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SIMPSON'S TYPE INEQUALITIES VIA THE KATUGAMPOLA FRACTIONAL INTEGRALS FOR s-CONVEX FUNCTIONS

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Abstract. In this paper, we introduce some Simpson's type integral inequalities via the Katugampola fractional integrals for functions whose first derivatives at certain powers are s-convex (in the second sense). The Katugampola fractional integrals are generalizations of the Riemann–Liouville and Hadamard fractional integrals. Hence, our results generalize some results in the literature related to the Riemann–Liouville fractional integrals. Results related to the Hadamard fractional integrals could also be derived from our results.

1. Introduction

The inequality below is known in the literature as the Simpson's inequality:

$$\left|\int_a^b f(t)dt - \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]\right| \leq \frac{(b-a)^4}{2880}\left\|f^{(4)}\right\|_\infty,$$

where $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable function on (a, b) and $\left\|f^{(4)}\right\|_\infty = \sup_{t\in(a,b)} \left|f^{(4)}(t)\right| < \infty$.

This inequality has been studied and generalized by many authors over the years. For more information on recent results about the Simpson's inequality, we refer the interested reader to the papers [1, 2, 6–8, 11, 14, 15].

Definition 1.1 ([3]). A function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex (in the second sense), for $s \in (0, 1]$, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

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for all $x, y \in [0, \infty)$ and $t \in [0, 1]$.

Remark 1.1. If $s = 1$ in Definition 1.1, then we have the definition of convex functions.

Recently, Cheng and Huang [5] obtained the following Simpson's type inequalities for s-convex functions via the Riemann–Liouville fractional integrals.

Theorem 1.1 ([5]). *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a diferentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'|$ is s-convex on $[a, b]$ for some fixed $s \in (0, 1]$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + J^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2^{s+1}}\Big(|f'(a)| + |f'(b)|\Big) I(\alpha, s),$$

where

$$I(\alpha, s) = \int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| [(1+t)^s + (1-t)^s] dt,$$

$J^\alpha_{b^-} f(x)$ and $J^\alpha_{a^+} f(x)$ denotes the right- and left-sided Riemann–Liouville fractional integrals of f at x respectively (see Definition 1.2).

Theorem 1.2 ([5]). *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a diferentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + J^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2}\left(\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right|^r dt\right)^{\frac{1}{r}} \left[\left(\frac{(2^{s+1}-1)|f'(b)|^q + |f'(a)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right.$$
$$\left. + \left(\frac{(2^{s+1}-1)|f'(a)|^q + |f'(b)|^q}{2^s(s+1)}\right)^{\frac{1}{q}}\right],$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

Theorem 1.3 ([5]). *Let $f : I \subset [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1([a, b])$, where $a, b \in I^\circ$ with $a < b$. If $|f'|^q$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$ and $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{2^{\alpha-1}\Gamma(\alpha+1)}{(b-a)^\alpha}\left[J^\alpha_{b^-} f\left(\frac{a+b}{2}\right) + J^\alpha_{a^+} f\left(\frac{a+b}{2}\right)\right]\right|$$
$$\leq \frac{b-a}{2} I_5(\alpha, s)\left\{I_6(\alpha, s)^{\frac{1}{q}} + I_7(\alpha, s)^{\frac{1}{q}}\right\},$$

where

$$I_5(\alpha, s) = \left(\int_0^1 \left|\frac{t^\alpha}{2} - \frac{1}{3}\right| dt\right)^{1-\frac{1}{q}},$$

$$I_6(\alpha,s)=\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s|f'(b)|^q+\left(\frac{1-t}{2}\right)^s|f'(a)|^q\right]dt$$

and

$$I_7(\alpha,s)=\int_0^1\left|\frac{t^\alpha}{2}-\frac{1}{3}\right|\left[\left(\frac{1+t}{2}\right)^s|f'(a)|^q+\left(\frac{1-t}{2}\right)^s|f'(b)|^q\right]dt.$$

The goal in this paper is to provide some Simpson's type inequalities for s-convex functions in the second sense via the Katugampola fractional integrals. Our results generalizes Theorems 1.1, 1.2, 1.3 and also some results in [11]. We complete this section with the definitions of the Riemann–Liouville, Hadamard and Katugampola fractional integrals.

