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ON DISTANCE SIGNLESS LAPLACIAN ESTRADA INDEX AND ENERGY OF GRAPHS

ABDOLLAH ALHEVAZ[1], MARYAM BAGHIPUR[1], AND SHARIEFUDDIN PIRZADA[2]

Abstract. For a connected graph G, the distance signless Laplacian matrix is defined as $D^Q(G) = \mathrm{Tr}(G) + D(G)$, where $D(G)$ is the distance matrix of G and $\mathrm{Tr}(G)$ is the diagonal matrix of vertex transmissions of G. The eigenvalues $\rho_1, \rho_2, \dots, \rho_n$ of $D^Q(G)$ are the distance signless Laplacian eigenvalues of the graph G. In this paper, we define the distance signless Laplacian Estrada index of the graph G as $D_E^Q E(G) = \sum_{i=1}^n e^{\left(\rho_i - \frac{2\sigma(G)}{n}\right)}$, where $\sigma(G)$ is the transmission of a graph G. We obtain upper and lower bounds for $D_E^Q E(G)$ and the distance signless Laplacian energy in terms of other graph invariants. Moreover, we derive some relations between $D_E^Q E(G)$ and the distance signless Laplacian energy of G.

1. Introduction and preliminaries

All graphs throughout this paper are finite, undirected, simple and connected. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \dots, v_n\}$ and edge set $E(G)$. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of vertex v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if each of its vertex has the same degree. We write $G \cong H$, where the graphs G and H are isomorphic. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v in G. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The *transmission* $\mathrm{Tr}_G(v)$ of a vertex

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v is defined to be the sum of the distances from v to all other vertices in G, that is, $\mathrm{Tr}_G(v) = \sum\limits_{u \in V(G)} d_{uv}$. A graph G is said to be *k-transmission regular* if $\mathrm{Tr}_G(v) = k$, for each $v \in V(G)$. The *transmission* of a graph G, denoted by $\sigma(G)$, is the sum of distances between all unordered pairs of vertices in G. For other undefined notations and terminology, the readers are referred to [33].

For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $\mathrm{Tr}_G(v_i)$ has been referred to as the *transmission degree* Tr_i [26] and hence the *transmission degree sequence* is given by $\{\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n\}$. Let $\mathrm{Tr}(G) = \mathrm{diag}(\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [2, 3] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = \mathrm{Tr}(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = \mathrm{Tr}(G) + D(G)$ is called the *distance signless Laplacian matrix of* G. If G is connected, then $D^Q(G)$ is symmetric, nonnegative and irreducible. Hence, all the eigenvalues of $D^Q(G)$ can be arranged as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, where ρ_1 is called the *distance signless Laplacian spectral radius* of G. (From now onwards, we will denote $\rho_1(G)$ by $\rho(G)$).

Based on investigations on geometric properties of biomolecules, Ernesto Estrada [13, 14] considered an expression of the form

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of a molecular graph G. The mathematical significance of this quantity was recognized short time later [22] and soon it became known under the name “Estrada index” [10]. The mathematical properties of the Estrada index have been intensively studied, see for example, [5,10,23]. There exists a vast literature related to Estrada index and its bounds and we refer the reader to the nice surveys [11, 21].

This graph-spectrum-based invariant has also an important role in chemistry, physics, and complex networks. For example, it has been used to measure the degree of folding of long chain polymeric molecules, including proteins [12, 13, 16]. It has found a number of applications in complex networks and characterizes the centrality [14], also serves as an insightful measure for investigating robustness of complex networks [39], for which EE has an acute discrimination on connectivity and changes monotonically with respect to the removal or addition of edges. For the application of the Estrada index in network theory see the book [15] and the papers [38, 39].

The pioneering papers [13,14] further proposes the study of graphs with an analogue of the Estrada index defined with respect to other (than adjacency) matrices. Because of the evident success of the graph Estrada index, this proposal has been put into effect and Estrada index based of the eigenvalues of other graph matrices have, one-by-one, been introduced: Estrada index based invariant with respect to distance matrix, as well as Estrada index based invariant with respect to Laplacian matrix, have been introduced and studied, see for example [6, 7, 24, 25, 27, 35–37, 40, 41]. Recently, in full analogy with the Estrada index, the signless Laplacian Estrada index of a connected graph G has been introduced and studied [4]. Further, in full analogy with the Estrada index, the distance Estrada index of a connected graph G has been introduced in [19]

$$DEE(G) = \sum_{i=1}^{n} e^{\mu_i},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the distance matrix of a graph G. Now, we define the distance signless Laplacian Estrada index $D_E^Q E(G)$, based on distance signless Laplacian matrix of the graph G as

$$D_E^Q E(G) = \sum_{i=1}^{n} e^{\left(\rho_i - \frac{2\sigma(G)}{n}\right)}, \tag{1.1}$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are the distance signlees Laplacian eigenvalues of a graph G. Let

$$M_k = \sum_{i=1}^{n} \left(\rho_i - \frac{2\sigma(G)}{n}\right)^k.$$

Then

$$\begin{aligned} M_0 &= n, \\ M_1 &= 0, \\ M_2 &= 2 \sum_{1 \le i < j \le n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}. \end{aligned} \tag{1.2}$$

Recalling the power series expansion of e^x, we can write the distance signless Laplacian Estrada index as

$$D_E^Q E(G) = \sum_{k \ge 0} \frac{M_k}{k!}. \tag{1.3}$$

The rest of the paper is organized as follows. In Section 2, we obtain some upper and lower bounds for the distance signless Laplacian Estrada index $D_E^Q E(G)$ involving different graph invariants, and also characterize the extremal graphs. In Section 3, we compute the distance signless Laplacian Estrada index of some classes of graphs, as well as giving some relations with the earlier distance Estrada index. Finally, in Section 4, we derive some relations between the distance signless Laplacian Estrada index and the distance signless Laplacian energy of G.

2. Bounds for the Distance Signless Laplacian Estrada Index

We start by giving some previously known results that will be needed in the proofs of our results in the sequel.

Lemma 2.1. ([1, Theorem 2.2]). *If the transmission degree sequence of G is $\{\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n\}$, then*

$$\rho(G) \geq 2\sqrt{\frac{\sum_{i=1}^{n} \mathrm{Tr}_i^2}{n}},$$

with equality if and only if G is transmission regular.

Lemma 2.2. ([42, Lemma 2.2]). *If G is a connected graph of order n, then*

$$\rho(G) \geq \frac{4\sigma(G)}{n},$$

with equality if and only if G is transmission regular.

The following lemma will be helpful in the sequel. Its proof is similar to [28, Lemma 2], and hence is excluded.

Lemma 2.3. *A connected graph G has two distinct distance signless Laplacian eigenvalues if and only if G is a complete graph.*

For non-increasing real sequences $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ of length n, we say that (x) is majorized by (y) or (y) majorizes (x), denoted by $(x) \preceq (y)$ if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{and} \quad \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \text{for all } k = 1, 2, \ldots, n-1.$$

The following observation can be found in [32].

Lemma 2.4 ([32])**.** *Let $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ be nonincreasing sequences of real numbers of length n. If $(x) \preceq (y)$, then for any convex function ψ, we have $\sum_{i=1}^{n} \psi(x_i) \leq \sum_{i=1}^{n} \psi(y_i)$. Furthermore, if $(x) \prec (y)$ and ψ is strictly convex, then $\sum_{i=1}^{n} \psi(x_i) < \sum_{i=1}^{n} \psi(y_i)$.*

Lemma 2.5 ([34])**.** *Let G be a connected graph of order n having distance signless Laplacian eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ and transmission degrees $\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n$. Then*

$$(\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n) \preceq (\rho_1, \rho_2, \ldots, \rho_n).$$

Now, we present some upper bounds for the distance signless Laplacian Estrada index involving different graph invariants.

Theorem 2.1. *Let G be a connected graph of order n. Then, for any integer $k_0 \geq 2$,*

$$\begin{aligned} D_E^Q E(G) \leq & n - 1 - \sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}} \\ & + \sum_{k=2}^{k_0} \frac{M_k(G) - \left(\sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}\right)^k}{k!} \end{aligned}$$

$$+\, e^{\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}, \tag{2.1}$$

with equality if and only if $G = K_1$.

Proof. We have

$$\begin{aligned}
D_E^Q E(G) &= \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\ge k_0+1}\frac{1}{k!}\sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k\\
&\le \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\ge k_0+1}\frac{1}{k!}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k\\
&\le \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\ge k_0+1}\frac{1}{k!}\left(\sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^2\right)^{\frac{k}{2}}\\
&= \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\ge k_0+1}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k}{k!}\\
&= \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+e^{\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}\\
&\quad -\sum_{k=0}^{k_0}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k}{k!},
\end{aligned}$$

and (2.1) follows. From the derivation of (2.1), it is evident that equality will be attained in (2.1) if and only if G has no non-zero eigenvalues, i.e., $G = K_1$. $\square$

Remark 2.1. Since

$$\begin{aligned}
M_k(G) &= \sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k\\
&\le \sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k \le \left(\sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^2\right)^{\frac{k}{2}} = (M_2(G))^{\frac{k}{2}}.
\end{aligned}$$

In the second inequality above, we use the following inequality: For nonnegative $a_1, a_2, \ldots, a_n$ and integer $k \ge 2$

$$\sum_{i=1}^{n} a_i^k \le \left(\sum_{i=1}^{n} a_i^2\right)^{\frac{k}{2}}. \tag{2.2}$$

Hence, $M_k(G)-\left(\sqrt{M_2(G)}\right)^k \le 0$. Then

$$\sum_{k=2}^{k_0}\frac{M_k(G)-\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k}{k!}\le 0.$$

Therefore, we have the following observation from Theorem 2.1,

$$\begin{aligned} D_E^Q E(G) \leq & n-1-\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}} \\ & + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}. \end{aligned}$$

Theorem 2.2. *Let G be a connected graph of order n. Then for any integer $k_0 \geq 2$*

$$\begin{aligned} D_E^Q E(G) \leq & n-2-\rho_1+\frac{2\sigma(G)}{n}-\sqrt{\xi} \\ & + \sum_{k=2}^{k_0}\frac{M_k(G)-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^k-\left(\sqrt{\xi}\right)^k}{k!}+e^{\rho_1-\frac{2\sigma(G)}{n}}+e^{\sqrt{\xi}}, \end{aligned} \tag{2.3}$$

where $\xi = 2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2$, with equality if and only if $G = K_1$.

Proof. We have

$$\begin{aligned} & D_E^Q E(G) - e^{\rho_1-\frac{2\sigma(G)}{n}} \\ = & \sum_{k=0}^{k_0}\frac{M_k(G)-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^k}{k!}+\sum_{k\geq k_0+1}\frac{1}{k!}\sum_{i=2}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k \\ \leq & \sum_{k=0}^{k_0}\frac{M_k(G)-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^k}{k!}+\sum_{k\geq k_0+1}\frac{1}{k!}\sum_{i=2}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k \\ \leq & \sum_{k=0}^{k_0}\frac{M_k(G)-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^k}{k!}+\sum_{k\geq k_0+1}\frac{1}{k!}\left(\sum_{i=2}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^2\right)^{\frac{k}{2}} \\ = & \sum_{k=0}^{k_0}\frac{M_k(G)-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^k}{k!} \\ & + \sum_{k\geq k_0+1}\frac{\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!} \\ = & \sum_{k=0}^{k_0}\frac{M_k(G)-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^k}{k!} \\ & + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}} \end{aligned}$$

$$-\sum_{k=0}^{k_0}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!},$$

where the first inequality follows from inequality:

$$\sum_{i=2}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k\le\sum_{i=2}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k.$$

Also, in the second inequality, we use the inequality (2.2). Further, bearing in mind the power-series expansion of $e^x=\sum_{k\ge 0}\frac{x^k}{k!}$, we have

$$\begin{aligned}
&e^{\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}}\\
&=\sum_{k=0}^{k_0}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!}\\
&+\sum_{k\ge k_0+1}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!}.
\end{aligned}$$

Hence, the last equality holds. Then the result follows. □

Theorem 2.3. *Let G be a connected graph of order n and diameter d. Then*

$$\frac{1}{2}\sqrt{2n(n^2+4n-3)}\le D_E^QE(G)\le n-1+e^{\sqrt{n(n-1)\left(d^2+\frac{n^2(n-1)}{4}-n+1\right)}}. \tag{2.4}$$

Equality holds on both sides of (2.4) *if and only if $G\cong K_1$.*

Proof. **Lower bound.** From (1.1), we get

$$D_E^QE^2(G)=\sum_{i=1}^{n}e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)}+2\sum_{i<j}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}e^{\left(\rho_j-\frac{2\sigma(G)}{n}\right)}. \tag{2.5}$$

By the arithmetic-geometric mean inequality, we get

$$\begin{aligned}
2\sum_{i<j}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}e^{\left(\rho_j-\frac{2\sigma(G)}{n}\right)}\ge& n(n-1)\left(\prod_{i<j}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}e^{\left(\rho_j-\frac{2\sigma(G)}{n}\right)}\right)^{\frac{2}{n(n-1)}} &(2.6)\\
=&n(n-1)\left[\left(\prod_{i=1}^{n}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}\right)^{n-1}\right]^{\frac{2}{n(n-1)}}\\
=&n(n-1)(e^{M_1})^{\frac{2}{n}}\\
=&n(n-1). &(2.7)
\end{aligned}$$

By means of a power-series expansion and $M_0 = n, M_1 = 0$ and

$$M_2 = 2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n},$$

we get

$$\begin{aligned}\sum_{i=1}^{n} e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)} =& \sum_{i=1}^{n}\sum_{k\ge 0}\frac{\left[2\left(\rho_i-\frac{2\sigma(G)}{n}\right)\right]^k}{k!}\\ =& n + 4\sum_{1\le i<j\le n}(d_{ij})^2 + 2\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{8\sigma^2(G)}{n} + \sum_{i=1}^{n}\sum_{k\ge 3}\frac{\left[2\left(\rho_i-\frac{2\sigma(G)}{n}\right)\right]^k}{k!}.\end{aligned}$$

We use a multiplier $r\in[0,4]$ to arrive at

$$\begin{aligned}\sum_{i=1}^{n} e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)} \ge& n + 4\sum_{1\le i<j\le n}(d_{ij})^2 + 2\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{8\sigma^2(G)}{n} + r\sum_{i=1}^{n}\sum_{k\ge 3}\frac{\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k}{k!}\\ =& n + 4\sum_{1\le i<j\le n}(d_{ij})^2 + 2\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{8\sigma^2(G)}{n} - rn - r\sum_{1\le i<j\le n}(d_{ij})^2\\ &-\frac{r}{2}\sum_{i=1}^{n}\mathrm{Tr}_i^2 + \frac{2\sigma^2(G)}{n} + rD_E^Q E(G)\\ =&(1-r)n - \frac{6\sigma^2(G)}{n} + (4-r)\sum_{1\le i<j\le n}(d_{ij})^2 + (2-\frac{r}{2})\sum_{i=1}^{n}\mathrm{Tr}_i^2\\ &+ rD_E^Q E(G),\end{aligned}$$

where from (1.3), we get

$$\begin{aligned}rD_E^Q E(G) =& r\sum_{i=1}^{n}\sum_{k\ge 0}\frac{\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k}{k!} = rn + r\sum_{1\le i<j\le n}(d_{ij})^2 + \frac{r}{2}\sum_{i=1}^{n}\mathrm{Tr}_i^2\\ &-\frac{2\sigma^2(G)}{n} + r\sum_{i=1}^{n}\sum_{k\ge 3}\frac{\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k}{k!},\end{aligned}$$

and hence the last but one equality follows.

Since $\sum_{1\le i<j\le n}(d_{ij})^2 \ge \frac{n(n-1)}{2}$ and $\sum_{i=1}^{n}\mathrm{Tr}_i^2 \ge n(n-1)^2$, also by Cauchy-Schwartz inequality we have $(2\sigma(G))^2 = (\sum_{i=1}^{n}\mathrm{Tr}_i)^2 \le n\sum_{i=1}^{n}\mathrm{Tr}_i^2$, and then, for $r\le 1$, we obtain

(2.8)
$$\sum_{i=1}^{n} e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)} \ge (1-r)n + (4-r)\frac{n(n-1)}{2} + \frac{1-r}{2}\left(n(n-1)^2\right) + rD_E^Q E(G).$$

By substituting (2.7) and (2.8) in (2.5), and solving for $D_E^Q E(G)$, we get

$$D_E^Q E(G) \geq \frac{1}{2}\left(r + \sqrt{r^2 - 2n(2r+3) + 2n^2(r+4) + 2n^3(1-r)}\right).$$

It is easy to see that for $n \geq 2$ the function

$$f(x) := \frac{1}{2}\left(x + \sqrt{x^2 - 2n(2x+3) + 2n^2(x+4) + 2n^3(1-x)}\right)$$

monotonically decreases in the interval $[0,1]$. As a result, the best bound for $D_E^Q E(G)$ is attained for $r = 0$. This gives us the first part of the proof.

Upper bound. We have

$$\begin{aligned}
D_E^Q E(G) =& n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{\left(\rho_i - \frac{2\sigma(G)}{n}\right)^k}{k!}\\
\leq& n + \sum_{i=1}^{n}\sum_{k\geq 1}\frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!}\\
=& n + \sum_{k\geq 1}\frac{1}{k!}\sum_{i=1}^{n}\left[\left(\rho_i - \frac{2\sigma(G)}{n}\right)^2\right]^{\frac{k}{2}}\\
\leq& n + \sum_{k\geq 1}\frac{1}{k!}\left[\sum_{i=1}^{n}\left(\rho_i - \frac{2\sigma(G)}{n}\right)^2\right]^{\frac{k}{2}} \qquad (2.9)\\
=& n + \sum_{k\geq 1}\frac{1}{k!}\left[2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}\right]^{\frac{k}{2}}\\
=& n - 1 + \sum_{k\geq 0}\frac{\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}\right)^k}{k!}\\
=& n - 1 + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}}.
\end{aligned}$$

Since $d_{ij} \leq d$ for $i \neq j$ and there are $\frac{n(n-1)}{2}$ pairs of vertices in G, we have $2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} \leq 2\frac{n(n-1)}{2}d^2 + \frac{n^3(n-1)^2}{4} - n(n-1)^2$, so that

$$D_E^Q E(G) \leq n - 1 + e^{\sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)}}.$$

Hence, we get the right-hand side of the (2.4).

Now, suppose that the equality in (2.4) holds, then all the inequalities in the above argument must hold as equalities. In particular, from (2.6), we get $\rho_1 = \rho_2 = \cdots = \rho_n = \frac{2\sigma(G)}{n}$ (see [17]). Since, by Lemma 2.2, $\rho_1 \geq \frac{4\sigma(G)}{n}$, a contradiction. Thus, the left- hand side equality in (2.4) holds if and only if G is an empty graph. Since G is a connected graph, this only happens in the case of $G \cong K_1$, then the graph G has all zero D^Q-eigenvalues. Again, let the right-hand side equality in (2.4) holds, then from (2.9), we get $\rho_1 = \rho_2 = \cdots = \rho_n = \frac{2\sigma(G)}{n}$. Similarly, we get $G \cong K_1$ and the proof is complete. $\square$

Now, we turn our attention to giving some lower bounds for the distance signless Laplacian Estrada index in terms of other graph invariants.

Theorem 2.4. *Let G be a connected graph of order n. Then*

$$D_E^Q E(G) \geq e^{\frac{2\sigma(G)}{n}} + (n-1)e^{\frac{-2\sigma(G)}{n(n-1)}}, \tag{2.10}$$

with equality if and only if $G = K_n$.

Proof. Starting with (1.1) and using the arithmetic-geometric mean inequality, we get

$$\begin{aligned} D_E^Q E(G) &= e^{\rho_1 - \frac{2\sigma(G)}{n}} + e^{\rho_2 - \frac{2\sigma(G)}{n}} + \cdots + e^{\rho_n - \frac{2\sigma(G)}{n}} \\ &\geq e^{\rho_1 - \frac{2\sigma(G)}{n}} + (n-1)\left(\prod_{i=2}^{n} e^{\rho_i - \frac{2\sigma(G)}{n}}\right)^{\frac{1}{n-1}} \qquad (2.11) \\ &= e^{\rho_1 - \frac{2\sigma(G)}{n}} + (n-1)\left(e^{\frac{2\sigma(G)}{n} - \rho_1}\right)^{\frac{1}{n-1}}. \qquad (2.12) \end{aligned}$$

Consider the following function

$$f(x) = e^x + (n-1)e^{\frac{-x}{n-1}}, \tag{2.13}$$

for $x \geq 0$. We have

$$f'(x) = e^x - e^{\frac{-x}{n-1}} \geq 0,$$

for $x \geq 0$. It is easy to see that $f(x)$ is an increasing function for $x \geq 0$. From (2.12) and Lemma 2.2, we obtain

$$D_E^Q E(G) \geq e^{\frac{2\sigma(G)}{n}} + (n-1)e^{\frac{-2\sigma(G)}{n(n-1)}}. \tag{2.14}$$

This completes the first part of the proof. Now, we suppose that the equality holds in (2.10). Then all inequalities in the above argument must be equalities. From (2.14), we have $\rho_1 = \frac{4\sigma(G)}{n}$, which implies that G is a transmission regular graph. From (2.11) and the arithmetic-geometric mean inequality, we get $\rho_2 = \rho_3 = \cdots = \rho_n$. Therefore, G has exactly two distinct distance signless Laplacian eigenvalues, and then by Lemma 2.3, G is the complete graph K_n.

Conversely, one can easily see that the equality holds in (2.10) for the complete graph K_n. This completes the proof. $\square$

Remark 2.2. For a graph G of order $n \geq 2$ and size m, it was shown in [43] that

$$EE(G) \geq e^{\frac{2m}{n}} + (n-1)e^{-\frac{2m}{n(n-1)}}, \tag{2.15}$$

with equality if and only if G is the empty graph or the complete graph. Since $\sigma(G) \geq \binom{n}{2} \geq m$ and the function $f(x)$ defined in (2.13) is increasing function, hence our given lower bound for distance signless Laplacian Estrada index in (2.10) is larger than the above lower bound in (2.15) for usual Estrada index. If G is the complete graph K_n, then $\sigma(G) = \binom{n}{2} = m$ and therefore the bounds coincide.

Let $M(G) = (\prod_{i=1}^{n} \mathrm{Tr}_i)^{\frac{1}{n}}$ be the geometric mean of the transmission degrees sequence. Then $\frac{2\sigma(G)}{n} \geq M(G)$ holds, and equality is attained if and only if $\mathrm{Tr}_1 = \cdots = \mathrm{Tr}_n$ (i.e., the graph G is transmission regular).

Lemma 2.6 ([44]). *Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right] \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} a_i^{\frac{1}{2}}\right)^2.$$

Theorem 2.5. *Let G be a connected graph of order $n \geq 2$. Then*

$$D_E^Q E(G) \geq e^{2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}}-\frac{2\sigma(G)}{n}} + (n-1)\left(e^{\frac{2\sigma(G)}{n}-\left(2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}}\right)}\right)^{\frac{1}{n-1}}, \tag{2.16}$$

with equality if and only if $G = K_n$.

Proof. Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} D_E^Q E(G) =& e^{\rho_1-\frac{2\sigma(G)}{n}} + e^{\rho_2-\frac{2\sigma(G)}{n}} + \cdots + e^{\rho_n-\frac{2\sigma(G)}{n}} \\ \geq& e^{\rho_1-\frac{2\sigma(G)}{n}} + (n-1)\left(\prod_{i=2}^{n} e^{\rho_i-\frac{2\sigma(G)}{n}}\right)^{\frac{1}{n-1}} \qquad (2.17) \\ =& e^{\rho_1-\frac{2\sigma(G)}{n}} + (n-1)\left(e^{\frac{2\sigma(G)}{n}-\rho_1}\right)^{\frac{1}{n-1}}. \end{aligned}$$

By Lemma 2.1, $\rho_1 \geq 2\sqrt{\frac{\sum_{i=1}^{n}\mathrm{Tr}_i^2}{n}}$. Setting $\sqrt{a_i} = \mathrm{Tr}_i$ in Lemma 2.6, we get

$$n^2\left[\frac{\sum_{i=1}^{n}\mathrm{Tr}_i^2}{n} - \left(\frac{2\sigma(G)}{n}\right)^2\right] \geq \sum_{i=1}^{n}\mathrm{Tr}_i^2 - n\left(\prod_{i=1}^{n}\mathrm{Tr}_i^2\right)^{\frac{1}{n}}.$$

Combining this with Lemma 2.1, yields

$$\rho_1 \geq 2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}}. \tag{2.18}$$

It is easy to see that $2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}} \geq \frac{4\sigma(G)}{n}$, and so,

$$2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}} - \frac{2\sigma(G)}{n} \geq \frac{2\sigma(G)}{n} \geq 0.$$

Similarly to Theorem 3.4, we get the result. When $G = K_n$, we have $\rho_1 = 2n-2, \rho_2 = \cdots = \rho_n = n-2$, $\sigma(G) = \frac{n(n-1)}{2}$ and $M(G) = n-1$. Hence, $D_E^Q E(G) = e^{n-1}+(n-1)e^{-1}$ and the equality holds.

Conversely, suppose that the equality holds. Then from (2.17), we have $\rho_2 = \cdots = \rho_n$. Clearly $4\sigma^2(G) = M^2(G)n$ if and only if $n = 1$. From (2.18), it follows that $\rho_1 > 0$ for $n \geq 2$. Thus G has exactly two distinct distance signless Laplacian eigenvalues, and so Lemma 2.3 implies that G is the complete graph K_n. $\square$

Let G be a k-transmission regular graph. Then $\sigma(G) = \frac{nk}{2}$ and $M(G) = k$ and hence we get the following observation.

Corollary 2.1. *Let G be a k-transmission regular graph. Then*

$$D_E^Q E(G) \geq e^k + (n-1)e^{\frac{-k}{n-1}},$$

with equality if and only if $G = K_n$.

We recall Holder inequality. Let $a_1, \ldots, a_n$, $b_1, \ldots, b_n$ be non-negative real numbers, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

Here, we give the lower bound for $D_E^Q E(G)$ in terms of n and $\sigma(G)$.

Theorem 2.6. *Let G be a connected graph of order n. Then*

$$D_E^Q E(G) > n + 2\left(\frac{\sigma(G)}{n}\right)^2.$$

Proof. By Holder inequality for $p = q = 2$, we have

$$2\sigma(G) = \sum_{i=1}^{n} \mathrm{Tr}_i \leq \sqrt{n}\left(\sum_{i=1}^{n} \mathrm{Tr}_i^2\right)^{\frac{1}{2}}.$$

Hence,

$$\sum_{i=1}^{n} \mathrm{Tr}_i^2 \geq \frac{4\sigma^2(G)}{n}. \tag{2.19}$$

Now, by Cauchy-Schwartz inequality, we have

$$\mathrm{Tr}_i^2 = \left(\sum_{j=1}^{n} d_{ij}\right)^2 \leq n\sum_{j=1}^{n} d_{ij}^2.$$

Hence,

$$\sum_{i=1}^{n} \mathrm{Tr}_i^2 \leq n\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2,$$

and then by (2.19) we get

$$\sum_{1\leq i<j\leq n} d_{ij}^2 \geq \frac{1}{2n}\sum_{i=1}^{n} \mathrm{Tr}_i^2 \geq \frac{1}{2n}.\frac{4\sigma^2(G)}{n} = \frac{2\sigma^2(G)}{n^2}.$$

Thus, we have

$$\begin{aligned} D_E^Q E(G) >& n + \sum_{1\le i<j\le n} (d_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{2\sigma^2(G)}{n} \\ \ge& n + \frac{2\sigma^2(G)}{n^2} + \frac{2\sigma^2(G)}{n} - \frac{2\sigma^2(G)}{n} \\ =& n + 2\left(\frac{\sigma(G)}{n}\right)^2. \end{aligned}$$ □

Corollary 2.2. *Let G be a connected graph of order n. Then*

$$D_E^Q E(G) > \frac{n^2+1}{2}.$$

Proof. Since $d_{ij} \ge 1$ for $i \ne j$ and there are $\frac{n(n-1)}{2}$ pairs of vertices in G, from the lower bound of Theorem 2.6, we get

$$D_E^Q E(G) > n + 2\left(\frac{\sigma(G)}{n}\right)^2 \ge n + 2\left(\frac{\frac{n(n-1)}{2}}{n}\right)^2 = \frac{n^2+1}{2}.$$ □

Hence, the result.

3. Distance Signless Laplacian Estrada Index of some Classes of Graphs

In this section we obtain the distance signless Laplacian Estrada index of some classes of graphs.

Lemma 3.1. *Let G be a k-transmission regular graph of order n. Then*

$$D_E^Q E(G) = DEE(G).$$

Proof. Note that the distance signless Laplacian spectrum of the graph G consists of $k+\mu_1 \ge k+\mu_2 \ge \cdots \ge k+\mu_n$, where $\mu_1 \ge \cdots \ge \mu_n$ is the distance spectrum of G. Also it is easy to see that $\sigma(G) = \frac{nk}{2}$. Then $D_E^Q E(G) = \sum_{i=1}^{n} e^{k+\mu_i-k} = DEE(G)$. □

The Cartesian product of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

Corollary 3.1. *Let G be an r-regular graph of diameter at most 2 with an adjacency matrix A and* $\mathrm{Spec}(G) = \{r, \lambda_2, \ldots, \lambda_n\}$. *Then, the distance signless Laplacian Estrada index of $H = G \times K_2$ is*

$$D_E^Q E(H) = e^{5n-2r-4} + e^{-n} + n - 1 + \sum_{i=2}^{n} e^{-2\lambda_i-4}.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $V(K_2) = \{w_1, w_2\}$. From the fact

$$d_H((v_i, w_j), (v_s, w_t)) = d_G(v_i, v_s) + d_{K_2}(w_j, w_t) = d_G(v_i, v_s) + 1,$$

we see that all vertices of H have the same transmission and $\mathrm{Tr}_H(v_i, w_j) = 5n - 2r - 4$. So $\mathrm{Tr}(H) = (5n - 2r - 4)I$. Then $\sigma(H) = \frac{n(5n-2r-4)}{2}$. Note that $H = G \times K_2$ has distance spectrum (see [30])

$$\mathrm{Spec}(H) = \begin{pmatrix} 5n - 2(r+2) & -2(\lambda_i + 2) & -n & 0 \\ \\ 1 & 1 & 1 & n-1 \end{pmatrix}$$

for $i = 2, \ldots, n$. Then

$$D_E^Q E(H) = e^{5n-2r-4} + e^{-n} + n - 1 + \sum_{i=2}^{n} e^{-2\lambda_i - 4}. \qquad \square$$

Given a graph G, the graph $G \nabla G$ is obtained by joining every vertex of G to every vertex of another copy of G.

Corollary 3.2. *Let G be an $r-$regular graph with an adjacency matrix A and $\mathrm{Spec}(G) = \{r, \lambda_2, \ldots, \lambda_n\}$. Then, the distance signless Laplacian Estrada index of $G \nabla G$ is*

$$D_E^Q E(G \nabla G) = e^{3n-r-2} + e^{n-r-2} + 2\sum_{i=2}^{n} e^{-2\lambda_i - 4}.$$

Proof. For $v \in G \nabla G$, it is easy to see that $\mathrm{Tr}(v) = d(v) + 2(n - d(v) - 1) + n = 3n - d(v) - 2 = 3n - r - 2$. Then $G \nabla G$ is a transmission regular graph and $\mathrm{Tr}(G \nabla G) = (3n - r - 2)I$. Note that the $G \nabla G$ has distance spectrum (see [30])

$$\mathrm{Spec}(G \nabla G) = \begin{pmatrix} 3n - r - 2 & n - r - 2 & -2(\lambda_i + 2) \\ \\ 1 & 1 & 2 \end{pmatrix},$$

for $i = 2, \ldots, n$. Then

$$D_E^Q E(G \nabla G) = e^{3n-r-2} + e^{n-r-2} + 2\sum_{i=2}^{n} e^{-2\lambda_i - 4}. \qquad \square$$

Next, we obtain the distance signless Laplacian Estrada index of the lexicographic product $G[H]$ of two graphs G and H. The following definition of the lexicographic product of G and H is from [9].

Definition 3.1. Let G and H be two graphs on vertex sets $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$, respectively. Then their *lexicographic product* $G[H]$ is a graph defined by $V(G[H]) = V(G) \times V(H)$, the Cartesian product of $V(G)$ and $V(H)$ in which $u = (u_1, v_1)$ is adjacent to $v = (u_2, v_2)$ if and only if either

(a) u_1 is adjacent to v_1 in G, or
(b) $u_1 = v_1$ and u_2 is adjacent to v_2 in G.

Corollary 3.3. *Let G be a k-transmission regular graph of order p. Let H be an r-regular graph of order n with adjacency eigenvalues $\{r, \lambda_2, \dots, \lambda_n\}$. Let $\{\mu_1, \dots, \mu_p\}$ be the eigenvalues of the distance matrix $D(G)$ of G. Then*

$$D_E^Q E(G[H]) = e^{2n-r-2} \sum_{i=1}^{p} e^{n\mu_i} + ne^{-4} \sum_{j=2}^{n} e^{-2\lambda_j}.$$

Proof. For $v \in G[H]$, it is easy to see that $\mathrm{Tr}(v) = r+2(n-r-1)+kn = kn+2n-r-2$. Then $G[H]$ is a transmission regular graph and $\mathrm{Tr}(G[H]) = (kn + 2n - r - 2)I$. Note that $G[H]$ has distance spectrum (see [29])

$$\mathrm{Spec}(G[H]) = \begin{pmatrix} n\mu_i + 2n - r - 2 & -2(\lambda_j + 2) \\ \\ 1 & n \end{pmatrix},$$

for $i = 1, \dots, p$ and $j = 2, \dots, n$. Then

$$D_E^Q E(G[H]) = e^{2n-r-2} \sum_{i=1}^{p} e^{n\mu_i} + ne^{-4} \sum_{j=2}^{n} e^{-2\lambda_j}.$$ □

Theorem 3.1. *Let G be an r-regular graph of order n, size m and diameter at most 2. If $\{2r, q_2, \dots, q_n\}$ are the eigenvalues of the signless Laplacian matrix $Q(G)$ of G, then*

$$D_E^Q E(G) = e^{2(n^2-n-m)} + \sum_{i=2}^{n} e^{2m-2n-nq_i}.$$

Proof. We know that the transmission of each vertex $v \in V(G)$ is $\mathrm{Tr}(v) = d(v) + 2(n - d(v) - 1) = 2n - d(v) - 1$ and so transmission $\sigma(G)$ of G is $\sigma(G) = n^2 - n - m$. Also

$$\begin{aligned} D^Q(G) = \mathrm{Tr}(G) + D(G) =& (2n-2)I - rI + 2J - 2I - A(G) \\ =& (2n-4)I + 2J - Q(G), \end{aligned}$$

where J is the all ones matrix. Then

$$\begin{aligned} D_E^Q E(G) =& \sum_{i=1}^{n} e^{\rho_i - \frac{2\sigma(G)}{n}} = e^{(4n-2r-4) - \frac{2(n^2-n-m)}{n}} + \sum_{i=2}^{n} e^{(2n-4-q_i) - \frac{2(n^2-n-m)}{n}} \\ =& e^{2(n^2-n-m)} + \sum_{i=2}^{n} e^{2m-2n-nq_i}. \end{aligned}$$ □

As an immediate consequence of the above theorem, we get the following.

Corollary 3.4. *Let G be an r-regular graph of order n, size m and diameter at most 2. If $\{r, \lambda_2, \dots, \lambda_n\}$ are the eigenvalues of the adjacency matrix $A(G)$ of G, then*

$$D_E^Q E(G) = e^{2(n^2-n-m)} + \sum_{i=2}^{n} e^{-n(\lambda_i+2)}.$$

4. Relations Between Distance Signless Laplacian Estrada Index and Distance Signless Laplacian Energy

The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. This quantity, introduced first time in [20] and having a clear connection to chemical problems, has now attracted much attention of mathematicians and mathematical chemists. We observe that several interesting results have been obtained for the energy of different graph structures. The pioneering paper [20] further proposes the study of energy in graphs with an analogue of the energy defined with respect to other (than adjacency) matrices assigned to the graphs. This proposal has been put into effect and extended: the energy of a graph with respect to Laplacian matrix as well as the energy of a graph with respect to distance matrix, have been studied (see [25, 30] for more details in this subject). Recently, Alhevaz et al. [1] have considered a new kind of energy with respect to the distance signless Laplacian matrix, the concept of distance signless Laplacian energy, denoted by $E_{D^Q}(G)$, and defined as

$$E_{D^Q}(G) = \sum_{i=1}^{n} \left| \rho_i - \frac{2\sigma(G)}{n} \right|.$$

In this section, we obtain some relations between $E_{D^Q}(G)$ and $D_E^Q E(G)$ for a simple connected graph G.

Theorem 4.1. *Let G be a connected graph of order n with diameter d. Then*

$$\begin{aligned} D_E^Q E(G) - E_{D^Q}(G) \leq & n - 1 - \sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)} \\ & + e^{\sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)}} \end{aligned} \tag{4.1}$$

or

$$D_E^Q E(G) \leq n - 1 + e^{E_{D^Q}(G)}. \tag{4.2}$$

Equality holds in (4.1) *or* (4.2) *if and only if* $G \cong K_1$.

Proof. From the proof of Theorem 2.3, we have

$$D_E^Q E(G) = n + \sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left(\rho_i - \frac{2\sigma(G)}{n}\right)^k}{k!} \leq n + \sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!}.$$

Taking into account the definition of the distance signless Laplacian energy, we get

$$D_E^Q E(G) \leq n + E_{D^Q}(G) + \sum_{i=1}^{n} \sum_{k \geq 2} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!},$$

which, as in Theorem 2.3, leads to

$$\begin{aligned} D_E^Q E(G) - E_{D^Q}(G) \leq & n + \sum_{i=1}^{n} \sum_{k\geq 2} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!} \\ \leq & n - 1 - \sqrt{2 \sum_{1\leq i<j\leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}} \\ & + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}}. \end{aligned}$$

One can easily see that the function $f(x) = e^x - x$ monotonically increases for $x \geq 0$. Therefore, the best upper bound for $D_E^Q E(G) - E_{D^Q}(G)$ is obtained for $2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} \leq 2\frac{n(n-1)}{2}d^2 + \frac{n^3(n-1)^2}{4} - n(n-1)^2$, and we get

$$\begin{aligned} D_E^Q E(G) - E_{D^Q}(G) \leq & n - 1 - \sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)} \\ & + e^{\sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)}}. \end{aligned}$$

Another way to obtain the relation between $D_E^Q E(G)$ and $E_{D^Q}(G)$ is as follows:

$$\begin{aligned} D_E^Q E(G) \leq & n + \sum_{i=1}^{n} \sum_{k\geq 1} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!} \\ \leq & n + \sum_{k\geq 1} \frac{1}{k!} \left(\sum_{i=1}^{n} \left|\rho_i - \frac{2\sigma(G)}{n}\right|\right)^k \\ = & n + \sum_{k\geq 1} \frac{\left(E_{D^Q}(G)\right)^k}{k!} \\ = & n - 1 + \sum_{k\geq 0} \frac{\left(E_{D^Q}(G)\right)^k}{k!}, \end{aligned}$$

implying

$$D_E^Q E(G) \leq n - 1 + e^{E_{D^Q}}(G).$$

Also, equality holds in (4.1) or (4.2) if and only $G \cong K_1$. ☐

Lemma 4.1 ([31])**.** *Let $x_1, \ldots, x_n$ be positive numbers. Then*

$$\frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Lemma 4.2 ([8])**.** *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} b_i\right) \leq n \sum_{i=1}^{n} a_i b_i.$$

Equality occurs if and only if $a_1 = \cdots = a_n$ *or* $b_1 = \cdots = b_n$.

Theorem 4.2. *Let G be a connected graph of order n. Then*

$$e^{-\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}} \le E_{DQ}(G) \le e^{\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}.$$

Proof. First we prove the given lower bound. By definition of the energy and by the arithmetic-geometric mean inequality, we have

$$\begin{aligned}E_{DQ}(G) = \sum_{i=1}^{n}\left|\rho_i - \frac{2\sigma(G)}{n}\right| &= n\left(\frac{1}{n}\sum_{i=1}^{n}\left|\rho_i - \frac{2\sigma(G)}{n}\right|\right)\\ &\ge n\left(\sqrt[n]{\left|\rho_1 - \frac{2\sigma(G)}{n}\right|\left|\rho_2 - \frac{2\sigma(G)}{n}\right|\dots\left|\rho_n - \frac{2\sigma(G)}{n}\right|}\right).\end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned}&n\left(\sqrt[n]{\left|\rho_1 - \frac{2\sigma(G)}{n}\right|\left|\rho_2 - \frac{2\sigma(G)}{n}\right|\dots\left|\rho_n - \frac{2\sigma(G)}{n}\right|}\right) \ge n\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}}\right)\\
&\ge n\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\\
&\ge n\left(\frac{n}{n\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right) \quad \text{(by Lemma 4.2)}\\
&\ge n\left(\frac{n}{n^2\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right) > n\left(\frac{n}{n^2\sum_{i=1}^{n}e^{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}}\right)\\
&= \frac{1}{\sum_{i=1}^{n}\sum_{k\ge 0}\frac{\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k}{k!}} = \frac{1}{\sum_{k\ge 0}\frac{1}{k!}\left(\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k\right)}\\
&\ge \frac{1}{\sum_{k\ge 0}\frac{1}{k!}\left(\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^2\right)^{\frac{k}{2}}} \quad \text{(by (2.2))}\\
&= \frac{1}{\sum_{k\ge 0}\frac{1}{k!}\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k} \quad \text{(by (1.2))}.\end{aligned}$$

Therefore, we have $E_{DQ}(G) \ge e^{-\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}$.

Now, we prove the given upper bound. We have,

$$E_{DQ}(G) = \sum_{i=1}^{n}\left|\rho_i - \frac{2\sigma(G)}{n}\right| < \sum_{i=1}^{n}e^{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}$$

$$
\begin{aligned}
&=\sum_{i=1}^{n}\sum_{k\geq 0}\frac{\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k}{k!}=\sum_{k\geq 0}\frac{1}{k!}\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k\\
&\leq\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^2\right)^{\frac{k}{2}}\quad(\text{ by inequality (2.2)})\\
&=\sum_{k\geq 0}\frac{1}{k!}\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}\right)^{\frac{k}{2}}\quad(\text{by Eq. (1.2)})\\
&=\sum_{k\geq 0}\frac{1}{k!}\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^{k}\\
&=e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}},
\end{aligned}
$$

and the proof is complete. □

Theorem 4.3. *Let G be a connected graph of order n. Then*

$$
E_{D^Q}(G)\geq\frac{1}{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}. \tag{4.3}
$$

Proof. By definition of the energy and by the arithmetic-geometric mean inequality, we have

$$
\begin{aligned}
E_{D^Q}(G)=\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|&=n\left(\frac{1}{n}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)\\
&\geq n\left(\sqrt[n]{\left|\rho_1-\frac{2\sigma(G)}{n}\right|\left|\rho_2-\frac{2\sigma(G)}{n}\right|\dots\left|\rho_n-\frac{2\sigma(G)}{n}\right|}\right).
\end{aligned}
$$

By Lemma 4.1 and Lemma 4.2, we have

$$
\begin{aligned}
&n\left(\sqrt[n]{\left|\rho_1-\frac{2\sigma(G)}{n}\right|\left|\rho_2-\frac{2\sigma(G)}{n}\right|\dots\left|\rho_n-\frac{2\sigma(G)}{n}\right|}\right)\geq n\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}}\right)\\
&\geq n\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\geq n\left(\frac{n}{n\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\\
&\geq n\left(\frac{n}{n^2\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\geq\frac{1}{\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k}\\
&\geq\frac{1}{\left(\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^2\right)^{\frac{k}{2}}}=\frac{1}{\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}\right)^{\frac{k}{2}}},
\end{aligned}
$$

Hence, for $k=2$, we arrive at (4.3). □

5. Conclusions

In this paper, we have defined the distance signless Laplacian Estrada index, where we have given some upper and lower bounds for $D_E^Q E(G)$ in terms of other graph invariants. Also, we have obtained the distance signless Laplacian Estrada index for some classes of graphs. Moreover, we derive some relations between $D_E^Q E(G)$ and the distance signless Laplacian energy of G. It would be interesting to give an expression for $D_E^Q E(G)$ in terms of the ordinary Estrada index in certain classes of graphs. Alternatively, one could possibly consider the range of values for $D_E^Q E(G)$ over some family of graphs of fixed order, for example, trees on n vertices.

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CERTAIN PROPERTIES OF APOSTOL-TYPE HERMITE-BASED-FROBENIUS-GENOCCHI POLYNOMIALS

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Abstract. This paper is well designed to set-up some new identities related to generalized Apostol-type Hermite-based-Frobenius-Genocchi polynomials and by applying the generating functions, we derive some implicit summation formulae and symmetric identities. Further a relationship between Array-type polynomials, Apostol-type Bernoulli polynomials and generalized Apostol-type Frobenius-Genocchi polynomials is also established.

1. Introduction

Let $a, b, c \in \mathbb{R}^+$, $a \neq b$ and $x \in \mathbb{R}$. The generalized Apostol-Bernoulli, Euler and Genocchi polynomials with the parameters are given by means of the following generating function as follows (see [1–17]):

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \tag{1.1}$$

where $|\lambda| = 1$, $\left|t \ln \frac{b}{a}\right| < 2\pi$,

$$\left(\frac{2}{\lambda b^t + a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} E_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \tag{1.2}$$

where $|\lambda| = 1$, $\left|t \ln \frac{b}{a}\right| < \pi$, and

$$\left(\frac{2t}{\lambda b^t + a^t}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} G_n^{(\alpha)}(x;\lambda;a,b,c)\frac{t^n}{n!}, \tag{1.3}$$

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where $|\lambda| = 1$, $\left|t \ln \frac{b}{a}\right| < \pi$.

It is clear from (1.1), (1.2) and (1.3) that $B_n^{(\alpha)}(x;\lambda;1,e,e) = B_n(x;\lambda)$, $E_n^{(\alpha)}(x;\lambda;1,e,e) = E_n(x;\lambda)$ and $G_n^{(\alpha)}(x;\lambda;1,e,e) = G_n(x;\lambda)$.

Recently, Kurt et al. [3] and Simsek (see [13, 14]) introduced the Apostol type Frobenius-Euler polynomials defined as follows.

Let $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x \in \mathbb{R}$. The generalized Apostol type Frobenius-Euler polynomials are defined by means of the following generating function:

$$\left(\frac{a^t - u}{\lambda b^t - u}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u,a,b,c,\lambda)\frac{t^n}{n!}. \tag{1.4}$$

For $x = 0$ and $\alpha = 1$ in (1.4), we get

$$\frac{a^t - u}{\lambda b^t - u} = \sum_{n=0}^{\infty} H_n(u,a,b;\lambda)\frac{t^n}{n!},$$

where $H_n(u,a,b;\lambda)$ denotes the generalized Apostol type Frobenius-Euler numbers (see [14, 16, 17]).

On setting $a = 1$, $b = e$, $\lambda = 1$ in (1.4), the result reduces to

$$\left(\frac{1-u}{e^t - u}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} H_n^{(\alpha)}(x;u)\frac{t^n}{n!}, \quad \alpha \in \mathbb{Z},$$

where $H_n^{(\alpha)}(x;u)$ is called classical Frobenius-Euler polynomial of order α.

Observe that $H_n^{(1)}(x,u) = H_n(x,u)$ which denotes the Frobenius-Euler polynomials and $H_n^{(\alpha)}(0;u) = H_n^{(\alpha)}(u)$, which denotes the Frobenius-Euler numbers of order α. $H_n(x;-1) = E_n(x)$, which denotes the Euler polynomials, (see [7, 11, 15]).

Very recently, Yaşar and Özarslan [17] introduced Frobenius-Genocchi polynomials defined by means of the following generating relation:

$$\frac{(1-\lambda)t}{e^t - \lambda}e^{xt} = \sum_{n=0}^{\infty} G_n^F(x;\lambda)\frac{t^n}{n!}. \tag{1.5}$$

Taking $\lambda = -1$ in (1.5), we get Genocchi polynomials

$$\frac{2t}{e^t + 1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \quad |t| < \pi.$$

Pathan and Khan [10] introduced the generalized Hermite-based Bernoulli polynomials ${}_HB_n^{(\alpha)}(x,y)$ of two variables defined by

$$\left(\frac{t}{e^t - 1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!},$$

which is essentially a generalization of Bernoulli numbers, Bernoulli polynomials, Hermite polynomials and Hermite-Bernoulli polynomials ${}_HB_n(x,y)$ introduced by Dattoli et al. [2, page 386, (1.6)] in the form

$$\left(\frac{t}{e^t - 1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}.$$

Definition 1.1. Let $c > 0$. The generalized 2-variable 1-parameter Hermite Kamp'e de Feriet polynomials $H_n(x, y; c)$ polynomials for nonnegative integer n are defined by

$$c^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y; c)\frac{t^n}{n!}. \tag{1.6}$$

This is an extended 2-variable Hermite Kampé de Fériet polynomials $H_n(x, y)$ defined by (see [5–7, 10])

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x, y)\frac{t^n}{n!}.$$

Note that $H_n(x, y; e) = H_n(x, y)$. In order to collect the powers of t we expand the left hand side of (1.6) to the representation

$$H_n(x, y; c) = n! \sum_{j=0}^{[\frac{n}{2}]} \frac{(\ln c)^{n-2j} x^{n-2j} y^j}{j!(n-2j)!}. \tag{1.7}$$

Simsek [13] constructed the λ-Stirling type number of second kind $S(n, \nu; a, b; \lambda)$ by mean of the following generating function:

$$\sum_{n=0}^{\infty} \mathcal{S}(n, \nu; a, b; \lambda)\frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^\nu}{\nu!}, \tag{1.8}$$

and the generalized array type polynomials is defined by Simsek (see [13, page 6, (3.1)])

$$\sum_{n=0}^{\infty} \mathcal{S}_\nu^n(x; a, b; \lambda)\frac{t^n}{n!} = \frac{(\lambda b^t - a^t)^\nu}{\nu!} b^{xt}.$$

Kurt and Simsek [3] introduced the polynomial $Y_n(x; \lambda; a)$, which is given by the following generating function:

$$\frac{t}{\lambda a^t - 1} a^{xt} = \sum_{n=0}^{\infty} Y_n(x; \lambda; a)\frac{t^n}{n!}, \quad a \geq 1. \tag{1.9}$$

We also note that for $x = 0$, above equation gives a relation as $Y_n(0; \lambda; a) = Y_n(\lambda; a)$ (see [13, 14]). Again if we set $x = 0$ and $a = 1$ in (1.9), we get

$$\frac{t}{\lambda - 1} = \sum_{n=0}^{\infty} Y_n(0, \lambda; 1)\frac{t^n}{n!}.$$

The paper is organized as follows. In Section 2, we introduce generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u, a, b, c; \lambda)$ and their properties. In Section 3, we derive some implicit summation formulae for generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials. In Section 4, we give general symmetry identities by using different analytical means and applying generating functions and last Section 5, we find relation between λ-type Stirling polynomials, Apostol-Bernoulli polynomials and generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials.

2. Generalized Apostol-Type Hermite-Based-Frobenius-Genocchi Polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$

The intent of this section is to define the generalized Apostol-type Hermite-based-Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$ with suitable properties.

Definition 2.1. For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, the generalized Apostol-type Hermite-based Frobenius-Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a.b.c; \lambda)$ of order α are defined by means of the following generating function:

$$\left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{xt+yt^2} = \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^n}{n!}. \tag{2.1}$$

Remark 2.1. For $y = 0$ (2.1) reduces to

$$\left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{xt} = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x; u; a, b, c; \lambda)\frac{t^n}{n!},$$

where $\mathcal{G}_n^{(\alpha)}(x; u; a, b, c; \lambda)$ is known as Apostol-type Frobenius Genocchi polynomials of order α (see [8]).

Remark 2.2. On setting $x = y = 0$ and $\alpha = 1$ in (2.1), we have

$$\left(\frac{(a^t - u)t}{\lambda b^t - u}\right) = \sum_{n=0}^{\infty} \mathcal{G}_n(u; a, b; \lambda)\frac{t^n}{n!},$$

where $\mathcal{G}_n^{\alpha}(u; a.b.c; \lambda)$ denotes the generalized Apostol-type Frobenius-Genocchi numbers.

Remark 2.3. If we set $a = 1$, $b = c = e$, $u = -1$, then (2.1) immediately reduces to Hermite-based Genocchi polynomials (see [6, 7])

$$\left(\frac{2t}{\lambda e^t + 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} {}_HG_n^{(\alpha)}(x, y; \lambda), \quad |t| < \pi.$$

Now we give some properties of the generalized Apostol-type Hermite-based- Frobenius Genocchi polynomials ${}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)$, which are stated in terms of theorems as follows.

