Kragujevac Journal of Mathematics
Volume 45(5) (2021), Pages 667–684.

RESULTS ON (ENGEL, SOLVABLE, NILPOTENT) FUZZY SUBPOLYGROUPS

ELAHE MOHAMMADZADEH[1] AND RAJAB ALI BORZOOEI[2]

Abstract. In this paper, first we define the notion of an Engel polygroup, to get further properties on Engel fuzzy subpolygroups. Moreover, we prove that every normal fuzzy subpolygroup of an Engel polygroup is Engel. Furthermore, we introduce the notions of solvable and nilpotent fuzzy subpolygroups and we get some of their properties. Finally we investigate the relations among solvable and nilpotent fuzzy subpolygroups with Engel fuzzy subpolygroups.

1. Introduction

Researches on Engel groups have centered mainly on the question, whether n-Engel groups are nilpotents. Clearly every 1-Engel group is Abelian. Levi [14] proved that 2-Engel groups are nilpotent of class at most 3. Heineken in [12] showed that every 3-Engel group G is nilpotent of class at most 4 if G has no element of order 2 or 5. L. Kappe and W. Kappe [13] gave a characterization of 3-Engel groups which is analogous to Levi's theorem on 2-Engel groups. Moreover, the study of fuzzy Engel groups was investigated in [2, 16, 17].

On the other hand, hyperstructure theory was first initiated by Marty [15] in 1934 when he defined hypergroups and started to analyze their properties. Since there are extensive applications in many branches of mathematics and applied sciences, the theory of algebraic hyperstructures has nowadays become a well-established branch in algebraic theory. Fuzzy subsets have been introduced in (1965) by L. A. Zadeh [22] as an extension of the classical notion of set. With appropriate definitions in the fuzzy setting most of the elementary results of group theory have been superseded

Key words and phrases. Engel group, Engel polygroup, (Engel, solvable, nilpotent) fuzzy subpolygroup.
2010 *Mathematics Subject Classification.* Primary: 43A62. Secondary: 20F45, 20N25.
DOI 10.46793/KgJMat2105.667M
Received: January 25, 2019.
Accepted: April 14, 2019.

with a starling generalized effect. Specially, the study of fuzzy hyperstructures is an interesting research topic of fuzzy sets. There is a considerable amount of work on the connections between fuzzy sets and hyperstructures. Fuzzy hyperstructures is a direct extension of the concept of fuzzy algebras. This approach can be extended to fuzzy hypergroups. In [23], the concept of a fuzzy subpolygroup is introduced. In [7], Borzooei and et. al introduced the notion of Engel (nilpotent) fuzzy subpolygroups and various properties of Engel fuzzy subpolygroups were proved.

Now, in this paper, first we introduce and study Engel polygroups and solvable fuzzy subpolygroups. Then, we investigate the important properties of such fuzzy hyperstructure. Moreover, we obtain a necessary and sufficient condition between solvable fuzzy subpolygroups and the solvable group $P/\sim$, the group of equivalence classes derived from a fuzzy subpolygroup of P. Finally, by the relation between these notions we get some interesting results on Engel fuzzy subpolygroups.

2. Preliminary

Let $X_1, X_2, \ldots, X_n$ be non-empty subsets of group G. Define the *commutator subgroup* of X_1 and X_2 by

$$[X_1, X_2] = \langle [x_1, x_2] \mid x_1 \in X_1, x_2 \in X_2 \rangle.$$

More generally, define

$$[X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n],$$

where $n \geq 2$ and $[X_1] = \langle X_1 \rangle$. Also, recall that $X_1^{X_2} = \langle x_1^{x_2} \mid x_1 \in X_1, x_2 \in X_2 \rangle$ [19]. Let G be any group and $x, y \in G$. Define the *n-commutator* $[x,_n y]$, for any $n \in \mathbb{N}$ and $x, y \in G$, by $[x,_0 y] = x$, $[x,_1 y] = xyx^{-1}y^{-1}$ and $[x,_n y] = [[x,_{n-1} y], y]$. Now, a group G is called an *Engel group* if for each $x, y \in G$, there is a positive integer $n = n(x, y)$, such that $[x,_n y] = e$, where e is the identity of the group G. Suppose $n = n(x, y)$ can be chosen independently of any $x, y \in G$, then we say that G is an *n-Engel group.*

We recall the notion of a nilpotent group. Let G be a group. *Lower central series* of G is defined by $G = l_1(G) \geq l_2(G) \geq \cdots$, where $l_1(G) = G$ and for each integer $n > 1$, $l_n(G) = [l_{n-1}(G), G]$. Then G is called *nilpotent* if there exists a non-negative integer m, such that $l_m(G) = \{e\}$. The smallest such integer is called the class of G. Also, *derived series* of G is defined by $\cdots \subseteq G^n \subseteq \cdots \subseteq G^0 = G$; where for each integer $n > 1$, $G^n = [G^{n-1}, G^{n-1}]$. Now, G is called *solvable* if there exists a non-negative integer m, such that $G^m = \{e\}$. The smallest such integer is called the class of G (see [19]).

Definition 2.1 ([9])**.** A *polygroup* is an algebraic structure $(P, \cdot, ^{-1}, e)$, where "$\cdot$" is a hyperoperation on P, "$^{-1}$" is an unitary operation on P and $e \in P$, such that the following axioms hold:

(i) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$;
(ii) $e \cdot x = x \cdot e = x$;
(iii) $x \in y \cdot z \Rightarrow y \in x \cdot z^{-1} \Rightarrow z \in y^{-1} \cdot x$,

for any $x, y, z \in P$.

A non-empty subset K of a polygroup P is called a *subpolygroup* of P, if $a, b \in K$ implies $a \cdot b \subseteq K$ and $a \in K$ implies $a^{-1} \in K$. A subpolygroup N of a polygroup P is called *normal*, if $a^{-1}Na \subseteq N$, for any $a \in P$ (see [9]). The *commutator* of two elements in a polygroup $\langle P, \cdot, e, ^{-1} \rangle$, is defined by $[x, y] = \{t \mid t \in x \cdot y \cdot x^{-1} \cdot y^{-1}\}$. If $A \subseteq P$, then $[A, y] = \{t \mid t \in A \cdot y \cdot A^{-1} \cdot y^{-1}\}$. Therefore,

$$[[x, y], y] = \{t \mid t \in [x, y] \cdot y \cdot [x, y]^{-1} \cdot y^{-1}\}$$

and, inductively, we define

$$[x,_n y] = [[x,_{n-1} y], y] = \{t \mid t \in [x,_{n-1} y] \cdot y \cdot [x,_{n-1} y]^{-1} \cdot y^{-1}\}.$$

Also, $A^x = \{t \mid t \in x \cdot A \cdot x^{-1}\}$ (see [3]).

Definition 2.2 ([1,3])**.** Let P be a polygroup. For any $s \in P$ and $k \geq 0$, we define:

(i) $L_{0,s}(P) = P$;
(ii) $L_{k+1,s}(P) = \{h \in P \mid x \cdot s \cap h \cdot s \cdot x \neq \phi, x \in L_{k,s}(P)\}$;
(iii) $L_0(P) = P$;
(iv) $L_{k+1}(P) = \{h \mid x \cdot y \cap h \cdot y \cdot x \neq \phi, x \in L_k(P) \;\; and \;\; y \in P\}$;
(v) $l_{0,s}(P) = P$;
(vi) $l_{k+1,s}(P) = \langle\{h \in P \mid h \in [x, s], x \in l_{k,s}(P)\}\rangle$;
(vii) $l_0(P) = P$;
(viii) $l_{k+1}(P) = \langle\{h \in P \mid h \in [x, y], x \in l_k(P), y \in P\}\rangle$;
(ix) $i_0(P) = P$, $i_{k+1}(P) = \langle\{h \in P \mid h \in [x, y], x, y \in i_k(P)\}\rangle$.

Theorem 2.1 ([3])**.** *Let P be a polygroup. Then for any $s \in P$ and $k \geq 0$*

$$L_{k+1,s}(P) = \{h \in P \mid h \in [x, s], x \in L_{k,s}(P)\}.$$

Let P be a polygroup and $\rho \subseteq P \times P$ be an equivalence relation on P. For non-empty subsets A and B of P, we define $A\overline{\overline{\rho}}B \Leftrightarrow$ (for all $a \in A$ and for all $b \in B$ we get $a\rho b$). Then the relation ρ is called a strongly regular on the left (on the right) if $x\rho y \Rightarrow a \cdot x\overline{\overline{\rho}}a \cdot y(x \cdot a\overline{\overline{\rho}}y \cdot a)$ for any $x, y, a \in P$. Moreover, ρ is called *strongly regular* if it is strongly regular on the right and on the left.

