4), we have

$$\begin{aligned}
\|Nu\|_{L^1} &= \int_0^1 \left|(Nu)(s)\right| ds = \int_0^1 \left| f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^C D^{\alpha}_{0^+} u(t)\Big)\right| ds \\
&\leq \sum_{i=0}^{3} \int_0^1 \left|\theta_i(s)\right|\left|u^{(i)}(s)\right| ds + \int_0^1 \left|\theta_4(s)\right|\left|{}^C D^{\alpha}_{0^+} u(s)\right| ds + \int_0^1 \left|\theta_5(s)\right| ds \\
&\leq \|u\|_X \sum_{i=0}^{4} \|\theta_i\|_{L^1} + \|\theta_5\|_{L^1}.
\end{aligned} \tag{3.20}$$

Therefore, (3.19) and (3.20), yields

$$\|u\|_X \leq \frac{18\mu\Gamma(\alpha-2)M + \Big(22+\Gamma(\alpha-2)\Big)\|\theta_5\|_{L^1}}{\mu\Big(1-\Lambda\Big)\Gamma(\alpha-2)}.$$

So, Ω_1 is bounded.

Let

$$\Omega_2 = \{u \in \ker L \,:\, Nu \in \operatorname{Im} L\}.$$

For $u \in \Omega_2$, then $u \in \ker L = \left\{u : u(t) = \sum_{k=1}^{3} \delta_k t^k,\ \delta_1, \delta_2, \delta_3 \in \mathbb{R}\right\}$ and $Q(Nu) = 0$, that is, $T_1 N\left(\sum_{k=1}^{3} \delta_k t^k\right) = T_2 N\left(\sum_{k=1}^{3} \delta_k t^k\right) = T_3 N\left(\sum_{k=1}^{3} \delta_k t^k\right) = 0$. From condition (H_5), we get $|\delta_1| \le M^*$, $|\delta_2| \le M^*$, $|\delta_3| \le M^*$. Hence, Ω_2 is bounded. Let

$$\Omega_3 = \{u \in \ker L : -\lambda J u + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\},$$

if the first part of (H_5) holds.

Or we'll set

$$\Omega_3 = \{u \in \ker L : -\lambda J u + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\}$$

if the second part of (H_5) holds.

Here $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by

$$J\left(\sum_{k=1}^{3} \delta_k t^k\right) = \omega_1 + \omega_2 t + \omega_3 t^2, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}, \tag{3.21}$$

where

$$\omega_1 = \frac{1}{\Delta}\Big(e_{11}|\delta_1| + e_{12}|\delta_2| + e_{13}|\delta_3|\Big),$$
$$\omega_2 = \frac{1}{\Delta}\Big(e_{21}|\delta_1| + e_{22}|\delta_2| + e_{23}|\delta_3|\Big),$$
$$\omega_3 = \frac{1}{\Delta}\Big(e_{31}|\delta_1| + e_{32}|\delta_2| + e_{33}|\delta_3|\Big).$$

Without loss of generality, we assume that the first part of (H_5) holds. In fact $u \in \Omega_3$, means that $u = \sum_{k=1}^{3} \delta_k t^k$ and $-\lambda J u + (1-\lambda) QNu = 0$. Then we obtain

$$-\lambda J\left(\sum_{k=1}^{3} \delta_k t^k\right) + (1-\lambda) QN\left(\sum_{k=1}^{3} \delta_k t^k\right) = 0. \tag{3.22}$$

If $\lambda = 0$, then $|\delta_1| \le M^*$, $|\delta_2| \le M^*$, $|\delta_3| \le M^*$. If $\lambda = 1$, then

$$\begin{cases} e_{11}|\delta_1| + e_{12}|\delta_2| + e_{13}|\delta_3| = 0, \\ e_{21}|\delta_1| + e_{22}|\delta_2| + e_{23}|\delta_3| = 0, \\ e_{31}|\delta_1| + e_{32}|\delta_2| + e_{33}|\delta_3| = 0. \end{cases} \tag{3.23}$$

The determinant of coefficients for (3.23) is $\Delta^2 \neq 0$. Thus, (3.23) only have zero solutions, that is $\delta_1 = \delta_2 = \delta_3 = 0$.

Otherwise, if $\lambda \neq 0$ and $\lambda \neq 1$, again from (3.21), (3.22) becomes

$$\begin{aligned} \lambda\left(\omega_1 + \omega_2 t + \omega_3 t^2\right) =& (1-\lambda)\left(Q_1 N\left(\sum_{k=1}^{3} \delta_k t^k\right) + Q_2 N\left(\sum_{k=1}^{3} \delta_k t^k\right) t \right.\\ &\left. + Q_3 N\left(\sum_{k=1}^{3} \delta_k t^k\right) t^2\right) \end{aligned}$$

Hence,

$$\lambda \omega_i = (1-\lambda) Q_i\left(\sum_{k=1}^{3} \delta_k t^k\right), \quad \text{for } i = 1, 2, 3.$$

Thus,

$$\lambda|\delta_i| = (1-\lambda)T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right), \quad \text{for } i=1,2,3.$$

Then, we get

$$\lambda\sum_{i=1}^{3}|\delta_i| = (1-\lambda)\sum_{i=1}^{3}T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right) < 0.$$

By the first part of (H_5), we have $|\delta_1| \leq M^*$, $|\delta_2| \leq M^*$, $|\delta_3| \leq M^*$. Here, Ω_3 is bounded.

Now, we shall prove that all the conditions of Theorem 2.1 are satisfied. Let Ω be a bounded open set of X containing $\bigcup_{i=1}^{3}\overline{\Omega}_i$. By Lemma 3.3, N is L-compact on $\overline{\Omega}$, because Ω_1 and Ω_2 are bounded sets, then

(1) $Lu \neq \lambda Nu$ for each $(u,\lambda) \in [(\operatorname{dom} L\backslash \ker L)\cap \partial\Omega]\times(0,1)$;
(2) $Nu \notin \operatorname{Im}L$ for each $u \in \ker L \cap \partial\Omega$.

At least we will prove that the hypothesis (3) of Theorem 2.1 is satisfied. Let

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu.$$

The set Ω_3 is bounded, then $H(u,\lambda) \neq 0$, for all $u \in \ker L\cap\partial\Omega$. Appealing to the homotopy property of the degree, we obtain

$$\begin{aligned}\deg\left(QN\mid_{\ker L}, \Omega\cap\ker L, 0\right) &= \deg\left(H(.,0), \Omega\cap\ker L, 0\right)\\ &= \deg\left(H(.,1), \Omega\cap\ker L, 0\right)\\ &= \deg\left(\pm J, \Omega\cap\ker L, 0\right) \neq 0.\end{aligned}$$

Then by Theorem 2.1, $Lu = Nu$ has at least one solution in $\operatorname{dom} L\cap\overline{\Omega}$, we conclude that the boundary value problem (1.1) has at least one solution in X. The proof is finished. □

Remark 3.1. It is very important to note that the condition $\Delta \neq 0$ is not necessary since L still Fredholm even if this condition is dropped. Indeed the role of Q in Mawhin's theory is purely auxiliary and conditions like that usually arise from the authors of hundreds of paper choosing $\operatorname{Im} Q$ just simply being $\ker L$. Avoiding such an assumption is just a matter of choosing Q differently, for more details see [14, 20, 21].

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4. Example

To illustrate our main results, we will present an example.

Example 4.1. Let us consider the following fractional boundary value problem

$$\begin{cases}\left(\phi(t)^C D_{0^+}^{\frac{7}{2}}u(t)\right)' = f\left(t, u(t), u'(t), u''(t), u'''(t), {}^C D_{0^+}^{\frac{7}{2}}u(t)\right), \quad t\in[0,1],\\ u(0)=0, \quad {}^C D_{0^+}^{\alpha}u(0)=0, \quad u'''(0) = -u'''(\frac{1}{6}) + 2u'''(\frac{1}{5}),\\ u''(0) = 4u''(\frac{1}{4}) - 3u''(\frac{1}{3}), \quad u'(1) = u'(\frac{1}{4}) - 3u'(\frac{1}{2}) + 3u'(\frac{3}{4}),\end{cases} \tag{4.1}$$

where $\phi(t) = e^{-12t}$ and

$$\begin{aligned}&100e^{12}f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^C D_{0^+}^{\frac{7}{2}} u(t)\Big)\\ =&\frac{|u'''(t)|}{1+(u'''(t))^2} + \cos {}^C D_{0^+}^{\frac{7}{2}} u(t)\Big(1-\sin u'(t)\Big)\Big(1-\sin u''(t)\Big)\\ &+\frac{2}{\pi}\arctan\Big(u(t){}^C D_{0^+}^{\frac{7}{2}} u(t)\Big).\end{aligned}$$

Corresponding to the problem (1.1), we have that $\alpha = \frac{7}{2},\ l=2,\ m=2,\ n=3,\ a_1 = -1,\ a_2 = 2,\ \xi_1 = \frac{1}{6},\ \xi_2 = \frac{1}{5},\ b_1 = 4,\ b_2 = -3,\ \eta_1 = \frac{1}{4},\ \eta_2 = \frac{1}{3},\ c_1 = 1,\ c_2 = -3,\ c_3 = 3,\ \rho_1 = \frac{1}{4},\ \rho_2 = \frac{1}{2},\ \rho_3 = \frac{3}{4},\ \mu = e^{-12}$. Then we get $a_1 + a_2 = b_1 + b_2 = c_1 + c_2 + c_3 = 1,\ b_1\eta_1 + b_2\eta_2 = 0,\ c_1\rho_1 + c_2\rho_2 + c_3\rho_3 = c_1\rho_1^2 + c_2\rho_2^2 + c_3\rho_3^2 = 1$. Thus, the condition (H_1) holds.

Also, we find

$$\begin{aligned}T_1(y) =& -\int_0^{\frac{1}{6}} e^{12s}\Big(\frac{1}{6}-s\Big)^{-\frac{1}{2}}\int_0^s y(r)drds + 2\int_0^{\frac{1}{5}} e^{12s}\Big(\frac{1}{5}-s\Big)^{-\frac{1}{2}}\int_0^s y(r)drds,\\ T_2(y) =& 4\int_0^{\frac{1}{4}} e^{12s}\Big(\frac{1}{4}-s\Big)^{\frac{1}{2}}\int_0^s y(r)drds - 3\int_0^{\frac{1}{3}} e^{12s}\Big(\frac{1}{3}-s\Big)^{\frac{1}{2}}\int_0^s y(r)drds,\\ T_3(y) =& \int_0^1 e^{12s}(1-s)^{\frac{3}{2}}\int_0^s y(r)drds - \int_0^{\frac{1}{4}} e^{12s}\Big(\frac{1}{4}-s\Big)^{\frac{3}{2}}\int_0^s y(r)drds\\ &+3\int_0^{\frac{1}{2}} e^{12s}\Big(\frac{1}{2}-s\Big)^{\frac{3}{2}}\int_0^s y(r)drds - 3\int_0^{\frac{3}{4}} e^{12s}\Big(\frac{3}{4}-s\Big)^{\frac{3}{2}}\int_0^s y(r)drds.\end{aligned}$$

By calculations, we get

$$\begin{aligned}d_{11} =& \frac{1881}{1420},\quad d_{12} = \frac{207}{1669},\quad d_{13} = \frac{143}{9103},\\ d_{21} =& -\frac{920}{1803},\quad d_{22} = -\frac{484}{6725},\quad d_{23} = -\frac{277}{20262},\\ d_{31} =& \frac{15770}{51},\quad d_{32} = \frac{6489}{50},\quad d_{33} = \frac{5427}{74}.\end{aligned}$$

Then, $\Delta = -\frac{655}{539} \neq 0$. Therefore, the condition (H_2) holds.

On the other hand, we have

$$\Big|f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^C D_{0^+}^{\frac{7}{2}} u(t)\Big)\Big| \le 0.01e^{-12}|u'''(t)| + 0.05e^{-12}.$$

We can get that the condition (H_3) holds, where

$$\theta_0(t) = \theta_1(t) = \theta_2(t) = \theta_4(t) = 0,\quad \theta_3(t) = 0.01e^{-12},\quad \theta_5(t) = 0.05e^{-12}$$

and $\Lambda = \frac{838}{3245} < 1$.

Let $M = 1$ and assume that $|u'''(t)| > 1$ holds for all $t \in [0, 1]$, we obtain

$$T_3(y) > 0.01e^{-12}\int_0^1 e^{12s}(1-s)^{\frac{3}{2}} sds - 0.06e^{-12}\int_0^{\frac{1}{4}} e^{12s}\Big(\frac{1}{4}-s\Big)^{\frac{3}{2}} sds$$

$$+ 0.03e^{-12}\int_0^{\frac{1}{2}} e^{12s}\Big(\frac{1}{2}-s\Big)^{\frac{3}{2}} sds - 0.18e^{-12}\int_0^{\frac{3}{4}} e^{12s}\Big(\frac{3}{4}-s\Big)^{\frac{3}{2}} sds.$$
$$=\frac{43818}{2900}e^{-12} > 0,$$

so condition (H_4) is satisfied.

Let $M^* = 1$ and $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ be such that $|\delta_1| > 1$, $|\delta_2| > 1$, $|\delta_3| > 1$, we have

$$\begin{aligned} N\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big) =& 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2} + 0.01e^{-12}\cos {}^C D_{0^+}^{\frac{7}{2}}\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big) \\ &\times \Big(1-\sin\Big(\delta_1 + 2\delta_2 t + 3\delta_3 t^2\Big)\Big) \times \Big(1-\sin\Big(2\delta_2 + 6\delta_3 t\Big)\Big) \\ &+\frac{0.02e^{-12}}{\pi}\arctan\Big(\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big){}^C D_{0^+}^{\frac{7}{2}}\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big)\Big) \\ =& 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}. \end{aligned}$$

Hence,

$$\begin{aligned} T_1 N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) &= 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}d_{11}, \\ T_2 N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) &= 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}d_{12}, \\ T_3 N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) &= 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}d_{13}. \end{aligned}$$

Thus,

$$\sum_{i=1}^{3} T_i N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) = 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}(d_{11}+d_{12}+d_{13}) > 0.$$

So, (H_5) hold. Then, all the assumptions of Theorem 3.1 hold. Thus, the problem (4.1) has at least one solution.

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APPLICATION OF THE HOPF MAXIMUM PRINCIPLE TO THE THEORY OF GEODESIC MAPPINGS

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Abstract. In the present paper we consider some applications the Hopf maximum principle and its generalization to the classical theory of geodesic mappings. As a result, a series of classical theorems on geodesic mappings become consequences of our statements which we shall prove in the present paper.

1. Introduction

The *Hopf maximum principle* is a maximum principle in the theory of second order elliptic differential equations and has been described as the "classic and bedrock result" of that theory. E. Hopf proved in 1927 that if a function satisfies a second order partial differential inequality of a certain kind in a connected domain of $\mathbb{R}^n$ and attains a maximum in the domain then the function is constant. The simple idea behind Hopf's proof, the comparison technique he introduced for this purpose, has led to an enormous range of important applications and generalizations (see [2, 3, 14]). In the present paper we consider some applications the Hopf maximum principle and its generalization to the classical theory of *geodesic mappings* or in other words *projective mappings* (see, for example, [5, p. 131–142], [9–11]). As a result, a series of classical theorems on geodesic mappings become consequences of our statements which we shall prove in the present paper.

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2. Geodesically Equivalent Riemannian Metrics on Complete and Compact Riemannian Manifolds

Two Riemannian metrics g and $\bar{g}$ on a connected domain $U \subset M$ of a same smooth manifold M are said to be *pointwise projectively equivalent* or in other words *pointwise geodesically equivalent*, if every geodesic of g in U is a reparametrized geodesic of $\bar{g}$. In addition, we say that g and $\bar{g}$ are *pointwise affine equivalent* in a connected domain $U \subset M$, if their Levi-Civita connections ∇ and $\bar{\nabla}$ of g and $\bar{g}$ respectively, coincide.

The volume element of g is the volume form $\mathrm{Vol}(g)$, which is defined whether or not M is oriented. In local coordinates, $\mathrm{Vol}(g) = \sqrt{\det g}\,|dx|$. In turn, for $\bar{g}$ we have $\mathrm{Vol}(\bar{g}) = \sqrt{\det \bar{g}}\,|dx|$. As well known (see [5, p. 133]), two metrics g and $\bar{g}$ are geodesically equivalent in a connected domain $U \subset M$ if and only if for the function

$$\varphi = \frac{1}{n+1} \log \left(\frac{\mathrm{Vol}(\bar{g})}{\mathrm{Vol}(g)} \right), \tag{2.1}$$

we have

$$(\nabla_Z \bar{g})(X, Y) = 2\bar{g}(X, Y)\, d\varphi(Z) + \bar{g}(X, Z)\, d\varphi(Y) + \bar{g}(Y, Z)\, d\varphi(X) \tag{2.2}$$

at every point x of $U \subset M$ and for any vectors $X, Y, Z \in T_x M$. As a consequence of these equations, we obtain the following equalities (see [5, p. 135])

$$\overline{\mathrm{Ric}} = \mathrm{Ric} + (n-1)\,(\nabla\, d\varphi - d\varphi \otimes d\varphi), \tag{2.3}$$

where Ric and $\overline{\mathrm{Ric}}$ are the Ricci tensors of g and $\bar{g}$, respectively. Now, if we set $\Delta\varphi = \mathrm{trace}_g \nabla\, d\varphi$, then from (2.3) have

$$\Delta\varphi = \frac{1}{n-1}\,(s^* - s) + g(d\varphi, d\varphi), \tag{2.4}$$

for $\|\varphi\|^2 = g(d\varphi, d\varphi)$, the scalar curvature $s = \mathrm{trace}_g\, \mathrm{Ric}$ of g and $s^* = \mathrm{trace}_g \overline{\mathrm{Ric}}$.

