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SOLVABILITY FOR MULTI-POINT BVP OF NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE WITH THREE DIMENSIONAL KERNELS

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Abstract. This work deals with the BVP multi-point existence of solutions of a nonlinear fractional differential equations at resonance, where the kernel's dimension of the fractional differential operator is equal to three. Our results are based on Mawhin's theory of coincidence. As application, we give an example to illustrate our results.

1. Introduction

The present work concerns a kind of fractional differential equation which can be written as $Lx = Nx$, where L is a linear Fredholm operator of index zero, and N is a nonlinear operator. It is well known that if the kernel of the linear part contains only zero, the corresponding boundary value problem is called non-resonant. In this case, L is invertible, the equation can be reduced to a fixed point problem for the $L^{-1}N$ operator. Otherwise, if L is a non-invertible, i.e., $\dim \ker L \geq 1$, then the problem is said to be at resonance, and then the problem can be solved by using the coincidence degree theory. The higher value of dim $kerL$ is the more difficult. Recently, many authors investigated the existence of solutions for fractional differential equations at resonance. For instance, see [3–6, 9–11, 15, 16, 18, 19, 32] and the references therein.

The case of $\dim \ker L = 1$ has been discussed by many authors [3, 4, 6, 9–11, 16, 18, 19, 32]. In [6], Z. Bai and Y. Zhang investigated the boundary value problem for a

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fractional differential equation with nonlinear growth with $\dim\ker L = 1$

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\Big(t, u(t), D^{\alpha-1}_{0+}u(t)\Big), & t \in [0,1],\\ u(0) = 0, \quad u(1) = \sigma u(\eta), \end{cases}$$

where D^{α}_{0+} is the standard Riemann-Liouville derivative, $1 < \alpha \leq 2$, $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is continuous and $\sigma \in (0, \infty)$, $\eta \in (0, 1)$ are given constants such that $\sigma\eta^{\alpha-1} = 1$.

Z. Hu et al. in [10] prove the existence of solutions of two-point boundary value problem for a fractional differential equation at resonance with $\dim\ker L = 1$

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\Big(t, u(t), u'(t)\Big), & t \in [0,1],\\ u(0) = 0, \quad u(1) = u'(1), \end{cases}$$

where D^{α}_{0+} is the Caputo fractional derivative, $1 < \alpha \leq 2$, $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the Caratheodory conditions.

L. Hu et al. studied in [11] a two-point boundary value problem for fractional differential equation at resonance with $\dim\ker L = 1$:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\Big(t, u(t), D^{\alpha-1}_{0+}u(t), D^{\alpha-2}_{0+}u(t), \dots, D^{\alpha-(N-1)}_{0+}u(t)\Big),\\ u(0) = D^{\alpha-2}_{0+}u(0) = \cdots = D^{\alpha-(N-1)}_{0+}u(0) = 0, \quad D^{\alpha-1}_{0+}u(0) = D^{\alpha-1}_{0+}u(1), \end{cases}$$

where $0 < t < 1$, $N - 1 < \alpha \leq N$, D^{α}_{0+} is Riemann-Liouville fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function.

For the case $\dim\ker L = 2$, Bai and Zhang established in [5] the existence of at least one solution for the m-point boundary value problem for fractional differential equation at resonance with $\dim\ker L = 2$

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\Big(t, u(t), D^{\alpha-2}_{0+}u(t), D^{\alpha-1}_{0+}u(t)\Big), & t \in (0,1),\\ I^{\alpha-1}_{0+}u(0) = 0, \quad D^{\alpha-1}_{0+}u(0) = D^{3-\alpha}_{0+}(\eta), \quad u(1) = \sum_{i=1}^{m}\alpha_i u(\eta_i), \end{cases}$$

where $2 < \alpha < 3$, $0 < \eta \leq 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_m < 1$, $m \geq 2$, $\sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-1} = \sum_{i=1}^{m}\alpha_i\eta_i^{\alpha-2} = 1$. D^{α}_{0+} and I^{α}_{0+} are the standard Riemann-Liouville fractional derivative and fractional integral respectively and $f : [0,1] \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the Caratheodory conditions. The results are obtained under the assumption that:

$$R = \frac{1}{\alpha}\eta^{\alpha}\frac{\Gamma(\alpha)\Gamma(\alpha-1)}{\Gamma(2\alpha-1)}\left[1 - \sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-2}\right] - \frac{1}{\alpha-1}\eta^{\alpha-1}\frac{(\Gamma(\alpha))^2}{\Gamma(2\alpha)}\left[1 - \sum_{i=1}^{m}\alpha_i\eta_i^{2\alpha-1}\right] \neq 0.$$

W. Jiang showed in [15] an existence result for the boundary value problem of fractional differential equation at resonance with $\dim\ker L = 2$:

$$\begin{cases} D^{\alpha}_{0+}u(t) = f\Big(t, u(t), D^{\alpha-1}_{0+}u(t)\Big), & t \in J = [0,1],\\ u(0) = 0, \quad D^{\alpha-1}_{0+}u(0) = \sum_{i=1}^{m}a_i D^{\alpha-1}_{0+}(\xi_i), \quad D^{\alpha-2}_{0+}u(0) = \sum_{j=1}^{n}b_j D^{\alpha-2}_{0+}(\eta_j), \end{cases}$$

where $2 < \alpha < 3$, D^{α}_{0+} is Riemann-Liouville fractional derivative, $0 < \xi_1 < \xi_2 < \cdots < \xi_m < 1$, $0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\sum_{i=1}^{m}a_i = 1$, $\sum_{j=1}^{n}b_j = 1$, $\sum_{j=1}^{n}b_j\eta_j = 1$,

$f:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ satisfies the Caratheodory conditions. The results are obtained under the assumption that

$$\frac{1}{3}\Big(1-\sum_{j=1}^{n} b_j\eta_j^3\Big)\sum_{i=1}^{m} a_i\xi_i-\frac{1}{2}\Big(1-\sum_{j=1}^{n} b_j\eta_j^2\Big)\sum_{i=1}^{m} a_i\xi_i^2\neq 0.$$

Motivated by the results cited above, we investigate the solvability of multi-point boundary value problem of nonlinear fractional differential equation at resonance with $\dim\ker L=3$

$$\begin{cases}\Big(\phi(t)^C D_{0+}^{\alpha}u(t)\Big)'=f\Big(t,u(t),u'(t),u''(t),u'''(t),{}^C D_{0+}^{\alpha}u(t)\Big), & t\in I,\\ u(0)=0,\quad {}^C D_{0+}^{\alpha}u(0)=0,\quad u'''(0)=\sum_{i=1}^{m}a_iu'''(\xi_i),\\ u''(0)=\sum_{j=1}^{l}b_ju''(\eta_j),\quad u'(1)=\sum_{k=1}^{n}c_ku'(\rho_k),\end{cases}\tag{1.1}$$

where ${}^C D_{0+}^{\alpha}$ is the Caputo fractional derivative, $3<\alpha\le 4$, $0<\xi_1<\cdots<\xi_m<1$, $0<\eta_1<\cdots<\eta_l<1$, $0<\rho_1<\cdots<\rho_n<1$, $a_i,b_j,c_k\in\mathbb{R}$, $i=1,\dots,m$, $j=1,\dots,l$, $k=1,\dots,n$, $I=[0,1]$, $\phi(t)\in C^1([0,1])$, $\mu=\min_{t\in I}\phi(t)>0$ and $f:[0,1]\times\mathbb{R}^5\to\mathbb{R}$ is a Caratheodory function, that is,

(i) for each $x\in\mathbb{R}^5$, the function $x\to f(t,x)$ is Lebesgue measurable;
(ii) for almost every $t\in[0,1]$, the function $t\to f(t,x)$ is continuous on $\mathbb{R}^5$;
(iii) for each $r>0$, there exists $\varphi_r(t)\in L^1\Big([0,1],\,\mathbb{R}\Big)$ such that, for a.e. $t\in[0,1]$ and every $|x|\le r$, we have $|f(t,x)|\le\varphi_r(t)$.