Theorem 2.2 ([8])**.** *If P is a polyrgroup and ρ is a strongly regular relation on P, then $(P/\rho, \otimes)$ is a group, where $\rho(x) \otimes \rho(y) = \rho(z)$ for any $z \in x \cdot y$.*

For any $n \geq 1$, we define the relation β_n on a polygroup P, as follows:

$$a\beta_n b \Leftrightarrow (\exists (x_1, \ldots, x_n) \in P^n) \; \{a, b\} \subseteq \prod_{i=1}^{n} x_i$$

and we let $\beta = \cup_{n \geq 1}\beta_n$. Suppose that β^* is the *transitive closure* of β. Then β^* is a *strongly regular relation* on P [8].

Let $(H, \cdot)$ and $(H', \star)$ be two polygroups. A function $f : H \to H'$ is called a *homomorphism* if $f(a \cdot b) \subseteq f(a) \star f(b)$ for any $a, b \in H$. We say that f is a *good homomorphism* if $f(a \cdot b) = f(a) \star f(b)$ for any $a, b \in H$.

Definition 2.3 ([9])**.** A polygroup P is said to be *nilpotent* if there exists $n \in \mathbb{N}$ such that $l_n(P) \subseteq w$ or equivalently $l_n(P).w = w$, where w is the kernel of $f : P \to \frac{P}{\beta^*}$. The smallest integer n such that $l_n(P).w = w$ is called the *nilpotency class* or for simplicity the class of P. Also, a polygroup P is said to be *solvable* if there exists $n \in \mathbb{N}$ such that $i_n(P) \subseteq w$. The smallest such integer is called the class of P.

A *fuzzy subset* μ of X is a function $\mu : X \to [0,1]$. Let f be a function from X into Y, and μ, ν be two fuzzy subsets of X, Y, respectively. Defined the fuzzy subset $f(\mu)$ of Y, by

$$(f(\mu))(y) = \begin{cases} \bigvee\limits_{x \in f^{-1}(y)} \mu(x), & f^{-1}(y) \neq \phi, \\ 0, & \text{otherwise}, \end{cases}$$

for any $y \in Y$, and fuzzy subset $f^{-1}(\nu)$ of X by $(f^{-1}(\nu))(x) = \nu(f(x))$ for any $x \in X$. The *intersection* $\mu_1 \cap \mu_2$ of fuzzy subsets μ_1 and μ_2 of X, is defined by $(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}$ for any $x \in X$. (Note that $\mu_1 \cap \mu_2$, is the largest fuzzy subset of X contained in the both of μ_1 and μ_2). Also $\mu_1 \times \mu_2$ is a fuzzy subset of $X \times X$, which is defined by $(\mu_1 \times \mu_2)(x_1, x_2) = \min\{\mu_1(x_1), \mu_2(x_2)\}$ for any $x_1, x_2 \in X$ (see [20, 22]).

Definition 2.4 ([20])**.** Let μ be a fuzzy subset of a group G. Then μ is called a *fuzzy subgroup* of G, if $\mu(xy) \geq \mu(x) \wedge \mu(y)$ and $\mu(x^{-1}) \geq \mu(x)$ for any $x, y \in G$. A fuzzy subgroup μ of G is called *normal* if $\mu(xy) = \mu(yx)$ for any $x, y \in G$.

Definition 2.5 ([23])**.** Let $(P, \cdot)$ be a polygroup and μ be a fuzzy subset of P. Then μ is called a *fuzzy subpolygroup* of P, when $z \in x \cdot y$ implies $\mu(z) \geq \min\{\mu(x), \mu(y)\}$ and $\mu(x^{-1}) \geq \mu(x)$ for any $x, y \in P$. Moreover, a fuzzy subpolygroup μ of P is called *normal* if $z \in x \cdot y$ and $z' \in y \cdot x$, then $\mu(z) = \mu(z')$ for any $x, y \in P$.

Theorem 2.3 ([23])**.** *Let μ be a fuzzy subpolygroup of polygroup P. Then $\mu(e) \geq \mu(x)$ and $\mu(x^{-1}) = \mu(x)$, for any $x \in P$. Moreover, μ is a normal fuzzy subpolygroup of P if and only if $\mu_t = \{x \mid \mu(x) \geq t\}$ is a normal subpolygroup of P for any $t \in [0, \mu(e)]$.*

Theorem 2.4 ([10])**.** *Let μ be a fuzzy subpolygroup of a polygroup P. Then the following conditions are equivalent, for any $x, y \in P$:*

(i) *μ is a normal fuzzy subpolygroup of P;*
(ii) *for any $z \in y \cdot x \cdot y^{-1}$, $\mu(z) = \mu(x)$;*
(iii) *for any $z \in y \cdot x \cdot y^{-1}$, $\mu(z) \geq \mu(x)$;*
(iv) *for any $z \in y^{-1} \cdot x^{-1} \cdot y \cdot x$, $\mu(z) \geq \mu(x)$.*

Theorem 2.5 ([10])**.** *Let P and P' be two polygroups, μ be a fuzzy subpolygroup of P, λ be a fuzzy subpolygroup of P' and $f : P \longrightarrow P'$ be a function. If f is a good homomorphism, then $f^{-1}(\lambda)(f(\mu))$ is a fuzzy subpolygroup of $P(P')$.*

Theorem 2.6 ([10])**.** *Let P_1 and P_2 be two polygroups and μ and ν be two fuzzy subpolygroups of P_1 and P_2, respectively. If $\mu(e_1) = \nu(e_2) = 1$ and $\mu \times \nu$ is a fuzzy subpolygroup of $P_1 \times P_2$, then μ and ν are fuzzy subpolygroups of P_1 and P_2, respectively.*

Notation. From now on, in this paper we let $(P, \cdot, ^{-1}, e)$ be a polygroup and $n \in \mathbb{N}$. For simplicity of notations, sometimes we may write xy instead of $x.y$.

3. Engel Polygroups

In this section, we introduce the notion of Engel polygroup and we obtain some results on Engel polygroups that are used in the other sections.

Definition 3.1. A polygroup P is said to be *n-Engel*($n \in \mathbb{N}$) if $l_{n,s}(P) \subseteq \omega$ or equivalently $l_{n,s}(P).\omega = \omega$ for any $s \in P$, where ω is the heart of P and

$$\begin{aligned} l_{0,s}(P) =& P, \\ l_{k+1,s}(P) =& \langle \{h \in P \mid h \in [x, s], x \in l_{k,s}(P)\} \rangle. \end{aligned}$$

Example 3.1. Let P be a polygroup by the following table:

.	e	a	b
e	e	a	b
a	a	e	b
b	b	b	$\{e, a, b\}$

.

Then $[e, a] = e,\ [a, a] = e,\ [b, a] = P$ and so, $l_{1,a}(P) = \langle \{h \in P \mid h \in [x, a], x \in P\} \rangle = P$. Similarly, we see that $l_{1,b}(P) = P = l_{1,e}(P)$. Therefore, for any $s \in P$, $l_{1,s}(P) = P = \omega$. Consequently, P is an 1-Engel polygroup.

Theorem 3.1. *Every polygroup of order less than* 7 *is* 1*-Engel.*

Proof. Suppose that P is a proper polygroup of order less than 7. Then $\frac{P}{\beta^*}$ is an Abelian group of order less than 6. Now, let $h \in l_{1,s}(P)$ where $s \in P$. Then there exists $x \in P$ such that $h \in [x, s]$. Thus,

$$\beta^*(h) = \beta^*([x, s]) = [\beta^*(x), \beta^*(s)] = \beta^*(e),$$

which implies that $h \in w$. Therefore, P is 1-Engel. $\square$

Theorem 3.2 ([9])**.** *Let $(G, \cdot)$ be a group and $P_G = G \cup \{a\}$, where $a \notin G$. Then $(P_G, \circ)$ is a polygroup, where operation "$\circ$" is defined as follows*

(1) $a \circ a = e$*;*
(2) $e \circ x = x \circ e = x$ *for every* $x \in P_G$*;*
(3) $x \circ x^{-1} = \{e, a\}$*, for every* $x \in P_G \setminus \{e, a\}$*;*
(4) $a \circ x = x \circ a = x$*, for every* $x \in P_G \setminus \{e, a\}$*;*
(5) $x \circ y = x \cdot y$*, for every* $(x, y) \in G^2$ *such that* $y \neq x^{-1}$.