Theorem 2.1. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the following result holds true*

$$\begin{aligned}(2u-1)\sum_{r=0}^{n}\binom{n}{r}{}_H\mathcal{G}_r(x, y; u; a, b, c; \lambda)\,\mathcal{G}_{n-r}(z; 1-u; a, b, c; \lambda)\\ = n(u-1){}_H\mathcal{G}_{n-1}(x+z, y; u; a, b, c; \lambda) + nu\,{}_H\mathcal{G}_{n-1}(x+z, y; 1-u, a, b, c; \lambda)\\ + \sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}\,{}_H\mathcal{G}_r(x+z, y; u; a, b, c; \lambda)\\ - \sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}{}_H\mathcal{G}_r(x+z, y; 1-u, a, b, c; \lambda).\end{aligned} \tag{2.2}$$

Proof. In order to prove (2.2), for $\alpha = 1$, we get

$$(2u-1)\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)c^{xt+yt^2}\left(\frac{(a^t-(1-u))t}{\lambda b^t-(1-u)}\right)c^{zt} \tag{2.3}$$
$$=t^2(a^t-u)(a^t-(1-u))c^{(x+z)t+yt^2}\left[\frac{1}{\lambda b^t-u}-\frac{1}{\lambda b^t-(1-u)}\right].$$

Employing the result of (2.1), (2.3) reduces as

$$(2u-1)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x,y;u;a,b,c;\lambda)\frac{t^r}{r!}\sum_{n=0}^{\infty}\mathcal{G}_n(z;1-u;a,b,c;\lambda)\frac{t^n}{n!} \tag{2.4}$$
$$=(a^t-(1-u)t)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;u,a,b,c;\lambda)\frac{t^r}{r!}-(a^t-u)t$$
$$\times\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;1-u;a,b,c;\lambda)\frac{t^r}{r!}.$$

Using [15, page 100, (1)] (2.4) reduces to

(2.5)
$$(2u-1)\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}{}_H\mathcal{G}_r(x,y;u;a,b,c;\lambda)\,{}_H\mathcal{G}_{n-r}(z,y;1-u;a,b,c;\lambda)\frac{t^n}{n!}$$
$$=(a^t-(1-u)t)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;u,a,b,c;\lambda)\frac{t^r}{r!}-(a^t-u)t$$
$$\times\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;1-u;a,b,c;\lambda)\frac{t^r}{r!}$$
$$=(u-1)\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;u,a,b,c;\lambda)\frac{t^{r+1}}{r!}+u\sum_{r=0}^{\infty}{}_H\mathcal{G}_r(x+z,y;1-u,a,b,c;\lambda)\frac{t^{r+1}}{r!}$$
$$+\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}{}_H\mathcal{G}_r(x+z,y;u;a,b,c;\lambda)\frac{t^n}{n!}$$
$$-\sum_{n=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}(\ln a)^{n-r}{}_H\mathcal{G}_r(x+z,y;1-u;a,b,c;\lambda)\frac{t^n}{n!}.$$

On comparing the coefficient of t^n from the above equation, we arrive at our desired result. $\square$

Theorem 2.2. *For $a,b,c\in\mathbb{R}^+$, $a\neq b$, $x,y\in\mathbb{R}$, $\lambda\in\mathbb{C}$, $k\in\mathbb{N}$, $\alpha\in\mathbb{Z}$, the following relationship holds true*

$$\sum_{k=0}^{n}{}_H\mathcal{G}_k^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\,{}_H\mathcal{G}_{(n-k)}^{(\alpha-m)}(x,y;u;a.b.c;\lambda)=\mathcal{G}_n^{(-m)}(u;a,b;\lambda). \tag{2.6}$$

Proof. In order to prove (2.6), replacing x with $-x$, y with $-y$ and α with $-\alpha$ in (2.1), we get get

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{(-\alpha)} c^{-(xt+yt^2)}. \tag{2.7}$$

Making use of the above equation in the left-hand side of (2.6), we can write

$$\sum_{k=0}^{\infty} {}_H\mathcal{G}_k^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha-m)}(x,y;u;a,b,c;\lambda)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{-m}.$$

We can write the above equation as

$$\begin{aligned}&\sum_{k=0}^{\infty} {}_H\mathcal{G}_k^{(-\alpha)}(-x,-y;u;a,b,c;\lambda)\frac{t^k}{k!}\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha-m)}(x,y;u;a,b,c;\lambda)\frac{t^n}{n!}\\ &=\sum_{n=0}^{\infty} \mathcal{G}_n^{(-m)}(u;a,b;\lambda)\frac{t^n}{n!}.\end{aligned}$$

Using [15, page 100, (1)] in the above equation and then comparing the coefficients of t^n, we immediately come to our desired result (2.6). ☐

Theorem 2.3. *For $n \geq 0$, $p,q \in \mathbb{R}$, the following formula for generalized Apostol type Frobenius-Genocchi-Hermite polynomials holds true*

$$\begin{aligned}{}_H\mathcal{G}_n^{(\alpha)}(px,qy;u,a,b,c;\lambda) =& \sum_{k=0}^{n}\frac{n!}{(n-k)!}{}_H\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b,c;\lambda)\\ &\times\sum_{j=0}^{[\frac{k}{2}]}\frac{((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^j}{(k-2j)!j!}.\end{aligned}$$

Proof. Rewrite the generating function (2.1), we have

$$\begin{aligned}&\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^n}{n!}\\ &=\left(\frac{(a^t-u)t}{\lambda b^t-u}\right)^{\alpha} c^{xt+yt^2}c^{(p-1)xt}c^{(q-1)yt^2}\\ &=\left(\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}((p-1)x\ln c)^k\frac{t^k}{k!}\right)\times\left(\sum_{j=0}^{\infty}((q-1)y\ln c)^j\frac{t^{2j}}{j!}\right)\\ &=\left(\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}((p-1)x\ln c)^k((q-1)y\ln c)^j\frac{t^{k+2j}}{k!j!}\right).\end{aligned}$$

Replacing k by $k-2j$ in above equation, we have

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^n}{n!}\right)$$

$$\times\left(\sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k}{2}]}((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^{j}\frac{t^{k}}{(k-2j)!j!}\right)$$

$$=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{[\frac{k}{2}]}{}_{H}\mathcal{G}_{n}^{(\alpha)}(x,y;u,a,b,c;\lambda)((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^{j}\frac{t^{n+k}}{(k-2j)!j!n!}.$$

Again replacing n by $n-k$ in above equation, we have

$$\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^{n}}{n!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{[\frac{k}{2}]}{}_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b,c;\lambda)((p-1)x\ln c)^{k-2j}((q-1)y\ln c)^{j}$$
$$\times\frac{t^{n}}{(k-2j)!j!(n-k)!}.$$

Finally, equating the coefficients of t^n on both sides, we acquire the result. □

Remark 2.4. By taking $c=e$ in Theorem 2.3, we get the following corollary.

Corollary 2.1. *For $p,q\in\mathbb{R}$, $x,y\in\mathbb{C}$ and $n\geq 0$, we have*

$$_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b;\lambda)$$
$$=\sum_{k=0}^{n}\frac{n!}{(n-k)!}{}_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b;\lambda)\sum_{j=0}^{[\frac{k}{2}]}\frac{((p-1)x)^{k-2j}((q-1)y)^{j}}{(k-2j)!j!}.$$

Theorem 2.4. *For $n\geq 0$, $p,q\in\mathbb{R}$ and $x,y\in\mathbb{C}$, we have*

$$\begin{aligned}&{}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b,c;\lambda)\\&=\sum_{k=0}^{n}\binom{n}{k}{}_{H}\mathcal{G}_{n-k}^{(\alpha)}(x,y;u,a,b,c;\lambda)H_{k}((p-1)x,(q-1)y;c).\end{aligned}\tag{2.8}$$

Proof. In order to proof above result, we set x as px and y as qy in (2.1),

$$\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{(\alpha)}(px,qy;u,a,b,c;\lambda)\frac{t^{n}}{n!}=\left(\frac{(a^{t}-u)t}{\lambda b^{t}-u}\right)^{\alpha}c^{xt+yt^{2}}c^{(p-1)xt}c^{(q-1)yt^{2}}$$
$$=\sum_{n=0}^{\infty}{}_{H}\mathcal{G}_{n}^{(\alpha)}(x,y;u,a,b,c;\lambda)\frac{t^{n}}{n!}\sum_{k=0}^{\infty}H_{k}((p-1)x,(q-1)y;c)\frac{t^{k}}{k!}.$$

By assistance of [15] and then on comparing the coefficients of t^n, we have arrive at our result. □

Theorem 2.5. *For $n\geq 0$, $p,q\in\mathbb{R}$ and $x,y\in\mathbb{C}$, we have*

$$_{H}\mathcal{G}_{n}^{(\alpha+\beta)}(x+z,y+z;u,a,b,c;\lambda)=\sum_{k=0}^{n}\binom{n}{k}{}_{H}\mathcal{G}_{k}^{(\alpha)}(x,z;u;a,b,c;\lambda)$$

$$\times {}_H\mathcal{G}^{(\beta)}_{n-k}(z,y;u;a,b,c;\lambda),$$

$$ {}_H\mathcal{G}^{(-\alpha)}_{n}(2x,2y;u^2;a^2,b^2,c^2;\lambda^2)=\sum_{k=0}^{n}\binom{n}{k}\,{}_H\mathcal{G}^{(-\alpha)}_{k}(x,y;u;a,b,c;\lambda)$$
$$\times {}_H H^{(-\alpha)}_{n-k}(x,y;-u;a,b,c;\lambda).$$

Proof. Proof of these identities can be solved by making use of (2.1) and (1.5) with some required calculations. □

3. Summation Formulae for Generalized Apostol-Type Hermite-Based-Frobenius-Genocchi Polynomials

Here in this section, we provide the implicit formulae for generalized Apostol-type Hermite-based-Frobinis-Genocchi polynomials.

Theorem 3.1. *For $a,b,c\in\mathbb{R}^+, a\neq b,\ x,y\in\mathbb{R},\ \lambda\in\mathbb{C},\ k\in\mathbb{N},\ \alpha\in\mathbb{Z}$, the following relation holds true*

$$ {}_H\mathcal{G}^{(\alpha)}_{k+l}(z,y;u;a,b,c;\lambda)=\sum_{n,m=0}^{k,l}\binom{l}{m}\binom{k}{n}(z-x)^{m+n}(\ln c)^{m+n} \tag{3.1}$$
$$\times {}_H\mathcal{G}^{(\alpha)}_{k-n+l-m}(x,y;u;a,b,c;\lambda).$$

Proof. Replacing t by $t+w$ in (2.1) and then using ([15], page 52, (2)), in the above equation, we get

$$\left(\frac{(a^{(t+w)}-u)(t+w)}{\lambda b^{t+w}-u}\right)^{\alpha}c^{y}(t+w)^2=c^{-x(t+w)}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}^{(\alpha)}_{k+l}(x,y;u;a,b,c;\lambda)\frac{t^k}{k!}\frac{w^l}{l!}. \tag{3.2}$$

Replacing x by z and then equating the obtained equation from the above equation (3.2), we get

$$c^{(z-x)(t+w)}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}^{(\alpha)}_{k+l}(x;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}=\sum_{k,l=0}^{\infty}{}_H\mathcal{G}^{(\alpha)}_{k+l}(z,y;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}.$$

Expanding the exponent part of left-hand side, the above equation converts as

$$\sum_{N=0}^{\infty}\frac{(\ln c)[(z-x)(t+w)]^N}{N!}\sum_{k,l=0}^{\infty}{}_H\mathcal{G}^{(\alpha)}_{k+l}(x,y;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!} \tag{3.3}$$
$$=\sum_{k,l=0}^{\infty}{}_H\mathcal{G}^{(\alpha)}_{k+l}(z,y;u;a,b,c;\lambda))\frac{t^k}{k!}\frac{w^l}{l!}.$$

On comparing the coefficients of equal powers of t and w after taking the reference of [15, page 52, (2) and page 100, (1)] to the above equation, we attain our required result. □

Corollary 3.1. *For $l=0$, the above result reduces to*

$$ {}_H\mathcal{G}^{(\alpha)}_{k}(z,y;u;a,b,c;\lambda)=\sum_{n=0}^{k}\binom{k}{n}(z-x)^n(\ln c)^n\,{}_H\mathcal{G}^{(\alpha)}_{k-n}(x,y;u;a,b,c;\lambda).$$

Theorem 3.2. *For $a, b, c \in \mathbb{R}^+, a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, $n \geq 0$, the following relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}^{(\alpha)}(u; a, b; \lambda)\, H_m(x, y; c). $$

Proof. From equation (2.1) and (1.7), we have

$$ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^n}{n!} = \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{xt+yt^2} $$

$$ = \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(u; a, b)\frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(x, y; c)\frac{t^m}{m!}. $$

On using [15, page 100, (1)], and then comparing the coefficient of equal powers, we have the required result. □

Theorem 3.3. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha)}(x + 1, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} (\ln c)^{n-m}\, {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b, c; \lambda). $$

Proof. Replacing x by $x + 1$, (2.1) reduces to

$$ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(x + 1, y; u; a, b, c; \lambda)\frac{t^n}{n!} = \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{(x+1)t+yt^2} $$

$$ = \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{(xt+yt^2)} c^t $$

$$ = \sum_{m=0}^{\infty} {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^m}{m!} \sum_{n=0}^{\infty} \frac{(\ln c)^n\, t^n}{n!}. $$

Using [15, page 100, (1)] and on comparing coefficient of t^n, we have the required result. □

Theorem 3.4. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in C$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha+1)}(x, y; u; a, b, c; \lambda) = \sum_{m=0}^{n} \binom{n}{m} \mathcal{G}_{n-m}(u; a, b; \lambda)\, {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b; \lambda). $$

Proof. Replacing α by $\alpha + 1$ in (2.1), we have

$$ \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha+1} c^{xt+yt^2} = \left(\frac{(a^t - u)t}{\lambda b^t - u}\right) \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{xt+yt^2} $$

$$ = \sum_{n=0}^{\infty} \mathcal{G}_n(u; a, b; \lambda)\frac{t^n}{n!} \sum_{m=0}^{\infty} {}_H\mathcal{G}_m^{(\alpha)}(x, y; u; a, b, c; \lambda)\frac{t^m}{m!}. $$

Making use of [15, page 100, (1)] and then on comparing coefficient of t^n, we lead to our required result. □

Theorem 3.5. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in C$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$, the relation holds true*

$$ {}_H\mathcal{G}_n^{(\alpha)}(y, x; u; a, b, c; \lambda) = \sum_{k=0}^{[\frac{n}{2}]} \frac{n!}{k!\,(n-2k)!} \mathcal{G}_{n-2k}^{(\alpha)}(y, u; a, b, c; \lambda)(x \ln c)^k. $$

Proof. Interchanging x and y in (2.1), we have

$$ \begin{aligned} \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{\alpha} c^{yt+xt^2} &= \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(y, x; u; a, b, c; \lambda) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(y; u; a, b, c; \lambda) \frac{t^n}{n!} \sum_{k=0}^{\infty} (x \ln c)^k \frac{t^{2k}}{k!}. \end{aligned} $$

Making use of [15, page 100, (3))] and then on comparing coefficient of t^n, we lead to our required result. $\square$

4. Symmetric Identities

In this section, we establish symmetric identities for generalized Apostol type Hermite-based Frobenius-Genocchi polynomials by applying the generating function (2.1). Such type of identities have been introduced by many authors namely Khan [6], Khan et al. [5, 7] and Pathan and Khan [10–12].

Theorem 4.1. *Let $a, b, c > 0$, $a \neq b$, $x, y \in \mathbb{R}$ and $n \geq 0$, the following relation holds true*

$$ \begin{aligned} &\sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_H\mathcal{G}_{n-k}^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda) {}_H\mathcal{G}_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda) \\ &= \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} \mathcal{G}_{n-k}^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda) {}_H\mathcal{G}_k^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda). \end{aligned} \tag{4.1} $$

Proof. In order to proof (4.1), we suppose a function $H(t)$ as

$$ H(t) = \left[\left(\frac{(A^{at} - u)at}{\lambda B^{at} - u}\right)\left(\frac{(A^{bt} - u)bt}{\lambda B^{bt} - u}\right)\right]^{\alpha} c^{2(abxt + a^2b^2yt^2)}. $$

The above expression is symmetric in a and b hence we can write above equation into two ways as follows:

$$ \begin{aligned} H(t) &= \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda) \frac{(at)^n}{n!} \sum_{k=0}^{\infty} {}_H\mathcal{G}_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda) \frac{(bt)^k}{k!} \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} b^k a^{n-k} {}_H\mathcal{G}_{n-k}^{(\alpha)}(bx, b^2y; u; A, B, c; \lambda) {}_H\mathcal{G}_k^{(\alpha)}(ax, a^2y; u; A, B, c; \lambda) \frac{t^n}{n!}. \end{aligned} \tag{4.2} $$

Again we can write

(4.3)

$$H(t)=\sum_{n=0}^{\infty}{}_H\mathcal{G}_n^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda)\frac{(bt)^n}{n!}\sum_{k=0}^{\infty}{}_H\mathcal{G}_k^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda)\frac{(at)^k}{k!}$$
$$=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}{}_H\mathcal{G}_{n-k}^{(\alpha)}(ax,a^2y;u;A,B,c;\lambda){}_H\mathcal{G}_k^{(\alpha)}(bx,b^2y;u;A,B,c;\lambda)\frac{t^n}{n!}.$$

Comparing (4.2) and (4.3), we arrive at our desired result. □

Corollary 4.1. *For $\alpha=1$ in Theorem 4.1, we have the following symmetric identity:*

$$\sum_{k=0}^{n}\binom{n}{k}b^ka^{n-k}{}_H\mathcal{G}_{n-k}(bx,b^2y;u;A,B,c;\lambda){}_H\mathcal{G}_k(ax,a^2y;u;A,B,c;\lambda)$$
$$=\sum_{k=0}^{n}\binom{n}{k}a^kb^{n-k}{}_H\mathcal{G}_{n-k}(ax,a^2y;u;A,B,c;\lambda){}_H\mathcal{G}_k(bx,b^2y;u;A,B,c;\lambda).$$

Theorem 4.2. *Let $a,b,c>0$, $a\neq b$, $x,y\in\mathbb{R}$ and $n\geq 0$, the following relation holds true:*

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{(i+j)}b^ka^{n-k}{}_H\mathcal{G}_{n-k}^{(\alpha)}\left(bx+\frac{b}{a}i+j,b^2y;u;A,B,c;\lambda\right)$$
$$\times\,\mathcal{G}_k^{(\alpha)}(az,0;u;A,B,c;\lambda)$$
$$=\sum_{k=0}^{n}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{(i+j)}\binom{n}{k}a^kb^{n-k}{}_H\mathcal{G}_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j,a^2y;u;A,B,c;\lambda\right)$$
$$\times\,\mathcal{G}_k^{(\alpha)}(bz,0;u;A,B,c;\lambda).$$

Proof. In order to prove above result, we suppose $I(t)$ is

$$I(t)=\left[\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)\right]^{\alpha}\frac{(1+\lambda(-1)^{a+1}c^{abt})^2}{(\lambda c^{at}+1)(\lambda c^{bt}+1)}c^{ab(x+z)t+a^2b^2yt^2}$$
$$=\left(\frac{(A^{at}-u)at}{\lambda B^{at}-u}\right)^{\alpha}c^{abxt+a^2b^2yt^2}\sum_{i=0}^{a-1}(-\lambda)^ic^{ibt}\left(\frac{(A^{bt}-u)bt}{\lambda B^{bt}-u}\right)^{\alpha}c^{abzt}\sum_{j=0}^{b-1}(-\lambda)^jc^{jat}.$$

Using [15, page 100, (1)] we have

$$I(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}(-\lambda)^{i+j}a^{n-k}b^k{}_H\mathcal{G}_{n-k}^{(\alpha)}(bx+\frac{b}{a}i+j,b^2y;u;A,B,c;\lambda)$$
$$\times\,\mathcal{G}_k^{(\alpha)}(az;u;A,B,c;\lambda)\frac{t^n}{n!}.$$

On the other hand, we have

$$I(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}(-\lambda)^{i+j}b^{n-k}a^k{}_H\mathcal{G}_{n-k}^{(\alpha)}\left(ax+\frac{a}{b}i+j,a^2y;u;A,B,c;\lambda\right)$$

$$\times\, \mathcal{G}_k^{(\alpha)}\left(bz; u; A, B, c; \lambda\right) \frac{t^n}{n!}.$$

On comparing both the results, we have the required relation. □

5. Relation Between λ-Type Striling Numbers of Second Kind, Apostol-Bernoulli Polynomial and Generalized Apostol-Type Hermite-Based-Frobenius-Genocchi Polynomial

This section deals with some relationships in between Array-type polynomials, Apostol-Bernoulli polynomial and generalized Apostol-type Hermite-based Frobenius-Genocchi polynomial.

Theorem 5.1. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x, y \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$ and ν be an integer, then we have*

$$\begin{aligned}(5.1)\qquad {}_H\mathcal{G}_{n-2\nu}^{(-\nu)}(x, y; u; a, b, b; \lambda) =&\frac{(\nu)!}{(-n)_{2\nu}} \sum_{k=0}^{n} \sum_{m=0}^{l} \binom{m}{k} \binom{n}{m} S\left(k, v, 1, b; \frac{\lambda}{u}\right)\\ &\times Y_{m-k}^{(\nu)}\left(\frac{1}{u}; a\right) H_{l-m}(x, y).\end{aligned}$$

Proof. In order to proof above result, we replace of c with b and α with $-\nu$ in equation (2.1), we get

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x, y; u; a, b, b; \lambda)\frac{t^n}{n!} = \left(\frac{(a^t - u)t}{\lambda b^t - u}\right)^{(-\nu)} b^{xt+yt^2}.$$

On arranging the above equation, we arrive at

$$\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x, y; u; a, b, b; \lambda)\frac{t^n}{n!} = (\nu!)\frac{\left(\frac{\lambda}{u}b^t - 1\right)^{\nu} b^{xt+yt^2}}{(\nu!)\left(\frac{a^t}{u} - 1\right)^{\nu} t^{\nu}}\frac{t^{\nu}}{t^{\nu}}.$$

By assistance of (1.8) and (1.9), above equation reduces to

$$\begin{aligned}(5.2)\qquad \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x, y; u; a, b, b; \lambda)\frac{t^{n+2\nu}}{n!} =&(\nu!)\sum_{k=0}^{\infty} S\left(n, v, 1, b; \frac{\lambda}{u}\right)\frac{t^k}{k!}\\ &\times \sum_{m=0}^{\infty} Y_m^{(\nu)}\left(\frac{1}{u}, 1; a\right)\frac{t^m}{m!}\sum_{l=0}^{\infty} H_l(x, y; b)\frac{t^l}{l!}.\end{aligned}$$

Using Lemma [15, page 100, (1)] we get

$$\begin{aligned}\sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x, y; u; a, b, b; \lambda)\frac{t^{n+2\nu}}{n!} =&\nu!\sum_{l=0}^{\infty}\sum_{k=0}^{m}\sum_{m=0}^{l}\binom{m}{k}\binom{l}{m} S\left(k, v, 1, b; \frac{\lambda}{u}\right)\\ &\times Y_{m-k}^{(\nu)}\left(\frac{1}{u}, 1; a\right) H_{l-m}(x, y; b)\frac{t^l}{l!}.\end{aligned}$$

Using [15, page 23, (22) and (23)] and replacing l by n, and then by comparing the coefficients of t^n we arrive at our required result. □

Theorem 5.2. *For $a, b, c \in \mathbb{R}^+$, $a \neq b$, $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, $k \in \mathbb{N}$, $\alpha \in \mathbb{Z}$ and ν be an integer, we have*

$$ {}_H\mathcal{G}_{n-2\nu}^{(-\nu)}(x,y;u;a,b,b;\lambda) = \frac{(\nu)!}{(-n)_{2\nu}} \sum_{k=0}^{n} \binom{n}{k} \mathcal{S}\left(k,\nu,1,b,\frac{\lambda}{u}\right) \times {}_H\mathcal{B}_{n-k}^{(\nu)}\left(x,y,\frac{1}{u},1,a,b\right). $$

Proof. Making replacement of c with b and α with $-\nu$ in (2.1), we get

$$ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!} = \left(\frac{(a^t-u)t}{\lambda b^t - u}\right)^{(-\nu)} b^{xt+yt^2}. $$

On arranging the above equation, we arrive at

$$ \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^n}{n!} = (\nu!)\frac{\left(\frac{\lambda}{u}b^t-1\right)^{\nu} b^{xt+yt^2}}{(\nu!)\left(\frac{a^t}{u}-1\right)^{\nu} t^{\nu}}\frac{t^{\nu}}{t^{\nu}}. $$

Using (1.8) and (1.1), the above equation converts into

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!} =& (\nu!)\sum_{k=0}^{\infty} \mathcal{S}\left(k,\nu,1,b;\frac{\lambda}{u}\right)\frac{t^k}{k!} \\ & \times \sum_{n=0}^{\infty} {}_H\mathcal{B}_n^{(\nu)}\left(x,y,\frac{1}{u},1,a,b\right)\frac{t^n}{n!}. \end{aligned} $$

Using [15, page 100, (1)] right-hand side, it converts as follows

$$ \begin{aligned} \sum_{n=0}^{\infty} {}_H\mathcal{G}_n^{(-\nu)}(x,y;u;a,b,b;\lambda)\frac{t^{n+2\nu}}{n!} =& \nu!\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k} \mathcal{S}\left(k,\nu,1,b,\frac{\lambda}{u}\right) \\ & \times {}_H\mathcal{B}_{n-k}^{(\nu)}\left(x,y,\frac{1}{u},1,a,b\right)\frac{t^n}{n!!}. \end{aligned} $$

Using [15, page 23, (22) and (23)] and replacing l with n, then by comparing the coefficients of t^n, we arrive at our required result. $\square$

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CONSTRUCTION OF L-BORDERENERGETIC GRAPHS

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Abstract. If a graph G of order n has the Laplacian energy same as that of complete graph K_n then G is said to be L-borderenergeic graph. It is interesting and challenging as well to identify the graphs which are L-borderenergetic as only few graphs are known to be L-borderenergetic. In the present work we have investigated a sequence of L-borderenergetic graphs and also devise a procedure to find sequence of L-borderenergetic graphs from the known L-borderenergetic graph.

1. Introduction

Throughout this paper, we begin with finite, undirected and simple graph G. For a standard terminology and notations in graph theory we follow Balakrishnan and Ranganathan [1], while the terms related to algebra are used in the sense of Lang [8]. Throughout this paper $\overline{G}, K_p$ and $\overline{K_p}$, respectively, denote complement of G, complete graph on p vertices and null graph with p vertices. The average vertex degree of G is denoted by $\overline{d}$ and defined as $\overline{d} = \frac{\sum d_i}{n}$, where d_i is degree of vertex v_i.

Let G be an undirected simple graph with vertices $v_1, v_2, \ldots, v_n$. The *adjacency matrix* denoted by $A(G)$ of G is defined to be $A(G) = [a_{ij}]$, such that, $a_{ij} = 1$ if v_i is adjacent, with v_j and 0 otherwise. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $A(G)$ are known as eigenvalues of graph G. The energy $E(G)$ of graph G is defined by

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \,.$$

The concept of graph energy was introduced by Gutman [6] in 1978. It is well known that the energy of complete graph is $2(n-1)$. In 1978 Gutman [6] conjectured that among all the graph with n vertices, the complete graph K_n has the maximum

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energy. This conjecture was disproved by Walikar et al. [12] by showing existence of graphs whose energy is greater than that of complete graphs. The graphs whose energy is $2(n-1)$ are termed as Borderenergetic according to Gong et al. [5].

Let $D(G)$ be the diagonal matrix of whose $(i,i)^{\text{th}}$ entry is the degree of a vertex v_i. The matrix $L(G) = D(G) - A(G)$ is called the *Laplacian* matrix of G. The eigenvalues of $L(G)$ are denoted by $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n$. It is well known that $L(G)$ is a positive semi definite and singular matrix. So, for $i = 1, 2, \dots, n-1$, $\mu_i \geq 0$ and $\mu_n = 0$. The collection of all Laplacian eigenvalues together with their multiplicities is known as *Laplacian spectra* (L-spectra). Hence,

$$\operatorname{spec}_L(G) = \begin{pmatrix} \mu_1 & \mu_2 & \cdots & \mu_{n-1} & \mu_n = 0 \\ m(\mu_1) & m(\mu_2) & \cdots & m(\mu_{n-1}) & m(\mu_n) \end{pmatrix}.$$

The concept of Laplacian energy of G was introduced by Gutman and Zhou [7], is defined by $LE(G) = \left|\mu_i - \bar{d}\right|$, where μ_i are the Laplacian eigenvalues of G and $\bar{d}$ is the average vertex degree of G.

Recently, a concept analogous to borderenergetic graphs in the context of Laplacian energy has been introduced by Tura [10] which is teremed as L-borderenergetic graphs. According to him, a graph G of order n is said to be L-borderenergetic if $LE(G) = LE(K_n) = 2(n-1)$. Let S_n^1 be the graph obtained from an n-order star S_n by adding an edge between any two pendant vertices. Obviously, S_n^1 is an unicyclic and threshold graph. Deng et al. [3] have shown that S_n^1 is L-borderenergetic graph. Same authors [3] have established several characterizations on L-borderenergetic graphs with maximum degree at most 4.

Obviously there does not exist L-borderenergetic graph on two vertices. Hou and Tao [9] have proved that a L-borderenergetic graph on n vertices has at least n edges. As the only graph with three vertices are the paths P_3 or K_3, there does not exist a borderenergetic graphs on three vertices. By applying computer search, Hou and Tou [9] have obtained total 185 non isomorphic, non complete L-borderenergetic graphs of order upto 10. Elumalai and Rostami [4] corrected this number to 307 (see Table 1).

Table 1.

order	4	5	6	7	8	9	10
number	2	1	11	5	33	23	232

It is very interesting to investigate a graph or graph families which are L-borderenergetic because very few graphs are known to be L-borderenergetic. Here we have devised a procedure to construct a sequence of L borderenergetic graphs. We begin the next section with a definition and some existing results for the advancement of the discussion.

2. Main Result

Definition 2.1. The *join* of G_1 and G_2 is a graph $G = G_1 \vee G_2$ with vertex set $V(G_1) \cup V(G_2)$ and an edge set consisting of all the edges of G_1 and G_2 together with the edges joining each vertex of G_1 with every vertex of G_2.

Proposition 2.1 ([2]). *Let G_1 and G_2 be graphs of n_1 and n_2 vertices, respectively. If $\alpha_1, \alpha_2, \ldots, \alpha_{n_1-1}, \alpha_{n_1} = 0$ and $\beta_1, \beta_2, \ldots, \beta_{n_2-1}, \beta_{n_2} = 0$ be L-spectra of G_1 and G_2, respectively. Then the L-spectra of $G_1 \vee G_2$ are*

$$n_2 + \alpha_1, n_2 + \alpha_2, \ldots, n_2 + \alpha_{n_1-1}, n_1 + \beta_1, n_1 + \beta_2, \ldots, n_1 + \beta_{n_2-1}, n_1 + n_2, 0.$$

Theorem 2.1. *Let G be a L-borderenergetic graph of order n with average vertex degree $\bar{d} \in \mathbb{Z}$. Then for $p \neq 0$, $G \vee \overline{K_p}$ is L-borderenergetic if $p = n - \bar{d}$.*

Proof. Let $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n = 0$ be L-spectra of G. As G is L-borderenergetic of order n, $LE(G) = 2n - 2$, which implies that

$$\sum_{i=1}^{n} \left|\mu_i - \bar{d}\right| = 2n - 2.$$

Hence,

$$\sum_{i=1}^{n-1} \left|\mu_i - \bar{d}\right| = 2n - 2 - \bar{d}. \tag{2.1}$$

By Proposition 2.1, L-spectra of $G \vee \overline{K_p}$ is

$$\operatorname{spec}_L(G) = \begin{pmatrix} \mu_1 + p & \mu_2 + p & \cdots & \mu_{n-1} + p & n & n + p & 0 \\ 1 & 1 & \cdots & 1 & p - 1 & 1 & 1 \end{pmatrix}.$$

If $\bar{d}'$ is average vertex degree of newly constructed graph $G \vee \overline{K_p}$, then

$$\bar{d}' = \frac{n\bar{d} + 2np}{n + p}.$$

Note that for each $1 \leq i \leq n - 1$

$$\begin{aligned} \mu_i + p - \bar{d}' &= \mu_i + p - \frac{n\bar{d} + 2np}{p + n} \\ &= \mu_i - \bar{d} + \left(p + \bar{d} - \frac{n\bar{d} + 2np}{p + n}\right) \\ &= \mu_i - \bar{d} - \frac{p(n - p - \bar{d})}{p + n}. \end{aligned}$$

Now,

$$LE(G \vee \overline{K_p}) = \sum_{i=1}^{n-1} \left|\mu_i + p - \bar{d}'\right| + (p - 1)\left|n - \bar{d}'\right| + \left|n + p - \bar{d}'\right| + \left|\bar{d}'\right|$$

$$
\begin{aligned}
&= \sum_{i=1}^{n-1}\left|\mu_i - \bar{d} - \frac{p(n-p-\bar{d})}{p+n}\right| + (p-1)\left|n - \frac{n\bar{d}+2np}{n+p}\right| \\
&\quad + \left|n+p-\frac{n\bar{d}+2np}{n+p}\right| + \left|\frac{n\bar{d}+2np}{n+p}\right| \\
&= \sum_{i=1}^{n-1}\left|\mu_i - \bar{d} - \frac{p(n-p-\bar{d})}{p+n}\right| + (p-1)\left|\frac{n(n-p-\bar{d})}{n+p}\right| \\
&\quad + \left|p+\frac{n(n-p-\bar{d})}{n+p}\right| + \left|n-\frac{n(n-p-\bar{d})}{n+p}\right|.
\end{aligned}
$$

If $p = n - \bar{d}$, then

$$LE(G \vee \overline{K_p}) = \sum_{i=1}^{n-1}\left|\mu_i - \bar{d}\right| + |p| + |n|.$$

Therefore, by (2.1), $LE(G \vee \overline{K_p}) = 2n - 2 - \bar{d} + p + n = 2n + 2p - 2 = 2(n+p-1)$. Hence, $G \vee \overline{K_p}$ is L-borderenergetic. $\square$

3. Sequence of L-Borderenergetic Graphs

In this section we construct an infinite sequence of L-borderenergetic graphs. We term the graph under consideration as underlying graph. To construct the sequence we take any L-borderenergetic graphs of order n with average vertex degree $\bar{d} \in \mathbb{Z}$ as underlying graph and then the sequence is obtained by joining $n - \bar{d}$ vertices at each iteration.

Let $G^{(0)}$ is any L-borderenergetic graph of order n with average vertex degree $\bar{d} \in \mathbb{Z}$. Consider an infinite sequence of graphs $\mathcal{H} = \{G^{(0)}, G^{(1)}, \dots, G^{(k)}, \dots\}$ such that

$$G^{(1)} = G^{(0)} \vee \overline{K_{n-\bar{d}}},\ G^{(2)} = G^{(1)} \vee \overline{K_{n-\bar{d}}}, \dots, G^{(k)} = G^{(k-1)} \vee \overline{K_{n-\bar{d}}}, \dots$$

Note that each $G^{(k)}$ is of order $n+k(n-\bar{d})$ with average vertex degree $d_k = \bar{d}+k(n-\bar{d})$.

Lemma 3.1. *Let $G^{(0)}$ be a graph of order n with average vertex degree $\bar{d} \in \mathbb{Z}$ with Laplacian eigenvalues $\mu_1, \mu_2, \dots, \mu_{n-1}, \mu_n = 0$. Then for any $G^{(k)} \in \mathcal{H}$, $k \geq 1$, the Laplacian spectrum of $G^{(k)}$ is*

$$
\begin{aligned}
&\operatorname{spec}_L(G^{(k)}) \\
&= \begin{pmatrix} \mu_1 + k(n-\bar{d}) & \cdots & \mu_{n-1} + k(n-\bar{d}) & n+(k-1)(n-\bar{d}) & n+k(n-\bar{d}) & 0 \\ 1 & \cdots & 1 & k(n-\bar{d}-1) & k & 1 \end{pmatrix}.
\end{aligned}
$$

Proof. We prove this result by taking induction on k. From Theorem 2.1, it is clear that result is true for $k = 1$. Assume that the result is true for $k = s - 1$. Then by induction hypothesis

$$
\begin{aligned}
&\operatorname{spec}_L(G^{(s-1)}) \\
&= \begin{pmatrix} \mu_1 + (s-1)(n-\bar{d}) & \cdots & \mu_{n-1} + (s-1)(n-\bar{d}) & n+(s-2)(n-\bar{d}) & n+(s-1)(n-\bar{d}) & 0 \\ 1 & \cdots & 1 & (s-1)(n-\bar{d}-1) & (s-1) & 1 \end{pmatrix}.
\end{aligned}
$$

For $k = s$, $G^{(s)} = G^{(s-1)} \vee \overline{K_{n-\bar{d}}}$, from Proposition 2.1,

$$\operatorname{spec}_L(G^{(s)}) = \begin{pmatrix} \mu_1 + s(n-\bar{d}) & \cdots & \mu_{n-1} + s(n-\bar{d}) & n + (s-1)(n-\bar{d}) & n + s(n-\bar{d}) & 0 \\ 1 & \cdots & 1 & s(n-\bar{d}-1) & s & 1 \end{pmatrix}.$$

Thus, the result is true for all $s \in \mathbb{N}$. Hence, by induction the result follows. $\square$

Theorem 3.1. *For each $r \geq 1$, $G^{(k)} \in \mathcal{H}$ is L-borderenergetic with $K_{n+k(n-\bar{d})}$ for each $k \geq 1$.*

Proof. We have already shown that the order and average vertex degree of $G^{(k)}$ are $n + k(n-\bar{d})$ and $d_k = \bar{d} + k(n-\bar{d})$, respectively, for each $k \geq 1$.

$$\begin{aligned} LE(G^{(k)}) =& \sum_{i=1}^{n-1} \left|\mu_i + k(n-\bar{d}) - \bar{d} - k(n-\bar{d})\right| \\ &+ k(n-\bar{d}-1)\left|n + (k-1)(n-\bar{d}) - \bar{d} - k(n-\bar{d})\right| \\ &+ k\left|n + k(n-\bar{d}) - \bar{d} - k(n-\bar{d})\right| + \left|\bar{d} + k(n-\bar{d})\right| \\ =& \sum_{i=1}^{n-1} \left|\mu_i - \bar{d}\right| + k\left(n-\bar{d}\right) + \bar{d} + k(n-\bar{d}) \\ =& 2n - 2 - \bar{d} + 2k(n-\bar{d}) + \bar{d} \\ =& 2(n + k(n-\bar{d}) - 1) = LE(K_{n+k(n-\bar{d})}). \end{aligned}$$

Hence, $G^{(k)}$ is L-borderenergetic with $K_{n+k(n-\bar{d})}$ for each $k \geq 1$. $\square$

4. Some More Sequences From Known L-Borderenergetic Graphs

In this section we construct two infinite sequences of L-borderenergetic graphs $\mathcal{G}_i = \{G_i^{(0)}, G_i^{(1)}, \ldots, G_i^{(k)}, \ldots\} \subseteq \mathcal{H}$ for $i = 1, 2$, by taking some known L-borderenergetic graphs as underlying graph.

4.1. **The sequence of S_n^1.** Let $G_1^{(0)} = S_n^1$ be the graph obtained form n-order star S_n by adding a single edge. Note that S_n^1 is a graph of order n with average degree 2,

$$\operatorname{spec}_L(S_n^1) = \begin{pmatrix} 0 & 1 & 3 & n \\ 1 & n-3 & 1 & 1 \end{pmatrix}, \quad LE(G_1^{(0)}) = 2(n-1),$$

and thus it is L-borderenergetic with K_n. Consider an infinite sequence or borderenergetic graphs $\mathcal{G}_1 = \{G_1^{(0)}, G_1^{(1)}, G_1^{(2)}, \ldots, G_1^{(k)}, \ldots\}$ such that

$$G_1^{(1)} = G_1^{(0)} \vee \overline{K_{n-2}},\ G_1^{(2)} = G_1^{(1)} \vee \overline{K_{n-2}},\ G_1^{(3)} = G_1^{(2)} \vee \overline{K_{n-2}}, \ldots$$

The parameters $n, \bar{d}, LE$ of the sequence of S_n^1 are depicted in following Table 2.

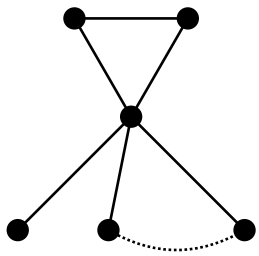

FIGURE 1. The graph S_n^1

TABLE 2.

G	**n**	$\overline{\mathbf{d}}$	***L-spectra***	$LE(G)$	***L-Borderenergetic With***
$G_1^{(0)}$	n	2	$0^1, 1^{(n-3)}, 3^1, n^1$	$2(n-1)$	K_n
$G_1^{(1)} = G_1^{(0)} \vee \overline{K_{n-2}}$	$2n-2$	n	$0^1, n^{(n-3)}, (n-1)^{(n-3)}, (n+1)^1, (2n-2)^2$	$2(2n-3)$	K_{2n-2}
$G_1^{(2)} = G_1^{(1)} \vee \overline{K_{n-2}}$	$3n-4$	$2n-2$	$0^1, (2n-2)^{(2n-6)}, (2n-3)^{(n-3)}, (2n-1)^1, (3n-4)^3$	$2(3n-5)$	K_{3n-3}
$G_1^{(3)} = G_1^{(2)} \vee \overline{K_{n-2}}$	$4n-6$	$3n-4$	$0^1, (3n-4)^{(3n-9)}, (3n-5)^{(n-3)}, (3n-3)^1, (4n-6)^4$	$2(4n-7)$	K_{4n-4}
$G_1^{(4)} = G_1^{(3)} \vee \overline{K_{n-2}}$	$5n-8$	$4n-6$	$0^1, (4n-6)^{(4n-12)}, (4n-7)^{(n-3)}, (4n-5)^1, (5n-8)^5$	$2(5n-9)$	K_{5n-5}
$G_1^{(5)} = G_1^{(4)} \vee \overline{K_{n-2}}$	$6n-10$	$5n-8$	$0^1, (4n-6)^{(5n-15)}, (4n-7)^{(n-3)}, (4n-5)^1, (5n-8)^6$	$2(6n-11)$	K_{6n-6}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

4.2. **The sequence of $K_{n-1} \odot K_n$.** For each integer $n \geq 3$, the graph $K_{n-1} \odot K_n$ is defined by

$$G = (K_{n-1} \cup K_{n-2}) \vee K_2.$$

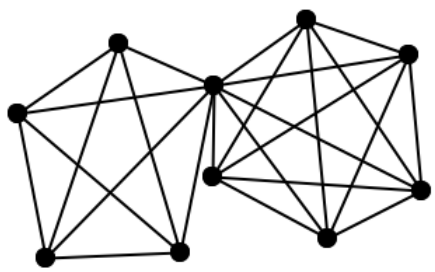

FIGURE 2. The graph $K_5 \odot K_6$

Tura [11] has proved that the $K_{n-1} \odot K_n$ is a graph with avrgare vertex degree $n-1$ and it is borderenergetic with K_{2n-2},

$$\mathrm{spec}_L(K_{n-1} \odot K_n) = \begin{pmatrix} 0 & 1 & n-1 & n & 2n-2 \\ 1 & 1 & n-3 & n-2 & 1 \end{pmatrix}, \quad LE(K_{n-1} \odot K_n) = 2(2n-3).$$

Consider an infinite sequence or borderenergetic graphs

$$\mathcal{G}_2 = \{G_2^{(0)}, G_2^{(1)}, G_2^{(2)}, \ldots, G_2^{(k)}, \ldots\},$$

such that

$$G_2^{(1)} = G_2^{(0)} \vee \overline{K_{n-1}}, \quad G_2^{(2)} = G_2^{(1)} \vee \overline{K_{n-1}}, \quad G_2^{(3)} = G_2^{(2)} \vee \overline{K_{n-1}}, \ldots$$

The parameters n, $\bar{d}$, LE of the sequence of borderenergetic graphs are depicted in following Table 3.

TABLE 3.

G	**n**	$\bar{d}$	L-**spectra**	**LE(G)**	L-**Borderenergetic With**
$G_2^{(0)}$	$2n-2$	$n-1$	$0^1, 1^1, (n-1)^{(n-3)}, n^{(n-2)}, (2n-2)^1$	$2(2n-3)$	K_{2n-2}
$G_2^{(1)} = G_2^{(0)} \vee \overline{K_{n-1}}$	$3n-3$	$2n-2$	$0^1, n^1, (2n-2)^{(2n-5)}, (2n-1)^{(n-2)}, (3n-3)^2$	$2(3n-4)$	K_{3n-3}
$G_2^{(2)} = G_2^{(1)} \vee \overline{K_{n-1}}$	$4n-4$	$3n-3$	$0^1, (2n-1)^1, (3n-3)^{(3n-7)}, (3n-2)^{(n-2)}, (4n-4)^3$	$2(4n-5)$	K_{4n-4}
$G_2^{(3)} = G_2^{(2)} \vee \overline{K_{n-1}}$	$5n-5$	$4n-4$	$0^1, (3n-2)^1, (4n-4)^{(4n-9)}, (4n-3)^{(n-2)}, (5n-5)^4$	$2(5n-6)$	K_{5n-5}
$G_2^{(4)} = G_2^{(3)} \vee \overline{K_{n-1}}$	$6n-6$	$5n-5$	$0^1, (4n-3)^1, (5n-5)^{(5n-11)}, (5n-4)^{(n-2)}, (6n-6)^5$	$2(6n-7)$	K_{6n-6}
$G_2^{(5)} = G_2^{(4)} \vee \overline{K_{n-1}}$	$7n-7$	$6n-6$	$0^1, (5n-4)^1, (6n-6)^{(6n-13)}, (6n-5)^{(n-2)}, (7n-7)^6$	$2(7n-8)$	K_{7n-7}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

5. CONCLUDING REMARKS

Here we have explored the concept of L-borderenergetic graphs which is analogous to the concept of borderenergetic graphs. We have investigated a sequence of L-borderenergetic graphs in the scenario when only handful graphs are known to be L-borderenergetic. The derived result is used for the construction of two sequences of L-borderenergetic graphs from the known L-borderenergetic graphs.

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ON THE APPLICATIONS OF BOCHNER-KODAIRA-MORREY-KOHN IDENTITY

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Abstract. This paper is devoted to studying some applications of the Bochner-Kodaira-Morrey-Kohn identity. For this study, we define a condition which is called (H_q) condition which is related to the Levi form on the complex manifold. Under the (H_q) condition and combining with the basic Bochner-Kodaira-Morrey-Kohn identity, we study the L^2 $\overline{\partial}$ Cauchy problems on domains in $\mathbb{C}^n$, Kähler manifold and in projective space. Also, we study this problem on a piecewise smooth strongly pseudoconvex domain in a complex manifold. Furthermore, the weighted L^2 $\overline{\partial}$ Cauchy problem is studied under the same condition in a Kähler manifold with semi-positive holomorphic bisectional curvature. On the other hand, we study the global regularity and the L^2 theory for the $\overline{\partial}$-operator with mixed boundary conditions on an annulus domain in a Stein manifold between an inner domain which satisfy (H_{n-q-1}) and an outer domain which satisfy (H_q).

1. Introduction

$\overline{\partial}$-equation has been a powerful method in complex geometry and complex algebraic geometry. The pioneer works of $\overline{\partial}$-equation are due to Kohn and Hörmander on the existence and boundary regularity of $\overline{\partial}$-equation on pseudoconvex domain. The L^2 estimate is a powerful method when solving $\overline{\partial}$-equation. In order to establish the L^2 estimate for $\overline{\partial}$ operator, a crucial step is to obtain a basic identity which is due to Bochner, Kodaira, Morrey, Kohn and Hörmander. Then after imposing some conditions such as the curvature of the complex vector bundle and the Levi form of the complex manifold, one can get a priori estimate which we call L^2 estimate for the $\overline{\partial}$ operator.

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In 1940, Bochner introduced his technique for getting topological information from the behavior of harmonic forms. In the late of 1950 and early of 1960, the approaches of Bochner-Kodaira and Oka is very deep approach based in partial differential equations. In 1966, Spencer defined the $\overline{\partial}$-Neumann problem and Kohn who finally formulated and solved the $\overline{\partial}$-Neumann problem on strictly pseudoconvex domains. Shortly after Kohn's work, Hörmander and Andreotti-Vesentini, independently and almost simultaneously, obtained weighted L^2 estimates for the inhomogeneous Cauchy-Riemann equations.

The paper consists of two parts. In the first part, we study the Cauchy-Riemann equations with compact support (L^2 Cauchy problem) on a domain in $\mathbb{C}^n$ which satisfy property (H_q). Moreover, we extend this result to the same domain in Kähler manifold for vector-valued forms of type (r,q), $q \geqslant 1$. Also, we study this problem on a piecewise smooth strongly pseudoconvex domain in a complex manifold. Furthermore, the weighted L^2 $\overline{\partial}$ Cauchy problem is studied on the same domains in an n-dimensional Kähler manifold with semi-positive holomorphic bisectional curvature. In the papers of Kohn-Rossi [13], Cao-Shaw-Wang [2], Abdelkader-Saber [1] and Saber [18]-[21] such equations are studying for various domains.

In the second part of this paper, we study the $\overline{\partial}$ equation on domains with mixed boundary conditions which was studied by Li and Shaw [16]. We generalize Li-Shaw's result to annulus in Stein manifolds under the conditions (H_q) defined here. Namely, we study the global boundary regularity and the L^2 theory for $\overline{\partial}_{mix}$-operator on an annulus domain in a Stein manifold between an internal domain which satisfies condition (H_{n-q-1}) and an external domain which satisfies condition (H_q). Making use of the method developed by Catlin, we study the L^2-estimate for the $\overline{\partial}_{mix}$-equation with mixed boundary conditions. This equation with various boundary conditions are the basic tools to work on analytic problems in several complex variables and complex geometry or many geometric. In the papers of Catlin [3], Cho [6] and Catlin-Cho [4], such equations played a crucial role for studying various extension problems for CR structures. In a paper of Catlin [3], Catlin proved that there is no cohomology obstruction for solving the $\overline{\partial}_{mix}$-equations. Other related studies for the $\overline{\partial}$-Dirichlet problem can be found in the work of Li-Shaw [16] and Huang-Li [11].

2. L^2 $\overline{\partial}$ Cauchy Problem in $\mathbb{C}^n$

Let $0 \leqslant r, q \leqslant n$, we can write an arbitrary (r,q)-form f as $f = \sum'_{I,J} f_{I,J}\, dz^I \wedge d\bar{z}^J$, where $I = (i_1, \ldots, i_r)$ and $J = (j_1, \ldots, j_q)$ are multiindices and $dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_r}$, $d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$. The notation $\sum'$ means the summation over strictly increasing multiindices. Let Ω be a relatively compact domain in $\mathbb{C}^n$ and let $C^\infty_{r,q}(\mathbb{C}^n)$ be the complex vector space of complex-valued differential (r,q)-forms of class C^∞ on $\mathbb{C}^n$. Let $C^\infty_{r,q}(\overline{\Omega}) = \left\{ f\,|_{\overline{\Omega}} :\ f \in C^\infty_{r,q}(\mathbb{C}^n) \right\}$ be the subspace of $C^\infty_{r,q}(\Omega)$ whose elements can be extended smoothly up to the boundary $b\Omega$. Let $\phi : \mathbb{C}^n \to \mathbb{R}^+$ be a plurisubharmonic

C^2-weight function and define the space

$$L^2(\Omega,\phi)=\left\{f:\Omega\longrightarrow\mathbb{C}:\int_\Omega |f|^2e^{-\phi}dV<\infty\right\},$$

where dV denotes the Lebesgue measure. Denote the inner product and the norm in $L^2(\Omega,\phi)$ by

$$\langle f,g\rangle_\phi=\int_\Omega f\,\overline{g}\,e^{-\phi}dV \text{ and } \|f\|_\phi=\int_\Omega |f|^2e^{-\phi}dV.$$

Recall that $L^2_{r,q}(\Omega,\phi)$ the space of (r,q)-forms with coefficients in $L^2(\Omega,\phi)$. If $f,g\in L^2_{r,q}(\Omega,\phi)$, the L^2-inner product and norms are defined by

$$\langle f,g\rangle_\phi=\int_\Omega f\wedge\star\overline{g}\,e^{-\phi} \text{ and } \|f\|^2_\phi=\langle f,f\rangle_\phi,$$

where $\star$ is the Hodge star operator (for detailed discussions of the Hodge star operator in the L^2-space see [5]).

Let $\overline{\partial}:L^2_{r,q-1}(\Omega,\phi)\to L^2_{r,q}(\Omega,\phi)$ be the closed operator which is the maximal extension of the differential operator and $\overline{\partial}^*$ be its L^2-adjoint. Here the $\overline{\partial}$ and $\overline{\partial}^*$-operators are defined as

$$\overline{\partial}f={\sum_{I,J}}'\sum_{\beta=1}^n\frac{\partial f_{I,J}}{\partial\overline{z}^\beta}\,d\overline{z}^\beta\wedge dz^I\wedge d\overline{z}^J,$$

$$\overline{\partial}^*f=(-1)^{r-1}{\sum_{I,K}}'\sum_{\alpha=1}^n\frac{\partial f_{I,\alpha K}}{\partial z^\alpha}\,dz^I\wedge d\overline{z}^K.$$

If $w\in\operatorname{dom}\overline{\partial}$ and $u\in\operatorname{dom}\overline{\partial}^*$, then

$$\langle\overline{\partial}w,u\rangle_\phi=\langle\overline{\partial}w,e^{-\phi}u\rangle=\langle w,\overline{\partial}^*(e^{-\phi}u)\rangle=\langle w,e^{\phi}\overline{\partial}^*(e^{-\phi}u)\rangle_\phi.$$

Thus, $\overline{\partial}^*_\phi=e^\phi\overline{\partial}^*(e^{-\phi})$. The complex Laplacian on (r,q)-forms is defined as $\Box_{r,q}=\overline{\partial}\,\overline{\partial}^*+\overline{\partial}^*\overline{\partial}$, for $q\geqslant 1$ and the $\overline{\partial}$-Neumann operator $N_{r,q}$ is - if it exists-the bounded inverse of $\Box_{r,q}$. Following ([12, I, page 127]), we set $\mathcal{B}_{r,q}(\overline{\Omega})=C^\infty_{r,q}(\overline{\Omega})\cap\operatorname{dom}\overline{\partial}^*$. Let $\mathcal{B}_{r,q}(U)$ denote the subset of $\mathcal{B}_{r,q}(\overline{\Omega})$ consisting of those forms whose support lies in $U\cap\overline{\Omega}$. We define the following norms on $\mathcal{B}_{r,q}(U)$:

$$E(u)={\sum_{I,J}}'\sum_{k=1}^n\int_\Omega\left|\frac{\partial u_{I,J}}{\partial\overline{z}^k}\right|^2dV+\|u\|^2, \tag{2.1}$$

$$D(u)=\|\overline{\partial}u\|^2+\|\overline{\partial}^*u\|^2+\|u\|^2.$$

Definition 2.1. A bounded domain Ω with C^2-smooth boundary is said to be satisfy condition (H_q) if there exists a defining function δ for Ω such that the sum s_q of any q eigenvalues of the matrix $\left(\frac{\partial^2\delta}{\partial z^\alpha\partial\overline{z}^\beta}\right)$ of the Levi form $\partial\overline{\partial}\delta(z)$ is semi-positive on $\overline{\Omega}$.