In this work, we will always suppose that the following conditions hold.

(H_1) $\sum_{i=1}^{m}a_i=\sum_{j=1}^{l}b_j=\sum_{k=1}^{n}c_k=1$, $\sum_{j=1}^{l}b_j\eta_j=0$, $\sum_{k=1}^{n}c_k\rho_k=\sum_{k=1}^{n}c_k\rho_k^2=1$.
(H_2)

$$\Delta=\begin{vmatrix} d_{11} & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ d_{31} & d_{32} & d_{33}\end{vmatrix}\neq 0,$$

where for $\nu=1,2,3$, we define

$$d_{\nu1}=\sum_{i=1}^{m}a_i\int_0^{\xi_i}\frac{s^{\nu}(\xi_i-s)^{\alpha-4}}{\nu\phi(s)}ds,\quad d_{\nu2}=\sum_{j=1}^{l}b_j\int_0^{\eta_j}\frac{s^{\nu}(\eta_j-s)^{\alpha-3}}{\nu\phi(s)}ds,$$
$$d_{\nu3}=\int_0^1\frac{s^{\nu}(1-s)^{\alpha-2}}{\nu\phi(s)}ds-\sum_{k=1}^{n}c_k\int_0^{\rho_k}\frac{s^{\nu}(\rho_k-s)^{\alpha-2}}{\nu\phi(s)}ds.$$

The rest of this work is organized as follows. In Section 2, we introduce some notations, definitions and lemmas which will be used later. In Section 3, we present and prove our main results by applying the coincidence degree continuation theorem. Finally, in Section 4 we provide an example.

2. Preliminaries

In this section, we present the necessary definitions and lemmas from fractional calculus theory. These definitions and properties can be found in recent literature, see for example [17, 26–28, 30].

Definition 2.1. Let $\alpha > 0$, and u a function $u : (0, \infty) \to \mathbb{R}$. The Riemann-Liouville fractional integral of order α of u is defined by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Remark 2.1. The notation $I_{0+}^{\alpha} u(t) \mid_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \varepsilon)$, $\varepsilon > 0$, of 0 as follows:

$$I_{0+}^{\alpha} u(t) \mid_{t=0} = \lim_{t \to 0+} I_{0+}^{\alpha} u(t).$$

Generally $[I_{0+}^{\alpha} u(t) \mid_{t=0}]$ is not necessarily zero. For instance, let $\alpha \in (0, 1)$, $u(t) = t^{-\alpha}$. Then

$$I_{0+}^{\alpha} t^{-\alpha}|_{t=0} = \lim_{t \to 0+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \lim_{t \to 0+} \Gamma(1-\alpha) = \Gamma(1-\alpha).$$

Definition 2.2. Let $\alpha > 0$. The Caputo fractional derivative of order α of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$${}^{C}D_{0+}^{\alpha} u(t) = I_{0+}^{n-\alpha} u^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of real number α, provided that the right-hand side is pointwise defined on $(0, \infty)$.

Lemma 2.1. *Let $\alpha, \eta > 0$, $n = [\alpha] + 1$, then the following relations hold*

$${}^{C}D_{0+}^{\alpha} t^{\eta} = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} t^{\eta-\alpha}, \quad \eta > n-1,$$

and

$${}^{C}D_{0+}^{\alpha} t^{k} = 0, \quad k = 0, \dots, n-1.$$

Lemma 2.2. *Let $\alpha, \beta \geq 0$ and $u \in L^1\big([0,1]\big)$. Then $I_{0+}^{\alpha} I_{0+}^{\beta} u(t) = I_{0+}^{\alpha+\beta} u(t)$ and ${}^{C}D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$ for all $t \in [0,1]$*

Lemma 2.3. *Let $\alpha > 0$, $n = [\alpha] + 1$. Then*

$$I_{0+}^{\alpha} \left({}^{C}D_{0+}^{\alpha} u(t)\right) = u(t) + \sum_{k=0}^{n-1} \delta_k t^k, \quad \delta_k \in \mathbb{R}.$$

Lemma 2.4. *Let $\alpha > 0$ and $n = [\alpha] + 1$. If ${}^{C}D_{0+}^{\alpha} u(t) \in C[0,1]$, then $u(t) \in C^{n-1}([0,1])$.*

Proof. Let $h(t) \in C[0,1]$, such that ${}^{C}D^{\alpha}_{0+}u(t) = h(t)$, then, from Lemma 2.2, we have

$$u(t) = I^{\alpha}_{0+}h(t) + \sum_{k=0}^{n-1} \delta_k t^k, \quad \delta_k \in \mathbb{R}.$$

It is easy to check that $u(t) \in C^{n-1}([0,1])$. □

Lemma 2.5. *Let $\alpha > 0,\ u \in L^1\Big([0,1],\mathbb{R}\Big)$. Then, for all $t \in [0,1]$, we have*

$$I^{\alpha+1}_{0+}u(t) \le \|I^{\alpha}_{0+}u\|_{L^1}.$$

Proof. Let $u \in L^1\Big([0,1],\mathbb{R}\Big)$, from Lemma 2.2, we have

$$I^{\alpha+1}_{0+}u(t) = I^{1}_{0+}I^{\alpha}_{0+}u(t) = \int_0^t I^{\alpha}_{0+}u(s)ds \le \int_0^1 |I^{\alpha}_{0+}u(s)|ds = \|I^{\alpha}_{0+}u\|_{L^1}.$$ □

Lemma 2.6 ([30])**.** *The fractional integral I^{α}_{0+}, $\alpha > 0$, is bounded in $L^1([0,1],\mathbb{R})$ with*

$$\|I^{\alpha}_{0+}u\|_{L^1} \le \frac{\|u\|_{L^1}}{\Gamma(\alpha+1)}.$$

Now, let us recall some notations about the coincidence degree continuation theorem. For more details see [25].

Definition 2.3. Let X and Y be real Banach spaces. A linear operator $L : \operatorname{dom} L \subset X \to Y$ is said to be a Fredholm operator of index zero if

(1) $\operatorname{Im} L$ is a closed subset of Y;
(2) $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

It follows from Definition 2.3 that there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\ker L = \operatorname{Im} P, \quad \operatorname{Im} L = \ker Q, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{Im} L \oplus \operatorname{Im} Q.$$

It follows that

$$L_p = L\ |_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible. We denote the inverse of this map by K_p.

