Kragujevac Journal of Mathematics
Volume 45(6) (2021), Pages 837–858.

ON DISTANCE SIGNLESS LAPLACIAN ESTRADA INDEX AND ENERGY OF GRAPHS

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Abstract. For a connected graph G, the distance signless Laplacian matrix is defined as $D^Q(G) = \mathrm{Tr}(G) + D(G)$, where $D(G)$ is the distance matrix of G and $\mathrm{Tr}(G)$ is the diagonal matrix of vertex transmissions of G. The eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ of $D^Q(G)$ are the distance signless Laplacian eigenvalues of the graph G. In this paper, we define the distance signless Laplacian Estrada index of the graph G as $D^Q_E E(G) = \sum_{i=1}^n e^{\left(\rho_i - \frac{2\sigma(G)}{n}\right)}$, where $\sigma(G)$ is the transmission of a graph G. We obtain upper and lower bounds for $D^Q_E E(G)$ and the distance signless Laplacian energy in terms of other graph invariants. Moreover, we derive some relations between $D^Q_E E(G)$ and the distance signless Laplacian energy of G.

1. Introduction and preliminaries

All graphs throughout this paper are finite, undirected, simple and connected. Let G be such a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G)$. The *order* of G is the number $n = |V(G)|$ and its *size* is the number $m = |E(G)|$. The set of vertices adjacent to $v \in V(G)$, denoted by $N(v)$, refers to the *neighborhood* of v. The *degree* of vertex v, denoted by $d_G(v)$ (we simply write d_v if it is clear from the context) means the cardinality of $N(v)$. A graph is called *regular* if each of its vertex has the same degree. We write $G \cong H$, where the graphs G and H are isomorphic. The *distance* between two vertices $u, v \in V(G)$, denoted by d_{uv}, is defined as the length of a shortest path between u and v in G. The *diameter* of G is the maximum distance between any two vertices of G. The *distance matrix* of G is denoted by $D(G)$ and is defined as $D(G) = (d_{uv})_{u,v \in V(G)}$. The *transmission* $\mathrm{Tr}_G(v)$ of a vertex

Key words and phrases. Distance signless Laplacian matrix, distance signless Laplacian Estrada index, distance Estrada index, transmission regular graph, distance signless Laplacian energy.
2010 *Mathematics Subject Classification.* Primary: 05C35. Secondary: 05C50, 15A18.
DOI 10.46793/KgJMat2106.837A
Received: January 07, 2019.
Accepted: June 04, 2019.

v is defined to be the sum of the distances from v to all other vertices in G, that is, $\mathrm{Tr}_G(v) = \sum_{u \in V(G)} d_{uv}$. A graph G is said to be *k-transmission regular* if $\mathrm{Tr}_G(v) = k$, for each $v \in V(G)$. The *transmission* of a graph G, denoted by $\sigma(G)$, is the sum of distances between all unordered pairs of vertices in G. For other undefined notations and terminology, the readers are referred to [33].

For a graph G with $V(G) = \{v_1, v_2, \ldots, v_n\}$, $\mathrm{Tr}_G(v_i)$ has been referred to as the *transmission degree* Tr_i [26] and hence the *transmission degree sequence* is given by $\{\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n\}$. Let $\mathrm{Tr}(G) = \mathrm{diag}(\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n)$ be the diagonal matrix of vertex transmissions of G. Aouchiche and Hansen [2, 3] introduced the Laplacian and the signless Laplacian for the distance matrix of a connected graph. The matrix $D^L(G) = \mathrm{Tr}(G) - D(G)$ is called the *distance Laplacian matrix* of G, while the matrix $D^Q(G) = \mathrm{Tr}(G) + D(G)$ is called the *distance signless Laplacian matrix of* G. If G is connected, then $D^Q(G)$ is symmetric, nonnegative and irreducible. Hence, all the eigenvalues of $D^Q(G)$ can be arranged as $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$, where ρ_1 is called the *distance signless Laplacian spectral radius* of G. (From now onwards, we will denote $\rho_1(G)$ by $\rho(G)$).

Based on investigations on geometric properties of biomolecules, Ernesto Estrada [13, 14] considered an expression of the form

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of the adjacency matrix of a molecular graph G. The mathematical significance of this quantity was recognized short time later [22] and soon it became known under the name "Estrada index" [10]. The mathematical properties of the Estrada index have been intensively studied, see for example, [5,10,23]. There exists a vast literature related to Estrada index and its bounds and we refer the reader to the nice surveys [11, 21].

This graph-spectrum-based invariant has also an important role in chemistry, physics, and complex networks. For example, it has been used to measure the degree of folding of long chain polymeric molecules, including proteins [12, 13, 16]. It has found a number of applications in complex networks and characterizes the centrality [14], also serves as an insightful measure for investigating robustness of complex networks [39], for which EE has an acute discrimination on connectivity and changes monotonically with respect to the removal or addition of edges. For the application of the Estrada index in network theory see the book [15] and the papers [38, 39].

The pioneering papers [13,14] further proposes the study of graphs with an analogue of the Estrada index defined with respect to other (than adjacency) matrices. Because of the evident success of the graph Estrada index, this proposal has been put into effect and Estrada index based of the eigenvalues of other graph matrices have, one-by-one, been introduced: Estrada index based invariant with respect to distance matrix, as well as Estrada index based invariant with respect to Laplacian matrix, have been introduced and studied, see for example [6, 7, 24, 25, 27, 35–37, 40, 41]. Recently, in full analogy with the Estrada index, the signless Laplacian Estrada index of a connected graph G has been introduced and studied [4]. Further, in full analogy with the Estrada index, the distance Estrada index of a connected graph G has been introduced in [19]

$$DEE(G) = \sum_{i=1}^{n} e^{\mu_i},$$

where $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of the distance matrix of a graph G. Now, we define the distance signless Laplacian Estrada index $D_E^Q E(G)$, based on distance signless Laplacian matrix of the graph G as

$$D_E^Q E(G) = \sum_{i=1}^{n} e^{\left(\rho_i - \frac{2\sigma(G)}{n}\right)}, \tag{1.1}$$

where $\rho_1, \rho_2, \ldots, \rho_n$ are the distance signlees Laplacian eigenvalues of a graph G. Let

$$M_k = \sum_{i=1}^{n} \left(\rho_i - \frac{2\sigma(G)}{n}\right)^k.$$

Then

$$\begin{aligned} M_0 &= n, \\ M_1 &= 0, \\ M_2 &= 2 \sum_{1 \le i < j \le n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}. \end{aligned} \tag{1.2}$$

Recalling the power series expansion of e^x, we can write the distance signless Laplacian Estrada index as

$$D_E^Q E(G) = \sum_{k \ge 0} \frac{M_k}{k!}. \tag{1.3}$$

The rest of the paper is organized as follows. In Section 2, we obtain some upper and lower bounds for the distance signless Laplacian Estrada index $D_E^Q E(G)$ involving different graph invariants, and also characterize the extremal graphs. In Section 3, we compute the distance signless Laplacian Estrada index of some classes of graphs, as well as giving some relations with the earlier distance Estrada index. Finally, in Section 4, we derive some relations between the distance signless Laplacian Estrada index and the distance signless Laplacian energy of G.