Proposition 2.1 (Bochner-Hörmander-Kohn-Morrey formula)**.** *Let Ω be a bounded domain in $\mathbb{C}^n$ with defining function δ and let ψ, ϕ be two real functions that are twice* *continuously differentiable on $\overline{\Omega}$, with $\psi \geqslant 0$. Then, for $u \in \mathcal{B}_{r,q}(\overline{\Omega})$ with $1 \leqslant q \leqslant n-1$, we have*

$$\begin{aligned}(2.2)\quad &\|\sqrt{\psi}\,\overline{\partial}u\|_\phi^2 + \|\sqrt{\psi}\,\overline{\partial}_\phi^* u\|_\phi^2\\ &= \sum_{I,K}{}' \sum_{\alpha,\beta=1}^{n} \int_{b\Omega} \psi \frac{\partial^2 \delta}{\partial z^\alpha \partial \overline{z}^\beta}\, u_{I,\alpha K}\, \overline{u}_{I,\beta K}\, e^{-\phi}\, dS\\ &+ \sum_{I,J}{}' \sum_{k=1}^{n} \int_\Omega \psi \left|\frac{\partial u_{I,J}}{\partial \overline{z}^k}\right|^2 e^{-\phi} dV + 2\mathrm{Re}\left(\sum_{I,K}{}' \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial z^\alpha}\, u_{I,\alpha K}\, dz^I \wedge d\overline{z}^K, \overline{\partial}_\phi^* u\right)_\phi\\ &+ \sum_{I,K}{}' \sum_{\alpha,\beta=1}^{n} \int_\Omega \left(\psi \frac{\partial^2 \phi}{\partial z^\alpha \partial \overline{z}^\beta} - \frac{\partial^2 \psi}{\partial z^\alpha \partial \overline{z}^\beta}\right) u_{I,\alpha K}\, \overline{u}_{I,\beta K}\, e^{-\phi}\, dV.\end{aligned}$$

Remark 2.1. When $\psi \equiv 1$ and $\phi \equiv 0$, one obtains the classical Kohn-Morrey formula ([12] and [9]).

Proposition 2.2. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Then, for $u \in \mathcal{B}_{r,q}(\overline{\Omega})$, one obtains*

$$(2.3)\qquad E(u) \leqslant C D(u).$$

Moreover, there exists a uniquely determined bounded linear operator $N_{r,q} : L^2_{r,q}(\Omega) \to L^2_{r,q}(\Omega)$, such that $\Box_{r,q} \circ N_{r,q} u = u$ for any $u \in L^2_{r,q}(\Omega)$.

Proof. Let $\mu_1 \leqslant \mu_2 \leqslant \cdots \leqslant \mu_n$ denote the eigenvalues of the matrix $\left(\frac{\partial^2 \delta}{\partial z^\alpha \partial \overline{z}^\beta}\right)$ and suppose that $\left(\frac{\partial^2 \delta}{\partial z^\alpha \partial \overline{z}^\beta}\right)$ is diagonalized. Then, in a suitable basis,

$$\begin{aligned}\sum_{I,K}{}' \sum_{\alpha,\beta=1}^{n-1} \frac{\partial^2 \delta}{\partial z^\alpha \partial \overline{z}^\beta} u_{I,\alpha K} \overline{u_{I,\beta K}} &= \sum_{I,K}{}' \sum_{\alpha=1}^{n} \mu_\alpha |u_{I,\alpha K}|^2\\ &= \sum_{\substack{|I|=r\\ J=(j_1,j_2,\ldots,j_q)}}{}' (\mu_1 + \mu_2 + \cdots + \mu_q)|u_{I,J}|^2 \geqslant s_q |u|^2.\end{aligned}$$

The second equality follows as. For $J = (j_1, j_2, \ldots, j_q)$ fixed, $|u_{I,J}|^2$ occurs precisely q times in the second sum, once as $|u_{I,j_1 K}|^2$, once as $|u_{I,j_2 K}|^2$, etc. At each occurrence, it is multiplied by μ_{jl} . By fixing $(j_1, j_2, \ldots, j_q)$ and set $u = d\overline{z}_{j_1} \wedge \cdots \wedge d\overline{z}_{j_q}$, we obtain

$$\sum_{I,K}{}' \sum_{\alpha,\beta=1}^{n-1} \frac{\partial^2 \delta}{\partial z^\alpha \partial \overline{z}^\beta} u_{I,\alpha K} \overline{u}_{I,\beta K} \geqslant 0.$$

Thus, the boundary integral in (2.2) is semi-positive. Also, by taking $\phi \equiv 0$ and replace ψ by $1 - e^\lambda$, where λ is an arbitrary twice continuously differentiable non-positive function, and after applying the Cauchy-Schwarz inequality to the term in (2.2) involving first derivatives of ψ, we find

$$\|\sqrt{1-e^\lambda}\,\overline{\partial}u\|^2 + \|\sqrt{1-e^\lambda}\,\overline{\partial}^* u\|^2 \geqslant \sum_{I,K}{}' \sum_{\alpha,\beta=1}^{n} \int_\Omega e^\lambda \frac{\partial^2 \lambda}{\partial z^\alpha \partial \overline{z}^\beta}\, u_{I,\alpha K}\, \overline{u}_{I,\beta K}\, dV - \|e^{\lambda/2}\,\overline{\partial}^* u\|^2.$$

Since $\psi + e^{\lambda} = 1$ and $\psi \leqslant 1$, it follows that

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \geqslant \sum_{I,K}{}' \sum_{\alpha,\beta=1}^{n} \int_\Omega e^{\lambda} \frac{\partial^2 \lambda}{\partial z^\alpha \partial \overline{z}^\beta} \, u_{I,\alpha K} \, \overline{u}_{I,\beta K} \, dV,$$

for every twice continuously differentiable non-positive function λ. Let z_0 be a point of Ω, and set $\lambda(z) = -1 + |z - z_0|^2/d^2$, where $d = \sup\limits_{z,z' \in \Omega} |z - z'|$ is the diameter of the bounded domain Ω. The preceding inequality then implies the fundamental estimate (2.3) which implies the following estimate

$$\|u\|^2 \leqslant \frac{d^2 e}{q} \left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \right).$$

Then, a bounded linear operator $N_{r,q} : L^2_{r,q}(\Omega) \to L^2_{r,q}(\Omega)$ exists, such that $\Box_{r,q} \circ N_{r,q} u = u$, for any $u \in L^2_{r,q}(\Omega)$. $\square$

Theorem 2.1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). For $f \in L^2_{r,q}(\mathbb{C}^n)$, $1 \leqslant q \leqslant n-1$, $\operatorname{supp} f \subset \overline{\Omega}$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{C}^n$, there exists $u \in L^2_{r,q-1}(\mathbb{C}^n)$, $\operatorname{supp} u \subset \overline{\Omega}$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}^n$.*

Proof. Let $f \in L^2_{r,q}(\mathbb{C}^n)$, $\operatorname{supp} f \subset \overline{\Omega}$, then $f \in L^2_{r,q}(\Omega)$. From Proposition 1.2, $N_{n-r,n-q}$ exists for $n - q \geqslant 1$. Since

$$u = - \star \overline{\overline{\partial} \, N_{n-r,n-q} \star \overline{f}}. \tag{2.4}$$

Thus, $\operatorname{supp} u \subset \overline{\Omega}$ and u vanishes on $b\Omega$. Now, we extend u to $\mathbb{C}^n$ by defining $u = 0$ in $\mathbb{C}^n \setminus \Omega$. As in Saber [18], the extended form u satisfies the equation $\overline{\partial} u = f$ in the distribution sense in $\mathbb{C}^n$. $\square$

3. The L^2 $\overline{\partial}$ Cauchy Problem for Vector-Valued Forms

Let X be an n-dimensional complex manifold with a Hermitian metric g. We assume that there is a smooth, real-valued function δ defined on a neighborhood U of $b\Omega$ in X. Assume that E is a holomorphic vector bundle, of rank p, over X and E^* its dual. An E-valued differential (r,q)-form u on X is given locally by a column vector ${}^t u = (u^1, u^2, \dots, u^p)$, where u^a; $1 \leqslant a \leqslant p$, are $\mathbb{C}$-valued differential forms of type (r,q) on X. For an orthonormal basis $e_1, e_2, \dots, e_p$ on the fiber $E_z = \pi^{-1}(z)$, over z, a Hermitian metric h along the fibers of E is expressed as $h = (h_{a\overline{b}})$; $h_{a\overline{b}} = h(e_a, e_b)$. Let θ be the connection of the Hermitian metric h (θ is given locally by the $(1,0)$-form $h^{-1}\partial h$). The space $L^2_{r,q}(\Omega, E)$ of square integrable differential forms of type (r,q) on Ω is a Hilbert space under the scalar product

$$\langle u, v \rangle = \int_\Omega {}^t((h)u) \wedge \star \overline{v} = \sum_{a=1}^{p} \int_\Omega ((h)u)^a \wedge \star \overline{(v)^a}.$$

Let $\ker(\overline{\partial}, E) = \{u \in \operatorname{dom}(\overline{\partial}, E) : \overline{\partial} u = 0\}$ and $\operatorname{Range}(\overline{\partial}, E) = \{\overline{\partial} u : u \in \operatorname{dom}(\overline{\partial}, E)\}$ be the kernel and the range of $\overline{\partial}$, respectively. Let $\#_E : C^\infty_{r,q}(X, E) \to$

$C^\infty_{q,r}(X,E^*)$ be the operator defined by $\#_E\, u = \overline{h}\overline{u}$, which commutes with the Hodge star operator. The corresponding operator $\#_{E^*} : C^\infty_{r,q}(X,E^*) \to C^\infty_{q,r}(X,E)$ is defined by $\#_{E^*}\, u = h^{-1}\text{-}u = \#_E^{-1}\, u$. Let $\Pi_{r,q} : L^2_{r,q}(\Omega,E) \to \ker(\square_{r,q},E)$ be the orthogonal projection from the space $L^2_{r,q}(\Omega,E)$ onto the space $\ker(\square_{r,q},E)$. Let $C^\infty_{r,q}(\overline{\Omega},E) = \left\{\varphi\,|_{\overline{\Omega}}\colon\, u \in C^\infty_{r,q}(X,E)\right\}$ be the subspace of $C^\infty_{r,q}(\Omega,E)$ whose elements can be extended smoothly up to $b\Omega$. As in [12, I, page 127], that even for vector bundles, we set $\mathcal{B}_{r,q}(\overline{\Omega},E) = C^\infty_{r,q}(\overline{\Omega},E) \cap \operatorname{dom}(\overline{\partial}^*,E)$. Let $\mathcal{B}_{r,q}(U,E)$ denote the subset of $\mathcal{B}_{r,q}(\overline{\Omega},E)$ consisting of those forms whose support lies in $U \cap \overline{\Omega}$. For each $\tau \geqslant 0$, we define the following norms on $\mathcal{B}_{r,q}(U,E)$:

$$\begin{aligned} \|u\|^2_\tau &= \langle u, e^{\tau|\delta|}u\rangle, \\ \widetilde{D}^2_\tau(u) &= \|\overline{\partial}u\|^2_\tau + \|\overline{\partial}^*_\tau u\|^2_\tau + \|u\|^2_\tau, \\ \widetilde{E}_\tau(u) &= \sum_{a=1}^{p} E_\tau(u^a), \end{aligned} \tag{3.1}$$

where E_τ is defined by (2.1) for complex-valued forms.

Theorem 3.1. *Let X be an n-dimensional complex manifold and let $\Omega \Subset X$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Let $E \to X$ be a holomorphic vector bundle over X. Then, for all $\tau \geqslant 0$, and $u = (u^1, u^2, \ldots, u^p) \in \mathcal{B}_{r,q}(U,E)$, we obtain*

$$\widetilde{E}^2_\tau(u) + \tau\|u\|^2_\tau \leqslant C\left(\widetilde{D}^2_\tau(u) + \tau^2 \sum_{a=1}^{p} \sideset{}{'}\sum_{\substack{|I|=r\\ n\in J}} \|u^a_{IJ}\|^2_\tau\right).$$

Proof. First, observe that for elements $u = (u^1, u^2, \ldots, u^p) \in \mathcal{B}_{r,q}(E)$ which have support in U, the norms $\|u\|^2_\tau$ and $\sum_{a=1}^{b} \|u^a\|^2_\tau$ are equivalent (independently of τ). If the metric h is represented by the matrix (h_{ab}), then

$$\|u\|^2 = \int_\Omega {}^t((h)u) \wedge \star\overline{u} = \sum_{a,b=1}^{n} \int_\Omega h_{ab}\, u^a \wedge \star\, \overline{(u)^b},$$

where the h_{ab} can be assumed to be C^∞ on $\overline{U}$. Now, for $u \in \mathcal{B}_{r,q}(U,E)$, $u = (u^1, u^2, \ldots, u^p) \in \mathcal{B}(U,E)$ with $u^a \in \mathcal{B}_{r,q}(U)$, $a = 1, 2, \ldots, p$. From (2.3), one obtains

$$\widetilde{E}^2_\tau(u) = \sum_{a=1}^{p} E_\tau(u^a) \leqslant C \sum_{a=1}^{p} \left(D^2_\tau(u^a) + \tau^2 \sideset{}{'}\sum_{\substack{|I|=r\\ n\in J}} \|u^a_{IJ}\|^2_\tau\right). \tag{3.2}$$

Now, since θ is a C^∞ form on $\overline{U}$ and $\vartheta_\tau u = \vartheta u - \star\theta \wedge \star u$, one obtains

$$C\sum_{a=1}^{p} \|\vartheta u^a\|^2_\tau \leqslant \|\vartheta u\|^2_\tau \leqslant \|\vartheta_\tau u\|^2_\tau + \|\theta \wedge \star u\|^2_\tau \leqslant \|\vartheta_\tau u\|^2_\tau + C'\|u\|^2_\tau.$$

Thus,

$$\begin{aligned}(1+C')\widetilde{D}^2_\tau(u) &\geqslant \|\overline{\partial}u\|^2_\tau + \|\vartheta_\tau u\|^2_\tau + C'\|u\|^2_\tau + \|u\|^2_\tau \\ &\geqslant C\sum_{a=1}^{p}\left(\|\overline{\partial}u^a\|^2_\tau + \|\vartheta u^a\|^2_\tau + \|u^a\|^2_\tau\right).\end{aligned}$$

Thus,

$$(1+C')\widetilde{D}^2_\tau(u) \geqslant C\sum_{a=1}^{p} D^2_\tau(u^a). \tag{3.3}$$

The inequalities (3.2) and (3.3) give the desired result. □

Theorem 3.2. *Let X be an n-dimensional complex manifold and let $\Omega \Subset X$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Let $E \to X$ be a holomorphic vector bundle over X. For $q \geqslant 1$, there exists a bounded linear operator $N_{r,q} : L^2_{r,q}(\Omega, E) \to L^2_{r,q}(\Omega, E)$ such that*

(i) Range $(N_{r,q}, E) \subset$ dom$(\square_{r,q}, E)$, $N_{r,q}\square_{r,q} = I - \Pi_{r,q}$ *on* dom$(\square_{r,q}, E)$*;*

(ii) for $u \in L^2_{r,q}(\Omega, E)$, we have $u = \overline{\partial}\,\overline{\partial}^* N_{r,q}u \oplus \overline{\partial}^*\overline{\partial} N_{r,q}u \oplus \Pi_{r,q}u$;

(iii) $N_{r,q}$ commutes with $\overline{\partial}$ and $\overline{\partial}^*$, $\Pi_{r,q}N_{r,q} = N_{r,q}\Pi_{r,q} = 0$;

(iv) $N_{r,q}(C^\infty_{r,q}(\overline{\Omega}, E) \subset C^\infty_{r,q}(\overline{\Omega}, E)$ and $\Pi_{r,q}(C^\infty_{r,q}(\overline{\Omega}, E)) \subset C^\infty_{r,q}(\overline{\Omega}, E)$.

Proof. Following Theorem 2.1, for $\tau = 0$ and $u \in$ dom$(\square_{r,q}, E)$ of degree $q \geqslant 1$, we have $\|u\|^2 \leqslant C\|\square_{r,q}u\|^2$. Since $\square_{r,q}$ is one to one on dom$(\square, E)$ from [9, (1.5.3)], then there exists a unique bounded inverse operator $N_{r,q}$: Range$(\square_{r,q}, E) \to$ dom$(\square_{r,q}, E) \cap (\ker(\square, E))^\perp$ such that $N_{r,q}\square_{r,q}u = u$ on dom$(\square_{r,q}, E)$. Thus, one can establish the existence theorem of the inverse of $\square_{r,q}$ the so called $\overline{\partial}$-Neumann operator $N_{r,q}$. □

Theorem 3.3. *Let X be an n-dimensional complex manifold and let $\Omega \Subset X$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Let $E \to X$ be a holomorphic vector bundle over X. Then, for $f \in L^2_{r,q}(X, E)$,* supp $f \subset \overline{\Omega}$, *$1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial}f = 0$ in the distribution sense in X, there exists $u \in L^2_{r,q-1}(X, E)$,* supp $u \subset \overline{\Omega}$ *such that $\overline{\partial}u = f$ in the distribution sense in X.*

Proof. Let $f \in L^2_{r,q}(X, E)$, supp $f \subset \overline{\Omega}$, then $f \in L^2_{r,q}(\Omega, E)$. From Theorem 2.2, $N_{n-r,n-q}$ exists for $n-q \geqslant 1$. Since $N_{n-r,n-q} = (\square_{n-r,n-q})^{-1}$ on Range$(\square_{n-r,n-q}, E^*)$ and Range$(N_{n-r,n-q}, E^*) \subset$ dom$(\square_{n-r,n-q}, E^*)$, then

$$N_{n-r,n-q}\#_E \star f \in \mathrm{dom}\,(\square_{n-r,n-q}, E^*) \subset L^2_{n-r,n-q}(\Omega, E^*),$$

for $q \leqslant n-1$. Thus, we can define $u \in L^2_{r,q-1}(\Omega, E)$ by

$$u = -\star \#_{E^*}\overline{\partial}\, N_{n-r,n-q}\#_E \star f. \tag{3.4}$$

Now, we extend u to X by defining $u = 0$ in $X \setminus \overline{\Omega}$. As in [1], the extended form u satisfies the equation $\overline{\partial}u = f$ in the distribution sense in X. □

4. The L^2 $\overline{\partial}$ Cauchy Problem on Piecewise Smooth Strongly Pseudoconvex Domains

A relatively compact open subset Ω of X has piecewise strongly pseudoconvex boundary $b\Omega$, if $b\Omega$ is covered by finitely many open subsets $\{U_j\}$, $1 \leqslant j \leqslant k$, of X and there are C^2 strictly plurisubharmonic functions δ_j on $\{U_j\}$, $1 \leqslant j \leqslant k$, such that $\Omega \cap (\bigcup_{j=1}^k U_j)$ is the set of all $x \in \bigcup_{j=1}^k U_j$ which, for every $1 \leqslant j \leqslant k$, satisfy $x \notin U_j$ or $\delta_j(x) < 0$.

The boundary $b\Omega$ need not be piecewise smooth, so we do not require any further conditions on the δ_j, $1 \leqslant j \leqslant k$. Following P. W. Darko [7], one have the following result.

Theorem 4.1. *Let X be an n-dimensional complex manifold with a C^∞ Hermitian metric. Let $\Omega \Subset X$ be a strongly pseudoconvex domain with piecewise smooth boundary. For $f \in L^2_{r,q}(X)$, $1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial} f = 0$ in the distribution sense in X, there exists $u \in L^2_{r,q-1}(X)$, such that $\overline{\partial} u = f$ in the of distributions and $\|u\|^2 \leqslant C\|f\|^2$, where C depends on Ω and r, q but not on f.*

Proposition 4.1. *Let X and Ω be the same as in Theorem 3.1. Then, there exists a uniquely bounded linear operator $N_{r,q} : L^2_{r,q}(\Omega) \to L^2_{r,q}(\Omega)$, such that $\Box_{r,q} \circ N_{r,q} u = u$ for any $u \in L^2_{r,q}(\Omega)$.*

Proof. Following Theorem 3.1 as in [5, Section 4.4], we have $\mathcal{H}_{r,q}(\Omega) = \{0\}$ for $q > 0$ and for every $f \in L^2_{r,q}(\Omega)$, $1 \leqslant q \leqslant n-1$, there exists $u \in \operatorname{dom} \overline{\partial} \cap \operatorname{dom} \overline{\partial}^*$ with $\overline{\partial}^* u \in \operatorname{dom} \overline{\partial}$ and $\overline{\partial} u \in \operatorname{dom} \overline{\partial}^*$, such that

$$\overline{\partial}\, \overline{\partial}^* u + \overline{\partial}^* \overline{\partial} u = f \ \text{ and } \ \|u\|^2 \leqslant C\|f\|^2,$$

where C depends on Ω and r, q but not on f. Theorem 3.1 implies the fundamental estimate

$$\|u\|^2 \leqslant C \left(\|\overline{\partial} u\|^2 + \|\overline{\partial}^* u\|^2 \right).$$

Then there exists a uniquely determined bounded linear operator $N_{r,q} : L^2_{r,q}(\Omega) \to L^2_{r,q}(\Omega)$, such that $\Box_{r,q} \circ N_{r,q} u = u$ for any $u \in L^2_{r,q}(\Omega)$. □

Theorem 4.2. *Let X and Ω be the same as in Theorem* 3.1*. For $f \in L^2_{r,q}(X)$, $1 \leqslant q \leqslant n-1$, $\operatorname{supp} f \subset \overline{\Omega}$, satisfying $\overline{\partial} f = 0$ in the distribution sense in X, there exists $u \in L^2_{r,q-1}(X)$, $\operatorname{supp} u \subset \overline{\Omega}$ such that $\overline{\partial} u = f$ in the distribution sense in X.*

Proof. Let $f \in L^2_{r,q}(X)$, $\operatorname{supp} f \subset \overline{\Omega}$, then $f \in L^2_{r,q}(\Omega)$. Since $N_{n-r,n-q}$ exists for $n - q \geqslant 1$. By defining u as in (2.4), then $\operatorname{supp} u \subset \overline{\Omega}$ and u vanishes on $b\Omega$. As in Saber [20], one can prove that the extended form u satisfies the equation $\overline{\partial} u = f$ in the distribution sense in X. □

5. The Weighted L^2 $\overline{\partial}$ Cauchy Problem

Let $(x_0, x_1, \ldots, x_n)$ be a (fixed) homogeneous coordinates of $\mathbb{P}^n$ and let ω be the Fubini-Study metric of the complex projective space of $\mathbb{P}^n$ determined by

$(x_0, x_1, \dots, x_n)$. If, for example, U_0 is the open set in $\mathbb{P}^n$ defined by $x_0 \neq 0$ and if $(z_1, z_2, \dots, z_n)$, where $z_i = x_i/x_0$ is the homogeneous coordinates of U_0, then the metric ω is written in the form

$$\omega = \frac{\sum_{i=1}^n |\, dz_i|^2}{1 + \sum_{i=1}^n |z_i|^2} - \frac{|\sum_{i=1}^n z_i \, d\overline{z}_i|^2}{(1 + \sum_{i=1}^n |z_i|^2)^2}, \quad \text{on } U_0.$$

This is well-known standard Kähler metric of $\mathbb{P}^n$. Let ∇ be the Levi-Civita connection of $\mathbb{P}^n$ with the standard Fubini-Study metric ω. The Levi-Civita connection, sometimes also known as the Riemannian connection or covariant derivative. Let $\{e_i\}$ be an orthonormal basis of vector fields. For any two vector fields u, v, the curvature operator of the connection ∇ is given by $\mathcal{R}(u,v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u,v]}$. with $\mathcal{R}_{ijkl} = g(\mathcal{R}(e_i, e_j)e_k, e_l)$. We also define the Ricci tensor $\mathcal{R}_{ij} = \sum_k \varepsilon_k \mathcal{R}_{ikkj}$ which turns out to be self-adjoint with respect to ω and the scalar curvature Θ as the trace of the Ricci tensor

$$\Theta = \sum_i \mathcal{R}_{ii} = \sum_{i,j} \varepsilon_i \varepsilon_j \mathcal{R}_{jiij}. \tag{5.1}$$

Let $\operatorname{dist}(z, b\Omega)$ be the Fubini distance from $z \in \Omega$ to the boundary $b\Omega$ and let $\delta : \mathbb{P}^n \to \mathbb{R}$ be a C^2 defining function for Ω normalized by $|d\delta| = 1$ on $b\Omega$ such that

$$\delta = \delta(z) = \begin{cases} ll - \operatorname{dist}(z, b\Omega), & \text{if } z \in \Omega, \\ \operatorname{dist}(z, b\Omega), & \text{if } z \in \mathbb{P}^n \backslash \Omega, \end{cases}$$

where δ is computed with respect to the Kähler metric ω on $\mathbb{P}^n$.

Proposition 5.1 (Bochner-Hörmander-Kohn-Morrey formula). *Let Ω be a bounded domain with C^2-smooth boundary $b\Omega$ and C^2-defining function $\delta(z)$. Then, for any $u \in C^\infty_{r,q}(\overline{\Omega}) \cap \operatorname{dom} \overline{\partial}^*_\phi$, with $1 \leqslant q \leqslant n-1$ and $\phi \in C^2(\overline{\Omega})$, we have*

$$\begin{aligned} \| \overline{\partial} u\|^2_\phi + \| \overline{\partial}^*_\phi u\|^2_\phi =& \langle \Theta u, \overline{u} \rangle_\phi + \sum_{I,J} \sum_{k=1}^n \int_\Omega \left| \frac{\partial u_{I,J}}{\partial \overline{z}^k} \right|^2 e^{-\phi} dV + \langle (i\partial\overline{\partial}\phi) u, \overline{u} \rangle_\phi \\ &+ \sum_{I,K} \sum_{\alpha,\beta=1}^n \int_{b\Omega} \frac{\partial^2 \delta}{\partial z^\alpha \partial \overline{z}^\beta} \, u_{I,\alpha K} \, \overline{u}_{I,\beta K} \, e^{-\phi} \, dS. \end{aligned} \tag{5.2}$$

Proof. This formula is known (cf. [9]) for some special cases. For the case $\phi = 0$, the stated formula was proved in Siu [22]. □

Proposition 5.2. ([17, Corollary 6.5]). *Let $\Omega \Subset \mathbb{P}^n$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Then, the Levi form of the function δ has at least $n - q + 1$ positive eigenvalues at each point of Ω.*

Proposition 5.3. *Suppose that Θ is the curvature term defined in (5.1) with respect to the Fubini-Study metric ω. Then, for any (r,q)-form u of $\Omega \Subset \mathbb{P}^n$ with $q \geqslant 1$,*

$$(\Theta u, \overline{u}) = q(2n+1)|u|^2, \quad \textit{when}\, u\, \textit{is a}\,(0,q)\textit{-form},$$
$$(\Theta u, \overline{u}) = 0, \quad \textit{for any}\,(n,q)\textit{-form}\, u,$$
$$(\Theta u, \overline{u}) \geqslant 0, \quad \textit{when}\, r \geqslant 1\, \textit{and}\, u\, \textit{is a}\,(r,q)\textit{-form}.$$

In fact, the assertion for $(0,q)$-forms and (n,q)-forms was computed in [23]. *Also, following Lemma* 3.3 *of Henkin-Iordan* [8] *and its proof showed that the curvature operator Θ acting on $L^2_{r,q}(\Omega)$ is a semi-positive operator.*

Theorem 5.1. *Let $\Omega \Subset \mathbb{P}^n$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). For any $1 \leqslant q \leqslant n$ and $t > 0$, there exists a bounded linear operator $N^t_{r,q} : L^2_{r,q}(\Omega) \to L^2_{r,q}(\Omega)$ satisfies the following properties:*

(i) $\mathrm{Range}\,(N^t_{r,q}) \subset \mathrm{dom}(\Box^t_{r,q})$, $N^t_{r,q}\Box^t_{r,q} = I$ *on* $\mathrm{dom}(\Box^t_{r,q})$;

(ii) for $f \in L^2_{r,q}(\Omega)$, we have $u = \overline{\partial}\,\overline{\partial}^*_t N^t_{r,q} f \,\oplus\, \overline{\partial}^*_t \overline{\partial} N^t_{r,q} f$,

(iii) $\overline{\partial} N^t_{r,q} = N^t_{r,q+1}\overline{\partial}$, $1 \leqslant q \leqslant n-1$ and $\overline{\partial}^*_t N^t_{r,q} = N^t_{r,q-1}\,\overline{\partial}^*_t$, $2 \leqslant q \leqslant n$,

(iv) $N^t_{r,q}$, $\overline{\partial} N^t_{r,q}$ and $\overline{\partial}^*_t N^t_{r,q}$ are bounded operators with respect to the L^2-norms.

Proof. By choosing $\phi_t = -t\log|\delta|$, $t > 0$ in Proposition 4.1, and using Proposition 4.2 and Proposition 4.3, the identity (5.2) implies the weighted L^2-existence for the $\overline{\partial}$. Also, for $u \in \mathrm{dom}(\Box^t_{r,q})$ of degree $q \geqslant 1$, we have for $t > 0$, $t\|u\|_t \leqslant \|\Box^t_{r,q} u\|_t$. Then, as in Theorem 2.2 there exists a unique bounded inverse operator $N^t_{r,q} : \mathrm{Range}(\Box^t_{r,q}) \to \mathrm{dom}\,(\Box^t_{r,q}) \cap (\ker(\Box^t_{r,q}))^\perp$, such that $N^t_{r,q}\Box^t_{r,q} f = f$ on $\mathrm{dom}\,(\Box^t_{r,q})$. Therefore, one can establish the existence theorem of the inverse of $\Box^t_{r,q}$ the so called weighted $\overline{\partial}$-Neumann operator $N^t_{r,q}$. □

Theorem 5.2. *Let $\Omega \Subset \mathbb{P}^n$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Then, for $f \in L^2_{r,q}(\delta^{-t})$, $1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial} f = 0$ in the distribution sense in $\mathbb{P}^n$ and f is supported in $\overline{\Omega}$, there exists $u \in L^2_{r,q-1}(\delta^{-t})$ such that $\overline{\partial} u = f$ in the distribution sense in $\mathbb{P}^n$ with u is supported in $\overline{\Omega}$ and*

$$\int_\Omega |u|^2 \delta^{-t} dV \leq C \int_\Omega |f|^2 \delta^{-t} dV,$$

for some $C > 0$.

Proof. Following Theorem 4.1, $N^t_{r,q}$ exists for forms in $L^2_{n-r,n-q}(\mathbb{P}^n, \delta^t)$. Let $\star_t$ denote the weighted Hodge-star operator with respect to the weighted norm of $L^2_{r,q}(\mathbb{P}^n, \delta^t)$. Then $\star_t = \delta^t \star = \star \delta^t$, where $\star$ is the Hodge star operator with the unweighted L^2 norm. Thus, we can define $u \in L^2_{r,q-1}(\mathbb{P}^n, \delta^{-t})$ by

$$u = -\star_t\, \overline{\overline{\partial}\, N_{n-r,n-q}\, \star_{-t}\, \overline{f}}.$$

Thus $\mathrm{supp}\, u \subset \overline{\Omega}$ and u vanishes on $b\Omega$. Now, we extend u to $\mathbb{P}^n$ by defining $u = 0$ in $\mathbb{P}^n \setminus \Omega$. As in Saber [20], the extended form u satisfies the equation $\overline{\partial} u = f$ in the distribution sense in $\mathbb{P}^n$. □

The above result can be extended easily to the following result.

Theorem 5.3. *Let X be an n-dimensional Kähler manifold with semi-positive holomorphic bisectional curvature and let $\Omega \Subset X$ be a bounded domain with C^2-smooth boundary satisfying condition (H_q). Then, for $f \in L^2_{r,q}(\delta^{-t})$, $1 \leqslant q \leqslant n-1$, satisfying $\overline{\partial} f = 0$ in the distribution sense in X and f is supported in $\overline{\Omega}$, there exists $u \in L^2_{r,q-1}(\delta^{-t})$ such that $\overline{\partial} u = f$ in the distribution sense in X with u is supported in $\overline{\Omega}$ and*

$$\int_\Omega |u|^2 \delta^{-t} dV \leq C \int_\Omega |f|^2 \delta^{-t} dV,$$

for some $C > 0$.

6. Global Boundary Regularity for $\overline{\partial}_{\text{MIX}}$

In this section, we study the global regularity for the $\overline{\partial}_{mix}$-equation with mixed boundary conditions. Throughout this section, Ω will denote the annulus in a Stein manifold X between Ω_1 and Ω_2 with C^3 boundary. Let $\delta \in C^3(\overline{\Omega}_2)$ be the defining function of Ω. We impose the $\overline{\partial}$-Dirichlet boundary condition on Ω_1 and the $\overline{\partial}$-Neumann boundary condition on Ω_2. We say that U satisfies the $\overline{\partial}$-Dirichlet condition along Ω_2 if $U_J|_{\Omega_2} \equiv 0$ whenever $J = (j_1, \dots, j_q)$ with $j_q \neq n$. We say U satisfies the $\overline{\partial}$-Neumann condition along Ω_1 if $U_J|_{\Omega_1} \equiv 0$ when $j_q = n$.

Definition 6.1. For $0 \leqslant r \leqslant n$, $0 \leqslant q \leqslant n$ and $u \in L^2_{r,q}(\Omega)$, $u \in \operatorname{dom} \overline{\partial}_{\text{mix}}$ if and only if there exists $f \in L^2_{r,q+1}(\Omega)$ and a sequence $\{u_\nu\} \in L^2_{r,q}(\Omega)$ which vanish near $b\Omega_2$ such that $u_\nu \to u$ in $L^2_{r,q}(\Omega)$ and $\overline{\partial} u_\nu \to u$ in $L^2_{r,q+1}(\Omega)$, then we say $u \in \operatorname{dom} \overline{\partial}_{\text{mix}}$ and $\overline{\partial}_{\text{mix}} u = f$.

Let $\overline{\partial}^*_{t,\text{mix}}$ be the Hilbert-space adjoint of $\overline{\partial}_{\text{mix}}$. Let $B^2_{r,q}(\Omega)$ denote the space of (r,q)-forms which are C^2-smooth in a neighborhood of $\overline{\Omega}$ and satisfies $\overline{\partial}$-Dirichlet condition on $b\Omega_2$ and $\overline{\partial}$-Neumann condition on $b\Omega_1$. Denote by $W^m_{r,q}(\Omega)$, $m \in \mathbb{R}$, the Hilbert spaces of (r,q)-forms with $W^m(\Omega)$-coefficients and their norms are denoted by $\|u\|_{W^m}$.

As in Lemma 6.4 of [3] and Lemma 4.3.2 in [5], the Hörmander-Friderichs smooth lemma also holds in this setting: Let $u \in \operatorname{dom} \overline{\partial}_{\text{mix}} \cap \operatorname{dom} \overline{\partial}^*_{\text{mix}} \cap L^2_{r,q}(\Omega, \phi)$, there exists $\{u_\nu\} \in L^2_{r,q}(\Omega, \phi)$ such that

$$\|u_\nu - u\| + \|\overline{\partial}_{\text{mix}} u_\nu - \overline{\partial}_{\text{mix}} u\| + \|\overline{\partial}^*_{\text{mix}} u_\nu - \overline{\partial}^*_{\text{mix}} u\| \to 0. \tag{6.1}$$

From now on we fix $\phi_t(z) = t|z|^2$ near $b\Omega_1$ and $\phi_t(z) = t(|z|^2 - \tau\delta)$ near $b\Omega_2$, where t and τ are positive constants which will be determined later. Let $\Box^t_{\text{mix}} = \overline{\partial}_{\text{mix}} \overline{\partial}^\star_{t,\text{mix}} + \overline{\partial}^\star_{t,\text{mix}} \overline{\partial}_{\text{mix}}$ be the complex Laplacian operator and take $f \in \operatorname{dom}(\Box^t_{\text{mix}})$ of degree $q \geqslant 1$, then we have for every $t > 0$.

The proof of the following proposition follows by using a partition of unity.

Proposition 6.1. *Let $\Omega = \Omega_1 \backslash \overline{\Omega}_2$ be an annulus domain in a Stein manifold X between an internal domain Ω_2 satisfies condition (H_{n-q-1}) and an external domain*

Ω_1 satisfies condition (H_q). There exist a positive constant t_* such that for any $t \geqslant t_*$, the harmonic space $\mathcal{H}^m_{r,q}(E)$ has finite dimension and there exists a positive constant C_t depending on t such that

$$\|u\|^2_{\phi_t} \leqslant C_t(\|\overline{\partial}_{mix}u\|^2_{\phi_t} + \|\overline{\partial}^*_{t,mix}u\|^2_t), \tag{6.2}$$

for $u \in \mathrm{dom}(\overline{\partial}_{mix}) \cap \mathrm{dom}(\overline{\partial}^*_{t,mix})$ with $q \geqslant 1$.

Proof. Assume first u is supported in a small neighborhood U of $p \in b\Omega$. Since u satisfies $\overline{\partial}$-Dirichlet condition on $b\Omega_2$, then $\star_t u$ satisfies $\overline{\partial}$-Neumann condition on $b\Omega_2$. Now $1 \leqslant n-q \leqslant n-2$, since $2 \leqslant q \leqslant n-1$. Thus, for $u \in B^2_{r,q}(\Omega)$, by a similar argument of Proposition 3.1 in Saber [19], we have

$$t\| \star_t u\|^2_{-t} \leqslant \|\overline{\partial} \star_t u\|^2_{-t} + \|\overline{\partial}^* \star_t u\|^2_{-t},$$

when t is sufficiently large. Since the Hodge star operator $\star$ is an isometry operator in L^2-space, we have

$$t\| \star_t u\|^2_{-t} \leqslant \| \star_t \overline{\partial} \star_t u\|^2_t + \| \star_t \overline{\partial}^* \star_t u\|^2_t. \tag{6.3}$$

Substituting the identity $\overline{\partial}_{\mathrm{mix}} = \star_t\overline{\partial}^*\star_t$ and $\overline{\partial}^*_{t,\mathrm{mix}} = \star_t\overline{\partial}\star_t$ to (6.3). It follows that

$$t\|u\|^2_t \leqslant \|\overline{\partial}^*_{t,\mathrm{mix}}u\|^2_t + \|\overline{\partial}_{\mathrm{mix}}u\|^2_t,$$

for all $u \in B^2_{r,q}(\Omega)$. Then (6.1) shows that (6.2) holds for all $u \in \mathrm{dom}\,\overline{\partial}_{\mathrm{mix}} \cap \mathrm{dom}\,\overline{\partial}^*_{t,\mathrm{mix}} \cap L^2_{r,q}(\Omega, \phi_t)$. □

Theorem 6.1. *Let X and Ω be the same as in Proposition* 5.1. *There exists a positive integer t^* such that, for $t \geqslant t^*$, $r \geqslant 0$, $q \geqslant 1$, there exists a bounded linear operator $N^t_{mix} : L^2_{r,q}(\Omega) \to L^2_{r,q}(\Omega)$ such that*

(i) $\mathrm{Range}(N^t_{mix}) \subset \mathrm{dom}(\Box^t_{mix})$, *$N^t_{mix}\Box^t_{mix} = I - \mathbb{H}_{mix}$ on* $\mathrm{dom}(\Box^t_{mix})$;

(ii) for $u \in L^2_{r,q}(\Omega)$, we have $u = \overline{\partial}_{mix}\overline{\partial}^*_{t,mix}N^t_{mix}u \oplus \overline{\partial}^*_{t,mix}\overline{\partial}_{mix}N^t_{mix}u \oplus \mathbb{H}_{mix}u$;

(iii) $N^t_{mix}\overline{\partial}_{mix} = \overline{\partial}_{mix}N^t_{mix}$ on $\mathrm{dom}(\overline{\partial}_{mix})$;

(iv) $N^t_{mix}\overline{\partial}^*_{t,mix} = \overline{\partial}^*_{t,mix}N^t_{mix}$ on $\mathrm{dom}(\overline{\partial}^*_{t,mix})$;

(v) N^t_{mix}, $\overline{\partial}_{mix}N^t_{mix}$ and $\overline{\partial}^*_{t,mix}N^t_{mix}$ are bounded operators with respect to the L^2-norms.

Proof. Following Proposition 5.1, one obtain that

$$t\|f\|_t \leq \|\Box^t_{\mathrm{mix}}f\|_t. \tag{6.4}$$

Since $\Box^t_{\mathrm{mix}}$ is a linear closed densely defined operator, then, from [9, Theorem 1.1.1], $\mathrm{Range}(\Box^t_{\mathrm{mix}})$ is closed. Thus, from (1.1.1) in [9] and the fact that $\Box_{r,q}$ is self adjoint, we have the Hodge decomposition

$$L^2_{r,q}(\Omega) = \overline{\partial}_{\mathrm{mix}}\overline{\partial}^\star_{t,\mathrm{mix}}\,\mathrm{dom}\,\Box^t_{\mathrm{mix}} \oplus \overline{\partial}^\star_{t,\mathrm{mix}}\overline{\partial}_{\mathrm{mix}}\,\mathrm{dom}\,\Box^t_{\mathrm{mix}}.$$

Since $\Box^t_{\mathrm{mix}}$ is one to one on $\mathrm{dom}\,\Box^t_{\mathrm{mix}}$ from (1.5.3) in [9], then there exists a unique bounded inverse operator $N^t_{\mathrm{mix}} : \mathrm{Range}\,\Box^t_{\mathrm{mix}} \to \mathrm{dom}\,\Box^t_{\mathrm{mix}} \cap (\ker \Box^t_{\mathrm{mix}})^\perp$ such that $N^t_{\mathrm{mix}}\Box^t_{\mathrm{mix}}f = f$ on $\mathrm{dom}\,\Box^t_{\mathrm{mix}}$. Thus, we can establish the existence theorem of the inverse of $\Box^t_{\mathrm{mix}}$ the so called weighted $\overline{\partial}$-Neumann operator N^t_{mix}. □

Corollary 6.1. *(i) If $f \in \ker \overline{\partial}_{mix}$, then $\overline{\partial}^*_{t,mix} N^t_{mix} f$ gives the solution u_t to the equation $\overline{\partial}_{mix} u_t = f$ of minimal $u_t \in L^2_{r,q-1}(\Omega)$-norm.*

(ii) If $f \in \ker \overline{\partial}^*_{t,mix}$, then $\overline{\partial}_{mix} N^t_{mix} f$ gives the solution u_t to the equation $\overline{\partial}^*_{t,mix} u_t = f$ of minimal $u_t \in L^2_{r,q+1}(\Omega)$-norm.

Using the elliptic regularization method which was used in [14], one can pass from the a priori estimates (6.4) to actual estimates and we can prove the following theorem.

Theorem 6.2. *For every integer $T \geqslant 0$ and real $t > T > 0$, N^t_{mix} is bounded from $W^m_{r,q}(\Omega)$ into itself.*

By Theorem 5.1 (v), Theorem 5.2 and the density of $C^\infty_{r,q}(\overline{\Omega})$ in $W^m_{r,q}(\Omega)$, the following is immediate.

Corollary 6.2. *If $f \in W^m_{r,q}(\Omega)$, $m = 0, 1, 2, 3, \ldots$ satisfies $\overline{\partial}_{mix} f = 0$, where $q \geqslant 1$, then there exits $u \in W^m_{r,q-1}(\Omega)$ so that $\overline{\partial}_{mix} u = f$ on Ω with estimate $\|u\|_{W^m} \leq C_m \|f\|_{W^m}$.*

Theorem 6.3. *For $f \in C^\infty_{r,q}(\overline{\Omega})$, with $\overline{\partial}_{mix} f = 0$, $q \geqslant 1$, there exists $u \in C^\infty_{r,q-1}(\overline{\Omega})$ such that $\overline{\partial}_{mix} u = f$.*

Proof. From Corollary 5.1, there is $u_k \in W^k_{r,q-1}(\Omega)$ satisfying $\overline{\partial}_{\mathrm{mix}} u_k = f$ for each positive integer k. We shall modify u_k to generate a new sequence that converges to a smooth solution. Since $u_k - u_{k+1}$ is in $W^k_{r,q-1}(\Omega) \cap \ker(\overline{\partial}_{\mathrm{mix}})$, there exists a $v_{k+1} \in W^{k+1}_{r,q-1}(\Omega) \cap \ker(\overline{\partial}_{\mathrm{mix}})$ such that

$$\|u_k - u_{k+1} - v_{k+1}\|_{W^k} \leq 2^{-k}, \quad k = 1, 2, 3, \ldots$$

Setting $\widetilde{u}_{k+1} = u_{k+1} + v_{k+1}$, then $\widetilde{u}_{k+1} \in W^{k+1}_{r,q-1}(\Omega)$ and $\overline{\partial}_{\mathrm{mix}} \widetilde{u}_k = f$. Inductively, we can choose a new sequence $\widetilde{u}_k \in W^k_{r,q-1}(\Omega)$ such that $\overline{\partial}_{\mathrm{mix}} \widetilde{u}_k = f$ and

$$\|\widetilde{u}_{k+1} - \widetilde{u}_k\|_{W^k} \leq 2^{-k}, \quad k = 1, 2, 3, \ldots$$

Set $u_\infty = \widetilde{u}_t + \sum_{k=t}^\infty (\widetilde{u}_{k+1} - \widetilde{u}_k)$, $t \in \mathbb{N}$. Then u_∞ is well defined and is in $W^k_{r,q-1}(\Omega)$ for every k. Thus, by the Sobolev embedding theorem, $u_\infty \in C^\infty_{r,q}(\overline{\Omega})$ and $\overline{\partial}_{\mathrm{mix}} u_\infty = f$. Thus the proof follows. □

Corollary 6.3. *We assume that $0 \leqslant r \leqslant n$, $2 \leqslant q \leqslant n$ and the boundary of Ω is smooth. Let N^t_{mix} be the weighted $\overline{\partial}_{mix}$-Neumann operator. For every $k \geqslant 0$, there exists S_k such that when $t \geqslant S_k$ we have that N^t_{mix}, $\overline{\partial}_{mix} N^t_{mix}$, $\overline{\partial}^*_{t,mix} N^t_{mix}$, $\overline{\partial}^*_{t,mix} N^t_{mix} \overline{\partial}_{mix}$ are exactly regular on $W^k_{r,q}(\Omega)$.*

Proof. When $f \in C^\infty_{r,q}(\overline{\Omega}) \cap \operatorname{dom} \overline{\partial}_{\mathrm{mix}}$ and $\operatorname{supp} f \Subset U \cap \overline{\Omega}$, where U is a special boundary chart, then from (6.4), we have that

$$t\|f\|^2_t \leqslant C^2 \|\Box^t_{\mathrm{mix}} f\|^2_t. \tag{6.5}$$

When $f \in C^\infty_{r,q}(\Omega)$ with supp f a compact subset in Ω, we have the following Garding's inequality

$$\|f\|^2_{W^1} \leqslant \|\overline{\partial}_{\mathrm{mix}} f\|^2_t + \|\overline{\partial}^*_{t,\mathrm{mix}} f\|^2_t + C_t\|f\|^2_t. \tag{6.6}$$

Combining (6.5) and (6.6) and with a similar argument as in Kohn [14], the result follows. □

Corollary 6.4. *Suppose that $f \in L^2_{r,q}(\Omega) \cap \ker \overline{\partial}_{mix}$, where $2 \leqslant q \leqslant n-1$. Then for each $k > 0$, there exists $f_n \in W^k_{r,q}(\Omega)$ with f_n satisfying $\overline{\partial}$-Dirichlet condition on $b\Omega_2$ such that $f_n \to f$ in $L^2_{r,q}(\Omega)$ and $\overline{\partial}_{mix} f_n = 0$.*

Corollary 6.5. *Suppose that $f \in C^\infty_{r,q}(\overline{\Omega}) \cap \ker \overline{\partial}_{mix}$, where $0 \leqslant p \leqslant n$, $2 \leqslant q \leqslant n-1$. Then, there exists $u \in C^\infty_{r,q-1}(\overline{\Omega}) \cap$ dom $\overline{\partial}_{mix}$ satisfying $\overline{\partial}$-Dirichlet condition on $b\Omega_2$ such that $f_n \longrightarrow f$ in $L^2_{r,q}(\Omega)$ and $\overline{\partial}_{mix} u = f$.*

7. L^2 theory for $\overline{\partial}_{\mathrm{mix}}$

We consider an operator $\overline{\partial}_{\mathrm{mix}}$ which satisfies that $\overline{\partial}_c \subseteq \overline{\partial}_{\mathrm{mix}} \subseteq \overline{\partial}$, where $\overline{\partial}$ is the maximal realization of the differential operator $\overline{\partial}$. As Theorem 2.2 in Li-Shaw [16], we prove the following theorem.

Theorem 7.1. *Let X and Ω be the same as in Proposition 5.1. Then, for $0 \leqslant r \leqslant n$, $2 \leqslant q \leqslant n$, the Dolbeault cohomology $H^{r,q}_{L^2}(\Omega)$ with $L^2(\Omega)$-coefficients vanishes, i.e.,*

$$H^{r,q}_{\overline{\partial}_{mix},\,L^2}(\Omega) = \frac{\{f \in L^2_{r,q}(\Omega) \,:\, \overline{\partial}_{mix} f = 0\}}{\{f \in L^2_{r,q}(\Omega) \,:\, f = \overline{\partial}_{mix} u,\ u \in L^2_{r,q-1}(\Omega)\}} = \{0\}.$$

Proof. Let $f \in L^2_{r,q}(\Omega)$ with $\overline{\partial}_{\mathrm{mix}} f = 0$. Extending f to be zero in Ω_2, denoted by f^0, we have that $f^0 \in L^2_{r,q}(\Omega_1)$ and $\overline{\partial} f^0 = 0$ in Ω_1. This follows from the assumption that Ω_2 has C^2 boundary and the strong $\overline{\partial}_c$ and weak $\overline{\partial}_c$ are equal. Here we only need the boundary Ω_2 to be Lipschitz. For a proof of such weak equal strong results, see e.g. Lemma 2.4 in Laurent-Thiébaut-Shaw [15]. Thus we have from the L^2 theory for bounded domains satisfies condition (H_q), there exists a solution $v \in L^2_{r,q-1}(\Omega_1)$ such that $\overline{\partial} v = f^0$ in Ω_1. From the elliptic regularity in the interior for $\overline{\partial}$, we can assume that the form v is in $W^1_{r,q-1}(\Omega_2)$. The form v satisfies $\overline{\partial} v = 0$ on Ω_2. Since $q > 1$ and the boundary of Ω_2 is C^2-smooth, there exists a solution $w \in W^1_{r,q-1}(\Omega_2)$ such that $\overline{\partial} w = v$ in Ω_2. This follows from a result of Ho [10] for sufficiently smooth boundary when the boundary is only C^2. Let $\tilde{w}$ be a W^1 extension of w to Ω_1. We set $u = v - \overline{\partial}\tilde{w}$ in Ω_1. Then u is in $L^2_{r,q-1}(\Omega_1)$ with $\overline{\partial} u = f$ in Ω_1. But $u = 0$ on Ω_2. This implies that $u \in$ dom $\overline{\partial}_{\mathrm{mix}}$ and $\overline{\partial}_{\mathrm{mix}} u = f$. □

Theorem 7.2. *Let $\Omega = \Omega_1 \backslash \overline{\Omega}_2$ between two bounded strictly pseudoconvex domains Ω_1 and Ω_2 in an n-dimensional Stein manifold X such that $\Omega_2 \Subset \Omega_1$. Then*

$$H^{r,1}_{\overline{\partial}_{mix},\,L^2}(\Omega) \neq 0.$$

Proof. The proof follows as in Lemma 2.3 in Li and Shaw [16]. □

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POSITIVE SOLUTIONS FOR FIRST-ORDER NONLINEAR CAPUTO-HADAMARD FRACTIONAL RELAXATION DIFFERENTIAL EQUATIONS

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ABSTRACT. This article concerns the existence and uniqueness of positive solutions of the first-order nonlinear Caputo-Hadamard fractional relaxation differential equation

$$\begin{cases} \mathfrak{D}_1^\alpha \left(x\left(t\right)-g\left(t,x\left(t\right)\right)\right)+wx\left(t\right)=f\left(t,x\left(t\right)\right), \ 1<t\leq e, \\ x\left(1\right)=x_0>g\left(1,x_0\right)>0, \end{cases}$$

where $0<\alpha\leq 1$. In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct appropriate mappings and employ the Krasnoselskii fixed point theorem and the method of upper and lower solutions to show the existence of a positive solution of this equation. We also use the Banach fixed point theorem to show the existence of a unique positive solution. Finally, an example is given to illustrate our results.

1. INTRODUCTION

Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1]–[13], [16] and the references therein.

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Zhang in [16] investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equation

$$\begin{cases} D^{\alpha}x(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = 0, \end{cases}$$

where D^{α} is the standard Riemann Liouville fractional derivative of order $0 < \alpha < 1$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solution and cone fixed-point theorem, the author obtained the existence and uniqueness of a positive solution.

The nonlinear fractional differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) = f(t, x(t)) +{}^{C} D^{\alpha-1}g(t, x(t)), & 0 < t \leq T, \\ x(0) = \theta_1 > 0, \quad x'(0) = \theta_2 > 0, \end{cases}$$

has been investigated in [4], where ${}^{C}D^{\alpha}$ is the standard Caputo's fractional derivative of order $1 < \alpha \leq 2$, $g, f : [0,T] \times [0,\infty) \to [0,\infty)$ are given continuous functions, g is non-decreasing on x and $\theta_2 \geq g(0, \theta_1)$. By employing the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the authors obtained positivity results.

In [6], Chidouh, Guezane-Lakoud and Bebbouchi discussed the existence and uniqueness of the positive solution of the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{C}D^{\alpha}x(t) + wx(t) = f(t, x(t)), & 0 < t \leq 1, \\ x(0) = x_0 > 0, \end{cases}$$

where $0 < \alpha \leq 1$, $w > 0$ and $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a given continuous function. By using the method of the upper and lower solutions and Schauder and Banach fixed point theorems, the existence and uniqueness of solutions has been established.