Definition 1.2 ([12])**.** The left- and right-sided Riemann–Liouville fractional integrals of order $\alpha>0$ of f are defined by

$$J_{a+}^\alpha f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt$$

and

$$J_{b-}^\alpha f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,$$

with $a<x<b$ and $\Gamma(\cdot)$ is the gamma function given by

$$\Gamma(x):=\int_0^\infty t^{x-1}e^{-t}dt,\quad \mathrm{Re}(x)>0,$$

with the property that $\Gamma(x+1)=x\Gamma(x)$.

Definition 1.3 ([13])**.** The left- and right-sided Hadamard fractional integrals of order $\alpha>0$ of f are defined by

$$H_{a+}^\alpha f(x):=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{f(t)}{t}dt$$

and

$$H_{b-}^\alpha f(x):=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\ln\frac{t}{x}\right)^{\alpha-1}\frac{f(t)}{t}dt.$$

In what follows, $X_c^p(a,b)$, $c\in\mathbb{R}$, $1\le p\le\infty$, denotes the set of all complex-valued Lebesgue measurable functions f for which $\|f\|_{X_c^p}<\infty$, where the norm is defined by

$$\|f\|_{X_c^p}=\left(\int_a^b|t^cf(t)|^p\frac{dt}{t}\right)^{1/p},\quad 1\le p<\infty,$$

and, for $p=\infty$, $\|f\|_{X_c^\infty}=\operatorname{esssup}_{a\le t\le b}|t^cf(t)|$.

In 2011, Katugampola [9] introduced a new fractional integral operator which generalizes the Riemann–Liouville and Hadamard fractional integrals as follows.

Definition 1.4. Let $[a,b] \subset \mathbb{R}$ be a finite interval. Then, the left- and right-sided Katugampola fractional integrals of order $\alpha > 0$ of $f \in X_c^p(a,b)$ are defined by

$$ {}^\rho I_{a+}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^x \frac{t^{\rho-1}}{(x^\rho - t^\rho)^{1-\alpha}} f(t) dt $$

and

$$ {}^\rho I_{b-}^\alpha f(x) := \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_x^b \frac{t^{\rho-1}}{(t^\rho - x^\rho)^{1-\alpha}} f(t) dt, $$

with $a < x < b$ and $\rho > 0$, if the integrals exist.

Remark 1.2. It is shown in [9] that the Katugampola fractional integral operators are well-defined on $X_c^p(a,b)$.

Theorem 1.4 ([9]). *Let $\alpha > 0$ and $\rho > 0$. Then, for $x > a$,*

(a) $\lim_{\rho \to 1} {}^\rho I_{a+}^\alpha f(x) = J_{a+}^\alpha f(x)$;

(b) $\lim_{\rho \to 0^+} {}^\rho I_{a+}^\alpha f(x) = H_{a+}^\alpha f(x)$.

Similar results also hold for right-sided operators.

For more information about the Katumgapola fractional integrals and related results, we refer the interested reader to the papers [4,9,10].

2. Main Results

To obtain our main results, we need the following lemma which is a generalization of [5, Lemma 2.1] and [11, Lemma 5].