Definition 2.4. Let L be a Fredholm operator of index zero. If Ω is an open bounded subset of X and $\operatorname{dom} L \bigcap \Omega \neq \emptyset$. The map $N : \overline{\Omega} \to X$ will be called L-compact on $\overline{\Omega}$ if

(1) $QN(\overline{\Omega})$ is bounded;
(2) $K_{P,Q}N = K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Theorem 2.1. *Let $L : \operatorname{dom} L \subset X \to Y$ be a Fredholm operator of index zero and $N : X \to Y$ L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(1) *$Lx \neq \lambda Nx$ for every $(x,\ \lambda) \in [(\operatorname{dom} L \backslash \ker L) \cap \partial\Omega] \times (0,1)$;*
(2) *$Nx \notin \operatorname{Im} L$ for every $x \in \ker L \cap \partial\Omega$;*
(3) *$\deg(QN\ |_{\ker L}, \Omega \cap \ker L, 0) \neq 0$, where $Q : Y \to Y$ is a projection such that $\operatorname{Im} L = \ker Q$.*

Then, the abstract equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

For our purpose and according to Lemma 2.4, the adequate functional space is:

$$X = \Big\{u : {}^{C}D^{\alpha}_{0+}u \in C([0,1],\mathbb{R}),\ u \text{ satisfies boundary value conditions of } (1.1)\Big\}$$

endowed with the norm:

$$\|u\|_X = \sum_{i=0}^{3} \|u^{(i)}\|_\infty + \|{}^{C}D^{\alpha}_{0+}u\|_\infty, \quad \text{where } \|u\|_\infty = \max_{t\in[0,1]} |u(t)|.$$

By means of the functional analysis theory, we can prove that $(X, \|.\|_X)$ is a Banach space.

Let $Y = L^1[0,1]$ be the Lebesgue space of real measurable functions $t \mapsto y(t)$ defined on $[0,1]$ and such that $t \mapsto |y(t)|$ is Lebesgue integrable. Y is a Banach space with the norm $\|y\|_{L^1} = \int_0^1 |y(t)|dt$. Define L to be the linear operator from $\operatorname{dom} L \cap X$ to Y

$$Lu = \left(\phi\, {}^{C}D^{\alpha}_{0+}u\right)', \quad u \in \operatorname{dom} L,$$

where

$$\operatorname{dom} L = \Big\{u \in X \ :\ {}^{C}D^{\alpha}_{0+}u(t) \text{ is absolutely continuous on } [0,\ 1]\Big\}$$

and define the operator $N : X \to Y$ as:

$$Nu(t) = f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^{C}D^{\alpha}_{0+}u(t)\Big), \quad t \in [0,1].$$

Then the boundary value problem (1.1) can be written in abstract form as:

$$Lu = Nu, \quad u \in \operatorname{dom} L.$$

To study the compactness of operator N, we need the following lemma.

Lemma 2.7. *$U \subset X$ is a relatively compact set in X if and only if U is uniformly bounded and equicontinuous. Here uniformly bounded means there exists $M > 0$ such that for every $u \in U$*

$$\|u\|_X = \sum_{i=0}^{3} \|u^{(i)}\|_\infty + \|{}^{C}D^{\alpha}_{0+}u\|_\infty \le M,$$

and equicontinuous means that for all $\varepsilon > 0$, exists $\delta > 0$, such that

$$|u^{(i)}(t_1) - u^{(i)}(t_2)| < \varepsilon, \quad \text{for all } u \in U,\ t_1, t_2 \in I,\ |t_1 - t_2| < \delta,\ i \in \{0,1,2,3\},$$

and

$$|{}^{C}D^{\alpha}_{0+}u(t_1) - {}^{C}D^{\alpha}_{0+}u(t_2)| < \varepsilon, \quad \text{for all } u \in U,\ t_1, t_2 \in I,\ |t_1 - t_2| < \delta.$$

3. Main Results

In this section we shall present and prove our main result.

Lemma 3.1. *Let $y \in Y$, $\phi \in C^1[0,1]$, $\min_{t\in I}\phi(t) > \mu > 0$, and suppose that (H_1) holds. Then $u \in X$ is the solution of the following fractional differential equation:*

$$\begin{cases} \left(\phi(t)^C D^{\alpha}_{0+}u(t)\right)' = y(t), \quad t \in I = [0,1], \\ u(0) = 0, \quad ^C D^{\alpha}_{0+}u(0) = 0, \quad u'''(0) = \sum_{i=1}^m a_i u'''(\xi_i), \\ u''(0) = \sum_{j=1}^l b_j u''(\eta_j),\ u'(1) = \sum_{k=1}^n c_k u'(\rho_k), \end{cases} \tag{3.1}$$

where u is given by

$$u(t) = \sum_{i=1}^3 \delta_i t^i + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)drds, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}, \tag{3.2}$$

and

$$T_1(y) = T_2(y) = T_3(y) = 0, \tag{3.3}$$

where T_1, T_2, $T_3 : Y \to Y$ are three linear operators defined as follow:

$$\begin{aligned} T_1(y) =& \sum_{i=1}^m a_i \int_0^{\xi_i}\frac{(\xi_i - s)^{\alpha-4}}{\phi(s)}\int_0^s y(r)drds, \\ T_2(y) =& \sum_{j=1}^l b_j \int_0^{\eta_j}\frac{(\eta_j - s)^{\alpha-3}}{\phi(s)}\int_0^s y(r)drds, \\ T_3(y) =& \int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)}\int_0^s y(r)drds - \sum_{k=1}^n c_k \int_0^{\rho_k}\frac{(\rho_k - s)^{\alpha-2}}{\phi(s)}\int_0^s y(r)drds. \end{aligned}$$

Proof. Let u be a solution of problem (3.1). Then we have

$$\phi(t)^C D^{\alpha}_{0+}u(t) = \delta + \int_0^t y(s)ds, \quad \delta \in \mathbb{R}.$$

The hypothesis $^C D^{\alpha}_{0+}u(0) = 0$ and $\min_{t\in I}\phi(t) > 0$, allow us to write

$$^C D^{\alpha}_{0+}u(t) = \frac{1}{\phi(t)}\int_0^t y(s)ds.$$

By Lemma 2.3, we get that

$$u(t) = \sum_{i=0}^3 \delta_i t^i + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)drds, \quad \delta_0, \delta_1, \delta_2, \delta_3 \in \mathbb{R}.$$

$u(0) = 0$, implies that

$$u(t) = \sum_{i=1}^3 \delta_i t^i + \frac{1}{\Gamma(\alpha)}\int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)}\int_0^s y(r)drds, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}.$$

By $u'''(0) = \sum_{i=1}^{m} a_i u'''(\xi_i)$ and $\sum_{i=1}^{l} a_i = 1$, we obtain

$$\sum_{i=1}^{l} a_i \int_0^{\xi_i} \frac{(\xi_i - s)^{\alpha-4}}{\phi(s)} \int_0^s y(r)drds = 0,$$

From the conditions $u''(0) = \sum_{j=1}^{l} b_j u''(\eta_j)$ and $\sum_{j=1}^{l} b_j = 1$, $\sum_{j=1}^{l} b_j \eta_j = 0$, we get

$$\sum_{j=1}^{l} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\alpha-3}}{\phi(s)} \int_0^s y(r)drds = 0.$$

Combining $u'(1) = \sum_{k=1}^{n} c_k u'(\rho_k)$, $\sum_{k=1}^{n} c_k = 1$ and $\sum_{k=1}^{n} c_k \rho_k = 1$, $\sum_{k=1}^{n} c_k \rho_k^2 = 1$, we find

$$\int_0^1 \frac{(1-s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds - \sum_{k=1}^{n} c_k \int_0^{\rho_k} \frac{(\rho_k - s)^{\alpha-2}}{\phi(s)} \int_0^s y(r)drds = 0.$$

Thus,

$$T_1(y) = T_2(y) = T_3(y) = 0.$$

On the other hand, we let

$$u(t) = \sum_{i=1}^{3} \delta_i t^i + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds,$$

where $\delta_1, \delta_2, \delta_3$ are arbitrary constants. It is clear that $u(0) = 0$, in view of Lemmas 2.1 and 2.2, we obtain

$${}^{C}D_{0^+}^{\alpha} u(t) = \frac{1}{\phi(t)} \int_0^t y(s)ds.$$

Thus, ${}^{C}D_{0^+}^{\alpha} u(0) = 0$ and $\left(\phi(t)\, {}^{C}D_{0^+}^{\alpha} u(t)\right)' = y(t)$ for all $t \in [0,1]$.