Ahmad and Ntouyas in [3] studied the existence and uniqueness of solutions to the following boundary value problem

$$\begin{cases} \mathfrak{D}_1^{\alpha}\left(\mathfrak{D}_1^{\beta}u(t) - g(t, u_t)\right) = f(t, u_t), & t \in [1, b], \\ u(t) = \phi(t), & t \in [1-r, 1], \\ \mathfrak{D}_1^{\beta}u(1) = \eta \in \mathbb{R}, \end{cases}$$

where $\mathfrak{D}_1^{\alpha}$ and $\mathfrak{D}_1^{\beta}$ are the Caputo-Hadamard fractional derivatives, $0 < \alpha, \beta < 1$. By employing the fixed point theorems, the authors obtained existence and uniqueness results.

In this paper, we are interested in the analysis of qualitative theory of the problems of the positive solutions to fractional differential equations. Inspired and motivated by the works mentioned above and the papers [1]–[13], [16] and the references therein, we concentrate on the positivity of solutions for the first-order nonlinear Caputo-Hadamard fractional relaxation differential equation

$$\left\{\begin{array}{l}\mathfrak{D}_1^\alpha\left(x\left(t\right)-g\left(t,x\left(t\right)\right)\right)+wx\left(t\right)=f\left(t,x\left(t\right)\right), \quad 1<t\leq e,\\ x\left(1\right)=x_0>g\left(1,x_0\right)>0,\end{array}\right. \tag{1.1}$$

where $0<\alpha\leq 1$, $w>0$, $g,f:[1,e]\times[0,\infty)\to[0,\infty)$ are continuous. To show the existence and uniqueness of the positive solution, we transform (1.1) into an integral equation and then by the method of upper and lower solutions and use the Krasnoselskii and Banach fixed point theorems.

This paper is organized as follows. In Section 2, we introduce some notations and lemmas, and state some preliminaries results needed in later sections. Also, we present the inversion of (1.1) and the Banach and Krasnoselskii fixed point theorems. For details on the Banach and Krasnoselskii theorems we refer the reader to [15]. In Sections 3 and 4, we give and prove our main results on positivity and we provide an example to illustrate our results.

2. Preliminaries

Let $X=C\left([1,e]\right)$ be the Banach space of all real-valued continuous functions defined on the compact interval $[1,e]$, endowed with the maximum norm. Define the cone

$$\mathcal{E}=\left\{x\in X: x(t)\geq 0 \text{ for all } t\in[1,e]\right\}.$$

We introduce some necessary definitions, lemmas and theorems which will be used in this paper. For more details, see [9, 13].

Definition 2.1 ([9])**.** The Hadamard fractional integral of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbb{R}$ is defined as

$$\mathfrak{I}_1^\alpha x(t)=\frac{1}{\Gamma(\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{\alpha-1}x(s)\frac{ds}{s},\quad \alpha>0.$$

Definition 2.2 ([9])**.** The Caputo-Hadamard fractional derivative of order $\alpha>0$ for a continuous function $x:[1,+\infty)\to\mathbb{R}$ is defined as

$$\mathfrak{D}_1^\alpha x(t)=\frac{1}{\Gamma(n-\alpha)}\int_1^t\left(\log\frac{t}{s}\right)^{n-\alpha-1}\delta^n(x)(s)\frac{ds}{s},\quad n-1<\alpha<n,$$

where $\delta^n=\left(t\frac{d}{dt}\right)^n$, $n\in\mathbb{N}$.

Lemma 2.1 ([9])**.** *Let $n-1<\alpha\leq n$, $n\in\mathbb{N}$ and $x\in C^n\left([1,T]\right)$. Then*

$$\left(\mathfrak{I}_1^\alpha\mathfrak{D}_1^\alpha x\right)(t)=x(t)-\sum_{k=0}^{n-1}\frac{x^{(k)}(1)}{\Gamma(k+1)}(\log t)^k.$$

Lemma 2.2 ([9])**.** *For all $\mu>0$ and $\nu>-1$*

$$\frac{1}{\Gamma(\mu)}\int_1^t\left(\log\frac{t}{s}\right)^{\mu-1}(\log s)^\nu\frac{ds}{s}=\frac{\Gamma(\nu+1)}{\Gamma(\mu+\nu+1)}(\log t)^{\mu+\nu}.$$

Definition 2.3 ([14]). The two-parameter function of the Mittag-Leffler type is defined by the series expansion

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0,\ \beta \in \mathbb{C},\ z \in \mathbb{C}.$$

For $\beta = 1$, we obtain the Mittag-Leffler function in one parameter

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0,\ z \in \mathbb{C}.$$

Lemma 2.3 ([14]). *The generalized Mittag-Leffler function $E_{\alpha,\beta}(-x)$ with $x \geq 0$ is completely monotonic if and only if $0 < \alpha \leq 1$ and $\beta \geq \alpha$. In other words, it yields*

$$(-1)^n \frac{d^n}{dx^n} E_{\alpha,\beta}(-x) \geq 0, \quad \text{for all } n \in \mathbb{N}.$$

Obviously, $0 \leq E_{\alpha,\beta}(-x) \leq \frac{1}{\Gamma(\beta)}$, where $x \geq 0$, $0 \leq \alpha \leq 1$ and $\beta \geq \alpha$.

The following lemma is fundamental to our results.

Lemma 2.4. *Let $x \in C([1,e])$, x' and $\frac{\partial g}{\partial t}$ exist, then x is a solution of (1.1) if and only if*

$$\begin{aligned} x(t) &= (x_0 - g(1,x_0)) E_{\alpha}(-w(\log t)^{\alpha}) + g(t,x(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) F(s,x(s)) \frac{ds}{s}, \quad 1 \leq t \leq e, \end{aligned} \tag{2.1}$$

where $F(t,x) = f(t,x) - wg(t,x)$.

Proof. It is easy to prove by the Laplace transform. □

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a positive solution of (1.1).

Definition 2.4. Let $(X, \|\cdot\|)$ be a Banach space and $\mathcal{H} : X \to X$. The operator $\mathcal{H}$ is a contraction operator if there is an $\lambda \in (0,1)$ such that $x, y \in X$ imply

$$\|\mathcal{H}x - \mathcal{H}y\| \leq \lambda \|x - y\|.$$

Theorem 2.1 (Banach [15]). *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space X and $\mathcal{H} : \mathcal{K} \to \mathcal{K}$ be a contraction operator. Then there is a unique $x \in \mathcal{K}$ with $\mathcal{H}x = x$.*

Theorem 2.2 (Krasnoselskii fixed point theorem [15]). *If $\mathcal{K}$ is a nonempty bounded, closed and convex subset of a Banach space X, $\mathcal{A}$ and $\mathcal{B}$ two operators defined on $\mathcal{K}$ with values in X such that*

i) *$\mathcal{A}x + \mathcal{B}y \in \mathcal{K}$ for all $x, y \in \mathcal{K}$;*
ii) *$\mathcal{A}$ is continuous and compact;*
iii) *$\mathcal{B}$ is a contraction.*

Then there exists $z \in \mathcal{K}$ such that $z = \mathcal{A}z + \mathcal{B}z$.

3. Existence of Positive Solutions

In this section, we consider the results of existence problem for many cases of (1.1). Moreover, we introduce the following conditions.

$(H1)$ $g, F : [1, e] \times [0, \infty) \to [0, \infty)$ are continuous functions and g is nondecreasing on x.

$(H2)$ There exists $L_g \in (0, 1)$ such that

$$|g(t, x) - g(t, y)| \leq L_g \|x - y\| .$$

$(H3)$ There exists $L_F > 0$ such that

$$|F(t, x) - F(t, y)| \leq L_F \|x - y\| .$$

We note that to apply Theorem 2.2 we need to construct two mappings, one is contraction and the other is completely continuous. Therefore, we express (2.1) as

$$x(t) = (\mathcal{A}x)(t) + (\mathcal{B}x)(t) = (\mathcal{H}x)(t), \tag{3.1}$$

where the operators $\mathcal{A}, \mathcal{B} : \mathcal{E} \to X$ are defined by

$$(\mathcal{A}x)(t) = \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}$$

and

$$(\mathcal{B}x)(t) = (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, x(t)).$$

We need the following lemmas to establish our results.

Lemma 3.1. *Assume that $(H1)$ holds. Then, the operator $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is completely continuous.*

Proof. By Lemma 2.3 and taking into account that F is continuous nonnegative function, we get that $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is continuous. The function $F : [1, e] \times B_\eta \to [0, \infty)$ is bounded, then there exists $\rho > 0$ such that $0 \leq F(t, x(t)) \leq \rho$, where $B_\eta = \{x \in \mathcal{E},\ \|x\| \leq \eta\}$. We obtain

$$\begin{aligned}
|(\mathcal{A}x)(t)| &= \left|\int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}\right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |F(s, x(s))| \frac{ds}{s} \\
&\leq \frac{\rho}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\
&\leq \frac{\rho(\log t)^\alpha}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus,

$$\|\mathcal{A}x\| \leq \frac{\rho}{\Gamma(\alpha+1)}.$$

Hence, $\mathcal{A}(B_\eta)$ is uniformly bounded.

Now, we will prove that $\mathcal{A}(B_\eta)$ is equicontinuous. Let $x \in B_\eta$, then for any $t_1, t_2 \in [1, e]$, $t_2 > t_1$, we have

$$\begin{aligned}
&\left|(\mathcal{A}x)(t_1) - (\mathcal{A}x)(t_2)\right| \\
=&\left|\int_1^{t_1} \left(\log \frac{t_1}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t_1}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}\right. \\
&\left.- \int_1^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t_2}{s}\right)^{\alpha}\right) F(s, x(s)) \frac{ds}{s}\right| \\
\leq&\frac{1}{\Gamma(\alpha)} \int_1^{t_1} \left|\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right| |F(s, x(s))| \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} |F(s, x(s))| \frac{ds}{s} \\
\leq&\frac{\rho}{\Gamma(\alpha)} \left(\int_1^{t_1} \left(\left(\log \frac{t_1}{s}\right)^{\alpha-1} - \left(\log \frac{t_2}{s}\right)^{\alpha-1}\right) \frac{ds}{s} + \int_{t_1}^{t_2} \left(\log \frac{t_2}{s}\right)^{\alpha-1} \frac{ds}{s}\right) \\
\leq&\frac{\rho}{\Gamma(\alpha+1)} \left((\log t_1)^{\alpha} - (\log t_2)^{\alpha} + 2\left(\log \frac{t_2}{t_1}\right)^{\alpha}\right) \\
\leq&\frac{2\rho}{\Gamma(\alpha+1)} \left(\log \frac{t_2}{t_1}\right)^{\alpha},
\end{aligned}$$

which is independent of x and tends to zero as $t_2 \to t_1$. Thus, $\mathcal{A}(B_\eta)$ is equicontinuous. So, the compactness of $\mathcal{A}$ follows by Ascoli Arzela's theorem. □

Lemma 3.2. *Assume that* $(H1)$ *and* $(H2)$ *hold. Then the operator* $\mathcal{B} : \mathcal{E} \to \mathcal{E}$ *is a contraction.*

Proof. By Lemma 2.3 and taking into account that g is continuous nonnegative function and $x_0 > g(1, x_0)$, we get that $\mathcal{B} : \mathcal{E} \to \mathcal{E}$. For $x, y \in \mathcal{E}$ we have

$$|(\mathcal{B}x)(t) - (\mathcal{B}y)(t)| = |g(t, x(t)) - g(t, y(t))| \leq L_g \|x - y\|.$$

Thus, $\|\mathcal{B}x - \mathcal{B}y\| \leq L_g \|x - y\|$. Hence, $\mathcal{B}$ is a contraction. □

Now, for any $x \in [a, b] \subset \mathbb{R}^+$, we define respectively the upper and lower control functions as follows

$$H(t, x) = \sup_{a \leq y \leq x} F(t, y), \quad h(t, x) = \inf_{x \leq y \leq b} F(t, y).$$

It is clear that these functions are nondecreasing on $[a, b]$.

Definition 3.1. Let $\overline{x}, \underline{x} \in \mathcal{E}$, $a \leq \underline{x} \leq \overline{x} \leq b$, satisfying

$$\begin{aligned}
\overline{x}(t) \geq& (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^{\alpha}) + g(t, \overline{x}(t)) \\
&+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) H(s, \overline{x}(s)) \frac{ds}{s}, \quad 1 \leq t \leq e,
\end{aligned}$$

and

$$\begin{aligned}\underline{x}(t) \leq{}& (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \underline{x}(t)) \\ &+ \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) h(s, \underline{x}(s)) \frac{ds}{s}, \quad 1 \leq t \leq e.\end{aligned}$$

Then the functions $\overline{x}$ and $\underline{x}$ are called a pair of upper and lower solutions for the equation (1.1).

Theorem 3.1. *Assume that* $(H1)$ *and* $(H2)$ *hold and* $\overline{x}$ *and* $\underline{x}$ *are respectively upper and lower solutions of* (1.1), *then* (1.1) *has at least one positive solution.*

Proof. Let

$$\mathcal{K} = \{x \in \mathcal{E} : \underline{x}(t) \leq x(t) \leq \overline{x}(t), \ t \in [1, e]\}.$$

As $\mathcal{K} \subset E$ and $\mathcal{K}$ is a nonempty bounded, closed and convex subset. By Lemma 3.1, $\mathcal{A} : \mathcal{K} \to \mathcal{E}$ is completely continuous. Also, from Lemma 3.2, $\mathcal{B} : \mathcal{K} \to \mathcal{E}$ is a contraction. Next, we show that if $x, y \in \mathcal{K}$, we have $\mathcal{A}x + \mathcal{B}y \in \mathcal{K}$. For any $x, y \in \mathcal{K}$, we have $\underline{x} \leq x, y \leq \overline{x}$, then

$$\begin{aligned}&(\mathcal{A}x)(t) + (\mathcal{B}y)(t) \\ &= (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, y(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) F(s, x(s)) \frac{ds}{s} \\ &\leq (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \overline{x}(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) H(s, \overline{x}(s)) \frac{ds}{s} \\ &\leq \overline{x}(t) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}&(\mathcal{A}x)(t) + (\mathcal{B}y)(t) \\ &= (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, y(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) F(s, x(s)) \frac{ds}{s} \\ &\geq (x_0 - g(1, x_0)) E_\alpha(-w(\log t)^\alpha) + g(t, \underline{x}(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) h(s, \underline{x}(s)) \frac{ds}{s} \\ &\geq \underline{x}(t). \end{aligned} \tag{3.3}$$

Thus, from (3.2) and (3.3), we obtain that $\mathcal{A}x + \mathcal{B}y \in \mathcal{K}$. We now see that all the conditions of the Krasnoselskii's fixed point theorem are satisfied. Thus there exists a fixed point x in $\mathcal{K}$. Therefore, (1.1) has at least one positive solution x in $\mathcal{K}$. $\square$

Corollary 3.1. *Assume that* $(H1)$ *and* $(H2)$ *hold and there exist* $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$ *such that*

$$\lambda_1 \leq g(t,x) \leq \lambda_2, \quad (t,x) \in [1,e] \times [0,+\infty), \tag{3.4}$$

and

$$\lambda_3 \leq F(t,x) \leq \lambda_4, \quad (t,x) \in [1,e] \times [0,+\infty). \tag{3.5}$$

Then (1.1) *has at least one positive solution* $x \in \mathcal{E}$*, moreover*

$$x(t) \geq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_1 + \lambda_3 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha) \tag{3.6}$$

and

$$x(t) \leq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 + \lambda_4 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha). \tag{3.7}$$

Proof. From (3.5) and the definition of control functions, we have

$$\lambda_3 \leq h(t,x) \leq H(t,x) \leq \lambda_4. \tag{3.8}$$

Now, let

$$\begin{aligned}\overline{x}(t) &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 + \lambda_4 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha) \\ &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 \\ &\quad + \lambda_4 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) \frac{ds}{s}.\end{aligned}$$

Taking into account (3.4) and (3.8), we have

$$\begin{aligned}\overline{x}(t) &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_2 \\ &\quad + \lambda_4 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) \frac{ds}{s} \\ &\geq (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + g(t,\overline{x}(t)) \\ &\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) H(s,\overline{x}(s)) \frac{ds}{s}.\end{aligned}$$

It is clear that $\overline{x}$ is the upper solution of (1.1).

Now, let

$$\begin{aligned}\underline{x}(t) &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_1 + \lambda_3 (\log t)^\alpha E_{\alpha,\alpha+1}(-w(\log t)^\alpha) \\ &= (x_0 - g(1,x_0)) E_\alpha(-w(\log t)^\alpha) + \lambda_1 \\ &\quad + \lambda_3 \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^\alpha\right) \frac{ds}{s}.\end{aligned}$$

By (3.4), (3.8) and the same way that we used to search the upper solution, we conclude also that $\underline{x}$ is the lower solution of (1.1). Therefore, from Theorem 3.1, we conclude that (1.1) has at least one positive solution $x \in \mathcal{E}$ which verifies the inequalities (3.6) and (3.7). $\square$

Corollary 3.2. *Assume that* $(H1)$ *and* $(H2)$ *hold and there exists* $a_1, a_2 > 0$ *such that*

$$a_1 \leq g(t,x), \; a_2 \leq F(t,x), \quad (t,x) \in [1,e] \times [0,+\infty), \tag{3.9}$$

and

$$\lim_{x \to +\infty} g(t,x) < +\infty, \quad \lim_{x \to +\infty} F(t,x) < +\infty, \tag{3.10}$$

then (1.1) *has at least one positive solution.*

Proof. By (3.10), there exist positive constants N_1, N_2, R_1 and R_2 such that

$$g(t,x) \leq N_1, \quad \text{for any } x \geq R_1,\, t \in [1,e], \tag{3.11}$$

and

$$F(t,x) \leq N_2, \quad \text{for any } x \geq R_2,\, t \in [1,e]. \tag{3.12}$$

Let $C_1 = \max\limits_{1 \leq t \leq e, 0 \leq x \leq R_1} g(t,x)$ and $C_2 = \max\limits_{1 \leq t \leq e, 0 \leq x \leq R_2} F(t,x)$. Then, by (3.11) and (3.12), we have

$$a_1 \leq g(t,x) \leq N_1 + C_1, \quad \text{for any } x \geq 0,\, t \in [1,e],$$

and

$$a_2 \leq F(t,x) \leq N_2 + C_2, \quad \text{for any } x \geq 0,\, t \in [1,e].$$

Thus, from Corollary 3.1, (1.1) has at least one positive solution x in $\mathcal{E}$ which satisfies the following inequalities

$$x(t) \geq (x_0 - g(1,x_0)) E_\alpha\left(-w(\log t)^\alpha\right) + a_1 + a_2 (\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right)$$

and

$$\begin{aligned} x(t) \leq{}& (x_0 - g(1,x_0)) E_\alpha\left(-w(\log t)^\alpha\right) + N_1 + C_1 \\ &+ (N_2 + C_2)(\log t)^\alpha E_{\alpha,\alpha+1}\left(-w(\log t)^\alpha\right). \end{aligned}$$

□

4. Uniqueness of Positive Solution

In this section, we shall prove the uniqueness of the positive solution using the contraction mapping principle.

Theorem 4.1. *Assume that* $(H1)$-$(H3)$ *hold and*

$$L_g + \frac{L_F}{\Gamma(\alpha+1)} < 1, \tag{4.1}$$

then (1.1) *has a unique positive solution* $x \in \mathcal{K}$.

Proof. From Theorem 3.1, it follows that (1.1) has at least one positive solution in $\mathcal{K}$. Hence, we need only to prove that the operator $\mathcal{H}$ defined in (3.1) is a contraction on X. In fact, since for any $x_1, x_2 \in \mathcal{K}$, $(H2)$ and $(H3)$ are verified, then we have

$$\begin{aligned}
&\left|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\right| \\
&\leq \left|g(t, x_1(t)) - g(t, x_2(t))\right| \\
&\quad + \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} E_{\alpha,\alpha}\left(-w\left(\log \frac{t}{s}\right)^{\alpha}\right) \left|F(s, x_1(s)) - F(s, x_2(s))\right| \frac{ds}{s} \\
&\leq L_g \left\|x_1 - x_2\right\| + \frac{(\log t)^{\alpha}}{\Gamma(\alpha+1)} L_F \left\|x_1 - x_2\right\| \\
&\leq \left(L_g + \frac{L_F}{\Gamma(\alpha+1)}\right) \left\|x_1 - x_2\right\|.
\end{aligned}$$

Thus,

$$\left\|\mathcal{H}x_1 - \mathcal{H}x_2\right\| \leq \left(L_g + \frac{L_F}{\Gamma(\alpha+1)}\right) \left\|x_1 - x_2\right\|.$$

Hence, the operator $\mathcal{H}$ is a contraction mapping by (4.1). Therefore, by the contraction mapping principle, we conclude that the problem (1.1) has a unique positive solution $x \in \mathcal{K}$. □

Finally, we give an example to illustrate our results.

Example 4.1. We consider the following nonlinear Caputo-Hadamard fractional relaxation differential equation

$$\left\{\begin{array}{l}
\mathfrak{D}_1^{1/3}\left(x(t) - \dfrac{x(t)+2}{x(t)+3}\right) + x(t) \\
= \dfrac{(t+6)x^2(t) + (4t+21)x(t) + 5t + 18}{(t+3)\left(x^2(t) + 4x(t) + 3\right)}, \quad 1 < t \leq e, \\
x(1) = 1,
\end{array}\right. \tag{4.2}$$

where

$$\alpha = \frac{1}{3}, \quad w = 1, \quad x_0 = 1, \quad g(t,x) = \frac{x+2}{x+3}, \quad g(1, x_0) = \frac{3}{4},$$
$$f(t,x) = \frac{(t+6)x^2 + (4t+21)x + 5t + 18}{(t+3)(x^2 + 4x + 3)}, \quad F(t,x) = \frac{1}{3+t}\left(\frac{t}{x+1} + 3\right).$$

Since g is nondecreasing on x and F is decreasing on x

$$\frac{2}{3} \leq g(t,x) \leq 1, \quad \frac{3}{3+e} \leq F(t,x) \leq 1,$$

for $(t,x) \in [1,e] \times [0,\infty)$. Hence, by Corollary 3.1, (4.2) has a positive solution, which verifies $\underline{x}(t) \leq x(t) \leq \overline{x}(t)$, where

$$\overline{x}(t) = \frac{1}{4} E_{1/3}\left(-(\log t)^{1/3}\right) + 1 + (\log t)^{1/3} E_{1/3,4/3}\left(-(\log t)^{1/3}\right)$$

and

$$\underline{x}(t) = \frac{1}{4}E_{1/3}\left(-(\log t)^{1/3}\right) + \frac{2}{3} + \frac{3}{3+e}(\log t)^{1/3} E_{1/3,4/3}\left(-(\log t)^{1/3}\right),$$

are respectively the upper and lower solutions of (4.2). Also, we have

$$L_g + \frac{L_F}{\Gamma(\alpha+1)} \simeq 0.64 < 1.$$

Then, by Theorem 4.1, (4.2) has a unique positive solution which is bounded by $\underline{x}$ and $\overline{x}$.

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EXISTENCE AND UNIQUENESS OF THE MILD SOLUTION OF AN ABSTRACT SEMILINEAR FRACTIONAL DIFFERENTIAL EQUATION WITH STATE DEPENDENT NONLOCAL CONDITION

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Abstract. The purpose of this paper is to investigate the existence and uniqueness of mild solutions to a semilinear Cauchy problem for an abstract fractional differential equation with state dependent nonlocal condition. Continuous dependence of solutions on initial conditions and local ϵ-approximate mild solution of the considered problem will be discussed.

1. Introduction

Many authors are interested in studying different classes of differential equations by using several forms of accompanying conditions. L. Byszewski [1] inaugurated the study of Cauchy problems for the abstract evolution differential equation $u'(t) + Au(t) = f(t, u(t))$, $t \in (t_0, t_0 + a]$, with the nonlocal condition $u(t_0) + g(t_1, t_2, \ldots, t_p, u(\cdot)) = u_0$. K. Deng [2] indicated that the nonlocal condition can be applied in physics with more precise measurements, accurate results and better effect than the usual initial condition. Deng used the nonlocal form $g(u) = \sum_{k=1}^{p} c_k u(t_k)$, where $c_k, k = 1, 2, \ldots, p$, are given constants. A. El-Sayed et al. [3] discussed the existence of solutions to the deviated-advanced nonlocal differential inclusion

$$\begin{aligned} x'(t) \in& F(t, x(t)) \text{ a.e. } t \in (0,1),\\ \sum_{k=1}^{m} a_k x(\phi(\tau_k)) =& \alpha \sum_{j=1}^{n} b_j x(\psi(\eta_j)), \quad a_k, b_j > 0, \tau_k, \eta_j \in (0,1), \end{aligned}$$

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where F is a set-valued function from $[0,1]\times\mathbb{R}$ into $P(\mathbb{R}^+)$ (the power set of $\mathbb{R}^+$), $\alpha > 0$ is a parameter and ϕ, ψ are, respectively, deviated and advanced given functions. In [3] some special forms of nonlocal conditions are displayed such as $\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = 0$, $\sum_{k=1}^{m} a_k x(\phi(\tau_k)) = \alpha x(\psi(\eta))$, $\tau_k, \eta \in (0,1)$, $\int_0^1 x(\phi(s))ds = 0$, $\int_0^1 x(\psi(s))ds = 0$ and $\int_0^1 x(\phi(s))ds = \alpha \int_0^1 x(\psi(s))ds$. E. Hernandez and D. O'Regan [8] investigated the existence and uniqueness of mild solutions for the class

$$u'(t) = Au(t) + F(t, u(\gamma(t))), \quad t \in [0,a],$$

with the state dependent nonlocal condition

$$u(0) = H(\sigma(u), u) \in X,$$

where A generates an analytic semigroup of linear operators on a Banach space X and $F(\cdot)$, $\gamma(\cdot)$, $H(\cdot)$ and $\sigma(\cdot)$ are suitable continuous functions. The state dependent nonlocal condition generalizes many types of nonlocal conditions. For instance, the conditions $u(0) = u_0$, $u(0) = \sum_{i=1}^{p} c_i u(t_i)$, with $0 \leq t_1 < \cdots < t_p \leq a$ and $u(0) = g(u)$, where $g \in C(C(J,X),X)$ can be considered as state dependent nonlocal conditions. For more details about the state dependent nonlocal conditions see [7]. For the history, applications and significant results on fractional derivatives and integrals, we refer the reader to [10, 12, 14–16, 19].

The aim of our manuscript is to discuss the existence and uniqueness of mild solutions to the state dependent nonlocal problem

$$\begin{aligned} &(1.1) && {}^cD^\alpha u(t) = Au(t) + F\left(t, u(t), u\left(\gamma(t)\right)\right), \quad t \in [0,b], \\ &(1.2) && u(0) = H\left(\sigma(u), u\right) \in X. \end{aligned}$$

${}^cD^\alpha$ denotes the Caputo fractional derivative of order $\alpha \in (0,1)$. The operator A is the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of operators on X and $F(\cdot)$, $\gamma(\cdot)$, $H(\cdot)$ and $\sigma(\cdot)$ are appropriate continuous functions satisfying some hypotheses. We illustrate our results by giving an illustrative example. Further, we discuss the continuous dependence of solutions on initial conditions and local ϵ-approximate mild solution of problem (1.1). The results obtained are based upon the method of semigroups, the contraction mapping principle and the Krasnoselskii's fixed point theorem.

The rest of this paper is organized as follows. In Section 2, we display some notations, main definitions and theorems which are used through out the paper. The main results will be given in Section 3 where we investigate the existence and uniqueness of mild solutions to problem (1.1)–(1.2). In Section 4, we discuss the continuous dependence of solutions on initial conditions and study local ϵ-approximate mild solution of problem (1.1).

2. Preliminaries

Here, we introduce some notations, main definitions and theorems which are crucial in what follows.

Let $J = [0, b]$, where $b > 0$, $(X, \|\cdot\|_X)$ be a Banach space, $B(X)$ be the space of all bounded linear operators from X into X, $C(J, X)$ be the set of all continuous functions $u : J \to X$ with the norm $\|u\|_C = \sup\{\|u(t)\| : u \in C(J, X), t \in J\}$, $C^n(J, X)$ be the set of all n-differentiable functions, with $u^{(n)} \in C(J, X)$, $AC(J, X)$ be the set of all absolutely continuous functions from J into X.

Let ϕ_η, $\eta > 0$, be the function $\phi_\eta(t) = t^{\eta-1}/\Gamma(\eta)$ for $t > 0$ and $\phi_\eta(t) = 0$ for $t \leq 0$. For $\eta = 0$, $\phi_0(t)$ is the Dirac delta function.

Let $A : D(A) \subset X \to X$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of uniformly bounded linear operators on X.

Let $\rho(A)$ be the resolvent set of A, i.e., the set of all complex numbers λ for which $\lambda I - A$ is invertible. The family $\{(\lambda I - A)^{-1}\}_{\lambda\in\rho(A)}$ of bounded linear operators is called the resolvent of A.

A function $\gamma(t) : J \to J$ is said to be a deviated function if $\gamma(t) \leq t$ for all $t \in J$. As an example of a deviated function, we have $\gamma(t) = \beta t$, $\beta \in (0, 1)$.

Farctional integral according to Riemann-Liouville approach and Caputo fractional derivative are given in what follows [10, 14].

Definition 2.1. The fractional integral of order $\alpha > 0$ with the lower limit 0 of the function $u : [0, \infty) \to X$ is defined by

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds = (\phi_\alpha * u)(t), \quad t \geq 0,$$

provided that the right-hand side is point-wise defined. The symbol $*$ stands for the convolution operation and $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.2. The Caputo derivative of order $\alpha \in (0, 1)$ with the lower limit 0 for a function $u \in AC(J, X)$ is defined by

$${}^cD^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} Du(s) ds = I^{1-\alpha} Du(t), \quad D = \frac{d}{dt}.$$

We recall some definitions and properties about C_0-semigroups [6, 13].

Definition 2.3. A family $\{T(t) : 0 \leq t < \infty\}$ of linear operators from X to X is called a C_0-semigroup if:

1. $\|T(t)\| \leq \infty$, i.e., $\sup\{\|T(t)u\| : u \in X, \|u\| \leq 1\} < \infty$ for each $t \geq 0$;
2. $T(t + s)u = T(t)T(s)u$ for all $u \in X$ and all $t, s \geq 0$;
3. $T(0)u = u$ for all $u \in X$;
4. $t \mapsto T(t)u$ is continuous for $t \geq 0$ for each $u \in X$.

Definition 2.4. For the C_0-semigroup $\{T(t)\}_{t\geq 0}$, the following holds.

1. There exist constants $N \geq 1$ and $\omega \geq 0$ such that $\|T(t)\| \leq Ne^{\omega t}$ for $0 \leq t < \infty$.
2. $\{T(t)\}_{t\geq 0}$ is called a C_0-contraction semigroup if $\|T(t)\| \leq 1$ for each $t \geq 0$.
3. $\{T(t)\}_{t\geq 0}$ is called a uniformly continuous semigroup if $t \mapsto T(t)$ is continuous in the uniform operator topology.

4. The linear operator $A : D(A) \subset X \to X$ is called the (infinitesimal) generator of $\{T(t)\}_{t\geq 0}$ where the domain $D(A)$ of A is the set of all functions $u \in X$ for which the limit $\lim_{t\to 0^+}(T(t)u-u)/t$ exists in X. The previous limit gives Au in X. The domain $D(A)$ is dense in X and A is closed.
5. $\{T(t)\}_{t\geq 0}$ is a compact semigroup, if and only if $\{T(t)\}_{t\geq 0}$ is continuous in the uniform operator topology and $(\lambda I - A)^{-1}$ is compact for $\lambda \in \rho(A)$.
6. If $\{T(t)\}_{t\geq 0}$ is compact, then $(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda s}T(s)ds$, $\mathrm{Re}\lambda > \omega$.

A useful Krasnoselskii's fixed point theorem [4] and Gronwall's inequality [18] are given in what follows.

Theorem 2.1. *Let X be a Banach space, Y be a bounded, closed and convex subset of X and K, Q be operators of Y into X such that $Ku+Qv \in Y$ for every pair $u, v \in Y$. If Q is a contraction and K is completely continuous, then the equation $Ku+Qu = u$ has a solution in Y.*

Theorem 2.2. *Suppose $\alpha > 0$, $a(t)$ is a nonnegative function locally integrable on J, $g(t)$ is a nonnegative, nondecreasing continuous function defined on J, $g(t) \leq c$ (constant), and $u(t)$ is nonnegative and locally integrable on J with*

$$u(t) \leq a(t) + g(t)\int_0^t (t-s)^{\alpha-1}u(s)ds,$$

then

$$u(t) \leq a(t) + \int_0^t \left[\sum_{n=1}^{\infty}\frac{(g(t)\Gamma(\alpha))^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}a(s)\right]ds, \quad t \in J. \tag{2.1}$$

3. Existence and Uniqueness of Solutions

In this part, we investigate the existence and uniqueness of continuous mild solutions to the nonlocal problem (1.1)–(1.2).

Consider the one-sided stable probability density [11, 19]

$$\psi_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-1-\alpha n}\frac{\Gamma(1+\alpha n)}{n!}\sin(\alpha n\pi), \quad \alpha \in (0,1), \theta \in (0,\infty), \tag{3.1}$$

whose Laplace transform is given by

$$\int_0^\infty e^{-\lambda\theta}\psi_\alpha(\theta)d\theta = e^{-\lambda^\alpha}, \tag{3.2}$$

and consider the probability density function

$$h_\alpha(\theta) = \frac{1}{\alpha}\theta^{-1-1/\alpha}\psi_\alpha(\theta^{-1/\alpha}), \quad \theta \in (0,\infty), \tag{3.3}$$

which satisfies

$$h_\alpha(\theta) \geq 0, \quad \int_0^\infty h_\alpha(\theta)d\theta = 1 \text{ and } \int_0^\infty \theta^\nu h_\alpha(\theta) = \frac{\Gamma(1+\nu)}{\Gamma(1+\alpha\nu)}, \quad \nu \in [0,1]. \tag{3.4}$$

We have relied on the following lemma to define a mild solution for problem (1.1)–(1.2).

Lemma 3.1. *The solution of the nonlocal problem* (1.1)–(1.2) *can be expressed by the integral equation*

$$
\begin{aligned}
u(t) &= \int_0^\infty h_\alpha(\theta)T(t^\alpha\theta)H\left(\sigma(u),u\right)d\theta \\
&\quad + \alpha\int_0^t\int_0^\infty \theta(t-s)^{\alpha-1}h_\alpha(\theta)T((t-s)^\alpha\theta)F(s,u(s),u(\gamma(s)))d\theta ds.
\end{aligned} \tag{3.5}
$$

Proof. Let $u(t)$ be a solution of problem (1.1). Operating I^α on both sides of (1.1), we obtain

$$
u(t) = u(0) + \phi_\alpha(t) * Au(t) + \phi_\alpha(t) * F(t,u(t),u(\gamma(t))). \tag{3.6}
$$

Let $U(\lambda) = \int_0^\infty e^{-\lambda s}u(s)ds$ and $P(\lambda) = \int_0^\infty e^{-\lambda s}F(s,u(s),u(\gamma(s)))ds,\ \lambda > 0$. Taking Laplace transform for (3.6), we get

$$
\begin{aligned}
U(\lambda) &= \frac{1}{\lambda}u(0) + \frac{1}{\lambda^\alpha}AU(\lambda) + \frac{1}{\lambda^\alpha}P(\lambda) \\
&= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}u(0) + (\lambda^\alpha I - A)^{-1}P(\lambda) \\
&= \lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}T(s)u(0)ds + \left(\int_0^\infty e^{-\lambda^\alpha s}T(s)ds\right)P(\lambda),
\end{aligned} \tag{3.7}
$$

where I is the identity operator defined on X. Using (3.2), direct calculation gives that

$$
\begin{aligned}
\lambda^{\alpha-1}\int_0^\infty e^{-\lambda^\alpha s}T(s)u(0)ds &= \int_0^\infty \alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}T(t^\alpha)u(0)dt \\
&= \int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}\left(e^{-(\lambda t)^\alpha}\right)T(t^\alpha)u(0)dt \\
&= \int_0^\infty\int_0^\infty \theta\psi_\alpha(\theta)e^{-\lambda t\theta}T(t^\alpha)u(0)d\theta dt \\
&= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)u(0)d\theta\right]dt
\end{aligned} \tag{3.8}
$$

and

$$
\begin{aligned}
&\left(\int_0^\infty e^{-\lambda^\alpha s}T(s)ds\right)P(\lambda) \\
&= \left(\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}T(t^\alpha)dt\right)P(\lambda) \\
&= \left(\int_0^\infty\int_0^\infty \alpha\psi_\alpha(\theta)e^{-\lambda t\theta}T\left(t^\alpha\right)t^{\alpha-1}d\theta dt\right)P(\lambda) \\
&= \left(\int_0^\infty e^{-\lambda t}\left(\alpha\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}d\theta\right)dt\right)P(\lambda) \\
&= \int_0^\infty e^{-\lambda t}\left[\left(\alpha\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}d\theta\right) * F(t,u(t),u(\gamma(t)))\right]dt
\end{aligned} \tag{3.9}
$$

$$=\int_0^\infty e^{-\lambda t}\left[\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}F(s,u(s),u(\gamma(s)))d\theta ds\right]dt.$$

Substituting (1.2), (3.8) and (3.9) into (3.7), we get

$$\begin{aligned}&U(\lambda)\\ =&\int_0^\infty e^{-\lambda t}\left[\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)H\left(\sigma(u),u\right)d\theta\right]dt\\ &+\int_0^\infty e^{-\lambda t}\left[\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}F(s,u(s),u(\gamma(s)))d\theta ds\right]dt.\end{aligned}$$

Inverting the Laplace transform, we obtain

$$\begin{aligned}u(t)=&\int_0^\infty \psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)H\left(\sigma(u),u\right)d\theta\\ &+\alpha\int_0^t\int_0^\infty \psi_\alpha(\theta)T\left(\frac{(t-s)^\alpha}{\theta^\alpha}\right)\frac{(t-s)^{\alpha-1}}{\theta^\alpha}F(s,u(s),u(\gamma(s)))d\theta ds.\end{aligned}$$

Using (3.3), we get (3.5). This completes the proof. □

Define the operators $\{S_\alpha(t)\}_{t\geq 0}$ and $\{R_\alpha(t)\}_{t\geq 0}$ for any $u\in X$, by

$$S_\alpha(t)u=\int_0^\infty h_\alpha(\theta)T(t^\alpha\theta)ud\theta \text{ and } R_\alpha(t)u=\alpha\int_0^\infty \theta h_\alpha(\theta)T(t^\alpha\theta)ud\theta. \tag{3.10}$$

Now, the mild solution of the nonlocal problem (1.1)–(1.2) can be defined by following.

Definition 3.1. A function $u(t)\in C(J,X)$ is called a mild solution of the nonlocal problem (1.1)–(1.2) if $u(0)=H\left(\sigma(u),u\right)$ and

$$u(t)=S_\alpha(t)H\left(\sigma(u),u\right)+\int_0^t (t-s)^{\alpha-1}R_\alpha(t-s)F(s,u(s),u(\gamma(s)))ds. \tag{3.11}$$

The following lemma gives some basic properties of S_α and R_α which are useful in the sequel [9, 17].

Lemma 3.2. *The operators $S_\alpha(t)$, $t\geq 0$, and $R_\alpha(t)$, $t\geq 0$, have the following properties.*

1. *For any fixed $t\geq 0$, the operators $S_\alpha(t)$ and $R_\alpha(t)$ are linear and bounded operators, which means that for any $u\in X$*

$$\|S_\alpha(t)u\|\leq M\|u\| \quad \textit{and} \quad \|R_\alpha(t)u\|\leq \frac{\alpha M}{\Gamma(1+\alpha)}\|u\|, \quad \textit{for all } t\in J, \tag{3.12}$$

where $M:=\sup_{t\in[0,\infty)}\|T(t)\|_{B(X)}<\infty$.

2. *For every $u\in X$, $t\mapsto S_\alpha(t)u$ and $t\mapsto R_\alpha(t)u$ are continuous functions from $[0,\infty)$ into X.*

3. *The operators $S_\alpha(t)$, $t\geq 0$, and $R_\alpha(t)$, $t\geq 0$, are strongly continuous in $[0,\infty)$, which means that for all $u\in X$ and $0\leq t_1<t_2\leq b$, we have*

$$\|S_\alpha(t_2)u-S_\alpha(t_1)u\|\to 0 \quad \textit{and} \quad \|R_\alpha(t_2)u-R_\alpha(t_1)u\|\to 0 \textit{ as } t_2\to t_1.$$

4. *If $T(t)$ is a compact operator for every $t > 0$, then the operators $S_\alpha(t)$ and $R_\alpha(t)$ are also compact for every $t > 0$.*

In order to discuss the existence and uniqueness of mild solutions to the nonlocal problem (1.1)–(1.2), consider the following assumptions:

(H_1) $T(t)$ is a compact operator for each $t > 0$;
(H_2) $\gamma : J \to J$ is a deviated continuous function, i.e., $\gamma(t) \leq t$, $t \in J$;
(H_3) $\sigma : C(J, X) \to J$ is a Lipschitz function with Lipschitz constant L_σ;
(H_4) $F : J \times X^2 \to X$ is continuous and there exist constants $p, q > 0$ such that

$$\|F(t, u_1, v_1) - F(t, u_2, v_2)\| \leq p\|u_1 - u_2\| + q\|v_1 - v_2\|, \quad \text{with } f = \max_{t \in J} \|F(t, 0, 0)\|;$$

(H_5) $H : J \times C(J, X) \to X$ is continuous and there exists $\lambda > 0$ such that

$$\|H(u_1, u_2) - H(v_1, v_2)\| \leq \|u_1 - v_1\| + \lambda\|u_2 - v_2\|$$

and $H(\cdot)$ is bounded, with $h = \sup_{u \in C(J,X)} \|H(\sigma(u), u)\|$.

Noting that for all $u, v \in C(J, X)$:
(a) from (H_3) and (H_5),

$$\|H(\sigma(u), u) - H(\sigma(v), v)\| \leq (\lambda + L_\sigma)\|u - v\|; \tag{3.13}$$

(b) from (H_4),

$$\begin{aligned} \|F(t, u, v)\| &\leq \|F(t, u, v) - F(t, 0, 0)\| + \|F(t, 0, 0)\| \\ &\leq p\|u\| + q\|v\| + f. \end{aligned} \tag{3.14}$$

For the existence of mild solutions to problem (1.1)–(1.2), we give the following theorem.

Theorem 3.1. *Let the assumptions (H_1)–(H_5) be satisfied. Then, the nonlocal problem (1.1)–(1.2) has at least one mild solution $u \in C(J, X)$ if*

$$\max\left\{(\lambda + L_\sigma)M, \frac{M(p+q)b^\alpha}{\Gamma(1+\alpha)}\right\} < 1.$$

Proof. Let $N(r)$ be the nonempty, closed and convex subset of $C(J, X)$ such that

$$N(r) = \left\{u \in C(J, X) : \|u\| \leq r,\, r = \frac{M[b^\alpha f + h\Gamma(1+\alpha)]}{\Gamma(1+\alpha) - M(p+q)b^\alpha}\right\}.$$

Let $W : C(J, X) \to C(J, X)$ be the operator given by $Wu(t) = Ku(t) + Qu(t)$, where

$$Ku(t) = \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u(s), u(\gamma(s)))ds \tag{3.15}$$

and

$$Qu(t) = S_\alpha(t) H(\sigma(u), u). \tag{3.16}$$

The proof will be given in four steps.

Step 1. $Ku + Qv \in N(r)$ whenever $u, v \in N(r)$.

Using (3.15) and (3.16) with applying (3.12), we have

$$\begin{aligned}
&\|Ku(t)+Qv(t)\| \\
\leq &\int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)F(s,u(s),u(\gamma(s)))\|ds+\|S_\alpha(t)H\left(\sigma(v),v\right)\| \\
\leq &\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s,u(s),u(\gamma(s)))\|ds+M\|H\left(\sigma(v),v\right)\|.
\end{aligned}$$

Using (3.14) and (H_5), we obtain

$$\|Ku(t)+Qv(t)\| \leq \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left[p\|u(s)\|+q\|u(\gamma(s))\|+f\right]ds+hM.$$

For $u\in N(r)$, we get

$$\|Ku+Qv\| \leq M\left(\frac{b^\alpha[f+(p+q)r]}{\Gamma(\alpha+1)}+h\right)=r. \tag{3.17}$$

Thus, $Ku+Qv\in N(r)$ whenever $u,v\in N(r)$.

Step 2. K is continuous.

Let $\{u_n\}_{n=1}^\infty$ be a sequence in $C(J,X)$ such that u_n tends to $u\in C(J,X)$ as n tends to ∞ for all $t\in J$.

Using (3.15) and (3.12) we have

$$\begin{aligned}
&\|Ku_n(t)-Ku(t)\| \\
\leq &\int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)\left[F(s,u_n(s),u_n(\gamma(s)))-F(s,u(s),u(\gamma(s)))\right]\|ds \\
\leq &\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s,u_n(s),u_n(\gamma(s)))-F(s,u(s),u(\gamma(s)))\|ds.
\end{aligned}$$

Applying (H_4), we obtain

$$\begin{aligned}
&\|Ku_n(t)-Ku(t)\| \\
\leq &\frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left[p\|u_n(s)-u(s)\|+q\|u_n(\gamma(s))-u(\gamma(s))\|\right]ds.
\end{aligned}$$

Then

$$\|Ku_n-Ku\| \leq \frac{Mb^\alpha(p+q)}{\Gamma(\alpha+1)}\|u_n-u\|,$$

which tends to zero as n tends to ∞. Thus, K is a continuous operator.

Step 3. K is compact.

From (3.12), (3.14) and (3.15), we have

$$\begin{aligned}
\|Ku(t)\| &\leq \int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)F(s,u(s),u(\gamma(s)))\|ds \\
&\leq \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s,u(s),u(\gamma(s)))\|ds
\end{aligned}$$

$$\leq \frac{\alpha M}{\Gamma(\alpha+1)} \int_0^t (t-s)^{\alpha-1} \left[p\|u(s)\| + q\|u(\gamma(s))\| + f\right] ds.$$

For $u, v \in N(r)$, we get

$$\|Ku\| \leq \frac{Mb^\alpha[f+(p+q)r]}{\Gamma(1+\alpha)}.$$

So, the class of functions $\{Ku(t)\}$ is uniformly bounded in $N(r)$.

Let $0 \leq t_1 \leq t_2 \leq b$. From (3.15), we have

$$\begin{aligned}
&\|Ku(t_2) - Ku(t_1)\| \\
\leq &\left\| \int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] R_\alpha(t_2-s) F(s, u(s), u(\gamma(s))) ds \right\| \\
&+ \left\| \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} R_\alpha(t_2-s) F(s, u(s), u(\gamma(s))) ds \right\| \\
&+ \left\| \int_0^{t_1} (t_1-s)^{\alpha-1} \left[R_\alpha(t_2-s) - R_\alpha(t_1-s)\right] F(s, u(s), u(\gamma(s))) ds \right\|.
\end{aligned}$$

Applying (3.12) and (3.14), we obtain

$$\begin{aligned}
&\|Ku(t_2) - Ku(t_1)\| \\
\leq &\frac{\alpha M[f+(p+q)r]}{\Gamma(1+\alpha)} \left[\int_0^{t_1} \left[(t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1}\right] ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds\right] \\
&+ \int_0^{t_1} (t_1-s)^{\alpha-1} \| \left(R_\alpha(t_2-s) - R_\alpha(t_1-s)\right) F(s, u(s), u(\gamma(s)))\| ds.
\end{aligned}$$

Since $\lim_{t_2 \to t_1} \|R_\alpha(t_2-s) - R_\alpha(t_1-s)\| = 0$ uniformly for $0 \leq s \leq t_1 \leq t_2 \leq b$, it is easy to see that $\|Ku(t_2) - Ku(t_1)\| \to 0$ as $t_2 \to t_1$. Thus, $\{Ku(t)\}$ is equicontinuous. By Arzela-Ascoli theorem, $\{Ku(t)\}$ is relatively compact and K is a compact operator.

Step 4. Q is a contraction.

Let $u, v \in N(r)$. From (3.16), we have

$$\|Qu(t) - Qv(t)\| \leq \|S_\alpha(t) \left(H(\sigma(u), u) - H(\sigma(v), v)\right)\|,$$

then by applying (3.12) and (3.13), we get $\|Qu - Qv\| \leq (\lambda + L_\sigma)M\|u - v\|$. Since $(\lambda + L_\sigma)M < 1$, Q is a contraction operator [5].

As a consequence of Krasonselskii's fixed point theorem, the operator W has at least one fixed point. Therefore, the nonlocal problem (1.1)–(1.2) has at least one mild solution $u \in N(r)$ which completes the proof. □

For the uniqueness of mild solutions to problem (1.1)–(1.2), we give the following theorem.

Theorem 3.2. *Let the assumptions* (H_1)-(H_5) *be satisfied. Then, the nonlocal problem* (1.1)–(1.2) *has a unique mild solution* $u \in C(J, X)$ *if*

$$M\left(\lambda + L_\sigma + \frac{b^\alpha(p+q)}{\Gamma(1+\alpha)}\right) < 1.$$

Proof. Consider the operator $W : C(J, X) \to C(J, X)$ such that

$$Wu(t) = S_\alpha(t)H\left(\sigma(u), u\right) + \int_0^t (t-s)^{\alpha-1}R_\alpha(t-s)F(s, u(s), u(\gamma(s)))ds. \tag{3.18}$$

The proof will be given in two steps.

Step 1. W maps $N(r)$ into itself. From (3.12) and (3.18), we have

$$\begin{aligned}\|Wu(t)\| &\le \|S_\alpha(t)H\left(\sigma(u), u\right)\| + \int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)F(s, u(s), u(\gamma(s)))\|ds \\ &\le M\|H\left(\sigma(u), u\right)\| + \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s, u(s), u(\gamma(s)))\|ds.\end{aligned}$$

Let $u \in N(r)$, with applying (3.14), we get

$$\|Wu\| \le M\left(\frac{b^\alpha[f + (p+q)r]}{\Gamma(1+\alpha)} + h\right) = r.$$

Therefore, $WN(r) \subseteq N(r)$.

Step 2. W is a contraction. Let $u, v \in N(r)$. Using (3.12), (3.13) and (3.18), we obtain

$$\begin{aligned}&\|Wu(t) - Wv(t)\| \\ \le&\|S_\alpha(t)\left[H\left(\sigma(u), u\right) - H\left(\sigma(v), v\right)\right]\| \\ &+ \int_0^t (t-s)^{\alpha-1}\|R_\alpha(t-s)\left[F(s, u(s)u(\gamma(s))) - F(s, v(s), v(\gamma(s)))\right]\|ds \\ \le&M\|H\left(\sigma(u), u\right) - H\left(\sigma(v), v\right)\| \\ &+ \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\|F(s, u(s)u(\gamma(s))) - F(s, v(s), v(\gamma(s)))\|ds \\ \le&M(\lambda + L_\sigma)\|u(t) - v(t)\| + \\ &+ \frac{\alpha M}{\Gamma(\alpha+1)}\int_0^t (t-s)^{\alpha-1}\left[p\|u(s) - v(s)\| + q\|u(\gamma(s)) - v(\gamma(s))\|\right]ds.\end{aligned}$$

Then

$$\|Wu - Wv\| \le M\left(\lambda + L_\sigma + \frac{b^\alpha(p+q)}{\Gamma(1+\alpha)}\right)\|u - v\|.$$

Since $M\left(\lambda + L_\sigma + \frac{b^\alpha(p+q)}{\Gamma(1+\alpha)}\right) < 1$, W is a contraction operator and it has a unique fixed point $u \in N(r)$ which is the unique mild solution of the nonlocal problem (1.1)–(1.2). Therefore, we get the required. $\square$

We finalize this section by the following example to illustrate our results.

Example 3.1. Let $X = L_2\left([0, \pi], \mathbb{R}\right)$, the space of all functions for which the 2^{nd} power of the absolute value is Lebesgue integrable. Consider a fractional partial differential equations of the form

$$\begin{cases} {}_tD^{0.4}x(t,z) + {}_zD^2x(t,z) = \dfrac{1}{19+e^t}\left(\dfrac{\|x(t)\|}{1+\|x(t)\|} + \dfrac{\|x(0.7t)\|}{1+\|x(0.7t)\|}\right), \quad t \in [0,1], \\ x(0,z) = 0.3x\left(\sigma(x), z\right) \in L_2\left([0,\pi], \mathbb{R}\right), \quad z \in [0,\pi], \end{cases}$$

where ${}_tD^{0.4}$ denotes Caputo fractional partial derivatives with $\alpha = 0.4$.

Let A be an operator defined by $Ax = -x''$ with the domain

$$D(A) = \left\{x(\cdot) \in L_2\left([0,\pi], \mathbb{R}\right) : x' \text{ is absolutely continuous}, x(0) = x(\pi) = 0\right\},$$

then A generates a C_0-semigroup $\{T(t)\}_{t\geq 0}$ which is compact [21], that is (H_1) holds. Operator A has the natural eigenvalues $m, -m^2$, with normalized eigenvectors $x_m(t) = (2/\pi)^{0.5}\sin(mt)$. For each $x \in L_2\left([0,\pi], \mathbb{R}\right)$, $T(t)v = \sum_{m=1}^{\infty} e^{-m^2t}\langle v, x_m\rangle x_m$. In particular, $T(\cdot)$ is a uniformly stable semigroup and $\|T(t)\|_{L_2[0,\pi]} \leq e^{-t}$. Our problem can be reformed as the nonlocal problem (1.1)–(1.2).

Let $\sigma \in C_{Lip}\left(C\left([0,1], L_2([0,\pi], \mathbb{R})\right), [0,1]\right)$, with $L_\sigma = 0.2$. Defining $H(\cdot)$ by $H(t,x) = 0.3x$, then $\|H(t,x) - H(t,y)\| \leq 0.3\|x-y\|$. Clearly, $\lambda = 0.3$. Let

$$F(t, x(t), y(t)) = \frac{1}{19+e^t}\left(\frac{\|x(t)\|}{1+\|x(t)\|} + \frac{\|y(t)\|}{1+\|y(t)\|}\right),$$

then $f = 0$ and

$$\begin{aligned} &\|F(t, x_1(t), y_1(t)) - F(t, x_2(t), y_2(t))\| \\ \leq &\frac{1}{20}\left(\left\|\frac{\|x_1(t)\|}{1+\|x_1(t)\|} - \frac{\|x_2(t)\|}{1+\|x_2(t)\|}\right\| + \left\|\frac{\|y_1(t)\|}{1+\|y_1(t)\|} - \frac{\|y_2(t)\|}{1+\|y_2(t)\|}\right\|\right) \\ \leq &\frac{1}{20}\left(\|x_1(t)\| - \|x_2(t)\| + \|y_1(t)\| - \|y_2(t)\|\right) \\ \leq &\frac{1}{20}\left(\|x_1(t) - x_2(t)\| + \|y_1(t) - y_2(t)\|\right). \end{aligned}$$

So, we have $p = q = 0.05$. Therefore, all conditions of Theorem 3.2 are satisfied and the considered problem has a unique continuous mild solution.