Now, we prove the following theorem.

Theorem 2.1. *Let g and $\bar{g}$ be two pointwise geodesically equivalent Riemannian metrics on a connected domain $U \subset M$ of an n-dimensional $(n \geq 2)$ smooth manifold M such that $s^* \geq s$ at every point of U, where s is the scalar curvature of g and $s^* = \mathrm{trace}_g \overline{\mathrm{Ric}}$ for the Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{g}$. The assumption that the function $\varphi = (n+1)^{-1} \log(\mathrm{Vol}(\bar{g})/\mathrm{Vol}(g))$ attains a local maximum value at some point $x \in U$ implies that g and $\bar{g}$ are geodesically equivalent on U if and only if they are pointwise affinely equivalent metrics. Furthermore, if there is at least one point of U at which $s^* > s$, then $\bar{g} = g$.*

Proof. We suppose now that g and $\bar{g}$ be two geodesically equivalent Riemannian metrics on a connected domain $U \subset M$ of an n-dimensional smooth manifold M such that $s^* \geq s$ where s is the scalar curvature of g and $s^* = \mathrm{trace}_g \overline{\mathrm{Ric}}$ for the Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{g}$. As a result, the function $\varphi = (n+1)^{-1} \log(\mathrm{Vol}(\bar{g})/\mathrm{Vol}(g))$ satisfies the inequality $\Delta\varphi \geq 0$ at each point of U, by (2.4). Therefore, φ is a *subharmonic function* (see [3, 14]). In this case, assumption that the function φ attains a local maximum value at some point then implies φ is a constant C in U, by the *Hopf's maximum principle* (see [3, Theorem 1]). Then from (2.2) we obtain that $\nabla\bar{g} = 0$ on U and hence g and $\bar{g}$ are affine equivalent on U. If $C > 0$, then gradφ is nowhere zero. Now, at a point where $s^* > s$, the left side of (2.4) is zero while the right side is positive. This contradiction shows that $C = 0$ and hence $\bar{g} = g$. Thus we have proved our Theorem 2.1. □

In particular, if $\overline{\mathrm{Ric}} \geq 0$ and $s \leq 0$ at an arbitrary point of U then $s^* \geq s$. In this case, $\Delta\varphi \geq 0$ at each point of U, by (2.4). Therefore, the following corollary is true.

Corollary 2.1. *Let g and $\bar{g}$ be two Riemannian metrics on a connected domain $U \subset M$ of an n-dimensional $(n \geq 2)$ compact smooth manifold M such that $s \leq 0$ for the scalar curvature s of g and $\overline{\mathrm{Ric}} \geq 0$ for the Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{g}$. Then the assumption that the function $\varphi = (n+1)^{-1}\log(\mathrm{Vol}(\bar{g})/\mathrm{Vol}(g))$ attains a local maximum value at some point $x \in U$ implies that g and $\bar{g}$ are pointwise geodesically equivalent if and only if they are pointwise affinely equivalent metrics. Furthermore, if there is at least one point $x \in U$ at which the Ricci tensor $\overline{\mathrm{Ric}}$ is positive in all directions or the scalar curvature s is negative, then $\bar{g} = g$.*

Let $U = M$ and M be a compact manifold. Then there exists a point $x \in M$ at which the function $\varphi = (n+1)^{-1}\log(\mathrm{Vol}(\bar{g})/\mathrm{Vol}(g))$ attains the maximum. As a result we can formulate the following statements that are corollaries of our Theorem 2.1 (see also Theorem 3 and Corollary 4 from [7] and with Theorem 1.3 from [4]).

Corollary 2.2. *Let g and $\bar{g}$ be two Riemannian metrics on an n-dimensional $(n \geq 2)$ compact smooth manifold M such that $s^* \geq s$ where s is the scalar curvature of g and $s^* = \mathrm{trace}_g\overline{\mathrm{Ric}}$ for the Ricci tensor $\overline{\mathrm{Ric}}$ of $\bar{g}$. Then g and $\bar{g}$ are pointwise geodesically equivalent if and only if they are pointwise affinely equivalent metrics. Furthermore, if there is at last point of M at which $s^* > s$, then $\bar{g} = g$.*

Corollary 2.3. *Let g and $\bar{g}$ be two geodesically equivalent Riemannian metrics on an n-dimensional compact smooth manifold M such that $s \leq 0$ and $\overline{\mathrm{Ric}} \geq 0$ where s is the scalar curvature of g and $\overline{\mathrm{Ric}}$ is the Ricci tensor of $\bar{g}$. Then g and $\bar{g}$ are pointwise geodesically equivalent if and only if they are pointwise affinely equivalent metrics. Furthermore, if there is at least one point of M at which the Ricci curvature $\overline{\mathrm{Ric}}$ is positive or the scalar curvature s is negative, then $\bar{g} = g$.*

Let g and $\bar{g}$ be two geodesically equivalent Riemannian metrics on a connected domain $U \subset M$ of an n-dimensional $(n \geq 2)$ smooth manifold M. We suppose that grad$\varphi = (\varphi_i)$ and $\bar{g}^{-1} = (\bar{g}^{jk})$ with respect to a local coordinate system $x^1, \ldots, x^n$ on U and denote by ξ the vector field with the local components $\xi^j = \varphi_k \bar{g}^{jk}$ for $i, j, k = 1, \ldots, n$. If the metric g is an Einstein metric then by direct calculations we obtain the formula (see also [12])

$$\Delta\varphi = \frac{2(n+3)}{n(n-1)}\, s \cdot \psi + 2\, g(\nabla\xi, \nabla\xi), \tag{2.5}$$

for $\psi = e^{4\varphi} g(\xi, \xi)$. This formula is an analogue of our formula (2.4). Therefore, we can prove an analogue of our Theorem 2.1.

Theorem 2.2. *Let g be an Einstein metric with the nonnegative scalar curvature s on a connected domain $U \subset M$ of an n-dimensional $(n \geq 3)$ smooth manifold M. If there exists another Riemannian metric $\bar{g}$ on U that pointwise geodesically equivalent to g and the function $\psi = e^{4\varphi} g(\xi, \xi)$ for the vector field ξ corresponding to* $\mathrm{grad}\varphi$ *under the duality defined by the metric $\bar{g}$ attains a local maximum value at some point $x \in U$, then the scalar curvature s is necessarily equal to zero and $\bar{g}$ is pointwise affine equivalent to g or $\bar{g} = g$ for the case $s > 0$.*

Let $U = M$ and M be a compact smooth manifold. Then there exists a point $x \in M$ at which the function ψ attains the maximum. As a result we can formulate the following theorem that is a corollary of our Theorem 2.2 (see also [12]).

Corollary 2.4. *Let M be an n-dimensional $(n \geq 3)$ compact smooth manifold M and g be an Einstein metric with nonnegative scalar curvature s on M. If there exists another Riemannian metric $\bar{g}$ on M that pointwise geodesically equivalent to g, then the scalar curvature s is necessarily equal to zero and $\bar{g}$ is pointwise affine equivalent to g or $\bar{g} = g$ for the case $s > 0$.*

3. Geodesically Equivalent Riemannian Metrics on Complete Noncompact Riemannian Manifolds

Li and Schoen have proved in [8] that there is no a non-constant, non-negative L^p-integrable $(0 < p < \infty)$ subharmonic function ψ on any complete Riemannian manifold (M, g) with non-negative Ricci tensor. In other word, if we suppose that $\mathrm{Ric} \geq 0$ and $\int_M \|\psi\|^p d\mathrm{Vol}_g < \infty$ for a complete Riemannian manifold (M, g), then $\psi = C$ for some constant C. In this case, we have $C^p \int_M d\mathrm{Vol}_g < \infty$. If $C > 0$, ψ is nowhere zero and the volume of (M, g) is finite. Side by side, we know from [14] that every complete non-compact Riemannian manifold (M, g) with non-negative Ricci tensor has infinite volume. This contradiction shows $C = 0$ and hence $\psi \equiv 0$. Therefore, we can formulate the following lemma.

Lemma 3.1. *Let (M, g) be a complete non-compact Riemannian manifold with non-negative Ricci tensor, then there is no nonzero non-negative $L^p(M, g)$-integrable $(0 < p < \infty)$ subharmonic function.*

On the other hand, if the scalar curvature s of an Einstein metric g is nonnegative then $\mathrm{Ric} = \frac{s}{n} g \geq 0$ and from (2.5) we obtain $\Delta\psi \geq 0$ and hence ψ is a non-negative subharmonic function.

Using the Lemma we can formulate the following statement.

Corollary 3.1. *Let (M, g) be a complete non-compact Einstein manifold with non-negative scalar curvature, and $\bar{g}$ be another Riemannian metric on M that pointwise geodesically equivalent to g. If the function $\psi = e^{4\varphi} g(\xi, \xi)$ for the vector field ξ*

corresponding to gradφ *under the duality defined by the metric* $\bar{g}$ *is* $L^p(M,g)$*-integrable* $(0 < p < \infty)$ *function then the scalar curvature* s *is necessarily equal to zero and* $\bar{g}$ *is pointwise affine equivalent to* g*.*

Remark 3.1. Other results on pointwise geodesically equivalent Riemannian metrics on compact and non-compact Riemannian manifolds can be found among others in papers from the following list [1, 6, 9, 12, 13].

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BOUNDEDNESS OF CERTAIN SYSTEM OF SECOND ORDER DIFFERENTIAL EQUATIONS

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Abstract. This work is concerned with the ultimate boundedness of solutions of the system of vector differential equations

$$\dot{X} = H(Y), \quad \dot{Y} = -F(X,Y)Y - G(X) + P(t,X,Y),$$

where $t \in \mathbb{R}^+$, $X = X(t)$, $Y = Y(t) \in \mathbb{R}^n$, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $G, H : \mathbb{R}^n \to \mathbb{R}^n$ and $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. By using a Lyapunov function as a basic technique, we prove that the solutions of the system of equations are ultimately bounded. In addition, result obtained includes and improves some related results in literature.

1. Introduction

For over five decades, many authors have dealt considerably with qualitative properties of solutions (namely, stability, boundedness, convergence, existence of periodic solutions) of first order and higher order ordinary differential equations using the direct method of Lyapunov (also known as the second method of Lyapunov) [1–16]. This method enables us to determine the qualitative properties of solutions of a differential equation without actually finding its analytic solution. The method entails the construction of a positive definite function, whose derivative with respect to t along the solution path is negative semi-definite. However, the construction of this function remains a general problem [10].

Using the Lyapunov's direct method, many authors have obtained boundedness results of solutions of scalar differential equations [1, 4, 9–11, 14, 16], and some others have extended these results to vector differential equations [2, 3, 5, 7, 8, 12, 13, 15].

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Recently, Omeike et al. [8] considered the system of equations

$$\dot{X} = Y, \quad \dot{Y} = -F(X,Y)Y - G(X) + P(t,X,Y), \tag{1.1}$$

where $X, Y : \mathbb{R}^+ \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, F is an $n \times n$ continuous symmetric positive definite matrix function for the arguments displayed explicitly, $\mathbb{R}$ denotes the real line, $-\infty < t < \infty$, $\mathbb{R}^n$ denotes the real n-dimensional Euclidean space equipped with the usual norm $\|\cdot\|$, and the dots (which appear in the (1.1)) as usual indicate differentiation with respect to t. (1.1) is a system derivable from the second order equation

$$\ddot{X} + F(X,\dot{X})\dot{X} + G(X) = P(t,X,\dot{X}),$$

by setting $\dot{X} = Y$. (1.1) is an n-dimensional analogue of a system of equation

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -f(x,y)y - g(x) + p(t,x,y), \end{aligned} \tag{1.2}$$

studied by Tejumola [11], an equation of motion in Mathematical Physics. Omeike et al. [8] extended the results obtained in Tejumola [11] to (1.1) and obtained conditions which guarantee boundedness of solutions. Tejumola [12] further studied (1.2) in the form

$$\begin{aligned} \dot{x} &= h(y), \\ \dot{y} &= -f(x,y)y - g(x) + p(t,x,y), \end{aligned} \tag{1.3}$$

for boundedness of solutions. By constructing an incomplete Lyapunov function (see E. N. Chukwu [4]) and augmenting with a signum function a boundedness result was proved. In this present work, we extend the result obtained by Tejumola [12] to the n-dimensional analogue of (1.3), given by

$$\begin{aligned} \dot{X} &= H(Y), \\ \dot{Y} &= -F(X,Y)Y - G(X) + P(t,X,Y), \end{aligned} \tag{1.4}$$

where $H : \mathbb{R}^n \to \mathbb{R}^n$ and X, Y, F, G and P are as described above. It is also assumed that F, G, H and P are continuous for the argument displayed explicitly. In addition, the existence and uniqueness of the solutions of (1.4) with any prescribed initial conditions will be assumed (see Picard-Lindelof theorem in [9]).

The motivation for the present work is derived from the works of Tejumola [11, 12] and Omeike et al. [8]. We prove that solutions of (1.4) are bounded. To the best of our knowledge, no author in the literature has extended the boundedness result obtained by Tejumola [12] to (1.4).

2. Notations

We shall use the notation as given in [2]. Throughout this paper δ's, Δ's and D's with or without suffixes will denote positive constants whose magnitudes depend on an $n \times n$ matrix function $F(X,Y)$ and vector functions $H(Y)$, $P(t,X,Y)$. The δ's,

Δ's and D's with numerical or alphabetical suffixes shall retain fixed magnitudes, while those without suffixes are not necessarily the same at each occurrence.

Also, we shall denote the scalar product $\langle X, Y\rangle$ of any vectors X, Y in $\mathbb{R}^n$, with respective components $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ by $\sum_{i=1}^n x_i y_i$. In particular, $\langle X, X\rangle = \|X\|^2$. Finally, by $\operatorname{sgn} X$, we mean $(\operatorname{sgn} x_1, \operatorname{sgn} x_2, \ldots, \operatorname{sgn} x_n)$, $x_i \neq 0$, and $\|\operatorname{sgn} X\| = \sqrt{n} > 0$.

3. Main Results

The following algebraic results will be required in the proofs of our main results.

Lemma 3.1. *Let A be a real symmetric positive definite $n \times n$ matrix. Then for $X \in \mathbb{R}^n$, $\delta_a\|X\|^2 \leq \langle AX, X\rangle \leq \Delta_a\|X\|^2$, where δ_a and Δ_a are, respectively, the least and greatest eigenvalues of the matrix A.*

Proof. See [6, 13]. □

Lemma 3.2. *Let $G(0) = 0 = H(0)$ and assume that the matrices A, $J_g(X)$ and $J_h(Y)$ are symmetric, positive definite and commute pairwise for all $X, Y \in \mathbb{R}^n$. Then*

$$\langle G(X), AX\rangle = \int_0^1 X^T A J_g(\sigma X) X d\sigma,$$
$$\langle H(Y), AY\rangle = \int_0^1 Y^T A J_h(\sigma Y) Y d\sigma,$$

where $J_g(X)$ and $J_h(Y)$ are respectively the Jacobian matrices $\frac{\partial g_i}{\partial x_j}$ and $\frac{\partial h_i}{\partial y_j}$ of $G(X)$ and $H(Y)$.

Proof. See [5, 13]. □

Lemma 3.3. *Let $G(0) = 0$ and assume that $J_g(X)$ is symmetric for all arbitrary $X \in \mathbb{R}^n$. Then*

$$\frac{d}{dt}\int_0^1 \langle G(\sigma X), X\rangle d\sigma = \langle G(X), \dot{X}\rangle,$$

for all $X = X(t) \in \mathbb{R}^n$.

Proof. See [5]. □

Our main theorems are the following.

Theorem 3.1. *Let a, L, β, Δ_f, Δ_g, Δ_h, δ_f, δ_g, δ_h be positive constants and let all the basic assumptions imposed on F, G, H and P hold, and that $G(0) = H(0) = 0$ hold. Suppose further that for any arbitrary $X, Y \in \mathbb{R}^n$*

(i) *$J_g(X), J_h(Y)$ are symmetric and positive definite;*

(ii) *the eigenvalues $\lambda_i(F(X, Y))$, $\lambda_i(J_g(X))$, $\lambda_i(J_h(Y))$ of $F(X, Y), J_g(X)$ and $J_h(Y)$ respectively satisfy*

$$0 < \delta_f \leq \lambda_i(F(X, Y)) \leq \Delta_f, \tag{3.1}$$

$$0 < \delta_g \leq \lambda_i(J_g(X)) \leq \Delta_g, \tag{3.2}$$

(3.3) $$0 < \delta_h \leq \lambda_i(J_h(Y)) \leq \Delta_h;$$

(iii)

(3.4) $$\|P(t, X, Y)\| \leq a,$$

where a is a positive constant.

Suppose further that

(iv)

(3.5) $$\alpha\langle G(X), \operatorname{sgn} X\rangle \to \infty \quad \textit{as} \quad \|X\| \to \infty,$$

where $\alpha = \operatorname{sgn}\langle G(X), \operatorname{sgn} X\rangle$.

