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ON $\check{\phi}$-SEMISYMMETRIC LP-KENMOTSU MANIFOLDS WITH A QSNM-CONNECTION ADMITTING RICCI SOLITONS

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Abstract. In the present work, we characterize Lorentzian para-Kenmotsu (briefly, LP-Kenmotsu) manifolds with a quarter-symmetric non-metric connection (briefly, QSNM-connection) $\widehat{\nabla}$ satisfying certain $\check{\phi}$-semisymmetric conditions admitting Ricci solitions. At the end of the paper, a 3-dimensional example of LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$ is given to verify some results of the present paper.

1. Introduction

In a $(2n+1)$-dimensional connected and C^{∞}-smooth semi-Riemannian manifold $(M,\check{g})$, the Levi-Civita connection $\check{\nabla}$, the Riemannian-Christoffel curvature tensor $\check{R}$, the projective curvature tensor $\check{P}$, the concircular curvature tensor $\check{V}$, the conformal curvature tensor $\check{C}$ and the D-conformal curvature tensor $\check{B}$ are defined by [5, 6]

(1.1)
$$\check{R}(\check{E},\check{F})\check{W} =\check{\nabla}_{\check{E}}\check{\nabla}_{\check{F}}\check{W}-\check{\nabla}_{\check{F}}\check{\nabla}_{\check{E}}\check{W}-\check{\nabla}_{[\check{E},\check{F}]}\check{W},$$
(1.2)
$$\check{P}(\check{E},\check{F})\check{W} =\check{R}(\check{E},\check{F})\check{W}-\frac{1}{2n}[\check{S}(\check{F},\check{W})\check{E}-\check{S}(\check{E},\check{W})\check{F}],$$
(1.3)
$$\check{V}(\check{E},\check{F})\check{W} =\check{R}(\check{E},\check{F})\check{W}-\frac{\check{r}}{2n(2n+1)}[\check{g}(\check{F},\check{W})\check{E}-\check{g}(\check{E},\check{W})\check{F}],$$

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(1.4)
$$\check{C}(\check{E},\check{F})\check{W} =\check{R}(\check{E},\check{F})\check{W} - \frac{1}{(2n-1)}[\check{S}(\check{F},\check{W})\check{E} - \check{S}(\check{E},\check{W})\check{F} + \check{g}(\check{F},\check{W})\check{Q}\check{E} - \check{g}(\check{E},\check{W})\check{Q}\check{F}] + \frac{\check{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{E} - \check{g}(\check{E},\check{W})\check{F}],$$

(1.5)
$$\begin{aligned}\check{B}(\check{E},\check{F})\check{W} =&\check{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\check{S}(\check{E},\check{W})\check{F} - \check{S}(\check{F},\check{W})\check{E} + \check{g}(\check{E},\check{W})\check{Q}\check{F}\\ &- \check{g}(\check{F},\check{W})\check{Q}\check{E} - \check{S}(\check{E},\check{W})\check{\eta}(\check{F})\xi + \check{S}(\check{F},\check{W})\check{\eta}(\check{E})\xi - \check{\eta}(\check{E})\check{\eta}(\check{W})\check{Q}\check{F}\\ &+ \check{\eta}(\check{F})\check{\eta}(\check{W})\check{Q}\check{E}] - \frac{k-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{F} - \check{g}(\check{F},\check{W})\check{E}] + \frac{k}{2(n-1)}\\ &\times [\check{g}(\check{E},\check{W})\check{\eta}(\check{F})\xi - \check{g}(\check{F},\check{W})\check{\eta}(\check{E})\xi + \check{\eta}(\check{E})\check{\eta}(\check{W})\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{E}],\end{aligned}$$

respectively, where $\check{r}$ is the scalar curvature, $\check{S}$ and $\check{Q}$ are the Ricci tensor and the Ricci operator, respectively such that $\check{S}(\check{E},\check{F}) = \check{g}(\check{Q}\check{E},\check{F})$ and $k = \frac{\check{r}+4n}{2n-1}$.

The connection $\widehat{\nabla}$ which is linear and defined on $(M,\check{g})$ is said to be a quarter-symmetric [11] if its torsion tensor $\check{T}$

$$\check{T}(\check{E},\check{F}) = \widehat{\nabla}_{\check{E}}\check{F} - \widehat{\nabla}_{\check{F}}\check{E} - [\check{E},\check{F}] = \check{\eta}(\check{F})\check{\phi}\check{E} - \check{\eta}(\check{E})\check{\phi}\check{F}, \tag{1.6}$$

where $\check{\phi}$ is a $(1,1)$-tensor field and $\check{\eta}$ is a 1-form. If moreover, $\widehat{\nabla}$ satisfies the condition

$$(\widehat{\nabla}_{\check{E}}\check{g})(\check{F},\check{W}) = -\check{\eta}(\check{F})\check{g}(\check{\phi}\check{E},\check{W}) - \check{\eta}(\check{W})\check{g}(\check{F},\check{\phi}\check{E}), \tag{1.7}$$

where $\check{E},\check{F},\check{W} \in \chi(M)$ and $\chi(M)$ is the set of all differentiable vector fields on M, then connection $\widehat{\nabla}$ is called a QSNM-connection. The authors in [2,3,7,12] have studied QSNM-connection in various manifolds.

In an LP-Kenmotsu manifold, a relation between the connections $\widehat{\nabla}$ and $\check{\nabla}$ is given by

$$\widehat{\nabla}_{\check{E}}\check{F} = \check{\nabla}_{\check{E}}\check{F} + \check{\eta}(\check{F})\check{\phi}\check{E}. \tag{1.8}$$

On a Riemannian manifold $(M,\check{g})$, a Ricci soliton $(\check{g},U,\check{\lambda})$ is a generalization of an Einstein metric such that (see [9,10]) $\check{\pounds}_U\check{g} + 2\check{S} + 2\check{\lambda}\check{g} = 0$, where $\check{S}$, $\check{\pounds}_U$ and $\check{\lambda}$ are the Ricci tensor, the Lie derivative operator along the vector field U on M and a real constant, respectively. A Ricci soliton is said to be shrinking, steady or expanding according as $\check{\lambda} < 0$, $\check{\lambda} = 0$ or $\check{\lambda} > 0$, respectively.