4. Continuous Dependence and ϵ-Approximate Mild Solution

In this section, we discuss the continuous dependence of solutions on initial conditions and the local ϵ-approximate mild solution of problem (1.1).

Theorem 4.1. *Let the assumptions $(H_1), (H_2)$ and (H_4) be satisfied and $u_1(t)$ and $u_2(t)$ be the solutions of problem (1.1) corresponding to $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Then*

$$\|u_1 - u_2\| \leq M\|u_1^0 - u_2^0\|\left(1 + \sum_{n=1}^{\infty}\frac{[M(p+q)]^n}{\Gamma(1+n\alpha)}b^{n\alpha}\right). \tag{4.1}$$

Proof. Let $u_1(t)$ and $u_2(t)$ be the solutions of problem (1.1) corresponding to $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Hence,

$$^cD^\alpha u_1(t) = Au_1(t) + F\left(t, u_1(t), u_1\left(\gamma(t)\right)\right), \quad u_1(0) = u_1^0,\ t \in J,$$

and

$$^cD^\alpha u_2(t) = Au_2(t) + F\left(t, u_2(t), u_2\left(\gamma(t)\right)\right), \quad u_2(0) = u_2^0,\ t \in J.$$

This implies

$$u_1(t) = S_\alpha(t)u_1^0 + \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s)))ds$$

and

$$u_2(t) = S_\alpha(t)u_2^0 + \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s)))ds.$$

So, we have

$$\begin{aligned} &\|u_1(t) - u_2(t)\| \\ \leq &\|S_\alpha(t)(u_1^0 - u_2^0)\| \\ &+ \int_0^t (t-s)^{\alpha-1} \|R_\alpha(t-s)[F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s)))]\| ds. \end{aligned}$$

Using $(H_4), (H_2)$ and (3.12), we get

$$\|u_1(t) - u_2(t)\| \leq M\|u_1^0 - u_2^0\| + \frac{\alpha M(p+q)}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \|u_1(s) - u_2(s)\| ds.$$

Applying Theorem 2.2, we obtain

$$\|u_1(t) - u_2(t)\| \leq M\|u_1^0 - u_2^0\| + \int_0^t \sum_{n=1}^{\infty} \frac{[M(p+q)]^n}{\Gamma(n\alpha)} (t-s)^{n\alpha-1} M \|u_1^0 - u_2^0\| ds.$$

Therefore, it is easy to get the required inequality. $\square$

One can note the following.
(a) Since

$$\frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha n + \alpha + 1)} \leq \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha n + 2)} = \frac{1}{\alpha n + 1},$$

then

$$\lim_{n\to\infty} \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha n + \alpha + 1)} = 0.$$

Therefore, by the ratio test,

$$\sum_{n=1}^{\infty} \frac{[M(p+q)]^n}{\Gamma(1+n\alpha)} b^{n\alpha}$$

is a convergent series.

(b) Inequality (4.1) shows continuous dependence of solutions of the problem (1.1) on initial conditions as well as it gives the uniqueness which follows by putting $u_1^0 = u_2^0$.

Definition 4.1. A solution of the integral inequality

$$\left\| u(t) - S_\alpha(t)u(0) - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u(s), u(\gamma(s))) ds \right\| \leq \epsilon$$

is called a local ϵ-approximate mild solution of problem (1.1).

Theorem 4.2. *Let the assumptions* $(H_1), (H_1)$ *and* (H_4) *be satisfied. Suppose that* $u_1(t)$ *and* $u_2(t)$ *are* ϵ*-approximate mild solutions of problem* (1.1) *corresponding to* $u_1(0) = u_1^0$ *and* $u_2(0) = u_2^0$*, respectively. Then*

$$\|u_1 - u_2\| \leq \left(\epsilon_1 + \epsilon_2 + M\|u_1^0 - u_2^0\|\right) \left(1 + \sum_{n=1}^{\infty} \frac{[M(p+q)]^n}{\Gamma(1+n\alpha)} b^{n\alpha}\right). \tag{4.2}$$

Proof. Let $u_1(t)$ and $u_2(t)$ be ϵ-approximate mild solutions of problem (1.1) corresponding to $u_1(0) = u_1^0$ and $u_2(0) = u_2^0$, respectively. Hence

$$\left\| u_1(t) - S_\alpha(t)u_1^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s))) ds \right\| \leq \epsilon_1$$

and

$$\left\| u_2(t) - S_\alpha(t)u_2^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s))) ds \right\| \leq \epsilon_2.$$

We know that

$$\|z\| - \|y\| \leq \|z - y\| \leq \|z\| + \|y\|, \quad \text{for all } z, y \in X,$$

so let $z = u_1(t) - u_2(t)$ and

$$\begin{aligned} y =& S_\alpha(t)(u_1^0 - u_2^0) \\ &+ \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) \left[F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s))) \right] ds. \end{aligned}$$

Hence,

$$\begin{aligned} &\|u_1(t) - u_2(t)\| \\ &- \Big\| S_\alpha(t)(u_1^0 - u_2^0) \\ &+ \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) [F(s, u_1(s), u_1(\gamma(s))) - F(s, u_2(s), u_2(\gamma(s)))] ds \Big\| \\ \leq& \Big\| \Big[u_1(t) - S_\alpha(t)u_1^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s))) ds \Big] \\ &- \Big[u_2(t) - S_\alpha(t)u_2^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s))) ds \Big] \Big\| \\ \leq& \left\| u_1(t) - S_\alpha(t)u_1^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_1(s), u_1(\gamma(s))) ds \right\| \\ &+ \left\| u_2(t) - S_\alpha(t)u_2^0 - \int_0^t (t-s)^{\alpha-1} R_\alpha(t-s) F(s, u_2(s), u_2(\gamma(s))) ds \right\| \\ \leq& \epsilon_1 + \epsilon_2. \end{aligned}$$

Then

$$
\begin{aligned}
&\|u_1(t)-u_2(t)\| \\
\leq &\epsilon_1+\epsilon_2+\|S_\alpha(t)(u_1^0-u_2^0)\| \\
&+\int_0^t (t-s)^{\alpha-1}\left\|R_\alpha(t-s)[F(s,u_1(s),u_1(\gamma(s)))-F(s,u_2(s),u_2(\gamma(s)))]\right\| ds.
\end{aligned}
$$

Using (3.12), (H_4) and (H_2), we obtain

$$
\begin{aligned}
&\|u_1(t)-u_2(t)\| \\
\leq &\epsilon_1+\epsilon_2+M\|u_1^0-u_2^0\|+\frac{\alpha M(p+q)}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}\|u_1(s)-u_2(s)\|ds.
\end{aligned}
$$

Applying Theorem 2.2, we have

$$
\begin{aligned}
\|u_1(t)-u_2(t)\| \leq &\epsilon_1+\epsilon_2+M\|u_1^0-u_2^0\| \\
&+\int_0^t \sum_{n=1}^{\infty}\frac{\left[\frac{\alpha M(p+q)\Gamma(\alpha)}{\Gamma(1+\alpha)}\right]^n}{\Gamma(n\alpha)}(t-s)^{n\alpha-1}\left(\epsilon_1+\epsilon_2+M\|u_1^0-u_2^0\|\right) ds.
\end{aligned}
$$

Therefore, we get the required. □

Remark 4.1. From Definition 4.1, if $\epsilon=0$, then $u(t)$ is a solution of the integral equation

$$
u(t)=S_\alpha(t)u(0)+\int_0^t (t-s)^{\alpha-1}R_\alpha(t-s)F(s,u(s),u(\gamma(s)))ds,
$$

which is a mild solution of problem (1.1).

Remark 4.2. From (4.2), $\epsilon_1=\epsilon_2=0$ implies $u_1(t)$ and $u_2(t)$ are the mild solutions of (1.1) corresponding to the initial conditions $u_1(0)=u_1^0$ and $u_2(0)=u_2^0$, respectively. Further, (4.2) reduced to (4.1), which gives continuous dependence of mild solutions of (1.1) corresponding to initial conditions.

Remark 4.3. (4.2) proves the uniqueness of mild solutions of (1.1) if $\epsilon_1=\epsilon_2=0$ and $u_1^0=u_2^0$.

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GEOMETRIC INVARIANTS UNDER THE MÖBIUS ACTION OF THE GROUP $SL(2;\mathbb{R})$

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ABSTRACT. In this paper we have introduced new invariant geometric objects in the homogeneous spaces of complex, dual and double numbers for the principal group $SL(2;\mathbb{R})$, in the Klein's Erlangen Program. We have considered the action as the Möbius action and have taken the spaces as the spaces of complex, dual and double numbers. Some new decompositions of $SL(2;\mathbb{R})$ have been used.

1. INTRODUCTION

In this paper, we have described and extended the geometry of the group $SL(2;\mathbb{R})$ on the two-dimensional space in the line of the Erlangen program defined by Felix Klein. The Erlangen program states that, have a geometric space and a transformation group, a geometry is the study of the invariance of geometric objects under a group action of that transformation group [11, 17]. A geometry is often referred to as a pair (G, X), where G is the transformation group and X is the geometric space. This pair is called Klein's geometry [14]. Vladimir V. Kisil has shown three geometries under linear fractional transformation of the group $SL(2;\mathbb{R})$ [8, 10]. By the geometric spaces taken, they are classified as elliptic, parabolic and hyperbolic cases [8]. The elliptic case is isomorphic to the upper half plane of the space of complex numbers. Similarly, parabolic case and hyperbolic cases are isomorphic to the upper half plane of spaces of dual and double numbers respectively, [16]. Similar work can be done for lower half plane also. Kisil in his paper [8] worked on the geometric objects which are lying strictly in the upper half planes of complex, dual and double numbers. He did not

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mention about other geometric objects. In this paper, we have taken the geometric objects of paper [8] which intercept the U-axis at two real points and those which have U-axis as a tangent and showed that they are also invariants under some restrictions.

The aim of our work is to extend the work of Kisil in the same line to include more invariant geometric objects in the existing $SL(2;\mathbb{R})$ geometry. The reason behind it is to fill the gaps in the existing geometries of $SL(2;\mathbb{R})$. Erlangen program of $SL(2;\mathbb{R})$ applications in mathematics and theoretical physics, e.g optics, classical mechanics (see Section 5), functional calculus [7] etc.

In this paper, we restrict ourselves to the geometry of the Lie group $SL(2;R)$. However interested readers may find construction of some other pairs of Klein's geometry in [14, 15] of the type (G, X), where G is a principal group and X be a geometric space.

In next section we have discussed some of the terminologies and results of our predecessors in this subject and in section 3 and 4 we mentioned our results. After that some applications in physics have been described in Section 5.

2. Preliminaries

In our paper, we shall use the following terminologies.

Definition 2.1 (Transformation group)**.** A transformation group G is a non-void set of mappings of a set X into itself with the following properties:

(a) the identity map is included in G;
(b) if $g_1 \in G$ and $g_2 \in G$, then $g_1 g_2 \in G$;
(c) if $g \in G$, then g^{-1} exists and belongs to G.

Definition 2.2 (Homogeneous space)**.** A topological space X together with an abstract group $(G, *)$, which acts on X transitively is said to be a homogeneous space [4].

Definition 2.3 (Isotropy subgroup)**.** For an abstract group G and for a group action of it on a set X, the set of elements $G_x = \{g \in G : \ g \cdot x = x\}$ forms a subgroup of G which is called the isotropy (fix) subgroup of G by x.

In his paper [8], Kisil have shown that if H is a one-dimensional subgroup of $SL(2;\mathbb{R})$, namely $K = \left\{ \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} : \ t \in \mathbb{R} \right\}$, $N = \left\{ \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} : \ \nu \in \mathbb{R} \right\}$ and $A = \left\{ \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} : \ \alpha\ (> 0) \in \mathbb{R} \right\}$, then $SL(2;\mathbb{R})/H$ $(H = K, N, A)$ is a homogeneous space.

Dual numbers and double numbers are defined by $\mathbb{O} = \{u + iv : i^2 = 0,\ (u, v) \in \mathbb{R}^2\}$ and $\mathbb{D} = \{u + iv : i^2 = 1,\ (u, v) \in \mathbb{R}^2\}$, respectively.

Complex numbers with dual and double numbers are denoted as

$$\mathbb{R}^\sigma = \{a + ib : \ i^2 = \sigma = -1, 0, 1,\ (a, b) \in \mathbb{R}^2\}.$$

The Möbius action is defined as $g : SL(2;\mathbb{R}) \times \mathbb{R}^\sigma \to \mathbb{R}^\sigma$ by

$$g \cdot z = \frac{az+b}{cz+d},$$

for $z \in \mathbb{R}^\sigma$ and $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$. It is a left action of $SL(2;\mathbb{R})$ on $\mathbb{R}^\sigma$, i.e., $g_1(g_2 \cdot z) = (g_1 g_2) \cdot z$.

To study the action we shall decompose $g \in SL(2;\mathbb{R})$ by the Iwasawa decomposition as $g = g_a g_n g_k$, where $g_a \in A$, $g_n \in N$ and $g_k \in K$ [8, 12].

From now on througout this paper we shall denote elliptic case $(SL(2;\mathbb{R})/K)$ which is isomorphic to the upper half plane of the space of complex numbers [8], parabolic case $(SL(2;\mathbb{R})/N)$ which is isomorphic to the upper half plane of dual numbers and hyperbolic case $(SL(2;\mathbb{R})/A)$ which is isomorphic to the upper half plane of double numbers namely and in short we denote it as EPH-cases.

In this paper we shall refer to **cycles** [8], as straight lines and one of the following. Circles in the elliptic case, parabolas (with vertical axis of symmetry) in the parabolic case and rectangular hyperbolas (with vertical axis of symmetry) in the hyperbolic case. Also, the word parabola and hyperbola in this paper always assume only one of the above described types.

The **center** of a cycle is referred to as :

1. center of a circle in the elliptic case;
2. focus of a parabola in the parabolic case and
3. center of a rectangular hyperbola in the hyperbolic case.

The **vertex** of a cycle is referred to as:

1. the lowest point of a circle;
2. vertex of a parabola and
3. vertex of a rectangular hyperbola, in each of the three EPH cases.

We can define the **radius** of a **cycle** as:

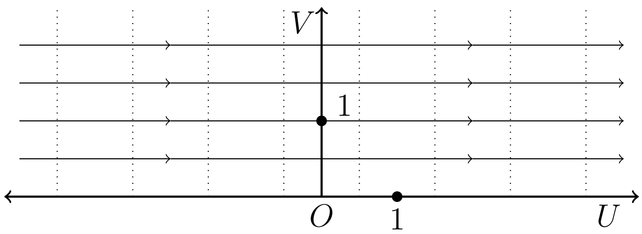
1. radius of a circle in the elliptic case;
2. distance between center and vertex of a parabola in the parabolic case;
3. distance between center and vertex of a hyperbola in the hyperbolic case.

In the next subsections 2.1 and 3.1, we shall discuss some of the results stated by Kisil [8]. We shall give their proof in details. Earlier most of the works (of subsections 2.1 and 3.1) have been proved using CAS (computer algebra system), by brute-force calculations [8, 10] or, given a short proof.

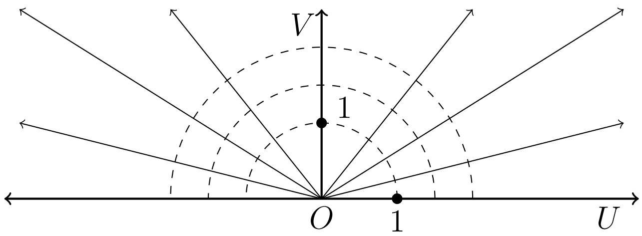
2.1. **Action of the subgroups.**

Lemma 2.1. *The action of the subgroup N under Möbius transformation on $\mathbb{R}^\sigma$ is $g_n \cdot (u,v) = (u+\nu,\ v)$, where $g_n = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ and $(u,v) \in \mathbb{R}^\sigma$, which defines shifts along the real axis U by ν (see [8]).*

Lemma 2.2. *The action of the subgroup A under Möbius transformation on $\mathbb{R}^\sigma$ is $g_n \cdot (u,v) = \alpha^{-2}(u,\ v)$, where $g_a = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$, $\alpha > 0$, and $(u,v) \in \mathbb{R}^\sigma$, which defines dilations by the factor α^{-2}, which fixes the origin $(0,0)$ (see [8]).*



(A) N-orbit



(B) A-orbit

FIGURE 1. Actions of different subgroups of $SL(2;\mathbb{R})$

Theorem 2.1. *A K-orbit in $\mathbb{R}^\sigma$ passing through the point $(0,t)$ has the following equation (see [8])*

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) + 1 = 0. \tag{2.1}$$

Proof. If (u,v) be any point on the $K-$orbit passing through $(0,t)$ then it can be determined as follows.

$$\begin{aligned} u + iv &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \cdot (it) = \frac{(it)\cos\theta - \sin\theta}{(it)\sin\theta + \cos\theta} \\ &= \frac{(-\sin\theta - it\cos\theta)(\cos\theta - it\sin\theta)}{\cos^2\theta - i^2\sin^2\theta}. \end{aligned}$$

There are three cases to follow.

(a) In the elliptic case $(i^2 = -1)$,

$$u + iv = \frac{-\sin\theta\cos\theta + t^2\sin\theta\cos\theta}{\cos^2\theta + t^2\sin^2\theta} + i\frac{t}{\cos^2\theta + t^2\sin^2\theta},$$

then $u = \frac{(t^2-1)\sin\theta\cos\theta}{\cos^2\theta + t^2\sin^2\theta}$ and $v = \frac{t}{\cos^2\theta + t^2\sin^2\theta}$.

From the above expressions of u and v, we get $u^2 + v^2 - v(t^{-1} + t) = -1$.

(b) In the parabolic case ($i^2 = 0$),

$$u + iv = -\frac{\sin\theta\cos\theta}{\cos^2\theta} = i\frac{t}{\cos^2\theta},$$

then $u = -\tan\theta$ and $v = t\sec^2\theta$. Thus, $u^2 - vt^{-1} = -1$.

(c) In the hyperbolic case ($i^2 = 1$),

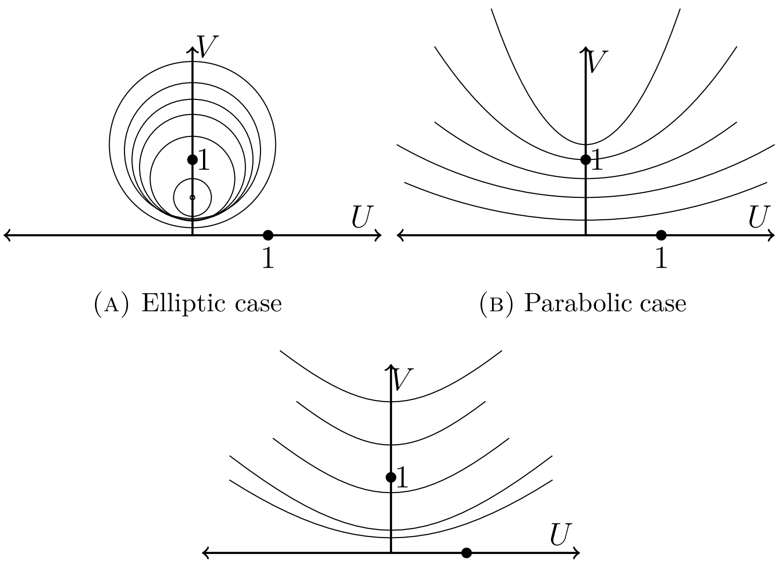
$$u + iv = -\frac{\sin\theta\cos\theta - t^2\sin\theta\cos\theta}{\cos^2\theta - t^2\sin^2\theta} + i\frac{t}{\cos^2\theta - t^2\sin^2\theta},$$

then $u = \frac{-(t^2+1)\sin\theta\cos\theta}{\cos^2\theta - t^2\sin^2\theta}$ and $v = \frac{t}{\cos^2\theta - t^2\sin^2\theta}$.

Thus, $u^2 - v^2 - v(t^{-1} - t) = -1$.

Combining the above three cases, we get the K-orbit as (Figure 2)

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) + 1 = 0, \quad \text{for } \sigma = i^2 = -1, 0, 1.$$



(A) Elliptic case

(B) Parabolic case

(C) Hyperbolic case

FIGURE 2. Orbits of the subgroup K in *EPH*-cases

$\square$

Remark 2.1. The shape of a geometric object in $\mathbb{R}^\sigma$ is solely dependent upon the action of the subgroup K.

Theorem 2.2. *The curvature of the K-orbit* (*at the vertex*) *in the elliptic, parabolic and hyperbolic cases are* (*see* [8])

$$\kappa = \frac{2t}{1 + \sigma t^2}, \quad \sigma = -1, 0, 1. \tag{2.2}$$

Proof. There are three cases to follow.

(a) We know that the curvature of a circle is inverse of its radius. As our K-orbit in the elliptic case is a circle with equation

$$u^2 + \left(v - \frac{t^{-1}+t}{2}\right)^2 = \left(\frac{t^{-1}-t}{2}\right)^2,$$

therefore its curvature would be $\kappa_e = \frac{2}{t^{-1}-t} = \frac{2t}{1-t^2}$.

(b) In the parabolic case, a K-orbit is a parabola with equation

$$u^2 = \frac{1}{t}(v-t).$$

For determining the curvature, let us suppose that its parametric equation is $u = \frac{r}{t}$ and $v = t + \frac{r^2}{t}$, where r being an arbitrary parameter. The differential coefficients of u and v with respect to r at vertex (i.e when $r = 0$) are $u' = \frac{1}{t},\ v' = 0,\ u'' = 0$ and $v'' = \frac{2}{t}$. Therefore, the curvature at the vertex is

$$\kappa_p \Big|_{r=0} = \left| \frac{u'v'' - u''v'}{(u'^2 + v'^2)^{\frac{3}{2}}} \right| = 2t.$$

(c) In the hyperbolic case, the K-orbit can be written as

$$\left(v + \frac{t^{-1}-t}{2}\right)^2 - u^2 = \left(\frac{t^{-1}+t}{2}\right)^2,$$

or, $(v-b)^2 - u^2 = r^2$ (say). We parametrize u and v as $u = r\tan\theta,\ v = b + r\sec\theta$, where θ is the parameter. The differential coefficients of u and v with respect to θ at the vertex (i.e., when $\theta = 0$) are $u' = r,\ v' = 0,\ u'' = 0$ and $v'' = r$. Its curvature at the vertex is

$$\kappa_h \Big|_{\theta=0} = \left| \frac{u'v'' - u''v'}{(u'^2 + v'^2)^{\frac{3}{2}}} \right| = \frac{1}{r} = \frac{2}{1+t^2}.$$

Combining the above three cases, we get the curvature of the K-orbit as

$$\kappa = \frac{2t}{1+\sigma t^2}, \quad \sigma = -1, 0, 1.$$

□

Next theorem has been proved in short earlier in the paper [8] we shall give its detailed proof.

Theorem 2.3. *Möbius transformation preserves cycles in the upper half plane,* [8].

Proof. We know by the Lemmas 2.1 and 2.2 that the subgroups N and A produces shifts and dilations, respectively. Also, the subgroup K has orbits which are either circles, parabolas or hyperbolas. We have to prove that Möbius transformations preserve the cycles in the upper half plane.

Our first observation is that the subgroups A and N obviously preserve all circles, parabolas, hyperbolas and straight lines in all $\mathbb{R}^\sigma$. Thus we use subgroups A and N to fit a given cycle exactly on a particular orbit of subgroup K shown on Figure

2 of the corresponding type. Let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$ be an arbitrary Möbius transformation. We shall show that gC is of the same type of cycle as C.

To this end, for an arbitrary cycle C, we can find $g'_n \in N$ which puts the centre of C on the V-axis (see Figure 3).

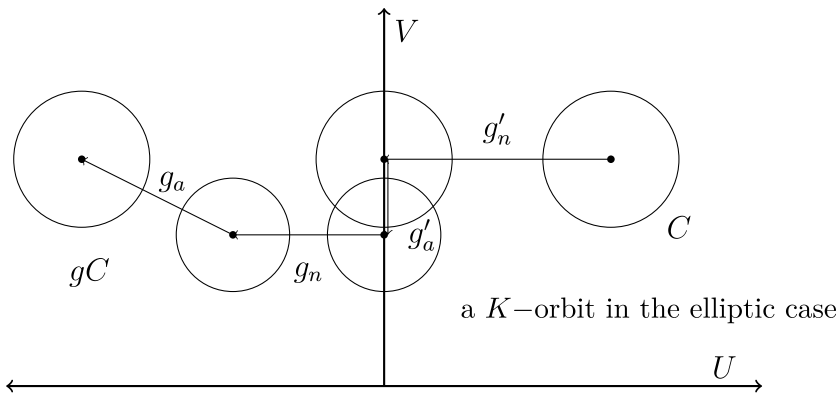


FIGURE 3. Pictorial representation of the proof of Theorem 2.3

Then there is a unique $g'_a \in A$ which scales it exactly to an orbit of K. To be more precise, we calculate the values of the shift factor and scaling factor in order to determine the factors of g'_n and g'_a for all the three EPH-cases.

Suppose we consider any cycle with centre (f, h) and vertex (u, v) and translate it to the V-axis by a shift factor f, then the co-ordinates of the center become $(0, h)$ and vertex become $(0, v)$. Therefore, the shift factor is f and of the form g'_n is $\begin{pmatrix} 1 & -f \\ 0 & 1 \end{pmatrix}$ in all three EPH-cases. Next we find the form of g'_a. Let us consider the cycle with the given vertex $(0, v_1)$ and curvature κ_1 at the point $(0, v_1)$. Now in order to fit this cycle into the orbit whose vertex and curvature are given by $(0, v)$ and κ, we calculate the scaling factor α, as follows. We have $v = \alpha v_1$ and $\kappa = \frac{\kappa_1}{\alpha}$, i.e., $\alpha = \frac{v}{v_1}$ and $v\kappa = v_1\kappa_1 = v_1\frac{2v_1}{1+\sigma v_1^2} = \frac{2}{\sigma + v_1^{-2}}$ (cf. Theorem 2.2). Therefore, $v_1 = \sqrt{\frac{v\kappa}{2-\sigma v\kappa}}$. This shows $\alpha = \frac{v}{v_1} = \sqrt{\frac{\kappa}{v(2-\sigma v\kappa)}}$, which is the scaling factor and $g'_a = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}$ for all the EPH cases.

Next, we show that this scaling factor α if exists then it is unique. Indeed, suppose that α and α' are both scaling factors used to bring the vertex $(0, v_1)$ to fit into the orbit with vertex $(0, v)$. Then we have $v = \alpha v_1$ and $v = \alpha' v_1$, i.e., $(\alpha - \alpha')v_1 = 0$. Now, we know that $v_1 \neq 0$, therefore, $\alpha - \alpha' = 0$, i.e., $\alpha = \alpha'$. This shows that the scaling factor is unique.

Let us take

$$g' = g(g'_a g'_n)^{-1} = \begin{pmatrix} \dfrac{a}{\sqrt{\alpha}} & \dfrac{ua+b}{\sqrt{\alpha}} \\ c\sqrt{\alpha} & \dfrac{uc+d}{\sqrt{\alpha}} \end{pmatrix}.$$

As $g' \in SL(2;\mathbb{R})$ using Iwasawa decomposition we get $g' = g_a g_n g_k$, for some $g_a \in A$, $g_n \in N$ and $g_k \in K$. Now,

$$\begin{aligned} gC &= g(g'_a g'_n)^{-1}(g'_a g'_n C) = (g_a g_n g_k)(g'_a g'_n C) \\ &= g_a g_n (g_k (g'_a g'_n C) = g_a g_n g'_a g'_n C \quad \text{(as K-orbits are K-invariant)}. \end{aligned}$$

Also, g_a, g_n, g'_a, g'_n do not change the shape of the cycle therefore gC is the same type of cycle as C. □

3. Introduction of Two New Subgroups A' and N' of $SL(2;\mathbb{R})$

In the earlier section 2.1 where we have seen the importance of K-subgroups in $SL(2;\mathbb{R})$ geometry. In this section we introduce two subgroups of matrices A' and N' defined by

$$A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}, \quad N' = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$$

in place of the subgroup K to obtain our results. We first describe their orbits.

Proposition 3.1. *The subgroups* $A' = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} : t \in \mathbb{R} \right\}$ *and* $N' = \left\{ \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$ *are conjugate to* $A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\}$ *and* $N = \left\{ \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} : \nu \in \mathbb{R} \right\}$*, respectively.*

Proof. This can be seen by the following two equations

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} \cosh t & \sinh t \\ \cosh t & \sinh t \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -\nu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix}.$$

Therefore, any matrix of A' is similar to a matrix of A and any matrix of N' is similar to a matrix of N. □

3.1. **Orbits of the two new subgroups A' and N'.**

Theorem 3.1. *The orbits of the subgroup A' are cycles*

$$u^2 - \sigma v^2 + (t^{-1} + \sigma t)v - 1 = 0, \qquad \textit{where } \sigma = i^2 = -1, 0, 1,$$

which intercept the U-axis at two real points $(1,0)$ and $(-1,0)$ and have center on the V-axis in all the three elliptic, parabolic and hyperbolic cases.

Proof. We have the following three cases.

(a) **Elliptic case**. If (u,v) be an arbitrary point on the A'-orbit which passes through $(0,t)$, then

$$\begin{aligned} u + iv &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \cdot (it) = \frac{it\cosh\theta + \sinh\theta}{it\sinh\theta + \cosh\theta} \\ &= \frac{(1+t^2)\cosh\theta\sinh\theta}{\cosh^2\theta + t^2\sinh^2\theta} + i\frac{t}{\cosh^2\theta + t^2\sinh^2\theta}. \end{aligned}$$

Eliminating θ, we get $u^2 + v^2 = 1 + (t - t^{-1})v$.

(b) **Parabolic case**. Similar to the previous case if (u,v) is an arbitrary point on the orbit which passes through $(0,t)$, then

$$u + iv = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \cdot (it) = \frac{it\cosh\theta + \sinh\theta}{it\sinh\theta + \cosh\theta} = \frac{\sinh\theta}{\cosh\theta} + i\frac{t}{\cosh^2\theta}.$$

Therefore, the orbit would be $u^2 = -\frac{1}{t}(v - t)$.

(b) **Hyperbolic case** Again by taking an arbitrary point (u,v) in the orbit and assuming that it passes through $(0,t)$, we have

$$\begin{aligned} u + iv &= \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix} \cdot (it) = \frac{it\cosh\theta + \sinh\theta}{it\sinh\theta + \cosh\theta} \\ &= \frac{(1-t^2)\cosh\theta\sinh\theta}{\cosh^2\theta - t^2\sinh^2\theta} + i\frac{t}{\cosh^2\theta - t^2\sinh^2\theta}. \end{aligned}$$

Eliminating θ, we get $u^2 - v^2 = 1 - (t + t^{-1})v$, which is a hyperbola intercepting the U-axis at two real points.

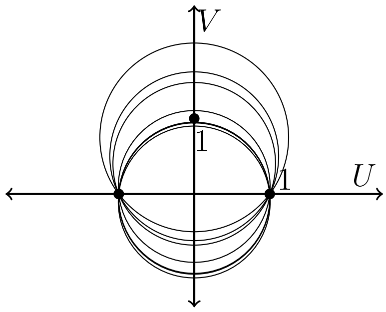
Combining the above three cases we get the A'-orbit as

$$u^2 - \sigma v^2 + (t^{-1} + \sigma t)v - 1 = 0, \quad \sigma = i^2 = -1, 0, 1. \tag{3.1}$$

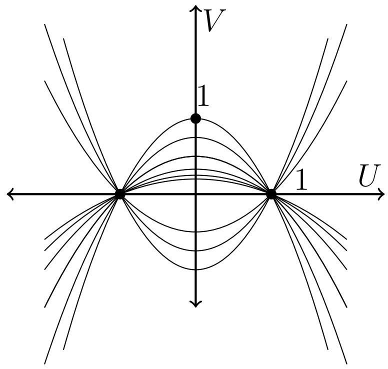
Putting $v = 0$ we get $u^2 = 1$, i.e., $u = \pm 1$. Thus, they are cycles intercepting the U-axis at $(1,0)$ and $(-1,0)$ unlike the K-orbit. $\square$

Theorem 3.2. *In an A'-orbit*

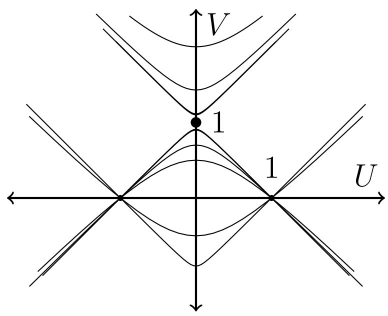
(a) *in the elliptic case, the relation between radius r_e and center $(0, v_e)$ is $r_e^2 - v_e^2 = 1$;*
(b) *in the parabolic case, the relation between vertex $(0, v_p)$ and center (focus) $(0, r_p/4)$ is $r_p v_p = 1$ and*



(A) Circles intercept the U-axis



(B) Parabolas intercepting the U-axis



(C) Rectangular hyperbolas intercepting the U-axis

FIGURE 4. The orbits of the subgroup A' in elliptic, parabolic and hyperbolic cases

(c) *in the hyperbolic case, the relation between center* $(0, v_h)$ *and* r_h *which is the distance between the center and vertex is* $v_h^2 - r_h^2 = 1$.

Proof. There are three cases to follow.

(a) **Elliptic case.** In this case, the A'-orbit is $u^2 + (v - v_e)^2 = r_e^2$, where r_e and v_e are of the form $r_e = \frac{t+t^{-1}}{2}$ and $v_e = \frac{t-t^{-1}}{2}$ (for some $t \in \mathbb{R}$). Therefore, by a simple calculation we can show $r_e^2 - v_e^2 = \left(\frac{t+t^{-1}}{2}\right)^2 - \left(\frac{t-t^{-1}}{2}\right)^2 = 1$.

(b) **Parabolic case.** In this case, the A'-orbit is $u^2 = r_p(v + v_p)^2$, we have center $(0, r_p/4)$ and vertex is $(0, v_p)$, then r_p and v_p are of the form $r_p = -\frac{1}{t}$ and $v_p = -t$ for some $t \in \mathbb{R}$. Therefore, $r_p v_p = 1$.

(c) **Hyperbolic case.** The A'-orbit in this case is $(v - v_h)^2 - u^2 = r_h^2$. We have $v_h = \frac{t+t^{-1}}{2}$ and $r_h = \frac{t-t^{-1}}{2}$ for some $t \in \mathbb{R}$, then $v_h^2 - r_h^2 = 1$. $\square$

Remark 3.1. The subgroups A' and N' come naturally in Möbius action of $SL(2;\mathbb{R})$. In the next proposition we show that.

Proposition 3.2. *The isotropy (fix) subgroups of* $(0,1) \in \mathbb{R}^\sigma$ *under the Möbius action of* $SL(2;\mathbb{R})$ *are* K, A' *and* N'.

Proof. Let $i^2 = -1, 0, 1$, for the three cases elliptic, parabolic and hyperbolic. Then if $\frac{ai+b}{ci+d} = i$, then by calculation, we get

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}, \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix},$$

for the three cases, respectively. □

Theorem 3.3. *The orbits of the subgroup* N' *are*

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) = 0, \quad \sigma = i^2 = -1, 0, 1.$$

They are circles, parabolas and rectangular hyperbolas which are tangent to the U*-axis and have center on the* V*-axis in the elliptic, parabolic and hyperbolic cases respectively.*

Proof. We have the following three cases.

(a) **Elliptic case.** If (u, v) be an arbitrary point on N'-orbit which passes through $(1, t)$ (If we take $(0, t)$ point, then it is only $(0, 0)$ in the parabolic case), then

$$u + iv = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot (1 + it) = \frac{1 + it}{\nu(1 + it) + 1} = \frac{1 + \nu + \nu t^2}{(1 + \nu)^2 + \nu^2 t^2} + i \frac{t}{(1 + \nu)^2 + \nu^2 t^2}.$$

Eliminating ν we get the orbit as

$$u^2 + v^2 - v(t^{-1} + t) = 0. \tag{3.2}$$

(b) **Parabolic case.** If the cycle passes through $(1, t)$, then any arbitrary point (u, v) on the N'-orbit would be

$$u + iv = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot (1 + it) = \frac{1 + it}{\nu + i\nu t + 1} = \frac{1}{1 + \nu} + i \frac{t}{(1 + \nu)^2}.$$

The orbit would be

$$u^2 - vt^{-1} = 0. \tag{3.3}$$

(c) **Hyperbolic case.** If (u, v) be any arbitrary point on the orbit and it passes through $(1, t)$, then

$$u + iv = \begin{pmatrix} 1 & 0 \\ \nu & 1 \end{pmatrix} \cdot (1 + it) = \frac{1 + it}{\nu + i\nu t + 1} = \frac{1 + \nu + \nu t^2}{(1 + \nu)^2 - \nu^2 t^2} + i \frac{t}{(1 + \nu)^2 - \nu^2 t^2}.$$

Eliminating ν we get the orbit as

$$u^2 - v^2 - v(t^{-1} - t) = 0.$$

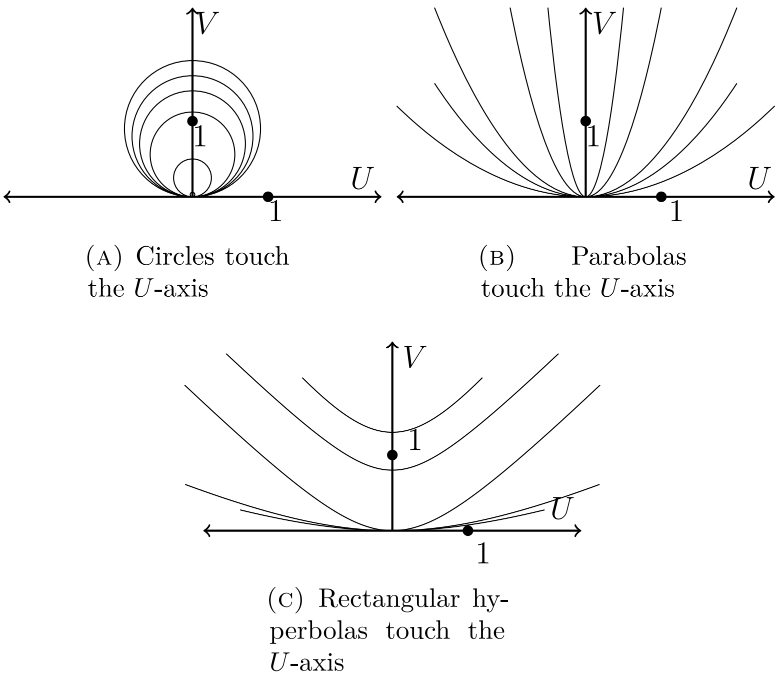


FIGURE 5. Cycles which are tangent to the U-axis

Combining the above three cases we get

$$u^2 - \sigma v^2 - v(t^{-1} - \sigma t) = 0, \quad \sigma = -1, 0, 1. \tag{3.4}$$

As such the orbits of N' are cycles which are tangent to the U-axis. □

Remark 3.2. The center of a N'-orbit passing through $(1, t)$ is of the form $\left(0, \frac{t^{-1}-\sigma t}{2}\right)$ for some $t \in \mathbb{R}$ and $\sigma = -1, 0, 1$, in EPH cases.

Proposition 3.3. *Orbits of the isotropy subgroups of K, N' and A' of $(0,1)$ in $\mathbb{R}^\sigma$ in elliptic, parabolic and hyperbolic cases are*

$$u^2 - \sigma v^2 - 2lv - \sigma = 0, \quad \text{where } l \in \mathbb{R}.$$

4. Main Result

In this section we develop our work.

4.1. **Decomposition of $SL(2;\mathbb{R})$ using the subgroups A' and N'.** In this subsection we define two other decomposition of $SL(2;\mathbb{R})$ in order to include new invariances of types of cycles for the transformation group $SL(2;\mathbb{R})$.

Theorem 4.1. *Any matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$, where $d \neq 0$ can be represented uniquely as $g = g_a g_n g_{n'}$, where $g_a \in A, g_n \in N$ and $g_{n'} \in N'$.*

Proof. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, let $g_a = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$, $g_n = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ and $g_{n'} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$. Then $g = g_a g_n g_{n'}$, i.e.,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{-1}(1+t\nu) & \alpha^{-1}\nu \\ \alpha t & \alpha \end{pmatrix}.$$

Therefore, $d = \alpha, t = \frac{c}{d}$ and $\nu = bd$.

The uniqueness can be obtained by using another decomposition with the same form and showing that the corresponding matrices are equal. □

Theorem 4.2. *Any matrix* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$, *where* $|d| > |c|$ *can be represented uniquely as* $g = g_a g_n g_{a'}$, *where* $g_a \in A, g_n \in N$ *and* $g_{a'} \in A'$.

Proof. Let $g_a = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$, $g_n = \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix}$ and $g_{a'} = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$ for $\alpha, t, \nu \in \mathbb{R}$ and $\alpha > 0$. Then $g = g_a g_n g_{a'}$, i.e.,

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & \alpha \end{pmatrix} \cdot \begin{pmatrix} 1 & \nu \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \\ &= \begin{pmatrix} \alpha^{-1}(\cosh t + \nu \sinh t) & \alpha^{-1}(\sinh t + \nu \cosh t) \\ \alpha \sinh t & \alpha \cosh t \end{pmatrix}. \end{aligned}$$

Thus, $\alpha = \sqrt{d^2 - c^2}$, $t = \tanh^{-1}\left(\frac{c}{d}\right)$ and $\nu = bd - ac$.

One can prove the uniqueness of the decomposition by taking another decomposition with the same form and showing that the corresponding matrices are equal. □

4.2. **Invariance of cycles in $\mathbb{R}^\sigma$.** We now introduce new types of cycles in the existing $SL(2;\mathbb{R})$ geometries [8,9] which have important applications (see Section 5).

Theorem 4.3. *If C is any arbitrary cycle which have U-axis as a tangent in the space of $\mathbb{R}^\sigma$ with center (u,v), then it is invariant under the Möbius action of* $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{R})$ *if* $u > -\frac{d}{c}$.

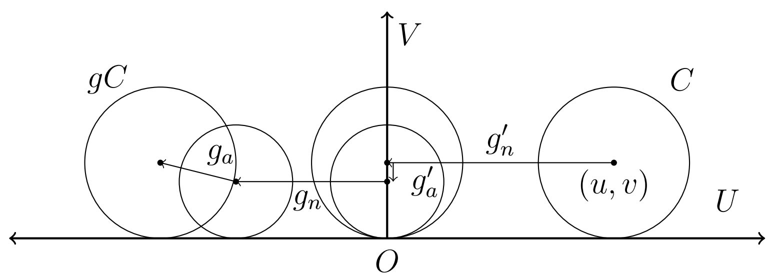


FIGURE 6. Pictorial representation of the proof of the Theorem 4.3

Proof. The Möbius action on the cycle C by the matrix $g'_n = \begin{pmatrix} 1 & -u \\ 0 & 1 \end{pmatrix}$ shifts its center (u, v) to $(0, v)$. Then applying Möbius action again on g'_nC by $g'_a = \begin{pmatrix} \frac{1}{\sqrt{\alpha}} & 0 \\ 0 & \sqrt{\alpha} \end{pmatrix}$, $\alpha > 0$, we scale its center to $(0, v\alpha^{-1})$.

As N'-orbit have center of the form $\frac{t^{-1}-\sigma t}{2}$, by Remark 3.3, as such if we want $g'_ag'_nC$ to be a N'-orbit then we must have $v\alpha^{-1} = \frac{t^{-1}-\sigma t}{2}$, $\sigma = -1, 0, 1$, $t \in \mathbb{R}$. Therefore, the scaling factor $\alpha = \frac{2vt}{1-\sigma t^2}$.

The uniqueness of g'_a can be proved by taking another $g''_a = \begin{pmatrix} \frac{1}{\sqrt{\alpha'}} & 0 \\ 0 & \sqrt{\alpha'} \end{pmatrix}$. By the same calculations we get $\alpha' = \frac{2vt}{1-\sigma t^2}$. Therefore, $g'_a = g''_a$.

Now,

$$g' = g(g'_ag'_n)^{-1} = \begin{pmatrix} a\sqrt{\alpha} & (au+b)\sqrt{\alpha^{-1}} \\ c\sqrt{\alpha} & (cu+d)\sqrt{\alpha^{-1}} \end{pmatrix} \in SL(2; \mathbb{R}).$$

If $g(g'_ag'_n)^{-1}$ have a decomposition of the form of Theorem 4.1, then $cu + d > 0$, i.e., $u > -\frac{d}{c}$.

We can decompose $g(g'_ag'_n)^{-1} = g_ag_ng_{n'}$ for some $g_a \in A$, $g_n \in N$ and $g_{n'} \in N'$. Now,

$$\begin{aligned} gC &= g(g'_ag'_n)^{-1}g'_ag'_nC = g_ag_ng_{n'}(g'_ag'_nC) \\ &= g_ag_ng'_ag'_nC \quad \text{(as } g'_ag'_nC \text{ is a } N'\text{−orbit therefore it is } N'\text{-invariant).} \end{aligned}$$

As g_a, g'_a, g_n, g'_n do not change the shape of C, by Remark 2.1, therefore, gC would be the same type of cycle as C geometrically. □

Theorem 4.4. *If C is an arbitrary cycle with center (u_λ, v_λ) and radius r_λ which intercepts the U-axis at two real points then it is invariant under the Möbius action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2; \mathbb{R})$ and if it satisfies $|c|\alpha < |cu_\lambda + d|$, where $\alpha = \sqrt{(v_\lambda^2 - r_\lambda^2)\sigma + r_\lambda v_\lambda(1 - |\sigma|)}$, $\sigma = -1, 0, 1$, and $\lambda = e, p, h$ in elliptic, parabolic and hyperbolic cases, respectively.*

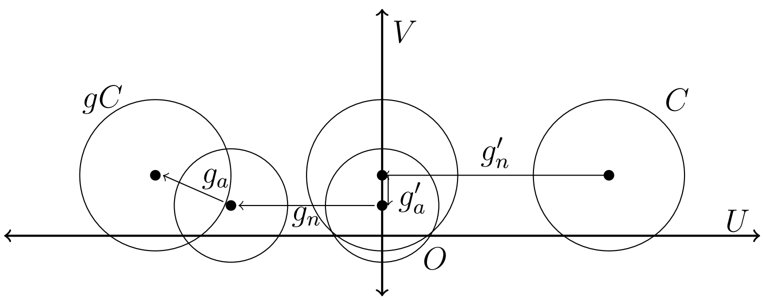


FIGURE 7. Pictorial representation of the proof of the Theorem 4.4

Proof. We consider the following three cases.

(a) **Elliptic case.** Let us take an arbitrary circle $(u-u_e)^2+(v-v_e^2)^2=r_e^2$ with center (u_e,v_e) and radius r_e which intercepts the U-axis at two real points therefore $|r_e|>|v_e|$.

First we translate the center of C to V-axis by a shift factor u_e, then the coordinates of the center become $(0,v_e)$ and the matrix which shifts is of the form $g'_n=\begin{pmatrix}1 & -u_e\\ 0 & 1\end{pmatrix}$. Next to fit the circle into an A'-orbit, we take Möbius action by the matrix $g'_a\in A$ defined as $g'_a=\begin{pmatrix}\frac{1}{\sqrt{\alpha_e}} & 0\\ 0 & \sqrt{\alpha_e}\end{pmatrix}$, $\alpha_e>0$. The radius and center (see Remark 2.1) become $r_e\alpha^{-1}$ and $(0,v_e\alpha^{-1})$. If $g'_ag'_nC$ is an A'-orbit, then using the Theorem 3.2 (of A'-orbit) we get $(r\alpha_e^{-1})^2-(v_e\alpha_e^{-1})^2=1$, i.e., $\alpha_e=\sqrt{r_e^2-v_e^2}$.

(b) **Parabolic case.** In this case we take an arbitrary parabola (with vertical axis of symmetry) $(u-u_p)^2=r_p(v+v_p)$ which intercepts the U-axis at two real points. Applying Möbius action by $g'_n=\begin{pmatrix}1 & -u_p\\ 0 & 1\end{pmatrix}$, we can shift the center to $(0,v_p)$, by Lemma 2.1, and applying $g'_a=\begin{pmatrix}\frac{1}{\sqrt{\alpha_p}} & 0\\ 0 & \sqrt{\alpha_p}\end{pmatrix}$, $\alpha_p>0$, we get the parabola as $u^2=r_p\alpha_p^{-1}(v+v_p\alpha_p^{-1})$. If this is an A'-orbit then by the Theorem 3.2 we get $(r_p\alpha_p^{-1})(v_p\alpha_p^{-1})=1$, i.e., $\alpha_p=\sqrt{r_pv_p}$.

(c) **Hyperbolic case.** We take an arbitrary hyperbola (with vertical axis of symmetry) $(v-v_h)^2-(u-u_h)^2=r_h^2$ with center (u_h,v_h) which intercepts U-axis at two real points then $|v_h|>|r_h|$. To fit this into a A'-orbit we apply Möbius actions by two matrices

$$g'_n=\begin{pmatrix}1 & -u_h\\ 0 & 1\end{pmatrix},\quad g'_a=\begin{pmatrix}\frac{1}{\sqrt{\alpha_h}} & 0\\ 0 & \sqrt{\alpha_h}\end{pmatrix},\quad \alpha_h>0.$$

Therefore, the equation transforms to the hyperbola $(v-v_h\alpha_h^{-1})^2-u^2=(r_h\alpha_h^{-1})^2$. To fit this into a A'-orbit using the Theorem 3.2, we get $(v_h\alpha_h^{-1})^2-(r_h\alpha_h^{-1})^2=1$, i.e., $\alpha_h=\sqrt{v_h^2-r_h^2}$.

Combining the above three cases we have $\alpha_\lambda=\sqrt{(v_\lambda^2-r_\lambda^2)\sigma+r_\lambda v_\lambda(1-|\sigma|)}$ for $\sigma=-1,0,1$ and $\lambda=e,p,h$.

Let us take α in place of α_e,α_p and α_h and u in place of u_e,u_p and u_h. Uniqueness of α can be proved by taking another α' and doing the same calculations we get $\alpha'=\sqrt{(v_\lambda^2-r_\lambda^2)\sigma+r_\lambda v_\lambda(1-|\sigma|)}$ for $\sigma=-1,0,1$ and $\lambda=e,p,h$. Therefore, $\alpha=\alpha'$ and α is unique.

Now,

$$g'=g(g'_ag'_n)^{-1}=\begin{pmatrix}a\sqrt{\alpha} & \frac{au+b}{\sqrt{\alpha}}\\ c\sqrt{\alpha} & \frac{cu+d}{\sqrt{\alpha}}\end{pmatrix}\in SL(2;\mathbb{R}).$$

If g' have decomposition of the form of Theorem 4.2, then $|c|\alpha < |cu + d|$. Thus, $g(g'_a g'_n)^{-1} = g_a g_n g_{a'}$ for some $g_a \in A, g_n \in N$ and $g_{a'} \in A'$. Therefore,

$$\begin{aligned} gC &= g(g'_a g'_n)^{-1} g'_a g'_n C = g_a g_n g_{a'}(g'_a g'_n C) \\ &= g_a g_n g'_a g'_n C \quad (\text{as } g'_a g'_n C \text{ is a } A'\text{-orbit therefore it is } A'\text{-invariant}). \end{aligned}$$

As g_a, g'_a, g_n, g'_n do not change shape of the cycle therefore gC is the same cycle as C. □

5. Discussion on Some Physical Applications of the Action of the Subgroups of the Group $SL(2;\mathbb{R})$

In optics, paraxial system is largely depend upon *paraxial groups* which are used to solve paraxial wave equation [1]. In fact, $SL(2;\mathbb{R})$ acts as a ray transfer matrix [3]. In chapter 2 of [3], the authors found the refractive matrix of the form

$$\mathcal{R} = \begin{pmatrix} 1 & 0 \\ \left(-\frac{n_1 - n_2}{r}\right) & 1 \end{pmatrix}.$$

That is two paraxial systems are dependent upon the relation $\begin{pmatrix} y_2 \\ V_2 \end{pmatrix} = \mathcal{R} \begin{pmatrix} y_1 \\ V_1 \end{pmatrix}$. This is the direct application of action of N' group.

In classical mechanics Galilean relativity principle [17] states that laws of mechanics will be invariant under the following linear transformation

$$\begin{pmatrix} t_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ x_1 \end{pmatrix},$$

where t is time and x is spatial component. The matrix belongs to the group N'.

These two examples show that if these ordered pairs obey the conditions of our theorems, in the previous section 4 then they are invariant under $SL(2;\mathbb{R})$ action.

6. Conclusion and Future Work

In the elliptic and parabolic cases the upper half plane is preserved but in the hyperbolic case it is not true, which has some implications in geometry, physics and analysis [8].

We have obtained new invariant objects in the three homogeneous spaces. It applies to all the fields where $SL(2;\mathbb{R})$ is used.

In future we can study the projective space of cycles. A generic cycle can be represented as $C = \begin{pmatrix} g + if & c \\ a & -g + if \end{pmatrix}$, where i is a hypercomplex unit and C is $a(x^2 - y^2) - 2gx - 2fy + c = 0$. A cycle C is transformed to gCg^{-1} under the Möbius action of $g \in SL(2;\mathbb{R})$ [2]. After extending the cycle group by us in the previous theorems we can now include more values in $(a, g, f, c) \in \mathbb{R}^4$ and investigate further.