Theorem 3.3. *Let G be an* 1*-Engel group. Then $\langle P_G, \circ, e, -1 \rangle$ is an* 1*-Engel polygroup.*

Proof. Let G be an 1-Engel group. By (1), $[a, a] = \{t \mid t \in a \circ a \circ a^{-1} \circ a^{-1} = e\}$. Then $e \in [a, a]$. Using (3) and (4), we have $e \in [a, y]$ in which $a \neq y \in P_G \setminus \{e, a\}$. Hence, $e \in [a, y]$ for any $y \in G \cup \{a\}$.

(I) Also, by hypotheses, for any $x, y \in G$ in which $y \neq x^{-1}$, we have $e = [x, y]$.

(II) So, by (I) and (II), $e \in [x, y]$ for any $x, y \in G \cup \{a\}$.

(III) Now, let $h \in l_{1,s}(P)$ where $s \in P_G$. Then $h \in [x, s]$ for some $x \in P_G$ and so by (III), $\beta^*(h) = [\beta^*(x), \beta^*(s)] = \beta^*(e)$, which implies that $h \in w$. Therefore, P_G is an 1-Engel polygroup. $\square$

Now, in the following theorem we give a method to construct a 1-Engel polygroup of order $n \in \mathbb{N}$.

Theorem 3.4. *For every $n \in \mathbb{N}$, there is a nontrivial 1-Engel polygroup of order $n+1$.*

Proof. For $n \in \mathbb{N}$, consider the Abelian group $\mathbb{Z}_n$. Clearly, $\mathbb{Z}_n$ is 1-Engel. Then by Theorem 3.3, $(P_{\mathbb{Z}_n}, \circ)$ is an 1-Engel polygroup of order $n+1$. $\square$

Theorem 3.5. *Let P_1 and P_2 be two polygroups. Then for any $k \geq 0$*

$$i_k(P_1 \times P_2) = i_k(P_1) \times i_k(P_2).$$

Proof. We prove our claim by induction on k. For $k = 0$, the proof is obvious. Now suppose that $(a, b) \in i_{k+1}(P_1 \times P_2)$. Then there exist $(u, v), (s, t) \in i_k(P_1 \times P_2)$ such that

$$(a, b) \in [(u, v), (s, t)] = [u, s] \times [v, t].$$

By using the hypotheses of induction, we conclude that $(u, v), (s, t) \in i_k(P_1) \times i_k(P_2)$. Thus for any $u, s \in i_k(P_1)$, we get $a \in [u, s]$ and for any $v, t \in i_k(P_2)$, we get $b \in [v, t]$. Hence $(a, b) \in i_{k+1}(P_1) \times i_{k+1}(P_2)$. Similarly, we obtain the converse. Therefore,

$$i_k(P_1 \times P_2) = i_k(P_1) \times i_k(P_2).$$ $\square$

Theorem 3.6. *Let P be a polygroup, $s \in P$ and N be a normal subpolygroup of P. Then*

$$l_{n,sN}\left(\frac{P}{N}\right) = \frac{l_{n,s}(P)N}{N}, \quad i_n\left(\frac{P}{N}\right) = \frac{i_n(P)N}{N}.$$

Proof. By induction on n we show that $l_{n,sN}\left(\frac{P}{N}\right) \subseteq \frac{l_{n,s}(P)N}{N}$ and $l_{n,sN}(\frac{P}{N}) \supseteq \frac{l_{n,s}(P)N}{N}$. For $n = 0$, the inclusions are obvious. Now, suppose that $yN \in l_{n+1,sN}(\frac{P}{N})$. Hence, there exists $aN \in l_{n,sN}(\frac{P}{N})$ such that $yN \in [aN, sN]$. By hypotheses of induction, we have $aN \in \frac{l_{n,s}(P)N}{N}$. Hence, there exists $a^{'} \in l_{n,s}(P)$ such that $aN = a^{'}N$. Thus, $yN \in [a^{'}N, sN] = [a^{'}, s]N$. So, there exist $a^{'} \in l_{n,s}(P)$ and $y^{'} \in [a^{'}, s]$ such that $yN = y^{'}N$. Hence, $yN \in \frac{l_{n+1,s}(P)N}{N}$. Conversely, if $yN \in \frac{l_{n+1,s}(P)N}{N}$, then there exists $y^{'} \in l_{n+1,s}(P)$ such that $yN = y^{'}N$. Therefore, $y^{'} \in [a, s]$, for some $a \in l_{n,s}(P)$. Thus, by hypotheses of induction, $aN \in \frac{l_{n,s}(P)N}{N} = l_{n,sN}\left(\frac{P}{N}\right)$ and $yN = y^{'}N \in [aN, sN]$ implies that $yN \in l_{n+1,sN}\left(\frac{P}{N}\right)$. Therefore, $l_{n,sN}\left(\frac{P}{N}\right) = \frac{l_{n,s}(P)N}{N}$. Similarly, we can prove that $i_n(\frac{P}{N}) = \frac{i_n(P)N}{N}$. $\square$

Corollary 3.1. (i) *If P is an n-Engel polygroup and N is a normal subpolygroup of P, then $\frac{P}{N}$ is n-Engel.*

(ii) If P is a solvable polygroup and N is a normal subpolygroup of P, then $\frac{P}{N}$ is solvabel.

Theorem 3.7. *Let P_1 and P_2 be two polygroups and $\phi : P_1 \to P_2$ be a good homomorphism. If ϕ is one to one and K is an n- Engel subpolygroup of P_1, then $\phi(K)$ is an n-Engel subpolygroup of P_2.*

Proof. By induction on n, we show that $l_{n,y}(\phi(K)) = \phi(l_{n,b}(K))$, where $\phi(b) = y$ and b, y are fix elements of K and $\phi(K)$, respectively. For $n = 0$, the proof is obvious. Now, let $z \in l_{n+1,y}(\phi(K))$. Then there exists $x \in l_{n,y}(\phi(K))$ such that $z \in [x, y]$. By hypotheses of induction, $x \in \phi(l_{n,b}(K))$. Also there exist $c, a \in K$ such that $z = \phi(c)$ and $x = \phi(a)$. Hence,

$$\phi(c) = z \in [\phi(a), \phi(b)] = \phi[a, b], \quad x = \phi(a) \in \phi(l_{n,b}(K)).$$

Thus for $a \in l_{n,b}(K)$, we get $c \in [a, b]$ that implies that $c \in l_{n+1,b}(K)$. Conversely, let $z \in \phi(l_{n+1,b}(K))$. Then for some $c \in l_{n+1,b}(K)$, $z = \phi(c)$. Using hypotheses of induction, $z = \phi(c) \in \phi[a, b] = [\phi(a), \phi(b)]$, where $a \in l_{n,b}(K)$, $y = \phi(b)$ and $\phi(a) \in l_{n,y}(\phi(K))$. Therefore, $z \in l_{n+1,y}(\phi(K))$. □

4. Results on Engel Fuzzy Subpolygroups

In this section, by considering the notion of Engel fuzzy subpolygroup, which is defined in [7], we state and prove some new related results.

Definition 4.1 ([7])**.** Let μ be a fuzzy subpolygroup of P and $n \in \mathbb{N}$. If for any $x, y \in P$ and $z \in [x,_n y]$, we have $\mu(z) = \mu(e)$, then μ is called an *n-Engel fuzzy subpolygroup* of P.

Theorem 4.1 ([7])**.** *Let P and P' be two polygroups with the identity elements e_1 and e_2, respectively, μ and λ be two n-Engel fuzzy subpolygroup of P and P', respectively, and $f : P \to P'$ be a function.*

(i) If f is a good homomorphism, then $f^{-1}(\lambda)$ is an n-Engel fuzzy subpolygroup of P.

(ii) If f is an onto good homomorphism, then $f(\mu)$ is an n-Engel fuzzy subpolygroup of P'.