Then there exists a finite constant K whose magnitude depends only on the constants a,L, β, Δ_f, Δ_g, Δ_h, δ_f, δ_g, δ_h, as well as the function $G(X)$ such that every solution $(X(t), Y(t))$ of (1.4) *ultimately satisfies*

(3.6) $$\|X(t)\| \leq K, \quad \|Y(t)\| \leq K.$$

Theorem 3.2. *In addition to the conditions* (i) *and* (ii) *of Theorem* 3.1*, suppose*

(i) *for all t, X and Y*

(3.7) $$\|P(t, X, Y)\| \leq \mu\|Y\|, \quad \mu > 0,$$

and

(ii)

(3.8) $$\lim_{\|X\|\to\infty} \alpha\langle G(X), \operatorname{sgn} X\rangle \to \infty.$$

Then there exists a finite positive constant K whose magnitude depends only on the constants a, L, β, μ, Δ_f, Δ_g, Δ_h, δ_f, δ_g, δ_h as well as the function $G(X)$, $H(Y)$ such that every solution $(X(t), Y(t))$ of (1.4) *ultimately satisfies* (3.6).

4. Proof of Main Results and Example

Proof of Theorem 3.1. Our method of proof, which makes use of the adaptation of the well-known Yoshizawa [16] technique, is the same as in [8].

Let the continuous function $U = U(X, Y)$ be defined by

(4.1) $$U = U_1 + U_2 + 1,$$

where

(4.2) $$U_1 = \int_0^1 \langle H(\sigma Y), Y\rangle d\sigma + \int_0^1 \langle G(\sigma X), X\rangle d\sigma$$

(4.3) $$U_2 = \begin{cases} \dfrac{L^{-1}}{\sqrt{n}}\alpha\langle Y, \operatorname{sgn} X\rangle, & \|Y\| \leq L, \\ \dfrac{1}{n}\langle \operatorname{sgn} X, \operatorname{sgn} Y\rangle, & \|Y\| \geq L, \end{cases} \qquad \text{if } \|X\| \geq 1,$$

or

$$U_2 = \begin{cases} L^{-1}\langle X, Y\rangle, & \|Y\| \le L, \\ \dfrac{1}{\sqrt{n}}\langle X, \operatorname{sgn} Y\rangle, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \le 1. \tag{4.4}$$

We shall show that $U(X, Y)$ satisfies

$$U(X, Y) \to +\infty \quad \text{as} \quad \|X\|^2 + \|Y\|^2 \to +\infty. \tag{4.5}$$

From the definition of U_2, we can show that $|U_2| \le 1$ as follows.

If $\|X\| \ge 1$, we obtain

$$\begin{aligned} |U_2| &= \begin{cases} \left|\dfrac{L^{-1}}{\sqrt{n}}\alpha\langle Y, \operatorname{sgn} X\rangle\right|, & \|Y\| \le L, \\ \left|\dfrac{1}{n}\langle \operatorname{sgn} X, \operatorname{sgn} Y\rangle\right|, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \ge 1, \\ &\le \begin{cases} \dfrac{L^{-1}}{\sqrt{n}}|\langle Y, \operatorname{sgn} X\rangle|, & \|Y\| \le L, \\ \dfrac{1}{n}|\langle \operatorname{sgn} X, \operatorname{sgn} Y\rangle|, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \ge 1, \\ &\le \begin{cases} \dfrac{L^{-1}}{\sqrt{n}}\|Y\|\|\operatorname{sgn} X\|, & \|Y\| \le L, \\ \dfrac{1}{n}\|\operatorname{sgn} X\|\|\operatorname{sgn} Y\|, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \ge 1, \\ &\le \begin{cases} \dfrac{L^{-1}}{\sqrt{n}} \times L \times \sqrt{n} = 1, & \|Y\| \le L, \\ \dfrac{1}{n} \times \sqrt{n} \times \sqrt{n} = 1, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \ge 1. \end{aligned}$$

Similarly, if $\|X\| \le 1$, we obtain

$$\begin{aligned} |U_2| &= \begin{cases} |L^{-1}\langle X, Y\rangle|, & \|Y\| \le L, \\ \left|\dfrac{1}{\sqrt{n}}\langle X, \operatorname{sgn} Y\rangle\right|, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \le 1, \\ &\le \begin{cases} L^{-1}|\langle X, Y\rangle|, & \|Y\| \le L, \\ \dfrac{1}{\sqrt{n}}|\langle X, \operatorname{sgn} Y\rangle|, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \le 1, \\ &\le \begin{cases} L^{-1}\|X\|\|Y\||, & \|Y\| \le L, \\ \dfrac{1}{\sqrt{n}}\|X\|\|\operatorname{sgn} Y\||, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \le 1, \\ &\le \begin{cases} L^{-1} \times 1 \times L = 1, & \|Y\| \le L, \\ \dfrac{1}{\sqrt{n}} \times 1 \times \sqrt{n} = 1, & \|Y\| \ge L, \end{cases} \quad \text{if } \|X\| \le 1. \end{aligned}$$

Thus, we have $|U_2| \le 1$.

Now, since $|U_2| \leq 1$, (4.1) yields $U \geq U_1$, and by Lemma 3.2, followed by Lemma 3.1 and inequalities (3.2) and (3.3), we have

$$U_1 \geq D_0(\|X\|^2 + \|Y\|^2),$$

where $D_0 = \min\{\delta_h, \delta_g\}$. Thus,

$$U(X,Y) \to \infty \quad \text{as} \quad \|X\|^2 + \|Y\|^2 \to \infty. \tag{4.6}$$

We are now left to show that $\dot{U}$ exists and that there are finite constants D_1, D_2 such that

$$\dot{U} \leq -D_1, \quad \text{if } \|X\|^2 + \|Y\|^2 \geq D_2. \tag{4.7}$$

From this and (4.5) it will then follow, just as in [8], that there is a constant $D > 0$ such that every solution $(X(t), Y(t))$ of (1.4) ultimately satisfies

$$\|X\|^2 + \|Y\|^2 \leq D,$$

and this verifies (3.6).

To verify (4.7), observe from (4.1) to (4.4) and (1.4) that by applying Lemma 3.3 to U_1, we obtain

$$\dot{U} = \dot{U}_1 + \dot{U}_2, \tag{4.8}$$

where

$$\dot{U}_1 = -\langle H(Y), F(X,Y)Y\rangle + \langle H(Y), P(t,X,Y)\rangle, \tag{4.9}$$

and

(4.10)
$$\dot{U}_2 = \begin{cases} \dfrac{L^{-1}}{\sqrt{n}}\alpha\langle -F(X,Y)Y - G(X) + P(t,X,Y), \operatorname{sgn} X\rangle, & \|Y\| \leq L, \\ 0, & \|Y\| \geq L, \end{cases} \quad \text{if } \|X\| \geq 1,$$

or

(4.11)
$$\dot{U}_2 = \begin{cases} L^{-1}\langle H(Y), Y\rangle + L^{-1}\langle -F(X,Y)Y - G(X) + P(t,X,Y), X\rangle, & \|Y\| \leq L, \\ \dfrac{1}{\sqrt{n}}\langle H(Y), \operatorname{sgn} Y\rangle, & \|Y\| \geq L, \end{cases}$$

if $\|X\| \leq 1$. Thus, if $\|Y\| \leq L$, $\dot{U}_2$ satisfies

(4.12)
$$\dot{U}_2 = -\frac{\alpha}{L\sqrt{n}}\langle F(X,Y)Y, \operatorname{sgn} X\rangle - \frac{\alpha}{L\sqrt{n}}\langle G(X), \operatorname{sgn} X\rangle + \frac{\alpha}{L\sqrt{n}}\langle P(t,X,Y), \operatorname{sgn} X\rangle,$$

if $\|X\| \geq 1$, or

$$\dot{U}_2 = \frac{1}{L}\langle H(Y), Y\rangle - \langle X, F(X,Y)Y\rangle - \langle X, G(X)\rangle + \langle P(t,X,Y), X\rangle \text{ if } \|X\| \leq 1. \tag{4.13}$$

But if $\|Y\| \geq L$, then

$$\dot{U}_2 = \begin{cases} 0, & \|X\| \geq 1, \\ \frac{1}{\sqrt{n}}\langle H(Y), \operatorname{sgn} Y\rangle, & \|X\| \leq 1. \end{cases} \tag{4.14}$$

In obtaining estimates for $\dot{U}$ we shall consider points outside of the closed bounded set defined by $\|X\| \leq 1$ and $\|Y\| \leq L$. It will be convenient to consider the following three regions in turn: (I) $\|X\| \geq 1$ and $\|Y\| \leq L$, (II) $\|X\| \leq 1$ and $\|Y\| \geq L$, and (III) $\|X\| \geq 1$ and $\|Y\| \geq L$. For the case (I), we have from (4.8), (4.9) and (4.12) that

$$\begin{aligned} \dot{U} = &- \langle H(Y), F(X,Y)Y\rangle + \langle H(Y), P(t,X,Y)\rangle - \frac{\alpha}{L\sqrt{n}}\langle F(X,Y)Y, \operatorname{sgn} X\rangle \\ &- \frac{\alpha}{L\sqrt{n}}\langle G(X), \operatorname{sgn} X\rangle + \frac{\alpha}{L\sqrt{n}}\langle P(t,X,Y), \operatorname{sgn} X\rangle, \end{aligned}$$

so that by (3.1)–(3.4), and setting $\beta = \sqrt{n}$,

$$\dot{U} \leq -\frac{1}{L\beta}\left(\alpha\langle G(X), \operatorname{sgn} X\rangle - \beta a \Delta_h L^2\right) + \Delta_f,$$

since $\|Y\| \leq L$. Thus, in view of (3.5), there exists a finite constant $D_3 (> 1)$, sufficiently large, such that

$$\dot{U} \leq -1 \quad \text{provided} \quad \|X\| \geq D_3. \tag{4.15}$$

As for the case (II): $\|X\| \leq 1$ and $\|Y\| \geq L$, we have from (4.8), (4.9) and (4.14) that

$$\dot{U} = -\langle H(Y), F(X,Y)Y\rangle + \langle H(Y), P(t,X,Y)\rangle + \frac{1}{\sqrt{n}}\langle H(Y), \operatorname{sgn} Y\rangle,$$

so that by (3.1), (3.3) and (3.4)

$$\dot{U} \leq -\left(\delta_h\delta_f\|Y\| - (a+1)\Delta_h\right)\|Y\|,$$

$$\dot{U} \leq -1, \quad \text{if } \|Y\| \geq \max\left\{\frac{\Delta_h^2(a+1)^2 + \delta_h\delta_f}{\delta_h\delta_f\Delta_h(a+1)}, L\right\} = D_4. \tag{4.16}$$

Case (III). $\|X\| \geq 1$ and $\|Y\| \geq L$ follow from case (II) since $\dot{U}_2 = 0$ if $\|X\| \geq 1$ and $\|Y\| \geq L$. The two results (4.15) and (4.16) together imply that

$$\dot{U} \leq -1 \quad \text{provided} \quad \|X\|^2 + \|Y\|^2 \geq D_3^2 + D_4^2.$$

This verifies (3.6) and Theorem 3.1 now follows. □

Proof of Theorem 3.2. The procedure here is the same as that used for Theorem 3.1 but only that $P(t,X,Y) \neq 0$ as in the proof of Theorem 3.1. The proof of Theorem 3.2 is immediate as soon as we show (4.6) and (4.7). The verification of (4.6) given in §4 carries over with obvious modifications.

To verify (4.7), our starting point will be the estimates (4.8)–(4.14), which are still valid in this case. Thus, in obtaining estimates for $\dot{U}$ we shall consider points outside of the closed bounded set defined by $\|X\| \leq 1$ and $\|Y\| \leq L$. It will be convenient to consider the following three regions in turn: (I) $\|X\| \geq 1$ and $\|Y\| \leq L$, (II) $\|X\| \leq 1$ and $\|Y\| \geq L$, and (III) $\|X\| \geq 1$ and $\|Y\| \geq L$. For the case (I), we have from (4.8), (4.9) and (4.12) that

$$\dot{U} = -\langle H(Y), F(X,Y)Y\rangle + \langle H(Y), P(t,X,Y)\rangle - \frac{\alpha}{L\sqrt{n}}\langle F(X,Y)Y, \operatorname{sgn} X\rangle \\ - \frac{\alpha}{L\sqrt{n}}\langle G(X), \operatorname{sgn} X\rangle + \frac{\alpha}{L\sqrt{n}}\langle P(t,X,Y), \operatorname{sgn} X\rangle,$$

so that by (3.1)–(3.3) and (3.7), and setting $\beta = \sqrt{n}$

$$\dot{U} \leq -\frac{1}{L\beta}\left\{\alpha\langle G(X), \operatorname{sgn} X\rangle - \beta\mu L(1+\Delta_h L^2)\right\} + \Delta_f,$$

since $\|Y\| \leq L$. Thus, in view of (3.8), there exists a finite constant $D_5(> 1)$, sufficiently large, such that $\dot{U} \leq -1$ provided $\|X\| \geq D_5$. As for the case (II): $\|X\| \leq 1$ and $\|Y\| \geq L$, we have from (4.8), (4.9) and (4.14) that

$$\dot{U} = -\langle H(Y), F(X,Y)Y\rangle + \langle H(Y), P(t,X,Y)\rangle + \frac{1}{\sqrt{n}}\langle H(Y), \operatorname{sgn} Y\rangle,$$

so that by (3.1), (3.3) and (3.7)

$$\dot{U} \leq -\left((\delta_h\delta_f - \mu\Delta_h)\|Y\| - \Delta_h\right)\|Y\|,$$

$$\dot{U} \leq -1 \text{ if } \|Y\| \geq \max\left\{\frac{\Delta_h^2 + (\delta_h\delta_f - \mu\Delta_h)}{\Delta_h(\delta_h\delta_f - \mu\Delta_h)}, L\right\} = D_6,$$

where $\delta_h\delta_f - \mu\Delta_h > 0$.

Case (III). $\|X\| \geq 1$ and $\|Y\| \geq L$, we have from (4.8), (4.9) and (4.14) that

$$\dot{U} = -\langle H(Y), F(X,Y)Y\rangle + \langle H(Y), P(t,X,Y)\rangle,$$

so that by (3.1), (3.3) and (3.7) we obtain

$$\dot{U} \leq -1 \text{ if } \|Y\| \geq \max\{(\delta_h\delta_f - \mu\Delta_h)^{-\frac{1}{2}}, L\}.$$

This verifies (3.6) and Theorem 3.2 now follows. □

Next, we present an illustrative example to demonstrate the applicability of the results proved in this section.

Example 4.1. As a special case of (1.4), let us have for $n = 2$ that

$$F(X,Y) = \begin{pmatrix} 2 + \frac{1}{x_1^2+y_1^2+1} & 1 \\ 1 & 2 + \frac{1}{x_1^2+y_2^2+1} \end{pmatrix}, \quad G(X) = \begin{pmatrix} 2x_1 + \sin x_1 \\ 2x_2 + \sin x_2 \end{pmatrix},$$

$$H(Y) = \begin{pmatrix} y_1 + \tan^{-1} y_1 \\ y_2 + \tan^{-1} y_2 \end{pmatrix} \quad \text{and} \quad P(t,X,Y) = \begin{pmatrix} \frac{1}{1+y_1^2} + \sin t \\ \exp^{-x_1^2} \end{pmatrix}.$$

Clearly, we have $\lambda_1(F(X,Y)) = 4 - \sqrt{5} + \frac{1}{x_1^2+y_1^2+1} + \frac{1}{x_2^2+y_2^2+1}$ and $\lambda_2(F(X,Y)) = 4 + \sqrt{5} + \frac{1}{x_1^2+y_1^2+1} + \frac{1}{x_2^2+y_2^2+1}$. Thus, $4 - \sqrt{5} < \lambda_1(F(X,Y)),\ \lambda_2(F(X,Y)) \ < 6 + \sqrt{5}$, with $\delta_f = 4 - \sqrt{5}$ and $\Delta_f = 6 + \sqrt{5}$

It can easily be seen that

$$J_g(X) = \begin{pmatrix} 2+\cos x_1 & 0 \\ 0 & 2+\cos x_2 \end{pmatrix},$$

$\lambda_1(J_g) = 2+\cos x_1$, $\lambda_2(J_g) = 2+\cos x_2$, with $\delta_g = 1$ and $\Delta_g = 3$,

$$J_h(Y) = \begin{pmatrix} 1+\frac{1}{1+y_1^2} & 0 \\ 0 & 1+\frac{1}{1+y_2^2} \end{pmatrix},$$

$\lambda_1(J_h) = 1+\frac{1}{1+y_1^2}$, $\lambda_2(J_h) = 1+\frac{1}{1+y_2^2}$, with $\delta_h = 1$ and $\Delta_h = 2$, and lastly, it is obvious that vector $P(t,X,Y)$ above satisfies

$$\|P(t,X,Y)\| \le \sqrt{5}.$$

It will be seen from the Figure 1 obtained by Maple 16, that the simulated solutions of the differential equation constructed are bounded. This further justifies our given results.