Lemma 2.1. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. Then the following identity holds:*

$$ \begin{aligned} &\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho) \right] \\ &- \frac{2^{\alpha-1}\rho^\alpha \Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha} \left[{}^\rho I_{a+}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I_{b-}^\alpha f\left(\frac{a^\rho + b^\rho}{2}\right) \right] \\ =&\frac{\rho(b^\rho - a^\rho)}{2} \left[\int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right) dt \right. \\ &\left. - \int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1-t^\rho}{2}a^\rho + \frac{1+t^\rho}{2}b^\rho\right) dt \right]. \end{aligned} $$

Proof. We start by considering the following computations which follows from change of variables and using the definiton of the Katugampola fractional integrals.

$$ \begin{aligned} &\int_0^1 t^{\alpha\rho-1} f\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right) dt \\ =&\int_0^1 t^{(\alpha-1)\rho} t^{\rho-1} f\left(\frac{1+t^\rho}{2}a^\rho + \frac{1-t^\rho}{2}b^\rho\right) dt \end{aligned} $$

$$
\begin{aligned}
&= \frac{2^{\alpha}}{(b^{\rho}-a^{\rho})^{\alpha}} \int_{a}^{\left(\frac{a^{\rho}+b^{\rho}}{2}\right)^{\frac{1}{\rho}}} \left(\frac{a^{\rho}+b^{\rho}}{2} - u^{\rho}\right)^{\alpha-1} u^{\rho-1} f\left(u^{\rho}\right) du \\
&= \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha)}{(b^{\rho}-a^{\rho})^{\alpha}} {}^{\rho}I^{\alpha}_{a+} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \tag{2.1}
\end{aligned}
$$

and, by similar argument as above, we have

$$
\int_0^1 t^{\alpha\rho-1} f\left(\frac{1-t^{\rho}}{2}a^{\rho} + \frac{1+t^{\rho}}{2}b^{\rho}\right) dt = \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha)}{(b^{\rho}-a^{\rho})^{\alpha}} {}^{\rho}I^{\alpha}_{b-} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right). \tag{2.2}
$$

Now, by using integration by parts and (2.1), we obtain

$$
\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{2}{\rho(a^{\rho}-b^{\rho})} \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) f\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right)\Bigg|_0^1 \\
&\quad + \frac{2\alpha\rho}{\rho(a^{\rho}-b^{\rho})} \int_0^1 \frac{t^{\alpha\rho-1}}{2} f\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{1}{3\rho(b^{\rho}-a^{\rho})} f(a^{\rho}) + \frac{2}{3\rho(b^{\rho}-a^{\rho})} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \\
&\quad - \frac{\alpha}{b^{\rho}-a^{\rho}} \int_0^1 t^{\alpha\rho-1} f\left(\frac{1+t^{\rho}}{2}a^{\rho} + \frac{1-t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{1}{3\rho(b^{\rho}-a^{\rho})} f(a^{\rho}) + \frac{2}{3\rho(b^{\rho}-a^{\rho})} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \\
&\quad - \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} {}^{\rho}I^{\alpha}_{a+} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right). \qquad (2.3)
\end{aligned}
$$

Similarly, by using integration by parts and (2.2), we obtain

$$
\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1}{3} - \frac{t^{\alpha\rho}}{2}\right) t^{\rho-1} f'\left(\frac{1-t^{\rho}}{2}a^{\rho} + \frac{1+t^{\rho}}{2}b^{\rho}\right) dt \\
&= \frac{-1}{3\rho(b^{\rho}-a^{\rho})} f(b^{\rho}) - \frac{2}{3\rho(b^{\rho}-a^{\rho})} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) \\
&\quad + \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} {}^{\rho}I^{\alpha}_{b-} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right). \qquad (2.4)
\end{aligned}
$$

Using (2.3) and (2.4), we obtain

$$
\begin{aligned}
I_1 - I_2 &= \frac{1}{3\rho(b^{\rho}-a^{\rho})} \left[f(a^{\rho}) + 4f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) + f(b^{\rho})\right] \\
&\quad - \frac{2^{\alpha}\rho^{\alpha-1}\Gamma(\alpha+1)}{(b^{\rho}-a^{\rho})^{\alpha+1}} \left[{}^{\rho}I^{\alpha}_{a+} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right) + {}^{\rho}I^{\alpha}_{b-} f\left(\frac{a^{\rho}+b^{\rho}}{2}\right)\right]. \qquad (2.5)
\end{aligned}
$$