2. Bounds for the Distance Signless Laplacian Estrada Index

We start by giving some previously known results that will be needed in the proofs of our results in the sequel.

Lemma 2.1. ([1, Theorem 2.2]). *If the transmission degree sequence of G is $\{\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n\}$, then*

$$\rho(G) \geq 2\sqrt{\frac{\sum_{i=1}^{n} \mathrm{Tr}_i^2}{n}},$$

with equality if and only if G is transmission regular.

Lemma 2.2. ([42, Lemma 2.2]). *If G is a connected graph of order n, then*

$$\rho(G) \geq \frac{4\sigma(G)}{n},$$

with equality if and only if G is transmission regular.

The following lemma will be helpful in the sequel. Its proof is similar to [28, Lemma 2], and hence is excluded.

Lemma 2.3. *A connected graph G has two distinct distance signless Laplacian eigenvalues if and only if G is a complete graph.*

For non-increasing real sequences $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ of length n, we say that (x) is majorized by (y) or (y) majorizes (x), denoted by $(x) \preceq (y)$ if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{and} \quad \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i, \quad \text{for all } k = 1, 2, \ldots, n-1.$$

The following observation can be found in [32].

Lemma 2.4 ([32])**.** *Let $(x) = (x_1, x_2, \ldots, x_n)$ and $(y) = (y_1, y_2, \ldots, y_n)$ be nonincreasing sequences of real numbers of length n. If $(x) \preceq (y)$, then for any convex function ψ, we have $\sum_{i=1}^{n} \psi(x_i) \leq \sum_{i=1}^{n} \psi(y_i)$. Furthermore, if $(x) \prec (y)$ and ψ is strictly convex, then $\sum_{i=1}^{n} \psi(x_i) < \sum_{i=1}^{n} \psi(y_i)$.*

Lemma 2.5 ([34])**.** *Let G be a connected graph of order n having distance signless Laplacian eigenvalues $\rho_1, \rho_2, \ldots, \rho_n$ and transmission degrees $\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n$. Then*

$$(\mathrm{Tr}_1, \mathrm{Tr}_2, \ldots, \mathrm{Tr}_n) \preceq (\rho_1, \rho_2, \ldots, \rho_n).$$

Now, we present some upper bounds for the distance signless Laplacian Estrada index involving different graph invariants.

Theorem 2.1. *Let G be a connected graph of order n. Then, for any integer $k_0 \geq 2$,*

$$\begin{aligned} D_E^Q E(G) \leq & n - 1 - \sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}} \\ & + \sum_{k=2}^{k_0} \frac{M_k(G) - \left(\sqrt{2 \sum_{1 \leq i < j \leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}} \right)^k}{k!} \end{aligned}$$

$$+\,e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}, \tag{2.1}$$

with equality if and only if $G = K_1$.

Proof. We have

$$\begin{aligned}
D_E^Q E(G) &= \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\geq k_0+1}\frac{1}{k!}\sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k\\
&\leq \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\geq k_0+1}\frac{1}{k!}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k\\
&\leq \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\geq k_0+1}\frac{1}{k!}\left(\sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^2\right)^{\frac{k}{2}}\\
&= \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+\sum_{k\geq k_0+1}\frac{\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k}{k!}\\
&= \sum_{k=0}^{k_0}\frac{M_k(G)}{k!}+e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}\\
&\quad -\sum_{k=0}^{k_0}\frac{\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k}{k!},
\end{aligned}$$

and (2.1) follows. From the derivation of (2.1), it is evident that equality will be attained in (2.1) if and only if G has no non-zero eigenvalues, i.e., $G = K_1$. ☐

Remark 2.1. Since

$$\begin{aligned}
M_k(G) &= \sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k\\
&\leq \sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k \leq \left(\sum_{i=1}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^2\right)^{\frac{k}{2}} = (M_2(G))^{\frac{k}{2}}.
\end{aligned}$$

In the second inequality above, we use the following inequality: For nonnegative $a_1, a_2, \ldots, a_n$ and integer $k \geq 2$

$$\sum_{i=1}^{n}a_i^k \leq \left(\sum_{i=1}^{n}a_i^2\right)^{\frac{k}{2}}. \tag{2.2}$$

Hence, $M_k(G)-\left(\sqrt{M_2(G)}\right)^k \leq 0$. Then

$$\sum_{k=2}^{k_0}\frac{M_k(G)-\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k}{k!}\leq 0.$$

Therefore, we have the following observation from Theorem 2.1,

$$D_E^Q E(G) \leq n - 1 - \sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}} + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}}.$$

Theorem 2.2. *Let G be a connected graph of order n. Then for any integer $k_0 \geq 2$*

$$\begin{aligned} D_E^Q E(G) \leq & n - 2 - \rho_1 + \frac{2\sigma(G)}{n} - \sqrt{\xi} \\ & + \sum_{k=2}^{k_0}\frac{M_k(G) - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^k - \left(\sqrt{\xi}\right)^k}{k!} + e^{\rho_1 - \frac{2\sigma(G)}{n}} + e^{\sqrt{\xi}}, \end{aligned} \tag{2.3}$$

where $\xi = 2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^2$, with equality if and only if $G = K_1$.