Proposition 4.1 ([7])**.** *Let μ be a normal fuzzy subpolygroup of P and relation $\sim$ on P is defined as follows:*

$$x \sim y \Leftrightarrow (\exists a \in xy^{-1})\ st.\ \mu(a) = \mu(e).$$

Then $\sim$ is a strongly regular relation on P.

Suppose that for any $x \in P$, $\mu[x]$ is the equivalence class containing x with respect to strongly regular relation $\sim$ on P and $\frac{P}{\sim}$ denoted the set of all equivalence classes $\mu[x]$, i.e., $\frac{P}{\sim} = \{\mu[x] \mid x \in P\}$.

Theorem 4.2 ([7]). $\left(\frac{P}{\sim}, \odot, ^{-1}, \mu[e]\right)$ *is a group, where*

$$\mu[x]^{-1} = \mu[x^{-1}], \quad \mu[x] \odot \mu[y] = \{\mu[z] \mid z \in xy\},$$

for any $x, y \in P$.

Theorem 4.3 ([7]). *Let μ be a normal fuzzy subpolygroup of a polygroup P. Then μ is a n-Engel fuzzy subpolygroup of P if and only if $\frac{P}{\sim}$ is a n-Engel group.*

Let μ be a normal fuzzy subpolygroup of P. Then $\{\mu(x) \mid x \in P\}$ is called the *order of μ.*

Theorem 4.4. *Any normal fuzzy subpolygroup of order less than* 6*, is an* 1*-Engel fuzzy subpolygroup of P.*

Proof. Let μ be a normal fuzzy subpolygroup of order less than 6. Then $\frac{P}{\sim}$ is a group of order less than 6. Hence it is Abelian, which implies that $\frac{P}{\sim}$ is an 1-Engel group. Now, by Theorem 4.3, μ is a 1-Engel fuzzy subpolygroup of P. $\square$

Let $\mu_* = \{x \mid \mu(x) = \mu(e)\}$. Clearly, μ_* is a normal subpolygroup of P.

Theorem 4.5. *If P is an n-Engel polygroup, then any normal fuzzy subpolygroups of P is n-Engel.*

Proof. Let P be n-Engel and μ be a normal fuzzy subpolygroup of P. First we show that $\frac{P}{\sim} \approx \frac{P}{\mu_*}$. Define

$$f : \frac{P}{\sim} \to \frac{P}{\mu_*} \text{ by } f(\mu[x]) = \mu_* x, \quad x \in P.$$

If $\mu[x] = \mu[y]$ for $x, y \in P$, then $x \sim y$ and so there exists $r \in xy^{-1}$ such that $\mu(r) = \mu(e)$, where e is the identity element of P. Now, we show that for any $x, y \in P$, if $x \sim y$, then $\mu(r) = \mu(e)$ for any $r \in xy^{-1}$. If $x \sim y$, then by the definition of $\sim$, there exists $a \in xy^{-1}$ such that $\mu(a) = \mu(e)$. Now, let $r \in xy^{-1}$ be an arbitrary element of P. Since μ is normal, we have $\mu(e) = \mu(a) = \mu(r)$ which implies that for any $r \in xy^{-1}$, $\mu(e) = \mu(r)$. Hence, $xy^{-1} \subseteq \mu_*$. Thus, $\mu_* x = \mu_* y$.

Conversely, if $\mu_* x = \mu_* y$, then $xy^{-1} \subseteq \mu_*$ and so for any $r \in xy^{-1}$, $\mu(e) = \mu(r)$, which implies that $x \sim y$. Consequently, f is an isomorphism by the fact that

$$\mu_* x \odot \mu_* y = \{\mu_* z \mid z \in xy\}, \quad \mu[x] \odot \mu[y] = \{\mu[z] \mid z \in xy\}.$$

Hence, $\frac{P}{\sim} \approx \frac{P}{\mu_*}$. Since P is n-Engel, by Corollary 3.1, $\frac{P}{\mu_*}$ is n-Engel and so $\frac{P}{\sim}$ is n-Engel. Therefore, by Theorem 4.3, μ is n-Engel. $\square$

Example 4.1. Let $P = \{e, a, b, c, d, f, g\}$. Then P with the following hyperoperation is a polygroup

$\cdot$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e,a\}$	g	f	d	c
c	c	c	f	$\{e,a\}$	g	b	d
d	d	d	g	f	$\{e,a\}$	c	b
f	f	f	c	d	b	g	$\{e,a\}$
g	g	g	d	b	c	$\{e,a\}$	f

Now, we define the fuzzy set μ on P, by

$$\mu(x) = \begin{cases} 0.75, & x \in \{e,a,f,g\}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, P is not an n-Engel polygroup. But, we show that μ is a normal n-Engel fuzzy subpolygroup of P. Since, for any $t \in [0,1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e,a,f,g\}$ or P, hence, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. Now, for any $z \in [x,_n s]$ where $x, s \in P$ we get $z \in l_n(P) = \{e,a,f,g\}$ and so $\mu(z) = \mu(e)$, which implies that μ is a normal n-Engel fuzzy subpolygroup of P.

Theorem 4.6. *Let μ be a normal fuzzy subpolygroup of $(P,\cdot,^{-1},e_1)$. Then $\left(\frac{P}{\mu_*},\cdot,^{-1},e_2\right)$ is an n-Engel polygroup if and only if μ is an n-Engel fuzzy subpolygroup of P.*

Proof. Let $\frac{P}{\mu_*}$ be an n-Engel polygroup and $\pi : P \to \frac{P}{\mu_*}$ be the natural epimorphism. Since $z \in \pi^{-1}(\pi(x))$, we get $\pi(z) = \pi(x)$ and so $\pi(e_1) \in \pi(z^{-1}\cdot z) = \pi(z^{-1}\cdot x)$. Thus, there exists $r \in z^{-1}\cdot x$ such that $e_2 = \pi(e_1) = \pi(r)$, which implies that $r \in \ker\pi = \mu_*$. Therefore, $\mu(r) = \mu(e_1)$ and so $z \sim x$. Hence, for any $x \in P$

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z\in\pi^{-1}(\pi(x))} \mu(z) = \bigvee_{z\sim x} \mu(z) \geq \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) \supseteq \mu$. Now, since $\frac{P}{\mu_*}$ is an n-Engel polygroup and $\pi(\mu)$ is a fuzzy subpolygroup of $\frac{P}{\mu_*}$, by Theorem 4.5, $\pi(\mu)$ is n-Engel and by Theorem 4.1, $\pi^{-1}(\pi(\mu))$ is an n-Engel. Now, we show that μ is n-Engel. For this, let $x \in [t,_n s]$, where $t \in P$, $s \in P$ and $f : \frac{P}{\sim} \to \frac{P}{\mu_*}$ be as in the proof of Theorem 4.5. Since $\pi^{-1}(\pi(\mu))$ is an n-Engel fuzzy subpolygroup of P, so $\pi^{-1}(\pi(\mu))(x) = \pi^{-1}(\pi(\mu))(e_1)$. Hence, $\bigvee_{z\sim x}\mu(z) = \mu(e_1)$. Then $x \sim e_1$ and so $\mu[x] = \mu[e_1]$. Hence by $f(\mu[x]) = \mu_* x$ we have $\mu_* x = \mu_* e_1$. Thus, $x \in \mu_*$, which implies that $\mu(x) = \mu(e_1)$. Therefore, μ is an n-Engel fuzzy subpolygroup of P.

Conversely, let μ be a normal n-Engel fuzzy subpolygroup of P. By Theorem 4.3, $\frac{P}{\sim}$ is an n-Engel group also, $\frac{P}{\sim} \cong \frac{P}{\mu_*}$ and so $\frac{P}{\mu_*}$ is an n-Engel group. □

Example 4.2. Let $D_3 = \langle a, b; a^3 = b^2 = e, ba = a^2 b\rangle$ be the dihedral group with six elements and $t_0, t_1 \in [0,1]$ such that $t_0 > t_1$. Define a fuzzy subgroup μ of D_3 as follows:

$$\mu(x) = \begin{cases} t_0, & \text{if } x \in < a >, \\ t_1 & \text{if } x \notin < a > . \end{cases}$$

Then $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a \rangle$. Thus, μ_* is a normal subgroup of D_3. Also, $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian, hence it is 1-Engel and so by Theorem 4.6, μ is an 1-Engel fuzzy subpolygroup of D_3.