The present work is arranged in the following manner. After Introduction, a brief introducton of LP-Kenmotsu manifolds is given in Section 2. In Section 3, we find the relation between the curvature tensors of an LP-Kenmotsu manifold with the connections $\check{\nabla}$ and $\widehat{\nabla}$. In Section 4, we study LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$ admitting Ricci solitons. $\check{\phi}$-projectively semisymmetric, $\check{\phi}$-concircularly semisymmetric, $\check{\phi}$-conformally semisymmetric and $\check{\phi}$-D-conformally semisymmetric LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$ admitting Ricci solitons have been studied in Section 5. At the end of the paper, a 3-dimensional example of *LP*-Kenmotsu manifolds with a connection $\widehat{\nabla}$ is given to verify some results of the present paper.

2. Preliminaries

A $(2n+1)$-dimensionsional differentiable manifold M with structure $(\check{\phi}, \xi, \check{\eta}, \check{g})$ is said to be a Lorentzian almost paracontact metric manifold, if it admits $\check{\phi}$: a tensor field of type $(1,1)$, ξ: a contravariant vector field, $\check{\eta}$: a 1-form and $\check{g}$: a Lorentzian metric satisfying [8]

$$\check{\eta}(\xi) = -1, \tag{2.1}$$

$$\check{\phi}^2\check{E} = \check{E} + \check{\eta}(\check{E})\xi, \tag{2.2}$$

$$\check{\phi}\xi = 0, \quad \check{\eta}(\check{\phi}Y) = 0, \tag{2.3}$$

$$\check{g}(\check{\phi}\check{E}, \check{\phi}\check{F}) = \check{g}(\check{E}, \check{F}) + \check{\eta}(\check{E})\check{\eta}(\check{F}),$$

$$\check{g}(\check{E}, \xi) = \check{\eta}(\check{E}), \tag{2.4}$$

$$\check{\Phi}(\check{F}, \check{E}) = \check{\Phi}(\check{E}, \check{F}) = \check{g}(\check{E}, \check{\phi}\check{F}), \tag{2.5}$$

for any $\check{E}, \check{F}$ on M.

For ξ: a killing vector field, the (para) contact structure is said to be a K-(para) contact. In this case, we have

$$\check{\nabla}_{\check{E}}\xi = \check{\phi}\check{E}. \tag{2.6}$$

A Lorentzian almost paracontact manifold M is called an *LP*-Sasakian manifold if

$$(\check{\nabla}_{\check{E}}\check{\phi})\check{F} = \check{g}(\check{E}, \check{F})\xi + \check{\eta}(\check{F})\check{E} + 2\check{\eta}(\check{E})\check{\eta}(\check{F})\xi,$$

for any $\check{E}, \check{F}$ on M.

Now, we define a new manifold called a Lorentzian para-Kenmostu (briefly, *LP*-Kenmotsu) manifold:

Definition 2.1. A Lorentzian almost paracontact manifold is called Lorentzian para-Kenmostu (briefy, *LP*-Kenmostu) manifold if [1]

$$(\check{\nabla}_{\check{E}}\check{\phi})\check{F} = -\check{g}(\check{\phi}\check{E}, \check{F})\xi - \check{\eta}(\check{F})\check{\phi}\check{E}, \tag{2.7}$$

for any $\check{E}, \check{F}$ on M.

In the Lorentzian para-Kenmostu manifold, we have

$$\check{\nabla}_{\check{E}}\xi = -\check{\phi}^2\check{E},$$

$$(\check{\nabla}_{\check{E}}\check{\eta})\check{F} = -\check{g}(\check{\phi}\check{E}, \check{\phi}\check{F}).$$

Moreover, on an *LP*-Kenmotsu, the following relations hold [1]:

$$\check{g}(\check{R}(\check{E}, \check{F})\check{W}, \xi) = \check{\eta}(\check{R}(\check{E}, \check{F})\check{W}) = \check{g}(\check{F}, \check{W})\check{\eta}(\check{E}) - \check{g}(\check{E}, \check{W})\check{\eta}(\check{F}),$$

$$\begin{aligned}
\check{R}(\xi,\check{E})\check{F} =& -\check{R}(\check{E},\xi)\check{F} = \check{g}(\check{E},\check{F})\xi - \check{\eta}(\check{F})\check{E},\\
\check{R}(\check{E},\check{F})\xi =& \check{\eta}(\check{F})\check{E} - \check{\eta}(\check{E})\check{F},\\
\check{R}(\xi,\check{E})\xi =& \check{E} + \check{\eta}(\check{E})\xi,\\
\check{S}(\check{E},\xi) =& (\dim M - 1)\check{\eta}(\check{E}), \quad \check{S}(\xi,\xi) = -(\dim M - 1),\\
\check{Q}\xi =& (\dim M - 1)\xi,
\end{aligned}$$

for any $\check{E}, \check{F}, \check{W}$ on M.

Definition 2.2. An LP-Kenmotsu manifold is called an η-Einstein manifold if its Ricci tensor satisfies [4] $\check{S}(\check{E},\check{F}) = a_1\check{g}(\check{E},\check{F}) + a_2\check{\eta}(\check{E})\check{\eta}(\check{F})$, where a_1 and a_2 are smooth functions on M.