Proof. We have

$$\begin{aligned} & D_E^Q E(G) - e^{\rho_1 - \frac{2\sigma(G)}{n}} \\ = & \sum_{k=0}^{k_0}\frac{M_k(G) - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^k}{k!} + \sum_{k\geq k_0+1}\frac{1}{k!}\sum_{i=2}^{n}\left(\rho_i - \frac{2\sigma(G)}{n}\right)^k \\ \leq & \sum_{k=0}^{k_0}\frac{M_k(G) - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^k}{k!} + \sum_{k\geq k_0+1}\frac{1}{k!}\sum_{i=2}^{n}\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k \\ \leq & \sum_{k=0}^{k_0}\frac{M_k(G) - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^k}{k!} + \sum_{k\geq k_0+1}\frac{1}{k!}\left(\sum_{i=2}^{n}\left(\rho_i - \frac{2\sigma(G)}{n}\right)^2\right)^{\frac{k}{2}} \\ = & \sum_{k=0}^{k_0}\frac{M_k(G) - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^k}{k!} \\ & + \sum_{k\geq k_0+1}\frac{\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!} \\ = & \sum_{k=0}^{k_0}\frac{M_k(G) - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^k}{k!} \\ & + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} - \left(\rho_1 - \frac{2\sigma(G)}{n}\right)^2}} \end{aligned}$$

$$-\sum_{k=0}^{k_0}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!},$$

where the first inequality follows from inequality:

$$\sum_{i=2}^{n}\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k\le\sum_{i=2}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|^k.$$

Also, in the second inequality, we use the inequality (2.2). Further, bearing in mind the power-series expansion of $e^x=\sum_{k\ge 0}\frac{x^k}{k!}$, we have

$$\begin{aligned}&e^{\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}}\\ &=\sum_{k=0}^{k_0}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!}\\ &+\sum_{k\ge k_0+1}\frac{\left(\sqrt{2\sum_{1\le i<j\le n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}-\left(\rho_1-\frac{2\sigma(G)}{n}\right)^2}\right)^k}{k!}.\end{aligned}$$

Hence, the last equality holds. Then the result follows. □

Theorem 2.3. *Let G be a connected graph of order n and diameter d. Then*

$$\frac{1}{2}\sqrt{2n(n^2+4n-3)}\le D_E^QE(G)\le n-1+e^{\sqrt{n(n-1)\left(d^2+\frac{n^2(n-1)}{4}-n+1\right)}}. \tag{2.4}$$

Equality holds on both sides of (2.4) *if and only if $G\cong K_1$.*

Proof. **Lower bound.** From (1.1), we get

$$D_E^QE^2(G)=\sum_{i=1}^{n}e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)}+2\sum_{i<j}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}e^{\left(\rho_j-\frac{2\sigma(G)}{n}\right)}. \tag{2.5}$$

By the arithmetic-geometric mean inequality, we get

$$\begin{aligned}2\sum_{i<j}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}e^{\left(\rho_j-\frac{2\sigma(G)}{n}\right)}&\ge n(n-1)\left(\prod_{i<j}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}e^{\left(\rho_j-\frac{2\sigma(G)}{n}\right)}\right)^{\frac{2}{n(n-1)}} \qquad (2.6)\\ &=n(n-1)\left[\left(\prod_{i=1}^{n}e^{\left(\rho_i-\frac{2\sigma(G)}{n}\right)}\right)^{n-1}\right]^{\frac{2}{n(n-1)}}\\ &=n(n-1)(e^{M_1})^{\frac{2}{n}}\\ &=n(n-1). \qquad (2.7)\end{aligned}$$

By means of a power-series expansion and $M_0 = n, M_1 = 0$ and

$$M_2 = 2\sum_{1\le i<j\le n}(d_{ij})^2 + \sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n},$$

we get

$$\begin{aligned}\sum_{i=1}^{n} e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)} =&\sum_{i=1}^{n}\sum_{k\ge 0}\frac{\left[2\left(\rho_i-\frac{2\sigma(G)}{n}\right)\right]^k}{k!}\\ =&n+4\sum_{1\le i<j\le n}(d_{ij})^2 + 2\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{8\sigma^2(G)}{n} + \sum_{i=1}^{n}\sum_{k\ge 3}\frac{\left[2\left(\rho_i-\frac{2\sigma(G)}{n}\right)\right]^k}{k!}.\end{aligned}$$

We use a multiplier $r\in[0,4]$ to arrive at

$$\begin{aligned}\sum_{i=1}^{n} e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)} \ge&n+4\sum_{1\le i<j\le n}(d_{ij})^2 + 2\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{8\sigma^2(G)}{n} + r\sum_{i=1}^{n}\sum_{k\ge 3}\frac{\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k}{k!}\\ =&n+4\sum_{1\le i<j\le n}(d_{ij})^2 + 2\sum_{i=1}^{n}\mathrm{Tr}_i^2 - \frac{8\sigma^2(G)}{n} - rn - r\sum_{1\le i<j\le n}(d_{ij})^2\\ &-\frac{r}{2}\sum_{i=1}^{n}\mathrm{Tr}_i^2 + \frac{2\sigma^2(G)}{n} + rD_E^QE(G)\\ =&(1-r)n - \frac{6\sigma^2(G)}{n} + (4-r)\sum_{1\le i<j\le n}(d_{ij})^2 + (2-\frac{r}{2})\sum_{i=1}^{n}\mathrm{Tr}_i^2\\ &+rD_E^QE(G),\end{aligned}$$

where from (1.3), we get

$$\begin{aligned}rD_E^QE(G) =&r\sum_{i=1}^{n}\sum_{k\ge 0}\frac{\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k}{k!} = rn + r\sum_{1\le i<j\le n}(d_{ij})^2 + \frac{r}{2}\sum_{i=1}^{n}\mathrm{Tr}_i^2\\ &-\frac{2\sigma^2(G)}{n} + r\sum_{i=1}^{n}\sum_{k\ge 3}\frac{\left(\rho_i-\frac{2\sigma(G)}{n}\right)^k}{k!},\end{aligned}$$

and hence the last but one equality follows.

Since $\sum_{1\le i<j\le n}(d_{ij})^2 \ge \frac{n(n-1)}{2}$ and $\sum_{i=1}^{n}\mathrm{Tr}_i^2 \ge n(n-1)^2$, also by Cauchy-Schwartz inequality we have $(2\sigma(G))^2 = (\sum_{i=1}^{n}\mathrm{Tr}_i)^2 \le n\sum_{i=1}^{n}\mathrm{Tr}_i^2$, and then, for $r\le 1$, we obtain

(2.8)
$$\sum_{i=1}^{n} e^{2\left(\rho_i-\frac{2\sigma(G)}{n}\right)} \ge (1-r)n + (4-r)\frac{n(n-1)}{2} + \frac{1-r}{2}\left(n(n-1)^2\right) + rD_E^QE(G).$$

By substituting (2.7) and (2.8) in (2.5), and solving for $D_E^Q E(G)$, we get

$$D_E^Q E(G) \geq \frac{1}{2}\left(r + \sqrt{r^2 - 2n(2r+3) + 2n^2(r+4) + 2n^3(1-r)}\right).$$

It is easy to see that for $n \geq 2$ the function

$$f(x) := \frac{1}{2}\left(x + \sqrt{x^2 - 2n(2x+3) + 2n^2(x+4) + 2n^3(1-x)}\right)$$

monotonically decreases in the interval $[0, 1]$. As a result, the best bound for $D_E^Q E(G)$ is attained for $r = 0$. This gives us the first part of the proof.