The desired identity is obtained by multiplying both sides of (2.5) by $\frac{\rho(b^\rho-a^\rho)}{2}$. This completes the proof. $\square$

Theorem 2.1. *Let $\alpha,\rho>0$ and let $f:[a^\rho,b^\rho]\to\mathbb{R}$ be a differentiable function on (a^ρ,b^ρ), with $0\le a<b$ such that $f'\in L_1([a^\rho,b^\rho])$. If $|f'|$ is s-convex for $s\in(0,1]$, then the following inequalities hold:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right]-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\right.\\
&\quad\left.\times\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right|\\
&\le\frac{b^\rho-a^\rho}{2^{s+1}}\mathcal{C}(\alpha,s)\Big(|f'(a^\rho)|+|f'(b^\rho)|\Big)\\
&\le\frac{b^\rho-a^\rho}{3(s+1)}\Big(|f'(a^\rho)|+|f'(b^\rho)|\Big),
\end{aligned}\tag{2.6}$$

where

$$\mathcal{C}(\alpha,s)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\Big((1+u)^s+(1-u)^s\Big)du.$$

Proof. Using Lemma 2.1 and the s-convexity of $|f'|$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right]\right.\\
&\quad\left.-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right|\\
&\le\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\Bigg(\left|f'\left(\frac{1+t^\rho}{2}a^\rho+\frac{1-t^\rho}{2}b^\rho\right)\right|\\
&\quad+\left|f'\left(\frac{1-t^\rho}{2}a^\rho+\frac{1+t^\rho}{2}b^\rho\right)\right|\Bigg)dt\\
&\le\frac{\rho(b^\rho-a^\rho)}{2}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\Bigg(\frac{(1+t^\rho)^s}{2^s}|f'(a^\rho)|+\frac{(1-t^\rho)^s}{2^s}|f'(b^\rho)|\\
&\quad+\frac{(1-t^\rho)^s}{2^s}|f'(a^\rho)|+\frac{(1+t^\rho)^s}{2^s}|f'(b^\rho)|\Bigg)dt\\
&=\frac{(b^\rho-a^\rho)}{2^{s+1}}\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\Big((1+u)^s+(1-u)^s\Big)\Big(|f'(a^\rho)|+|f'(b^\rho)|\Big)du\\
&=\frac{(b^\rho-a^\rho)}{2^{s+1}}\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left((1+u)^s+(1-u)^s\right)du\left(|f'(a^\rho)|+|f'(b^\rho)|\right)\\
&=\frac{(b^\rho-a^\rho)}{2^{s+1}}\mathcal{C}(\alpha,s)\left(|f'(a^\rho)|+|f'(b^\rho)|\right),
\end{aligned}$$

where

$$\mathcal{C}(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| ((1+u)^s + (1-u)^s)\, du.$$

This proves the first inequality in (2.6). To obtain the second inequality in (2.6), we observe that $\left|\frac{1}{3} - \frac{u^\alpha}{2}\right| \le \frac{1}{3}$ for all $u \in [0,1]$. Thus,

$$\mathcal{C}(\alpha, s) \le \frac{1}{3}\int_0^1 \Big((1+u)^s + (1-u)^s\Big) du = \frac{2^{s+1}}{3(s+1)}.$$

This completes the proof. □

Remark 2.1. If $\rho = 1$, then the first inequality in Theorem 2.1 coincides with the inequality in Theorem 1.1 and the second inequality coincides with the inequality in Corollary 8 in [11].