Theorem 4.7. *Let μ and ν be two fuzzy subpolygroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is an n-Engel fuzzy subpolygroup of P, then ν is an n-Engel fuzzy subpolygroup of P, too.*

Proof. Let μ and ν be two fuzzy subgroups of P, where $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Now let μ be an n-Engel and $x \in [h,_n s]$, where $h \in P$ and $s \in P$. Then, $\mu(x) = \mu(e) = \nu(e)$ and so by hypotheses $\nu(e) = \mu(x) \leq \nu(x)$. Thus, $\nu(x) = \nu(e)$, which implies that ν is an n-Engel fuzzy subpolygroup of P. □

Definition 4.2 ([6])**.** Let μ be a fuzzy set on P. Then the lower level subset of μ is defined by,

$$\overline{\mu}_t = \{x \in P; \mu(x) \leq t\}, \quad \text{where } t \in [0,1].$$

Now the fuzzy set $A_{\overline{\mu}_t}$ is defined by

$$A_{\overline{\mu}_t}(x) = \begin{cases} \mu(x), & \text{if } x \in \overline{\mu}_t, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $A_{\overline{\mu}_t} \subseteq \mu$.

Corollary 4.1. *Let $A_{\overline{\mu}_t}$ be an n-Engel fuzzy subpolygroup of P. Then μ is an n-Engel fuzzy subpolygroup of P, too.*

Proof. Let μ be an Engel fuzzy subpolygroup of P. Clearly, $A_{\overline{\mu}_t}$ is a fuzzy supolygroup of P. Since $A_{\overline{\mu}_t} \subseteq \mu$, by Theorem 4.7, $A_{\overline{\mu}_t}$ is Engel fuzzy subpolygroup of P. □

Suppose that μ is a fuzzy subset of P. Support of μ is defined by $\text{supp}(\mu) = \{x \in P \mid \mu(x) > 0\}$.

Definition 4.3 ([4])**.** Let μ and ν be fuzzy subpolygroups of P and H, respectively. Then a good isomorphism $f : \text{supp}(\mu) \to \text{supp}(\nu)$ is called a *fuzzy good isomorphism* from μ to ν, if there exists a positive real number k such that $\mu(x) = k\nu(f(x))$ for any $x \in \text{supp}(\mu) \setminus \{e\}$. In this case we write $\mu \simeq \nu$.

Theorem 4.8. *Let μ and ν be two fuzzy subpolygroups of $(P, \cdot, ^{-1}, e_1)$ and $(H, \cdot, ^{-1}, e_2)$, respectively, and $\mu \simeq \nu$. If μ is n-Engel, then ν is an Engel fuzzy subpolygroup of* $\text{supp}(\nu)$.

Proof. Let $z \in [x,_n y]$, where $x, y \in \text{supp}(\nu)$. Since $\mu \simeq \nu$, then there exists a positive real number k such that $\mu(x) = k\nu(f(x))$ for any $x \in \text{supp}(\mu) \setminus \{e_1\}$ and $x = f(a)$, $y = f(b)$ for some $a, b \in \text{supp}(\mu)$. So, $z \in [x,_n y] = [f(a),_n f(b)] = f[a,_n b]$. Therefore, $z = f(c)$, for some $c \in [a,_n b]$ and so, by hypotheses $\mu(c) = \mu(e_1)$. Thus,

$k\nu(z) = k\nu(f(c)) = \mu(c) = \mu(e_1) = k\nu(f(e_1)) = k\nu(e_2)$ and so, $\nu(z) = \nu(e_2)$, which implies that ν is n-Engel. □

5. Nilpotent Fuzzy Subpolygroups

In this section, by considering the notion of nilpotent fuzzy subpolygroup, we state and prove some results on this structure.

Definition 5.1 ([7])**.** Let μ be a fuzzy subpolygroup of P. Then μ is called a *nilpotent fuzzy subpolygroup of class n* ($n \in \mathbb{N}$), if $z \in l_n(P)$ implies that $\mu(z) = \mu(e)$.

Theorem 5.1. *Any nilpotent fuzzy subpolygroup of class $n = 1$ is a normal fuzzy subpolygroup.*

Proof. Let μ be a nilpotent fuzzy subpolygroup of class $n = 1$. Then for any $z \in l_1(P)$, $\mu(e) = \mu(z)$. Now, the proof follows by Theorem 2.4. □

By the following example we see that the converse of Theorem 5.1, is not true in general.

Example 5.1. Let $P = \{e, a, b, c, d, f, g\}$. Then P with the following hyperoperation is a polygroup

$\cdot$	e	a	b	c	d	f	g
e	e	a	b	c	d	f	g
a	a	e	b	c	d	f	g
b	b	b	$\{e,a\}$	g	f	d	c
c	c	c	f	$\{e,a\}$	g	b	d
d	d	d	g	f	$\{e,a\}$	c	b
f	f	f	c	d	b	g	$\{e,a\}$
g	g	g	d	b	c	$\{e,a\}$	f

.

We define the fuzzy set μ on P, by

$$\mu(x) = \begin{cases} 0.75, & x \in \{e, a\}, \\ 0.5, & x \in \{f, g\}, \\ 0, & \text{otherwise}. \end{cases}$$

We show that, μ is a normal fuzzy subpolygroup of P which is not nilpotent of class $n = 1$. First for any $t \in [0, 1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e, a, f, g\}$, $\{e, a\}$ or P and since for any $x \in P$, $x^{-1}\{e, a\}x \subseteq \{e, a\}$, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. But for $g = [c, f] \in l_1(P) = \{e, a, f, g\}$ we get $\mu(g) \neq \mu(e)$ which implies that μ is not nilpotent of class $n = 1$.

Theorem 5.2. *Let P_1 and P_2 be two polygroups with the identity elements e_1 and e_2, respectively. Suppose that μ and λ be two nilpotent fuzzy subpolygroups of P_1 and P_2, respectively, and $\phi : P_1 \to P_2$ be a function.*

(i) *If ϕ is a good homomorphism, then $\phi^{-1}(\lambda)$ is a nilpotent fuzzy subpolygroup of P_1.*

(ii) *If ϕ is an isohomomorphism, then $\phi(\mu)$ is a nilpotent fuzzy subpolygroup of P_2.*

Proof. (i) The proof is clear. (ii) First note that $l_n(\phi(P_1)) = \phi(l_n(P_1))$ (see [9]). Now, let μ be a nilpotent fuzzy subpolygroup of P_1 and $y \in l_{n,}(P_2)$. Then,

$$y \in l_n(P_2) = l_n(\phi(P_1)) = \phi(l_n(P_1))$$

and so there exists $z \in l_n(P_1)$ such that $y = \phi(z)$. By hypotheses, $\mu(z) = \mu(e_1)$. Thus,

$$\phi(\mu)(y) = \bigvee_{x \in \phi(y)} \mu(x) = \mu(z) = \mu(e_1) = \phi(\mu)(e_2).$$

Hence, $\phi(\mu)(y) = \phi(\mu)(e_2)$, which implies that $\phi(\mu)$ is nilpotent. □

Theorem 5.3 ([7])**.** *Let μ be a fuzzy subpolygroup of P. Then μ is a nilpotent fuzzy subpolygroup of P if and only if $\frac{P}{\sim}$ is a nilpotent group.*

Note that if P is a nilpotent polygroup and N is a normal subpolygroup of P, then $\frac{P}{N}$ is nilpotent (see [9]).

Theorem 5.4. *If P is a nilpotent polygroup, then any normal fuzzy subpolygroup of P is nilpotent.*

Proof. Let P be a nilpotent of class n and μ be a fuzzy subpolygroup of P. Since $\frac{P}{\sim} \approx \frac{P}{\mu_*}$ and P is nilpotent, $\frac{P}{\mu_*}$ is nilpotent and so $\frac{P}{\sim}$ is nilpotent. Therefore, by Theorem 5.3, μ is nilpotent. □

Example 5.2. Let μ be as Example 4.1. We show that, P is not nilpotent and μ is a nilpotent normal fuzzy subpolygroup of P. First note that $l_n(P) = \{e, a, f, g\}$ (see [9]) and so P is not nilpotent. Also, for any $t \in [0, 1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e, a, f, g\}$ or P. Therefore, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. But for any $z \in l_n(P)$, $\mu(z) = \mu(e)$, which implies that μ is nilpotent.