Upper bound. We have

$$\begin{aligned}
D_E^Q E(G) =& n + \sum_{i=1}^{n} \sum_{k\geq 1} \frac{\left(\rho_i - \frac{2\sigma(G)}{n}\right)^k}{k!} \\
\leq& n + \sum_{i=1}^{n} \sum_{k\geq 1} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!} \\
=& n + \sum_{k\geq 1} \frac{1}{k!} \sum_{i=1}^{n} \left[\left(\rho_i - \frac{2\sigma(G)}{n}\right)^2\right]^{\frac{k}{2}} \\
\leq& n + \sum_{k\geq 1} \frac{1}{k!} \left[\sum_{i=1}^{n} \left(\rho_i - \frac{2\sigma(G)}{n}\right)^2\right]^{\frac{k}{2}} \qquad (2.9)\\
=& n + \sum_{k\geq 1} \frac{1}{k!} \left[2 \sum_{1\leq i<j\leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}\right]^{\frac{k}{2}} \\
=& n - 1 + \sum_{k\geq 0} \frac{\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}\right)^k}{k!} \\
=& n - 1 + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}}.
\end{aligned}$$

Since $d_{ij} \leq d$ for $i \neq j$ and there are $\frac{n(n-1)}{2}$ pairs of vertices in G, we have $2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} \leq 2\frac{n(n-1)}{2}d^2 + \frac{n^3(n-1)^2}{4} - n(n-1)^2$, so that

$$D_E^Q E(G) \leq n - 1 + e^{\sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)}}.$$

Hence, we get the right-hand side of the (2.4).

Now, suppose that the equality in (2.4) holds, then all the inequalities in the above argument must hold as equalities. In particular, from (2.6), we get $\rho_1 = \rho_2 = \cdots = \rho_n = \frac{2\sigma(G)}{n}$ (see [17]). Since, by Lemma 2.2, $\rho_1 \geq \frac{4\sigma(G)}{n}$, a contradiction. Thus, the left- hand side equality in (2.4) holds if and only if G is an empty graph. Since G is a connected graph, this only happens in the case of $G \cong K_1$, then the graph G has all zero D^Q-eigenvalues. Again, let the right-hand side equality in (2.4) holds, then from (2.9), we get $\rho_1 = \rho_2 = \cdots = \rho_n = \frac{2\sigma(G)}{n}$. Similarly, we get $G \cong K_1$ and the proof is complete. $\square$

Now, we turn our attention to giving some lower bounds for the distance signless Laplacian Estrada index in terms of other graph invariants.

Theorem 2.4. *Let G be a connected graph of order n. Then*

$$D_E^Q E(G) \geq e^{\frac{2\sigma(G)}{n}} + (n-1)e^{\frac{-2\sigma(G)}{n(n-1)}}, \tag{2.10}$$

with equality if and only if $G = K_n$.

Proof. Starting with (1.1) and using the arithmetic-geometric mean inequality, we get

$$\begin{aligned} D_E^Q E(G) =& e^{\rho_1 - \frac{2\sigma(G)}{n}} + e^{\rho_2 - \frac{2\sigma(G)}{n}} + \cdots + e^{\rho_n - \frac{2\sigma(G)}{n}} \\ \geq& e^{\rho_1 - \frac{2\sigma(G)}{n}} + (n-1)\left(\prod_{i=2}^{n} e^{\rho_i - \frac{2\sigma(G)}{n}}\right)^{\frac{1}{n-1}} \qquad (2.11) \\ =& e^{\rho_1 - \frac{2\sigma(G)}{n}} + (n-1)\left(e^{\frac{2\sigma(G)}{n} - \rho_1}\right)^{\frac{1}{n-1}}. \qquad (2.12) \end{aligned}$$

Consider the following function

$$f(x) = e^x + (n-1)e^{\frac{-x}{n-1}}, \tag{2.13}$$

for $x \geq 0$. We have

$$f'(x) = e^x - e^{\frac{-x}{n-1}} \geq 0,$$

for $x \geq 0$. It is easy to see that $f(x)$ is an increasing function for $x \geq 0$. From (2.12) and Lemma 2.2, we obtain

$$D_E^Q E(G) \geq e^{\frac{2\sigma(G)}{n}} + (n-1)e^{\frac{-2\sigma(G)}{n(n-1)}}. \tag{2.14}$$

This completes the first part of the proof. Now, we suppose that the equality holds in (2.10). Then all inequalities in the above argument must be equalities. From (2.14), we have $\rho_1 = \frac{4\sigma(G)}{n}$, which implies that G is a transmission regular graph. From (2.11) and the arithmetic-geometric mean inequality, we get $\rho_2 = \rho_3 = \cdots = \rho_n$. Therefore, G has exactly two distinct distance signless Laplacian eigenvalues, and then by Lemma 2.3, G is the complete graph K_n.

Conversely, one can easily see that the equality holds in (2.10) for the complete graph K_n. This completes the proof. $\square$

Remark 2.2. For a graph G of order $n \geq 2$ and size m, it was shown in [43] that

$$EE(G) \geq e^{\frac{2m}{n}} + (n-1)e^{-\frac{2m}{n(n-1)}}, \tag{2.15}$$

with equality if and only if G is the empty graph or the complete graph. Since $\sigma(G) \geq \binom{n}{2} \geq m$ and the function $f(x)$ defined in (2.13) is increasing function, hence our given lower bound for distance signless Laplacian Estrada index in (2.10) is larger than the above lower bound in (2.15) for usual Estrada index. If G is the complete graph K_n, then $\sigma(G) = \binom{n}{2} = m$ and therefore the bounds coincide.

Let $M(G) = \left(\prod_{i=1}^{n} \mathrm{Tr}_i\right)^{\frac{1}{n}}$ be the geometric mean of the transmission degrees sequence. Then $\frac{2\sigma(G)}{n} \geq M(G)$ holds, and equality is attained if and only if $\mathrm{Tr}_1 = \cdots = \mathrm{Tr}_n$ (i.e., the graph G is transmission regular).