3. Curvature Tensor of LP-Kenmotsu Manifolds with a Connection $\widehat{\nabla}$

The curvature tensor $\widehat{R}$ of an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is defined by

$$\widehat{R}(\check{E},\check{F})\check{W} = \widehat{\nabla}_{\check{E}}\widehat{\nabla}_{\check{F}}\check{W} - \widehat{\nabla}_{\check{F}}\widehat{\nabla}_{\check{E}}\check{W} - \widehat{\nabla}_{[\check{E},\check{F}]}\check{W}. \tag{3.1}$$

From (1.8), (2.1), (2.4), (2.6), (2.7) and (3.1), we obtain

$$\widehat{R}(\check{E},\check{F})\check{W} = \check{R}(\check{E},\check{F})\check{W} - \check{g}(\check{E},\check{W})\check{\phi}\check{F} + \check{g}(\check{F},\check{W})\check{\phi}\check{E}, \tag{3.2}$$

where $\check{R}(\check{E},\check{F})\check{W}$ is given by (1.1). Contracting $\check{E}$ in (3.2), we get

$$\widehat{S}(\check{F},\check{W}) = \check{S}(\check{F},\check{W}) + \check{g}(\check{F},\check{W})\check{\psi} - \check{g}(\check{\phi}\check{F},\check{W}). \tag{3.3}$$

From (3.3), it follows that

$$\widehat{Q}\check{F} = \check{Q}\check{F} + \check{\psi}\check{F} - \check{\phi}\check{F},$$

Contracting again $\check{F}$ and $\check{W}$ in (3.3), we obtain

$$\widehat{r} = \check{r} + 2n\check{\psi}, \tag{3.4}$$

where $\widehat{Q}$ is the Ricci operator, $\widehat{S}$ is the Ricci tensor and $\widehat{r}$ is the scalar curvature with respect to $\widehat{\nabla}$.

Lemma 3.1. *In a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$, we have*

$$\begin{aligned}
\widehat{R}(\check{E},\check{F})\xi =& \check{\eta}(\check{F})\check{E} - \check{\eta}(\check{E})\check{F} + \check{\eta}(\check{F})\check{\phi}\check{E} - \check{\eta}(\check{E})\check{\phi}\check{F},\\
\widehat{R}(\xi,\check{E})\check{F} =& -\widehat{R}(\check{E},\xi)\check{F} = \check{g}(\check{E},\check{F})\xi - \check{\eta}(\check{F})\check{E} - \check{\eta}(\check{F})\check{\phi}\check{E},\\
\widehat{R}(\xi,\check{E})\xi =& \check{\eta}(\check{E})\xi + \check{E} + \check{\phi}\check{E},
\end{aligned} \tag{3.5}$$

$$\widehat{S}(\check{E},\xi) = (2n+\check{\psi})\check{\eta}(\check{E}), \quad \widehat{S}(\xi,\xi) = -(2n+\check{\psi}), \tag{3.6}$$

$$\widehat{\nabla}_{\check{E}}\xi = -\check{E} - \check{\eta}(\check{E})\xi - \check{\phi}\check{E}, \tag{3.7}$$

$$\widehat{Q}\xi = (2n+\check{\psi})\xi, \tag{3.8}$$

for any $\check{E}, \check{F}$ *on* M.

4. Ricci soliton on LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$

Suppose that an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ admits a Ricci soliton $(\check{g}, \xi, \check{\lambda})$. Then in view of (1.9), we have

$$(\widehat{\pounds}_{\xi}\check{g})(\check{F}, \check{W}) + 2\widehat{S}(\check{F}, \check{W}) + 2\check{\lambda}\check{g}(\check{F}, \check{W}) = 0. \tag{4.1}$$

By using (3.7) and (1.6), we find

$$(\widehat{\pounds}_{\xi}\check{g})(\check{F}, \check{W}) = -2[\check{g}(\check{F}, \check{W}) + \check{\eta}(\check{F})\check{\eta}(\check{W})]. \tag{4.2}$$

Combining (4.1) and (4.2), we obtain

$$\widehat{S}(\check{F}, \check{W}) = (1 - \check{\lambda})\check{g}(\check{F}, \check{W}) + \check{\eta}(\check{F})\check{\eta}(\check{W}). \tag{4.3}$$

Taking $\check{W} = \xi$ in (4.3) and then using (2.1), (2.3), we get

$$\widehat{S}(\check{F}, \xi) = -\check{\lambda}\check{\eta}(\check{F}). \tag{4.4}$$

Thus from (3.6) and (4.4), it follows that

$$\check{\lambda} = -(2n + \check{\psi}). \tag{4.5}$$

Hence, (4.3) together with (4.5) leads to the following theorem.

Theorem 4.1. *If an LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admits a Ricci soliton $(\check{g}, \xi, \check{\lambda})$, then M is an η-Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

Now, assumig that $(\check{g}, U, \check{\lambda})$ is a Ricci soliton on an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ such that U is pointwise collinear with ξ, i.e., $U = \beta\xi$, where β is a function. Then (1.9) holds and we have

$$\beta\check{g}(\widehat{\nabla}_{\check{E}}\xi, \check{F}) + (\check{E}\beta)\check{\eta}(\check{F}) + \beta\check{g}(\check{E}, \widehat{\nabla}_{\check{F}}\xi) + (\check{F}\beta)\check{\eta}(\check{E}) + 2\widehat{S}(\check{E}, \check{F}) + 2\check{\lambda}\check{g}(\check{E}, \check{F}) = 0,$$

which in view of (3.7) and (1.6) becomes

$$-2\beta[\check{g}(\check{E}, \check{F}) + \check{\eta}(\check{E})\check{\eta}(\check{F})] + (\check{E}\beta)\check{\eta}(\check{F}) + (\check{F}\beta)\check{\eta}(\check{E}) + 2\widehat{S}(\check{E}, \check{F}) + 2\check{\lambda}\check{g}(\check{E}, \check{F}) = 0. \tag{4.6}$$