Theorem 5.5. *Let μ be a normal fuzzy subpolygroup of $(P, \cdot, ^{-1}, e_1)$. Then $\left(\frac{P}{\mu_*}, \cdot, ^{-1}, e_2\right)$ is a nilpotent polygroup if and only if μ is a nilpotent fuzzy subpolygroup of P.*

Proof. Let $\frac{P}{\mu_*}$ be a nilpoten polygroup and $\pi : P \to \frac{P}{\mu_*}$ be the natural epimomorphism. Since $z \in \pi^{-1}(\pi(x))$, we have $\pi(z) = \pi(x)$ and so $\pi(e_1) \in \pi(z^{-1} \cdot z) = \pi(z^{-1} \cdot x)$. Then, there exists $r \in z^{-1} \cdot x$ such that $e_2 = \pi(e_1) = \pi(r)$, which implies that $r \in \ker \pi = \mu_*$. Thus, $\mu(r) = \mu(e_1)$ and so $z \sim x$. Hence, for any $x \in P$

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z \in \pi^{-1}(\pi(x))} \mu(z) = \bigvee_{z \sim x} \mu(z) \geq \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) \supseteq \mu$. Now since $\frac{P}{\mu_*}$ is a nilpotent polygroup and $\pi(\mu)$ is a fuzzy subpolygroup of $\frac{P}{\mu_*}$, then by Theorem 5.4, $\pi(\mu)$ is nilpotent and by Theorem 5.2, $\pi^{-1}(\pi(\mu))$ is nilpotent. Now, we show that μ is nilpotent. For this, let $x \in l_n(p)$ and $f : \frac{P}{\sim} \to \frac{P}{\mu_*}$ be as in the proof of Theorem 4.5. Since $\pi^{-1}(\pi(\mu))$ is nilpotent, so $\pi^{-1}(\pi(\mu))(x) = \pi^{-1}(\pi(\mu))(e_1)$. Hence, $\bigvee_{z \sim x} \mu(z) = \mu(e_1)$ and so $x \sim e_1$. Then

$\mu[x] = \mu[e_1]$ and by $f(\mu[x]) = \mu_* x$, we have $\mu_* x = \mu_* e_1$. Thus, $x \in \mu_*$, which implies that $\mu(x) = \mu(e_1)$. Therefore, μ is a nilpotent fuzzy subpolygroup of P.

Conversely, let μ be a normal nilpotent fuzzy subpolygroup of P. By Theorem 5.3, $\frac{P}{\sim}$ is nilpotent. Also, $\frac{P}{\sim} \cong \frac{P}{\mu_*}$ and so $\frac{P}{\mu_*}$ is nilpotent. □

Example 5.3. In Example 4.2, $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a \rangle$. Thus μ_* is a normal subgroup of D_3. Also $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian, hence it is nilpotent and so by Theorem 5.5, μ is a nilpotent fuzzy subpolygroup.

Theorem 5.6. *Let μ and ν be two fuzzy subpolygroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is a nilpotent fuzzy subpolygroup of class n, then ν is a nilpotent fuzzy subpolygroup of class n.*

Proof. Let μ and ν be two fuzzy subgroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Now let μ be nilpotent of class n and $x \in l_n(P)$. Therefore, $\mu(x) = \mu(e) = \nu(e)$ and so by hypotheses $\nu(e) = \mu(x) \leq \nu(x)$. Thus, $\nu(x) = \nu(e)$, which implies that ν is nilpotent of class at most n. □

Corollary 5.1. *Let $A_{\overline{\mu}_t}$ be a nilpotent fuzzy subpolygroup of P. Then μ is nilpotent, too.*

Proof. Let $A_{\overline{\mu}_t}$ be a nilpotent fuzzy subpolygroup of P. Since $A_{\overline{\mu}_t} \subseteq \mu$, by Theorem 5.6, μ is nilpotent. □

6. Solvable Fuzzy Subpolygroups

In this section, we introduce the notion of solvable fuzzy subpolygroup on a polygroup and we state and prove some new results on it. Specially, we get the relation between solvable fuzzy subpolygroups and Engel fuzzy subpolygroups (nilpotent fuzzy subpolygroups).

Definition 6.1. Let μ be a fuzzy subpolygroup of P. Then μ is called a *solvable fuzzy subpolygroup* of P if there exists $n \in \mathbb{N}$ such that for any $z \in i_n(P)$, $\mu(z) = \mu(e)$.

In the following example we have a solvable fuzzy subpolygroup.

Example 6.1. Let $P = \{e, a, b, c, d\}$. Then P with the following hyperoperation is a polygroup

.	e	a	b	c	d
e	e	a	b	c	d
a	a	e	b	c	d
b	b	b	{ e,a }	d	c
c	c	c	d	{ e,a }	b
d	d	d	c	b	{ e,a }

.

We define the fuzzy subset μ on P, by

$$\mu(x) = \begin{cases} 0.75, & x \in \{e, a\}, \\ 0.5, & x = b, \\ 0, & \text{otherwise}. \end{cases}$$

Then we show that, μ is a solvable fuzzy subpolygroup. First for any $t \in [0,1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to $\{e,a,b\}$, $\{e,a\}$ or P. Hence, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. Since for any $x, y \in P$, $[x,y] = e$ *or* $\{e,a\}$ then for any $z \in i_1(P)$, $\mu(z) = \mu(e)$ and so, μ is solvable.

In the following, we are ready to obtain a necessary and sufficient condition between solvable fuzzy subpolygroups and the solvable group $P/\sim$, the group of equivalence classes derived from the fuzzy subpolygroup of P. Now, we use notation $i_k(H)$ instead of derived series G^k, where $k \in N$ and H is a group. Also, for simplity we write $\mu[x]\mu[y]$ instead of $\mu[x] \odot \mu[y]$.

Lemma 6.1. *For any* $0 \leq k$

$$i_k\left(\frac{P}{\sim}\right) = \langle\{\mu[t] \mid t \in i_k(P)\}\rangle.$$

Proof. We do the proof by induction on k. For $k = 0$, we have

$$i_0\left(\frac{P}{\sim}\right) = \frac{P}{\sim} = \langle\{\mu[t] \mid t \in i_0(P) = P\}\rangle.$$

Now, let it is true for k. We claim that

$$i_{k+1}\left(\frac{P}{\sim}\right) \supseteq \langle\{\mu[t] \mid t \in i_{k+1}(P)\}\rangle.$$

For this, suppose that $\mu[a] \in \langle\{\mu[t] \mid t \in i_{k+1}(P)\}\rangle$. Then $a \in i_{k+1}(P)$ and so there exist $x, s \in i_k(P)$ such that $a \in [x,s]$. By hypotheses of induction we conclude that $\mu[x], \mu[s] \in i_k(\frac{P}{\sim})$. Thus, $\mu[a] = [\mu[x], \mu[s]]$ in which $\mu[x], \mu[s] \in i_k(\frac{P}{\sim})$. Hence, $\mu[a] \in i_{k+1}\left(\frac{P}{\sim}\right)$. Also,

$$i_{k+1}\left(\frac{P}{\sim}\right) \subseteq \langle\{\mu[t] \mid t \in i_{k+1}(P)\}\rangle.$$

Since for $\mu[a] \in \frac{P}{\sim} \in i_{k+1}(\frac{P}{\sim})$, we have $\mu[a] = [\mu[x], \mu[s]]$ in which $\mu[x], \mu[s] \in i_k\left(\frac{P}{\sim}\right)$. Using hypotheses of induction $x, s \in i_k(P)$ (1). Thus $\mu[a] = \mu[x]\mu[s](\mu[x])^{-1}(\mu[s])^{-1}$, which implies that $\mu[x]\mu[s] = \mu[a]\mu[s]\mu[x]$. Thus, there exist $c \in xs$ and $d \in asx$ such that $\mu[c] = \mu[d]$. Since P is a polygroup, then there exists $u \in P$ such that $c \in xs \cap usx$ (2). Then

$$\mu[a]\mu[s]\mu[x] = \mu[d] = \mu[c] = \mu[x]\mu[s] = \mu[c] = \mu[u]\mu[s]\mu[x].$$

Hence, $\mu[a] = \mu[u]$ (3). By (2) and (1), we have $u \in i_{k+1}(P)$. Now, using (3) and previous relation we have

$$\mu[a] = \mu[u] \in \langle\{\mu[t] \mid t \in i_{k+1}(P)\}\rangle. \qquad \square$$

Theorem 6.1. *Let* μ *be a normal fuzzy subpolygroup of a polygroup* P. *Then* μ *is a solvable fuzzy subpolygroup if and only if* $\frac{P}{\sim}$ *is a solvable group.*

Proof. ($\Rightarrow$) Suppose that μ is a solvable fuzzy subpolygroup of P and $k \in \mathbb{N}$. Then by Lemmas 6.1, it is enough to show that $\langle\{\mu[t] \mid t \in i_k(P)\}\rangle = \{\mu[e]\}$. If $t \in i_k(P)$, then by hypotheses $\mu(t) = \mu(e)$ and so $t \sim e$, which implies that $\mu[t] = \mu[e]$. Therefore, $\frac{P}{\sim}$ is a solvable group.