If (3.3) holds, we can calculate the following equations

$$u'''(0) - \sum_{i=1}^{m} a_i u'''(\xi_i) = \frac{T_1(y)}{\Gamma(\alpha-3)} = 0, \quad u''(0) - \sum_{j=1}^{l} b_j u''(\eta_j) = \frac{T_2(y)}{\Gamma(\alpha-2)} = 0,$$

$$u'(1) - \sum_{k=1}^{n} c_k u'(\rho_k) = \frac{T_3(y)}{\Gamma(\alpha-1)} = 0,$$

so, u is the solution of the problem (3.1), this completes the proof. □

Lemma 3.2. *Assume that* (H_1) *and* (H_2) *hold. Let* $\phi \in C^1([0,1])$, $\min_{t\in[0,1]} \phi(t) > \mu > 0$, *then* $L : \operatorname{dom} L \subset X \to Y$ *is a Fredholm operator of index zero, and the inverse linear operator* $K_p = L_p^{-1} : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ *is defined by*

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r)drds. \tag{3.4}$$

It satisfies

$$\|K_p y\|_X \le \frac{4 + \Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)} \|y\|_{L^1}. \tag{3.5}$$

Proof. It is clear that $\ker L = \Big\{u \,:\, u(t) = \sum_{k=1}^{3} \delta_k t^k, \delta_1, \delta_2, \delta_3 \in \mathbb{R}\Big\}$. Furthermore, Lemma 3.1 implies

$$\mathrm{Im} L = \Big\{y \in Y \,:\, T_1(y) = T_2(y) = T_3(y) = 0\Big\}. \tag{3.6}$$

Consider continuous linear mapping $Q : Y \to Y$ defined by

$$Qy = Q_1(y) + Q_2(y)t + Q_3(y)t^2, \tag{3.7}$$

where Q_1, Q_2, $Q_3 : Y \to Y$ are three linear operators defined as follows

$$\begin{aligned} Q_1(y) &= \frac{1}{\Delta}\Big(e_{11}T_1(y) + e_{12}T_2(y) + e_{13}T_3(y)\Big), \\ Q_2(y) &= \frac{1}{\Delta}\Big(e_{21}T_1(y) + e_{22}T_2(y) + e_{23}T_3(y)\Big), \\ Q_3(y) &= \frac{1}{\Delta}\Big(e_{31}T_1(y) + e_{32}T_2(y) + e_{33}T_3(y)\Big), \end{aligned}$$

e_{ij}, $i, j = 1, 2, 3$, are the algebraic complements of d_{ij}.

We will prove that $\ker Q = \mathrm{Im}\, L$. Obviously, $\mathrm{Im}\, L \subset \ker Q$. As well, if $y \in \ker Q$, then

$$\begin{cases} e_{11}T_1(y) + e_{12}T_2(y) + e_{13}T_3(y) = 0, \\ e_{21}T_1(y) + e_{22}T_2(y) + e_{23}T_3(y) = 0, \\ e_{31}T_1(y) + e_{32}T_2(y) + e_{33}T_3(y) = 0. \end{cases} \tag{3.8}$$

The determinant of coefficients for (3.8) is $\Delta^2 \neq 0$. We find $T_1(y) = T_2(y) = T_3(y) = 0$ and that implies $y \in \mathrm{Im}\, L$. So, $\ker Q \subset \mathrm{Im}\, L$. Now, we prove $Q^2 y = Qy$, $y \in Y$. For $y \in Y$, we have

$$\begin{aligned} Q_1^2(y) &= \frac{1}{\Delta}\Big[e_{11}T_1\Big(Q_1(y)\Big) + e_{12}T_2\Big(Q_1(y)\Big) + e_{13}T_3\Big(Q_1(y)\Big)\Big] \\ &= \frac{1}{\Delta}\Big(e_{11}d_{11} + e_{12}d_{21} + e_{13}d_{31}\Big)Q_1 y \\ &= Q_1 y, \\ Q_1\Big(Q_2(y)t\Big) &= \frac{1}{\Delta}\Big[e_{11}T_1\Big(Q_2(y)t\Big) + e_{12}T_2\Big(Q_2(y)t\Big) + e_{13}T_3\Big(Q_2(y)t\Big)\Big] \\ &= \frac{1}{\Delta}\Big(e_{11}d_{12} + e_{12}d_{22} + e_{13}d_{32}\Big)Q_2 y \\ &= 0, \\ Q_1\Big(Q_3(y)t^2\Big) &= \frac{1}{\Delta}\Big[e_{11}T_1\Big(Q_3(y)t^2\Big) + e_{12}T_2\Big(Q_3(y)t^2\Big) + e_{13}T_3\Big(Q_3(y)t^2\Big)\Big] \\ &= \frac{1}{\Delta}\Big(e_{11}d_{13} + e_{12}d_{23} + e_{13}d_{33}\Big)Q_3 y \\ &= 0. \end{aligned}$$

Similarly, we obtain

$$Q_2\Big(Q_1(y)\Big) = 0, \quad Q_2\Big(Q_2(y)t\Big) = Q_2 y, \quad Q_2\Big(Q_3(y)t^2\Big) = 0,$$

$$Q_3\Big(Q_1(y)\Big) = 0, \quad Q_3\Big(Q_2(y)t\Big) = 0, \quad Q_3\Big(Q_3(y)t^2\Big) = Q_3 y.$$

Therefore, we get

$$\begin{aligned} Q^2 g =& Q_1\Big(Q_1(y)\Big) + Q_1\Big(Q_2(y)t\Big) + Q_1\Big(Q_3(y)t^2\Big) + Q_2\Big(Q_1(y)\Big)t + Q_2\Big(Q_2(y)t\Big)t \\ &+ Q_2\Big(Q_3(y)t^2\Big)t + Q_3\Big(Q_1(y)\Big)t^2 + Q_3\Big(Q_2(y)t\Big)t^2 + Q_3\Big(Q_3(y)t^2\Big)t^2 \\ =& Q_1(y) + Q_2(y)t + Q_3(y)t^2 \\ =& Qg. \end{aligned}$$

This implies that the operator Q is a projector.