Replacing $\check{F}$ by ξ in (4.6) and using (2.1), (2.4) and (3.6), we find

$$-(\check{E}\beta) + [(\xi\beta) + 2(2n + \check{\psi}) + 2\check{\lambda}]\check{\eta}(\check{E}) = 0, \tag{4.7}$$

which by taking $\check{E} = \xi$ and using (2.1) yields

$$(\xi\beta) + (2n + \check{\psi}) + \check{\lambda} = 0. \tag{4.8}$$

Combining the equations (4.7) and (4.8), we find

$$d\beta = [(2n + \check{\psi}) + \check{\lambda}]\check{\eta}. \tag{4.9}$$

Now, applying d on (4.9), we get

$$[(2n + \check{\psi}) + \check{\lambda}]\check{\eta} = 0 \implies \check{\lambda} = -(2n + \check{\psi}), \quad d\check{\eta} \neq 0. \tag{4.10}$$

Thus, from (4.9) and (4.10), we obtain $d\beta = 0$, i.e., β is a constant. Therefore, (4.6) reduces to

$$\widehat{S}(\check{E}, \check{F}) = (\beta - \check{\lambda})\check{g}(\check{E}, \check{F}) + \beta\check{\eta}(\check{E})\check{\eta}(\check{F}). \tag{4.11}$$

Hence, (4.10) together with (4.11) leads the following theorem.

Theorem 4.2. *If an LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admits a Ricci soliton $(\check{g}, U, \check{\lambda})$ such that U is pointwise collinear with ξ, then U is a constant multiple of ξ and M is an η-Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

5. Ricci soliton on $\check{\phi}$-semisymmetric LP-Kenmotsu manifolds with a connection $\widehat{\nabla}$

Definition 5.1. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-projectively semisymmetric if (see [13]) $\widehat{P}(\check{E}, \check{F}) \cdot \check{\phi} = 0$ for all $\check{E}, \check{F}$ on M.

Analogous to the equation (1.2), the projective curvature tensor with a connection $\bar{\nabla}$ is given by

$$\widehat{P}(\check{E}, \check{F})\check{W} = \widehat{R}(\check{E}, \check{F})\check{W} - \frac{1}{2n}[\widehat{S}(\check{F}, \check{W})\check{E} - \widehat{S}(\check{E}, \check{W})\check{F}]. \tag{5.1}$$

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-projectively semisymmetric, therefore

$$(\widehat{P}(\check{E}, \check{F}) \cdot \check{\phi})\check{W} = \widehat{P}(\check{E}, \check{F})\check{\phi}\check{W} - \check{\phi}\widehat{P}(\check{E}, \check{F})\check{W} = 0, \tag{5.2}$$

for all $\check{E}, \check{F}, \check{W}$ on M. From (5.1), we find

$$\widehat{P}(\check{E}, \check{F})\check{\phi}\check{W} = \widehat{R}(\check{E}, \check{F})\check{\phi}\check{W} - \frac{1}{2n}[\widehat{S}(\check{F}, \check{\phi}\check{W})\check{E} - \widehat{S}(\check{E}, \check{\phi}\check{W})\check{F}], \tag{5.3}$$

$$\check{\phi}\widehat{P}(\check{E}, \check{F})\check{W} = \check{\phi}\widehat{R}(\check{E}, \check{F})\check{W} - \frac{1}{2n}[\widehat{S}(\check{F}, \check{W})\check{\phi}\check{E} - \widehat{S}(\check{E}, \check{W})\check{\phi}\check{F}]. \tag{5.4}$$

By combining (5.2), (5.3) and (5.4), we have

$$\begin{aligned} &\widehat{R}(\check{E}, \check{F})\check{\phi}\check{W} - \check{\phi}\widehat{R}(\check{E}, \check{F})\check{W} - \frac{1}{2n}[\widehat{S}(\check{F}, \check{\phi}\check{W})\check{E} - \widehat{S}(\check{E}, \check{\phi}\check{W})\check{F}] \\ &+ \frac{1}{2n}[\widehat{S}(\check{F}, \check{W})\check{\phi}\check{E} - \widehat{S}(\check{E}, \check{W})\check{\phi}\check{F}] = 0. \end{aligned} \tag{5.5}$$

Taking $\check{F} = \xi$ in (5.5) and using (2.3), (3.5) and (3.6), we find

$$-\check{g}(\check{E}, \check{\phi}\check{W})\xi + \frac{1}{2n}\widehat{S}(\check{E}, \check{\phi}\check{W})\xi - \check{\eta}(\check{W})\check{\phi}\check{E} - \check{\eta}(\check{W})\check{\phi}^2\check{E} + \frac{(2n+\check{\psi})}{2n}\check{\eta}(\check{W})\check{\phi}\check{E} = 0.$$

Taking inner product of the above equation with ξ and making use of (2.1) and (2.3) yields $\widehat{S}(\check{E}, \check{\phi}\check{W}) = 2n\check{g}(\check{E}, \check{\phi}\check{W})$, which by setting $\check{W} = \check{\phi}\check{W}$ and using (2.2) gives

$$\widehat{S}(\check{E}, \check{W}) = 2n\check{g}(\check{E}, \check{W}) - \check{\psi}\check{\eta}(\check{E})\check{\eta}(\check{W}). \tag{5.6}$$

Now, taking $\check{W}=\xi$ in (5.6), we find

$$\widehat{S}(\check{E},\xi)=(2n+\check{\psi})\check{\eta}(\check{E}). \tag{5.7}$$

Thus, from (4.4) and (5.7), we obtain

$$\check{\lambda}=-(2n+\check{\psi}). \tag{5.8}$$

Hence, (5.6) together with (5.8) leads to the following theorem.