($\Leftarrow$) Let $\frac{P}{\sim}$ is solvable. We show that if $z \in i_k(P)$, then $\mu(z) = \mu(e)$. If $z \in i_k(P)$, then $z \in [x, s]$ where $x, s \in i_{k-1}(P)$. Hence, $\mu[z] = [\mu[x], \mu[s]]$, which by hypotheses implies that $\mu[z] = \mu[e]$ and so $z \sim e$. Then there exists $r \in ze^{-1}$ such that $\mu(r) = \mu(e)$ and so $\mu(z) = \mu(r) = \mu(e)$. Therefore, μ is an a solvable fuzzy subpolygroup. □

Theorem 6.2. *Let P be a solvable polygroup. Then any normal fuzzy subpolygroup of P is solvable.*

Proof. Let P be solvable polygroup and μ be a fuzzy subpolygroup of P. Since $\frac{P}{\sim} \approx \frac{P}{\mu_*}$ and P is solvable, by Corollary 3.1, $\frac{P}{\mu_*}$ is solvable and so $\frac{P}{\sim}$ is solvable, too. Therefore, by Theorem 6.1, μ is solvable. □

Example 6.2. Let A_5 be the alternating group of degree 5 and $P = A_5 \cup \{a\}$ be a polygroup as Theorem 3.2. We define the fuzzy subset μ on P, by $\mu(x) = 1$, for any $x \in P$. It is clear that P is not solvable (see [9]). But, for any $t \in [0, 1]$, $\mu_t = \{x \mid \mu(x) \geq t\}$ is equal to P. Hence, by Theorem 2.3, μ is a normal fuzzy subpolygroup of P. Now, since for any $z \in P$, $\mu(z) = \mu(e)$, we get that μ is solvable.

By the same manipulation of Theorem 5.2, we have the following theorem.

Theorem 6.3. *Let P_1 and P_2 be two polygroups with the identity elements e_1 and e_2, respectively. Suppose that μ and λ be two solvable fuzzy subpolygroup of P_1 and P_2, respectively, and $\phi : P_1 \to P_2$ be a function.*

(i) *If ϕ is a good homomorphism, then $\phi^{-1}(\lambda)$ is a solvable fuzzy subpolygroup of P_1.*
(ii) *If ϕ is an isohomomorphism, then $\phi(\mu)$ is a solvable fuzzy subpolygroup of P_2.*

Theorem 6.4. *Let μ be a normal fuzzy subpolygroup of $(P, \cdot, ^{-1}, e_1)$. Then $(\frac{P}{\mu_*}, \cdot, ^{-1}, e_2)$ is a solvable polygroup if and only if μ is a solvable fuzzy subpolygroup.*

Proof. Let $\frac{P}{\mu_*}$ be a solvable polygroup and $\pi : P \to \frac{P}{\mu_*}$ be the natural epimorphism. Since $z \in \pi^{-1}(\pi(x))$, we have $\pi(z) = \pi(x)$ and so $\pi(e_1) \in \pi(z^{-1} \cdot z) = \pi(z^{-1} \cdot x)$. Thus, there exists $r \in z^{-1} \cdot x$ such that $e_2 = \pi(e_1) = \pi(r)$ which implies that $r \in \ker \pi = \mu_*$. Hence, $\mu(r) = \mu(e_1)$ and so $z \sim x$. Then, for any $x \in P$,

$$\pi^{-1}(\pi(\mu))(x) = \pi(\mu)(\pi(x)) = \bigvee_{z \in \pi^{-1}(\pi(x))} \mu(z) = \bigvee_{z \sim x} \mu(z) \geq \mu(x),$$

and so $\pi^{-1}(\pi(\mu)) \supseteq \mu$. Now, since $\frac{P}{\mu_*}$ is a solvable polygroup and $\pi(\mu)$ is a fuzzy subpolygroup of $\frac{P}{\mu_*}$, by Theorem 6.2, $\pi(\mu)$ is solvable and by Theorem 6.3, $\pi^{-1}(\pi(\mu))$ is solvable. Now, we show that μ is solvable. For this let $x \in i_n(p)$ and $f : \frac{P}{\sim} \longrightarrow \frac{P}{\mu_*}$ be as in the proof of Theorem 4.5. Since $\pi^{-1}(\pi(\mu))$ is solvable, so $\pi^{-1}(\pi(\mu))(x) =$

$\pi^{-1}(\pi(\mu))(e_1)$. Then $\bigvee_{z\sim x}\mu(z) = \mu(e_1)$ and so $x \sim e_1$. Hence, $\mu[x] = \mu[e_1]$. Now, by $f(\mu[x]) = \mu_* x$ we have $\mu_* x = \mu_* e_1$. Thus $x \in \mu_*$ which implies that $\mu(x) = \mu(e_1)$. Therefore, μ is a solvable fuzzy subpolygroup.

Conversely, let μ be a normal solvable fuzzy subpolygroup of P. By Theorem 6.1, $\frac{P}{\sim}$ is solvable also, $\frac{P}{\sim} \cong \frac{P}{\mu_*}$ and so $\frac{P}{\mu_*}$ is solvable. □

Example 6.3. In Example 4.2, $\mu(e) = t_0$ and so $\mu_* = \{x \mid \mu(x) = \mu(e)\} = \langle a\rangle$. Thus μ_* is a normal subgroup of D_3. Also, $\frac{D_3}{\mu_*} \approx \mathbb{Z}_2$. Since $\mathbb{Z}_2$ is Abelian, it is solvable and so by Theorem 6.4, μ is a solvable fuzzy subgroup.

Theorem 6.5. *Let μ and ν be two fuzzy subpolygroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. If μ is a solvable fuzzy subpolygroup, then ν is a solvable fuzzy subpolygroup.*

Proof. Let μ and ν be two fuzzy subgroups of P such that $\mu \subseteq \nu$ and $\mu(e) = \nu(e)$. Now let μ be solvable and $x \in i_n(P)$. Hence, $\mu(x) = \mu(e) = \nu(e)$ and so by hypotheses $\nu(e) = \mu(x) \leq \nu(x)$. Therefore, $\nu(x) = \nu(e)$, which implies that ν is solvable. □

Corollary 6.1. *If $A_{\overline{\mu}_t}$ is a solvable fuzzy subpolygroup of P, then μ is solvable, too.*

Proof. Let $A_{\overline{\mu}_t}$ be a solvable fuzzy subpolygroup of P. Since $A_{\overline{\mu}_t} \subseteq \mu$, by Theorem 6.5, μ is solvable. □

Theorem 6.6. *Let μ be a nilpotent fuzzy subpolygroup of P. Then μ is a solvable fuzzy subpolygroup.*

Proof. First we prove that $i_j(P) \subseteq l_j(P)$, for any non negative integer j. We do the proof by induction on j. The proof is clear for $j = 0$. Now let $i_j(P) \subseteq l_j(P)$, for any $j \leq n$ and $x \in i_n(P)$. Then $x \in [a, b]$, for some $a, b \in i_{n-1}(P)$. By hypotheses of induction, $a \in l_{n-1}(P)$ and $b \in P$. Thus, $x \in l_n(P)$. Hence $i_j(P) \subseteq l_j(P)$, for any non negative integer j. Now, let $x \in i_n(P)$ and μ be a nilpotnt fuzzy subpolygroup of class $n \in \mathbb{N}$. Since $x \in i_n(P) \subseteq l_n(P)$ so by hypotheses $\mu(x) = \mu(e)$. Therefore, μ is solvable. □

We recall that if G is a group and $a \in G$, then the order of a is the least positive integer n such that $a^n = e$. Also, a group G is of exponent n ($n \in \mathbb{N}$), if the order of any $x \in G$ is n.