We can use our theory to find invariant metric [5] such as

$$ds^2 = \frac{du^2 - \sigma dv^2}{v^2}, \quad \sigma = -1, 0, 1.$$

In future, we can enrich Erlangen program of $SL(2;\mathbb{R})$ by including new invariants in all the three spaces. This theory can be used to make function theories on $\mathbb{R}^2$ [6, 13]. We can also extend our study of invariants to higher dimensional Lie groups, e.g., $SL(3;\mathbb{R})$, $SL(3;\mathbb{C})$, etc.

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SOME RESULTS FOR ENDOMORPHISMS IN PRIME RINGS

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Abstract. In this article, we present some commutativity theorems for a prime ring $\mathcal{R}$ equipped with endomorphisms α, β, γ and δ satisfying any one of the following identities:

(1) $[\alpha(x), \beta(y)] + \gamma([x, y]) + \delta(x \circ y) = 0$ for all $x, y \in \mathcal{R}$;

(2) $\alpha(x) \circ \beta(y) + \gamma([x, y]) = 0$ for all $x, y \in \mathcal{R}$.

Moreover, we provide examples to show that the assumed restrictions cannot be relaxed.

1. Introduction

Let $\mathcal{R}$ be a ring with center $Z(\mathcal{R})$. For any $x, y \in \mathcal{R}$, $[x, y]$ will denote the commutator $xy - yx$ while $x \circ y$ will represent the anti-commutator $xy + yx$. Recall that a ring $\mathcal{R}$ is said to be prime if $a\mathcal{R}b = \{0\}$ implies that either $a = 0$ or $b = 0$. A ring $\mathcal{R}$ is said to be 2-torsion free if $2a = 0$ (where $a \in \mathcal{R}$) implies $a = 0$. It is straight forward to see that a prime ring with characteristic different from two is 2-torsion free. A mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be centralizing on $\mathcal{R}$ if $[f(x), x] \in Z(\mathcal{R})$ holds for all $x \in \mathcal{R}$. In the special case if $[f(x), x] = 0$ for all $x \in \mathcal{R}$, f is said to be commuting on $\mathcal{R}$. An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation of $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. A derivation d is said to be inner if there exists $a \in \mathcal{R}$ such that $d(x) = ax - xa$ for all $x \in \mathcal{R}$. Following Bresar [6], an additive mapping $F : \mathcal{R} \to \mathcal{R}$ is called a generalized derivation if there exists a derivation $d : \mathcal{R} \to \mathcal{R}$ such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in \mathcal{R}$. The concept of generalized derivations includes both the concept of derivation and the concept of left multiplier (i.e., an additive mapping $F : \mathcal{R} \to \mathcal{R}$ satisfying $F(xy) = F(x)y$ for all $x, y \in \mathcal{R}$).

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Recently, a considerable number of researchers have investigated the ideals in prime rings as well as the commutativity of prime rings that consider derivations and generalized derivations, see for example [1–3] and [4].

Over the last four decade, several authors have proved results on commutativity of prime rings or semiprime rings that admitting automorphisms, derivations or generalized derivations which are centralizing or commuting on appropriate subset of $\mathcal{R}$ (see [2–5] etc.).

In this paper, we investigate the commutativity of a prime ring $\mathcal{R}$ admitting endomorphisms α, β, γ and δ satisfying any one of the following properties:

(a) $[\alpha(x), \beta(y)] + \gamma([x, y]) + \delta(x \circ y) = 0$ for all $x, y \in \mathcal{R}$;
(b) $\alpha(x) \circ \beta(y) + \gamma([x, y]) = 0$ for all $x, y \in \mathcal{R}$.

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2. Some preliminaries

This section, includes some well known basic identities which will be used for developing the proof of our main results:

(a) $[x, yz] = y[x, z] + [x, y]z$ for all $x, y, z \in \mathcal{R}$;
(b) $[xy, z] = x[y, z] + [x, z]y$ for all $x, y, z \in \mathcal{R}$;
(c) $x \circ (yz) = (x \circ y)z - y[x, z] = y(x \circ z) + [x, y]z$ for all $x, y, z \in \mathcal{R}$;
(d) $(xy) \circ z = x(y \circ z) - [x, z]y = (x \circ z)y + x[y, z]$ for all $x, y, z \in \mathcal{R}$.

3. Some Results for Prime Rings

Theorem 3.1. *Let $\mathcal{R}$ be a prime ring with* $\mathrm{char}(\mathcal{R}) \neq 2$*, α, β, γ and δ endomorphisms of $\mathcal{R}$ such that*

$$[\alpha(x), \beta(y)] + \gamma([x, y]) + \delta(x \circ y) = 0, \quad \textit{for all}\, x, y \in \mathcal{R}.$$

If β, γ are onto, then $\delta = 0$ and $\mathcal{R}$ is commutative.

Proof. Suppose that

$$[\alpha(x), \beta(y)] + \gamma([x, y]) + \delta(x \circ y) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.1}$$

Replacing y by yx in (3.1), we get

$$\beta(y)[\alpha(x), \beta(x)] + [\alpha(x), \beta(y)]\beta(x) + \gamma([x, y])\gamma(x) + \delta(x \circ y)\delta(x) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.2}$$

For $y = x$, (3.1) implies that

$$[\alpha(x), \beta(x)] + 2\delta(x^2) = 0, \quad \text{for all } x \in \mathcal{R}. \tag{3.3}$$

Using (3.1) and (3.3), then (3.2) can be rewritten as

$$2\beta(y)\delta(x^2) = \gamma([x, y])(\gamma(x) - \beta(x)) + \delta(x \circ y)(\delta(x) - \beta(x)), \quad \text{for all } x, y \in \mathcal{R}. \tag{3.4}$$

For $y = x$, (3.4) gives

$$\beta(x)\delta(x^2) = \delta(x^2)(\delta(x) - \beta(x)) = \delta(x^2)\delta(x) - \delta(x^2)\beta(x), \quad \text{for all } x \in \mathcal{R}. \tag{3.5}$$

Taking xy in place of y in (3.4), it is obvious to see that

$$\begin{aligned} 2\beta(x)\beta(y)\delta(x^2) =& \gamma(x)\gamma([x, y])(\gamma(x) - \beta(x)) \\ &+ \delta(x)\delta(x \circ y)(\delta(x) - \beta(x)), \quad \text{for all } x, y \in \mathcal{R}. \end{aligned} \tag{3.6}$$

Left-multiplying (3.4) by $\beta(x)$, we have also

$$\begin{aligned} 2\beta(x)\beta(y)\delta(x^2) =& \beta(x)\gamma([x, y])(\gamma(x) - \beta(x)) \\ &+ \beta(x)\delta(x \circ y)(\delta(x) - \beta(x)), \quad \text{for all } x, y \in \mathcal{R}. \end{aligned} \tag{3.7}$$

By identifying (3.6) and (3.7), we can easily arrive at

$$(\gamma(x) - \beta(x))\gamma([x, y])(\gamma(x) - \beta(x)) + (\delta(x) - \beta(x))\delta(x \circ y)(\delta(x) - \beta(x)) = 0.$$

For $x = y$, using $\operatorname{char}(\mathcal{R}) \neq 2$, then

$$(\delta(x) - \beta(x))\delta(x^2)(\delta(x) - \beta(x)) = 0, \quad \text{for all } x \in \mathcal{R}. \tag{3.8}$$

Using (3.5) and (3.8), we obtain

$$(\delta(x) - \beta(x))\beta(x)\delta(x^2) = 0, \quad \text{for all } x \in \mathcal{R}, \tag{3.9}$$

and

$$(\delta(x) - \beta(x))\delta(x^2)\delta(x) = (\delta(x) - \beta(x))\delta(x^2)\beta(x), \quad \text{for all } x \in \mathcal{R}.$$

Right-multiplying (3.4) by $\beta(x)\delta(x^2)$ and using (3.9), we get

$$2\beta(y)\delta(x^2)\beta(x)\delta(x^2) = \gamma([x, y])(\gamma(x) - \beta(x))\beta(x)\delta(x^2), \quad \text{for all } x, y \in \mathcal{R}. \tag{3.10}$$

Replacing y by xy in (3.10), we can easily arrive at

$$(\gamma(x) - \beta(x))\gamma([x, y])(\gamma(x) - \beta(x))\beta(x)\delta(x^2) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.11}$$

Using (3.10) and (3.11), we find that

$$(\gamma(x) - \beta(x))\beta(y)\delta(x^2)\beta(x)\delta(x^2) = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

Since β is onto, we get

$$(\gamma(x) - \beta(x))\mathcal{R}\delta(x^2)\beta(x)\delta(x^2) = \{0\}, \quad \text{for all } x \in \mathcal{R}.$$

By primeness of $\mathcal{R}$, we obtain

$$\gamma(x) = \beta(x) \text{ or } \delta(x^2)\beta(x)\delta(x^2) = 0 \quad \text{for all } x \in \mathcal{R}. \tag{3.12}$$

Suppose there exists $x_0 \in \mathcal{R}$ such that $\gamma(x_0) = \beta(x_0)$, then (3.4) becomes

$$2\beta(y)\delta(x_0^2) = \delta(x_0 \circ y)(\delta(x_0) - \beta(x_0)), \quad \text{for all } y \in \mathcal{R}. \tag{3.13}$$

In (3.13) we substitute x_0y for y and using $\operatorname{char}(\mathcal{R}) \neq 2$, to get

$$(\delta(x_0) - \beta(x_0))\beta(y)\delta(x_0^2) = 0, \quad \text{for all } y \in \mathcal{R}.$$

Since β is onto, we obtain $(\delta(x_0) - \beta(x_0))\mathcal{R}\delta(x_0^2) = \{0\}$. By primeness of $\mathcal{R}$, we conclude that either $\delta(x_0) = \beta(x_0)$ or $\delta(x_0^2) = 0$.

If $\delta(x_0) = \beta(x_0)$, according to our assumption after (3.12) it follows from (3.4) that $2\beta(y)\delta(x_0^2) = 0$ for all $y \in \mathcal{R}$. Since β is onto and $\mathrm{char}(\mathcal{R}) \neq 2$, we conclude that $\delta(x_0^2) = 0$. In both cases, we have $\delta(x_0^2) = 0$ and by (3.12), we get

$$\delta(x^2)\beta(x)\delta(x^2) = 0, \quad \text{for all } x \in \mathcal{R}.$$

Using (3.5), we conclude that

$$0 = \delta(x^2)\beta(x)\delta(x^2)\delta(x^2) = \delta(x^2)(\delta(x^2)\delta(x) - \delta(x^2)\beta(x))\delta(x^2),$$

which leads to $0 = \delta(x^2)\delta(x^2)\delta(x)\delta(x^2) = \delta(x^7) = (\delta(x))^7$ for all $x \in \mathcal{R}$. By a well-know result of Lovitzki [7] a prime rings cannot be nil of bounded index. Then $\delta = 0$. In this case, equation (3.4) becomes

$$\gamma([x,y])(\gamma(x) - \beta(x)) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.14}$$

Taking ty in place of y in (3.14), and using it again, we obtain

$$\gamma([x,y])\gamma(t)(\gamma(x) - \beta(x)) = 0, \quad \text{for all } x, y, t \in \mathcal{R}.$$

Since γ is onto, we get $\gamma([x,y])\mathcal{R}(\gamma(x) - \beta(x)) = \{0\}$, for all $x, y \in \mathcal{R}$. In view of the primeness of $\mathcal{R}$, the last equation reduces to

$$\gamma([x,y]) = 0 \quad \text{or} \quad \gamma(x) = \beta(x), \quad \text{for all } x, y \in \mathcal{R}. \tag{3.15}$$

If there exists $x_0 \in \mathcal{R}$ such that $\gamma([x_0, y]) = 0$ for all $y \in \mathcal{R}$, it is clear that $\gamma(x_0) \in Z(\mathcal{R})$ because γ is onto and so, $[\alpha(x), \beta(x_0)] = 0$ for all $x \in \mathcal{R}$.

By hypothesis, we have

$$[\alpha(x), \beta(yx_0)] + \gamma([x, yx_0]) = 0, \quad \text{for all } x, y \in \mathcal{R},$$

which leads to

$$\beta(y)[\alpha(x), \beta(x_0)] + [\alpha(x), \beta(y)]\beta(x_0) + \gamma([x,y])\gamma(x_0) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.16}$$

Since $[\alpha(x), \beta(x_0)] = 0$ for all $x \in \mathcal{R}$, (3.16) becomes

$$[\alpha(x), \beta(y)]\beta(x_0) + \gamma([x,y])\gamma(x_0) = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

Using (3.1), the last equation yields

$$\gamma([x,y])(\gamma(x_0) - \beta(x_0)) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.17}$$

Replacing y by yt in (3.17) and using it with the fact that γ is onto, we conclude that $\gamma([x,y])\mathcal{R}(\gamma(x_0) - \beta(x_0)) = \{0\}$, for all $x, y \in \mathcal{R}$. Since $\mathcal{R}$ is prime, we obtain $\gamma([x,y]) = 0$ for all $x, y \in \mathcal{R}$ or $\gamma(x_0) = \beta(x_0)$. Therefore, $[x,y] = 0$ for all $x, y \in \mathcal{R}$ or $\gamma(x_0) = \beta(x_0)$. In this case, (3.15) forces that $\mathcal{R}$ is commutative or $\gamma(x) = \beta(x)$ for all $x \in \mathcal{R}$.

Now assume that the second case, then (3.1) becomes

$$[\alpha(x), \beta(y)] + \beta([x,y]) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.18}$$

Taking xy instead of x in (3.18), we obtain

$$\beta([x,y])(\beta(y) - \alpha(y)) = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

Putting xr in place of x where $r \in \mathcal{R}$, we can easily arrive at

$$\beta([x,y])\mathcal{R}(\beta(y)-\alpha(y)) = \{0\}, \quad \text{for all } x, y \in \mathcal{R}.$$

In light of primeness of $\mathcal{R}$, we arrive at

$$\beta([x,y]) = 0 \text{ or } \alpha(y) = \beta(y), \quad \text{for all } x, y \in \mathcal{R}. \tag{3.19}$$

If there exists $y_0 \in \mathcal{R}$ such that $\alpha(y_0) = \beta(y_0)$, by (3.18) we have

$$\begin{aligned} 0 =& [\alpha(y_0), \beta(x)]) + \beta([y_0, x]) = [\beta(y_0), \beta(x)]) + \beta([y_0, x]) \\ =& 2\beta([y_0, x]), \quad \text{for all } x \in \mathcal{R}. \end{aligned}$$

Since $\mathrm{char}(\mathcal{R}) \neq 2$, we get $\beta([y_0, x]) = 0$ for all $x \in \mathcal{R}$. Then (3.19) becomes $\beta([x,y]) = 0$ for all $x, y \in \mathcal{R}$. Since β is onto, then $[x,y] = 0$ for all $x, y \in \mathcal{R}$, which forces that $\mathcal{R}$ is commutative. ☐

Corollary 3.1. *Let $\mathcal{R}$ be a prime ring with $\mathrm{char}(\mathcal{R}) \neq 2$ and α, β endomorphisms of $\mathcal{R}$ such that β is onto, then the following assertions are equivalent:*

(a) $[\alpha(x), \beta(y)] + \beta([x,y]) = 0$ *for all* $x, y \in \mathcal{R}$*;*
(b) $\mathcal{R}$ *is commutative.*

Proof. Just replace γ by β and δ with the null application in Theorem 3.1. ☐

Corollary 3.2. *Let $\mathcal{R}$ be a prime ring with $\mathrm{char}(\mathcal{R}) \neq 2$ and α an endomorphism of $\mathcal{R}$, then the following assertions are equivalent:*

(a) $\alpha(x) + x \in Z(\mathcal{R})$ *for all* $x \in \mathcal{R}$*;*
(b) $\mathcal{R}$ *is commutative.*

Proof. If we put $\beta = id_{\mathcal{R}}$, we get the required result. ☐

Theorem 3.2. *Let $\mathcal{R}$ be a prime ring with $\mathrm{char}(\mathcal{R}) \neq 2$, α is an automorphism of $\mathcal{R}$ and β, γ epimorphisms of $\mathcal{R}$, then the following assertions are equivalent:*

(a) $\alpha(x) \circ \beta(y) + \gamma([x,y]) = 0$ *for all* $x, y \in \mathcal{R}$*;*
(b) $\mathcal{R}$ *is commutative.*

Proof. It is obvious that (b)⇒(a).

(a)⇒(b) Suppose that

$$\alpha(x) \circ \beta(y) + \gamma([x,y]) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.20}$$

Replacing y by yx in (3.20) and using identity (c), we get

$$\beta(y)[\alpha(x), \beta(x)] = (\alpha(x) \circ \beta(y))\beta(x) + \gamma([x,y])\gamma(x), \quad \text{for all } x, y \in \mathcal{R}. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$\beta(y)[\alpha(x), \beta(x)] = \gamma([x,y])(\gamma(x) - \beta(x)), \quad \text{for all } x, y \in \mathcal{R}. \tag{3.22}$$

Putting xy in place of y in (3.22), we find that

$$(\gamma(x) - \beta(x))\gamma([x,y])(\gamma(x) - \beta(x)) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.23}$$

Invoking (3.22), (3.23) yields

$$(\gamma(x) - \beta(x))\beta(y)[\alpha(x), \beta(x)] = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

Since β is onto, we obtain

$$(\gamma(x) - \beta(x))\mathcal{R}[\alpha(x), \beta(x)] = \{0\}, \quad \text{for all } x \in \mathcal{R}.$$

By primeness of $\mathcal{R}$, we get

$$\gamma(x) = \beta(x) \quad \text{or} \quad [\alpha(x), \beta(x)] = 0, \quad \text{for all } x \in \mathcal{R}. \tag{3.24}$$

If there exists $x_0 \in \mathcal{R}$ such that $[\alpha(x_0), \beta(x_0)] = 0$, then (3.22) gives $\gamma([x_0, y])(\gamma(x_0) - \beta(x_0)) = 0$ for all $y \in \mathcal{R}$. Replacing y by yr, we get $\gamma([x_0, y])\gamma(r)(\gamma(x_0) - \beta(x_0)) = 0$ for all $y, r \in \mathcal{R}$. Since γ is onto, we obtain $\gamma([x_0, y])\mathcal{R}(\gamma(x_0) - \beta(x_0)) = \{0\}$ for all $y \in \mathcal{R}$. By primeness of $\mathcal{R}$, one can easily verify that $\gamma([x_0, y]) = 0$ for all $y \in \mathcal{R}$ or $\gamma(x_0) = \beta(x_0)$.

Suppose the first case and using (3.20), we get $\alpha(x_0) \circ \beta(y) = 0$ for all $y \in \mathcal{R}$. Replacing y by yt and using identity (c), we obtain $\beta(y)[\alpha(x_0), \beta(t)] = 0$ for all $y, t \in \mathcal{R}$. Since $\mathcal{R}$ is prime and β is onto, we get $\alpha(x_0) \in Z(\mathcal{R})$, and therefore, (3.20) forces that $2\alpha(x_0)\mathcal{R}\beta(y) = 0$ for all $y \in \mathcal{R}$. Using the fact that $\mathcal{R}$ is prime and $\mathrm{char}(\mathcal{R}) \neq 2$, we get $\alpha(x_0) = 0$. Since α is an automorphism of $\mathcal{R}$, we obtain $x_0 = 0$. In this case, (3.24) becomes $\gamma(x) = \beta(x),$ for all $x \in \mathcal{R}$. Replacing y by xy in (3.20) and using it, we get

$$\alpha(x) \circ \beta(x)\beta(y) + \beta(x)(-\alpha(x) \circ \beta(y)) = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

Developing the last expression, we arrive at

$$[\alpha(x), \beta(x)]\beta(y) = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

Using the fact that $\mathcal{R}$ is prime and β is onto, we obtain $[\alpha(x), \beta(x)] = 0,$ for all $x, y \in \mathcal{R}$. For $y = x$, (3.20) with the last expression give $\alpha(x)\beta(x) = \beta(x)\alpha(x) = 0$ for all $x \in \mathcal{R}$.

Replacing y by yx in (3.20) and using it again, we obtain

$$\alpha(x) \circ \beta(y)\beta(x) + \beta([x, y])\beta(x) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.25}$$

Developing (3.25) by using identity (c), we conclude that

$$[\alpha(x), \beta(y)]\beta(x) + \beta([x, y])\beta(x) = 0, \quad \text{for all } x, y \in \mathcal{R}. \tag{3.26}$$

Putting yt in place of y and using identity (a) with (3.26), we can easily arrive at

$$([\alpha(x), \beta(y)] + \beta([x, y]))\beta(t)\beta(x) = 0, \quad \text{for all } x, y, t \in \mathcal{R}. \tag{3.27}$$

Since β is onto, equation (3.27) reduces to

$$([\alpha(x), \beta(y)] + \beta([x, y]))\mathcal{R}\beta(x) = \{0\}, \quad \text{for all } x, y \in \mathcal{R}.$$

By primeness of $\mathcal{R}$, we obtain

$$[\alpha(x), \beta(y)] + \beta([x, y]) = 0 \quad \text{or} \quad \beta(x) = 0, \quad \text{for all } x, y \in \mathcal{R}.$$

It is clear that both cases give the following equation

$$[\alpha(x),\beta(y)]+\beta([x,y])=0,\quad \text{for all } x,y\in\mathcal{R}. \tag{3.28}$$

As (3.28) is the same as (3.18), arguing as in the proof of Theorem 3.1, we conclude that $\mathcal{R}$ is commutative. $\square$

In Examples 3.1, 3.2, we show that the condition "$\mathcal{R}$ is prime" is necessary in Theorems 3.1, 3.2.

Example 3.1. Let us defined $\mathcal{R}$ and $\alpha,\beta,\gamma:\mathcal{R}\to\mathcal{R}$ as follow:

$$\mathcal{R}=\left\{\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)\middle|\, x,y,z\in\mathbb{Z}\right\},\quad \alpha\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)=\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & -z & 0\end{array}\right),$$

$$\beta=id_{\mathcal{R}},\quad \gamma\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)=\left(\begin{array}{ccc} x & -y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)\text{ and } \delta=0.$$

It is clear that $\mathcal{R}$ is a ring which is not prime and $\mathrm{char}(\mathcal{R})\neq 2$. Moreover, α is an endomorphism of $\mathcal{R}$ and β, γ epimorphisms of $\mathcal{R}$ such that $[\alpha(x),\beta(y)]+\gamma([x,y])=0$ for all $x,y\in\mathcal{R}$, but $\mathcal{R}$ is noncommutative.

Example 3.2. Let us defined $\mathcal{R}$ and $\alpha,\beta,\gamma:\mathcal{R}\to\mathcal{R}$ as follow:

$$\mathcal{R}=\left\{\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)\middle|\, x,y,z\in\mathbb{Z}\right\},\quad \alpha=id_{\mathcal{R}},$$

$$\beta=\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)=\left(\begin{array}{ccc} -x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right),\quad \gamma\left(\begin{array}{ccc} x & y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right)=\left(\begin{array}{ccc} x & -y & 0\\ 0 & 0 & 0\\ 0 & z & 0\end{array}\right).$$

It is clear that $\mathcal{R}$ is a ring which is not prime and $\mathrm{char}(\mathcal{R})\neq 2$. Moreover, α is an automorphism of $\mathcal{R}$ and β, γ epimorphisms of $\mathcal{R}$ such that $\alpha(x)\circ\beta(y)+\gamma([x,y])=0$ for all $x,y\in\mathcal{R}$, but $\mathcal{R}$ is noncommutative.

The following example proves that the condition "$\mathrm{char}(\mathcal{R})\neq 2$" in Theorem 3.2 is not superfluous.

Example 3.3. Let us define $\mathcal{R}=M_2(\mathbb{Z}_2)$ and $\alpha=\beta=\gamma=id_{\mathcal{R}}$. It is clear that $\mathcal{R}$ is a noncommutative prime ring such that $\mathrm{char}(\mathcal{R})=2$. Moreover, α is an automorphism of $\mathcal{R}$ and β, γ epimorphisms of $\mathcal{R}$ such that

(a) $[\alpha(x),\beta(y)]+\gamma([x,y])=0$ for all $x,y\in\mathcal{R}$;
(b) $\alpha(x)\circ\beta(y)+\gamma([x,y])=0$ for all $x,y\in\mathcal{R}$.

But $\mathcal{R}$ is noncommutative.

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A NEW METHOD TO SOLVE DUAL SYSTEMS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY LEGENDRE WAVELETS

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Abstract. The method that will be presented, is numerical solution based on the Legendre wavelets for solving dual systems of fractional integro-differential equations (FIDEs). First of all we make the operational matrix of fractional order integration. The application of this matrix is transforming FIDEs to a system of algebric equations. By this changing, we are able to solve it by a simple solution. In this way, the Legendre wavelets and their operator matrix are the most important keys of our solution. After explaining the method we test on some illustrative examples which numerical solutions of these examples demonstrate the validity and applicability of suggested method.

1. Introduction

Nowadays using fractional calculus has valuable usages in some fields of science and engineering. The study of dual systems of FIDEs have many applications in engineering, biomechanice and other scientific divisions. Dual systems of FIDEs also appear in modeling some of chemical and material engineering processes [8, 13, 15]. In most cases obtaining an analytical solution of FIDEs is impossible or so difficult. Thus, various procedures for obtaining approximate solutions of this kind of equations have attracted the attentions of many researchers.

In recent years, several numerical methods have been devoted for solving FIDEs but they are not properly applied to solve dual systems of FIDEs [1, 4, 18]. The greatest information that we can obtain from this case, is studing of papers that have

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been presented by various methods to arrive to an approximate solution. One of these methods is wavelet method [20]. Wavelets are generally a family of oscillatory functions which can be used to obtain approximate solutions of unknown functions [12]. There are many methods for solving FIDEs, by helping of wavelets; for instance take a look at [2,9,11,22]. The application of wavelets is significant in many scientific disciplines, such as time-frequency analysis, signal processing and numerical analysis [3].

This paper is based on Legendre wavelets that are a special type of wavelets that successfully have passed the exams in system analysis, system identification, optimal control and numerical solutions of differential and integral equations. Legendre wavelets are based on Legendre polynomials. From numerical point of view, wavelets have a closer and more accurate approximation than Legendre polynomials [17]. In the study of various methods for numerical solution of systems of FIDEs, we find that the wavelets method has been used less. Therefore, we have chosen the method of the Legendre wavelets for numerical solution of systems of FIDEs. We now apply the Legendre wavelets method to solve the following dual system [21]:

$$\begin{cases} D^r f(x) = u_1(x, f(x), g(x)) + \int_0^x u_2(t, f(t), g(t))dt, \\ D^s g(x) = v_1(x, f(x), g(x)) + \int_0^x v_2(t, f(t), g(t))dt, \end{cases}$$

where $x, t \in [0, 1]$, $r, s \in (0, 1]$, and D^r, D^s display the Caputo derivative operator.

2. Legendre Wavelets and their Functional Properties

2.1. Legendre wavelets. Legendre wavelets are defined on $[0, 1)$ as [10]:

$$\psi_{nm}(x) = \begin{cases} 2^{\frac{k}{2}}\sqrt{m + \frac{1}{2}}L_m(2^k x - \widehat{n}), & \frac{\widehat{n}-1}{2^k} \le x < \frac{\widehat{n}+1}{2^k}, \\ 0, & \text{otherwise}, \end{cases}$$

where $n = 1, 2, \dots, 2^{k-1}$, $\widehat{n} = 2n-1$, $m = 0, 1, 2, \dots, M-1$, $k, M \in \mathbb{N}$, m is the degree of the Legendre polynomials and $L_m(x)$ are the well-known Legendre polynomials of order m that are defined on the interval $[-1, 1]$ and satisfy the following recursive formula

$$\begin{aligned} &L_0(x) = 1, \quad L_1(x) = x, \\ &L_{m+1}(x) = \Big(\frac{2m+1}{m+1}\Big)xL_m(x) - \Big(\frac{m}{m+1}\Big)L_{m-1}(x), \quad m = 1, 2, \dots \end{aligned}$$

2.2. Function approximation. The Legendre wavelet series representation of the function $f(x)$ defined over $[0, 1)$ is given by

$$f(x) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} a_{nm}\psi_{nm}(x) = A^T\Psi(x), \tag{2.1}$$

where $a_{nm} = \langle f(x), \psi_{nm}(x)\rangle$ and $\langle \cdot, \cdot \rangle$ denotes the inner product. If the infinite series in (2.1) is finited, (2.1) can be written as

$$f(x) \approx \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} a_{nm}\psi_{nm}(x) = A^T \Psi(x), \tag{2.2}$$

where A and $\Psi(x)$ are $2^{k-1}M \times 1$ matrices given by

$$A = \Big[a_{10}, a_{11}, \ldots, a_{1(M-1)}, a_{20}, a_{21}, \ldots, a_{2(M-1)}, \ldots, a_{2^{k-1}0}, a_{2^{k-1}1}, \ldots, a_{2^{k-1}(M-1)}\Big]^T$$

$$\Psi(x) = \Big[\psi_{10}(x), \psi_{11}(x), \ldots, \psi_{1(M-1)}(x), \psi_{20}(x), \psi_{21}(x), \ldots, \psi_{2(M-1)}(x), \ldots,$$
$$\psi_{2^{k-1}0}(x), \psi_{2^{k-1}1}(x), \ldots, \psi_{2^{k-1}(M-1)}(x)\Big]^T.$$

For simplicity, (2.2) can be rewritten as

$$f(x) \approx \sum_{i=1}^{n'} a_i\psi_i(x) = A_{n'}^T \Psi_{n'}(x) = \widehat{f}(x),$$

where $a_i = a_{nm}$, $\psi_i = \psi_{nm}$, $n' = 2^{k-1}M$, $i = M(n-1)+m+1$. Obtain the collocation points as

$$x_i = \frac{i-0.5}{n'}, \quad i = 1, 2, \cdots, 2^{k-1}M.$$

We define the Legendre wavelets matrix as

$$\phi_{n'\times n'} = \left[\Psi\left(\frac{1}{2n'}\right), \Psi\left(\frac{3}{2n'}\right), \Psi\left(\frac{5}{2n'}\right), \ldots, \Psi\left(\frac{i-0.5}{n'}\right)\right].$$

3. Operational Matrix of the Integration for Legendre Wavelets

3.1. **Preliminaries and natations.** In this section, we first present some definitions and basic concepts that have the most applications in this paper [19].

Definition 3.1. The Reimann-Liouville fractional integral operator of order $\gamma \geq 0$ is a function defined as

$$I^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(\gamma)} \displaystyle\int_0^x (x-t)^{\gamma-1} f(t)dt, & \gamma > 0, \\ f(x), & \gamma = 0, \end{cases}$$

where $\Gamma(\gamma)$ is the gamma function as

$$\Gamma(\gamma) = \int_0^\infty t^{\gamma-1}e^{-t}dt.$$

Definition 3.2. The Caputo fractional derivative of order $\gamma > 0$ is defined as

$$D^\gamma f(x) = \begin{cases} \dfrac{1}{\Gamma(n-\gamma)} \displaystyle\int_0^x (x-t)^{n-\gamma-1} f^{(n)}(t)dt, & \gamma > 0,\ n-1 < \gamma < n, \\ \dfrac{d^{(n)} f(x)}{dx^n}, & \gamma = n, \end{cases}$$

where $x \in [0, \infty)$, and $n = 1, 2, 3, \ldots$

For $x > 0$ the Caputo derivative and Reimann-Liouville integral operator have the following relationships

$$\begin{aligned} D^\gamma I^\gamma f(x) =& f(x), \\ I^\gamma D^\gamma f(x) =& f(x) - \sum_{m=0}^{n-1} \frac{f^{(m)}(0^+)}{m!} x^m, \quad n-1 < \gamma < n. \end{aligned} \tag{3.1}$$

3.2. **Operational matrix of the fractional integration.** Here the main goal is to get the fractional-order Legendre wavelets operational matrix of integration. For this purpose, we have to define the set of Block puls functions (BPFs) as follows [16]

$$b_i(x) = \begin{cases} 1, & \dfrac{i-1}{n'} \leq x < \dfrac{i}{n'}, \\ 0, & \text{otherwise}, \end{cases}$$

where $i = 1, 2, \ldots, n'$, and $n' = 2^{k-1}M$.

The BPFs have two properties which will be used later

$$b_i(x)b_j(x) = \begin{cases} b_i(x), & i = j, \\ 0, & i \neq j, \end{cases}$$

$$\int_0^x b_i(x)b_j(x)dx = \begin{cases} \dfrac{1}{n'}, & i = j, \\ \\ 0, & i \neq j. \end{cases}$$

Definition 3.3. Let $C = [c_1, c_2, \ldots, c_{n'}]^T$ and $D = [d_1, d_2, \ldots, d_{n'}]^T$ be two matrices $n' \times n'$, then we define that $C \otimes D = [c_1d_1, c_2d_2, \ldots, c_{n'}d_{n'}]^T$.

Lemma 3.1. *Suppose that $g(x)$ and $h(x)$ are two functions defined on $L^2[0,1]$ as we have $g(x) = G^T B_{n'}(x)$ and $h(x) = H^T B_{n'}(x)$, where $G^T = [g_1, g_2, \ldots, g_{n'}]$, $H^T = [h_1, h_2, \ldots, h_{n'}]$ and $B_{n'}(x) = [b_1, b_2, \ldots, b_{n'}]^T$, then we have*

$$g(x)h(x) \approx G^T B_{n'}(x) H^T B_{n'}(x) = (G^T \otimes H^T) B_{n'}(x), \tag{3.2}$$

$$g(x)^2 \approx (G^T B_{n'}(x))^2 = (G^T)^2 B_{n'}(x). \tag{3.3}$$

Proof. By using the properties of BPFs, the proof is obvious. □

The fractional integration of order γ in Reimann-Liouvill concept can be expressed as [5]

$$I^\gamma B_{n'}(x) \approx R^\gamma B_{n'}(x), \tag{3.4}$$

where R^{γ} is the BPFs operational matrix with

$$R^{\gamma}=\frac{1}{n'^{\gamma}}\frac{1}{\Gamma(\gamma+2)}\begin{bmatrix}1 & \xi_1 & \xi_2 & \cdots & \xi_{n'-1}\\ 0 & 1 & \xi_1 & \cdots & \xi_{n'-2}\\ 0 & 0 & 1 & \cdots & \xi_{n'-3}\\ 0 & 0 & 0 & \cdots & \xi_{n'-4}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & 1\end{bmatrix},$$

and $\xi_k=(k+1)^{\gamma+1}-2k^{\gamma+1}+(k-1)^{\gamma+1}$, $k=1,2,\ldots,n'-1$.

We now derive the Legendre wavelets operational matrix of the fractional integration. The integration of Legendre wavelets $\Psi_{n'}(x)$ can be obtained as

$$I\Psi_{n'}(x)=\int_0^x \Psi_{n'}(\tau)d\tau\approx q_{n'\times n'}\Psi_{n'}(x), \tag{3.5}$$

where the n'-square matrix $q_{n'\times n'}$ is called Legendre wavelets operational matrix and $q^{\gamma}_{n'\times n'}$ is called Legendre wavelets fractional integral operational matrix and achived by

$$I^{\gamma}\Psi_{n'}(x)\approx q^{\gamma}_{n'\times n'}\Psi(x)_{n'}, \tag{3.6}$$

the Legendre wavelets can be expanded into n'-set BPFs as

$$\Psi_{n'}(x)\approx\phi_{n'\times n'}B_{n'}(x), \tag{3.7}$$

we get [6] from (3.4), (3.6) and (3.7)

$$\begin{aligned}q^{\gamma}_{n'\times n'}\Psi_{n'}(x)\approx& I^{\gamma}\Psi_{n'}(x)\approx I^{\gamma}\phi_{n'\times n'}B_{n'}(x)=\phi_{n'\times n'}I^{\gamma}B_{n'}(x)\approx\phi_{n'\times n'}R^{\gamma}B_{n'}(x)\\ \approx&\phi_{n'\times n'}R^{\gamma}\phi^{-1}_{n'\times n'}\psi_{n'}(x).\end{aligned}$$

Finally, we conclude from (3.6) $q^{\gamma}_{n'\times n'}\approx\phi_{n'\times n'}R^{\gamma}\Psi^{-1}_{n'\times n'}$.

In general, the matrix $\phi_{n'\times n'}$ counted in the below form

$$\phi_{n'\times n'}=\begin{bmatrix}L & 0 & 0 & \cdots & 0\\ 0 & L & 0 & \cdots & 0\\ 0 & 0 & L & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & L\end{bmatrix},$$

where L is a $M\times M$ matrix given by [7]

$$L=\begin{bmatrix}\psi_{10}\left(\frac{1}{2n'}\right) & \psi_{10}\left(\frac{3}{2n'}\right) & \cdots & \psi_{10}\left(\frac{i-0.5}{n'}\right)\\ \psi_{11}\left(\frac{1}{2n'}\right) & \psi_{11}\left(\frac{3}{2n'}\right) & \cdots & \psi_{11}\left(\frac{i-0.5}{n'}\right)\\ \vdots & \vdots & \ddots & \vdots\\ \psi_{2^{k-1}(M-1)}\left(\frac{1}{2n'}\right) & \psi_{2^{k-1}(M-1)}\left(\frac{3}{2n'}\right) & \cdots & \psi_{2^{k-1}(M-1)}\left(\frac{i-0.5}{n'}\right)\end{bmatrix}.$$

The six basis functions are by

$$\begin{cases}\psi_{10}(x)=\sqrt{2}, \\ \psi_{11}(x)=\sqrt{6}(4x-1), & 0\leq x<\frac{1}{2}, \\ \psi_{12}(x)=\sqrt{10}(24x^2-12x+1), \end{cases}$$

$$\begin{cases}\psi_{20}(x)=\sqrt{2}, \\ \psi_{21}(x)=\sqrt{6}(4x-1), & \frac{1}{2}\leq x<1, \\ \psi_{22}(x)=\sqrt{10}(24x^2-36x+13). \end{cases}$$

Here, we present the matrices R^{γ}, $\phi_{n'\times n'}$ and q^{γ} for $k=2$, $M=3$, $n=1,2$, $m=0,1,2$, $\gamma=0.6$ and using the collocation points $x_i=\frac{i-0.5}{n'}$, $i=1,2,\dots,n'$, $n'=2^{k-1}M$. Clearly, we have:

$$R^{0.6}=\begin{bmatrix} 0.23872 & 0.24622 & 0.17586 & 0.14847 & 0.13201 & 0.12061 \\ 0 & 0.23872 & 0.24622 & 0.17586 & 0.14847 & 0.13201 \\ 0 & 0 & 0.23872 & 0.24622 & 0.17586 & 0.14847 \\ 0 & 0 & 0 & 0.23872 & 0.24622 & 0.17586 \\ 0 & 0 & 0 & 0 & 0.23872 & 0.24622 \\ 0 & 0 & 0 & 0 & 0 & 0.23872 \end{bmatrix},$$

$$\phi_{6\times 6}=\begin{bmatrix} 1.41421 & 1.41421 & 1.41421 & 0 & 0 & 0 \\ -1.63299 & 0 & 1.63299 & 0 & 0 & 0 \\ 0.52705 & -1.58114 & 0.52705 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.41421 & 1.41421 & 1.41421 \\ 0 & 0 & 0 & -1.63299 & 0 & 1.63299 \\ 0 & 0 & 0 & 0.52705 & -1.58114 & 0.52705 \end{bmatrix},$$

$$q^{0.6}=\begin{bmatrix} 0.45856 & 0.18277 & -0.02360 & 0.47845 & -0.07337 & 0.01977 \\ -0.14723 & 0.15079 & 0.12261 & 0.06705 & -0.03495 & 0.01469 \\ -0.05571 & -0.09082 & 0.10681 & -0.04913 & 0.00096 & 0.00190 \\ 0 & 0 & 0 & 0.45856 & 0.18277 & -0.02360 \\ 0 & 0 & 0 & -0.14723 & 0.15079 & 0.12261 \\ 0 & 0 & 0 & -0.05571 & -0.09082 & 0.10681 \end{bmatrix}.$$

3.3. **Error analysis.** The following theorem presents the error analysis of the Legendre wavelets approximation function. By increasing values of k and M the error gets closer to zero. As you will see, solved examples confirm this sentence. So, we say surely the mentioned method and its approximation function will be successfully responsive for solving examples of the discussed subject.

Theorem 3.1 ([14]). *Suppose $f(x)\in C^2[0,1]$ and $\widehat{f}(x)$ is the best approximation of $f(x)$, then we have for these two functions defined in* (2.1) *and* (2.2)*:*

$$\begin{aligned}\|e_f\|_2=\|\text{error}(f(x))\|_2=\|f(x)-\widehat{f}(x)\|_2&=o\left(\frac{1}{M!2^{Mk}}\right), \\ &=\frac{c}{M!2^{Mk}}_{c>0} \quad \textit{as } k\to\infty,\ M\to\infty.\end{aligned}$$

4. Numerical Examples

In this section, we are going to solve two numerical examples by using the proposed method in Section 3, Also, we will compare their approximate and exact solution from graphical and numerical point of view. The numerical results show these performance of the mentioned method.

Example 4.1 ([21]).

$$\begin{cases} D^r f(x) = -\dfrac{1}{2}f^2(x) - g(x) + \dfrac{1}{2} - \displaystyle\int_0^x g(t)f(t)dt, & 0 < r \leq 1, \\ D^s g(x) = g^2(x) + f^2(x) - \displaystyle\int_0^x g(t)dt, & 0 < s \leq 1, \end{cases} \tag{4.1}$$

with the initial conditions $f(0) = 1$ and $g(0) = 0$. Exact solutions for the above coupled systems when $r = s = 1$ are obtained by $f(x) = \cos x$ and $g(x) = \sin x$, the exact solutions of $f(x)$ and $g(x)$ for $r, s \in (0, 1)$ are unknown.

Let

$$\begin{cases} D^r f(x) \approx A_{n'}^T \Psi_{n'}(x), \\ D^s g(x) \approx E_{n'}^T \Psi_{n'}(x), \end{cases} \tag{4.2}$$

where $A_{n'}^T = [a_1, a_2, a_3, \ldots, a_{n'}]$ and $E_{n'}^T = [e_1, e_2, e_3, \ldots, e_{n'}]$. By using the initial conditions and (3.1), (3.6), (3.7) and (4.2), we have

$$\begin{cases} f(x) = & I^r D^r f(x) + f(0) \approx A_{n'}^T q_{n'\times n'}^r \Psi_{n'}(x) + 1 \\ \quad \approx & A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x) + [1, \ldots, 1]_{n'\times n'}, \\ g(x) = & I^s D^s g(x) + g(0) \approx E_{n'}^T q_{n'\times n'}^s \Psi_{n'}(x) \approx E_{n'} q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x). \end{cases} \tag{4.3}$$

Then, by using (3.2), (3.3), (3.5) and (4.3), we obtain

$$\begin{aligned} f^2(x) \approx & (A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'})^2 B_{n'}(x) + 2A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x) + [1, 1, \ldots, 1]_{n'\times n'}, \\ g^2(x) \approx & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'})^2 B_{n'}(x), \end{aligned}$$

(4.4)

$$\begin{aligned} \int_0^x g(t)dt \approx & \int_0^x E_{n'}^T q_{n'\times n'}^s \Psi_{n'}(t)dt \approx E_{n'}^T q_{n'\times n'}^{1+s} \phi_{n'\times n'} B_{n'}(x), \\ g(x)f(x) \approx & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x))(A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x) + 1) \\ = & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) B_{n'}(x) + E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x), \\ \int_0^x g(t)f(t)dt \approx & \int_0^x (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) B_{n'}(t)dt \\ & + \int_0^x E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(t)dt \\ = & (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) \int_0^x B_{n'}(t)dt \\ & + E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \int_0^x B_{n'}(t)dt \end{aligned}$$

$$
\begin{aligned}
&\approx(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}\otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})\int_0^x \phi^{-1}_{n'\times n'}\Psi_{n'}(t)dt\\
&\quad+(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'})\int_0^x \phi^{-1}_{n'\times n'}\Psi_{n'}(t)dt\\
&\approx(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}\otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})\phi^{-1}_{n'\times n'}q_{n'\times n'}\phi_{n'\times n'}B_{n'}(x)\\
&\quad+E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x).
\end{aligned}
\tag{4.5}
$$

By replacing (3.7), (4.2)–(4.4) and (4.5) into (4.1), and also according to the properties of BPFs, we conclude

$$
\begin{cases}
A^T_{n'}\phi_{n'\times n'} = & -\dfrac{1}{2}(A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 - A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}\\
& -E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} - (E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}\otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}\\
& +E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'})\phi^{-1}_{n'\times n'}q_{n'\times n'}\phi_{n'\times n'},\\
E^T_{n'}\phi_{n'\times n'} = & (E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'})^2 + (A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 + 2A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}\\
& -E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'} + [1,1,\ldots,1]_{1\times n'}.
\end{cases}
\tag{4.6}
$$

(4.6) is now a system of nonlinear algebric equations which is a transformed type of (4.1). It has $2n'$ unknown coefficients, A_i and E_i, which we can find them and the numerical solutions of $f(x)$ and $g(x)$ by solving this system by presented numerical method.

The approximate solutions obtained by using the proposed method and also absolute error value for different values k, M, r, s and x in the Tables $1-3$ have been shown. From Tables 1-3 and Figures 1-5 we can see that by increasing k and M the numerical solutions converge to the exact solutions, specially when $r, s \to 1$.

TABLE 1. Numerical results of the Example 4.1 for $k = 2$, $M = 6$, $n' = 2^{k-1}M = 12$, $i = 1, 2, 3, \ldots, n'$, and different values r and s.

$x_i = \frac{i-0.5}{n'}$	$r = 0.7,\ s = 0.7$ $f(x),\ g(x)$	$r = 0.8,\ s = 0.8$ $f(x),\ g(x)$	$r = 0.9,\ s = 0.9$ $f(x),\ g(x)$	$r = 1,\ s = 1$ $f(x),\ g(x)$	Ecact solution $f(x),\ g(x)$
$x_1 = 0.04167$	$0.98729, 0.11185$	$0.99339, 0.08097$	$0.99660, 0.05817$	$0.99827, 0.41546$	$0.99913, 0.04166$
$x_2 = 0.12500$	$0.95593, 0.24847$	$0.97390, 0.19959$	$0.98486, 0.15830$	$0.99136, 0.12435$	$0.99220, 0.12467$
$x_3 = 0.20833$	$0.91290, 0.34846$	$0.94327, 0.29703$	$0.96398, 0.24904$	$0.97758, 0.20629$	$0.97838, 0.20683$
$x_4 = 0.29167$	$0.86384, 0.42851$	$0.90475, 0.38190$	$0.93530, 0.33342$	$0.95704, 0.28680$	$0.95777, 0.28755$
$x_5 = 0.37500$	$0.81126, 0.49298$	$0.86008, 0.45627$	$0.89968, 0.41195$	$0.92988, 0.36531$	$0.93051, 0.36627$
$x_6 = 0.45833$	$0.75698, 0.54381$	$0.81063, 0.52091$	$0.85786, 0.48458$	$0.89628, 0.44128$	$0.89679, 0.44245$
$x_7 = 0.54167$	$0.70249, 0.58196$	$0.75760, 0.57604$	$0.81054, 0.55109$	$0.85648, 0.51418$	$0.85685, 0.51556$
$x_8 = 0.62500$	$0.64909, 0.60799$	$0.70209, 0.62168$	$0.75844, 0.61113$	$0.81076, 0.58351$	$0.81096, 0.58510$
$x_9 = 0.70833$	$0.59794, 0.62222$	$0.64516, 0.65768$	$0.70224, 0.66435$	$0.75944, 0.64878$	$0.75945, 0.65057$
$x_{10} = 0.79167$	$0.55009, 0.62485$	$0.58783, 0.68380$	$0.64269, 0.71035$	$0.70287, 0.70954$	$0.70266, 0.71153$
$x_{11} = 0.87500$	$0.50650, 0.61613$	$0.53112, 0.69978$	$0.58050, 0.74871$	$0.64145, 0.76535$	$0.64100, 0.76754$
$x_{12} = 0.95833$	$0.46802, 0.59639$	$0.47601, 0.70530$	$0.51646, 0.77900$	$0.57560, 0.81583$	$0.57488, 0.81823$

TABLE 2. Numerical results of the Example 4.1 for $k = 6$, $M = 3$, $n' = 2^{k-1}M = 96$, $i = 1, 2, 3, \ldots, n'$, and different values r and s

$x_i = \frac{i-0.5}{n'}$	$r = 0.85,\ s = 0.85$ $f(x),\ g(x)$	$r = 0.9,\ s = 0.9$ $f(x),\ g(x)$	$r = 0.95,\ s = 0.95$ $f(x),\ g(x)$	$r = 1,\ s = 1$ $f(x),\ g(x)$	Ecact solution $f(x),\ g(x)$
$x_8 = 0.07813$	$0.99150, 0.12072$	$0.99393, 0.10459$	$0.99568, 0.09043$	$0.99694, 0.07804$	$0.99695, 0.07805$
$x_{16} = 0.16146$	$0.97100, 0.22197$	$0.97769, 0.19983$	$0.98292, 0.17943$	$0.98698, 0.16075$	$0.98699, 0.16076$
$x_{24} = 0.24479$	$0.94160, 0.31225$	$0.95307, 0.28786$	$0.96250, 0.26449$	$0.97018, 0.24234$	$0.97019, 0.24235$
$x_{32} = 0.32813$	$0.90481, 0.39397$	$0.22105, 0.36986$	$0.93492, 0.34583$	$0.94664, 0.32226$	$0.94665, 0.32227$
$x_{40} = 0.41146$	$0.86177, 0.46779$	$0.88241, 0.44602$	$0.90064, 0.42322$	$0.91653, 0.39993$	$0.91654, 0.39995$
$x_{48} = 0.49479$	$0.81349, 0.53383$	$0.83787, 0.51619$	$0.86010, 0.49628$	$0.88006, 0.47483$	$0.88007, 0.47485$
$x_{56} = 0.57813$	$0.76090, 0.59198$	$0.78813, 0.58007$	$0.81378, 0.56461$	$0.83748, 0.54643$	$0.83749, 0.54645$
$x_{64} = 0.66146$	$0.70490, 0.64202$	$0.73389, 0.63733$	$0.76217, 0.62776$	$0.78910, 0.61424$	$0.78910, 0.61427$
$x_{72} = 0.74479$	$0.64638, 0.68365$	$0.67585, 0.68758$	$0.70579, 0.68530$	$0.73523, 0.67779$	$0.73523, 0.67782$
$x_{80} = 0.82813$	$0.58625, 0.71652$	$0.61477, 0.73042$	$0.64520, 0.73680$	$0.67626, 0.73663$	$0.67626, 0.73666$
$x_{88} = 0.91146$	$0.52539, 0.74023$	$0.55138, 0.76544$	$0.58097, 0.78185$	$0.61260, 0.79036$	$0.61259, 0.79040$
$x_{96} = 0.99979$	$0.46474, 0.75438$	$0.48647, 0.79219$	$0.51373, 0.82002$	$0.54469, 0.83861$	$0.54468, 0.83865$

TABLE 3. Absolute error relevant to Tables 1 and 2 when $r = s = 1$

x_i	e_f	e_g	x_i	e_f	e_g
x_1	$8.6311e-04$	$1.0824e-04$	x_8	$1.3409e-05$	$3.1776e-06$
x_2	$8.4048e-04$	$3.2442e-04$	x_{16}	$1.2897e-05$	$6.5587e-06$
x_3	$7.9451e-04$	$5.3968e-04$	x_{24}	$1.2015e-05$	$9.9213e-06$
x_4	$7.2457e-04$	$7.5337e-04$	x_{32}	$1.0753e-05$	$1.3255e-05$
x_5	$6.3022e-04$	$9.6491e-04$	x_{40}	$9.1071e-06$	$1.6552e-05$
x_6	$5.1122e-04$	$1.1739e-03$	x_{48}	$7.0736e-06$	$1.9807e-05$
x_7	$3.6749e-04$	$1.3801e-03$	x_{56}	$4.6529e-06$	$2.3020e-05$
x_8	$1.9913e-04$	$1.5838e-03$	x_{64}	$1.8473e-06$	$2.6198e-05$
x_9	$6.3200e-06$	$1.7858e-03$	x_{72}	$1.3399e-06$	$2.9357e-05$
x_{10}	$2.1069e-04$	$1.9875e-03$	x_{80}	$4.9047e-06$	$3.2527e-05$
x_{11}	$4.5169e-04$	$2.1913e-03$	x_{88}	$8.8442e-06$	$3.5751e-05$
x_{12}	$7.1658e-04$	$2.4006e-03$	x_{96}	$1.3158e-05$	$3.9088e-05$

Example 4.2 ([21]).

$$
\begin{cases} D^r f(x) = \dfrac{1}{3}g(x)f(x) - g(x) + 1 - \int_0^x [g(t) - 2f(t)]dt, & 0 < r \leq 1, \\ D^s g(x) = \dfrac{1}{3}g(x)f(x) + \dfrac{1}{2}f^2(x) + 2f(x) - \int_0^x [g(t) + f(t)]dt, & 0 < s \leq 1, \end{cases} \tag{4.7}
$$

with the initial conditions $f(0) = 0$ and $g(0) = 0$, exact solutions for the above coupled systems when $r = s = 1$ are obtained by $f(x) = x$ and $g(x) = x^2$. The exact solutions

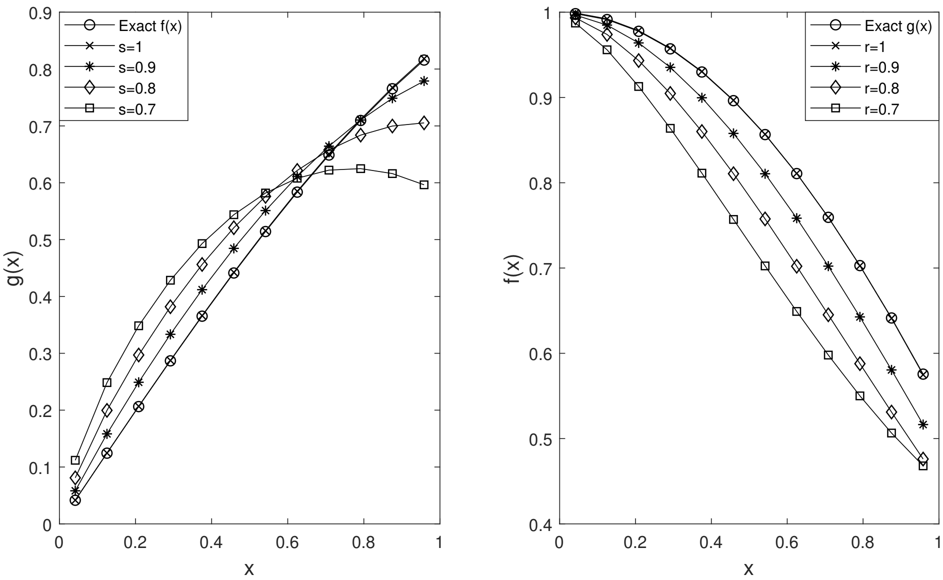

FIGURE 1. Numerical solution for different values of r and s when $k = 2$, $M = 6$ and $n' = 12$.