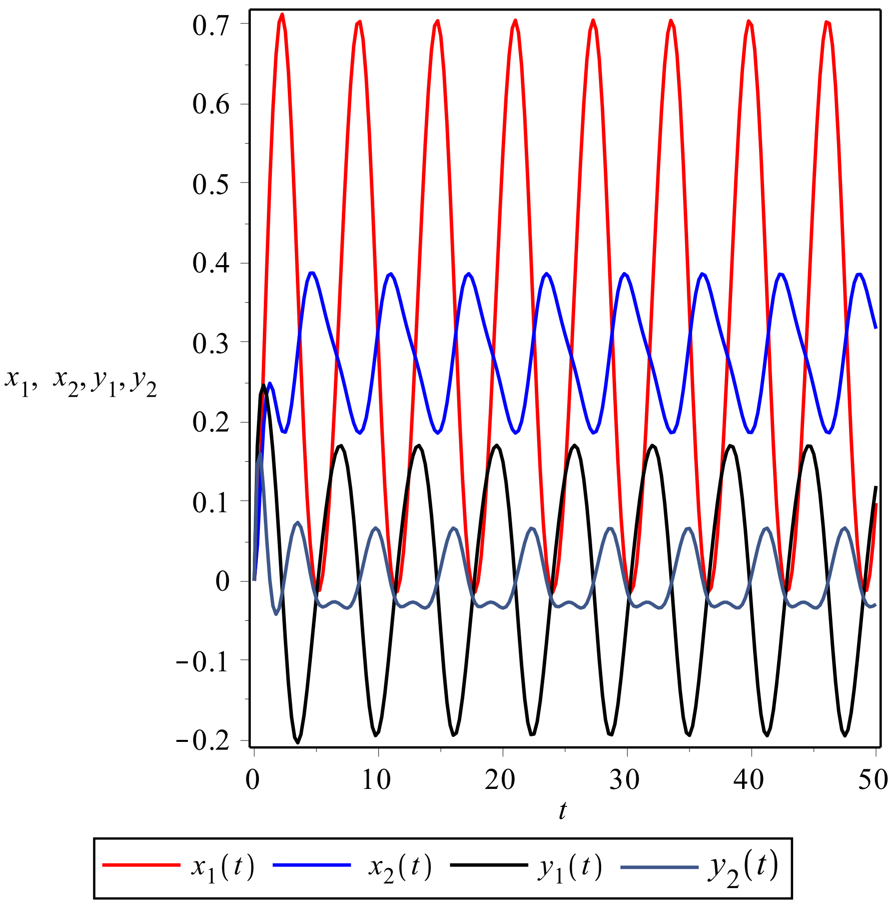

FIGURE 1. Solution paths of the given example.

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HARDY-TYPE INEQUALITIES FOR AN EXTENSION OF THE RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVE OPERATORS

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Abstract. In this paper we present variety of Hardy-type inequalities and their refinements for an extension of Riemann-Liouville fractional derivative operators. Moreover, we use an extension of extended Riemann-Liouville fractional derivative and modified extension of Riemann-Liouville fractional derivative using convex and monotone convex functions. Furthermore, mean value theorems and n-exponential convexity of the related functionals is discussed.

1. Introduction

The Hardy integral inequality is one of the most significant inequality in analysis with respect to its applications. In the recent years many researchers discover the new generalizations and refinements by involving fractional calculus operators (see [1, 4, 16]). Recently Iqbal et al. [8, 9] study applications of Hardy-type and refined Hardy-type inequalities involving different kinds of fractional integral operators. Here we give such type of inequalities for more general forms of Riemann-Liouville fractional integral operators using convex and monotone convex functions.

Let $(\Sigma_1, \Omega_1, \mu_1)$ and $(\Sigma_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Let $U(f, k)$ denote the class of functions $g : \Omega_1 \to \mathbb{R}$ with the representation

$$g(x) = \int_{\Omega_2} k(x, t) f(t) d\mu_2(t),$$

and A_k be an integral operator defined by

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$$A_k f(x) := \frac{g(x)}{K(x)} = \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \tag{1.1}$$

where $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ is measurable and non-negative kernel, $f : \Omega_2 \to \mathbb{R}$ is measurable function and

$$0 < K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t), \quad x \in \Omega_1. \tag{1.2}$$

The following definition is presented in [13].

Definition 1.1. Let I be an interval in $\mathbb{R}$. A function $\Phi : I \to \mathbb{R}$ is called convex if

$$\Phi(\lambda x + (1-\lambda)y) \leq \lambda \Phi(x) + (1-\lambda)\Phi(y), \tag{1.3}$$

for all points $x, y \in I$ and all $\lambda \in [0,1]$. The function Φ is strictly convex if inequality (1.3) holds strictly for all distinct points in I and $\lambda \in (0,1)$.

The upcoming theorem is given in [11].

Theorem 1.1. *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with positive σ-finite measures, u be a weight function on Ω_1, k be a non-negative measurable function on $\Omega_1 \times \Omega_2$ and K be defined on Ω_1 by* (1.2)*. Suppose $K(x) > 0$ for all $x \in \Omega_1$, that the function $x \mapsto u(x)\frac{k(x,t)}{K(x)}$ is integrable on Ω_1 for each fixed $t \in \Omega_2$ and that v is defined on Ω_2 by*

$$v(t) := \int_{\Omega_1} u(x) \frac{k(x,t)}{K(x)} \, d\mu_1(x) < \infty. \tag{1.4}$$

If Φ is a convex function on the interval $I \subseteq \mathbb{R}$, then the inequality

$$\int_{\Omega_1} u(x) \Phi(A_k f(x)) \, d\mu_1(x) \leq \int_{\Omega_2} v(t) \Phi(f(t)) \, d\mu_2(t) \tag{1.5}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ such that $\operatorname{Im} f \subseteq I$, where A_k is defined by (1.1)*.*

Substitute $k(x,t)$ by $k(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, 2$, are measurable functions in Theorem 1.1, we obtain [6, Theorem 2.1].

Definition 1.2. Let $\Phi : I \to \mathbb{R}$ be a convex function, then the sub-differential of Φ in x is denoted by $\partial\Phi(x)$ and is defined as

$$\partial\Phi(x) = \{y \in \mathbb{R} : y \text{ is the slope of a support line at } x\}.$$

Next result is given in [4].

Theorem 1.2. *Let the assumptions of Theorem* 1.1 *be satisfied. Moreover, if* Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$*, then the inequality*

$$\begin{aligned}&\int_{\Omega_2} v(t)\Phi(f(t))\, d\mu_2(t) - \int_{\Omega_1} u(x)\Phi(A_k f(x))\, d\mu_1(x)\\ \geq &\int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} k(x,t)\, \big|\, |\Phi(f(t)) - \Phi(A_k f(x))|\\ &- |\varphi(A_k f(x))| \cdot |f(t) - A_k f(x)|\, \big|\, d\mu_2(t)\, d\mu_1(x)\end{aligned}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ *such that* $f(t) \in I$ *for all* $t \in \Omega_2$.

If Φ *is a monotone convex function on an interval* $I \subseteq \mathbb{R}$*, then the inequality*

$$\begin{aligned}&\int_{\Omega_2} v(t)\Phi\left(f(t)\right) d\mu_2(t) - \int_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x)\\ \geq &\left| \int_{\Omega_1} \frac{u(x)}{K(x)} \int_{\Omega_2} sgn(f(t) - A_k f(x)) k(x,t)\, \left[\Phi(f(t)) - \Phi(A_k f(x))\right.\right.\\ &\left.\left. - |\varphi(A_k f(x))| \cdot (f(t) - A_k f(x))\right] d\mu_2(t)\, d\mu_1(x) \right|,\end{aligned}$$

holds for all measurable functions $f : \Omega_2 \to \mathbb{R}$ *such that* $f(t) \in I$ *for all fixed* $t \in \Omega_2$*, where* $A_k f$ *is defined by* (1.1).

Next mean value theorem is given in [5].

Theorem 1.3. *Let* $(\Omega_1, \Sigma_1, \mu_1)$*,* $(\Omega_2, \Sigma_2, \mu_2)$ *be measure spaces with* σ*-finite measures and* $u : \Omega_1 \to \mathbb{R}$ *be a weight function. Let* I *be a compact interval of* $\mathbb{R}$*,* $\tilde{h} \in C^2(I)$ *and* $f : \Omega_2 \to \mathbb{R}$ *a measurable function such that* $\operatorname{Im} f \subseteq I$*. Then there exists* $\eta \in I$ *such that*

$$\begin{aligned}&\int_{\Omega_2} v(t)\tilde{h}(f(t))\, d\mu_2(t) - \int_{\Omega_1} u(x)\tilde{h}(A_k f(x))\, d\mu_1(x)\\ =&\frac{\tilde{h}''(\eta)}{2} \left[\int_{\Omega_2} v(t) f^2(t)\, d\mu_2(t) - \int_{\Omega_1} u(x)(A_k f(x))^2\, d\mu_1(x) \right],\end{aligned}$$

where $A_k f$ *and* v *are defined by* (1.1) *and* (1.4)*, respectively.*

The definition of exponentially convex function is given in [3] by Bernstein.

Definition 1.3. A function $\Phi \colon (a,b) \to \mathbb{R}$ is *exponentially convex* if it is continuous and

$$\sum_{i,j=1}^{n} t_i t_j \Phi(x_i + x_j) \geq 0,$$

for all $n \in \mathbb{N}$ and all sequences $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ of real numbers, such that $x_i + x_j \in (a,b)$, $1 \leq i,j \leq n$.

Lemma 1.1. *Let $s \in \mathbb{R}$ and let the function $\varphi_s \colon (0,\infty) \to \mathbb{R}$ be defined by*

$$\varphi_s(x) = \begin{cases} \dfrac{x^s}{s(s-1)}, & s \neq 0,1, \\ -\log x, & s = 0, \\ x \log x, & s = 1. \end{cases} \tag{1.6}$$

Then $\varphi_s''(x) = x^{s-2}$, that is, φ_s is a convex function.

The upcoming theorem is presented in [5].

Theorem 1.4. *Let the conditions of Theorem 1.1 be satisfied and φ_s be defined by (1.6). Let f be a positive function. Then the function $\xi : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi(s) = \int\limits_{\Omega_2} v(t)\varphi_s(f(t))\, d\mu_2(t) - \int\limits_{\Omega_1} u(x)\varphi_s(A_k f(x))\, d\mu_1(x)$$

is exponentially convex.

Theorem 1.5. *Let the conditions of Theorem 1.3 be satisfied. Moreover, $k, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$ and*

$$\int\limits_{\Omega_2} v(t)\, \tilde{h}(f(t))\, d\mu_2(t) - \int\limits_{\Omega_1} u(x)\, \tilde{h}(A_k f(x))\, d\mu_1(x) \neq 0.$$

Then there exists $\eta \in I$ such that it holds

$$\frac{k''(\eta)}{\tilde{h}''(\eta)} = \frac{\int\limits_{\Omega_2} v(t)\, k(f(t))\, d\mu_2(t) - \int\limits_{\Omega_1} u(x)\, k(A_k f(x))\, d\mu_1(x)}{\int\limits_{\Omega_2} v(t)\, \tilde{h}(f(t))\, d\mu_2(t) - \int\limits_{\Omega_1} u(x)\, \tilde{h}(A_k f(x))\, d\mu_1(x)}.$$

By Theorem 1.1, and bearing in mind (1.5), we define the following positive linear functional:

$$\Delta(\Phi) = \int\limits_{\Omega_2} v(t)\Phi\left(f(t)\right) d\mu_2(t) - \int\limits_{\Omega_1} u(x)\Phi\left(A_k f(x)\right) d\mu_1(x). \tag{1.7}$$

Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a function. Then for distinct points $z_i \in I$, $i = 0,1,2$, the divided differences of first and second order are defined by

$$\begin{aligned} [z_i, z_{i+1}; f] &= \frac{f(z_{i+1}) - f(z_i)}{z_{i+1} - z_i}, \quad i = 0,1, \\ [z_0, z_1, z_2; f] &= \frac{[z_1, z_2; f] - [z_0, z_1; f]}{z_2 - z_0}. \end{aligned} \tag{1.8}$$

The values of the divided differences are independent of the order of points z_0, z_1, z_2 and may be extended to include the cases when some or all points are equal, that is $[z_0, z_0; f] = \lim_{z_1 \to z_0} [z_0, z_1; f] = f'(z_0)$, provided that f' exists.

Now, passing through the limit $z_1 \to z_0$ and replacing z_2 by z in (1.8), we have

$$[z_0, z_0, z; f] = \lim_{z_1 \to z_0} [z_0, z_1, z; f] = \frac{f(z) - f(z_0) - (z - z_0) f'(z_0)}{(z - z_0)^2}, \quad z \neq z_0,$$

provided that f' exists. Also, passing to the limit $z_i \to z$, $i = 0,1,2$, in (1.8), we have

$$[z,z,z;f] = \lim_{z_i \to z}[z_0,z_1,z_2;f] = \frac{f''(z)}{2},$$

provided that f'' exists.

One can observe that if for all $z_0, z_1 \in I$, $[z_0, z_1, f] \geq 0$, then f is increasing on I and if for all $z_0, z_1, z_2 \in I$, $[z_0, z_1, z_2; f] \geq 0$, then f is convex on I.

Next, we recall the notion of n-exponential convexity given in [15].

Definition 1.4. For any open interval I of $\mathbb{R}$, the function $\Phi : I \to \mathbb{R}$ is n-exponentially convex in the Jensen sense on I if

$$\sum_{i,j=1}^{n} t_i t_j \Phi\left(\frac{\zeta_i + \zeta_j}{2}\right) \geq 0$$

holds for all choices of $t_i \in \mathbb{R},\ \zeta_i \in I,\ i = 1,\ldots,n$.

A function $\Phi : I \to \mathbb{R}$ is n-exponentially convex on I if it is n-exponentially convex in the Jensen sense and continuous on I.

The following theorem is given in [7].

Theorem 1.6. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points $z_0, z_1, z_2 \in I$. Let Δ be linear functionals defined by* (1.7). *Then the function $p \mapsto \Delta(\Phi_p)$ is n-exponentially convex in the Jensen sense on J, if it is continuous on J.*

2. Hardy-Type Inequalities for Fractional Derivative

We begin with the well known definition of Riemann-Liouville fractional derivative od order μ is defined ([10, 19]) by

$$\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}dt, \quad \mathrm{Re}(\mu) > 0. \tag{2.1}$$

For the case $m-1 < \mathrm{Re}(\mu) < m$, $\mathrm{Re}(\mu) > 0$, where $m = 1,2,\ldots$, it follows

$$\mathfrak{D}_x^{\mu}\{f(x)\} = \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\{f(x)\} = \frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}dt\right\} \tag{2.2}$$

and

$$\mathfrak{D}_x^{\mu}\{x^{\sigma}\} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\mu+1)}x^{\sigma-\mu}, \quad \mathrm{Re}(\sigma) > -1.$$

The extended Riemann-Liouville fractional derivative of order μ is defined in [14] by

$$\mathfrak{D}_x^{\mu}\{f(x);p\} = \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}\exp\left(-\frac{px^2}{t(x-t)}\right)dt, \quad \mathrm{Re}(\mu) > 0. \tag{2.3}$$

For the case $m-1<\mathrm{Re}(\mu)<m$, where $m=1,2,\dots$, it follows

$$\begin{aligned}(2.4)\quad \mathfrak{D}_x^{\mu}\{f(x);p\} &= \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\{f(x);p\}\\ &=\frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}\exp\left(-\frac{px^2}{t(x-t)}\right)dt\right\},\quad \mathrm{Re}(\mu)>0.\end{aligned}$$

An extension of fractional derivative operator established in [2] is given by

$$(2.5)\quad \mathfrak{D}_x^{\mu}\{f(x);p,q\}=\frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}\exp\left(-\frac{px}{t}-\frac{qx}{(x-t)}\right)dt,\quad \mathrm{Re}(\mu)>0.$$

For example

$$\mathfrak{D}_x^{\mu}\{f(x);p,q\}_{x=1}=\frac{B_{p,q}(\nu+1,\mu)}{\Gamma(-\mu)},$$

where $B_{p,q}(\nu+1,\mu)$ is the extended beta functions (see [12]) defined by

$$B_{p,q}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t}-\frac{q}{1-t}}dt,\quad x,y,p,q\in\mathbb{C},\mathrm{Re}(p)>0,\mathrm{Re}(q)>0.$$

For $p=q$ we denote $B_{p,q}$ by B_p and for $p=q=0$ we get the classical beta function defined by

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt,\quad \mathrm{Re}(x)>0,\mathrm{Re}(y)>0.$$

Theorem 2.1. *Let* $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$ *and* $\mathrm{Re}(\mu)>0$. *Let* $\mathfrak{D}_x^{\mu}\{f(x);p,q\}$ *denotes the extension of Riemann-Liouville fractional derivative of order* μ *and let* u *be a weight function defined on* $(0,b)$. *For each fixed* $t\in(0,b)$, *define a function* $\tilde{v}$ *by*

$$(2.6)\qquad \tilde{v}(t)=\int_t^b u(x)\frac{(x-t)^{-\mu-1}\exp\left(-\frac{px}{t}-\frac{qx}{(x-t)}\right)}{x^{-\mu}B_{p,q}(1,-\mu)}dx<\infty.$$

If Φ *is a convex function on the interval* $I\in\mathbb{R}$, *then the inequality*

$$(2.7)\qquad \int_0^b u(x)\Phi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx\le\int_0^b\tilde{v}(t)\Phi(f(t))dt$$

holds true for all measurable functions $f\in L(a,b)$.

Proof. Applying Theorem 1.1 with $\Omega_1=\Omega_2=(0,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$,

$$(2.8)\qquad \tilde{k}(x,t)=\begin{cases}\frac{1}{\Gamma(-\mu)}(x-t)^{-\mu-1}\exp\left(-\frac{px}{t}-\frac{qx}{(x-t)}\right), & 0\le t\le x,\\ 0, & x<t\le b,\end{cases}$$

$$\tilde{K}(x) = \frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} \exp\left(-\frac{px}{t} - \frac{qx}{(x-t)}\right) dt = \frac{x^{-\mu} B_{p,q}(1,-\mu)}{\Gamma(-\mu)}$$

and

$$A_k f(x) = \frac{\Gamma(-\mu) x^{\mu} \mathfrak{D}_x^{\mu}\{f(x); p, q\}}{B_{p,q}(1,-\mu)},$$

we get inequality (2.7). □

Substitute $\tilde{k}(x,t)$ by $\tilde{k}(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, 2$, are measurable functions in Theorem 2.1 we obtain the following result.