Definition 6.2. If μ is a fuzzy subpolygroup of P and $a \in P$, then the *order of a with respect to* μ is the least positive integer n such that for any $r \in a^n$, $\mu(r) = \mu(e)$. We denote the order of a with respect to μ by $\circ(\mu(a))$. Also, μ is of exponent n, if the order of any $a \in P$ is n.

Theorem 6.7. *Let μ be a fuzzy polygroup of P and $x \in P$. If for any $r \in x^m$ we have $\mu(r) = \mu(e)$ for some integer m, then $\circ(\mu(a)) \mid m$.*

Proof. Let $\circ(\mu(a)) = n$. By the Euclidean algorithm, there exist integers s and t such that $m = ns + t$, where $0 \leq t < n$. Then for $r \in x^t = x^m \cdot (x^n)^{-s}$, there exist

$h \in x^m$ and $g \in (x^n)^{-s}$ such that $r \in hg$ and so $\mu(r) \geq \mu(h) \wedge \mu(g) \geq \mu(e) \wedge \mu(g) = \mu(g)$. Since $g \in (x^n)^{-s} = (x^n)^{-1} \cdot (x^n)^{-1} \cdots (x^n)^{-1}$, we get $g \in p_1.p_2 \ldots p_s$, in which $p_1, p_2, \ldots, p_s \in (x^n)^{-1}$ and so by hypotheses $\mu(g) \geq \mu(e)$. Consequently, $\mu(r) = \mu(e)$. Hence, $t = 0$, by the minimality of n. □

Theorem 6.8 ([11, 14])**.** (i) *Every* 3*-Engel group of exponent* 4*, is solvable.*
(ii) *Each group of exponent* 3 *is* 2*-Engel.*

Theorem 6.9. (i) *Let* μ *be a* 3*-Engel normal fuzzy subpolygroup of exponent* 4*. Then* μ *is solvable.*
(ii) *Each normal fuzzy subpolygroup of exponent* 3 *is* 2*-Engel.*

Proof. (i) Let μ be a 3-Engel normal fuzzy subpolygroup on P such that for any $z \in x^4$, $\mu(z) = \mu(e)$. Then, by Theorem 4.3, $\underset{\sim}{P}$ is a 3-Engel group and $\mu[e] = \mu[z] = (\mu[x])^4$. Therefore, by Theorem 6.8 (i), $\underset{\sim}{P}$ is solvable and so, by Theorem 6.1, μ is solvable.

(ii) By Theorem 4.2, $\mu[e] = \mu[z] = (\mu[x])^3$. Thus, $\underset{\sim}{P}$ is of exponent 3 and so by Theorem 6.8(ii), $\underset{\sim}{P}$ is 2-Engel. Therefore, by Theorem 4.3, μ is 2-Engel. □

Theorem 6.10 ([18])**.** *Every* 3*-Engel solvable group with no element of order* 2*, is nilpotent.*

Theorem 6.11. *Let* μ *be a* 3*-Engel solvabel normal fuzzy subpolygroup on* P *such that for any* $z \in x^2$*,* $\mu(z) \neq \mu(e)$*. Then* μ *is nilpotent.*

Proof. Let μ be a 3-Engel solvable normal fuzzy subpolygroup of P such that for any $z \in x^2$, $\mu(z) \neq \mu(e)$. Then, by Theorems 4.3 and 6.1 $\underset{\sim}{P}$ is a 3-Engel solvabel group and $\mu(e) \neq \mu(z) = (\mu(x))^2$. Therefore, by Theorem 6.10, $\underset{\sim}{P}$ is nilpotent and so, by Theorem 5.3, μ is nilpotent. □

7. Conclusions

In this paper, we defined the notion of Engel polygroups. This help us to get usefull results on Engel fuzzy subpolygroups. On the other hand, we prove that every normal fuzzy subpolygroup of an Engel polygroup is Engel. Also, some connections between Engel (nilpotent, solvable) fuzzy subpolygroups and Engel (nilpotent, solvable) groups are stablished and studied. Finally, we prove some results on 3-Engel fuzzy subpolygroups. Specially, we prove that every 3-Engel normal fuzzy subpolygroup of exponent 4, is solvable.

Acknowledgements. The authors wish to express their appreciation for several excellent suggestions for improvements in this paper made by the editor and referees.

In this research, E. Mohammadzadeh has been supported by a grant from Payame Noor University, Iran.

References

[1] H. Aghabozorgi, B. Davvaz and M. Jafarpour, *Nilpotent groups derived from hypergroups*, J. Algebra **382** (2013) 177–184.

[2] R. Ameri, R. A. Borzooei and E. Mohammadzadeh, *Engel fuzzy subgroups*, Ital. J. Pure Appl. Math. **34** (2015), 251–262.

[3] R. Ameri and E. Mohammadzadeh, *Engel groups derived from hypergroups*, European J. Combin. **44** (2015), 191–197.

[4] R. Ameri and H. Hedayati, *Fuzzy isomorphism and quotient of fuzzy subpolygroups*, Quasigroup Related Systems **13** (2015), 175–184.

[5] S. Bachmuth and H. Y. Mochizuki, *Third Engel groups and the Macdonald-Neumann conjecture*, Bull. Austral. Math. Soc. **5** (1971), 379–386.

[6] R. Biswas, *Fuzzy subgroups and anti fuzzy subgroups*, Fuzzy Sets and Systems **35** (1990), 121–124.

[7] R. A. Borzooei, E. Mohammadzadeh and V. Fotea, *On Engel fuzzy subpolygroups*, New Math. Nat. Comput. **13**(2) (2017), 165–206.

[8] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani Editore, Tricesimo, 1993.

[9] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific, Basel, 2013.

[10] B. Davvaz and I. Cristea, *Fuzzy Algebraic Hyperstructures. An Introduction*, Springer-Verlag, Berlin, 2015.

[11] N. D. Gupta and K. W. Weston, *On groups of exponent four*, J. Algebra **17** (1971), 59–66.

[12] H. Heineken, *Engelsche elemente der lange drei*, Illinois J. Math. **5** (1961), 681–707.

[13] L. C. Kappe and W. P. Kappe, *On three-Engle groups*, Bull. Austral. Math. Soc. **7** (1972), 391–405.

[14] F. W. Levi, *Groups in which the commutator operation satisfies certain algebraic conditions*, J. Indian Math. Soc. **6** (1942), 87–97.

[15] F. Marty, *Sur une generalization de la notion de groupe*, 8th Congress Math. Scandenaves, Stockholm, Sweden, 1934, 45–49.

[16] E. Mohammadzadeh and R. A. Borzooei, *Nilpotent fuzzy subgroups*, Mathematics **6**(27) (2018), DOI 10.3390/math6020027.

[17] E. Mohammadzadeh, R. A. Borzooei and Y. B. Jun, *Results on Engel fuzzy subgroups*, Algebraic Structures and Their Applications **4**(2) (2017), 1–14.

[18] D. J. S. Robinson, *Finitiness Conditions and Generalized Soluble Groups (Part 2)*, Springer-Verlag, New York, Berlin, 1972.

[19] D. J. S. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1980.

[20] A. Rosenfeld, *Fuzzy groups*, J. Math. Anal. Appl. **35** (1971), 512–517.

[21] M. Suzuki, *Group Theory I*, Springer-Verlag, Berlin, New York, 1982.

[22] L. A. Zadeh, *Fuzzy sets*, Information and Control **8** (1965), 338–353.

[23] M. Zahedi, M. Bolurian and A. Hasankhani, *On polygroups and fuzzy subpolygroups*, J. Fuzzy Math. **1** (1995), 1–15.

[24] M. Zorn, *Nilpotency of finite groups,* Bull. Amer. Math. Soc. **42** (1936), 485–486.

[1]Department of Mathematics, Faculty of Science,
Payame Noor University,
193953697, Tehran, Iran
Email address: mohamadzadeh36@gmail.com

[2]Department of Mathematics,
Shahid Beheshti University,
Tehran, Iran
Email address: borzooei@sbu.ac.ir