Take $y \in Y$ in the form $y = (y - Qy) + Qy$. Then $(y - Qy) \in \ker Q = \operatorname{Im} L$ and $Qy \in \operatorname{Im} Q$. Thus, $Y = \operatorname{Im} Q + \operatorname{Im} L$. And for any $y \in \operatorname{Im} Q \cap \operatorname{Im} L$ from $y \in \operatorname{Im} Q$, there exist constants $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ such that $y(t) = \sum_{k=1}^{3} \delta_k t^k$, from $y \in \operatorname{Im} L$, we obtain

$$\begin{cases} d_{11}\delta_1 + d_{12}\delta_2 + d_{13}\delta_3 = 0, \\ d_{21}\delta_1 + d_{22}\delta_2 + d_{23}\delta_3 = 0, \\ d_{31}\delta_1 + d_{32}\delta_2 + d_{33}\delta_3 = 0. \end{cases} \tag{3.9}$$

The determinant of coefficients for (3.9) is $\Delta \neq 0$. Therefore, (3.9) has an unique solution $\delta_1 = \delta_2 = \delta_3 = 0$, which implies $\operatorname{Im} Q \cap \operatorname{Im} L = 0$. Then, we have

$$Y = \operatorname{Im} Q \oplus \ker Q = \operatorname{Im} Q \oplus \operatorname{Im} L. \tag{3.10}$$

Thus, $\dim \ker L = 3 = \dim \operatorname{Im} Q = \operatorname{codim} \ker Q = \operatorname{codim} \operatorname{Im} L$, this means that L is a Fredholm operator of index zero.

Let $P : X \to X$ be a mapping defined by

$$Pu(t) = \sum_{k=1}^{3} \frac{u^{(k)}(0)}{k!} t^k. \tag{3.11}$$

We note that P is a linear continuous projector and $\operatorname{Im} P = \ker L$. It follows from $u = (u - Pu) + Pu$ that $X = \ker P + \ker L$. By simple calculation, we obtain that $\ker L \cap \ker P = \{0\}$. Hence,

$$X = \ker L \oplus \ker P. \tag{3.12}$$

Define $K_p : \operatorname{Im} L \to \operatorname{dom} L \cap \ker P$ as follows:

$$(K_p y)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s y(r) dr ds.$$

Now, we will prove that K_p is the inverse of $L\, |_{\operatorname{dom} L \cap \ker P}$. In fact, for $u \in \operatorname{dom} L \cap \ker P$, we have

$$(K_p L)u(t) = I_{0^+}^{\alpha} \left(\frac{I_{0^+}^{1}\Big(\phi\, {}^C D_{0^+}^{\alpha} u\Big)'}{\phi} \right)(t) = I_{0^+}^{\alpha}\, {}^C D_{0^+}^{\alpha} u(t) = u(t) + \sum_{k=0}^{3} \frac{u^{(k)}(0)}{k!} t^k.$$

In view of $u \in \operatorname{dom} L \cap \ker P$, $u(0) = 0$ and $Pu = 0$. Thus,

$$(K_p L)u(t) = u(t), \tag{3.13}$$

and for $y \in \operatorname{Im} L$, we find

$$(LK_p)y(t) = L(K_p y)(t) = \left[\phi(t)\, {}^C D_{0^+}^{\alpha} \left(I_{0^+}^{\alpha} \Big(\frac{I_{0^+}^1 y}{\phi} \Big)(t) \right)\right]' = y(t).$$

Thus, $K_p = \left(L \mid_{\operatorname{dom} L \cap \ker P} \right)^{-1}$. Again for each $y \in \operatorname{Im} L$, and from Lemmas 2.2, 2.5 and 2.6, we have

$$\begin{aligned}
\|K_p y\|_X &= \sum_{i=0}^{3} \max_{t\in[0,1]} \left| (K_p y)^{(i)}(t) \right| + \max_{t\in[0,1]} \left| {}^C D_{0^+}^{\alpha} (K_p y)(t) \right| \\
&= \sum_{i=0}^{3} \max_{t\in[0,1]} \left| I_{0^+}^{\alpha - i} \Big(\frac{I_{0^+}^1 y}{\phi} \Big)(t) \right| + \max_{t\in[0,1]} \left| {}^C D_{0^+}^{\alpha} I_{0^+}^{\alpha} \Big(\frac{I_{0^+}^1 y}{\phi} \Big)(t) \right| \\
&\le \sum_{i=0}^{3} \|y\|_{L^1} \max_{t\in[0,1]} \left| I_{0^+}^{\alpha - i} \frac{1}{\phi}(t) \right| + \|y\|_{L^1} \max_{t\in[0,1]} \left| I_{0^+}^{1} \frac{1}{\phi}(t) \right| \\
&\le \sum_{i=0}^{3} \|y\|_{L^1} \max_{t\in[0,1]} \left| I_{0^+}^{\alpha - i} \frac{1}{\mu}(t) \right| + \|y\|_{L^1} \max_{t\in[0,1]} \left| I_{0^+}^{1} \frac{1}{\mu}(t) \right| \\
&\le \sum_{i=0}^{3} \frac{\|y\|_{L^1}}{\mu \Gamma(\alpha + 1 - i)} + \frac{\|y\|_{L^1}}{\mu} \\
&\le \frac{4 + \Gamma(\alpha - 2)}{\mu \Gamma(\alpha - 2)} \|y\|_{L^1}.
\end{aligned}$$

□

Lemma 3.3. *Suppose that* Ω *is an open bounded subset of* X *such that* $\operatorname{dom} L \cap \overline{\Omega} \neq \emptyset$. *Then,* N *is* L*-compact on* $\overline{\Omega}$.

Proof. It is clear that $QN(\overline{\Omega})$ and $K_p(I - Q)N(\overline{\Omega})$ are bounded, due to the fact that f realize the caratheodory conditions.

Using the Lebesgue dominated convergence theorem, we can easily find that QN and $K_{P,Q} N = K_p(I - Q)N : \overline{\Omega} \to X$ are continuous. By the hypothesis (iii) on the function f, there exists a constant $A > 0$, such that $|(I - Q)N(u(t))| \le A$, for all $u \in \Omega$, $t \in [0, 1]$. For $i = 0, 1, 2, 3$, $0 \le t_1 \le t_2 \le 1$, and $u \in \Omega$, we put $M(t) = (I - Q)Nu(t)$. One has

$$\begin{aligned}
&\left| \Big(K_{P,Q} Nu\Big)^{(i)}(t_2) - \Big(K_{P,Q} Nu\Big)^{(i)}(t_1) \right| \\
=&\frac{1}{\Gamma(\alpha - i)} \left| \int_0^{t_2} \frac{(t_2 - s)^{\alpha - i - 1}}{\phi(s)} \int_0^s M(r)drds - \int_0^{t_1} \frac{(t_1 - s)^{\alpha - i - 1}}{\phi(s)} \int_0^s M(r)drds \right| \\
\le&\frac{A}{\mu \Gamma(\alpha - i)} \left\{ \int_0^{t_1} (t_2 - s)^{\alpha - i - 1} - (t_1 - s)^{\alpha - i - 1} ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha - i - 1} ds \right\} \\
=&\frac{A}{\mu \Gamma(\alpha + 1 - i)} (t_2^{\alpha - i} - t_1^{\alpha - i}),
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
&\left|{}^{C}D^{\alpha}_{0^+}K_{P,Q}\,Nu(t_2) - {}^{C}D^{\alpha}_{0^+}K_{P,Q}\,Nu(t_1)\right|\\
=&\left|\frac{1}{\phi(t_2)}\int_0^{t_2} M(s)ds - \frac{1}{\phi(t_1)}\int_0^{t_1} M(s)ds\right|\\
=&\left|\left(\frac{1}{\phi(t_2)} - \frac{1}{\phi(t_1)}\right)\int_0^{t_1} M(s)ds + \frac{1}{\phi(t_2)}\int_{t_1}^{t_2} M(s)ds\right|\\
\leq&\frac{A}{\mu^2}\Big|\phi(t_2)-\phi(t_1)\Big| + \frac{A}{\mu}\Big(t_2 - t_1\Big).
\end{aligned}$$