Theorem 2.2. *Let* $\mathrm{Re}(p) > 0$*,* $\mathrm{Re}(q) > 0$ *and* $\mathrm{Re}(\mu) > 0$*. Let* $\mathfrak{D}_x^{\mu}\{f(x); p, q\}$ *denotes the extension of Riemann-Liouville fractional derivative of order* μ *and let* u *be a weight function defined on* $(0, b)$*. For each fixed* $t \in (0, b)$*, define a function*

$$\tilde{p}(t) := \frac{f_2(t)}{\Gamma(-\mu)} \int_t^b u(x) \frac{(x-t)^{-\mu-1} \exp\left(-\frac{px}{t} - \frac{qx}{(x-t)}\right)}{\mathfrak{D}_x^{\mu}\{f_2(x); p, q\}} dx < \infty.$$

If $\Phi : I \to \mathbb{R}$ *is a convex function and* $\frac{\mathfrak{D}_x^{\mu}\{f_1(x);p,q\}}{\mathfrak{D}_x^{\mu}\{f_2(x);p,q\}}, \frac{f_1(t)}{f_2(t)} \in I$*, then the inequality*

$$\int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f_1(x); p, q\}}{\mathfrak{D}_x^{\mu}\{f_2(x); p, q\}}\right) dx \le \int_0^b \tilde{p}(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right) dt \tag{2.9}$$

holds true.

New refined weighted Hardy-type inequality for extension of Riemann-Liouville fractional derivative (2.5) is given in the next theorem.

Theorem 2.3. *Let the assumptions of Theorem* 2.1 *be satisfied. Moreover, if* Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \mathrm{Int}\, I$*, then the inequality*

$$\begin{aligned}
&\int_0^b \tilde{v}(t)\Phi(f(t))dt - \int_0^b u(x)\Phi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x); p, q\}}{B_{p,q}(1,-\mu)}\right) dx \\
&\ge \int_0^b \frac{u(x)}{x^{-\mu} B_{p,q}(1,-\mu)} \int_0^x (x-t)^{-\mu-1} \exp\left(-\frac{px}{t} - \frac{qx}{(x-t)}\right) \\
&\times \left| \left| \Phi(f(t)) - \Phi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x); p, q\}}{B_{p,q}(1,-\mu)}\right) \right| \right. \\
&- \left. \left| \varphi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x); p, q\}}{B_{p,q}(1,-\mu)}\right) \right| \left| f(t) - \frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x); p, q\}}{B_{p,q}(1,-\mu)} \right| \right| dtdx
\end{aligned}$$

holds for all measurable functions $f:\Omega_2\to\mathbb{R}$. *If* Φ *is a monotone convex function on an interval* $I\subseteq\mathbb{R}$, *then the inequality*

$$\begin{aligned}&\int_0^b \tilde{v}(t)\Phi\left(f(t)\right)dt-\int_0^b u(x)\Phi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx\\ \geq&\left|\int_0^b \frac{u(x)}{x^{-\mu}B_{p,q}(1,-\mu)}\int_0^x \operatorname{sgn}\left(f(t)-\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)\right.\\ &\times(x-t)^{-\mu-1}\exp\left(-\frac{px}{t}-\frac{qx}{(x-t)}\right)\left[\Phi(f(t))-\Phi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)\right.\\ &\left.\left.-\left|\varphi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)\right|\left(f(t)-\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)\right]dt\,dx\right|\end{aligned}$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$.

Proof. Similar to Theorem 2.1 by applying Theorem 1.2. □

Next we give the mean value theorems for extension of Riemann-Liouville fractional derivative of order μ.

Theorem 2.4. *Let the assumptions of Theorem* 2.1 *be satisfied. Let* I *be a compact interval of* $\mathbb{R}$, $\tilde{h}\in C^2(I)$ *and* $f:(0,b)\to\mathbb{R}$ *a measurable function such that* $\operatorname{Im} f\subseteq I$. *Then there exists* $\eta\in I$ *such that*

$$\begin{aligned}&\int_0^b \tilde{v}(t)\tilde{h}(f(t))\,dt-\int_0^b u(x)\tilde{h}\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx\\ =&\frac{\tilde{h}''(\eta)}{2}\left[\int_0^b \tilde{v}(t)f^2(t)\,dt-\int_0^b u(x)\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)^2dx\right],\end{aligned}$$

where $\tilde{v}$ *is defined by* (2.6).

Proof. Similar to proof of Theorem 2.1, by applying Theorem 1.3. □

Theorem 2.5. *Let the assumptions of Theorem* 2.4 *be satisfied. Moreover,* $k,\tilde{h}\in C^2(I)$ *such that* $\tilde{h}''(x)\neq 0$ *for every* $x\in I$ *and*

$$\int_0^b \tilde{v}(t)\,\tilde{h}(f(t))\,dt-\int_0^b u(x)\,\tilde{h}\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx\neq 0.$$

Then there exists $\eta\in I$ *such that it holds*

$$\frac{k''(\eta)}{\tilde{h}''(\eta)}=\frac{\int_0^b \tilde{v}(t)\,k(f(t))\,dt-\int_0^b u(x)\,k\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx}{\int_0^b \tilde{v}(t)\,\tilde{h}\left(f(t)\right)\,dt-\int_0^b u(x)\,\tilde{h}\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx}.$$

Proof. Similar to proof of Theorem 2.1, by applying Theorem 1.5. □

Theorem 2.6. *Let the conditions of Theorem* 2.1 *be satisfied and φ_s be defined by* (1.6). *Let f be a positive function. Then the function $\xi : \mathbb{R} \to [0, \infty)$ defined by*

$$\xi(s) = \int_0^b \tilde{v}(t)\varphi_s(f(t))\,dt - \int_0^b u(x)\varphi_s\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)dx \tag{2.10}$$

is exponentially convex.

Proof. Applying Theorem 1.4, with $\Omega_1 = \Omega_2 = (a,b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$ and $\tilde{k}(x,t)$ given in (2.8), we get the exponential convexity of linear functional (2.10). □

3. Hardy-Type Inequalities for Extension of Extended Riemman-Liouvill Fractional Derivative

Recently Rehaman et al. [17] define an extension of extended Riemman-Liouvill fractional derivative of order μ as

$$\begin{aligned}\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\} =& \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right] \\ &\times {}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]dt, \quad \mathrm{Re}(\mu) > 0.\end{aligned} \tag{3.1}$$

For the case $m-1 < \mathrm{Re}(\mu) < m$, where $m = 1, 2, \dots$, it follows

$$\begin{aligned}\mathfrak{D}_x^{\mu}\{f(x);p;\lambda;\rho\} =& \frac{d^m}{dx^m}\mathfrak{D}_x^{\mu-m}\{f(x);p;q;\lambda;\rho\} \\ =& \frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^x f(t)(x-t)^{-\mu+m-1}\right. \\ &\left.\times {}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]dt\right\},\end{aligned}$$

where $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$. It is clear that $\lambda = \rho$, then (3.1) reduces to (2.5).

Theorem 3.1. *Let* $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}\,\mu) > 0$, $\mathrm{Re}(\lambda) > 0$ *and* $\mathrm{Re}(\rho) > 0$. *Let $\mathfrak{D}_x^{\mu}\{f(x);p,q,\lambda,\rho\}$ be the extension of extended Riemman-Liouvill fractional derivative of order μ. Let u be a weight function defined on $(0,b)$, then $\bar{v}$ is defined by*

$$\bar{v}(t) = \int_t^b u(x)\frac{(x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]}{\int_0^x (x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]}dx < \infty. \tag{3.2}$$

If Φ is a convex function on the interval I, then the inequality

$$\int_0^b u(x)\,\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx \leq \int_0^b \bar{v}(t)\,\Phi\left(f(t)\right)dt \tag{3.3}$$

holds true.

Proof. Applying Theorem 1.1, with $\Omega_1=\Omega_2=(0,b)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$,

$$\bar{k}(x,t)=\begin{cases}\frac{1}{\Gamma(-\mu)}(x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\,{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right], & 0\leq t\leq x,\\ 0, & x<t\leq b,\end{cases}$$

$$\bar{K}(x)=\frac{1}{\Gamma(-\mu)}\int_0^x (x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\,{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right],$$

and $\bar{v}$ as in (3.2), we get inequality (3.3). □

Substitute $\bar{k}(x,t)$ by $\bar{k}(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$ where $f_i:\Omega_2\to\mathbb{R}$, $i=1,2$, are measurable functions in Theorem 3.1 we obtain the following result.

Theorem 3.2. *Let $D_x^{\mu}\{f(x);p,q,\lambda,\rho\}$ be the fractional derivative operator of order μ. Let u be a weight function defined on $(0,b)$ and for each fixed $t\in(0,b)$ define $\bar{p}$ on $(0,b)$ as*

$$\bar{p}(t):=\frac{f_2(t)}{\Gamma(-\mu)}\int_t^b u(x)\frac{(x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\,{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]}{\mathfrak{D}_x^{\mu}\{f_2(x);p,q;\lambda;\rho\}(x)}dx<\infty.$$

If $\Phi:I\to\mathbb{R}$ is a convex function, then the inequality

$$\int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f_1(x);p,q;\lambda;\rho\}}{\mathfrak{D}_x^{\mu}\{f_2(x);p,q;\lambda;\rho\}}\right)dx\leq\int_0^b \bar{p}(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right)dt \tag{3.4}$$

holds true for all $f_i\in L^1[a,b]$.

Theorem 3.3. *Let* $\mathrm{Re}(p)>0$, $\mathrm{Re}(q)>0$, $\mathrm{Re}(\mu)>0$, $\mathrm{Re}(\lambda)>0$ *and* $\mathrm{Re}(\rho)>0$. *Let $D_x^{\mu}\{f(x);p,q,\lambda,\rho\}$ be the extension of extended Riemman-Liouvill fractional derivative of order μ. Let u be a weight function defined on $(0,b)$. Moreover, if Φ is a convex function on an interval $I\subseteq\mathbb{R}$ and $\varphi:I\to\mathbb{R}$ is any function, such that $\varphi(x)\in\partial\Phi(x)$ for all $x\in\mathrm{Int}I$ and $\bar{v}$ as in* (3.2)*, then the inequality*

$$\int_0^b \bar{v}(t)\Phi\left(f(t)\right)dt-\int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx$$

$$\geq\frac{1}{\Gamma(-\mu)}\int_0^b\frac{u(x)}{\bar{K}(x)}\int_a^x (x-t)^{-\mu-1}\,{}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\,{}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]$$

$$\times \left| \left| \Phi\left(f(t)\right) - \Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right| - \left| \varphi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right| \cdot \left| f(t) - \left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right| \right| dtdx$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$*, such that* $f(t)\in I$ *for all* $t\in(a,b)$*.*

If Φ *is a monotone convex function on an interval* $I\subseteq\mathbb{R}$*, then the inequality*

$$\int_0^b \bar{v}(t)\Phi\left(f(t)\right)dt - \int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx$$
$$\geq \left| \frac{1}{\Gamma(-\mu)} \int_0^b \frac{u(x)}{\bar{K}(x)} \int_a^x \operatorname{sgn}\left(f(t) - \frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right.$$
$$\times (x-t)^{-\mu-1}\, {}_1F_1\left[\lambda;\rho;-\frac{px}{t}\right]\, {}_1F_1\left[\lambda;\rho;-\frac{qx}{(x-t)}\right]$$
$$\times \left[\Phi\left(f(t)\right) - \Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right.$$
$$\left.\left. - \left| \varphi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right| \cdot \left(f(t) - \frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right) \right] dt\,dx \right|$$

holds for all measurable functions $f:(0,b)\to\mathbb{R}$*.*

Proof. Similar to proof of Theorem 3.1, by applying Theorem 1.2. □

Theorem 3.4. *Let* $\mathrm{Re}(p)>0$*,* $\mathrm{Re}(q)>0$*,* $\mathrm{Re}(\mu)>0$*,* $\mathrm{Re}(\lambda)>0$ *and* $\mathrm{Re}(\rho)>0$*. Let* $D_x^{\mu}\{f(x);p,q,\lambda,\rho\}$ *be the extension of extended Riemman-Liouvill fractional derivative of order* μ*, and* I *a compact interval of* $\mathbb{R}$*,* $\tilde{h}\in C^2(I)$ *and let* $f:(0,b)\to\mathbb{R}$ *be a measurable function such that* $\operatorname{Im} f\subseteq I$*. Then for the weight function* u *defined on* $(0,b)$ *there exists* $\eta\in I$ *such that*

$$\int_0^b \bar{v}(t)\tilde{h}\left(f(t)\right)dt - \int_0^b u(x)\tilde{h}\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx$$
$$= \frac{\tilde{h}''(\eta)}{2}\left[\int_0^b \bar{v}(t)f^2(t)\,dt - \int_0^b u(x)\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)^2 dx\right],$$

where $\bar{v}$ *is defined by* (3.2)*.*

Proof. Similar to proof of Theorem 3.1, by applying Theorem 1.3. □

Theorem 3.5. *Let* $\mathrm{Re}(p)>0$*,* $\mathrm{Re}(q)>0$*,* $\mathrm{Re}(\mu)>0$*,* $\mathrm{Re}(\lambda)>0$ *and* $\mathrm{Re}(\rho)>0$*. Let the extension of extended Riemman-Liouvill fractional derivative* $D_x^{\mu}\{f(x);p,q,\lambda,\rho\}$ *of order* μ *and* I *a compact interval of* $\mathbb{R}$*,* $k,\tilde{h}\in C^2(I)$ *such that* $\tilde{h}''(x)\neq 0$ *for every*

$x \in I$. Moreover, $f:(0,b) \to \mathbb{R}$ a measurable function with $Imf \subseteq I$, u a weight function, $\bar{v}$ as in (3.2) and

$$\int_0^b \bar{v}(t)\,\tilde{h}\,(f(t))\,dt - \int_0^b u(x)\,\tilde{h}\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx \neq 0.$$

Then there exists $\eta \in I$ such that the following equality holds true

$$\frac{k''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_0^b \bar{v}(t)k\,(f(t))\,dt - \int_0^b u(x)\,k\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx}{\int_0^b \bar{v}(t)\,\tilde{h}\,(f(t))\,dt - \int_0^b u(x)\,\tilde{h}\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx}.$$

Proof. Similar to proof of Theorem 3.1. □

Theorem 3.6. *Let* $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$, $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(\lambda) > 0$ *and* $\mathrm{Re}(\rho) > 0$. *Let the fractional derivative operator $D_x^{\mu}\{f(x);p,q,\lambda,\rho\}$ of order μ and f a positive function and let u be a weight function defined on (a,b), $\bar{v}$ be as in (4.4). Then the function $\xi:\mathbb{R}\to[0,\infty)$ defined by*

$$\xi(s) = \int_0^b \bar{v}(t)\varphi_s\,(f(t))\,dt - \int_0^b u(x)\varphi_s\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)dx$$

is exponentially convex.

Proof. Similar to proof of Theorem 3.1, by applying Theorem 1.4. □

4. Inequalities for Modified Extension of Riemman-Liouvill Fractional Derivative

The following definition is given in [18].

Definition 4.1.

$$\mathfrak{D}_{z,p}^{\mu,\alpha}\{f(z)\} = \frac{1}{\Gamma(-\mu)}\int_0^z f(t)(z-t)^{-\mu-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)dt, \quad \mathrm{Re}(\mu) > 0, \tag{4.1}$$

where

$$E_{\alpha}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+1)}. \tag{4.2}$$

For the case $m-1 < \mathrm{Re}(\mu) < m$, where $m = 1,2,\dots$, it follows

$$\begin{aligned}\mathfrak{D}_{z,p}^{\mu,\alpha}\{f(z)\} &= \frac{d^m}{dx^m}\mathfrak{D}_{z,p}^{\mu-m,\alpha}\{f(z)\}\\ &= \frac{d^m}{dx^m}\left\{\frac{1}{\Gamma(-\mu+m)}\int_0^z f(t)(z-t)^{-\mu+m-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)dt\right\},\end{aligned} \tag{4.3}$$

where $\mathrm{Re}(\mu) > 0$, $\mathrm{Re}(p) > 0$, $\mathrm{Re}(q) > 0$.

Remark 4.1. Obviously if $\alpha = 1$, then (4.1) and (4.3) reduces to the extended fractional derivative (2.3) and (2.4), respectively. Similarly, if we set $\alpha = 1$ and $p = 0$, we get (2.1) and (2.2), respectively.

Very recently Shadab et al. [20] introduce new and modified extension of beta function as:

$$B_p^\alpha(\sigma_1, \sigma_2) = \int_0^1 t^{\sigma_1 - 1}(1-t)^{\sigma_2 - 1} E_\alpha\left(-\frac{p}{t(1-t)}\right),$$

where $\mathrm{Re}(\sigma_1) > 0$, $\mathrm{Re}(\sigma_2) > 0$ and $E_\alpha(\cdot)$ is defined by (4.2).

Theorem 4.1. *Let $\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(z)\}$ denotes the new and modified extension of Riemann-Liouville fractional derivative of order μ and let u be a weight function defined on $(0, b)$, then $\hat{v}$ is defined by*

$$\hat{v}(t) = \int_t^b u(x) \frac{(x-t)^{-\mu-1} E_\alpha\left(-\frac{pz^2}{t(z-t)}\right)}{x^{-\mu} B_p^\alpha(1, -\mu)} dx < \infty. \tag{4.4}$$

If Φ is a convex function on the interval I, then the inequality

$$\int_0^b u(x) \Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu} B_p^\alpha(1, -\mu)}\right) dx \le \int_0^b \hat{v}(t) \Phi(f(t))\, dt \tag{4.5}$$

holds true.