Lemma 2.6 ([44])**.** *Let $a_1, a_2, \ldots, a_n$ be non-negative numbers. Then*

$$n\left[\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{\frac{1}{n}}\right] \leq n\sum_{i=1}^{n} a_i - \left(\sum_{i=1}^{n} a_i^{\frac{1}{2}}\right)^2.$$

Theorem 2.5. *Let G be a connected graph of order $n \geq 2$. Then*

$$D_E^Q E(G) \geq e^{2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}}-\frac{2\sigma(G)}{n}} + (n-1)\left(e^{\frac{2\sigma(G)}{n}-\left(2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}}\right)}\right)^{\frac{1}{n-1}}, \tag{2.16}$$

with equality if and only if $G = K_n$.

Proof. Using the arithmetic-geometric mean inequality, we obtain

$$\begin{aligned} D_E^Q E(G) =& e^{\rho_1-\frac{2\sigma(G)}{n}} + e^{\rho_2-\frac{2\sigma(G)}{n}} + \cdots + e^{\rho_n-\frac{2\sigma(G)}{n}} \\ \geq& e^{\rho_1-\frac{2\sigma(G)}{n}} + (n-1)\left(\prod_{i=2}^{n} e^{\rho_i-\frac{2\sigma(G)}{n}}\right)^{\frac{1}{n-1}} \\ =& e^{\rho_1-\frac{2\sigma(G)}{n}} + (n-1)\left(e^{\frac{2\sigma(G)}{n}-\rho_1}\right)^{\frac{1}{n-1}}. \end{aligned} \tag{2.17}$$

By Lemma 2.1, $\rho_1 \geq 2\sqrt{\frac{\sum_{i=1}^{n} \mathrm{Tr}_i^2}{n}}$. Setting $\sqrt{a_i} = \mathrm{Tr}_i$ in Lemma 2.6, we get

$$n^2\left[\frac{\sum_{i=1}^{n} \mathrm{Tr}_i^2}{n} - \left(\frac{2\sigma(G)}{n}\right)^2\right] \geq \sum_{i=1}^{n} \mathrm{Tr}_i^2 - n\left(\prod_{i=1}^{n} \mathrm{Tr}_i^2\right)^{\frac{1}{n}}.$$

Combining this with Lemma 2.1, yields

$$\rho_1 \geq 2\sqrt{\frac{4\sigma^2(G) - M^2(G)n}{n(n-1)}}. \tag{2.18}$$

It is easy to see that $2\sqrt{\frac{4\sigma^2(G)-M^2(G)n}{n(n-1)}} \geq \frac{4\sigma(G)}{n}$, and so,

$$2\sqrt{\frac{4\sigma^2(G) - M^2(G)n}{n(n-1)}} - \frac{2\sigma(G)}{n} \geq \frac{2\sigma(G)}{n} \geq 0.$$

Similarly to Theorem 3.4, we get the result. When $G = K_n$, we have $\rho_1 = 2n-2, \rho_2 = \cdots = \rho_n = n-2$, $\sigma(G) = \frac{n(n-1)}{2}$ and $M(G) = n-1$. Hence, $D_E^Q E(G) = e^{n-1} + (n-1)e^{-1}$ and the equality holds.

Conversely, suppose that the equality holds. Then from (2.17), we have $\rho_2 = \cdots = \rho_n$. Clearly $4\sigma^2(G) = M^2(G)n$ if and only if $n = 1$. From (2.18), it follows that $\rho_1 > 0$ for $n \geq 2$. Thus G has exactly two distinct distance signless Laplacian eigenvalues, and so Lemma 2.3 implies that G is the complete graph K_n. $\square$

Let G be a k-transmission regular graph. Then $\sigma(G) = \frac{nk}{2}$ and $M(G) = k$ and hence we get the following observation.

Corollary 2.1. *Let G be a k-transmission regular graph. Then*

$$D_E^Q E(G) \geq e^k + (n-1)e^{\frac{-k}{n-1}},$$

with equality if and only if $G = K_n$.

We recall Holder inequality. Let $a_1, \dots, a_n$, $b_1, \dots, b_n$ be non-negative real numbers, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{\frac{1}{p}} \left(\sum_{i=1}^{n} b_i^q\right)^{\frac{1}{q}}.$$

Here, we give the lower bound for $D_E^Q E(G)$ in terms of n and $\sigma(G)$.

Theorem 2.6. *Let G be a connected graph of order n. Then*

$$D_E^Q E(G) > n + 2\left(\frac{\sigma(G)}{n}\right)^2.$$

Proof. By Holder inequality for $p = q = 2$, we have

$$2\sigma(G) = \sum_{i=1}^{n} \mathrm{Tr}_i \leq \sqrt{n}\left(\sum_{i=1}^{n} \mathrm{Tr}_i^2\right)^{\frac{1}{2}}.$$

Hence,

$$\sum_{i=1}^{n} \mathrm{Tr}_i^2 \geq \frac{4\sigma^2(G)}{n}. \tag{2.19}$$

Now, by Cauchy-Schwartz inequality, we have

$$\mathrm{Tr}_i^2 = \left(\sum_{j=1}^{n} d_{ij}\right)^2 \leq n\sum_{j=1}^{n} d_{ij}^2.$$

Hence,

$$\sum_{i=1}^{n} \mathrm{Tr}_i^2 \leq n\sum_{i=1}^{n}\sum_{j=1}^{n} d_{ij}^2,$$

and then by (2.19) we get

$$\sum_{1\leq i<j\leq n} d_{ij}^2 \geq \frac{1}{2n}\sum_{i=1}^{n} \mathrm{Tr}_i^2 \geq \frac{1}{2n}.\frac{4\sigma^2(G)}{n} = \frac{2\sigma^2(G)}{n^2}.$$

Thus, we have

$$\begin{aligned} D_E^Q E(G) >& n + \sum_{1\le i<j\le n} (d_{ij})^2 + \frac{1}{2}\sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{2\sigma^2(G)}{n} \\ \ge& n + \frac{2\sigma^2(G)}{n^2} + \frac{2\sigma^2(G)}{n} - \frac{2\sigma^2(G)}{n} \\ =& n + 2\left(\frac{\sigma(G)}{n}\right)^2. \end{aligned}$$

□

Corollary 2.2. *Let G be a connected graph of order n. Then*

$$D_E^Q E(G) > \frac{n^2+1}{2}.$$

Proof. Since $d_{ij} \ge 1$ for $i \ne j$ and there are $\frac{n(n-1)}{2}$ pairs of vertices in G, from the lower bound of Theorem 2.6, we get

$$D_E^Q E(G) > n + 2\left(\frac{\sigma(G)}{n}\right)^2 \ge n + 2\left(\frac{\frac{n(n-1)}{2}}{n}\right)^2 = \frac{n^2+1}{2}.$$

□

Hence, the result.