Theorem 5.1. *If a* $(2n+1)$*-dimensional* LP*-Kenmotsu manifold* M *with a connection* $\widehat{\nabla}$ *admitting Ricci soliton is* $\check{\phi}$*-projectively semisymmetric, then* M *is an* $\eta-$*Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to* $\check{\psi}<-2n$, $\check{\psi}>-2n$ *or* $\check{\psi}=-2n$.

Definition 5.2. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-concircularly semisymmetric if $\widehat{V}(\check{E},\check{F})\cdot\check{\phi}=0$ for all $\check{E},\check{F}$ on M.

Analogous to the equation (1.3), the concircular curvature tensor with a connection $\widehat{\nabla}$ is given by

$$\widehat{V}(\check{E},\check{F})\check{W}=\widehat{R}(\check{E},\check{F})\check{W}-\frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F},\check{W})\check{E}-\check{g}(\check{E},\check{W})\check{F}]. \tag{5.9}$$

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-concircularly semisymmetric, therefore

$$(\widehat{V}(\check{E},\check{F})\cdot\check{\phi})\check{W}=\widehat{V}(\check{E},\check{F})\check{\phi}\check{W}-\check{\phi}\widehat{V}(\check{E},\check{F})\check{W}=0, \tag{5.10}$$

for all $\check{E},\check{F},\check{W}$ on M. From (5.9), it follows that

$$\widehat{V}(\check{E},\check{F})\check{\phi}\check{W}=\widehat{R}(\check{E},\check{F})\check{\phi}\check{W}-\frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F},\check{\phi}\check{W})\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\check{F}], \tag{5.11}$$

$$\check{\phi}\widehat{V}(\check{E},\check{F})\check{W}=\check{\phi}\widehat{R}(\check{E},\check{F})\check{W}-\frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F},\check{W})\check{\phi}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\check{F}]. \tag{5.12}$$

Combining (5.10), (5.11) and (5.12), we have

$$\begin{aligned}&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W}-\check{\phi}\widehat{R}(\check{E},\check{F})\check{W}-\frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F},\check{\phi}\check{W})\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\check{F}]\\&+\frac{\widehat{r}}{2n(2n+1)}[\check{g}(\check{F},\check{W})\check{\phi}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\check{F}]=0,\end{aligned}$$

which, by taking $\check{F}=\xi$ and using (2.3), (2.4) and (3.5), takes the form

$$\left[\frac{\widehat{r}}{2n(2n+1)}-1\right]\check{g}(\check{E},\check{\phi}\check{W})\xi+\left[\frac{\widehat{r}}{2n(2n+1)}-1\right]\check{\eta}(\check{W})\check{\phi}\check{E}+\check{\eta}(\check{W})\check{\phi}^2\check{E}=0. \tag{5.13}$$

Taking inner product of (5.13) with ξ and making use of (2.1) and (2.3), we get

$$\widehat{r}=2n(2n+1),\qquad \check{g}(\check{E},\check{\phi}\check{W})\neq 0.$$

This leads to the following theorem.

Theorem 5.2. *If a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-concircular semisymmetric, then the scalar curvature is constant.*

Definition 5.3. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-conformally semisymmetric if $\widehat{C}(\check{E},\check{F})\cdot\check{\phi}=0$ for all $\check{E},\check{F}$ on M.

Analogous to the equation (1.4), the conformal curvature tensor with a connection $\widehat{\nabla}$ is given by

(5.14)
$$\begin{aligned}\widehat{C}(\check{E},\check{F})\check{W} =&\widehat{R}(\check{E},\check{F})\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{W})\check{E}-\widehat{S}(\check{E},\check{W})\check{F}\\&+\check{g}(\check{F},\check{W})\widehat{Q}\check{E}-\check{g}(\check{E},\check{W})\widehat{Q}\check{F}]+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{E}-\check{g}(\check{E},\check{W})\check{F}].\end{aligned}$$

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-conformally semisymmetric, therefore

$$(\widehat{C}(\check{E},\check{F})\cdot\check{\phi})\check{W}=\widehat{C}(\check{E},\check{F})\check{\phi}\check{W}-\check{\phi}\widehat{C}(\check{E},\check{F})\check{W}=0, \tag{5.15}$$

for all $\check{E},\check{F},\check{W}$ on M. From (5.14), it follows that

$$\begin{aligned}\widehat{C}(\check{E},\check{F})\check{\phi}\check{W} =&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{\phi}\check{W})E\\&-\widehat{S}(\check{E},\check{\phi}\check{W})\check{F}+\check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F}]\\&+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{\phi}\check{W})\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\check{F}],\end{aligned} \tag{5.16}$$

$$\begin{aligned}\check{\phi}\widehat{C}(\check{E},\check{F})\check{W} =&\check{\phi}\widehat{R}(\check{E},\check{F})\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{W})\check{\phi}\check{E}\\&-\widehat{S}(\check{E},\check{W})\check{\phi}\check{F}+\check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F}]\\&+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{\phi}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\check{F}].\end{aligned} \tag{5.17}$$