Corollary 2.1. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \le a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|$ is convex, then the following inequalities hold:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right] \\
&\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\Bigg| \\
&\le \frac{b^\rho - a^\rho}{4}\mathcal{C}(\alpha, 1)\left(|f'(a^\rho)| + |f'(b^\rho)|\right) \\
&\le \frac{b^\rho - a^\rho}{6}\left(|f'(a^\rho)| + |f'(b^\rho)|\right).
\end{aligned}$$

Proof. The result follows directly if we take $s = 1$ in Theorem 2.1. □

Theorem 2.2. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \le a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is s-convex for $s \in (0,1]$ and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right] \\
&\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\Bigg| \\
&\le \frac{b^\rho - a^\rho}{2}\left(\int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right. \\
&\quad \left. + \left(\frac{1}{2^s(s+1)}|f'(a^\rho)|^q + \frac{2^{s+1}-1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq\frac{b^\rho-a^\rho}{6}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|f'(a^\rho)|^q+\frac{1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right.\\
&\left.\quad+\left(\frac{1}{2^s(s+1)}|f'(a^\rho)|^q+\frac{2^{s+1}-1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right],
\end{aligned}
\tag{2.7}
$$

where $\frac{1}{r}+\frac{1}{q}=1$.

Proof. Using Lemma 2.1, the Hölder's inequality and the s-convexity of $|f'|^q$, we obtain

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right]\right.\\
&\left.\quad-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right|\\
\leq&\frac{\rho(b^\rho-a^\rho)}{2}\left[\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left|f'\left(\frac{1+t^\rho}{2}a^\rho+\frac{1-t^\rho}{2}b^\rho\right)\right|dt\right.\\
&\left.\quad+\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left|f'\left(\frac{1-t^\rho}{2}a^\rho+\frac{1+t^\rho}{2}b^\rho\right)\right|dt\right]\\
=&\frac{b^\rho-a^\rho}{2}\left[\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left|f'\left(\frac{1+u}{2}a^\rho+\frac{1-u}{2}b^\rho\right)\right|du\right.\\
&\left.\quad+\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left|f'\left(\frac{1-u}{2}a^\rho+\frac{1+u}{2}b^\rho\right)\right|du\right]\\
\leq&\frac{b^\rho-a^\rho}{2}\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|f'\left(\frac{1+u}{2}a^\rho+\frac{1-u}{2}b^\rho\right)\right|^q du\right)^{\frac{1}{q}}\right.\\
&\left.\quad+\left(\int_0^1\left|f'\left(\frac{1-u}{2}a^\rho+\frac{1+u}{2}b^\rho\right)\right|^q du\right)^{\frac{1}{q}}\right]\\
\leq&\frac{b^\rho-a^\rho}{2}\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\int_0^1\left(\frac{(1+u)^s}{2^s}|f'(a^\rho)|^q+\frac{(1-u)^s}{2^s}|f'(b^\rho)|^q\right)du\right)^{\frac{1}{q}}\right.\\
&\left.\quad+\left(\int_0^1\left(\frac{(1-u)^s}{2^s}|f'(a^\rho)|^q+\frac{(1+u)^s}{2^s}|f'(b^\rho)|^q\right)du\right)^{\frac{1}{q}}\right]\\
=&\frac{b^\rho-a^\rho}{2}\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}}\left[\left(\frac{2^{s+1}-1}{2^s(s+1)}|f'(a^\rho)|^q+\frac{1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right.\\
&\left.\quad+\left(\frac{1}{2^s(s+1)}|f'(a^\rho)|^q+\frac{2^{s+1}-1}{2^s(s+1)}|f'(b^\rho)|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

This proves the first inequality of (2.7). The second inequality follows from the first inequality by using the fact that $\left|\frac{1}{3} - \frac{u^\alpha}{2}\right| \le \frac{1}{3}$ for all $u \in [0,1]$. $\square$

Remark 2.2. If $\rho = 1$, then the first inequality in Theorem 2.2 coincides with the inequality in Theorem 1.2 and the second inequality concides with the inequality in Corollary 12 in [11].