Proof. Applying Theorem 1.1, with $\Omega_1 = \Omega_2 = (0, b)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$\hat{k}(x,t) = \begin{cases} \frac{1}{\Gamma(-\mu)}(x-t)^{-\mu-1} E_\alpha\left(-\frac{pz^2}{t(z-t)}\right), & 0 \le t \le x, \\ 0, & x < t \le b, \end{cases}$$

$$\hat{K}(x) = \frac{1}{\Gamma(-\mu)} \int_0^x (x-t)^{-\mu-1} E_\alpha\left(-\frac{pz^2}{t(z-t)}\right) dt = \frac{1}{\Gamma(-\mu)} x^{-\mu} B_p^\alpha(1, -\mu)$$

and $\hat{v}$ as in (4.4), we get inequality (4.5). ☐

Substitute $\hat{k}(x,t)$ by $\hat{k}(x,t)f_2(t)$ and f by $\frac{f_1}{f_2}$, where $f_i : \Omega_2 \to \mathbb{R}$, $i = 1, 2$, are measurable functions in Theorem 4.1 we obtain the following result.

Theorem 4.2. *Let $\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(z)\}$ denotes the new and modified extension of Riemann-Liouville fractional derivative of order μ and let u be a weight function defined on $(0, b)$ and for each fixed $t \in (0, b)$ define $\hat{p}$ on $(0, b)$ as*

$$\hat{p}(t) := \frac{f_2(t)}{\Gamma(-\mu)} \int_t^b u(x) \frac{(x-t)^{-\mu-1} E_\alpha\left(-\frac{pz^2}{t(z-t)}\right)}{\mathfrak{D}_{x,p}^{\mu,\alpha}\{f_2(x)\}} dx < \infty.$$

If $\Phi : I \to \mathbb{R}$ is a convex function then the inequality

$$\int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_{x,p}^{\mu,\alpha}\{f_1(x)\}}{\mathfrak{D}_{x,p}^{\mu,\alpha}\{f_2(x)\}}\right) dx \le \int_a^b \hat{p}(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right) dt \tag{4.6}$$

holds true for all $f_i \in L^1[a,b]$.

Refinement of Theorem 4.1 is given in the upcoming theorem.

Theorem 4.3. *Let* $\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(z)\}$ *denotes the new and modified extension of Riemann-Liouville fractional derivative of order* μ *and let* u *be a weight function defined on* (a,b). *Moreover, if* Φ *is a convex function on an interval* $I \subseteq \mathbb{R}$ *and* $\varphi : I \to \mathbb{R}$ *is any function, such that* $\varphi(x) \in \partial\Phi(x)$ *for all* $x \in \operatorname{Int} I$ *and* $\hat{v}$ *as in* (4.4), *then the inequality*

$$\begin{aligned}
&\int_0^b \hat{v}(t)\Phi\left(f(t)\right)dt - \int_0^b u(x)\Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)dx\\
\geq &\int_0^b \frac{u(x)}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\int_a^x (x-t)^{-\mu-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)\\
&\times\left|\left|\Phi\left(f(t)\right)-\Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right|\right.\\
&-\left.\left|\varphi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right|\cdot\left|f(t)-\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right|\right|dtdx
\end{aligned}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$. *If* Φ *is a monotone convex function on an interval* $I \subseteq \mathbb{R}$, *then the inequality*

$$\begin{aligned}
&\int_0^b \hat{v}(t)\Phi\left(f(t)\right)dt - \int_0^b u(x)\Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)dx\\
\geq &\left|\int_0^b \frac{u(x)}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\int_0^x sgn\left(f(t)-\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right.\\
&\times(x-t)^{-\mu-1}E_{\alpha}\left(-\frac{pz^2}{t(z-t)}\right)\left[\Phi\left(f(t)\right)-\Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right.\\
&-\left.\left.\left|\varphi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right|\cdot\left(f(t)-\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)\right]dtdx\right|
\end{aligned}$$

holds for all measurable functions $f : (0,b) \to \mathbb{R}$.

Proof. Same as proof of Theorem 4.1, by applying Theorem 1.2. □

Next we give the mean value theorems.

Theorem 4.4. *Let* $\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(z)\}$ *denotes the new and modified extension of Riemann-Liouville fractional derivative of order* μ, I *be a compact interval of* $\mathbb{R}$, $\tilde{h} \in C^2(I)$ *and let* $f : (0,b) \to \mathbb{R}$ *be a measurable function such that* $\operatorname{Im} f \subseteq I$. *Then for the weight* *function u defined on $(0,b)$ there exists $\eta \in I$ such that*

$$\int_0^b \hat{v}(t)\tilde{h}\left(f(t)\right) dt - \int_0^b u(x)\tilde{h}\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right) dx$$

$$=\frac{\tilde{h}''(\eta)}{2}\left[\int_0^b \hat{v}(t)f^2(t)\,dt - \int_0^b u(x)\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right)^2 dx\right],$$

where $\hat{v}$ is defined by (4.4).

Proof. Similar to proof of Theorem 4.1, by applying Theorem 1.3. □

Theorem 4.5. *Let $\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(z)\}$ denotes the new and modified extension of Riemann-Liouville fractional derivative of order μ, I be a compact interval of $\mathbb{R}$, $k, \tilde{h} \in C^2(I)$ such that $\tilde{h}''(x) \neq 0$ for every $x \in I$. Moreover $f : (a,b) \to \mathbb{R}$ a measurable function with $\operatorname{Im} f \subseteq I$, u be a weight function, $\hat{v}$ as in* (4.4) *and*

$$\int_0^b \hat{v}(t)\,\tilde{h}\left(f(t)\right) dt - \int_0^b u(x)\,\tilde{h}\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right) dx \neq 0.$$

Then there exists $\eta \in I$ such that the following equality holds true

$$\frac{k''(\eta)}{\tilde{h}''(\eta)} = \frac{\int_0^b \hat{v}(t)k\left(f(t)\right) dt - \int_0^b u(x)\,k\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right) dx}{\int_0^b \hat{v}(t)\,\tilde{h}\left(f(t)\right) dt - \int_0^b u(x)\,\tilde{h}\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right) dx}.$$

Theorem 4.6. *Let $\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(z)\}$ denotes the new and modified extension of Riemann-Liouville fractional derivative of order μ and let u be a weight function defined on (a,b), $\hat{v}$ be as in* (4.4). *Then the function $\xi : \mathbb{R} \to [0,\infty)$ defined by*

$$\xi(s) = \int_0^b \hat{v}(t)\varphi_s\left(f(t)\right) dt - \int_0^b u(x)\varphi_s\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right) dx$$

is exponentially convex.

Proof. Similar to proof of Theorem 4.1, by applying Theorem 1.4. □

Now we shall discuss the exponentially convexity of the liner functional. Under the assumptions of the Theorem 2.1, Theorem 3.1 and Theorem 4.1 we define a linear functionals by taking the positive difference of the inequalities stated in (2.7), (3.3) and (4.5), respectively as:

$$\xi_1(\Phi) = \int_0^b \tilde{v}(t)\Phi(f(t))dt - \int_0^b \Phi\left(\frac{\Gamma(-\mu)x^{\mu}\mathfrak{D}_x^{\mu}\{f(x);p,q\}}{B_{p,q}(1,-\mu)}\right)u(x)\,dx, \tag{4.7}$$

$$\xi_2(\Phi) = \int_0^b \bar{v}(t)\Phi\left(f(t)\right)dt - \int_0^b \Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f(x);p,q;\lambda;\rho\}}{\bar{K}(x)}\right)u(x)\,dx \tag{4.8}$$

and

$$\xi_3(\Phi) = \int_0^b \hat{v}(t)\Phi\left(f(t)\right) dt - \int_0^b \Phi\left(\frac{\Gamma(-\mu)\mathfrak{D}_{x,p}^{\mu,\alpha}\{f(x)\}}{x^{-\mu}B_p^{\alpha}(1,-\mu)}\right) u(x)\, dx. \tag{4.9}$$

We also define a linear functionals by taking the positive difference of the left-hand side and right-hand side of the inequalities (2.9), (3.4) and (4.6), respectively as:

$$\xi_4(\Phi) = \int_0^b \tilde{p}(t)\Phi\left(\frac{f_1(t)}{f_2(t)}\right) dt - \int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f_1(x);p,q\}}{\mathfrak{D}_x^{\mu}\{f_2(x);p,q\}}\right) dx,$$

$$\xi_5(\Phi) = \int_0^b \bar{p}(t)\Phi\left(\frac{f_1(x)}{f_2(x)}\right) dt - \int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f_1(x);p,q,\lambda,\rho\}}{\mathfrak{D}_x^{\mu}\{f_2(x);p,q,\lambda,\rho\}}\right) dx$$

and

$$\xi_6(\Phi) = \int_0^b \hat{p}(t)\Phi\left(\frac{f_1(x)}{f_2(x)}\right) dt - \int_0^b u(x)\Phi\left(\frac{\mathfrak{D}_x^{\mu}\{f_1(x);p,q\}}{\mathfrak{D}_x^{\mu}\{f_2(x);p,q\}}\right) dx. \tag{4.10}$$

Theorem 4.7. *Let $\Gamma = \{\Phi_p : p \in J\}$ be a family of functions defined on I, such that the function $p \mapsto [z_0, z_1, z_2; \Phi_p]$ is n-exponentially convex in the Jensen sense on J for every three distinct points z_0, z_1, $z_2 \in I$. Let ξ_i, $i = 1, 2, \dots, 6$, be linear functionals defined by* (4.7)–(4.10)*, respectively. Then the function $p \mapsto \xi_i(\Phi_p)$, $i = 1, 2, \dots, 6$, is n-exponentially convex in the Jensen sense on J. If the function $p \mapsto \xi_i(\Phi_p)$ is continuous on J, then it is n-exponentially convex on J.*

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ON $\breve{\phi}$-SEMISYMMETRIC LP-KENMOTSU MANIFOLDS WITH A QSNM-CONNECTION ADMITTING RICCI SOLITONS

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Abstract. In the present work, we characterize Lorentzian para-Kenmotsu (briefly, LP-Kenmotsu) manifolds with a quarter-symmetric non-metric connection (briefly, QSNM-connection) $\widehat{\nabla}$ satisfying certain $\breve{\phi}$-semisymmetric conditions admitting Ricci solitions. At the end of the paper, a 3-dimensional example of LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$ is given to verify some results of the present paper.

1. Introduction

In a $(2n+1)$-dimensional connected and C^∞-smooth semi-Riemannian manifold $(M,\breve{g})$, the Levi-Civita connection $\breve{\nabla}$, the Riemannian-Christoffel curvature tensor $\breve{R}$, the projective curvature tensor $\breve{P}$, the concircular curvature tensor $\breve{V}$, the conformal curvature tensor $\breve{C}$ and the D-conformal curvature tensor $\breve{B}$ are defined by [5, 6]

(1.1)
$$\breve{R}(\breve{E},\breve{F})\breve{W}=\breve{\nabla}_{\breve{E}}\breve{\nabla}_{\breve{F}}\breve{W}-\breve{\nabla}_{\breve{F}}\breve{\nabla}_{\breve{E}}\breve{W}-\breve{\nabla}_{[\breve{E},\breve{F}]}\breve{W},$$
(1.2)
$$\breve{P}(\breve{E},\breve{F})\breve{W}=\breve{R}(\breve{E},\breve{F})\breve{W}-\frac{1}{2n}[\breve{S}(\breve{F},\breve{W})\breve{E}-\breve{S}(\breve{E},\breve{W})\breve{F}],$$
(1.3)
$$\breve{V}(\breve{E},\breve{F})\breve{W}=\breve{R}(\breve{E},\breve{F})\breve{W}-\frac{\breve{r}}{2n(2n+1)}[\breve{g}(\breve{F},\breve{W})\breve{E}-\breve{g}(\breve{E},\breve{W})\breve{F}],$$

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(1.4)

$$\check{C}(\check{E},\check{F})\check{W} = \check{R}(\check{E},\check{F})\check{W} - \frac{1}{(2n-1)}[\check{S}(\check{F},\check{W})\check{E} - \check{S}(\check{E},\check{W})\check{F} + \check{g}(\check{F},\check{W})\check{Q}\check{E} - \check{g}(\check{E},\check{W})\check{Q}\check{F}] + \frac{\check{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{E} - \check{g}(\check{E},\check{W})\check{F}],$$

(1.5)

$$\begin{aligned}\check{B}(\check{E},\check{F})\check{W} =& \check{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\check{S}(\check{E},\check{W})\check{F} - \check{S}(\check{F},\check{W})\check{E} + \check{g}(\check{E},\check{W})\check{Q}\check{F}\\ &- \check{g}(\check{F},\check{W})\check{Q}\check{E} - \check{S}(\check{E},\check{W})\check{\eta}(\check{F})\xi + \check{S}(\check{F},\check{W})\check{\eta}(\check{E})\xi - \check{\eta}(\check{E})\check{\eta}(\check{W})\check{Q}\check{F}\\ &+ \check{\eta}(\check{F})\check{\eta}(\check{W})\check{Q}\check{E}] - \frac{k-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{F} - \check{g}(\check{F},\check{W})\check{E}] + \frac{k}{2(n-1)}\\ &\times [\check{g}(\check{E},\check{W})\check{\eta}(\check{F})\xi - \check{g}(\check{F},\check{W})\check{\eta}(\check{E})\xi + \check{\eta}(\check{E})\check{\eta}(\check{W})\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{E}],\end{aligned}$$

respectively, where $\check{r}$ is the scalar curvature, $\check{S}$ and $\check{Q}$ are the Ricci tensor and the Ricci operator, respectively such that $\check{S}(\check{E},\check{F}) = \check{g}(\check{Q}\check{E},\check{F})$ and $k = \frac{\check{r}+4n}{2n-1}$.

The connection $\widehat{\nabla}$ which is linear and defined on $(M,\check{g})$ is said to be a quarter-symmetric [11] if its torsion tensor $\check{T}$

$$\check{T}(\check{E},\check{F}) = \widehat{\nabla}_{\check{E}}\check{F} - \widehat{\nabla}_{\check{F}}\check{E} - [\check{E},\check{F}] = \check{\eta}(\check{F})\check{\phi}\check{E} - \check{\eta}(\check{E})\check{\phi}\check{F}, \tag{1.6}$$

where $\check{\phi}$ is a $(1,1)$-tensor field and $\check{\eta}$ is a 1-form. If moreover, $\widehat{\nabla}$ satisfies the condition

$$(\widehat{\nabla}_{\check{E}}\check{g})(\check{F},\check{W}) = -\check{\eta}(\check{F})\check{g}(\check{\phi}\check{E},\check{W}) - \check{\eta}(\check{W})\check{g}(\check{F},\check{\phi}\check{E}), \tag{1.7}$$

where $\check{E},\check{F},\check{W} \in \chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on M, then connection $\widehat{\nabla}$ is called a QSNM-connection. The authors in [2, 3, 7, 12] have studied QSNM-connection in various manifolds.

In an LP-Kenmotsu manifold, a relation between the connections $\widehat{\nabla}$ and $\check{\nabla}$ is given by

$$\widehat{\nabla}_{\check{E}}\check{F} = \check{\nabla}_{\check{E}}\check{F} + \check{\eta}(\check{F})\check{\phi}\check{E}. \tag{1.8}$$

On a Riemannian manifold $(M,\check{g})$, a Ricci soliton $(\check{g},U,\check{\lambda})$ is a generalization of an Einstein metric such that (see [9, 10]) $\check{\pounds}_U\check{g} + 2\check{S} + 2\check{\lambda}\check{g} = 0$, where $\check{S}$, $\check{\pounds}_U$ and $\check{\lambda}$ are the Ricci tensor, the Lie derivative operator along the vector field U on M and a real constant, respectively. A Ricci soliton is said to be shrinking, steady or expanding according as $\check{\lambda} < 0$, $\check{\lambda} = 0$ or $\check{\lambda} > 0$, respectively.

The present work is arranged in the following manner. After Introduction, a brief introducton of LP-Kenmotsu manifolds is given in Section 2. In Section 3, we find the relation between the curvature tensors of an LP-Kenmotsu manifold with the connections $\check{\nabla}$ and $\widehat{\nabla}$. In Section 4, we study LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$ admitting Ricci solitons. $\check{\phi}$-projectively semisymmetric, $\check{\phi}$-concircularly semisymmetric, $\check{\phi}$-conformally semisymmetric and $\check{\phi}$-D-conformally semisymmetric LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$ admitting Ricci solitons have been studied in Section 5. At the end of the paper, a 3-dimensional example of *LP*-Kenmotsu manifolds with a connection $\widehat{\nabla}$ is given to verify some results of the present paper.

2. Preliminaries

A $(2n+1)$-dimensionsional differentiable manifold M with structure $(\check{\phi}, \xi, \check{\eta}, \check{g})$ is said to be a Lorentzian almost paracontact metric manifold, if it admits $\check{\phi}$: a tensor field of type $(1,1)$, ξ: a contravariant vector field, $\check{\eta}$: a 1-form and $\check{g}$: a Lorentzian metric satisfying [8]

$$\check{\eta}(\xi) = -1, \tag{2.1}$$

$$\check{\phi}^2\check{E} = \check{E} + \check{\eta}(\check{E})\xi, \tag{2.2}$$

$$\check{\phi}\xi = 0, \quad \check{\eta}(\check{\phi}Y) = 0, \tag{2.3}$$

$$\check{g}(\check{\phi}\check{E}, \check{\phi}\check{F}) = \check{g}(\check{E}, \check{F}) + \check{\eta}(\check{E})\check{\eta}(\check{F}),$$

$$\check{g}(\check{E}, \xi) = \check{\eta}(\check{E}), \tag{2.4}$$

$$\check{\Phi}(\check{F}, \check{E}) = \check{\Phi}(\check{E}, \check{F}) = \check{g}(\check{E}, \check{\phi}\check{F}), \tag{2.5}$$

for any $\check{E}, \check{F}$ on M.