Combining (5.15), (5.16) and (5.17), we have

$$\begin{aligned}&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W}-\check{\phi}\widehat{R}(\check{E},\check{F})\check{W}-\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{\phi}\check{W})\check{E}-\widehat{S}(\check{E},\check{\phi}\check{W})\check{F}\\&+\check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F}]+\frac{1}{(2n-1)}[\widehat{S}(\check{F},\check{W})\check{\phi}\check{E}-\widehat{S}(\check{E},\check{W})\check{\phi}\check{F}\\&+\check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F}]+\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{\phi}\check{W})\check{E}-\check{g}(\check{E},\check{\phi}\check{W})\check{F}]\\&-\frac{\widehat{r}}{2n(2n-1)}[\check{g}(\check{F},\check{W})\check{\phi}\check{E}-\check{g}(\check{E},\check{W})\check{\phi}\check{F}]=0,\end{aligned}$$

which by replacing $\check{E} = \xi$ and making use of (2.3), (2.4), (3.5), (3.6) and (3.8) takes the form

$$\left[\frac{2n+\check{\psi}}{2n-1} - \frac{\widehat{r}}{2n(2n-1)} - 1\right](\check{g}(\check{E},\check{\phi}\check{W})\xi + \check{\eta}(\check{W})\check{\phi}E) + \frac{1}{(2n-1)}\widehat{S}(\check{E},\check{\phi}\check{W})\xi + \frac{1}{(2n-1)}\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E} - \check{\eta}(\check{W})\check{\phi}^2\check{E} = 0. \tag{5.18}$$

Now, taking inner product of (5.18) with ξ and making use of (2.1) and (2.3), we obtain

$$\widehat{S}(\check{E},\check{\phi}\check{W}) = \left[(2n-1) + \frac{\widehat{r}}{2n} - (2n+\check{\psi})\right]\check{g}(\check{E},\check{\phi}\check{W}). \tag{5.19}$$

By replacing $\check{W}$ by $\check{\phi}\check{W}$ in (5.19) and then using (2.2), (2.4), (3.6), we get

$$\begin{aligned}\widehat{S}(\check{E},\check{W}) = &\left[(2n-1) + \frac{\widehat{r}}{2n} - (2n+\check{\psi})\right]\check{g}(\check{E},\check{W}) \\ &+ \left[(2n-1) + \frac{\widehat{r}}{2n} - 2(2n+\check{\psi})\right]\check{\eta}(\check{E})\check{\eta}(\check{W}).\end{aligned} \tag{5.20}$$

Taking $\check{W} = \xi$ in (5.20), we find

$$\widehat{S}(\check{E},\xi) = (2n+\check{\psi})\check{\eta}(\check{E}). \tag{5.21}$$

Thus, from (4.4) and (5.21), we obtain

$$\check{\lambda} = -(2n+\check{\psi}). \tag{5.22}$$

Hence, (5.20) together with (5.22) leads to the following theorem.

Theorem 5.3. *If a $(2n+1)$-dimensional LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admitting Ricci soliton is $\check{\phi}$-conformally semisymmetric, then M is an $\eta-$Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi} < -2n$, $\check{\psi} > -2n$ or $\check{\psi} = -2n$.*

Definition 5.4. An LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is called $\check{\phi}$-D-conformally semisymmetric if $\widehat{B}(\check{E},\check{F})\cdot\check{\phi} = 0$ for all $\check{E},\check{F}$ on M.

Analogous to the equation (1.5), the $D-$conformal curvature tensor with a connection $\widehat{\nabla}$ is given by

$$\begin{aligned}\widehat{B}(\check{E},\check{F})\check{W} =& \widehat{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{W})\check{F} - \widehat{S}(\check{F},\check{W})\check{E} \\ &+ \check{g}(\check{E},\check{W})\widehat{Q}\check{F} - \check{g}(\check{F},\check{W})\widehat{Q}\check{E} - \widehat{S}(\check{E},\check{W})\check{\eta}(\check{F})\xi \\ &+ \widehat{S}(\check{F},\check{W})\check{\eta}(\check{E})\xi - \check{\eta}(\check{E})\check{\eta}(\check{W})\widehat{Q}\check{F} + \check{\eta}(\check{F})\check{\eta}(\check{W})\widehat{Q}\check{E}] \\ &- \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{F} - \check{g}(\check{F},\check{W})\check{E}] + \frac{\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{W})\check{\eta}(\check{F})\xi \\ &- \check{g}(\check{F},\check{W})\check{\eta}(\check{E})\xi + \check{\eta}(\check{E})\check{\eta}(\check{W})\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{E}],\end{aligned} \tag{5.23}$$

where $\widehat{k} = \frac{\widehat{r}+4n}{(2n-1)}$.

Suppose that a $(2n+1)$-dimensional LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is $\check{\phi}$-D-conformally semisymmetric, therefore

$$(\widehat{B}(\check{E},\check{F})\cdot\check{\phi})\check{W} = \widehat{B}(\check{E},\check{F})\check{\phi}\check{W} - \check{\phi}\widehat{B}(\check{E},\check{F})\check{W} = 0, \tag{5.24}$$

for all $\check{E}, \check{F}, \check{W}$ on M. From (5.23), it follows that

$$\begin{aligned}\widehat{B}(\check{E},\check{F})\check{\phi}\check{W} =& \widehat{R}(\check{E},\check{F})\check{\phi}\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{\phi}\check{W})\check{F} - \widehat{S}(\check{F},\check{\phi}\check{W})\check{E} + \check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F}\\ &- \check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E} - \widehat{S}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\xi + \widehat{S}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})\xi]\\ &- \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{F} - \check{g}(\check{F},\check{\phi}\check{W})\check{E}] + \frac{\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\xi\\ &- \check{g}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})\xi],\end{aligned} \tag{5.25}$$

$$\begin{aligned}\check{\phi}\widehat{B}(\check{E},\check{F})W =& \check{\phi}\widehat{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{W})\check{\phi}\check{F} - \widehat{S}(\check{F},\check{W})\check{\phi}\check{E}\\ &+ \check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E} - \check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{F}\\ &+ \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E}] - \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{\phi}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\check{E}]\\ &+ \frac{\widehat{k}}{2(n-1)}[\check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\check{E}].\end{aligned} \tag{5.26}$$