Since $t^{\alpha-i}$ and $\phi(t)$ are uniformly continuous on $[0,1]$, we get that $K_p(I-Q)N : \overline{\Omega} \to X$ is compact. The lemma is proved. ☐

Theorem 3.1. *Let f be a Caratheodory function, $\phi \in C^1[0,1]$, $\min_{t\in[0,1]}\phi(t) > \mu > 0$. (H_1) and (H_2) hold. In addition, assume that the following conditions hold.*

(H_3) *There exist non-negative functions $\theta_i(t) \in Y$, $i = 0,\dots,5$, such that*

$$\left|f\Big(t, x_0, x_1, x_2, x_3, x_4\Big)\right| \leq \sum_{i=0}^{4}\theta_i(t)|x_i| + \theta_5(t),$$

where

$$\Lambda = \frac{22+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\sum_{i=0}^{4}\|\theta_i\|_{L^1} < 1.$$

(H_4) *There exists a constant $M > 0$ such that for $u \in \operatorname{dom} L\backslash \ker L$, if $|u^{'}(t)| > M$ or $|u^{''}(t)| > M$ or $|u^{'''}(t)| > M$ for all $t \in [0,1]$, then $T_1(Nu) \neq 0$ or $T_2(Nu) \neq 0$ or $T_3(Nu) \neq 0$.*

(H_5) *There exists a constant $M^* > 0$ such that for any $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$, if $|\delta_1| > M^*$, $|\delta_2| > M^*$, $|\delta_3| > M^*$, then either*

$$\sum_{i=1}^{3} T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right) < 0$$

or

$$\sum_{i=1}^{3} T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right) > 0.$$

Then (1.1) *has at least one solution.*

Proof. Consider the set

$$\Omega_1 = \left\{u \in \operatorname{dom} L\backslash \ker L \,:\, Lu = \lambda Nu,\ \lambda \in [0,1]\right\}.$$

Then for $u \in \Omega_1$, $Lu = \lambda Nu$, thus $\lambda \neq 0$, $Nu \in \operatorname{Im} L = \ker Q \subset Y$. Hence, $Q(Nu) = 0$ that is, $T_1(Nu) = T_2(Nu) = T_3(Nu) = 0$. We get from (H_4) the existence of $t_1, t_2, t_3 \in [0,1]$, such that $|u'(t_1)| \leq M$, $|u''(t_2)| \leq M$, $|u'''(t_3)| \leq M$.

If $t_1 = t_2 = t_3 = 0$, we have that $|u'(0)| \le M$, $|u''(0)| \le M$, $|u'''(0)| \le M$. Otherwise, if $\max\{t_1, t_2, t_3\} \ne 0$, by $Lu = \lambda Nu$, we obtain

$$u(t) = \sum_{k=1}^{3} \frac{u^{(k)}(0)}{k!} t^k + \frac{\lambda}{\Gamma(\alpha)} \int_0^t \frac{(t-s)^{\alpha-1}}{\phi(s)} \int_0^s Nu(r) dr ds.$$

Then

$$u'''(t) = u'''(0) + \frac{\lambda}{\Gamma(\alpha-3)} \int_0^t \frac{(t-s)^{\alpha-4}}{\phi(s)} \int_0^s Nu(r) dr ds.$$

If $t_3 \ne 0$, we get

$$u'''(t_3) = u'''(0) + \frac{\lambda}{\Gamma(\alpha-3)} \int_0^{t_3} \frac{(t_3-s)^{\alpha-4}}{\phi(s)} \int_0^s Nu(r) dr ds,$$

together with $|u'''(t_3)| \le M$, we have

$$|u'''(0)| \le |u'''(t_3)| + \frac{1}{\Gamma(\alpha-3)} \int_0^{t_3} \frac{(t_3-s)^{\alpha-4}}{\phi(s)} \int_0^s |Nu(r)| dr ds \le M + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}.$$

Therefore,

$$|u'''(0)| \le M + \frac{\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.14}$$

If $t_2 \ne 0$, then

$$u''(t_2) = u''(0) + u'''(0)t_2 + \frac{\lambda}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s Nu(r) dr ds,$$

from (3.14) and $|u''(t_2)| \le M$, we find

$$\begin{aligned} |u''(0)| &\le |u''(t_2)| + |u'''(0)| + \frac{1}{\Gamma(\alpha-2)} \int_0^{t_2} \frac{(t_2-s)^{\alpha-3}}{\phi(s)} \int_0^s |Nu(r)| dr ds \\ &\le 2M + \frac{2\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \end{aligned}$$

Consequently,

$$|u''(0)| \le 2M + \frac{2\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.15}$$

If $t_1 \ne 0$, then

$$u'(t_1) = u'(0) + u''(0)t_1 + \frac{u'''(0)}{2} t_1^2 + \frac{\lambda}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s Nu(r) dr ds,$$

according to (3.14), (3.15) and $|u'(t_1)| \le M$, we get

$$\begin{aligned} |u'(0)| &\le |u'(t_1)| + |u''(0)| + |u'''(0)| + \frac{1}{\Gamma(\alpha-1)} \int_0^{t_1} \frac{(t_1-s)^{\alpha-2}}{\phi(s)} \int_0^s |Nu(r)| dr ds \\ &\le 4M + \frac{4\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \end{aligned}$$

So,

$$|u'(0)| \leq 4M + \frac{4\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.16}$$

Again for $u \in \Omega_1$, we get

$$\begin{aligned}\|Pu\|_X &= \sum_{i=0}^{3} \max_{t\in[0,1]} \left|(Pu)^{(i)}(t)\right| + \max_{t\in[0,1]} \left|{}^C D_{0^+}^{\alpha}(Pu)(t)\right| \\ &\leq 2|u'(0)| + 3|u''(0)| + 4|u'''(0)|.\end{aligned}$$

From (3.14), (3.15) and (3.16), we obtain

$$\|Pu\|_X \leq 18M + \frac{18\|Nu\|_{L^1}}{\mu\Gamma(\alpha-2)}. \tag{3.17}$$

Again for all $u \in \Omega_1$, we have $(I-P)u \in \operatorname{dom} L \cap \ker P$. Thus, by (3.13) and (3.5), we find

$$\begin{aligned}\|(I-P)u\|_X = \|K_p L(I-P)u\|_X &\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|L(I-P)u\|_{L^1} \\ &\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Lu\|_{L^1} \\ &\leq \frac{4+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Nu\|_{L^1}.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), we obtain

$$\|u\|_X \leq \|Pu\|_X + \|(I-P)u\|_X \leq 18M + \frac{22+\Gamma(\alpha-2)}{\mu\Gamma(\alpha-2)}\|Nu\|_{L^1}. \tag{3.19}$$

On the other hand, from (H_4), we have

$$\begin{aligned}\|Nu\|_{L^1} &= \int_0^1 \left|(Nu)(s)\right| ds = \int_0^1 \left|f\Big(t, u(t), u'(t), u''(t), u'''(t), {}^C D_{0^+}^{\alpha} u(t)\Big)\right| ds \\ &\leq \sum_{i=0}^{3} \int_0^1 \left|\theta_i(s)\right|\left|u^{(i)}(s)\right| ds + \int_0^1 \left|\theta_4(s)\right|\left|{}^C D_{0^+}^{\alpha} u(s)\right| ds + \int_0^1 \left|\theta_5(s)\right| ds \\ &\leq \|u\|_X \sum_{i=0}^{4} \|\theta_i\|_{L^1} + \|\theta_5\|_{L^1}.\end{aligned} \tag{3.20}$$

Therefore, (3.19) and (3.20), yields

$$\|u\|_X \leq \frac{18\mu\Gamma(\alpha-2)M + \Big(22+\Gamma(\alpha-2)\Big)\|\theta_5\|_{L^1}}{\mu\Big(1-\Lambda\Big)\Gamma(\alpha-2)}.$$

So, Ω_1 is bounded.