Corollary 2.2. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \le a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is convex and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right] \\
&\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\Bigg| \\
\le&\frac{b^\rho - a^\rho}{2}\left(\int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right|^r\right)^{\frac{1}{r}} \left[\left(\frac{3|f'(a^\rho)|^q + |f'(b^\rho)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(b^\rho)|^q + |f'(a^\rho)|^q}{4}\right)^{\frac{1}{q}}\right] \\
\le&\frac{b^\rho - a^\rho}{6}\left[\left(\frac{3|f'(a^\rho)|^q + |f'(b^\rho)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{3|f'(b^\rho)|^q + |f'(a^\rho)|^q}{4}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

Proof. The result follows directly if we take $s = 1$ in Theorem 2.2. $\square$

Theorem 2.3. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \le a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is s-convex for $s \in (0,1]$ and $q > 1$, then the following inequalities hold:*

$$\begin{aligned}
&\Bigg|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right] \\
&\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\Bigg| \\
\le&\frac{b^\rho - a^\rho}{2}\Big(\mathcal{M}_0(\alpha)\Big)^{\frac{1}{r}}\left[\left(\frac{1}{2^s}\Big(\mathcal{M}_1(\alpha,s)|f'(a^\rho)|^q + \mathcal{M}_2(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{1}{2^s}\Big(\mathcal{M}_2(\alpha,s)|f'(a^\rho)|^q + \mathcal{M}_1(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right] \\
\le&\frac{b^\rho - a^\rho}{2}\left(\frac{1}{3}\right)^{\frac{1}{r}}\left[\left(\frac{1}{2^s}\left(\frac{2^{s+1}-1}{3(s+1)}|f'(a^\rho)|^q + \frac{1}{3(s+1)}|f'(b^\rho)|^q\right)\right)^{\frac{1}{q}}\right. \\
&\left. + \left(\frac{1}{2^s}\left(\frac{1}{3(s+1)}|f'(a^\rho)|^q + \frac{2^{s+1}-1}{3(s+1)} f'(b^\rho)|^q\right)\right)^{\frac{1}{q}}\right], \qquad (2.8)
\end{aligned}$$

where $\frac{1}{r}+\frac{1}{q}=1$, *with*

$$\mathcal{M}_0(\alpha)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|du,$$

$$\mathcal{M}_1(\alpha,s)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|(1+u)^s du$$

and

$$\mathcal{M}_2(\alpha,s)=\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|(1-u)^s du.$$

Proof. Using Lemma 2.1, the Hölder's inequality and the s-convexity of $|f'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho)+4f\left(\frac{a^\rho+b^\rho}{2}\right)+f(b^\rho)\right]\right.\\
&\left.-\frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho-a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+}f\left(\frac{a^\rho+b^\rho}{2}\right)+{}^\rho I^\alpha_{b^-}f\left(\frac{a^\rho+b^\rho}{2}\right)\right]\right|\\
\leq&\frac{\rho(b^\rho-a^\rho)}{2}\left[\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left|f'\left(\frac{1+t^\rho}{2}a^\rho+\frac{1-t^\rho}{2}b^\rho\right)\right|dt\right.\\
&\left.+\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left|f'\left(\frac{1-t^\rho}{2}a^\rho+\frac{1+t^\rho}{2}b^\rho\right)\right|dt\right]\\
=&\frac{b^\rho-a^\rho}{2}\left[\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left|f'\left(\frac{1+u}{2}a^\rho+\frac{1-u}{2}b^\rho\right)\right|du\right.\\
&\left.+\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left|f'\left(\frac{1-u}{2}a^\rho+\frac{1+u}{2}b^\rho\right)\right|du\right]\\
\leq&\frac{b^\rho-a^\rho}{2}\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|du\right)^{\frac{1}{r}}\left[\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left(\frac{(1+u)^s}{2^s}|f'(a^\rho)|^q\right.\right.\right.\\
&\left.\left.+\frac{(1-u)^s}{2^s}|f'(b^\rho)|^q\right)du\right)^{\frac{1}{q}} \qquad (2.9)\\
&\left.+\left(\int_0^1\left|\frac{1}{3}-\frac{u^\alpha}{2}\right|\left(\frac{(1-u)^s}{2^s}|f'(a^\rho)|^q+\frac{(1+u)^s}{2^s}|f'(b^\rho)|^q\right)du\right)^{\frac{1}{q}}\right]\\
=&\frac{b^\rho-a^\rho}{2}\Big(\mathcal{M}_0(\alpha)\Big)^{\frac{1}{r}}\left[\left(\frac{1}{2^s}\Big(\mathcal{M}_1(\alpha,s)|f'(a^\rho)|^q+\mathcal{M}_2(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{1}{2^s}\Big(\mathcal{M}_2(\alpha,s)|f'(a^\rho)|^q+\mathcal{M}_1(\alpha,s)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where