Combining (5.24), (5.25) and (5.26), we have

$$\begin{aligned}&\widehat{R}(\check{E},\check{F})\check{\phi}\check{W} - \check{\phi}\widehat{R}(\check{E},\check{F})\check{W} + \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{\phi}\check{W})\check{F} - \widehat{S}(\check{F},\check{\phi}\check{W})\check{E}\\ &+ \check{g}(\check{E},\check{\phi}\check{W})\widehat{Q}\check{F} - \check{g}(\check{F},\check{\phi}\check{W})\widehat{Q}\check{E} - \widehat{S}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\xi + \widehat{S}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})\xi]\\ &- \frac{1}{2(n-1)}[\widehat{S}(\check{E},\check{W})\check{\phi}\check{F} - \widehat{S}(\check{F},\check{W})\check{\phi}\check{E} + \check{g}(\check{E},\check{W})\check{\phi}\widehat{Q}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\widehat{Q}\check{E}\\ &- \check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{F} + \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E}] - \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{F} - \check{g}(\check{F},\check{\phi}\check{W})\check{E}]\\ &+ \frac{\widehat{k}-2}{2(n-1)}[\check{g}(\check{E},\check{W})\check{\phi}\check{F} - \check{g}(\check{F},\check{W})\check{\phi}\check{E}] + \frac{\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\check{\eta}(\check{F})\\ &- \check{g}(\check{F},\check{\phi}\check{W})\check{\eta}(\check{E})]\xi - \frac{\widehat{k}}{2(n-1)}[\check{\eta}(\check{E})\check{\eta}(\check{W})\check{\phi}\check{F} - \check{\eta}(\check{F})\check{\eta}(\check{W})\check{\phi}\check{E}] = 0.\end{aligned} \tag{5.27}$$

By taking $\check{F} = \xi$ in (5.27) and then using (2.1), (2.3), (3.5), (3.6) and (3.8) takes the form

$$\frac{4+\check{\psi}-2\widehat{k}}{2(n-1)}[\check{g}(\check{E},\check{\phi}\check{W})\xi+\check{\eta}(\check{W})\check{\phi}\check{E}]+\frac{1}{n-1}\widehat{S}(\check{E},\check{\phi}\check{W})\xi-\check{\eta}(\check{W})\check{\phi}^2\check{E} \tag{5.28}$$
$$+\frac{1}{n-1}\check{\eta}(\check{W})\check{\phi}\widehat{Q}\check{E}=0.$$

Inner product of (5.28) with ξ and making use of (2.1) and (2.3) gives

$$\widehat{S}(\check{E},\check{\phi}\check{W})=\left[\frac{\check{\psi}}{2}+2-\widehat{k}\right]\check{g}(\check{E},\check{\phi}\check{W}). \tag{5.29}$$

Now, we replace $\check{W}$ by $\check{\phi}\check{W}$ in (5.29) and using (2.2), (2.4) and (3.6), we get

$$\widehat{S}(\check{E},\check{W})=\left[\frac{\check{\psi}}{2}+2-\widehat{k}\right]\check{g}(\check{E},\check{W})-\left[\frac{\check{\psi}}{2}+\widehat{k}+2n-2\right]\check{\eta}(\check{E})\check{\eta}(\check{W}). \tag{5.30}$$

Taking $\check{W}=\xi$ in (5.30), we find

$$\widehat{S}(\check{E},\xi)=(2n+\check{\psi})\check{\eta}(\check{E}). \tag{5.31}$$

Thus, from (4.4) and (5.31), we obtain

$$\check{\lambda}=-(2n+\check{\psi}). \tag{5.32}$$

Hence, (5.30) together with (5.32) leads to the following theorem.

Theorem 5.4. *If a $(2n+1)$-dimensional LP-Kenmotsu manifold M with a connection $\widehat{\nabla}$ admitting Ricci soliton is $\check{\phi}$-D-conformally semisymmetric, then M is an $\eta-$Einstein manifold and its Ricci solition will be expanding, shrinking or steady according to $\check{\psi}<-2n$, $\check{\psi}>-2n$ or $\check{\psi}=-2n$.*

Example 5.1. Let on a 3-dimensional manifold $M=\{(\breve{w}_1,\breve{w}_2,\breve{w}_3)\in R^3 : w>0\}$, where $(\breve{w}_1,\breve{w}_2,\breve{w}_3)$ are the standard coordinates of R^3, the linearly independent vector fields that at each point of M are given by

$$v^1=\frac{\breve{w}_3\partial}{\partial\breve{w}_1},\quad v^2=\frac{w\partial}{\partial\breve{w}_2},\quad v^3=\frac{w\partial}{\partial\breve{w}_3}=\xi.$$

Suppose the Lorentzian metric $\check{g}$ is defined by

$$\check{g}(v^1,v^1)=\check{g}(v^2,v^2)=1,\quad \check{g}(v^3,v^3)=-1,\quad \check{g}(v^1,v^2)=\check{g}(v^2,v^3)=\check{g}(v^1,v^3)=0.$$

Suppose the 1-form $\check{\eta}$ is defined by $\check{\eta}(\check{E})=\check{g}(\check{E},v^3)=\check{g}(\check{E},\xi)$ for all $\check{E}$ on M, and the $(1,1)$-tensor field $\check{\phi}$ is defined by

$$\check{\phi}v^1=-v^1,\quad \check{\phi}v^2=-v^2,\quad \check{\phi}v^3=0.$$

Then, using the linearity of $\check{g}$ and $\check{\phi}$, we have

$$\check{\eta}(\xi)=-1,\quad \check{\phi}^2\check{E}=\check{E}+\check{\eta}(\check{E})\xi,\quad \check{g}(\check{\phi}\check{E},\check{\phi}\check{F})=\check{g}(\check{E},\check{F})+\check{\eta}(\check{E})\check{\eta}(\check{F}),$$