Let

$$\Omega_2 = \left\{u \in \ker L \,:\, Nu \in \operatorname{Im} L\right\}.$$

For $u \in \Omega_2$, then $u \in \ker L = \left\{u : u(t) = \sum_{k=1}^{3} \delta_k t^k,\ \delta_1, \delta_2, \delta_3 \in \mathbb{R}\right\}$ and $Q(Nu) = 0$, that is, $T_1 N\left(\sum_{k=1}^{3} \delta_k t^k\right) = T_2 N\left(\sum_{k=1}^{3} \delta_k t^k\right) = T_3 N\left(\sum_{k=1}^{3} \delta_k t^k\right) = 0$. From condition (H_5), we get $|\delta_1| \le M^*$, $|\delta_2| \le M^*$, $|\delta_3| \le M^*$. Hence, Ω_2 is bounded. Let

$$\Omega_3 = \{u \in \ker L : -\lambda J u + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\},$$

if the first part of (H_5) holds.

Or we'll set

$$\Omega_3 = \{u \in \ker L : -\lambda J u + (1-\lambda) QNu = 0,\ \lambda \in [0,1]\}$$

if the second part of (H_5) holds.

Here $J : \ker L \to \operatorname{Im} Q$ is the linear isomorphism given by

$$J\left(\sum_{k=1}^{3} \delta_k t^k\right) = \omega_1 + \omega_2 t + \omega_3 t^2, \quad \delta_1, \delta_2, \delta_3 \in \mathbb{R}, \tag{3.21}$$

where

$$\begin{aligned}
\omega_1 &= \frac{1}{\Delta}\Big(e_{11}|\delta_1| + e_{12}|\delta_2| + e_{13}|\delta_3|\Big),\\
\omega_2 &= \frac{1}{\Delta}\Big(e_{21}|\delta_1| + e_{22}|\delta_2| + e_{23}|\delta_3|\Big),\\
\omega_3 &= \frac{1}{\Delta}\Big(e_{31}|\delta_1| + e_{32}|\delta_2| + e_{33}|\delta_3|\Big).
\end{aligned}$$

Without loss of generality, we assume that the first part of (H_5) holds. In fact $u \in \Omega_3$, means that $u = \sum_{k=1}^{3} \delta_k t^k$ and $-\lambda J u + (1-\lambda) QNu = 0$. Then we obtain

$$-\lambda J\left(\sum_{k=1}^{3} \delta_k t^k\right) + (1-\lambda) QN\left(\sum_{k=1}^{3} \delta_k t^k\right) = 0. \tag{3.22}$$

If $\lambda = 0$, then $|\delta_1| \le M^*$, $|\delta_2| \le M^*$, $|\delta_3| \le M^*$. If $\lambda = 1$, then

$$\begin{cases}
e_{11}|\delta_1| + e_{12}|\delta_2| + e_{13}|\delta_3| = 0,\\
e_{21}|\delta_1| + e_{22}|\delta_2| + e_{23}|\delta_3| = 0,\\
e_{31}|\delta_1| + e_{32}|\delta_2| + e_{33}|\delta_3| = 0.
\end{cases} \tag{3.23}$$

The determinant of coefficients for (3.23) is $\Delta^2 \ne 0$. Thus, (3.23) only have zero solutions, that is $\delta_1 = \delta_2 = \delta_3 = 0$.

Otherwise, if $\lambda \ne 0$ and $\lambda \ne 1$, again from (3.21), (3.22) becomes

$$\begin{aligned}
\lambda\left(\omega_1 + \omega_2 t + \omega_3 t^2\right) =& (1-\lambda)\left(Q_1 N\left(\sum_{k=1}^{3} \delta_k t^k\right) + Q_2 N\left(\sum_{k=1}^{3} \delta_k t^k\right) t\right.\\
&\left.+ Q_3 N\left(\sum_{k=1}^{3} \delta_k t^k\right) t^2\right)
\end{aligned}$$

Hence,

$$\lambda \omega_i = (1-\lambda) Q_i\left(\sum_{k=1}^{3} \delta_k t^k\right), \quad \text{for } i = 1, 2, 3.$$

Thus,

$$\lambda|\delta_i| = (1-\lambda)T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right), \quad \text{for } i=1,2,3.$$

Then, we get

$$\lambda\sum_{i=1}^{3}|\delta_i| = (1-\lambda)\sum_{i=1}^{3}T_i N\left(\sum_{k=1}^{3}\delta_k t^k\right) < 0.$$

By the first part of (H_5), we have $|\delta_1| \leq M^*$, $|\delta_2| \leq M^*$, $|\delta_3| \leq M^*$. Here, Ω_3 is bounded.

Now, we shall prove that all the conditions of Theorem 2.1 are satisfied. Let Ω be a bounded open set of X containing $\bigcup_{i=1}^{3}\overline{\Omega}_i$. By Lemma 3.3, N is L-compact on $\overline{\Omega}$, because Ω_1 and Ω_2 are bounded sets, then

(1) $Lu \neq \lambda Nu$ for each $(u,\lambda) \in [(\operatorname{dom} L\setminus \ker L)\cap \partial\Omega] \times (0,1)$;
(2) $Nu \notin \operatorname{Im}L$ for each $u \in \ker L \cap \partial\Omega$.

At least we will prove that the hypothesis (3) of Theorem 2.1 is satisfied. Let

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu.$$

The set Ω_3 is bounded, then $H(u,\lambda) \neq 0$, for all $u \in \ker L \cap \partial\Omega$. Appealing to the homotopy property of the degree, we obtain

$$\begin{aligned}\deg\left(QN\mid_{\ker L}, \Omega\cap\ker L, 0\right) &= \deg\left(H(.,0), \Omega\cap\ker L, 0\right)\\ &= \deg\left(H(.,1), \Omega\cap\ker L, 0\right)\\ &= \deg\left(\pm J, \Omega\cap\ker L, 0\right) \neq 0.\end{aligned}$$

Then by Theorem 2.1, $Lu = Nu$ has at least one solution in $\operatorname{dom} L\cap\overline{\Omega}$, we conclude that the boundary value problem (1.1) has at least one solution in X. The proof is finished. ☐

Remark 3.1. It is very important to note that the condition $\Delta \neq 0$ is not necessary since L still Fredholm even if this condition is dropped. Indeed the role of Q in Mawhin's theory is purely auxiliary and conditions like that usually arise from the authors of hundreds of paper choosing $\operatorname{Im} Q$ just simply being $\ker L$. Avoiding such an assumption is just a matter of choosing Q differently, for more details see [14, 20, 21].

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4. Example

To illustrate our main results, we will present an example.