$$\mathcal{M}_0(\alpha) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| du,$$

$$\mathcal{M}_1(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1+u)^s du$$

and

$$\mathcal{M}_2(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1-u)^s du.$$

This proves the first inequality of (2.8). For the second inequality, since $\left|\frac{1}{3} - \frac{u^\alpha}{2}\right| \leq \frac{1}{3}$ for all $u \in [0,1]$, it follows that

$$\mathcal{M}_0(\alpha) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| du \leq \frac{1}{3},$$

$$\mathcal{M}_1(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1+u)^s du \leq \frac{1}{3}\int_0^1 (1+u)^s du = \frac{2^{s+1}-1}{3(s+1)}$$

and

$$\mathcal{M}_2(\alpha, s) = \int_0^1 \left|\frac{1}{3} - \frac{u^\alpha}{2}\right| (1-u)^s du \leq \frac{1}{3}\int_0^1 (1-u)^s du = \frac{1}{3(s+1)}.$$

This completes the proof of the theorem. ☐

Remark 2.3. If $\rho = 1$, then the first inequality in Theorem 2.3 coincides with the inequality in Theorem 1.3.

Corollary 2.3. *Let $\alpha, \rho > 0$ and let $f : [a^\rho, b^\rho] \to \mathbb{R}$ be a differentiable function on (a^ρ, b^ρ), with $0 \leq a < b$ such that $f' \in L_1([a^\rho, b^\rho])$. If $|f'|^q$ is convex and $q > 1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a^\rho) + 4f\left(\frac{a^\rho + b^\rho}{2}\right) + f(b^\rho)\right]\right.\\
&\left.\quad - \frac{2^{\alpha-1}\rho^\alpha\Gamma(\alpha+1)}{(b^\rho - a^\rho)^\alpha}\left[{}^\rho I^\alpha_{a^+} f\left(\frac{a^\rho + b^\rho}{2}\right) + {}^\rho I^\alpha_{b^-} f\left(\frac{a^\rho + b^\rho}{2}\right)\right]\right|\\
\leq &\frac{b^\rho - a^\rho}{2}\Big(\mathcal{M}_0(\alpha)\Big)^{\frac{1}{r}}\left[\left(\frac{1}{2}\Big(\mathcal{M}_1(\alpha,1)|f'(a^\rho)|^q + \mathcal{M}_2(\alpha,1)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right.\\
&\left.\quad + \left(\frac{1}{2}\Big(\mathcal{M}_2(\alpha,1)|f'(a^\rho)|^q + \mathcal{M}_1(\alpha,1)|f'(b^\rho)|^q\Big)\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\frac{1}{r} + \frac{1}{q} = 1$.

Proof. The result follows directly if we take $s = 1$ in Theorem 2.3. ☐

3. Conclusion

We have introduced some new integral inequalities of Simpson's type for s-convex functions using the Katugampola fractional integrals. Our results generalize some results in the literature related to the Riemann–Liouville fractional integrals as pointed out in the paper. We have new results for the case $\rho \neq 1$. In particular, if we take the limit as $\rho \to 0^+$, then our results could be stated using the Hadamard fractional integrals. The details are left for the interested reader.

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