For ξ: a killing vector field, the (para) contact structure is said to be a K-(para) contact. In this case, we have

$$\check{\nabla}_{\check{E}}\xi = \check{\phi}\check{E}. \tag{2.6}$$

A Lorentzian almost paracontact manifold M is called an *LP*-Sasakian manifold if

$$(\check{\nabla}_{\check{E}}\check{\phi})\check{F} = \check{g}(\check{E}, \check{F})\xi + \check{\eta}(\check{F})\check{E} + 2\check{\eta}(\check{E})\check{\eta}(\check{F})\xi,$$

for any $\check{E}, \check{F}$ on M.

Now, we define a new manifold called a Lorentzian para-Kenmostu (briefly, *LP*-Kenmotsu) manifold:

Definition 2.1. A Lorentzian almost paracontact manifold is called Lorentzian para-Kenmostu (briefy, *LP*-Kenmostu) manifold if [1]

$$(\check{\nabla}_{\check{E}}\check{\phi})\check{F} = -\check{g}(\check{\phi}\check{E}, \check{F})\xi - \check{\eta}(\check{F})\check{\phi}\check{E}, \tag{2.7}$$

for any $\check{E}, \check{F}$ on M.

In the Lorentzian para-Kenmostu manifold, we have

$$\check{\nabla}_{\check{E}}\xi = -\check{\phi}^2\check{E},$$

$$(\check{\nabla}_{\check{E}}\check{\eta})\check{F} = -\check{g}(\check{\phi}\check{E}, \check{\phi}\check{F}).$$

Moreover, on an *LP*-Kenmotsu, the following relations hold [1]:

$$\check{g}(\check{R}(\check{E}, \check{F})\check{W}, \xi) = \check{\eta}(\check{R}(\check{E}, \check{F})\check{W}) = \check{g}(\check{F}, \check{W})\check{\eta}(\check{E}) - \check{g}(\check{E}, \check{W})\check{\eta}(\check{F}),$$

$$
\begin{aligned}
\check{R}(\xi,\check{E})\check{F} =& -\check{R}(\check{E},\xi)\check{F} = \check{g}(\check{E},\check{F})\xi - \check{\eta}(\check{F})\check{E},\\
\check{R}(\check{E},\check{F})\xi =& \check{\eta}(\check{F})\check{E} - \check{\eta}(\check{E})\check{F},\\
\check{R}(\xi,\check{E})\xi =& \check{E} + \check{\eta}(\check{E})\xi,\\
\check{S}(\check{E},\xi) =& (\dim M - 1)\check{\eta}(\check{E}), \quad \check{S}(\xi,\xi) = -(\dim M - 1),\\
\check{Q}\xi =& (\dim M - 1)\xi,
\end{aligned}
$$

for any $\check{E},\check{F},\check{W}$ on M.

Definition 2.2. An LP-Kenmotsu manifold is called an η-Einstein manifold if its Ricci tensor satisfies [4] $\check{S}(\check{E},\check{F}) = a_1\check{g}(\check{E},\check{F}) + a_2\check{\eta}(\check{E})\check{\eta}(\check{F})$, where a_1 and a_2 are smooth functions on M.

3. Curvature Tensor of LP-Kenmotsu Manifolds with a Connection $\widehat{\nabla}$

The curvature tensor $\widehat{R}$ of an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is defined by

$$\widehat{R}(\check{E},\check{F})\check{W} = \widehat{\nabla}_{\check{E}}\widehat{\nabla}_{\check{F}}\check{W} - \widehat{\nabla}_{\check{F}}\widehat{\nabla}_{\check{E}}\check{W} - \widehat{\nabla}_{[\check{E},\check{F}]}\check{W}. \tag{3.1}$$

From (1.8), (2.1), (2.4), (2.6), (2.7) and (3.1), we obtain

$$\widehat{R}(\check{E},\check{F})\check{W} = \check{R}(\check{E},\check{F})\check{W} - \check{g}(\check{E},\check{W})\check{\phi}\check{F} + \check{g}(\check{F},\check{W})\check{\phi}\check{E}, \tag{3.2}$$

where $\check{R}(\check{E},\check{F})\check{W}$ is given by (1.1). Contracting $\check{E}$ in (3.2), we get

$$\widehat{S}(\check{F},\check{W}) = \check{S}(\check{F},\check{W}) + \check{g}(\check{F},\check{W})\check{\psi} - \check{g}(\check{\phi}\check{F},\check{W}). \tag{3.3}$$

From (3.3), it follows that

$$\widehat{Q}\check{F} = \check{Q}\check{F} + \check{\psi}\check{F} - \check{\phi}\check{F},$$

Contracting again $\check{F}$ and $\check{W}$ in (3.3), we obtain

$$\widehat{r} = \check{r} + 2n\check{\psi}, \tag{3.4}$$

where $\widehat{Q}$ is the Ricci operator, $\widehat{S}$ is the Ricci tensor and $\widehat{r}$ is the scalar curvature with respect to $\widehat{\nabla}$.

Lemma 3.1. *In a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$, we have*

$$
\begin{aligned}
\widehat{R}(\check{E},\check{F})\xi =& \check{\eta}(\check{F})\check{E} - \check{\eta}(\check{E})\check{F} + \check{\eta}(\check{F})\check{\phi}\check{E} - \check{\eta}(\check{E})\check{\phi}\check{F},\\
\widehat{R}(\xi,\check{E})\check{F} =& -\widehat{R}(\check{E},\xi)\check{F} = \check{g}(\check{E},\check{F})\xi - \check{\eta}(\check{F})\check{E} - \check{\eta}(\check{F})\check{\phi}\check{E},\\
\widehat{R}(\xi,\check{E})\xi =& \check{\eta}(\check{E})\xi + \check{E} + \check{\phi}\check{E},
\end{aligned} \tag{3.5}
$$

$$\widehat{S}(\check{E},\xi) = (2n+\check{\psi})\check{\eta}(\check{E}), \quad \widehat{S}(\xi,\xi) = -(2n+\check{\psi}), \tag{3.6}$$

$$\widehat{\nabla}_{\check{E}}\xi = -\check{E} - \check{\eta}(\check{E})\xi - \check{\phi}\check{E}, \tag{3.7}$$

$$\widehat{Q}\xi = (2n+\check{\psi})\xi, \tag{3.8}$$

for any $\check{E}, \check{F}$ on M.

4. Ricci soliton on LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$

Suppose that an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ admits a Ricci soliton $(\check{g}, \xi, \check{\lambda})$. Then in view of (1.9), we have

$$(\widehat{\pounds}_{\xi}\check{g})(\check{F}, \check{W}) + 2\widehat{S}(\check{F}, \check{W}) + 2\check{\lambda}\check{g}(\check{F}, \check{W}) = 0. \tag{4.1}$$

By using (3.7) and (1.6), we find

$$(\widehat{\pounds}_{\xi}\check{g})(\check{F}, \check{W}) = -2[\check{g}(\check{F}, \check{W}) + \check{\eta}(\check{F})\check{\eta}(\check{W})]. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\widehat{S}(\check{F}, \check{W}) = (1 - \check{\lambda})\check{g}(\check{F}, \check{W}) + \check{\eta}(\check{F})\check{\eta}(\check{W}). \tag{4.3}$$

Taking $\check{W} = \xi$ in (4.3) and then using (2.1), (2.3), we get

$$\widehat{S}(\check{F}, \xi) = -\check{\lambda}\check{\eta}(\check{F}). \tag{4.4}$$

Thus from (3.6) and (4.4), it follows that

$$\check{\lambda} = -(2n + \check{\psi}). \tag{4.5}$$

Hence, (4.3) together with (4.5) leads to the following theorem.

Theorem 4.1. *If an LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admits a Ricci soliton $(\check{g}, \xi, \check{\lambda})$, then M is an η-Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

Now, assumig that $(\check{g}, U, \check{\lambda})$ is a Ricci soliton on an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ such that U is pointwise collinear with ξ, i.e., $U = \beta\xi$, where β is a function. Then (1.9) holds and we have

$$\beta\check{g}(\widehat{\nabla}_{\check{E}}\xi, \check{F}) + (\check{E}\beta)\check{\eta}(\check{F}) + \beta\check{g}(\check{E}, \widehat{\nabla}_{\check{F}}\xi) + (\check{F}\beta)\check{\eta}(\check{E}) + 2\widehat{S}(\check{E}, \check{F}) + 2\check{\lambda}\check{g}(\check{E}, \check{F}) = 0,$$

which in view of (3.7) and (1.6) becomes

$$-2\beta[\check{g}(\check{E}, \check{F}) + \check{\eta}(\check{E})\check{\eta}(\check{F})] + (\check{E}\beta)\check{\eta}(\check{F}) + (\check{F}\beta)\check{\eta}(\check{E}) + 2\widehat{S}(\check{E}, \check{F}) + 2\check{\lambda}\check{g}(\check{E}, \check{F}) = 0. \tag{4.6}$$

Replacing $\check{F}$ by ξ in (4.6) and using (2.1), (2.4) and (3.6), we find

$$-(\check{E}\beta) + [(\xi\beta) + 2(2n + \check{\psi}) + 2\check{\lambda}]\check{\eta}(\check{E}) = 0, \tag{4.7}$$

which by taking $\check{E} = \xi$ and using (2.1) yields

$$(\xi\beta) + (2n + \check{\psi}) + \check{\lambda} = 0. \tag{4.8}$$

Combining the equations (4.7) and (4.8), we find

$$d\beta = [(2n + \check{\psi}) + \check{\lambda}]\check{\eta}. \tag{4.9}$$

Now, applying d on (4.9), we get

$$[(2n + \check{\psi}) + \check{\lambda}]\check{\eta} = 0 \implies \check{\lambda} = -(2n + \check{\psi}), \quad d\check{\eta} \neq 0. \tag{4.10}$$

Thus, from (4.9) and (4.10), we obtain $d\beta = 0$, i.e., β is a constant. Therefore, (4.6) reduces to

$$\widehat{S}(\check{E},\check{F}) = (\beta - \check{\lambda})\check{g}(\check{E},\check{F}) + \beta\check{\eta}(\check{E})\check{\eta}(\check{F}). \tag{4.11}$$

Hence, (4.10) together with (4.11) leads the following theorem.

Theorem 4.2. *If an LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admits a Ricci soliton $(\check{g}, U, \check{\lambda})$ such that U is pointwise collinear with ξ, then U is a constant multiple of ξ and M is an η-Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

5. Ricci soliton on $\check{\phi}$-semisymmetric LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$

Definition 5.1. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-projectively semisymmetric if (see [13]) $\widehat{P}(\check{E},\check{F}) \cdot \check{\phi} = 0$ for all $\check{E},\check{F}$ on M.

Analogous to the equation (1.2), the projective curvature tensor with a connection $\bar{\nabla}$ is given by

$$\widehat{P}(\check{E},\check{F})\check{W} = \widehat{R}(\check{E},\check{F})\check{W} - \frac{1}{2n}[\widehat{S}(\check{F},\check{W})\check{E} - \widehat{S}(\check{E},\check{W})\check{F}]. \tag{5.1}$$

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-projectively semisymmetric, therefore

$$(\widehat{P}(\check{E},\check{F}) \cdot \check{\phi})\check{W} = \widehat{P}(\check{E},\check{F})\check{\phi}\check{W} - \check{\phi}\widehat{P}(\check{E},\check{F})\check{W} = 0, \tag{5.2}$$

for all $\check{E},\check{F},\check{W}$ on M. From (5.1), we find

$$\widehat{P}(\check{E},\check{F})\check{\phi}\check{W} = \widehat{R}(\check{E},\check{F})\check{\phi}\check{W} - \frac{1}{2n}[\widehat{S}(\check{F},\check{\phi}\check{W})\check{E} - \widehat{S}(\check{E},\check{\phi}\check{W})\check{F}], \tag{5.3}$$

$$\check{\phi}\widehat{P}(\check{E},\check{F})\check{W} = \check{\phi}\widehat{R}(\check{E},\check{F})\check{W} - \frac{1}{2n}[\widehat{S}(\check{F},\check{W})\check{\phi}\check{E} - \widehat{S}(\check{E},\check{W})\check{\phi}\check{F}]. \tag{5.4}$$

By combining (5.2), (5.3) and (5.4), we have

$$\begin{aligned}\widehat{R}(\check{E},\check{F})\check{\phi}\check{W} - \check{\phi}\widehat{R}(\check{E},\check{F})\check{W} &- \frac{1}{2n}[\widehat{S}(\check{F},\check{\phi}\check{W})\check{E} - \widehat{S}(\check{E},\check{\phi}\check{W})\check{F}]\\ &+ \frac{1}{2n}[\widehat{S}(\check{F},\check{W})\check{\phi}\check{E} - \widehat{S}(\check{E},\check{W})\check{\phi}\check{F}] = 0.\end{aligned} \tag{5.5}$$

Taking $\check{F} = \xi$ in (5.5) and using (2.3), (3.5) and (3.6), we find

$$-\check{g}(\check{E},\check{\phi}\check{W})\xi + \frac{1}{2n}\widehat{S}(\check{E},\check{\phi}\check{W})\xi - \check{\eta}(\check{W})\check{\phi}\check{E} - \check{\eta}(\check{W})\check{\phi}^2\check{E} + \frac{(2n+\check{\psi})}{2n}\check{\eta}(\check{W})\check{\phi}\check{E} = 0.$$

Taking inner product of the above equation with ξ and making use of (2.1) and (2.3) yields $\widehat{S}(\check{E},\check{\phi}\check{W}) = 2n\check{g}(\check{E},\check{\phi}\check{W})$, which by setting $\check{W} = \check{\phi}\check{W}$ and using (2.2) gives

$$\widehat{S}(\check{E},\check{W}) = 2n\check{g}(\check{E},\check{W}) - \check{\psi}\check{\eta}(\check{E})\check{\eta}(\check{W}). \tag{5.6}$$

Now, taking $\check{W} = \xi$ in (5.6), we find

$$\widehat{S}(\check{E}, \xi) = (2n + \check{\psi})\check{\eta}(\check{E}). \tag{5.7}$$

Thus, from (4.4) and (5.7), we obtain

$$\check{\lambda} = -(2n + \check{\psi}). \tag{5.8}$$

Hence, (5.6) together with (5.8) leads to the following theorem.

Theorem 5.1. *If a* $(2n+1)$*-dimensional* LP*-Kenmotsu manifold* M *with a connection* $\widehat{\nabla}$ *admitting Ricci soliton is* $\check{\phi}$*-projectively semisymmetric, then* M *is an* $\eta-$*Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to* $\check{\psi} < -2n$, $\check{\psi} > -2n$ *or* $\check{\psi} = -2n$.

Definition 5.2. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-concircularly semisymmetric if $\widehat{V}(\check{E}, \check{F}) \cdot \check{\phi} = 0$ for all $\check{E}, \check{F}$ on M.

Analogous to the equation (1.3), the concircular curvature tensor with a connection $\widehat{\nabla}$ is given by

$$\widehat{V}(\check{E}, \check{F})\check{W} = \widehat{R}(\check{E}, \check{F})\check{W} - \frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F}, \check{W})\check{E} - \check{g}(\check{E}, \check{W})\check{F}]. \tag{5.9}$$

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-concircularly semisymmetric, therefore

$$(\widehat{V}(\check{E}, \check{F}) \cdot \check{\phi})\check{W} = \widehat{V}(\check{E}, \check{F})\check{\phi}\check{W} - \check{\phi}\widehat{V}(\check{E}, \check{F})\check{W} = 0, \tag{5.10}$$

for all $\check{E}, \check{F}, \check{W}$ on M. From (5.9), it follows that

$$\widehat{V}(\check{E}, \check{F})\check{\phi}\check{W} = \widehat{R}(\check{E}, \check{F})\check{\phi}\check{W} - \frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F}, \check{\phi}\check{W})\check{E} - \check{g}(\check{E}, \check{\phi}\check{W})\check{F}], \tag{5.11}$$

$$\check{\phi}\widehat{V}(\check{E}, \check{F})\check{W} = \check{\phi}\widehat{R}(\check{E}, \check{F})\check{W} - \frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F}, \check{W})\check{\phi}\check{E} - \check{g}(\check{E}, \check{W})\check{\phi}\check{F}]. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12), we have

$$\begin{aligned} &\widehat{R}(\check{E}, \check{F})\check{\phi}\check{W} - \check{\phi}\widehat{R}(\check{E}, \check{F})\check{W} - \frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F}, \check{\phi}\check{W})\check{E} - \check{g}(\check{E}, \check{\phi}\check{W})\check{F}] \\ &+ \frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F}, \check{W})\check{\phi}\check{E} - \check{g}(\check{E}, \check{W})\check{\phi}\check{F}] = 0, \end{aligned}$$

which, by taking $\check{F} = \xi$ and using (2.3), (2.4) and (3.5), takes the form

$$\left[\frac{\widehat{r}}{2n(2n+1)} - 1\right]\check{g}(\check{E}, \check{\phi}\check{W})\xi + \left[\frac{\widehat{r}}{2n(2n+1)} - 1\right]\check{\eta}(\check{W})\check{\phi}\check{E} + \check{\eta}(\check{W})\check{\phi}^2\check{E} = 0. \tag{5.13}$$

Taking inner product of (5.13) with ξ and making use of (2.1) and (2.3), we get

$$\widehat{r} = 2n(2n+1), \quad \check{g}(\check{E}, \check{\phi}\check{W}) \neq 0.$$

This leads to the following theorem.