Example 4.1. Let us consider the following fractional boundary value problem

$$\begin{cases}\left(\phi(t)^C D_{0^+}^{\frac{7}{2}} u(t)\right)' = f\left(t, u(t), u'(t), u''(t), u'''(t), {}^C D_{0^+}^{\frac{7}{2}} u(t)\right), \quad t\in[0,1],\\ u(0)=0, \quad {}^C D_{0^+}^{\alpha} u(0)=0, \quad u'''(0) = -u'''(\frac{1}{6}) + 2u'''(\frac{1}{5}),\\ u''(0) = 4u''(\frac{1}{4}) - 3u''(\frac{1}{3}), \quad u'(1) = u'(\frac{1}{4}) - 3u'(\frac{1}{2}) + 3u'(\frac{3}{4}),\end{cases} \tag{4.1}$$

where $\phi(t)=e^{-12t}$ and

$$
\begin{aligned}
&100e^{12}f\left(t,u(t),u'(t),u''(t),u'''(t),{}^{C}D_{0+}^{\frac{7}{2}}u(t)\right)\\
=&\frac{|u'''(t)|}{1+(u'''(t))^2}+\cos{}^{C}D_{0+}^{\frac{7}{2}}u(t)\Big(1-\sin u'(t)\Big)\Big(1-\sin u''(t)\Big)\\
&+\frac{2}{\pi}\arctan\left(u(t){}^{C}D_{0+}^{\frac{7}{2}}u(t)\right).
\end{aligned}
$$

Corresponding to the problem (1.1), we have that $\alpha=\frac{7}{2},\ l=2,\ m=2,\ n=3,\ a_1=-1,\ a_2=2,\ \xi_1=\frac{1}{6},\ \xi_2=\frac{1}{5},\ b_1=4,\ b_2=-3,\ \eta_1=\frac{1}{4},\ \eta_2=\frac{1}{3},\ c_1=1,\ c_2=-3,\ c_3=3,\ \rho_1=\frac{1}{4},\ \rho_2=\frac{1}{2},\ \rho_3=\frac{3}{4},\ \mu=e^{-12}$. Then we get $a_1+a_2=b_1+b_2=\ c_1+c_2+c_3=1,\ b_1\eta_1+b_2\eta_2=0,\ c_1\rho_1+c_2\rho_2+c_3\rho_3=c_1\rho_1^2+c_2\rho_2^2+c_3\rho_3^2=1$. Thus, the condition (H_1) holds.

Also, we find

$$
\begin{aligned}
T_1(y)=&-\int_0^{\frac{1}{6}}e^{12s}\Big(\frac{1}{6}-s\Big)^{-\frac{1}{2}}\int_0^s y(r)drds+2\int_0^{\frac{1}{5}}e^{12s}\Big(\frac{1}{5}-s\Big)^{-\frac{1}{2}}\int_0^s y(r)drds,\\
T_2(y)=&4\int_0^{\frac{1}{4}}e^{12s}\Big(\frac{1}{4}-s\Big)^{\frac{1}{2}}\int_0^s y(r)drds-3\int_0^{\frac{1}{3}}e^{12s}\Big(\frac{1}{3}-s\Big)^{\frac{1}{2}}\int_0^s y(r)drds,\\
T_3(y)=&\int_0^1 e^{12s}(1-s)^{\frac{3}{2}}\int_0^s y(r)drds-\int_0^{\frac{1}{4}}e^{12s}\Big(\frac{1}{4}-s\Big)^{\frac{3}{2}}\int_0^s y(r)drds\\
&+3\int_0^{\frac{1}{2}}e^{12s}\Big(\frac{1}{2}-s\Big)^{\frac{3}{2}}\int_0^s y(r)drds-3\int_0^{\frac{3}{4}}e^{12s}\Big(\frac{3}{4}-s\Big)^{\frac{3}{2}}\int_0^s y(r)drds.
\end{aligned}
$$

By calculations, we get

$$
\begin{aligned}
&d_{11}=\frac{1881}{1420},\quad d_{12}=\frac{207}{1669},\quad d_{13}=\frac{143}{9103},\\
&d_{21}=-\frac{920}{1803},\quad d_{22}=-\frac{484}{6725},\quad d_{23}=-\frac{277}{20262},\\
&d_{31}=\frac{15770}{51},\quad d_{32}=\frac{6489}{50},\quad d_{33}=\frac{5427}{74}.
\end{aligned}
$$

Then, $\Delta=-\frac{655}{539}\neq 0$. Therefore, the condition (H_2) holds.

On the other hand, we have

$$
\left|f\Big(t,u(t),u'(t),u''(t),u'''(t),{}^{C}D_{0+}^{\frac{7}{2}}u(t)\Big)\right|\le 0.01e^{-12}|u'''(t)|+0.05e^{-12}.
$$

We can get that the condition (H_3) holds, where

$$
\theta_0(t)=\theta_1(t)=\theta_2(t)=\theta_4(t)=0,\quad \theta_3(t)=0.01e^{-12},\quad \theta_5(t)=0.05e^{-12}
$$

and $\Lambda=\frac{838}{3245}<1$.

Let $M=1$ and assume that $|u'''(t)|>1$ holds for all $t\in[0,1]$, we obtain

$$
T_3(y)>0.01e^{-12}\int_0^1 e^{12s}(1-s)^{\frac{3}{2}}sds-0.06e^{-12}\int_0^{\frac{1}{4}}e^{12s}\Big(\frac{1}{4}-s\Big)^{\frac{3}{2}}sds
$$

$$
\begin{aligned}
&+0.03e^{-12}\int_0^{\frac{1}{2}} e^{12s}\Big(\frac{1}{2}-s\Big)^{\frac{3}{2}} sds - 0.18e^{-12}\int_0^{\frac{3}{4}} e^{12s}\Big(\frac{3}{4}-s\Big)^{\frac{3}{2}} sds.\\
=&\frac{43818}{2900}e^{-12} > 0,
\end{aligned}
$$

so condition (H_4) is satisfied.

Let $M^* = 1$ and $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ be such that $|\delta_1| > 1$, $|\delta_2| > 1$, $|\delta_3| > 1$, we have

$$
\begin{aligned}
N\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big) =& 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2} + 0.01e^{-12}\cos {}^C D_{0^+}^{\frac{7}{2}}\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big)\\
&\times \Big(1-\sin\Big(\delta_1 + 2\delta_2 t + 3\delta_3 t^2\Big)\Big) \times \Big(1 - \sin\Big(2\delta_2 + 6\delta_3 t\Big)\Big)\\
&+\frac{0.02e^{-12}}{\pi}\arctan\Big(\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big){}^C D_{0^+}^{\frac{7}{2}}\Big(\delta_1 t + \delta_2 t^2 + \delta_3 t^3\Big)\Big)\\
=& 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
T_1 N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) &= 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}d_{11},\\
T_2 N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) &= 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}d_{12},\\
T_3 N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) &= 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}d_{13}.
\end{aligned}
$$

Thus,

$$
\sum_{i=1}^{3} T_i N\Big(\sum_{k=1}^{3}\delta_k t^k\Big) = 0.06e^{-12}\frac{|\delta_3|}{1+36\delta_3^2}(d_{11}+d_{12}+d_{13}) > 0.
$$

So, (H_5) hold. Then, all the assumptions of Theorem 3.1 hold. Thus, the problem (4.1) has at least one solution.

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