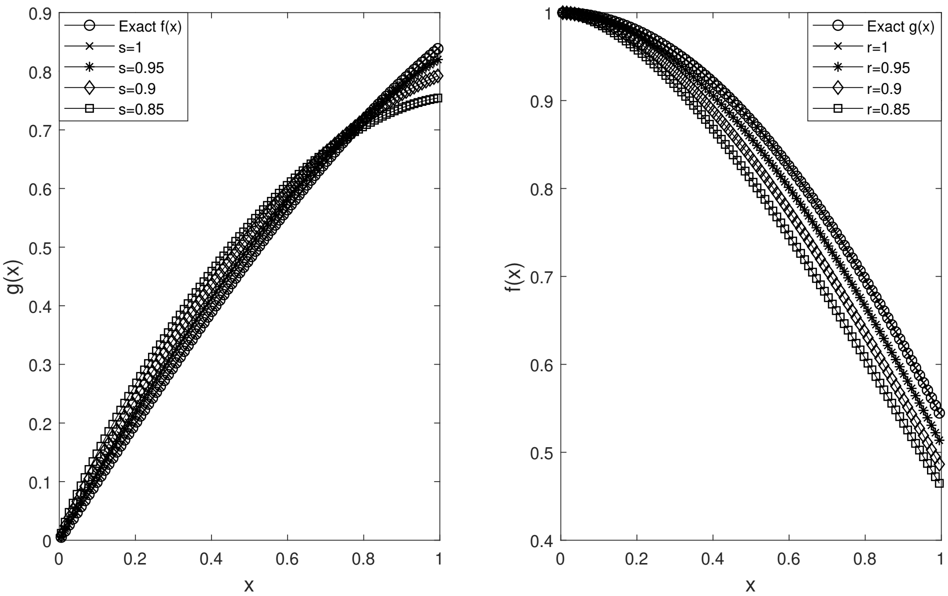

FIGURE 2. Numerical solution for different values of r and s when $k = 6$, $M = 3$ and $n' = 96$.

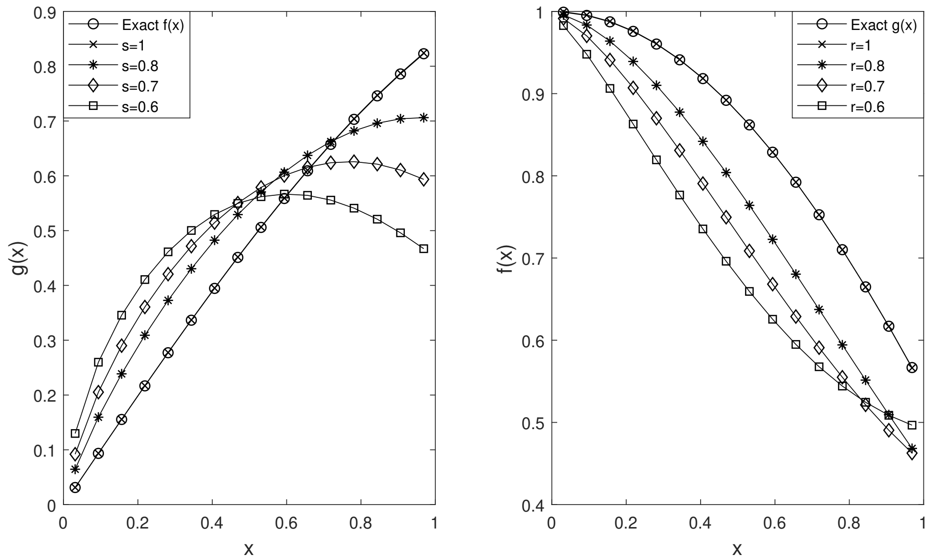

FIGURE 3. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

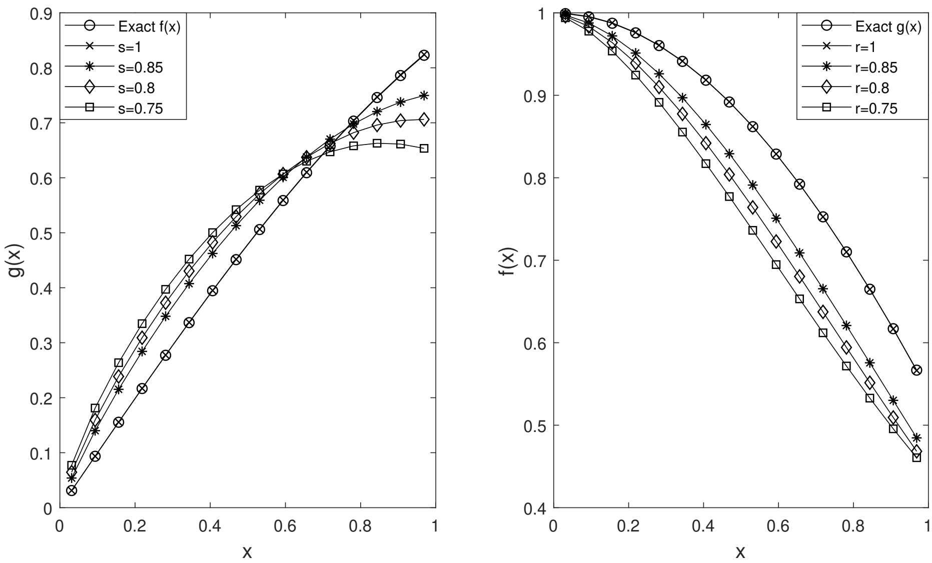

FIGURE 4. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

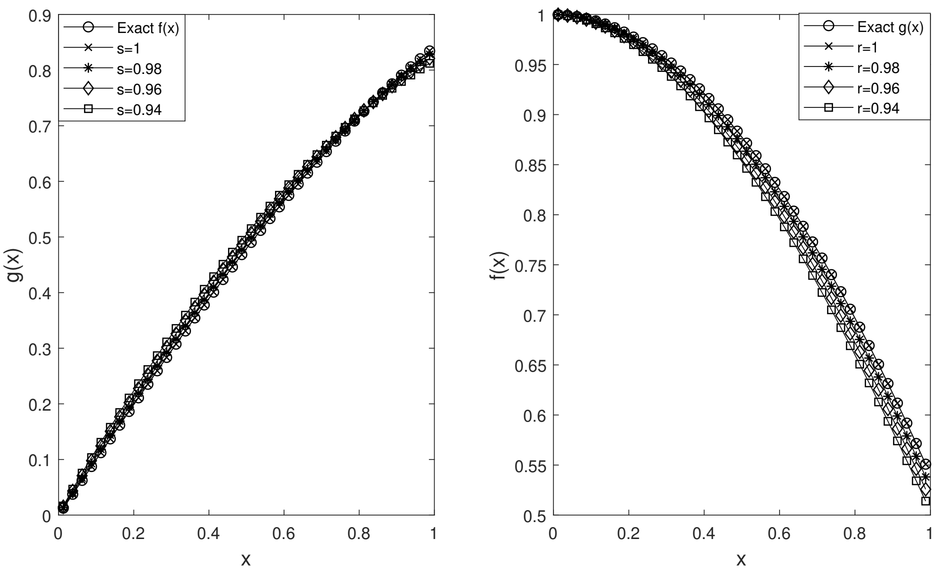

FIGURE 5. Numerical solution for different values of r and s when $k = 4$, $M = 5$ and $n' = 40$.

of $f(x)$ and $g(x)$ for $r, s \in (0, 1)$ are unknown. Let

$$\begin{cases} D^r f(x) \approx A_{n'}^T \Psi_{n'}(x), \\ D^s g(x) \approx E_{n'}^T \Psi_{n'}(x), \end{cases} \tag{4.8}$$

where $A_{n'}^T = [a_1, a_2, a_3, \ldots, a_{n'}]$ and $E_{n'}^T = [e_1, e_2, e_3, \ldots, e_{n'}]$.

By using the initial conditions and (3.2), (3.6), (3.7) and (4.8) we have

$$\begin{cases} f(x) = I^r D^r f(x) + f(0) \approx A_{n'}^T q_{n'\times n'}^r \Psi_{n'}(x) \approx A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x), \\ g(x) = I^s D^s g(x) + g(0) \approx E_{n'}^T q_{n'\times n'}^s \Psi_{n'}(x) \approx E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x). \end{cases} \tag{4.9}$$

So, by using (3.2), (3.3), (3.5) and (4.9), we obtain

$$\begin{aligned} g(x)f(x) \approx& (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} B_{n'}(x))(A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x)) \\ =& (E_{n'}^T q_{n'\times n'}^s \phi_{n'\times n'} \otimes A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'}) B_{n'}(x), \end{aligned} \tag{4.10}$$

$$f^2(x) \approx (A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'} B_{n'}(x))^2 = (A_{n'}^T q_{n'\times n'}^r \phi_{n'\times n'})^2 B_{n'}(x), \tag{4.11}$$

$$\begin{aligned} \int_0^x f(t)dt \approx& \int_0^x A_{n'}^T q_{n'\times n'}^r \Psi_{n'}(t)dt = A_{n'}^T q_{n'\times n'}^r \int_0^x \Psi_{n'}(t)dt \\ \approx& A_{n'}^T q_{n'\times n'}^r q_{n'\times n'}^1 \Psi(x) \approx A_{n'}^T q_{n'\times n'}^{1+r} \phi_{n'\times n'} B_{n'}(x), \end{aligned} \tag{4.12}$$

$$\int_0^x g(t)dt \approx \int_0^x E_{n'}^T q_{n'\times n'}^s \Psi_{n'}(t)dt \approx E_{n'}^T q_{n'\times n'}^{1+s} \phi_{n'\times n'} B_{n'}(x). \tag{4.13}$$

By replacing (3.7) and (4.8)–(4.13) into (4.7), we obtain

$$
\begin{cases}
A^T_{n'}\phi_{n'\times n'}B_{n'}(x) = & \frac{1}{3}(E_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})B_{n'}(x) \\
& -E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) + [1,1,1,\ldots,1]_{1\times n'}B_{n'}(x) \\
& -E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) + 2A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x), \\
E^T_{n'}\phi_{n'\times n'} = & \frac{1}{3}(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})B_{n'}(x) \\
& +\frac{1}{2}(A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2B_{n'}(x) + 2A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) \\
& -E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x) - A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}B_{n'}(x).
\end{cases}
\tag{4.14}
$$

According to the properties of BPFs and (4.14) we have

$$
\begin{cases}
A^T_{n'}\phi_{n'\times n'} = & \frac{1}{3}(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}) - E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \\
& +[1,1,1,\ldots,1]_{1\times n'} - E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'} + 2A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}, \\
E^T_{n'}\phi_{n'\times n'} = & \frac{1}{3}(E^T_{n'}q^s_{n'\times n'}\phi_{n'\times n'} \otimes A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'}) + \frac{1}{2}(A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'})^2 \\
& +2A^T_{n'}q^r_{n'\times n'}\phi_{n'\times n'} - E^T_{n'}q^{1+s}_{n'\times n'}\phi_{n'\times n'} - A^T_{n'}q^{1+r}_{n'\times n'}\phi_{n'\times n'}.
\end{cases}
$$

Table 4. Numerical results of the Example 4.2 for $k = 2$, $M = 6$, $n' = 12$ and different values r and s.

$x_i = \frac{i-0.5}{n'}$	$r = 0.7$, $s = 0.7$ $f(x)$, $g(x)$	$r = 0.8$, $s = 0.8$ $f(x)$, $g(x)$	$r = 0.9$, $s = 0.9$ $f(x)$, $g(x)$	$r = 1$, $s = 1$ $f(x)$, $g(x)$	Ecact solution $f(x)$, $g(x)$
$x_1 = 0.04167$	$0.11157, 0.02570$	$0.08101, 0.01331$	$0.05826, 0.00681$	$0.04162, 0.00345$	$0.04167, 0.00174$
$x_2 = 0.12500$	$0.24734, 0.08992$	$0.19985, 0.05289$	$0.15887, 0.03048$	$0.12485, 0.01729$	$0.12500, 0.01563$
$x_3 = 0.20833$	$0.34747, 0.17865$	$0.29849, 0.11560$	$0.25105, 0.07284$	$0.20808, 0.04499$	$0.20833, 0.04340$
$x_4 = 0.29167$	$0.43049, 0.28039$	$0.38662, 0.19511$	$0.33858, 0.13148$	$0.29130, 0.08656$	$0.29167, 0.0851$
$x_5 = 0.37500$	$0.50203, 0.39010$	$0.46725, 0.28816$	$0.42265, 0.20501$	$0.37452, 0.14198$	$0.37500, 0.14063$
$x_6 = 0.45833$	$0.56500, 0.50429$	$0.54192, 0.39228$	$0.50386, 0.29232$	$0.45773, 0.21126$	$0.45833, 0.21007$
$x_7 = 0.54167$	$0.62123, 0.62019$	$0.61160, 0.50541$	$0.58259, 0.39247$	$0.54094, 0.29439$	$0.54167, 0.29340$
$x_8 = 0.62500$	$0.67207, 0.73553$	$0.67698, 0.62570$	$0.65908, 0.50455$	$0.62415, 0.39137$	$0.62500, 0.39063$
$x_9 = 0.70833$	$0.71856, 0.84840$	$0.73855, 0.75142$	$0.73351, 0.62770$	$0.70736, 0.50221$	$0.70833, 0.50174$
$x_{10} = 0.79167$	$0.76163, 0.95723$	$0.79675, 0.88091$	$0.80600, 0.76104$	$0.79057, 0.62689$	$0.79167, 0.62674$
$x_{11} = 0.87500$	$0.80212, 1.06071$	$0.85194, 1.01260$	$0.87667, 0.90368$	$0.87377, 0.76541$	$0.87500, 0.76563$
$x_{12} = 0.95833$	$0.84082, 1.15789$	$0.90445, 1.14498$	$0.94558, 1.05471$	$0.95697, 0.91778$	$0.95833, 0.91840$

TABLE 5. Numerical results of the Example 4.2 for $k = 6$, $M = 3$, $n' = 96$, for $i = 8, 16, 24, \ldots, 96$, and different values r and s.

$x_i = \frac{i-0.5}{n'}$	$r = 0.85$ $s = 0.85$ $f(x),\ g(x)$	$r = 0.90$ $s = 0.90$ $f(x),\ g(x)$	$r = 0.95$ $s = 0.95$ $f(x),\ g(x)$	$r = 1$ $s = 1$ $f(x),\ g(x)$	Exact solution $f(x),\ g(x)$
$x_8 = 0.07813$	$0.12072, 0.01712$	$0.10466, 0.01221$	$0.09051, 0.00867$	$0.07812, 0.00613$	$0.07813, 0.00610$
$x_{16} = 0.16146$	$0.22252, 0.05872$	$0.20055, , 0.45011$	$0.18013, 0.03434$	$0.16146, 0.02609$	$0.16146, 0.02607$
$x_{24} = 0.24479$	$0.31477, 0.11879$	$0.29056, 0.09507$	$0.26714, 0.07568$	$0.24479, 0.05995$	$0.24479, 0.05992$
$x_{32} = 0.32813$	$0.40070, 0.19461$	$0.37657, 0.16072$	$0.35221, 0.13191$	$0.32812, 0.10769$	$0.32813, 0.10767$
$x_{40} = 0.41146$	$0.48172, 0.28420$	$0.45943, 0.24076$	$0.43578, 0.20252$	$0.41145, 0.16932$	$0.41146, 0.16930$
$x_{48} = 0.49479$	$0.55863, 0.38597$	$0.53961, 0.33418$	$0.51805, 0.28702$	$0.49478, 0.24484$	$0.49479, 0.24482$
$x_{56} = 0.57813$	$0.63198, 0.49845$	$0.61741, 0.44006$	$0.59916, 0.38499$	$0.57811, 0.33424$	$0.57813, 0.33423$
$x_{64} = 0.66146$	$0.70212, 0.62030$	$0.69306, 0.55752$	$0.67920, 0.49602$	$0.66144, 0.43754$	$0.66146, 0.43753$
$x_{72} = 0.74479$	$0.76936, 0.75018$	$0.76670, 0.68568$	$0.78823, 0.61968$	$0.74478, 0.55472$	$0.74479, 0.55471$
$x_{80} = 0.82813$	$0.83390, 0.88680$	$0.83845, 0.82367$	$0.83628, 0.75556$	$0.82811, 0.68579$	$0.82813, 0.68579$
$x_{88} = 0.91146$	$0.89595, 1.02884$	$0.90839, 0.97059$	$0.91339, 0.90321$	$0.91144, 0.83075$	$0.91146, 0.83076$
$x_{96} = 0.99479$	$0.95568, 1.17499$	$0.97659, 1.12551$	$0.98958, 1.06215$	$0.99477, 0.98960$	$0.99479, 0.98961$

TABLE 6. Absolute error relevant to Tables 4 and 5 when $r = s = 1$

x_i	e_f	e_g	x_i	e_f	e_g
x_1	$4.9582e-05$	$1.7179e-03$	x_8	$1.4528e-06$	$2.6604e-05$
x_2	$1.5141e-04$	$1.6686e-03$	x_{16}	$3.0835e-06$	$2.5657e-05$
x_3	$2.5823e-04$	$1.5927e-03$	x_{24}	$4.7883e-06$	$2.4283e-05$
x_4	$3.6945e-04$	$1.4885e-03$	x_{32}	$6.5581e-06$	$2.2451e-05$
x_5	$4.8451e-04$	$1.3539e-03$	x_{40}	$8.3847e-06$	$2.0132e-05$
x_6	$6.0291e-04$	$1.1872e-03$	x_{48}	$1.0261e-05$	$1.7294e-05$
x_7	$7.2421e-04$	$9.8598e-04$	x_{56}	$1.2179e-05$	$1.3901e-05$
x_8	$8.4799e-04$	$7.4813e-04$	x_{64}	$1.4134e-05$	$9.9173e-06$
x_9	$9.7387e-04$	$4.7122e-04$	x_{72}	$1.6120e-05$	$5.3045e-06$
x_{10}	$1.1015e-03$	$1.5269e-04$	x_{80}	$1.8131e-05$	$2.1047e-08$
x_{11}	$1.2305e-03$	$2.1017e-04$	x_{88}	$2.0162e-05$	$5.9769e-06$
x_{12}	$1.3606e-03$	$6.2023e-04$	x_{96}	$2.2208e-05$	$1.2736e-05$

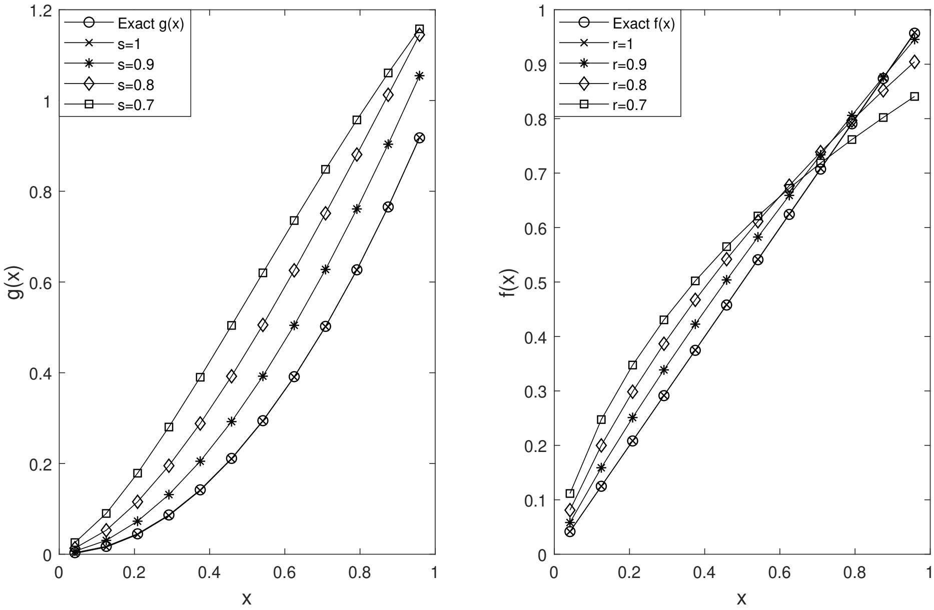

FIGURE 6. Numerical solution for different values of r and s, when $k = 2$, $M = 6$ and $n' = 12$.

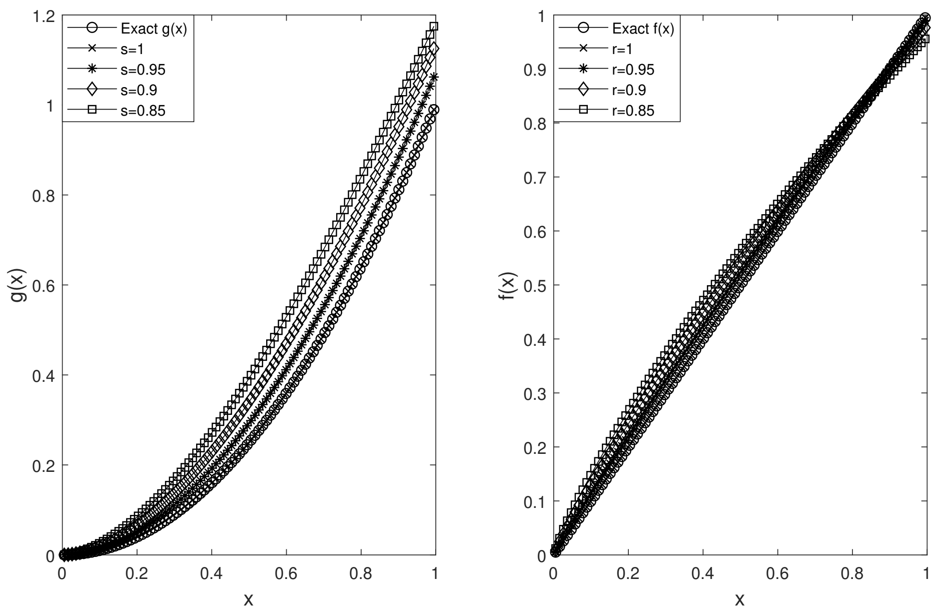

FIGURE 7. Numerical solution for different values of r and s when $k = 6$, $M = 3$ and $n' = 96$.

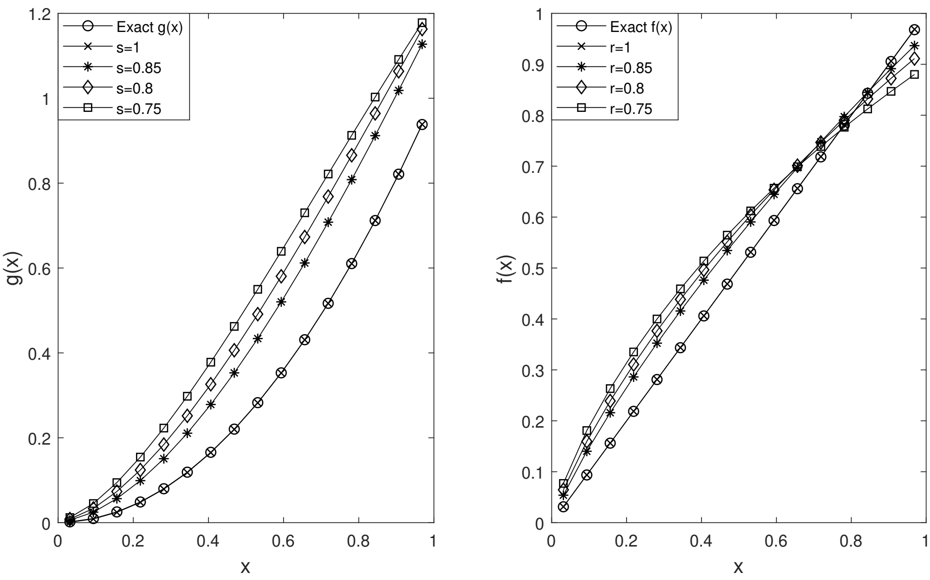

FIGURE 8. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

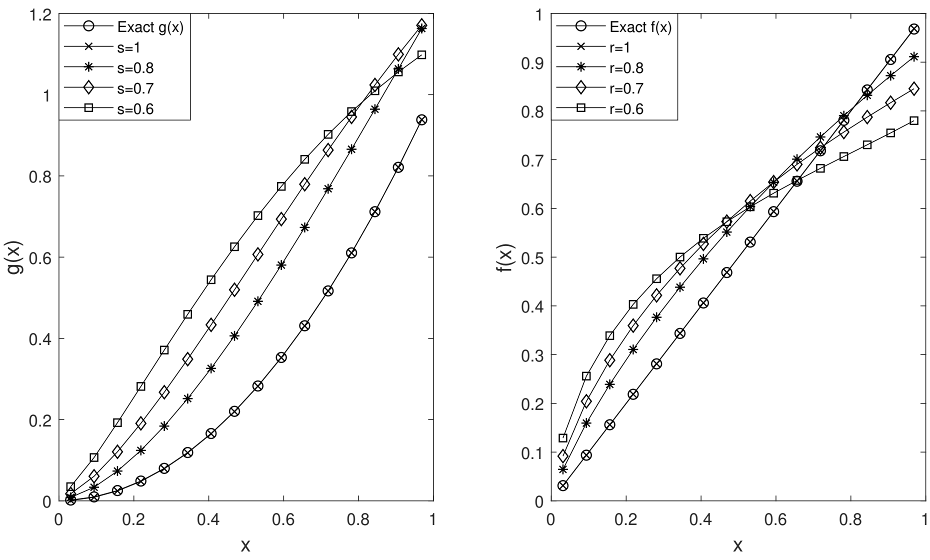

FIGURE 9. Numerical solution for different values of r and s when $k = 3$, $M = 4$ and $n' = 16$.

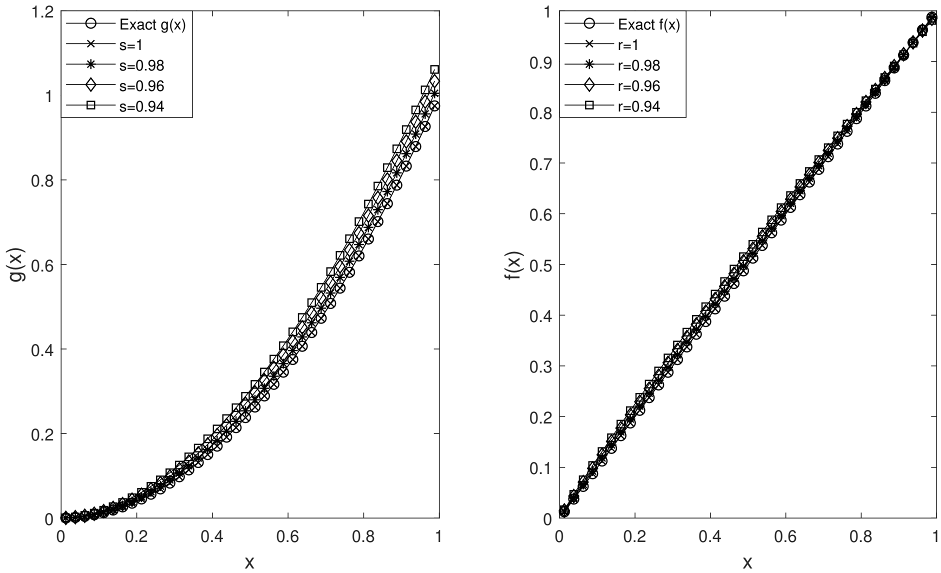

Figure 10. Numerical solution for different values of r and s when $k = 4$, $M = 5$ and $n' = 40$.

5. Conclusion

The main purpose of the presented article is introducing Legendre wavelets method for resolving coupled systems of FIDEs. As you saw, the numerical results obtained here, confirm its high accuracy degree.

The most noticeable profit of the mentioned method is converting complicated equations to simple ones, like we performed on examples. One of the best benefits of this procedure is having high exactness that you may have been recognized it according to the tables and figures.

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THE MINIMUM EDGE COVERING ENERGY OF A GRAPH

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Abstract. In this paper, we introduce a new kind of graph energy, the minimum edge covering energy, $E_{C_E}(G)$. It depends both on the underlying graph G, and on its particular minimum edge covering C_E. Upper and lower bounds for $E_{C_E}(G)$ are established. The minimum edge covering energy and some of the coefficients of the polynomial of well-known families of graphs like Star, Path and Cycle Graphs are computed.

1. Introduction

In the study of spectral graph theory, we use the spectra of the certain matrix associated with the graph, such as the adjacency matrix, the Laplacian matrix, and other related matrices. Some useful information about the graph can be obtained from the spectra of these various matrices. Let G be a simple graph and let its vertex set be $V(G) = \{v_1, v_2, \dots, v_n\}$. The adjacency matrix $A(G)$ of the graph G is a square matrix of order n whose (i, j)-entry is equal to unity if the vertices v_i and v_j are adjacent and it is equal to zero otherwise. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of $A(G)$, assumed in nonincreasing order, are the eigenvalues of the graph G. The concept of energy of a graph was introduced by I. Gutman [1] in the year 1978 as the sum of the absolute values of its eigenvalues:

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

One of the remarkable chemical applications of spectral graph theory is based on the close correspondence between the graph eigenvalues and the molecular orbital energy

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levels of π-electron in conjugated hydrocarbons. An interesting quantity in Huckel theory is the sum of the energies of all the electrons in a molecule, so-called total π-electron energy E. For more details on the mathematical aspects of the theory of graph energy see [2–5]. The basic properties including various upper and lower bounds for energy of a graph have been established [6,7], and it has found remarkable chemical applications in the molecular orbital theory of conjugated molecules [8]. Recently C. Adiga et al. [8] defined the minimum covering energy, $E_C(G)$ of a graph which depends on its particular minimum cover C. Further, incidence energy, matching energy, minimum dominating energy, Laplacian minimum dominating energy and minimum dominating distance energy of a graph G can be found in [9–16]. Motivated by these papers, we study the minimum edge covering E_{C_E} of a graph G. We compute some properties of the characteristic polynomial of a minimum edge covering matrix of a graph G. Upper and lower bounds for $E_{C_E}(G)$ are established. It is possible that the minimum edge covering energy that we are considering in this paper may have some applications in chemistry as well as in other areas. Let $G = (V, E)$ be a simple finite graph that is has no loops no multiple and directed edges. Graph G has n vertices, m edges with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$.

Definition 1.1. A subset C of V is called a covering set of G if every edge of G is incident to at least one vertex of C. Any covering set with minimum cardinality is called a minimum covering set.

Definition 1.2. A subset of E is called an edge covering set of G if every vertex of G is incident to at least one edge of it. Any edge covering set with minimum cardinality is called a minimum edge covering set. Let C_E be a minimum edge covering set of a graph G. The minimum edge covering matrix of G is a $m \times m$ matrix $A_{C_E}(G) = (e_{ij})_{m\times m}$, where

$$e_{ij} = \begin{cases} 1, & \text{if } e_i, e_j \text{ are adjacent}, \\ 1, & \text{if } i = j \text{ and } e_i \in C_E, \\ 0, & \text{otherwise}. \end{cases}$$

The characteristic polynomial of $A_{C_E}(G)$ is denoted by $f_m(G, \lambda) = \det(\lambda I - A_{C_E}(G)) = c_0\lambda^m + c_1\lambda^{m-1} + c_2\lambda^{m-2} + \cdots + c_m$. The matrix $A_{C_E}(G)$ is real and symmetric. Then the eigenvalues of $A_{C_E}(G)$ are real numbers and are labeled in non-increasing order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m$. The minimum edge covering energy of G is defined as

$$E_{C_E}(G) = \sum_{i=1}^{m} |\lambda_i|.$$

2. Problem Formulation and some Basic Properties of Minimum Edge Covering Energy

Following theorem obtains the coefficients of polynomial without applying determinant expansion.

Theorem 2.1. *Let G be a graph with vertex set V, edge set E, and the minimum edge covering set C_E. Let $f_m(G,\lambda) = \det(\lambda I - A_{C_E}(G)) = c_0\lambda^m + c_1\lambda^{m-1} + c_2\lambda^{m-2} + \cdots + c_m$ be the characteristic polynomial of G. Then*

(i) $c_0 = 1$;

(ii) $c_1 = -|C_E|$;

(iii) $c_2 = \binom{|C_E|}{2} - \left(\frac{1}{2}\sum_{i=1}^{m}\deg(e_i)\right)$;

(iv) $c_3 = -\binom{|C_E|}{3} + |C_E|\left(\frac{1}{2}\sum_{i=1}^{m}\deg(e_i)\right) - \sum_{i\in C_E}\deg(e_i) - 2(K_{1,3} + C_3)$, *where C_3 is the number of triangles and $K_{1,3}$ is the number of star graphs with four vertices in G.*

Proof. (i) Directly from the definition of $f_m(G,\lambda)$, it follows that $c_0 = 1$.

(ii) Since the sum of diagonal elements of $A_{C_E}(G)$ is equal to $|C_E|$, sum of determinants of all 1×1 principal submatrices of $A_{C_E}(G)$ is the trace of $A_{C_E}(G)$ which evidently equal to $|C_E|$. Thus, $(-1)^1 c_1 = |C_E|$.

(iii) $(-1)^2 c_2$ is equal to sum of determinats of all the 2×2 principal submatrices of $A_{C_E}(G)$, that is

$$\begin{aligned} c_2 &= \sum_{1\le i<j\le m}\begin{vmatrix} e_{ii} & e_{ij} \\ e_{ji} & e_{jj}\end{vmatrix} = \sum_{1\le i<j\le m}(e_{ii}e_{jj} - e_{ij}e_{ji}) \\ &= \sum_{1\le i<j\le m} e_{ii}e_{jj} - \sum_{1\le i<j\le m} e_{ij}^2 = \binom{|C_E|}{2} - \frac{1}{2}\sum_{i=1}^{m}\deg(e_i). \end{aligned}$$

(iv)

$$\begin{aligned} c_3 =& (-1)^3\sum_{1\le i<j<k\le m}\begin{vmatrix} e_{ii} & e_{ij} & e_{ik} \\ e_{ji} & e_{jj} & e_{jk} \\ e_{ki} & e_{kj} & e_{kk}\end{vmatrix} \\ =& -\sum_{1\le i<j<k\le m}[e_{ii}(e_{jj}e_{kk} - e_{kj}e_{jk}) - e_{ij}(e_{ji}e_{kk} - e_{ki}e_{jk}) \\ &+ e_{ik}(e_{ji}e_{kj} - e_{ki}e_{jj})] \\ =& -\sum_{1\le i<j<k\le m} e_{ii}e_{jj}e_{kk} + \sum_{1\le i<j<k\le m}[e_{ii}e_{jk} + e_{jj}e_{ik} + e_{kk}e_{ij}] \\ &- \sum_{1\le i<j<k\le m} e_{ij}e_{jk}e_{ki} - \sum_{1\le i<j<k\le m} e_{ik}e_{kj}e_{ji} \\ =& \binom{|C_E|}{3} + |C_E|\left(\frac{1}{2}\sum_{i=1}^{m}\deg(e_i)\right) - \sum_{i\in C_E}\deg(e_i) - 2(k_{1,3} + C_3). \end{aligned}$$

Thus,

$$c_3 = \binom{|C_E|}{3} + |C_E|\left(\frac{1}{2}\sum_{i=1}^{m}\deg(e_i)\right) - \sum_{i\in C_E}\deg(e_i) - 2(k_{1,3} + C_3). \qquad \square$$

Corollary 2.1. *Let G be a path P_n with n vertices and m edges, then*

(i) $c_0 = 1$;

(ii) $c_1 = -\left\lceil \frac{m+1}{2} \right\rceil$, $\lceil x \rceil$ *is the smallest integer number greater than or equals to x;*

(iii) $c_2 = \binom{\left\lceil \frac{m+1}{2} \right\rceil}{2} - (m-1)$;

(iv) $c_3 = \binom{\left\lceil \frac{m+1}{2} \right\rceil}{3} + \left\lceil \frac{m+1}{2} \right\rceil (m-1) - \sum_{i \in C_E} \deg(e_i)$.

Corollary 2.2. *Let G be a cycle C_n with n vertices and m edges, then*

(i) $c_0 = 1$;

(ii) $c_1 = -\left\lceil \frac{m}{2} \right\rceil$;

(iii) $c_2 = \binom{\left\lceil \frac{m}{2} \right\rceil}{2} - m$;

(iv) $c_3 = -\binom{\left\lceil \frac{m}{2} \right\rceil}{3} + \left\lceil \frac{m}{2} \right\rceil (m-2)$, $m \geq 4$.

Remark 2.1. Let G be a path P_n with n vertices and m edges, then

$$\sum_{i=1}^{m} \deg(e_i) = 2(m-1).$$

Theorem 2.2. *If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are eigenvalues of $A_{C_E}(G)$,then*

$$\sum_{i=1}^{m} \lambda_i^2 = |C_E| + \sum_{i=1}^{m} \deg(e_i).$$

Proof. The sum of squares of the eigenvalues of $A_{C_E}(G)$ is just the trace of $(A_{C_E}(G))^2$ Therefore,

$$\sum_{i=1}^{m} \lambda_i^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} e_{ij}e_{ji} = 2\sum_{i<j}(e_{ij})^2 + \sum_{i=1}^{m}(e_{ii})^2 = |C_E| + \sum_{i=1}^{m} \deg(e_i). \qquad \square$$

Theorem 2.3 (Parity theorem)**.** *Let G be a graph with a minimum edge covering set C_E. If the minimum edge covering energy $E_{C_E}(G)$ of G is a rational number, then*

$$E_{C_E} \equiv |C_E| \pmod 2. \tag{2.1}$$

Proof. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be positive, and the rest of the minimum edge covering eigenvalues non-positive. Thus,

$$E_{C_E}(G) = \sum_{i=1}^{m} |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \cdots + \lambda_m),$$

implies $E_{C_E}(G) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - |C_E|$. Since $\lambda_1, \lambda_2, \ldots, \lambda_r$ are algebraic integers, so is their sum. Hence, $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ must be an integer if $E_{C_E}(G)$ is rational. $\square$

3. The Minimum Edge Covering Energy of Some Graphs

Theorem 3.1. *The minimum edge covering energy of a star graph $K_{1,n-1}$ is m for $n \geq 2$. The polynomial of it is $\lambda^m - m\lambda^{m-1}$.*

Proof. Let $K_{1,n-1}$ be a star graph with vertex set $V = \{v_0, v_1, v_2, \dots, v_{n-1}\}$ and center v_0, then

$$A_{c_E}(G) = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix},$$

that $A_{c_E}(G)$ is $m \times m$. Then its characteristic polynomial is

$$f_m(k_{1,n-1}, \lambda) = \det \begin{pmatrix} \lambda - 1 & -1 & \cdots & -1 \\ -1 & \lambda - 1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & \lambda - 1 \end{pmatrix}. \tag{3.1}$$

By computing determinant of upper triangular of matrix (3.1) we will have

$$f_m(k_{1,n-1}, \lambda) = \lambda^m - m\lambda^{m-1}.$$

Then

$$\mathrm{Spec}(k_{1,n-1}) = \begin{pmatrix} m & 0 \\ 1 & m-1 \end{pmatrix},$$

$$E_{C_E}(G) = \sum_{i=1}^{m} |\lambda_i| = m. \qquad \square$$

Theorem 3.2. *Let G be a path or a cycle graph with a minimum edge covering set C_E. Then $E_{C_E}(G) \simeq 2m - |C_E|$ or $|E_{C_E}(G) - (2m - |C_E|)| \leq 1$.*

Proof. Let $\lambda_1, \lambda_2, \dots, \lambda_r$ be positive eigenvalues, and the rest of the minimum edge covering eigenvalues non-positive. Thus,

$$E_{C_E}(G) = \sum_{i=1}^{m} |\lambda_i| = (\lambda_1 + \lambda_2 + \cdots + \lambda_r) - (\lambda_{r+1} + \lambda_{r+2} + \cdots + \lambda_m),$$

implies $E_{C_E}(G) = 2(\lambda_1 + \lambda_2 + \cdots + \lambda_r) - |C_E|$. Since $\lambda_1, \lambda_2, \dots, \lambda_r$ are algebraic integers, so sum $(\lambda_1 + \lambda_2 + \cdots + \lambda_r) \simeq m$. Hence, $E_{C_E}(G) \simeq 2m - |C_E|$. $\square$

Theorem 3.3 (Upper bound)**.** *Let G be a graph with n vertices, m edges, and let C_E be a minimum edge covering set of G. Then*

$$E_{C_E}(G) \leq \sqrt{m\Big(\sum_{i=1}^{m} \deg(e_i) + |C_E|\Big)}.$$

Proof. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \cdots \geq \lambda_m$ be the eigenvalues of $A_{C_E}(G)$. Bearing in mind the Cauchy-Schwarz inequality

$$\left(\sum_{i=1}^{m} a_i b_j\right)^2 \leq \left(\sum_{i=1}^{m} a_i^2\right)\left(\sum_{i=1}^{m} b_i^2\right),$$

we put $a_i = 1$ and $b_i = |\lambda_i|$ and we have

$$[E_{C_E}(G)]^2 = \left(\sum_{i=1}^{m} |\lambda_i|\right)^2 \leq m\left(\sum_{i=1}^{m} |\lambda_i|^2\right) = m\sum_{i=1}^{m} \lambda_i^2 = m\left(\sum_{i=1}^{m} \deg(e_i) + |C_E|\right). \square$$

Theorem 3.4 (Lower bound)**.** *Let G be a graph with n vertices and m edges, and let C_E be a minimum edge covering set of G. If $D = |\det A_{C_E}(G)|$, then*

$$[E_{C_E}(G)]^2 \geq \sum_{i=1}^{m} \deg(e_i) + |C_E| + m(m-1)\sqrt[m]{\left(\prod_{i=1}^{m} \lambda_i\right)^2}, \tag{3.2}$$

with equality if G is a star of order n.

Proof. We have

$$[E_{C_E}(G)]^2 = \left(\sum_{i=1}^{m} |\lambda_i|\right)^2 = \left(\sum_{i=1}^{m} |\lambda_i|\right)\left(\sum_{i=1}^{m} |\lambda_i|\right) = \sum_{i=1}^{m} |\lambda_i|^2 + \sum_{i \neq j} |\lambda_i||\lambda_j|.$$

Now, by inequality between the arithmetic mean and geometric mean, we have

$$\frac{1}{m(m-1)} \sum_{i \neq j} |\lambda_i||\lambda_j| \geq \left(\prod_{i \neq j} |\lambda_i||\lambda_j|\right)^{\frac{1}{m(m-1)}}.$$

Thus,

$$\begin{aligned}
[E_{C_E}(G)]^2 &\geq \sum_{i=1}^{m} |\lambda_i|^2 + m(m-1)\Big(\prod_{i \neq j} |\lambda_i||\lambda_j|\Big)^{\frac{1}{m(m-1)}} \\
&\geq \sum_{i=1}^{m} |\lambda_i|^2 + m(m-1)\Big(\prod_{i=1}^{m} |\lambda_i|^{2(m-1)}\Big)^{\frac{1}{m(m-1)}} \\
&= \sum_{i=1}^{m} |\lambda_i|^2 + m(m-1)\Big|\prod_{i=1}^{m} \lambda_i\Big|^{\frac{2}{m}} \\
&= \sum_{i=1}^{m} \deg(e_i) + |C_E| + m(m-1)D^{\frac{2}{m}}.
\end{aligned}$$

Since in the star graphs the multiplicity of $\lambda = 0$ is $m-1$, so $\Pi_{i=1}^{m}\lambda_i = \det A_{C_E}(G) = 0$, $|C_E| = m$ and $\sum_{i=1}^{m} \deg(e_i) = m(m-1)$. Then, by placing the above values in (3.2), equality cases hold

$$m^2 = [E_{C_E}(K_{1,n-1})]^2 \geq m(m-1) + m + m(m-1)\sqrt[m]{\left(\prod_{i=1}^{m} \lambda_i\right)^2} = m^2. \qquad \square$$

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PSEUDO COMMUTATIVE DOUBLE BASIC ALGEBRAS

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Abstract. In this paper, we study the concept of pseudo commutative double basic algebras and investigate some related results. We prove that there are relations among pseudo commutative double basic algebras and other logical algebras such as pseudo hoops, pseudo BCK-algebras and double MV-algebras. We obtain a close relation between pseudo commutative double basic algebras and pseudo residuted l-groupoids. Then we investigate the properties of the boolean center of pseudo commutative double basic algebras and we use the boolean elements to define and study algebras on subintervals of pseudo commutative double basic algebras.

1. Introduction

C. C. Chang introduced the concept of MV-algebra in 1958 [16] to prove the completeness theorem of infinite valued Łukasiewicz propositional calculus. Basic algebras were introduced by I. Chajda et al. as a generalization of MV-algebras and orthomodular lattices, see [8, 10] and [11]. They are very useful in non-classical logics including the logic of quantum mechanics. Every basic algebra is in fact a bounded lattice with the so-called section antitone involutions.

I. Chajda introduced the concept of double basic algebra in 2009 [12] as a generalization of basic algebra. Double basic algebras are determined by two binary, two unary and a nullary operation satisfying similar axioms to those defining basic algebras. Hence the class of these algebras forms a variety. He proved that there exists a one-to-one correspondence between the variety of double basic algebras and the class of lattices with section antitone bijections. Also, he obtained the relation between double MV-algebras and double basic algebras.

Key words and phrases. Pseudo commutative double basic algebra, double MV-algebra, pseudo residuted l-groupoid, boolean element.

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The concept of interval algebra was introduced in [9] for MV-algebras. Chajda and Kuhr extended this concept also for double basic algebras in [12]. In the case of a double basic algebra, they endowed the subinterval $[a, 1]$ with the structure of double basic algebra for all elements a in a double basic algebra.

In this paper, we will obtain some properties of (pseudo commutative) double basic algebras. We find under what conditions pseudo commutative double basic algebras are double MV-algebras. We will study the relation between pseudo commutative double basic algebras and pseudo residuated l-gropoids. Finally, we will obtain some properties of boolean elements of a pseudo commutative double basic algebra and we will prove that a boolean center is subalgebra of a pseudo commutative double basic algebra. We will prove that, if a, b are boolean elements of a pseudo commutative double basic algebra and $a \leq b$, then the subintervals of the form $[0, a]$ and $[a, b]$ can also be endowed with a double basic algebra structure.

2. Preliminaries

In this section, we recall some definitions and theorems which will be needed in this paper.

Definition 2.1 ([8,10])**.** A basic algebra is an algebra $(A, \oplus, \neg, 0, 1)$ of type $(2, 1, 0, 0)$ satisfying the following identities:

(BA1) $x \oplus 0 = x$;
(BA2) $\neg\neg x = x$;
(BA3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$;
(BA4) $\neg(\neg(\neg(x \oplus y) \oplus y) \oplus z) \oplus (x \oplus z) = 1$.

Orthomodular lattices and MV-algebras are examples of basic algebras. It is obvious that a basic algebra is an MV-algebra if and only if the operation $\oplus$ is associative and commutative. It was recently shown that a basic algebra is an MV-algebra if and only if it is associative (see [15]).

Definition 2.2 ([12])**.** A double basic algebra is an algebra $A = (A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ satisfying the following identities:

(P1) $x \oplus 0 = x$, $x \boxplus 0 = x$;
(P2) $(x^-)^\sim = x$, $(x^\sim)^- = x$;
(P3) $(x^\sim \oplus y)^- \boxplus y = (y^\sim \oplus x)^- \boxplus x = (x^- \boxplus y)^\sim \oplus y = (y^- \boxplus x)^\sim \oplus x$;
(P4) $(((x \boxplus y)^\sim \oplus y)^- \boxplus z)^\sim \oplus (x \boxplus z) = 1$, $(((x \oplus y)^- \boxplus y)^\sim \oplus z)^- \boxplus (x \oplus z) = 1$.

The connections between basic and double basic algebras are obvious. Suppose that $(A, \oplus, \neg, 0, 1)$ is a basic algebra. Then $(A, \oplus, \oplus, \neg, \neg, 0, 1)$ is a double basic algebra. Conversely, if we are given a double basic algebra in which $\oplus$ coincides with $\boxplus$, then the negations $^-$ and $^\sim$ coincide too, hence the double basic algebra becomes a basic algebra (see [13]).

Proposition 2.1 ([13])**.** *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a double basic algebra. Then the reduct $(A, \boxplus, ^-, 0, 1)$ is a basic algebra if and only if A satisfies the identity $x \oplus y = x \boxplus y$.*

We call a double basic algebra $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ pseudo-commutative if it satisfies the identity $x \oplus y = y \boxplus x$ for all $x, y \in A$.

Given a double basic algebra $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$, define a binary relation $\leq$ on A by $x \leq y$ if and only if $x^\sim \oplus y = 1$ if and only if $x^- \boxplus y = 1$. Then $(A, \leq)$ is a lattice and the following identities hold:

$$x \vee y = (x^\sim \oplus y)^- \boxplus y \quad \text{and} \quad x \wedge y = (x^- \vee y^-)^\sim = (x^\sim \vee y^\sim)^-.$$

If a double basic algebra is pseudo-commutative, then the induced lattice is distributive (see [12]).

Proposition 2.2 ([12, 13]). *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a double basic algebra. Then the following hold for all $x, y, z \in A$:*

(1) $0 \oplus x = x$, $0 \boxplus x = x$;
(2) $1 \oplus x = 1 = x \oplus 1$, $1 \boxplus x = 1 = x \boxplus 1$;
(3) $x^\sim \oplus x = 1$, $x^- \boxplus x = 1$;
(4) $x \leq y$ *if and only if* $y^- \leq x^-$ *if and only if* $y^\sim \leq x^\sim$;
(5) $x \leq y$ *implies* $x \oplus z \leq y \oplus z$ *and* $x \boxplus z \leq y \boxplus z$;
(6) $y \leq x \oplus y$ *and* $y \leq x \boxplus y$;
(7) *if* $x \oplus y = z$ *and* $y \leq x^-$, *then* $x = (z^- \boxplus y)^\sim$;
(8) *if* $x \boxplus y = z$ *and* $y \leq x^\sim$, *then* $x = (z^\sim \oplus y)^-$;
(9) $(x \wedge y^\sim) \oplus y = x \oplus y$, $(x \wedge y^-) \boxplus y = x \boxplus y$;
(10) *if* x^-, y *are comparable, then* $x \oplus y = x \oplus (y \wedge x^-)$;
(11) *if* $x^\sim$, y *are comparable, then* $x \boxplus y = x \boxplus (y \wedge x^\sim)$;
(12) $(x \wedge y) \oplus z = (x \oplus z) \wedge (y \oplus z)$, $(x \wedge y) \boxplus z = (x \boxplus z) \wedge (y \boxplus z)$;
(13) $1^\sim = 0 = 1^-$, $0^\sim = 1 = 0^-$.

Let $(L, \wedge, \vee, (f_a)_{a \in L}, 0, 1)$ be a bounded lattice equipped with a set $\{f_a, a \in L\}$ of partial mappings such that every f_a is defined just on the section $[a, 1]$ and, moreover

(i) every f_a is a bijection of $[a, 1]$ onto $[a, 1]$;
(ii) both f_a and f_a^{-1} are antitone, i.e., for $x, y \in [a, 1]$ with $x \leq y$, we have $f_a(y) \leq f_a(x)$ and $f_a^{-1}(y) \leq f_a^{-1}(x)$.

It was proved in [12] that to any bounded lattice with section antitone bijections can be assigned a double basic algebra. Also conversely, every double basic algebra induces a bounded lattice equipped with antitone bijections in every section.

Definition 2.3 ([12]). A double MV-algebra is an algebra $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ of type $(2, 2, 1, 1, 0, 0)$ satisfying the following identities for all $x, y, z \in A$:

(D1) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ and $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$;
(D2) $x \oplus y = y \boxplus x$;
(D3) $x \oplus 0 = x$ and $x \boxplus 0 = x$;
(D4) $x^{-\sim} = x$ and $x^{\sim-} = x$;
(D5) $x \oplus 1 = 1$ and $x \boxplus 1 = 1$;
(D6) $(x^\sim \oplus y)^- \boxplus y = (y^\sim \oplus x)^- \boxplus x = (x^- \boxplus y)^\sim \oplus y = (y^- \boxplus x)^\sim \oplus x$.

Proposition 2.3 ([12]). *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a double basic algebra. The following conditions are equivalent:*
(a) *A is a double MV-algebra;*
(b) *A is pseudo-commutative and associative both in $\oplus$ and $\boxplus$;*
(c) *A satisfies the Exchange identity $(x \oplus (y \boxplus z) = y \boxplus (x \oplus z))$.*

Pseudo hoops were introduced by Bosbach in [2] and [3] under the name of complementary semigroups. It was proved that a bounded pseudo-hoop is a meet semilattice ordered residuated, integral and divisible monoid.

Definition 2.4 ([2]). A pseudo-hoop is an algebra $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ of type $(2, 2, 2, 0)$ such that, for all $x, y, z \in A$:
(psHOOP1) $x \odot 1 = 1 \odot x = x$;
(psHOOP2) $x \rightarrow x = x \rightsquigarrow x = 1$;
(psHOOP3) $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
(psHOOP4) $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
(psHOOP5) $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

Pseudo-BCK algebras were introduced by G. Georgescu and A. Iorgulescu ([19]) as non-commutative generalizations of BCK-algebras.

Definition 2.5 ([19]). A pseudo-BCK algebra (more precisely, reversed left-pseudo-BCK algebra) is a structure $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ where $\leq$ is a binary relation on A, $\rightarrow$ and $\rightsquigarrow$ are binary operations on A and 1 is an element of A satisfying, for all $x, y, z \in A$, the axioms:
(psBCK1) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$;
(psBCK2) $x \leq (x \rightarrow y) \rightsquigarrow y$, $x \leq (x \rightsquigarrow y) \rightarrow y$;
(psBCK3) $x \leq x$;
(psBCK4) $x \leq 1$;
(psBCK5) if $x \leq y$ and $y \leq x$, then $x = y$; (psBCK6) $x \leq y$ if and only if $x \rightarrow y = 1$ if and only if $x \rightsquigarrow y = 1$.