3. Distance Signless Laplacian Estrada Index of some Classes of Graphs

In this section we obtain the distance signless Laplacian Estrada index of some classes of graphs.

Lemma 3.1. *Let G be a k-transmission regular graph of order n. Then*

$$D_E^Q E(G) = DEE(G).$$

Proof. Note that the distance signless Laplacian spectrum of the graph G consists of $k+\mu_1 \ge k+\mu_2 \ge \cdots \ge k+\mu_n$, where $\mu_1 \ge \cdots \ge \mu_n$ is the distance spectrum of G. Also it is easy to see that $\sigma(G) = \frac{nk}{2}$. Then $D_E^Q E(G) = \sum_{i=1}^{n} e^{k+\mu_i-k} = DEE(G)$. □

The Cartesian product of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (u_1, u_2) and (v_1, v_2) are adjacent if and only if $u_1 = v_1$ and $u_2v_2 \in E(H)$ or $u_2 = v_2$ and $u_1v_1 \in E(G)$.

Corollary 3.1. *Let G be an r-regular graph of diameter at most 2 with an adjacency matrix A and* $\mathrm{Spec}(G) = \{r, \lambda_2, \ldots, \lambda_n\}$. *Then, the distance signless Laplacian Estrada index of $H = G \times K_2$ is*

$$D_E^Q E(H) = e^{5n-2r-4} + e^{-n} + n - 1 + \sum_{i=2}^{n} e^{-2\lambda_i-4}.$$

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$, $V(K_2) = \{w_1, w_2\}$. From the fact

$$d_H((v_i, w_j), (v_s, w_t)) = d_G(v_i, v_s) + d_{K_2}(w_j, w_t) = d_G(v_i, v_s) + 1,$$

we see that all vertices of H have the same transmission and $\mathrm{Tr}_H(v_i, w_j) = 5n - 2r - 4$. So $\mathrm{Tr}(H) = (5n - 2r - 4)I$. Then $\sigma(H) = \frac{n(5n-2r-4)}{2}$. Note that $H = G \times K_2$ has distance spectrum (see [30])

$$\mathrm{Spec}(H) = \begin{pmatrix} 5n - 2(r+2) & -2(\lambda_i + 2) & -n & 0 \\ 1 & 1 & 1 & n-1 \end{pmatrix}$$

for $i = 2, \ldots, n$. Then

$$D_E^Q E(H) = e^{5n-2r-4} + e^{-n} + n - 1 + \sum_{i=2}^{n} e^{-2\lambda_i - 4}. \qquad \square$$

Given a graph G, the graph $G\nabla G$ is obtained by joining every vertex of G to every vertex of another copy of G.

Corollary 3.2. *Let G be an $r-$regular graph with an adjacency matrix A and $\mathrm{Spec}(G) = \{r, \lambda_2, \ldots, \lambda_n\}$. Then, the distance signless Laplacian Estrada index of $G\nabla G$ is*

$$D_E^Q E(G\nabla G) = e^{3n-r-2} + e^{n-r-2} + 2\sum_{i=2}^{n} e^{-2\lambda_i - 4}.$$

Proof. For $v \in G\nabla G$, it is easy to see that $\mathrm{Tr}(v) = d(v) + 2(n - d(v) - 1) + n = 3n - d(v) - 2 = 3n - r - 2$. Then $G\nabla G$ is a transmission regular graph and $\mathrm{Tr}(G\nabla G) = (3n - r - 2)I$. Note that the $G\nabla G$ has distance spectrum (see [30])

$$\mathrm{Spec}(G\nabla G) = \begin{pmatrix} 3n - r - 2 & n - r - 2 & -2(\lambda_i + 2) \\ 1 & 1 & 2 \end{pmatrix},$$

for $i = 2, \ldots, n$. Then

$$D_E^Q E(G\nabla G) = e^{3n-r-2} + e^{n-r-2} + 2\sum_{i=2}^{n} e^{-2\lambda_i - 4}. \qquad \square$$

Next, we obtain the distance signless Laplacian Estrada index of the lexicographic product $G[H]$ of two graphs G and H. The following definition of the lexicographic product of G and H is from [9].

Definition 3.1. Let G and H be two graphs on vertex sets $V(G) = \{u_1, u_2, \ldots, u_p\}$ and $V(H) = \{v_1, v_2, \ldots, v_n\}$, respectively. Then their *lexicographic product* $G[H]$ is a graph defined by $V(G[H]) = V(G) \times V(H)$, the Cartesian product of $V(G)$ and $V(H)$ in which $u = (u_1, v_1)$ is adjacent to $v = (u_2, v_2)$ if and only if either

(a) u_1 is adjacent to v_1 in G, or
(b) $u_1 = v_1$ and u_2 is adjacent to v_2 in G.

Corollary 3.3. *Let G be a k-transmission regular graph of order p. Let H be an r-regular graph of order n with adjacency eigenvalues $\{r, \lambda_2, \dots, \lambda_n\}$. Let $\{\mu_1, \dots, \mu_p\}$ be the eigenvalues of the distance matrix $D(G)$ of G. Then*

$$D_E^Q E(G[H]) = e^{2n-r-2} \sum_{i=1}^{p} e^{n\mu_i} + ne^{-4} \sum_{j=2}^{n} e^{-2\lambda_j}.$$

Proof. For $v \in G[H]$, it is easy to see that $\mathrm{Tr}(v) = r+2(n-r-1)+kn = kn+2n-r-2$. Then $G[H]$ is a transmission regular graph and $\mathrm{Tr}(G[H]) = (kn + 2n - r - 2)I$. Note that $G[H]$ has distance spectrum (see [29])

$$\mathrm{Spec}(G[H]) = \begin{pmatrix} n\mu_i + 2n - r - 2 & -2(\lambda_j + 2) \\ \\ 1 & n \end{pmatrix},$$

for $i = 1, \dots, p$ and $j = 2, \dots, n$. Then

$$D_E^Q E(G[H]) = e^{2n-r-2} \sum_{i=1}^{p} e^{n\mu_i} + ne^{-4} \sum_{j=2}^{n} e^{-2\lambda_j}. \qquad \square$$

Theorem 3.1. *Let G be an r-regular graph of order n, size m and diameter at most 2. If $\{2r, q_2, \dots, q_n\}$ are the eigenvalues of the signless Laplacian matrix $Q(G)$ of G, then*

$$D_E^Q E(G) = e^{2(n^2-n-m)} + \sum_{i=2}^{n} e^{2m-2n-nq_i}.$$

Proof. We know that the transmission of each vertex $v \in V(G)$ is $\mathrm{Tr}(v) = d(v) + 2(n - d(v) - 1) = 2n - d(v) - 1$ and so transmission $\sigma(G)$ of G is $\sigma(G) = n^2 - n - m$. Also

$$\begin{aligned} D^Q(G) = \mathrm{Tr}(G) + D(G) =& (2n-2)I - rI + 2J - 2I - A(G) \\ =& (2n-4)I + 2J - Q(G), \end{aligned}$$

where J is the all ones matrix. Then

$$\begin{aligned} D_E^Q E(G) =& \sum_{i=1}^{n} e^{\rho_i - \frac{2\sigma(G)}{n}} = e^{(4n-2r-4) - \frac{2(n^2-n-m)}{n}} + \sum_{i=2}^{n} e^{(2n-4-q_i) - \frac{2(n^2-n-m)}{n}} \\ =& e^{2(n^2-n-m)} + \sum_{i=2}^{n} e^{2m-2n-nq_i}. \end{aligned} \qquad \square$$

As an immediate consequence of the above theorem, we get the following.