Theorem 5.2. *If a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-concircular semisymmetric, then the scalar curvature is constant.*

Definition 5.3. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-conformally semisymmetric if $\widehat{C}(\check{E},\check{F})\cdot\check{\phi}=0$ for all $\check{E},\check{F}$ on M.

Analogous to the equation (1.4), the conformal curvature tensor with a connection $\widehat{\nabla}$ is given by

(5.14)
$$\begin{aligned}\widehat{C}(\check{E},\check{F})\check{W} =&\widehat{R}(\check{E},\check{F})\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{W})\check{E}-\widehat{S}(\check{E},\check{W})\check{F}\\&+\check{g}(\check{F},\check{W})\widehat{Q}\check{E}-\check{g}(\check{E},\check{W})\widehat{Q}\check{F}]+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{E}-\check{g}(\check{E},\check{W})\check{F}].\end{aligned}$$

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-conformally semisymmetric, therefore

$$(\widehat{C}(\check{E},\check{F})\cdot\check{\phi})\check{W}=\widehat{C}(\check{E},\check{F})\check{\phi}\check{W}-\check{\phi}\widehat{C}(\check{E},\check{F})\check{W}=0, \tag{5.15}$$

for all $\check{E},\check{F},\check{W}$ on M. From (5.14), it follows that

$$\begin{aligned}\widehat{C}(\check{E},\check{F})\check{\phi}\check{W} =&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{\phi}\check{W})E\\&-\widehat{S}(\check{E},\check{\phi}\check{W})\check{F}+\check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E}-\check{g}(E,\check{\phi}\check{W})\widehat{Q}\check{F}]\\&+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{\phi}\check{W})\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\check{F}],\end{aligned} \tag{5.16}$$

$$\begin{aligned}\check{\phi}\widehat{C}(\check{E},\check{F})\check{W} =&\check{\phi}\widehat{R}(\check{E},\check{F})\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{W})\check{\phi}\check{E}\\&-\widehat{S}(\check{E},\check{W})\check{\phi}\check{F}+\check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F}]\\&+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{\phi}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\check{F}].\end{aligned} \tag{5.17}$$

Combining (5.15), (5.16) and (5.17), we have

$$\begin{aligned}&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W}-\check{\phi}\widehat{R}(\check{E},\check{F})\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{\phi}\check{W})\check{E}-\widehat{S}(\check{E},\check{\phi}\check{W})\check{F}\\&+\check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F}]+\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{W})\check{\phi}\check{E}-\widehat{S}(\check{E},\check{W})\check{\phi}\check{F}\\&+\check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F}]+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{\phi}\check{W})\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\check{F}]\\&-\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{\phi}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\check{F}]=0,\end{aligned}$$

which by replacing $\check{E} = \xi$ and making use of (2.3), (2.4), (3.5), (3.6) and (3.8) takes the form

$$\left[\frac{2n+\check{\psi}}{2n-1} - \frac{\widehat{r}}{2n(2n-1)} - 1\right](\check{g}(\check{E},\check{\phi}\check{W})\xi + \check{\eta}(\check{W})\check{\phi}E) + \frac{1}{(2n-1)}\widehat{S}(\check{E},\check{\phi}\check{W})\xi + \frac{1}{(2n-1)}\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E} - \check{\eta}(\check{W})\check{\phi}^2\check{E} = 0. \tag{5.18}$$

Now, taking inner product of (5.18) with ξ and making use of (2.1) and (2.3), we obtain

$$\widehat{S}(\check{E},\check{\phi}\check{W}) = \left[(2n-1) + \frac{\widehat{r}}{2n} - (2n+\check{\psi})\right]\check{g}(\check{E},\check{\phi}\check{W}). \tag{5.19}$$

By replacing $\check{W}$ by $\check{\phi}\check{W}$ in (5.19) and then using (2.2), (2.4), (3.6), we get

$$\begin{aligned}\widehat{S}(\check{E},\check{W}) = &\left[(2n-1) + \frac{\widehat{r}}{2n} - (2n+\check{\psi})\right]\check{g}(\check{E},\check{W}) \\ &+ \left[(2n-1) + \frac{\widehat{r}}{2n} - 2(2n+\check{\psi})\right]\check{\eta}(\check{E})\check{\eta}(\check{W}).\end{aligned} \tag{5.20}$$

Taking $\check{W} = \xi$ in (5.20), we find

$$\widehat{S}(\check{E},\xi) = (2n+\check{\psi})\check{\eta}(\check{E}). \tag{5.21}$$

Thus, from (4.4) and (5.21), we obtain

$$\check{\lambda} = -(2n+\check{\psi}). \tag{5.22}$$

Hence, (5.20) together with (5.22) leads to the following theorem.

Theorem 5.3. *If a $(2n+1)$-dimensional LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admitting Ricci soliton is $\check{\phi}$-conformally semisymmetric, then M is an $\eta-$Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

Definition 5.4. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-D-conformally semisymmetric if $\widehat{B}(\check{E},\check{F}) \cdot \check{\phi} = 0$ for all $\check{E},\check{F}$ on M.

Analogous to the equation (1.5), the $D-$conformal curvature tensor with a connection $\widehat{\nabla}$ is given by

$$\begin{aligned}\widehat{B}(\check{E},\check{F})\check{W} =&\widehat{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{W})\check{F} - \widehat{S}(\check{F},\check{W})\check{E} \\ &+ \check{g}(\check{E},\check{W})\widehat{Q}\check{F} - \check{g}(\check{F},\check{W})\widehat{Q}\check{E} - \widehat{S}(\check{E},\check{W})\check{\eta}(\check{F})\xi \\ &+ \widehat{S}(\check{F},\check{W})\check{\eta}(\check{E})\xi - \check{\eta}(\check{E})\check{\eta}(\check{W})\widehat{Q}\check{F} + \check{\eta}(\check{F})\check{\eta}(\check{W})\widehat{Q}\check{E}] \\ &- \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{F} - \check{g}(\check{F},\check{W})\check{E}] + \frac{\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{W})\check{\eta}(\check{F})\xi \\ &- \check{g}(\check{F},\check{W})\check{\eta}(\check{E})\xi + \check{\eta}(\check{E})\check{\eta}(\check{W})\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{E}],\end{aligned} \tag{5.23}$$

where $\widehat{k} = \frac{\widehat{r}+4n}{(2n-1)}$.

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-D-conformally semisymmetric, therefore

$$(\widehat{B}(\check{E},\check{F})\cdot\check{\phi})\check{W} = \widehat{B}(\check{E},\check{F})\check{\phi}\check{W} - \check{\phi}\widehat{B}(\check{E},\check{F})\check{W} = 0, \tag{5.24}$$

for all $\check{E},\check{F},\check{W}$ on M. From (5.23), it follows that

$$\begin{aligned}\widehat{B}(\check{E},\check{F})\check{\phi}\check{W} =& \widehat{R}(\check{E},\check{F})\check{\phi}\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{\phi}\check{W})\check{F} - \widehat{S}(\check{F},\check{\phi}\check{W})\check{E} + \check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F}\\ &- \check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E} - \widehat{S}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\xi + \widehat{S}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})\xi]\\ &- \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{F} - \check{g}(\check{F},\check{\phi}\check{W})\check{E}] + \frac{\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\xi\\ &- \check{g}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})\xi],\end{aligned} \tag{5.25}$$

$$\begin{aligned}\check{\phi}\widehat{B}(\check{E},\check{F})W =& \check{\phi}\widehat{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{W})\check{\phi}\check{F} - \widehat{S}(\check{F},\check{W})\check{\phi}\check{E}\\ &+ \check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E} - \check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{F}\\ &+ \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E}] - \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{\phi}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\check{E}]\\ &+ \frac{\widehat{k}}{2(n-1)}[\check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\check{E}].\end{aligned} \tag{5.26}$$

Combining (5.24), (5.25) and (5.26), we have

$$\begin{aligned}&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W} - \check{\phi}\widehat{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{\phi}\check{W})\check{F} - \widehat{S}(\check{F},\check{\phi}\check{W})\check{E}\\ &+ \check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F} - \check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E} - \widehat{S}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\xi + \widehat{S}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})\xi]\\ &- \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{W})\check{\phi}\check{F} - \widehat{S}(\check{F},\check{W})\check{\phi}\check{E} + \check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E}\\ &- \check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{F} + \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E}] - \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{F} - \check{g}(\check{F},\check{\phi}\check{W})\check{E}]\\ &+ \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{\phi}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\check{E}] + \frac{\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\\ &- \check{g}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})]\xi - \frac{\widehat{k}}{2(n-1)}[\check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\check{E}] = 0.\end{aligned} \tag{5.27}$$

By taking $\check{F} = \xi$ in (5.27) and then using (2.1), (2.3), (3.5), (3.6) and (3.8) takes the form

$$\frac{4+\check{\psi}-2\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\xi + \check{\eta}(\check{W})\check{\phi}\check{E}] + \frac{1}{n-1}\widehat{S}(\check{E},\check{\phi}\check{W})\xi - \check{\eta}(\check{W})\check{\phi}^2\check{E} + \frac{1}{n-1}\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E} = 0. \tag{5.28}$$

Inner product of (5.28) with ξ and making use of (2.1) and (2.3) gives

$$\widehat{S}(\check{E},\check{\phi}\check{W}) = \left[\frac{\check{\psi}}{2} + 2 - \widehat{k}\right]\check{g}(\check{E},\check{\phi}\check{W}). \tag{5.29}$$

Now, we replace $\check{W}$ by $\check{\phi}\check{W}$ in (5.29) and using (2.2), (2.4) and (3.6), we get

$$\widehat{S}(\check{E},\check{W}) = \left[\frac{\check{\psi}}{2} + 2 - \widehat{k}\right]\check{g}(\check{E},\check{W}) - \left[\frac{\check{\psi}}{2} + \widehat{k} + 2n - 2\right]\check{\eta}(\check{E})\check{\eta}(\check{W}). \tag{5.30}$$

Taking $\check{W} = \xi$ in (5.30), we find

$$\widehat{S}(\check{E},\xi) = (2n+\check{\psi})\check{\eta}(\check{E}). \tag{5.31}$$

Thus, from (4.4) and (5.31), we obtain

$$\check{\lambda} = -(2n+\check{\psi}). \tag{5.32}$$

Hence, (5.30) together with (5.32) leads to the following theorem.

Theorem 5.4. *If a $(2n+1)$-dimensional LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admitting Ricci soliton is $\check{\phi}$-D-conformally semisymmetric, then M is an $\eta-$Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

Example 5.1. Let on a 3-dimensional manifold $M = \{(\breve{w}_1, \breve{w}_2, \breve{w}_3) \in R^3 : w > 0\}$, where $(\breve{w}_1, \breve{w}_2, \breve{w}_3)$ are the standard coordinates of R^3, the linearly independent vector fields that at each point of M are given by

$$v^1 = \frac{\breve{w}_3\partial}{\partial\breve{w}_1}, \quad v^2 = \frac{w\partial}{\partial\breve{w}_2}, \quad v^3 = \frac{w\partial}{\partial\breve{w}_3} = \xi.$$

Suppose the Lorentzian metric $\check{g}$ is defined by

$$\check{g}(v^1,v^1) = \check{g}(v^2,v^2) = 1, \quad \check{g}(v^3,v^3) = -1, \quad \check{g}(v^1,v^2) = \check{g}(v^2,v^3) = \check{g}(v^1,v^3) = 0.$$

Suppose the 1-form $\check{\eta}$ is defined by $\check{\eta}(\check{E}) = \check{g}(\check{E},v^3) = \check{g}(\check{E},\xi)$ for all $\check{E}$ on M, and the $(1,1)$-tensor field $\check{\phi}$ is defined by

$$\check{\phi}v^1 = -v^1, \quad \check{\phi}v^2 = -v^2, \quad \check{\phi}v^3 = 0.$$

Then, using the linearity of $\check{g}$ and $\check{\phi}$, we have

$$\check{\eta}(\xi) = -1, \quad \check{\phi}^2\check{E} = \check{E} + \check{\eta}(\check{E})\xi, \quad \check{g}(\check{\phi}\check{E},\check{\phi}\check{F}) = \check{g}(\check{E},\check{F}) + \check{\eta}(\check{E})\check{\eta}(\check{F}),$$

for all $\check{E}, \check{F}$ on M. Thus, $(\check{\phi}, \xi, \check{\eta}, \check{g})$ defines a Lorentzian almost paracontact metric structure on M. Also, we have

$$[v^1, v^2] = 0, \quad [v^1, v^3] = -v^1, \quad [v^2, v^3] = -v^2. \tag{5.33}$$

From the Koszul's formula for $\check{g}$, we calculate

$$\begin{aligned} &\check{\nabla}_{v^1} v^1 = -\, v^3, \quad \check{\nabla}_{v^1} v^2 = 0, \quad \check{\nabla}_{v^1} v^3 = -v^1, \quad \check{\nabla}_{v^2} v^1 = 0, \\ &\check{\nabla}_{v^2} v^2 = -\, v^3, \quad \check{\nabla}_{v^2} v^3 = -v^2, \quad \check{\nabla}_{v^3} v^1 = 0, \quad \check{\nabla}_{v^3} v^2 = 0, \quad \check{\nabla}_{v^3} v^3 = 0. \end{aligned} \tag{5.34}$$

Also, one can easily verify that

$$\check{\nabla}_{\check{E}} \xi = -\check{E} - \check{\eta}(\check{E})\xi \quad \text{and} \quad (\check{\nabla}_{\check{E}} \check{\phi})\check{F} = -\check{g}(\check{\phi}\check{E}, \check{F})\xi - \check{\eta}(\check{F})\check{\phi}\check{E}.$$

Therefore, M is an LP-Kenmotsu manifold. From (1.1), (5.33) and (5.34), we obtain

$$\begin{aligned} &\check{R}(v^1, v^2)v^1 = -\, v^2, \quad \check{R}(v^2, v^3)v^1 = 0, \quad \check{R}(v^1, v^3)v^1 = -v^3, \\ &\check{R}(v^1, v^2)v^2 = v^1, \quad \check{R}(v^1, v^3)v^2 = 0, \quad \check{R}(v^2, v^3)v^2 = -v^3, \\ &\check{R}(v^1, v^2)v^3 = 0, \quad \check{R}(v^1, v^3)v^3 = -v^1, \quad \check{R}(v^2, v^3)v^3 = -v^2, \end{aligned} \tag{5.35}$$

from which it is clear that $\check{R}(\check{E}, \check{F})\check{W} = \check{g}(\check{F}, \check{W})\check{E} - \check{g}(\check{E}, \check{W})\check{F}$. Hence, $(M, \check{\phi}, \xi, \check{\eta}, \check{g})$ is an LP-Kenmotsu manifold of unit constant curvature. By virtue of (1.8) and (5.35), we obtain

$$\begin{aligned} &\widehat{\nabla}_{v^1} v^1 = -\, v^3, \quad \widehat{\nabla}_{v^2} v^1 = 0, \quad \widehat{\nabla}_{v^3} v^1 = 0, \quad \widehat{\nabla}_{v^1} v^2 = 0, \quad \widehat{\nabla}_{v^2} v^2 = -v^3, \\ &\widehat{\nabla}_{v^3} v^2 = 0, \quad \widehat{\nabla}_{v^1} v^3 = 0, \quad \widehat{\nabla}_{v^2} v^3 = 0, \quad \widehat{\nabla}_{v^3} v^3 = 0. \end{aligned}$$

From (3.2) and (5.35), we can easily obtain

$$\begin{aligned} &\widehat{R}(v^1, v^2)v^1 = 0, \quad \widehat{R}(v^1, v^3)v^1 = -v^3, \quad \widehat{R}(v^2, v^3)v^1 = 0, \\ &\widehat{R}(v^1, v^2)v^2 = 0, \quad \widehat{R}(v^1, v^3)v^2 = 0, \quad \widehat{R}(v^2, v^3)v^2 = -v^3, \\ &\widehat{R}(v^1, v^2)v^3 = 0, \quad \widehat{R}(v^1, v^3)v^3 = 0, \quad \widehat{R}(v^2, v^3)v^3 = 0. \end{aligned} \tag{5.36}$$

From (5.35) and (5.36), we calculate the Ricci tensors as follows:

$$\check{S}(v^1, v^1) = \check{S}(v^2, v^2) = 2, \quad \check{S}(v^3, v^3) = -2,$$

and

$$\widehat{S}(v^1, v^1) = \widehat{S}(v^2, v^2) = 1, \quad \widehat{S}(v^3, v^3) = 0.$$

Therefore, we find $\check{r} = 6$ and $\widehat{r} = 2$, where $\check{\psi} = -2$. Hence, (3.4) is satisfied. From (2.5), (1.6) and (1.7), we find

$$\begin{aligned} &\check{\Phi}(v^1, v^1) = \check{\Phi}(v^2, v^2) = -1, \quad \check{\Phi}(v^3, v^3) = 0, \\ &\check{T}(v^i, v^j) = 0, \quad i = j = 1, 2, 3, \\ &\check{T}(v^1, v^2) = 0, \quad \check{T}(v^1, v^3) = v^1, \quad \check{T}(v^2, v^3) = v^2, \\ &(\widehat{\nabla}_{v^1} \check{g})(v^1, v^3) = (\widehat{\nabla}_{v^2} \check{g})(v^2, v^3) = -1, \quad (\widehat{\nabla}_{v^3} \check{g})(v^1, v^2) = 0, \end{aligned}$$

respectively. Thus, the connection $\widehat{\nabla}$ defined on M is a QSNM. Now, by putting $F = W = v^i$ in (4.3) and summing up, we find $2 = 3(1 - \check{\lambda}) - 1$ implies $\check{\lambda} = 0$. Thus, a Ricci soliton on an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is steady for $\check{\psi} = -2n = -2$.

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