for all $\check{E}, \check{F}$ on M. Thus, $(\check{\phi}, \xi, \check{\eta}, \check{g})$ defines a Lorentzian almost paracontact metric structure on M. Also, we have

$$[v^1, v^2] = 0, \quad [v^1, v^3] = -v^1, \quad [v^2, v^3] = -v^2. \tag{5.33}$$

From the Koszul's formula for $\check{g}$, we calculate

$$\begin{aligned}
&\check{\nabla}_{v^1} v^1 = -\, v^3, \quad \check{\nabla}_{v^1} v^2 = 0, \quad \check{\nabla}_{v^1} v^3 = -v^1, \quad \check{\nabla}_{v^2} v^1 = 0,\\
&\check{\nabla}_{v^2} v^2 = -\, v^3, \quad \check{\nabla}_{v^2} v^3 = -v^2, \quad \check{\nabla}_{v^3} v^1 = 0, \quad \check{\nabla}_{v^3} v^2 = 0, \quad \check{\nabla}_{v^3} v^3 = 0.
\end{aligned} \tag{5.34}$$

Also, one can easily verify that

$$\check{\nabla}_{\check{E}} \xi = -\check{E} - \check{\eta}(\check{E})\xi \quad \text{and} \quad (\check{\nabla}_{\check{E}} \check{\phi})\check{F} = -\check{g}(\check{\phi}\check{E}, \check{F})\xi - \check{\eta}(\check{F})\check{\phi}\check{E}.$$

Therefore, M is an LP-Kenmotsu manifold. From (1.1), (5.33) and (5.34), we obtain

$$\begin{aligned}
\check{R}(v^1, v^2)v^1 &= -\, v^2, \quad \check{R}(v^2, v^3)v^1 = 0, \quad \check{R}(v^1, v^3)v^1 = -v^3,\\
\check{R}(v^1, v^2)v^2 &= v^1, \quad \check{R}(v^1, v^3)v^2 = 0, \quad \check{R}(v^2, v^3)v^2 = -v^3,\\
\check{R}(v^1, v^2)v^3 &= 0, \quad \check{R}(v^1, v^3)v^3 = -v^1, \quad \check{R}(v^2, v^3)v^3 = -v^2,
\end{aligned} \tag{5.35}$$

from which it is clear that $\check{R}(\check{E}, \check{F})\check{W} = \check{g}(\check{F}, \check{W})\check{E} - \check{g}(\check{E}, \check{W})\check{F}$. Hence, $(M, \check{\phi}, \xi, \check{\eta}, \check{g})$ is an LP-Kenmotsu manifold of unit constant curvature. By virtue of (1.8) and (5.35), we obtain

$$\begin{aligned}
&\widehat{\nabla}_{v^1} v^1 = -\, v^3, \quad \widehat{\nabla}_{v^2} v^1 = 0, \quad \widehat{\nabla}_{v^3} v^1 = 0, \quad \widehat{\nabla}_{v^1} v^2 = 0, \quad \widehat{\nabla}_{v^2} v^2 = -v^3,\\
&\widehat{\nabla}_{v^3} v^2 = 0, \quad \widehat{\nabla}_{v^1} v^3 = 0, \quad \widehat{\nabla}_{v^2} v^3 = 0, \quad \widehat{\nabla}_{v^3} v^3 = 0.
\end{aligned}$$

From (3.2) and (5.35), we can easily obtain

$$\begin{aligned}
\widehat{R}(v^1, v^2)v^1 &= 0, \quad \widehat{R}(v^1, v^3)v^1 = -v^3, \quad \widehat{R}(v^2, v^3)v^1 = 0,\\
\widehat{R}(v^1, v^2)v^2 &= 0, \quad \widehat{R}(v^1, v^3)v^2 = 0, \quad \widehat{R}(v^2, v^3)v^2 = -v^3,\\
\widehat{R}(v^1, v^2)v^3 &= 0, \quad \widehat{R}(v^1, v^3)v^3 = 0, \quad \widehat{R}(v^2, v^3)v^3 = 0.
\end{aligned} \tag{5.36}$$

From (5.35) and (5.36), we calculate the Ricci tensors as follows:

$$\check{S}(v^1, v^1) = \check{S}(v^2, v^2) = 2, \quad \check{S}(v^3, v^3) = -2,$$

and

$$\widehat{S}(v^1, v^1) = \widehat{S}(v^2, v^2) = 1, \quad \widehat{S}(v^3, v^3) = 0.$$

Therefore, we find $\check{r} = 6$ and $\widehat{r} = 2$, where $\check{\psi} = -2$. Hence, (3.4) is satisfied. From (2.5), (1.6) and (1.7), we find

$$\begin{aligned}
\check{\Phi}(v^1, v^1) &= \check{\Phi}(v^2, v^2) = -1, \quad \check{\Phi}(v^3, v^3) = 0,\\
\check{T}(v^i, v^j) &= 0, \quad i = j = 1, 2, 3,\\
\check{T}(v^1, v^2) &= 0, \quad \check{T}(v^1, v^3) = v^1, \quad \check{T}(v^2, v^3) = v^2,\\
(\widehat{\nabla}_{v^1}\check{g})(v^1, v^3) &= (\widehat{\nabla}_{v^2}\check{g})(v^2, v^3) = -1, \quad (\widehat{\nabla}_{v^3}\check{g})(v^1, v^2) = 0,
\end{aligned}$$

respectively. Thus, the connection $\widehat{\nabla}$ defined on M is a QSNM. Now, by putting $F = W = v^i$ in (4.3) and summing up, we find $2 = 3(1-\check{\lambda}) - 1$ implies $\check{\lambda} = 0$. Thus, a Ricci soliton on an LP-Kenmotsu manifold with a connection $\widehat{\nabla}$ is steady for $\check{\psi} = -2n = -2$.

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