Corollary 3.4. *Let G be an r-regular graph of order n, size m and diameter at most 2. If $\{r, \lambda_2, \dots, \lambda_n\}$ are the eigenvalues of the adjacency matrix $A(G)$ of G, then*

$$D_E^Q E(G) = e^{2(n^2-n-m)} + \sum_{i=2}^{n} e^{-n(\lambda_i+2)}.$$

4. Relations Between Distance Signless Laplacian Estrada Index and Distance Signless Laplacian Energy

The energy $E(G)$ of a graph G is equal to the sum of the absolute values of the eigenvalues of the adjacency matrix of G. This quantity, introduced first time in [20] and having a clear connection to chemical problems, has now attracted much attention of mathematicians and mathematical chemists. We observe that several interesting results have been obtained for the energy of different graph structures. The pioneering paper [20] further proposes the study of energy in graphs with an analogue of the energy defined with respect to other (than adjacency) matrices assigned to the graphs. This proposal has been put into effect and extended: the energy of a graph with respect to Laplacian matrix as well as the energy of a graph with respect to distance matrix, have been studied (see [25, 30] for more details in this subject). Recently, Alhevaz et al. [1] have considered a new kind of energy with respect to the distance signless Laplacian matrix, the concept of distance signless Laplacian energy, denoted by $E_{D^Q}(G)$, and defined as

$$E_{D^Q}(G) = \sum_{i=1}^{n} \left| \rho_i - \frac{2\sigma(G)}{n} \right|.$$

In this section, we obtain some relations between $E_{D^Q}(G)$ and $D_E^Q E(G)$ for a simple connected graph G.

Theorem 4.1. *Let G be a connected graph of order n with diameter d. Then*

$$\begin{aligned} D_E^Q E(G) - E_{D^Q}(G) \leq & n - 1 - \sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)} \\ & + e^{\sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)}} \end{aligned} \tag{4.1}$$

or

$$D_E^Q E(G) \leq n - 1 + e^{E_{D^Q}(G)}. \tag{4.2}$$

Equality holds in (4.1) *or* (4.2) *if and only if* $G \cong K_1$.

Proof. From the proof of Theorem 2.3, we have

$$D_E^Q E(G) = n + \sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left(\rho_i - \frac{2\sigma(G)}{n}\right)^k}{k!} \leq n + \sum_{i=1}^{n} \sum_{k \geq 1} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!}.$$

Taking into account the definition of the distance signless Laplacian energy, we get

$$D_E^Q E(G) \leq n + E_{D^Q}(G) + \sum_{i=1}^{n} \sum_{k \geq 2} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!},$$

which, as in Theorem 2.3, leads to

$$\begin{aligned} D_E^Q E(G) - E_{D^Q}(G) \leq & n + \sum_{i=1}^{n} \sum_{k\geq 2} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!} \\ \leq & n - 1 - \sqrt{2 \sum_{1\leq i<j\leq n} (d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}} \\ & + e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n}}}. \end{aligned}$$

One can easily see that the function $f(x) = e^x - x$ monotonically increases for $x \geq 0$. Therefore, the best upper bound for $D_E^Q E(G) - E_{D^Q}(G)$ is obtained for $2\sum_{1\leq i<j\leq n}(d_{ij})^2 + \sum_{i=1}^{n} \mathrm{Tr}_i^2 - \frac{4\sigma^2(G)}{n} \leq 2\frac{n(n-1)}{2} d^2 + \frac{n^3(n-1)^2}{4} - n(n-1)^2$, and we get

$$\begin{aligned} D_E^Q E(G) - E_{D^Q}(G) \leq & n - 1 - \sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n + 1\right)} \\ & + e^{\sqrt{n(n-1)\left(d^2 + \frac{n^2(n-1)}{4} - n+1\right)}}. \end{aligned}$$

Another way to obtain the relation between $D_E^Q E(G)$ and $E_{D^Q}(G)$ is as follows:

$$\begin{aligned} D_E^Q E(G) \leq & n + \sum_{i=1}^{n} \sum_{k\geq 1} \frac{\left|\rho_i - \frac{2\sigma(G)}{n}\right|^k}{k!} \\ \leq & n + \sum_{k\geq 1} \frac{1}{k!} \left(\sum_{i=1}^{n} \left|\rho_i - \frac{2\sigma(G)}{n}\right| \right)^k \\ = & n + \sum_{k\geq 1} \frac{(E_{D^Q}(G))^k}{k!} \\ = & n - 1 + \sum_{k\geq 0} \frac{(E_{D^Q}(G))^k}{k!}, \end{aligned}$$

implying

$$D_E^Q E(G) \leq n - 1 + e^{E_{D^Q}}(G).$$

Also, equality holds in (4.1) or (4.2) if and only $G \cong K_1$. □

Lemma 4.1 ([31]). *Let $x_1, \ldots, x_n$ be positive numbers. Then*

$$\frac{n}{\frac{1}{x_1} + \ldots + \frac{1}{x_n}} \leq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Lemma 4.2 ([8]). *Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be real numbers. Then*

$$\left(\sum_{i=1}^{n} a_i\right) \cdot \left(\sum_{i=1}^{n} b_i\right) \leq n \sum_{i=1}^{n} a_i b_i.$$

Equality occurs if and only if $a_1 = \cdots = a_n$ *or* $b_1 = \cdots = b_n$.