3. Some Properties of Double Basic Algebras

In any double basic algebra $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$, we define four new derived binary operations as follows:

$$x \odot y = (x^\sim \oplus y^\sim)^-, \quad x \boxdot y = (x^- \boxplus y^-)^\sim,$$
$$x \rightsquigarrow y = x^\sim \oplus y, \quad x \rightarrow y = x^- \boxplus y,$$

for all $x, y \in A$. It can be easily shown that
(C1) $x^- = x \rightarrow 0$, $x^\sim = x \rightsquigarrow 0$;
(C2) $x \oplus y = x^- \rightsquigarrow y$, $x \boxplus y = x^\sim \rightarrow y$;
(C3) $x \oplus y = (x^- \odot y^-)^\sim$, $x \boxplus y = (x^\sim \boxdot y^\sim)^-$;
(C4) $x \vee y = (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x = (x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$;
(C5) $x \vee y = (x \odot y^-) \boxplus y = (y \odot x^-) \boxplus x = (x \boxdot y^\sim) \oplus y = (y \boxdot x^\sim) \oplus x$;

(C6) $x \wedge y = (x \oplus y^-) \boxdot y = (y \oplus x^-) \boxdot x = (x \boxplus y^\sim) \odot y = (y \boxplus x^\sim) \odot x$;
(C7) $(((x \rightsquigarrow y) \to y) \rightsquigarrow z) \to (x \rightsquigarrow z) = (((x \to y) \rightsquigarrow y) \to z) \rightsquigarrow (x \to z) = 1$.

Proposition 3.1. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a double basic algebra. Then for all $x, y, z \in A$, the following statements hold:*

(1) $x \leq y$ *if and only if* $x \to y = 1$ *if and only if* $x \rightsquigarrow y = 1$*;*
(2) $x \to 1 = x \to x = x \rightsquigarrow x = x \rightsquigarrow 1 = 1$*;*
(3) $1 \to x = 1 \rightsquigarrow x = x$*;*
(4) $y \leq x \to y$ *and* $y \leq x \rightsquigarrow y$*;*
(5) $x \odot 1 = 1 \odot x = x$, $x \boxdot 1 = 1 \boxdot x = x$*;*
(6) $x \odot y \leq y$, $x \boxdot y \leq y$*;*
(7) *if* $x \leq y$*, then* $x \odot z \leq y \odot z$ *and* $x \boxdot z \leq y \boxdot z$*;*
(8) $x \leq y$ *implies that* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$*;*
(9) $x \leq y \to z$ *if and only if* $y \leq x \rightsquigarrow z$*;*
(10) $((x \rightsquigarrow y) \to y) \rightsquigarrow y = x \rightsquigarrow y$, $((x \to y) \rightsquigarrow y) \to y = x \to y$*;*
(11) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$, $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$.

Proof. The proof of parts (1)-(7) is straightforward by Proposition 2.2 and the definition of the operators.

(8) Suppose that $x \leq y$. Then $y^\sim \leq x^\sim$ by Proposition 2.2 part (4). Applying Proposition 2.2 part (5), we get $y^\sim \oplus z \leq x^\sim \oplus z$. By (P2) and (C2), we obtain $y \rightsquigarrow z \leq x \rightsquigarrow z$. Similarly, we can prove $y \to z \leq x \to z$.

(9) Suppose that $x \leq y \to z$. Using part (8), we have $(y \to z) \rightsquigarrow z \leq x \rightsquigarrow z$. By (C4), we have $y \leq y \vee z = (y \to z) \rightsquigarrow z$. Hence, $y \leq x \rightsquigarrow z$. Similarly, we can show that $y \leq x \rightsquigarrow z$ implies $x \leq y \to z$.

(10) Using (C4), we have $x \leq (x \rightsquigarrow y) \to y$ and $x \leq (x \to y) \rightsquigarrow y$. Applying part (8), we obtain $((x \rightsquigarrow y) \to y) \rightsquigarrow y \leq x \rightsquigarrow y$ and $((x \to y) \rightsquigarrow y) \to y \leq x \to y$. On the other hand, by (C4), we have $x \rightsquigarrow y \leq ((x \rightsquigarrow y) \to y) \rightsquigarrow y$ and $x \to y \leq ((x \to y) \rightsquigarrow y) \to y$.

(11) We have that $x, y \leq x \vee y$. By part (8), we get $(x \vee y) \to z \leq x \to z$ and $(x \vee y) \to z \leq y \to z$. Thus $(x \vee y) \to z$ is a lower bound of $\{x \to z, y \to z\}$. Let u be an arbitrary lower bound of $\{x \to z, y \to z\}$. Applying part (9), we get $x \leq u \rightsquigarrow z$ and $y \leq u \rightsquigarrow z$. Thus $x \vee y \leq u \rightsquigarrow z$. Using part (9), we have $u \leq (x \vee y) \to z$. Hence, $(x \vee y) \to z = (x \to z) \wedge (y \to z)$. Similarly, $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$. $\square$

Proposition 3.2. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra. Then the following properties hold, for every $x, y, z \in A$:*

(1) $x \vee y \leq x \oplus y$, $x \vee y \leq x \boxplus y$*;*
(2) *if* $x \leq y$*, then* $z \to x \leq z \to y$, $z \rightsquigarrow x \leq z \rightsquigarrow y$*;*
(3) $x \to y = y^- \rightsquigarrow x^-$, $x \rightsquigarrow y = y^\sim \to x^\sim$*;*
(4) $x \to y^\sim = y \rightsquigarrow x^-$*;*
(5) *if* $x \leq y$*, then* $z \odot x \leq z \odot y$, $z \boxdot x \leq z \boxdot y$*;*
(6) $(x \to y) \odot y^- = x^- \wedge y^-$, $(x \rightsquigarrow y) \boxdot y^\sim = x^\sim \wedge y^\sim$*;*

(7) $(x \vee y) \odot z = (x \odot z) \vee (y \odot z)$, $(x \vee y) \boxdot z = (x \boxdot z) \vee (y \boxdot z)$;
(8) $(x \to y) \boxdot x = x \wedge y = (x \rightsquigarrow y) \odot x$;
(9) $x \le y \to z$ *if and only if* $x \boxdot y \le z$;
(10) $x \le y \rightsquigarrow z$ *if and only if* $x \odot y \le z$;
(11) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$, $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$;
(12) $(y \vee z) \oplus x = (y \oplus x) \vee (z \oplus x)$, $(y \vee z) \boxplus x = (y \boxplus x) \vee (z \boxplus x)$;
(13) $(x \wedge y) \odot z = (x \odot z) \wedge (y \odot z)$, $(x \wedge y) \boxdot z = (x \boxdot z) \wedge (y \boxdot z)$.

Proof. The proofs of (1)-(4) are easy.

(5) Suppose $x \le y$. By Proposition 2.2 part (4), we have $y^- \le x^-$ and $y^\sim \le x^\sim$. Using Proposition 2.2 part (5), we obtain $y^- \oplus z^- \le x^- \oplus z^-$ and $y^\sim \boxplus z^\sim \le x^\sim \boxplus z^\sim$. Since A is pseudo commutative, then $z^- \boxplus y^- \le z^- \boxplus x^-$ and $z^\sim \oplus y^\sim \le z^\sim \oplus x^\sim$. By Proposition 2.2 part (4), we have $(z^- \boxplus x^-)^\sim \le (z^- \boxplus y^-)^\sim$ and $(z^\sim \oplus x^\sim)^- \le (z^\sim \oplus y^\sim)^-$, that is $z \odot x \le z \odot y$, $z \boxdot x \le z \boxdot y$.

(6) We have $(x \to y) \odot y^- = ((x^- \boxplus y)^\sim \oplus y)^- = (x \vee y)^- = x^- \wedge y^-$. Similarly, we can prove $(x \to y) \boxdot y^\sim = x^\sim \wedge y^\sim$.

(7) It follows from Proposition 2.2 part (12), (P2) and definition $\odot$.

(8) Since A is pseudo commutative and by (C6), we have

$$(x \to y) \boxdot x = (x^- \boxplus y) \boxdot x = (y \oplus x^-) \boxdot x = x \wedge y;$$
$$(x \rightsquigarrow y) \odot x = (x^\sim \oplus y) \odot x = (y \boxplus x^\sim) \odot x = x \wedge y.$$

(9) Let $x \le y \to z$. By Proposition 3.1 part (7), we get $x \boxdot y \le (y \to z) \boxdot y = y \wedge z \le z$. Conversely, suppose $x \boxdot y \le z$. Then

$$x \le (x \vee y^-) = (x \boxdot y) \oplus y^- \le z \oplus y^- = y^- \boxplus z = y \to z.$$

(10) The proof is similar to part (9).

(11) Since $y \wedge z \le y, z$, then $x \to (y \wedge z) \le x \to y$ and $x \to (y \wedge z) \le x \to z$ by part (2). Hence, $x \to (y \wedge z)$ is a lower bound of $\{x \to y, x \to z\}$. Let u be an arbitrary lower bound of $\{x \to y, x \to z\}$. Then $u \le x \to y$ and $u \le x \to z$. By part (9), we have $u \boxdot x \le y$ and $u \boxdot x \le z$. So, $u \boxdot x \le y \wedge z$. Again, applying part (9), we obtain $u \le x \to (y \wedge z)$. Hence, $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$. Similarly, $x \rightsquigarrow (y \wedge z) = (x \rightsquigarrow y) \wedge (x \rightsquigarrow z)$.

(12) Since $y, z \le y \vee z$, then $y \oplus x \le (y \vee z) \oplus x$ and $z \oplus x \le (y \vee z) \oplus x$ by Proposition 2.4 part (5). Hence, $(y \oplus x) \vee (z \oplus x) \le (y \vee z) \oplus x$. Conversely, let $u := (y \oplus x) \vee (z \oplus x)$. Then $y \oplus x \le u$ and $z \oplus x \le u$. By Proposition 3.1 part (7) and (C6), we have $z \wedge x^\sim = (z \oplus x) \boxdot x^\sim \le u \boxdot x^\sim$ and $y \wedge x^\sim = (y \oplus x) \boxdot x^\sim \le u \boxdot x^\sim$. Since A is a distributive lattice, then $(z \vee y) \wedge x^\sim = (y \wedge x^\sim) \vee (z \wedge x^\sim) \le u \boxdot x^\sim$. By Proposition 2.2 part (5) and (C5), we get $((z \vee y) \wedge x^\sim) \oplus x \le (u \boxdot x^\sim) \oplus x = u \vee x = u$. Using Proposition 2.2 part (12) and part (3), we obtain $((z \vee y) \oplus x) \wedge (x^\sim \oplus x) = (z \vee y) \oplus x \le u$. Hence, $(z \vee y) \oplus x \le (y \oplus x) \vee (z \oplus x)$.

(13) It follows from part (12) and definitions $\odot$ and $\boxdot$. ☐

Corollary 3.1. *Let* $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ *be a pseudo commutative double basic algebra. Then* $y \odot x = x \boxdot y$ *for all* $x, y \in A$.

Proof. Since $x \boxdot y \leq x \boxdot y$, then $x \leq y \to (x \boxdot y)$ by Proposition 3.2 part (9). We obtain $y \leq x \rightsquigarrow (x \boxdot y)$ by Proposition 3.1 part (9). Using Proposition 3.2 part (10), we get $y \odot x \leq x \boxdot y$. Similarly, we can prove $x \boxdot y \leq y \odot x$. Hence $x \boxdot y = y \odot x$. $\square$

Proposition 3.3. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra.*

(1) *If $(x \odot y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$ for any $x, y, z \in A$, then $\oplus$ is associative.*
(2) *If $(x \boxdot y) \to z = x \to (y \to z)$ for any $x, y, z \in A$, then $\boxplus$ is associative.*

Proof. (1) We have $(x^\sim \oplus y^\sim) \oplus z = (x \odot y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z) = x^\sim \oplus (y^\sim \oplus z)$. Hence $\oplus$ is associative. The proof of part (2) is similar. $\square$

Corollary 3.2. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra such that $(A, \wedge, \vee, \odot, \rightsquigarrow, \to, 1)$ and $(A, \wedge, \vee, \boxdot, \to, \rightsquigarrow, 1)$ be pseudo hoops. Then A is a double MV-algebra.*

Proof. Let $(A, \wedge, \vee, \odot, \rightsquigarrow, \to, 1)$ and $(A, \wedge, \vee, \boxdot, \to, \rightsquigarrow, 1)$ be pseudo hoops. Then $\oplus$ and $\boxplus$ are associative by Proposition 3.3. Since $\oplus$ and $\boxplus$ are associative and A is a pseudo commutative, then A is a double MV-algebra. $\square$

Proposition 3.4. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra such that satisfies the following identities for all $x, y, z \in A$*

$$x \to y \leq (y \to z) \rightsquigarrow (x \to z), \quad x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z).$$

Then $\oplus$ and $\boxplus$ are associative.

Proof. Suppose that A satisfies $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ and $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$. Then $(A, \leq, \to, \rightsquigarrow, 1)$ is a pseudo-BCK algebra. By Proposition 1.2 part (3) in [17], we have $x \rightsquigarrow (y \to z) = y \to (x \rightsquigarrow z)$ and $x \to (y \rightsquigarrow z) = y \rightsquigarrow (x \to z)$. Thus, $x^\sim \oplus (y^- \boxplus z) = y^- \boxplus (x^\sim \oplus z)$. So A satisfies Exchange identity. Hence, $\oplus$ and $\boxplus$ are associative by Proposition 2.3. $\square$

Corollary 3.3. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra such that $(A, \leq, \to, \rightsquigarrow, 1)$ is a pseudo BCK-algebra. Then A is a double MV-algebra.*

Proof. Similarly to the proof of Proposition 3.4, we can prove the associativity of $\oplus$ and $\boxplus$. Since A is a pseudo commutative, then A is a double MV-algebra by Proposition 2.3. $\square$

The relation between (commutative) basic algebras and residuated groupoids have been studied in [7]. In the following, we will study the relation between pseudo commutative double basic algebras and pseudo residuated l-groupoids.

Definition 3.1. A pseudo residuated l-groupoid is an algebra $(A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ of type $(2, 2, 2, 2, 2, 0, 0)$ such that

(R1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;
(R2) $(A, *, 1)$ is a groupoid with 1, i.e. it satisfies $x * 1 = 1 * x = x$;

(R3) $x * y \leq z$ if and only if $x \leq y \to z$ if and only if $y \leq x \rightsquigarrow z$ for any $x, y, z \in A$ (pseudo residuation).

Proposition 3.5. *Let* $(A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ *be a pseudo residuated l-groupoid. Then*

(1) *if* $x \leq y$*, then* $x * z \leq y * z$ *and* $z * x \leq z * y$*;*
(2) $(x \to y) * x \leq x \wedge y$*,* $x * (x \rightsquigarrow y) \leq x \wedge y$*;*
(3) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $y \rightsquigarrow z \leq x \rightsquigarrow z$*;*
(4) $1 \to x = x$*,* $1 \rightsquigarrow x = x$*.*

Proof. (1) Suppose that $x \leq y$. Since $y * z \leq y * z$ and $z * y \leq z * y$, then $y \leq z \to y * z$ and $y \leq z \rightsquigarrow z * y$. Thus $x \leq z \to y * z$ and $x \leq z \rightsquigarrow z * y$. Therefore $x * z \leq y * z$ and $z * x \leq z * y$.

The proofs of (2)-(4) are easy. □

Proposition 3.6. *Let* $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ *be a pseudo commutative double basic algebra. Then* $(A, \wedge, \vee, \boxdot, \to, \rightsquigarrow, 0, 1)$ *and* $(A, \wedge, \vee, \odot, \rightsquigarrow, \to, 0, 1)$ *are pseudo residuated l-groupoids.*

Proof. Since A is a pseudo commutative double basic algebra, then $(A, \wedge, \vee, 0, 1)$ is a bounded distributive lattice. By Proposition 3.1 part (5), $(A, \odot, 1)$ and $(A, \boxdot, 1)$ are groupoids with 1. Applying Proposition 3.2 part (9) and Proposition 3.1 part (9), we get $x \boxdot y \leq z$ if and only if $x \leq y \to z$ if and only if $y \leq x \rightsquigarrow z$.

Hence, $(A, \wedge, \vee, \boxdot, \to, \rightsquigarrow, 0, 1)$ is a pseudo residuated l-groupoid.

Again, By Proposition 3.2 part (10) and Proposition 3.1 part (9), we have

$$x \odot y \leq z \text{ if and only if } x \leq y \rightsquigarrow z \text{ if and only if } y \leq x \to z.$$

Hence, $(A, \wedge, \vee, \odot, \rightsquigarrow, \to, 1)$ is a pseudo residuated l-groupoid. □

For the case of basic algebra, the next theorems are proved in [7]. Here, we formulate and prove them for the case of pseudo commutative double basic algebras as well.

Lemma 3.1. *Let* $(A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ *be a pseudo residuated l-groupoid satisfying the pseudo double negation laws and*

(i) $x * (y^\sim \to x^\sim) = x \wedge y$*;*
(ii) $(y^- \rightsquigarrow x^-) * x = x \wedge y$*;*

then the following hold:

(1) $x \to y = y^- \rightsquigarrow x^-$*,* $x \rightsquigarrow y = y^\sim \to x^\sim$*;*
(2) $x * (y * z) = 0$ *implies* $(x * y) * z = 0$*;*
(3) $x \to y = (x * y^\sim)^-$*,* $x \rightsquigarrow y = (y^- * x)^\sim$*;*

where $x^- = x \to 0$ *and* $x^\sim = x \rightsquigarrow 0$*.*

Proof. (1) By assumption, we have $x * (y^\sim \to x^\sim) = x \wedge y \leq y$ and $(y^- \rightsquigarrow x^-) * x = x \wedge y \leq y$. Thus, $y^\sim \to x^\sim \leq x \rightsquigarrow y$ and $y^- \rightsquigarrow x^- \leq x \to y$.

Using the pseudo double negation laws and substituting y^- for x and x^- for y in $y^\sim \to x^\sim \leq x \rightsquigarrow y$, this yields $x \to y \leq y^- \rightsquigarrow x^-$.

In $y^- \rightsquigarrow x^- \leq x \to y$, applying the pseudo double negation laws and substituting $y^\sim$ for x and $x^\sim$ for y, this yields $x \rightsquigarrow y \leq y^\sim \to x^\sim$. Hence, $x \to y = y^- \rightsquigarrow x^-$ and $x \rightsquigarrow y = y^\sim \to x^\sim$.

(2) Suppose $x * (y * z) = 0$. Then $y * z \leq x \rightsquigarrow 0 = x^\sim$. So, $y \leq z \to x^\sim$. By part (1) and pseudo double negation laws, we get $y \leq x \rightsquigarrow z^-$. Thus, $x * y \leq z \to 0$. Hence, $(x * y) * z = 0$.

(3) Since $x * y^\sim \leq x * y^\sim$, then $x \leq y^\sim \to (x * y^\sim)$. Applying part (1) and pseudo double negation laws, we have $x \leq (x * y^\sim)^- \rightsquigarrow y$. So, $(x * y^\sim)^- * x \leq y$. Hence, $(x * y^\sim)^- \leq x \to y$.

On the other hand, using part (1) and then assumption (ii), we obtain

$$((x \to y) * x) * y^\sim = ((y^- \rightsquigarrow x^-) * x) * y^\sim = (x \wedge y) * y^\sim \leq y * y^\sim = 0.$$

Hence, $((x \to y) * x) * y^\sim = 0$. By part (2), we get $(x \to y) * (x * y^\sim) = 0$. Thus, $x \to y \leq (x * y^\sim)^-$. Therefore, $x \to y = (x * y^\sim)^-$. Similarly, we can prove $x \rightsquigarrow y = (y^- * x)^\sim$. □

Theorem 3.1. *Let $(A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ be a pseudo residuated l-groupoid satisfying the conditions of Lemma* 3.1*. Then $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ is a pseudo commutative double basic algebra where $x^- = x \to 0$, $x^\sim = x \rightsquigarrow 0$, $x \oplus y = (y^- * x^-)^\sim$ and $x \boxplus y = (x^\sim * y^\sim)^-$.*

Proof. We will check the axioms (P1)-(P4).

(P1) We have $x \oplus 0 = (0^- * x^-)^\sim = x^{-\sim} = x$ and $x \boxplus 0 = (x^\sim * 0^\sim)^- = x^{\sim -} = x$.

(P2) It holds by assumption (pseudo double negation laws).

(P3) By Lemma 3.1 part (3), definitions of $\oplus$ and $\boxplus$, pseudo double negation laws and then assumption (i), we get

$$\begin{aligned}(x^\sim \oplus y)^- \boxplus y =&(y^- * x) \boxplus y = (x \rightsquigarrow y)^- \boxplus y = ((x \rightsquigarrow y) * y^\sim)^- \\ =&((x^{\sim -} \rightsquigarrow y^{\sim -}) * y^\sim)^- = (x^\sim \wedge y^\sim)^- = x \vee y.\end{aligned}$$

Similarly, we can prove $(y^\sim \oplus x)^- \boxplus x = (x^- \boxplus y)^\sim \oplus y = (y^- \boxplus x)^\sim \oplus x = x \vee y$.

(P4) It is easy to prove $(x \wedge y^-) \boxplus z \leq x \boxplus z$. Hence,

$$\begin{aligned}1 &= ((x \wedge y^-) \boxplus z) \rightsquigarrow (x \boxplus z) \\ &= ((y^- * (x^\sim \to y^{-\sim})) \boxplus z) \rightsquigarrow (x \boxplus z) \\ &= ((y^- * (x^\sim \to y)) \boxplus z) \rightsquigarrow (x \boxplus z) \\ &= ((y^- * (x \boxplus y)) \boxplus z) \rightsquigarrow (x \boxplus z) \\ &= (((x \boxplus y)^\sim \oplus y)^- \boxplus z) \rightsquigarrow (x \boxplus z) \\ &= (((x \boxplus y)^\sim \oplus y)^- \boxplus z)^\sim \oplus (x \boxplus z).\end{aligned}$$

Similarly, we can prove $(((x \oplus y)^- \boxplus y)^\sim \oplus z)^- \boxplus (x \oplus z) = 1$. Hence, $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ is a double basic algebra. By Lemma 3.11 part (1) and (3), we have

$$x \oplus y = (y^- * x^-)^\sim = x^- \rightsquigarrow y = y^\sim \to x = (y^\sim * x^\sim)^- = y \boxplus x.$$

Hence, $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ is a pseudo commutative. □

Lemma 3.2. *Let* $(A, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ *be a pseudo residuated l-groupoid satisfying the following identities:*

$$x \vee y = (y^- \rightsquigarrow x^-) \rightsquigarrow y = (y^\sim \rightarrow x^\sim) \rightarrow y;$$

then the following hold:

(1) $x^{-\sim} = x$, $x^{\sim -} = x$;
(2) $x \rightarrow y = y^- \rightsquigarrow x^-$, $x \rightsquigarrow y = y^\sim \rightarrow x^\sim$;
(3) $x * (y * z) = 0$ *implies* $(x * y) * z = 0$;
(4) $x \rightarrow y = (x * y^\sim)^-$, $x \rightsquigarrow y = (y^- * x)^\sim$;
(5) $(x \rightarrow y) * x = x \wedge y$, $x * (x \rightsquigarrow y) = x \wedge y$.

Proof. (1) We have $1 \rightsquigarrow x^- = x^- \leq x \rightarrow 0$ by Proposition 3.5 part (4). Applying Proposition 3.5 part (3) and assumption, we get $(x \rightarrow 0) \rightsquigarrow 0 \leq (1 \rightsquigarrow x^-) \rightsquigarrow 0 = x \vee 0 = x$. Hence, $x^{-\sim} \leq x$.

On the other hand, $(x \rightarrow 0) * x \leq 0$ by Proposition 3.5 part (2). Hence, $x \leq (x \rightarrow 0) \rightsquigarrow 0 = x^{-\sim}$. Therefore, $x^{-\sim} = x$. Similarly, we can prove $x^{\sim -} = x$.

(2) Since $x \leq x \vee y = (y^- \rightsquigarrow x^-) \rightsquigarrow y$, then $(y^- \rightsquigarrow x^-) * x \leq y$. Thus, $y^- \rightsquigarrow x^- \leq x \rightarrow y$. Also, using the pseudo double negation laws and substituting $y^\sim$ for x and $x^\sim$ for y in $y^- \rightsquigarrow x^- \leq x \rightarrow y$, we obtain $x \rightsquigarrow y \leq y^\sim \rightarrow x^\sim$.

Similarly, we can show $y^\sim \rightarrow x^\sim \leq x \rightsquigarrow y$ and $x \rightarrow y \leq y^- \rightsquigarrow x^-$. Hence, $x \rightarrow y = y^- \rightsquigarrow x^-$, $x \rightsquigarrow y = y^- \rightarrow x^-$.

The proof of part (3) and part (4) is similar to the proof of part (2) and part (3) in Lemma 3.1, respectively.

(5) Using part (1), part (4) and then assumption, we get

$$(x \rightarrow y) * x = ((x \rightarrow y) * x^{-\sim})^{-\sim} = ((x \rightarrow y) \rightarrow x^-)^\sim = (x^- \vee y^-)^\sim = x \wedge y.$$

Similarly, we can prove that $x * (x \rightsquigarrow y) = x \wedge y$. □

Proposition 3.7. *Let* $(A, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ *be a pseudo residuated l-groupoid. Then A satisfies the conditions of Lemma* 3.2 *if and only if it satisfies the conditions of Lemma* 3.1*.*

Proof. Let $(A, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ be a pseudo residuated l-groupoid satisfying conditions of Lemma 3.1. Using Lemma 3.1 part (1) and part (3), we obtain

$$x \vee y = (y^- \wedge x^-)^\sim = (y^- * (x \rightarrow y))^\sim = (x \rightarrow y) \rightsquigarrow y = (y^- \rightsquigarrow x^-) \rightsquigarrow y;$$
$$x \vee y = (y^\sim \wedge x^\sim)^- = ((x \rightarrow y) * y^\sim)^- = (x \rightsquigarrow y) \rightarrow y = (y^\sim \rightarrow x^\sim) \rightarrow y.$$

Conversely, suppose that A satisfies the conditions of Lemma 3.2. Applying Lemma 3.2 part (2) and then part (5), we obtain

$$x * (y^\sim \rightarrow x^\sim) = x * (x \rightsquigarrow y) = x \wedge y, \quad (y^- \rightsquigarrow x^-) * x = (x \rightarrow y) * x = x \wedge y.$$

Also, by part (1) of Lemma 3.2, A satisfies the pseudo double negation laws. □

Corollary 3.4. *Let* $(A, \wedge, \vee, *, \rightarrow, \rightsquigarrow, 0, 1)$ *be a pseudo residuated l-groupoid satisfying the following identities:*

$$x \vee y = (y^- \rightsquigarrow x^-) \rightsquigarrow y = (y^\sim \to x^\sim) \to y.$$

Then $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ *is a pseudo commutative double basic algebra where* $x^- = x \to 0$*,* $x^\sim = x \rightsquigarrow 0$*,* $x \oplus y = (y^- * x^-)^\sim$ *and* $x \boxplus y = (x^\sim * y^\sim)^-$*.*

Proof. The proof follows from Proposition 3.7 and Theorem 3.1. □

By Proposition 3.6, if $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ is a pseudo commutative double basic algebra, then $F((A, \oplus, \boxplus, ^-, ^\sim, 0, 1)) := (A, \wedge, \vee, \boxdot, \to, \rightsquigarrow, 0, 1)$ is a pseudo residuated l-groupoid where $x \boxdot y = (x^- \boxplus y^-)^\sim$, $x \to y = (x \boxdot y^\sim)^-$, $x \rightsquigarrow y = (y^- \boxdot x)^\sim$.

Moreover, if $(A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ is a pseudo residuated l-groupoid that satisfies the conditions of Lemma 3.1 (or Lemma 3.2), then $G((A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)) := (A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ is a pseudo commutative double basic algebra, where $x^- = x \to 0$, $x^\sim = x \rightsquigarrow 0$, $x \oplus y = (y^- * x^-)^\sim$ and $x \boxplus y = (x^\sim * y^\sim)^-$.

The category whose objects are double basic algebras and whose morphisms are homomorphisms of double basic algebras is called the category of double basic algebras. Let $\mathcal{A}$ be its subcategory whose object are pseudo commutative double basic algebras. The category of pseudo residuated l-groupoids can be defined similarly. Let $\mathcal{B}$ be its subcategory whose objects are pseudo residuated l-groupoids satisfying conditions of Lemma 3.1. It is clear that $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$ are functors. In the next theorem, we study a relation between these functors.

Theorem 3.2. (1) *Let* $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ *be a pseudo commutative double basic algebra. Then* $G(F((A, \oplus, \boxplus, ^-, ^\sim, 0, 1))) = (A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$*.*

(2) *Let* $(A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1)$ *be a pseudo residuated l-groupoid that satisfies the conditions of Lemma* 3.1*. Then*

$$F(G((A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1))) = (A, \wedge, \vee, *, \to, \rightsquigarrow, 0, 1).$$

(3) *The category* $\mathcal{A}$ *and the category* $\mathcal{B}$ *are categorically isomorphic.*

Proof. (1) By Proposition 3.6, Proposition 3.2 part (3) and part (8),

$$F((A, \oplus, \boxplus, ^-, ^\sim, 0, 1)) := (A, \wedge, \vee, \boxdot, \to, \rightsquigarrow, 0, 1)$$

is a pseudo residuated l-groupoid satisfying conditions of Lemma 3.1. Hence, $G(F((A, \oplus, \boxplus, ^-, ^\sim, 0, 1)))$ is a pseudo commutative double basic algebra by Theorem 3.1. Now, suppose that $\oplus', \boxplus', ^{-'}$ and $^{\sim'}$ are the operations derived by $\boxdot$, $\to$ and $\rightsquigarrow$ on the pseudo residuated l-groupoid $F((A, \oplus, \boxplus, ^-, ^\sim, 0, 1))$, respectively. We will prove that $\oplus' = \oplus$, $\boxplus' = \boxplus$, $^{-'} = ^-$ and $^{\sim'} = ^\sim$. Let $x, y \in A$ be arbitrary. We have $x^{-'} = x \to 0 = x^- \boxplus 0 = x^-$ and $x^{\sim'} = x \rightsquigarrow 0 = x^\sim \oplus 0 = x^\sim$. Using Lemma 3.1 part (3) and part (1), we get

$$\begin{aligned} x \boxplus' y =& (x^{\sim'} \boxdot y^{\sim'})^{-'} = (x^\sim \boxdot y^\sim)^- = x \boxplus y, \\ x \oplus' y =& (x^{-'} \boxdot y^{-'})^{\sim'} = (y^- \boxdot x^-)^\sim = x^- \rightsquigarrow y = y^\sim \to x = (y^\sim \odot x^\sim)^- = y \boxplus x \\ =& x \oplus y. \end{aligned}$$

(2) By Theorem 3.1, $G((A,\wedge,\vee,*,\to,\rightsquigarrow,0,1)) := (A,\oplus,\boxplus,^-,^\sim,0,1)$ is a pseudo commutative double basic algebra. Hence, $F(G((A,\wedge,\vee,*,\to,\rightsquigarrow,0,1)))$ is a pseudo residuated l-groupoid satisfying conditions of Lemma 3.1. Let $F(G((A,\wedge,\vee,*,\to,\rightsquigarrow,0,1))) = (A,\wedge',\vee',*',\to',\rightsquigarrow',0,1)$. We have

$$\begin{aligned}
x*'y &= x \boxdot y = (x^- \boxplus y^-)^\sim = (x^{-\sim} * y^{-\sim})^{-\sim} = x*y,\\
x \to' y &= x^- \boxplus y = (x * y^\sim)^- = x \to y,\\
x \rightsquigarrow' y &= x^\sim \oplus y = (y^- * x)^- = x \rightsquigarrow y,\\
x \wedge' y &= x *' (y^{\sim'} \to x^{\sim'}) = x * (y^\sim \to x^\sim) = x \wedge y,\\
x \vee' y &= (x^{-'} \wedge' y^{-'})^{\sim'} = (x^- \wedge y^-)^\sim.
\end{aligned}$$

(3) The proof follows from (i) and (ii). ☐

By Proposition 3.6, if $(A,\oplus,\boxplus,^-,^\sim,0,1)$ is a pseudo commutative double basic algebra, then $H((A,\oplus,\boxplus,^-,^\sim,0,1)) := (A,\wedge,\vee,\odot,\rightsquigarrow,\to,0,1)$ is a pseudo residuated l-groupoid, (that is $x\odot y \le z$ if and only if $x \le y \rightsquigarrow z$ if and only if $y \le x \to z$ for any $x,y,z \in A$) where $x \odot y = (x^\sim \oplus y^\sim)^-$, $x \rightsquigarrow y = x^\sim \oplus y$, $x \to y = y \oplus x^-$. Then clearly, $H : \mathcal{A} \to \mathcal{B}$ is a functor.

Proposition 3.8. *Let $(A,\wedge,\vee,*,\rightsquigarrow,\to,0,1)$ be a pseudo residuated l-groupoid satisfying pseudo double negation laws and*

(i)' $(y^\sim \to x^\sim) * x = x \wedge y$;
(ii)' $x * (y^- \rightsquigarrow x^-) = x \wedge y$.

Then $(A,\oplus,\boxplus,^-,^\sim,0,1)$ is a pseudo commutative double basic algebra, where $x^- = x \to 0$, $x^\sim = x \rightsquigarrow 0$, $x \oplus y = (x^- * y^-)^\sim$ and $x \boxplus y = (y^\sim * x^\sim)^-$.

Proof. Similar to Lemma 3.1, we can prove

(1) $x \to y = y^- \rightsquigarrow x^-$, $x \rightsquigarrow y = y^\sim \to x^\sim$;
(2) $x * (y * z) = 0$ implies $(x * y) * z = 0$;
(3) $x \to y = (y^\sim * x)^-$, $x \rightsquigarrow y = (x * y^-)^\sim$;

and use them to prove that $(A,\oplus,\boxplus,^-,^\sim,0,1)$ is a pseudo commutative double basic algebra such that $x^- = x \to 0$, $x^\sim = x \rightsquigarrow 0$, $x \oplus y = (x^- * y^-)^\sim$ and $x \boxplus y = (y^\sim * x^\sim)^-$. ☐

Theorem 3.3. (1) *Let $(A,\oplus,\boxplus,^-,^\sim,0,1)$ be a pseudo commutative double basic algebra. Then $G(H((A,\oplus,\boxplus,^-,^\sim,0,1))) = (A,\oplus,\boxplus,^-,^\sim,0,1)$.*

(2) *Let $(A,\wedge,\vee,*,\rightsquigarrow,\to,0,1)$ be a pseudo residuated l-groupoid that satisfies the conditions of Proposition* 3.14*, then*

$$H(G((A,\wedge,\vee,*,\rightsquigarrow,\to,0,1))) = (A,\wedge,\vee,*,\rightsquigarrow,\to,0,1).$$

Proof. (1) By Proposition 3.6, $H((A,\oplus,\boxplus,^-,^\sim,0,1)) := (A,\wedge,\vee,\odot,\rightsquigarrow,\to,0,1)$ is a pseudo residuated l-groupoid. Using Proposition 3.2 part (3) and part (4), we have

(i)' $(y^\sim \to x^\sim) \odot x = (x \rightsquigarrow y) \odot x = x \wedge y$;
(ii)' $x \odot (y^\sim \to x^\sim) = (x \rightsquigarrow y) \boxdot x = x \wedge y$.

Hence, by Proposition 3.8, $G(H((A,\oplus,\boxplus,^{-},^{\sim},0,1))) = (A,\oplus',\boxplus',^{-'},^{\sim'},0,1)$ is a pseudo commutative double basic algebra where $x^{-'} = x \to 0$, $x^{\sim'} = x \rightsquigarrow 0$, $x \oplus' y = (x^{-'} \odot y^{-'})^{\sim'}$ and $x \boxplus' y = (y^{\sim'} \odot x^{\sim'})^{-'}$. We will show that $^{-'} =^{-}$, $^{\sim'} =^{\sim}$, $\oplus' = \oplus$ and $\boxplus' = \boxplus$. It is obvious that $^{-'} =^{-}$ and $^{\sim'} =^{\sim}$. We have

$$\begin{aligned} x \oplus' y =&(x^{-'} \odot y^{-'})^{\sim'} = (x^{-} \odot y^{-})^{\sim} = x \oplus y, \\ x \boxplus' y =&(y^{\sim'} \odot x^{\sim'})^{-'} = (y^{\sim} \odot x^{\sim})^{-} = (x^{\sim} \boxdot y^{\sim})^{-} = x \boxplus y. \end{aligned}$$

Hence, $G(H((A,\oplus,\boxplus,^{-},^{\sim},0,1))) = (A,\oplus,\boxplus,^{-},^{\sim},0,1)$.

(2) $G((A,\wedge,\vee,*,\rightsquigarrow,\to,0,1)) := (A,\oplus,\boxplus,^{-},^{\sim},0,1)$ is a pseudo commutative double basic algebra like Proposition 3.7. Hence, $H(G((A,\wedge,\vee,*,\to,\rightsquigarrow,0,1)))$ is a pseudo residuated l-groupoid satisfying conditions of Proposition 3.7. Suppose that

$$H(G((A,\wedge,\vee,*,\to,\rightsquigarrow,0,1))) = (A,\wedge',\vee',*',\rightsquigarrow',\to',0,1).$$

We have

$$\begin{aligned} x *' y =&x \odot y = (x^{\sim} \oplus y^{\sim})^{-} = (x^{-\sim} * y^{-\sim})^{\sim -} = x * y, \\ x \to' y =&x^{\sim} \oplus y = (x * y^{-})^{\sim} = x \rightsquigarrow y, \\ x \rightsquigarrow' y =&x^{-} \boxplus y = (y^{\sim} * x)^{-} = x \to y, \\ x \wedge' y =&x *' (y^{\sim'} \to x^{\sim'}) = (y^{-} \rightsquigarrow x^{-}) * x = x \wedge y, \\ x \vee' y =&(x^{-'} \wedge' y^{-'})^{\sim'} = (x^{\sim} \wedge y^{\sim})^{-} = x \vee y. \end{aligned}$$ ☐

Proposition 3.9. *Let* $(A,\oplus,\boxplus,^{-},^{\sim},0,1)$ *be a double MV-algebra. Then* $(A,\wedge,\vee,\odot,\rightsquigarrow,\to,1)$ *and* $(A,\wedge,\vee,\boxdot,\to,\rightsquigarrow,1)$ *are pseudo hoops.*

Proof. Let $(A,\oplus,\boxplus,^{-},^{\sim},0,1)$ be a double MV-algebra. Then $(A,\oplus,\boxplus,^{-},^{\sim},0,1)$ is a pseudo commutative double basic algebra. Hence, $(A,\wedge,\vee,\boxdot,\to,\rightsquigarrow,0,1)$ is a pseudo residuated l-groupoid by Proposition 3.6. Since $\boxplus$ is associative, then $\boxdot$ is associative. By Proposition 3.2 part (8) $(((x \boxdot y) \to z) \boxdot x) \boxdot y = ((x \boxdot y) \to z) \boxdot (x \boxdot y) \leq z$ and $(((x \boxdot y) \rightsquigarrow z) \odot y) \odot x = ((x \boxdot y) \rightsquigarrow z) \odot (y \odot x) = ((x \boxdot y) \rightsquigarrow z) \odot (x \boxdot y) \leq z$. So, $(x \boxdot y) \to z \leq x \to (y \to z)$ and $(x \boxdot y) \rightsquigarrow z \leq y \rightsquigarrow (x \rightsquigarrow z)$ by Proposition 3.2 part (9) and part (10). On the other hand

$$\begin{aligned} (x \to (y \to z)) \boxdot (x \boxdot y) =&((x \to (y \to z)) \boxdot x) \boxdot y \leq (y \to z) \boxdot y \leq z, \\ (y \rightsquigarrow (x \rightsquigarrow z)) \odot (x \boxdot y) =&(((y \rightsquigarrow (x \rightsquigarrow z)) \odot y) \odot x) \leq (x \rightsquigarrow z) \odot x \leq z. \end{aligned}$$

Hence, $(A,\wedge,\vee,\odot,\rightsquigarrow,\to,1)$ is a pseudo hoop. Similarly, we can prove $(x \odot y) \rightsquigarrow z = x \rightsquigarrow (y \rightsquigarrow z)$ and $(x \odot y) \to z = y \to (x \to z)$. Hence, $(A,\wedge,\vee,\odot,\rightsquigarrow,\to,1)$ is a pseudo hoop. ☐

Corollary 3.5. *Let* $(A,\oplus,\boxplus,^{-},^{\sim},0,1)$ *be a double MV-algebra. Then* $(A,\leq,\to,\rightsquigarrow,1)$ *is a pseudo BCK-algebra.*

Proof. We know that every pseudo-hoop is a pseudo-BCK algebra which is a meet semilattice satisfying the pseudo-divisibility property. Since $(A,\wedge,\vee,\odot,\rightsquigarrow,\to,1)$ is a pseudo hoop by Proposition 3.9, then $(A,\leq,\to,\rightsquigarrow,1)$ is a pseudo BCK-algebra. ☐

4. Boolean Center of Pseudo Commutative Double Basic Algebras

Let $(L, \wedge, \vee, 0, 1)$ be a bounded lattice. Recall that (see [1,21]) an element $a \in L$ is said to be complemented if there is an element $b \in L$ such that $a \vee b = 1$ and $a \wedge b = 0$. If such an element b exists, then it is called a complement of a. Complements are generally not unique unless the lattice is distributive. Hence in pseudo commutative double basic algebras the complements are unique.

Proposition 4.1. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra and $a \in A$ an element which has a complement $b \in A$. Then $a^- = a^\sim = b$ and $b^- = b^\sim = a$.*

Proof. We have $1 = a \vee b \leq a \boxplus b$. Hence, $a \boxplus b = 1$. Thus, $a^\sim \leq b$. On the other hand, we have that $a^\sim = 0 \boxplus a^\sim = (b \wedge a) \boxplus a^\sim = (b \boxplus a^\sim) \wedge (a \boxplus a^\sim) = (b \boxplus a^\sim) \wedge (a^\sim \oplus a) = (b \boxplus a^\sim) \wedge 1 = (b \boxplus a^\sim)$ by Proposition 2.4 part (12) and part (3). Since $(b \vee a^\sim) \leq (b \boxplus a^\sim) = a^\sim$, then $b \leq a^\sim$. Therefore, $a^\sim = b$ and $a = a^{\sim -} = b^-$. Similarly, we can prove $a^- = b$ and $b^\sim = a$. $\square$

Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra. We denote by $B(A)$ the boolean algebra associated with the bounded distributive lattice $L(A) = (A, \wedge, \vee, 0, 1)$. The set $B(A)$ is called the boolean center of A and elements of B(A) are called the boolean elements of A.

Proposition 4.2. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra. Then the following are equivalent:*

(1) $a \in B(A)$;
(2) $a \wedge a^- = 0$;
(3) $a \oplus a = a$;
(4) $a \boxplus a = a$;
(5) $a \wedge a^\sim = 0$;
(6) $a \vee a^- = 1$;
(7) $a \odot a = a$;
(8) $a \boxdot a = a$;
(9) $a \vee a^\sim = 1$.

Proof. (1)⇒(2) Since $a \in B(A)$, then a has a complement $b \in A$. By Proposition 4.1, we have $b = a^-$. Hence, $a \wedge a^- = 0$.

(2)⇒(3) By Proposition 2.2 part (12) and part (3), we obtain $a = 0 \oplus a = (a \wedge a^\sim) \oplus a = (a \oplus a) \wedge (a^\sim \oplus a) = a \oplus a$.

(3)⇒(4) Since A is pseudo commutative, it is clear.

(4)⇒(5) Let $a \boxplus a = a$. By Proposition 2.2 part (3), we have

$$a \wedge a^\sim = (a^- \vee a)^\sim = ((a \boxplus a)^\sim \oplus a)^- = (a^\sim \oplus a)^- = 1^- = 0.$$

(5)⇒(6) We have $0 = a \wedge a^\sim = (a^- \vee a^{\sim -})^\sim = (a^- \vee a)^\sim$. Hence $a^- \vee a = (a \wedge a^\sim)^- = 0^- = 1$.

(6)⇒(7) Using Proposition 3.1 part (5) and Proposition 3.2 part (7), we get $a = 1 \odot a = (a \vee a^{-}) \odot a = (a \odot a) \vee (a \odot a^{-}) = (a \odot a) \vee 0 = a \odot a$.

(7)⇒(8) It is clear.

(8)⇒(9) Suppose $a \boxdot a = a$. Then $a^{-} \boxplus a^{-} = a^{-}$. Similar to (4)⇒ (5), we can prove $a^{-} \wedge a^{-\sim} = 0$. Hence, $a \vee a^{\sim} = 1$.

(9)⇒(1) Since $a \vee a^{\sim} = 1$, then $a \wedge a^{-} = 0$. Similar to (2)⇒ (3), we can show that $a \oplus a = a$. Thus, $a \boxplus a = a$. Then, similar to (4) ⇒ (5), we obtain $a \wedge a^{\sim} = 0$. □

Proposition 4.3. *Let* $(A, \oplus, \boxplus, ^{-}, ^{\sim}, 0, 1)$ *be a pseudo commutative double basic algebra and* $a \in B(A)$. *Then the following hold:*

(1) $a^{-} = a \to a^{-} = a \rightsquigarrow a^{-} = a \to a^{\sim} = a \rightsquigarrow a^{\sim} = a^{\sim}$;

(2) $a^{-} \to a = a^{-} \rightsquigarrow a = a^{\sim} \to a = a^{\sim} \rightsquigarrow a = a$.

Proof. (1) We have $1 = a \vee a^{-} = (a \rightsquigarrow a^{-}) \to a^{-} = (a \to a^{-}) \rightsquigarrow a^{-}$ by (C4). Hence, $a \rightsquigarrow a^{-} \leq a^{-}$ and $a \to a^{-} \leq a^{-}$. Since $a^{-} \leq a \rightsquigarrow a^{-}$ and $a^{-} \leq a \to a^{-}$, then $a \to a^{-} = a \rightsquigarrow a^{-} = a^{-}$. By Proposition 3.2 part (4), we have $a \to a^{\sim} = a \rightsquigarrow a^{-} = a^{-}$. Similarly, we can prove $a \to a^{\sim} = a \rightsquigarrow a^{\sim} = a^{\sim}$.

(2) It follows from (1) and Proposition 3.2 part (3). □

Proposition 4.4. *Let* $(A, \oplus, \boxplus, ^{-}, ^{\sim}, 0, 1)$ *be a pseudo commutative double basic algebra,* $a \in B(A)$ *and* $x, y \in A$. *Then the following hold:*

(1) $a \oplus x = x \oplus a = a \vee x = a \boxplus x = x \boxplus a$;

(2) $a \odot x = x \odot a = a \wedge x = a \boxdot x = x \boxdot a$;

(3) $a \vee (x \odot y) = (a \vee x) \odot (a \vee y)$, $a \vee (x \boxdot y) = (a \vee x) \boxdot (a \vee y)$;

(4) $a \wedge (x \oplus y) = (a \wedge x) \oplus (a \wedge y)$, $a \wedge (x \boxplus y) = (a \wedge x) \boxplus (a \wedge y)$.

Proof. (1) Since A is pseudo commutative, we have $a \vee x \leq a \oplus x$. Using Proposition 3.2 part (13) and (C6), we get $(a \oplus x) \odot (a \vee x)^{\sim} = (a \oplus x) \odot (a^{\sim} \wedge x^{\sim}) = ((a \oplus x) \odot a^{\sim}) \wedge ((a \oplus x) \odot x^{\sim}) = ((a \oplus x) \odot a^{\sim}) \wedge (a \wedge x^{\sim}) \leq (a^{-} \wedge a) = 0$. Hence $(a \oplus x)^{-} \boxplus (a \vee x) = 1$ implies that $a \oplus x \leq a \vee x$. Since A is pseudo commutative, then $a \oplus x = a \vee x = x \oplus a$. Similarly, we can prove $a \vee x = a \boxplus x = x \boxplus a$.

(2) It follows from part (1) and the definitions of $\odot$ and $\boxdot$.

(3) By Proposition 3.2 part (7), Proposition 4.2 part (7) and Proposition 3.1 part (6), we have $(a \vee x) \odot (a \vee y) = (a \odot (a \vee y)) \vee (x \odot (a \vee y)) = (a \odot a) \vee (a \odot y) \vee (x \odot a) \vee (x \odot y) = a \vee (x \odot y)$.

(4) Using Proposition 2.2 part (12) Proposition 4.2 part (3) and Proposition 2.2 part (6), we obtain

$$\begin{aligned}(a \wedge x) \oplus (a \wedge y) =& (a \oplus (a \wedge y)) \vee (x \oplus (a \wedge y)) \\ =& (a \oplus a) \wedge (a \oplus y) \wedge (x \oplus a) \wedge (x \oplus y) \\ =& a \wedge (x \oplus y).\end{aligned}$$ □

Proposition 4.5. *Let* $(A, \oplus, \boxplus, ^{-}, ^{\sim}, 0, 1)$ *be a pseudo commutative double basic algebra. Then* $B(A)$ *is a boolean subalgebra of* A.

Proof. Clearly $0, 1 \in B(A)$. Let $a, b \in B(A)$. By Proposition 4.1, we have $a \wedge a^- = 0$, $a \vee a^- = 1$, $b \wedge b^- = 0$, $b \vee b^- = 1$. We will prove that $a^- \vee b^-$ is the complement of $a \wedge b$ and

$$\begin{aligned}(a \wedge b) \wedge (a^- \vee b^-) =&((a \wedge b) \wedge a^-) \vee ((a \wedge b) \wedge b^-) = 0 \vee 0 = 0,\\ (a \wedge b) \vee (a^- \vee b^-) =&(a \vee (a^- \vee b^-)) \wedge (b \vee (a^- \vee b^-)) = 1 \wedge 1 = 1.\end{aligned}$$

Similarly, we can show that $a^- \vee b^-$ is the complement of a $a \wedge b$. Hence $a \wedge b, a \vee b \in B(A)$ and $(B(A), \wedge, \vee)$ is a distributive lattice. By Proposition 4.4 part (1), we have $a \oplus b = a \vee b \in B(A)$ and $a \boxplus b = a \vee b \in B(A)$. It is clear that $a^-, a^\sim \in B(A)$. Hence $B(A)$ is closed under the operations $\oplus$, $\boxplus$, $^-$ and $^\sim$. $\square$

Corollary 4.1. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra. Then $(B(A), \wedge, \vee, ^-)$ is a De Morgan lattice.*

Proposition 4.6. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra and $a \in B(A)$. Then $([0, a], \oplus_0^a, \boxplus_0^a, ^{-_0^a}, ^{\sim_0^a}, 0, a)$ is a pseudo commutative double basic algebra, where*

$$\begin{aligned}x^{-_0^a} :=&x^- \odot a, \quad x^{\sim_0^a} := x^\sim \boxdot a,\\ x \oplus_0^a y :=&x \oplus y, \quad x \boxplus_0^a y := x \boxplus y.\end{aligned}$$

Proof. Let $x, y \in [0, a]$ be arbitrary. Using pseudo commutativity, Proposition 2.2 part (5) and Proposition 4.2 part (3) and part (4), we have $0 \leq x \oplus y \leq a \oplus a = a$, $0 \leq x \boxplus y \leq a \boxplus a = a$. Thus $x \oplus y, x \boxplus y \in [0, a]$. By Proposition 3.1 part (6), we obtain $x^{-_0^a} := x^- \odot a \leq a$ and $x^{\sim_0^a} := x^\sim \boxdot a \leq a$. So, $x^{-_0^a}, x^{\sim_0^a} \in [0, a]$. We will check condition (P2) from the definition of a double basic algebra.

Applying Corollary 3.1, Proposition 4.4 part (1), Proposition 3.2 part (7) and Proposition 4.2, we get

$$\begin{aligned}(x^{-_0^a})^{\sim_0^a} =&(x^- \odot a)^\sim \boxdot a = (x \oplus a^\sim) \boxdot a = a \odot (x \vee a^\sim) = (a \odot x) \vee (a \odot a^\sim)\\ =&(a \wedge x) \vee (a \wedge a^\sim) = x \vee 0 = x,\\ (x^{\sim_0^a})^{-_0^a} =&(x^\sim \boxdot a)^- \odot a = (x \boxplus a^-) \odot a = a \boxdot (x \vee a^-) = (a \boxdot x) \vee (a \boxdot a^-)\\ =&(a \wedge x) \vee (a \wedge a^-) = x \vee 0 = x.\end{aligned}$$

It is easy to prove the other identities. $\square$

Proposition 4.7. *Let $(A, \oplus, \boxplus, ^-, ^\sim, 0, 1)$ be a pseudo commutative double basic algebra with $a, b \in B(A)$ and $a \leq b$. Then $([a, b], \oplus_a^b, \boxplus_a^b, ^{-_a^b}, ^{\sim_a^b}, a, b)$ is a pseudo commutative double basic algebra, where*

$$\begin{aligned}x^{-_a^b} :=&(x^- \odot b) \boxplus a, \quad x^{\sim_a^b} := (x^\sim \boxdot b) \oplus a,\\ x \oplus_a^b y :=&((x^- \boxplus a)^\sim \oplus y) \wedge b, \quad x \boxplus_a^b y := ((x^\sim \oplus a)^- \boxplus y) \wedge b.\end{aligned}$$

Proof. It could be easily proven by Proposition 4.6 and Theorem 4 in [12]. $\square$

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RETRACTED PAPER: WARPED PRODUCT POINTWISE SEMI-SLANT SUBMANIFOLDS

ION MIHAI[1], SIRAJ UDDIN[2], AND ADELA MIHAI[3]

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The following article has been retracted from publication in Kragujevac Journal of Matematics by agreement between the authors and the journal Editor-in-Chief.

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During the publishing process of the above manuscript some concerns of the validity of certain conjectures were raised. For this reason, the manuscript in its current form has been retracted and will be republished after corrections are made to resolve the concerns raised.

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