Theorem 4.2. *Let* G *be a connected graph of order* n. *Then*

$$e^{-\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^n \mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}} \leq E_{D^Q}(G) \leq e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^n \mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}.$$

Proof. First we prove the given lower bound. By definition of the energy and by the arithmetic-geometric mean inequality, we have

$$\begin{aligned} E_{D^Q}(G) = \sum_{i=1}^n \left|\rho_i - \frac{2\sigma(G)}{n}\right| &= n\left(\frac{1}{n}\sum_{i=1}^n \left|\rho_i - \frac{2\sigma(G)}{n}\right|\right) \\ &\geq n\left(\sqrt[n]{\left|\rho_1 - \frac{2\sigma(G)}{n}\right|\left|\rho_2 - \frac{2\sigma(G)}{n}\right|\dots\left|\rho_n - \frac{2\sigma(G)}{n}\right|}\right). \end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned} & n\left(\sqrt[n]{\left|\rho_1 - \frac{2\sigma(G)}{n}\right|\left|\rho_2 - \frac{2\sigma(G)}{n}\right|\dots\left|\rho_n - \frac{2\sigma(G)}{n}\right|}\right) \geq n\left(\frac{n}{\sum_{i=1}^n \frac{1}{\left|\rho_i - \frac{2\sigma(G)}{n}\right|}}\right) \\ \geq & n\left(\frac{n}{\sum_{i=1}^n \frac{1}{\left|\rho_i - \frac{2\sigma(G)}{n}\right|}\sum_{i=1}^n \left|\rho_i - \frac{2\sigma(G)}{n}\right|}\right) \\ \geq & n\left(\frac{n}{n\sum_{i=1}^n \frac{1}{\left|\rho_i - \frac{2\sigma(G)}{n}\right|}\left|\rho_i - \frac{2\sigma(G)}{n}\right|}\right) \quad \text{(by Lemma 4.2)} \\ \geq & n\left(\frac{n}{n^2\sum_{i=1}^n \left|\rho_i - \frac{2\sigma(G)}{n}\right|}\right) > n\left(\frac{n}{n^2\sum_{i=1}^n e^{\left|\rho_i - \frac{2\sigma(G)}{n}\right|}}\right) \\ = & \frac{1}{\sum_{i=1}^n\sum_{k\geq 0}\frac{\left(\left|\rho_i - \frac{2\sigma(G)}{n}\right|\right)^k}{k!}} = \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{i=1}^n\left(\left|\rho_i - \frac{2\sigma(G)}{n}\right|\right)^k\right)} \\ \geq & \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{i=1}^n\left(\left|\rho_i - \frac{2\sigma(G)}{n}\right|\right)^2\right)^{\frac{k}{2}}} \quad \text{(by (2.2))} \\ = & \frac{1}{\sum_{k\geq 0}\frac{1}{k!}\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^n \mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^k} \quad \text{(by (1.2))}. \end{aligned}$$

Therefore, we have $E_{D^Q}(G) \geq e^{-\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^n \mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}}$.

Now, we prove the given upper bound. We have,

$$E_{D^Q}(G) = \sum_{i=1}^n \left|\rho_i - \frac{2\sigma(G)}{n}\right| < \sum_{i=1}^n e^{\left|\rho_i - \frac{2\sigma(G)}{n}\right|}$$

$$
\begin{aligned}
&=\sum_{i=1}^{n}\sum_{k\geq 0}\frac{\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k}{k!}=\sum_{k\geq 0}\frac{1}{k!}\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k\\
&\leq\sum_{k\geq 0}\frac{1}{k!}\left(\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^2\right)^{\frac{k}{2}}\quad (\text{ by inequality (2.2)})\\
&=\sum_{k\geq 0}\frac{1}{k!}\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}\right)^{\frac{k}{2}}\quad (\text{by Eq. (1.2)})\\
&=\sum_{k\geq 0}\frac{1}{k!}\left(\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}\right)^{k}\\
&=e^{\sqrt{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}},
\end{aligned}
$$

and the proof is complete. □

Theorem 4.3. *Let G be a connected graph of order n. Then*

$$
E_{D^Q}(G)\geq\frac{1}{2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}}. \tag{4.3}
$$

Proof. By definition of the energy and by the arithmetic-geometric mean inequality, we have

$$
\begin{aligned}
E_{D^Q}(G)=\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|&=n\left(\frac{1}{n}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)\\
&\geq n\left(\sqrt[n]{\left|\rho_1-\frac{2\sigma(G)}{n}\right|\left|\rho_2-\frac{2\sigma(G)}{n}\right|\dots\left|\rho_n-\frac{2\sigma(G)}{n}\right|}\right).
\end{aligned}
$$

By Lemma 4.1 and Lemma 4.2, we have

$$
\begin{aligned}
&n\left(\sqrt[n]{\left|\rho_1-\frac{2\sigma(G)}{n}\right|\left|\rho_2-\frac{2\sigma(G)}{n}\right|\dots\left|\rho_n-\frac{2\sigma(G)}{n}\right|}\right)\geq n\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}}\right)\\
&\geq n\left(\frac{n}{\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\geq n\left(\frac{n}{n\sum_{i=1}^{n}\frac{1}{\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\\
&\geq n\left(\frac{n}{n^2\sum_{i=1}^{n}\left|\rho_i-\frac{2\sigma(G)}{n}\right|}\right)\geq\frac{1}{\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^k}\\
&\geq\frac{1}{\left(\sum_{i=1}^{n}\left(\left|\rho_i-\frac{2\sigma(G)}{n}\right|\right)^2\right)^{\frac{k}{2}}}=\frac{1}{\left(2\sum_{1\leq i<j\leq n}(d_{ij})^2+\sum_{i=1}^{n}\mathrm{Tr}_i^2-\frac{4\sigma^2(G)}{n}\right)^{\frac{k}{2}}},
\end{aligned}
$$

Hence, for $k=2$, we arrive at (4.3). □

5. Conclusions

In this paper, we have defined the distance signless Laplacian Estrada index, where we have given some upper and lower bounds for $D_E^Q E(G)$ in terms of other graph invariants. Also, we have obtained the distance signless Laplacian Estrada index for some classes of graphs. Moreover, we derive some relations between $D_E^Q E(G)$ and the distance signless Laplacian energy of G. It would be interesting to give an expression for $D_E^Q E(G)$ in terms of the ordinary Estrada index in certain classes of graphs. Alternatively, one could possibly consider the range of values for $D_E^Q E(G)$ over some family of graphs of fixed order, for example, trees on n vertices.

Acknowledgements. We are very grateful to the editor and the anonymous referee for the valuable comments that have greatly improved the presentation of the paper. The research of A. Alhevaz was in part supported by a grant from Shahrood University of Technology, and the research of S. Pirzada is supported by SERB-DST, New Delhi under the research project number MTR/2017/000084.

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